

Arnold Sommerfeld Theoretical Physics

Mechanics

Mechanics of Deformable Bodies

Electrodynamics

Optics

Thermodynamics

Statistical Mechanics

Partial Differential

Equations in Physics



A. Sommerfeld.

MECHANICS

Lectures on Theoretical Physics, Vol. I

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FOREWORD TO SOMMERFELD'S COURSE

P. P. EWALD

POLYTECHNIC INSTITUTE OF BROOKLYN, NEW YORK

The author of this Course on Theoretical Physics, Arnold Sommerfeld, was one of the central figures in achieving the transformation through which physics passed in the two decades from 1910 to 1930. Without his inspired and untiring efforts both the tumultuous advance and the wide-spread dissemination of the quantum theory of the atom would not have been what they were. Sommerfeld's Institute for Theoretical Physics in Munich became a school from which issued a steady stream of research papers by German and foreign, young and mature students of atomic theory. His famous book "*Atombau und Spektrallinien*," followed later by the companion volume "*Wellenmechanik*," was for a long time the only full and authoritative account of this fundamental subject; its successive editions unroll an impressive survey of the rapid development of atomic theory following Niels Bohr's first papers.

Both by his training and previous research Sommerfeld was firmly rooted in the mathematical methods of classical physics, and to this fact he owes much of his mastery of the newly created methods of quantum physics, especially after the advent of Schrödinger's wave mechanics in 1926. It was therefore natural for Sommerfeld to give his students a thorough training in classical methods, all the more so since he himself took great delight in the aesthetic beauty of classical theory. The harmony between mathematical formalism, its physical interpretation, and experimental materialization was cast in relief in Sommerfeld's lectures and deeply impressed his students.

Sommerfeld was over seventy, and retired after forty years of academic teaching, when he committed his lectures to paper. He did it with a sense of double obligation: to preserve through a crisis the achievements that had carried physics to great triumphs and to bequeath to the young generation of physicists the valuable analytical tools that had been shaped on the classical problems. Sommerfeld had taken an active part in perfecting these tools from 1895 onwards, when he wrote his doctoral thesis on *Arbitrary Functions in Physics*. Among his earliest brilliant work was the construction of a strict solution for the diffraction of a wave by an edge; he extended the methods that Riemann used in the theory of functions, with the result that a solution of the diffraction problem by an image method in multi-valued space was obtained. The reader will find this discussed in Volume V on *Optics*. Extending from Sommerfeld's early period in Göttingen to the beginning of the quantum period in Munich, and in co-authorship with the great mathematician Felix Klein in Göttingen, the preparation of the four

volume standard work on the theory of rotating rigid bodies, *Theorie des Kreisels*, proceeded. This work was intended to demonstrate the intimate connection between "pure" and "applied" mathematics by bringing a great variety of mathematical topics, such as the theory of functions, elliptic functions, quaternions, Klein-Caley parameters, etc., to bear on this problem of dynamics of a rigid body. While holding the chair of Technical Mechanics at the Technische Hochschule in Aachen, 1899-1905, Sommerfeld became deeply interested in engineering problems. His papers on the hydrodynamics of lubrication, on the interaction between electrical generators working on the same power line, on the braking of trains, and on other topics all adopt a general approach that gives them lasting value. With the advent of wireless telegraphy a series of papers by Sommerfeld and his pupils began on the mode of emission and of propagation of radio waves. They offer excellent examples of the mathematical methods in which Sommerfeld was a master. In particular, the diffraction problem of these waves round the earth was brought down to the discussion of complex integrals, which form a strict solution (see Volume 6, Chapter 6).

It would be out of place to elaborate a full list of the achievements with which Sommerfeld enriched physical theory. The reader may be referred to some of the articles listed below. But a few words may be added on Sommerfeld as a teacher, and on the significance of the course of lectures now being published in translation.

The courses of theoretical physics held in Munich were of two kinds, general ones and specialized ones. The former were given four hours, or more precisely periods of 45-50 minutes, a week through a 13-week winter semester and an 11-week summer semester. These six courses form the subject of the present six volumes. Each served as an introduction for students who had taken the demonstration courses on experimental physics (given, in Munich, by Röntgen, later by W. Wien). In experimental physics the student acquired a factual survey of the phenomena and of their quantitative evaluation, based on a fundamentally non-mathematical treatment. In the courses on theoretical physics the elementary ground was gone over again, but with a view to developing the mathematical handling and to constructing an integrating theory which could then be extended to advanced problems. The latter might change from one series of courses to the next, and the inclusion of topical subjects in the second halves of these courses made them most interesting even to advanced students who had already gone over the subject in their previous work. In addition to the lectures, two hours a week were devoted to discussing problems.

The specialized courses were two-hour-a-week lecture courses on subjects which could be dealt with only briefly in the general course, or had topical interest. Those given by Sommerfeld were usually connected with his own current research and often contained parts which appeared a little later as

original papers. The interpretation of the Lorentz transformation as a rotation in four dimensional space (Volume 3, §26), the transition from wave optics to geometrical optics (Volume 5, §35), the discussion of the signal velocity in a dispersive medium (Volume 5, §22) are some examples. In later general courses such subjects were often included, to the exclusion of other less interesting parts of previous courses.

Besides the lecture courses, a seminar and colloquium offered instruction in advanced topics ; here the students had to review the assigned subject and to deliver the talk, which meant several weeks' intensive study.

From the student's point of view the great attraction of Sommerfeld's lectures lay in their clarity : the approach from the physical side, the formulation of the mathematical problem, the simple and yet general explanation of the mathematical methods used, and the thorough discussion of the result again in terms of physical experiment. His firm, well-distributed writing on the blackboard, and the evidence of his diagrams, helped the student to survey at the end of each period all the subjects that had been covered. Besides, the standard of the course was high enough to tax the powers of the better students and to demand vigilant cooperation. This was all the more important in a university system where there was no check on attendance and only a voluntary one on performance. In giving an original discussion of a problem in the exercises even the beginner would attract the attention of Sommerfeld or of the assistant in charge, and he would be stimulated by the appreciative understanding his effort received.

Sommerfeld had an extraordinary flair for genuine endeavor and performance, irrespective of the age of his students. That is why scientists of the rank of Debye, Pauli, Heisenberg (to name only those who now are Nobel prize men) became attached to him in their early years of study. But also the good average student was well looked after and given smaller problems or small responsibilities to exercise his forces. The indolent student soon turned away on his own account. Thus Sommerfeld's students formed in a way a select group, but their number remained large enough to create a breeze that helped the inexperienced newcomers rapidly to unfurl their own sails. May the translation of Sommerfeld's lectures carry some of this breeze afield and assist other groups in preparing to sail the ocean of discovery.

SOME ARTICLES ON THE AUTHOR'S WORK

Anon., *Current Biographies*, 1950, pp. 537-538 (with portrait).

P. Kirkpatrick, *Am. J. Physics* (1949). 17, 5, 312-316. (Presentation of the Oersted Medal to Sommerfeld by the American Association of Physics Teachers.)

M. Born, *Proc. Roy. Soc., London. A.* (1952). (Obituary.)

P. P. Ewald, *Nature* (1951). 168, 364-366. (Obituary Notice.)

W. Heisenberg, *Naturwissenschaften* (1951). 38, 337.

M. v. Lane, *Naturwissenschaften* (1951). 38, 513-518. (A full appraisal of Sommerfeld's work.)

PREFACE TO THE FIRST EDITION, SEPTEMBER 1942

The encouragement of some of my former students and the repeated suggestion of the publishers decided me to publish my general course on theoretical physics which I gave regularly for thirty-two years at the University of Munich.

This was an introductory course, and was attended not only by the physics majors of the University and the Polytechnic Institute (Technische Hochschule), but also by candidates for teachers' degrees in mathematics and physics, by students of astronomy and some few of physical chemistry—all usually in their third and fourth years. The lectures were held four times a week and supplemented by a two-hour problem period. Special courses on modern physics, which were given concurrently with these, have not been included in this series of books; their subject matter found its way into my scientific papers, summarizing articles, and other books. While it is true that quantum mechanics always hovers in the background and reference is made to it now and then, the actual substance of these lectures is classical physics.

The order of the courses, which has been kept in their publication, was

1. Mechanics
2. Mechanics of Deformable Bodies
3. Electrodynamics
4. Optics
5. Thermodynamics and Statistical Mechanics
6. Partial Differential Equations in Physics

The courses on mechanics were given in alternate years by myself and by my colleagues in mathematics. Concurrent courses in hydrodynamics, electrodynamics, and thermodynamics were taught by younger members of the staff. Vector analysis was given in a separate course so that its systematic development could be omitted from my lectures.

In print, as in my classes, I will not detain myself with the mathematical foundations, but proceed as rapidly as possible to the physical problems themselves. My aim is to give the reader a vivid picture of the vast and varied material that comes within the scope of theory from a suitably chosen mathematical and physical vantage point. With this in mind I shall not be too concerned if I have left some gaps in the systematic justification and axiomatic structure of the work. In any case I do not wish to frighten the hearer of my lectures with drawn-out investigations of a mathematical or logical nature and distract his attention from what is physically interesting. It is my belief that this attitude proved its worth in my courses; it has therefore been retained in the printed lectures. Whereas the lectures of Planck are irreproachable in systematic formulation, I believe that I can claim for mine a greater variety of subject matter and a more flexible handling

of the mathematical apparatus. Moreover, I gladly refer the reader to the more complete and often more thorough treatment of Planck, especially in thermodynamics and statistical mechanics.

The problems collected at the end of each volume should be regarded as supplementary to the text. They were handed in by the students and then presented orally during the problem periods. Elementary numerical problems, to be found so prolifically in texts and collections of problems, have, in general, not been included. The problems are numbered by chapter. Sections are numbered through in each volume and equations in each section. References within each volume to earlier equations can thus be made merely by giving the section, and equation numbers. To make it simpler to find a given section the upper inside corners of every pair of pages bear the section and chapter number.

Looking back on my years of teaching I wish to acknowledge with gratitude my special indebtedness to two men, Röntgen and Felix Klein. Röntgen not only created the external conditions for my professional activity by calling me to a privileged sphere of action; he also stood by my side and actively furthered the increasing scope of my work over a period of many years. Even earlier Felix Klein had imparted to my mathematical thinking that turn of mind which is best adapted to applications; through his mastery in the art of lecturing he exerted a strong indirect influence on my own teaching. In particular let me mention that the last part of this course was announced for the first time while I was still instructor in Göttingen and imbued with the mathematical tradition of that university symbolized in the three names Riemann—Dirichlet—Klein. At that time my course was less comprehensive than the present Volume VI, but it found much resonance in the audience. When the course was repeated in later years my students often told me that only here had they really grasped the handling and application of mathematical results, e.g., Fourier methods, applications of the theory of functions, boundary value problems.

In conclusion, let me send out these volumes with the wish that they arouse the reader's interest in our beautiful science and give him as much pleasure as the courses gave to those attending them and to me during my many years of teaching activity.

Munich, September 1942

Arnold Sommerfeld

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INTRODUCTION

Mechanics is the backbone of mathematical physics. Though it is true that we no longer require physics to explain all phenomena in terms of mechanical models, as was common during the last century, we are nevertheless convinced that the principles of mechanics, such as those of momentum, energy, and least action, are of the greatest importance in all branches of physics.

We call this book "Mechanics," not "Analytical Mechanics" as the mathematicians are wont to do. The latter name has its origin in the great work of Lagrange (1788), who attempted to mold the whole system of mechanics into a consistent language of mathematical equations and was proud of the fact that "one would not find a single diagram in his work." We, on the contrary, shall draw as much as possible on illustration and comparison. The reader will find in this volume many concrete applications in astronomy, physics, and even to some degree in engineering, which should help to make the principles clearer.

The exact title of the book should be "Mechanics of Systems of a Finite Number of Degrees of Freedom"; that of the second volume would accordingly be "Mechanics of Systems with an Infinite Number of Degrees of Freedom." Since, however, the concept of degrees of freedom is not too well known and can here be explained only at the beginning of the second chapter, we shall be satisfied with the customary title "Mechanics," a title hardly subject to misunderstanding.

We commence with Newton's fundamental analysis in his "*Philosophiæ Naturalis Principia Mathematica*" (London, 1687); not that Newton lacked important predecessors, such as Archimedes, Galileo, Kepler, and Huygens, to mention only a few. It was, nevertheless, Newton who first created a firm foundation for general mechanics. Even today, apart from some changes and reinforcements, the foundation laid down by him provides us with the most natural and didactically simplest approach to general mechanics.

We shall at first investigate the mechanics of the single mass point or particle.

CHAPTER I

MECHANICS OF A PARTICLE

§ 1. Newton's Axioms

The laws of motion will be introduced in axiomatic form; they summarize in precise form the whole body of experience.

First law: *Every material body remains in its state of rest or of uniform rectilinear motion unless compelled by forces acting on it to change its state.*¹

We shall at first withhold explanation of the concept of force introduced in this law. We notice that the states of *rest* and of *uniform (rectilinear) motion* are treated on equal footing and are regarded as natural states of the body. The law postulates a tendency of the body to remain in such a natural state; this tendency is called the *inertia* of the body. One often speaks of *Galileo's law of inertia* instead of Newton's first law in referring to the above axiom. We must say in this connection that while it is perfectly true that Galileo arrived at this law long before Newton (as a limiting result of his experiments with sliding bodies on planes of vanishing inclination), we find it characteristic of Newton that the law holds top position in his system. Newton's word "body" will, for the time being, be replaced by the words "particle" or "mass point."

To formulate the first law mathematically we shall make use of definitions 1 and 2 preceding it in the "Principia."

Definition 2: *The quantity of motion is the measure of the same, arising from the velocity and the quantity of matter conjunctly.*²

The "quantity of motion" is hence the product of two factors, the velocity, whose meaning is geometrically evident,³ and the "quantity of

¹ We mention here, and in connection with what is to follow, the book *Die Mechanik in ihrer Entwicklung* (8th ed., F. A. Brockhaus, Leipzig, 1923; translated into English under the title *The Science of Mechanics*, Open Court Publishing Co., LaSalle, Ill., 1942) by Ernst Mach. The study of this excellent critical history is recommended to all students of mechanics, especially since in our book we must restrict ourselves to the concepts of mechanics in a form ready for use and cannot delve into the origin and gradual clarification of these concepts. This should not be interpreted to mean, however, that we agree with Mach's positivistic philosophy as it is developed in Chapter IV, 4, of his book, with its attendant overemphasis of the Economy Principle, the denial of atomic theory and the preference for formal continuity theories.

² *Newton's Principia*, translated by Andrew Motte.—TRANSLATOR.

³ Evident, that is, once a reference system has been chosen in which the velocity is to be measured.

matter," which is to be explained physically. Newton attempts the latter in his definition 1, in which he says that the quantity of matter is measured by its density and volume conjunctly. This is obviously only a mock definition, since density itself cannot be defined in any other way than by the amount of matter in unit volume. In the same definition 1, Newton also states that instead of "quantity of matter" he will use the word *mass*. We shall follow him in this, but shall postpone the physical definition of mass (as well as that of force) until later.

The quantity of motion accordingly becomes the product of mass and velocity. Like the latter it is a directed magnitude, a *vector*. We write⁴

$$(1) \quad \mathbf{p} = m\mathbf{v}$$

and formulate the first law of motion in its final form:

$$(2) \quad \mathbf{p} = \text{constant in the absence of forces.}$$

We shall put the law of inertia thus formulated at the head of our mechanics. It is the result of an evolution extending over many centuries, and is by no means as self-evident as it appears to us today. The philosopher Kant, for instance, says in his paper, "Thoughts on the True Estimation of Living Forces," written in 1747, long after Newton: "There exist two kinds of motions; those which have ceased after a certain time, and those which persist." The motions that in Kant's opinion cease by themselves are, according to modern ideas — and those of Newton — motions which are attenuated by frictional forces and finally destroyed.

The expression "quantity of motion" is unfortunately chosen in that it does not take into account the vector character of $m\mathbf{v}$. Thus a better term would be the word "impulse," which conveys the idea of a push of a certain magnitude in a definite direction that the given $m\mathbf{v}$ is able to impart, by collision, to some body initially at rest. Since the term "impulse" is, however, used in a somewhat different sense in mechanics, we shall have to retain the name "quantity of motion," or, in modern language, *momentum* for the vector \mathbf{p} . Instead of the law of inertia and Newton's first law of motion we can then speak of the *law of conservation of momentum*.

We shall now discuss Newton's second law, the real law of motion: *The change in motion is proportional to the force acting and takes place in the direction of the straight line along which the force acts.*

⁴ We assume that the reader is familiar with the elements of vector algebra. Since, however, vector operations originated in close association with mechanics (including the mechanics of fluids), we shall often have occasion to explain vector concepts simultaneously with mechanical concepts.

As regards notation, vectors will be designated throughout by bold-faced letters; thus ω for the angular velocity when regarded as a (axial) vector. In diagrams, overhead arrows will occasionally be used.

By "change in motion" Newton undoubtedly means the change with time of the previously defined momentum \mathbf{p} , hence the vector $\dot{\mathbf{p}}$ (the dot is Newton's notation for the "fluxion" $\dot{\mathbf{p}} = \frac{d\mathbf{p}}{dt}$). If we designate the force by the letter \mathbf{F} , our second law can then be written as

$$(3) \quad \dot{\mathbf{p}} = \mathbf{F}.$$

Since we called \mathbf{p} the momentum, this law expresses the manner in which momentum changes with time, and can, for brevity, be called simply the *law of momentum*.

Unfortunately the law is often, especially in the mathematical literature, designated as "Newton's law of acceleration." It is of course true that if we treat m as a constant, (3) combined with (1) is identical to

$$(3a) \quad m\dot{\mathbf{v}} = \mathbf{F}: \text{Mass} \cdot \text{Acceleration} = \text{Force}.$$

But mass is not always constant; it is variable in the theory of relativity for example, where Newton's formulation (3) prophetically turns out to be the correct one. We shall treat a series of examples with variable mass in § 4, where we shall have a closer look at the interrelation between formulations (3) and (3a). Incidentally, the mechanical system which is next in simplicity to that of a single mass point, namely, the rotating rigid body, leads us to an equation of motion along the lines of (3), in the form "rate of change of moment of momentum (angular momentum) = moment of force (torque)"; a description in terms of angular acceleration, similar to (3a), is not possible. An effect similar to the non-constancy of mass in relativity must be taken into account: the moment of inertia, here replacing the mass, changes with changing location of the axis of rotation in the body.

We must now seek to get a clear idea of the concept of *force*. Kirchhoff⁵ wanted to degrade it to a quantity defined by the product of mass and acceleration. Hertz⁶, too, tried to eliminate and replace it by coupling the system under consideration with other, generally hidden systems interacting with the former. Hertz carried out this program with admirable consistency. His method, however, hardly produced fruitful results; and it is especially unsuitable for the beginner.

We are of the opinion that we have at least a *qualitative* notion of "force" which we acquire quite directly through the feeling we experience when using our muscles. In addition the earth has provided us with the comparison standard of gravity, with which we can measure all other forces *quantitatively*. For this purpose we need merely balance the effect

⁵ Gustav Kirchhoff, Vol. I of his *Vorlesungen über mathematische Physik*, p. 22.

⁶ Heinrich Hertz, *Miscellaneous Papers*, Vol. III, *Principles of Mechanics*, Macmillan, New York, 1896.

of a given force by a suitable weight. (By means of a pulley and string we can let the vertical force of gravity act in a direction opposed to the given force.) If, in addition, we procure a number of equally heavy bodies, a "set of weights," we obtain a tentative scale with which to measure forces quantitatively.

The same is true for the concept of force as for all other physical concepts and names: word definitions have very little meaning; physically significant definitions are obtained as soon as we prescribe a way of measuring the quantity in question. Such a prescription need not contain the details of practical procedure, but merely state a way to measure the quantity in principle.

The above prescription, making use of gravity, has given a concrete content to the right member of our law of momentum (3); it has thereby become a real physical statement. It is true that the left member still contains the mass m , up to now undefined. This does not mean that the definition of mass is the only content of the law. For the law brings out that it is \dot{p} , not p itself or perhaps β which is determined by the force. We shall see in § 4 how the definition of mass is obtained in case it is variable, the relativistic mass serving as example.

Third law: *Action always equals reaction, or: the forces two bodies exert on each other are always equal and opposite in direction.*

This is the principle of action and reaction. It says that for every pressure there is a pressure in the opposite direction. Forces always occur paired in nature. The falling stone attracts the earth just as strongly as the earth attracts the stone.

This law makes possible the transition from the mechanics of single mass points to that of compound systems; it is therefore fundamental to the entire field of structural statics, to name but one example.

We shall call the rule of the *parallelogram of forces* our **Fourth law**, even though in Newton it appears merely as an addition or corollary to the other laws of motion. The fourth law states that two forces applied to the same mass point compound to act like the diagonal of the parallelogram formed by them: *forces add like vectors*. This seems self-evident since we equated the force F to the vector \dot{p} in the Second Law. Actually, however, the Fourth Law, as Mach emphasizes, contains the axiom that each force acting on a mass point causes it to change its motion as if this force were the only one acting there. The parallelogram of forces hence establishes axiomatically the independence of the effects of several forces acting together at the same point, or, more generally, the *principle of superposition of forces*. Of course the last statement as well as the laws of motion preceding it are nothing but an idealization and a precise formulation of our whole body of experience.

Having introduced the concept of force, we shall at this point introduce that of *work* with the definition

$$(4) \quad dW = \mathbf{F} \cdot d\mathbf{s} = F \, ds \cos(\mathbf{F}, d\mathbf{s}).$$

Thus the work does not equal "force times distance" as often stated, but "component of force along path times path length" or "force times component of path length along force."

From the statement, "forces add vectorially," follows immediately the complementary statement that "work adds algebraically." Indeed

$$\mathbf{F}_1 + \mathbf{F}_2 + \cdots = \mathbf{F}$$

leads to

$$(5) \quad \mathbf{F}_1 \cdot d\mathbf{s} + \mathbf{F}_2 \cdot d\mathbf{s} + \cdots = \mathbf{F} \cdot d\mathbf{s}$$

by scalar multiplication with the distance $d\mathbf{s}$. Here \mathbf{F} is the resultant force. The definition of the scalar product contained in (4) automatically sees to it that, for example, in the first product of (5) only ds_1 , the component of the distance in the direction of the force \mathbf{F}_1 , occurs. Hence we can also write instead of (5) that

$$(6) \quad dW_1 + dW_2 + \cdots = dW,$$

as stated above.

Related to the concept of work is that of *power*; power is the work done in unit time.

In concluding these introductory remarks we shall have to agree on how to measure the mechanical quantities that we have introduced. Here we have a choice of two systems of units, the *physical* (or *absolute*) and the *practical* (or *gravitational*) metric systems. The difference between them is that in the absolute system the gram (or kilogram) serves as unit *mass*, whereas in the gravitational system the kilogram (or gram) serves as unit *force*. In the latter case we speak of a kilogram-weight and write

$$1 \text{ kg-weight} = g \cdot \text{kg-mass}.$$

The gravitational acceleration g is, however, a function of the location on the earth, being greater at the poles than at the equator because of a smaller distance from the center of the earth as well as because of diminished centrifugal force. Hence the kg-weight is dependent on location; a kg-weight sample cannot be transported. The gravitational system is therefore unsuitable for precise measurements. The physical system has, in contrast, been distinguished by the title, "absolute system of units." We have nevertheless become so accustomed to the gravitational system that in many cases where we should really say "mass," the word "weight" has once and for all made its way into our scientific language. Thus we talk of specific

weight when we should say specific mass or density; and of atomic and molecular weights — which surely have nothing to do with the acceleration due to gravity.

Gauss, the originator of absolute measurements, decided in favor of the absolute system after some hesitation. Initially he, too, was in favor of introducing force as the basic unit, since it played a more direct role than mass in his measurements of terrestrial magnetism. On the other hand he wanted these measurements to encompass the entire surface of our globe; he therefore saw himself forced to adopt as unit a quantity whose value would not depend on location.

Below we have put the two systems next to each other and at the same time introduced the derived units dyne, erg, joule, watt, and horse power (HP):

<u>Absolute system (CGS)</u>	<u>Gravitational system (MKS)</u>
cm, g-mass, sec	kg-weight, m, sec
1 kg-weight = $9.81 \cdot 10^5$ g cm sec ⁻² = $9.81 \cdot 10^5$ dyne	1 g-mass = $\frac{1 \text{ kg}}{1000} \frac{1}{g} \text{ sec}^2 \text{ m}^{-1}$
1 erg = 1 dyne · 1 cm	1 unit of work = 1 kg · 1 m
1 joule = 10^7 erg	
1 mkg-weight = $1000 \cdot g \cdot 100$ erg = $9.81 \cdot 10^7$ erg = 9.81 joule	
1 watt = 1 joule sec ⁻¹	1 unit of power = 1 kg m sec ⁻¹
1 kilowatt = 1000 joule sec ⁻¹ = $\frac{1 \text{ HP}}{0.736} = 1.36 \text{ HP}$	1 HP = 75 kg m sec ⁻¹ = $75 \cdot 1000 \cdot 100 \cdot 981 \text{ erg sec}^{-1}$ = $75 \cdot 9.81 \text{ watt} = 0.736 \text{ kw}$

It should be noted that according to a decision of the pertinent international commissions the CGS system was to be replaced by an absolute MKS system beginning with the year 1940. In this new system the meter takes the place of the centimeter, and the kilogram that of the gram as unit of mass, while the second is retained as unit of time. This is in agreement with a proposal of G. Giorgi, which shows its advantages fully only in electrodynamics with the addition of a fourth independent electrical unit (see Vol. III of this series). In mechanics the proposed change would have the advantage that in the definition of the joule and the watt the bothersome powers of ten are eliminated. With the new larger units M and K the units of work and power become

$$1 \text{ M}^2\text{KS}^{-2} = 10^7 \text{ cm}^2 \text{ g sec}^{-2} = 1 \text{ joule},$$

$$1 \text{ M}^2\text{KS}^{-3} = 10^7 \text{ cm}^2 \text{ g sec}^{-3} = 1 \text{ watt}.$$

The unit of force in the new system, called the newton, is thus

$$1 \text{ newton} = 1 \text{ MKS}^{-2} = 10^5 \text{ cm g sec}^{-2} = 10^5 \text{ dyne.}$$

This, too, can be regarded as an advantage of the Giorgi system, since now the new unit of force is brought closer to the convenient gravitational unit, the kg-weight, whereas the old unit of force, the dyne, is inconveniently small for most practical uses.

§ 2. *Space, Time and Reference Systems*⁷

Newton's views about space and time seem to us moderns quite unrealistic and appear to contradict his declared intention to base his analysis on fact alone. He states:

"Absolute space, in its own nature, without regard to anything external, remains always similar and immovable.

"Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without regard to anything external, and by another name is called duration."

From these two quotations one would conclude that Newton worried but little where absolute time was to be taken from, and how an immovable absolute space was to be distinguished from one moving uniformly with respect to it. This is all the more surprising since he put the states of rest and of uniform motion on the same footing in his first law. On the other hand Newton tried to clarify the distinction between absolute and relative motion by his famous pail experiment.⁸ In this experiment a pail is suspended from a twisted thread and filled with water. The pail is then suddenly released and as the thread untwists acquires a rotation about its axis of symmetry. The surface of the water remains at first level, although the relative velocity between pail and water is great. Gradually the water is set in motion by friction with the walls of the pail, climbs up the wall, and its surface assumes the familiar hollow paraboloidal shape. Finally a steady state is reached in which the relative motion between pail and water is zero; the "absolute" motion of the water in space has, on the other hand, increased to a maximum, and with it the curvature of the surface.

Actually the experiment only shows that the rotating pail does not furnish a suitable reference system from which the motion of the water can be understood. Is the earth such an unsuitable reference system?

⁷ The beginner to whom the following somewhat abstract considerations seem unfamiliar may postpone the study of this section and of § 4 until a later time.

⁸ "I have performed this experiment myself," says Newton, probably with reference to the natural philosophers, perhaps his compatriot Francis Bacon, who was wont to describe the results of experiments he had not performed.

It, too, rotates and furthermore describes an orbit around the sun. In general, what are the requirements that an ideal reference system has to satisfy in mechanics? By a reference system is meant a frame in space and time which will enable us to read off the position of mass points and the passage of time; we might take a Cartesian system of coordinates x, y, z , and a time scale, t .

In practice we shall have to rely on the astronomers for this choice. The fixed stars furnish sufficiently constant directions for our coordinate axes, and the sidereal day furnishes a sufficiently constant interval of time. Theoretically, on the other hand, we are forced to recognize a disagreeable tautology: that reference frame is an ideal one in which the Galilean law of inertia holds with sufficient accuracy, for a sufficiently force-free body. Thus the first law is degraded to a formal identity or to the rank of definition. The only positive, not purely formal content that the law retains is the assertion that reference systems of the required properties do exist. All our experience indicates that one such system is approximated by astronomical determinations of position and time.

We mean essentially the same thing when we say that the laws of mechanics presuppose the existence of an *inertial frame*, i.e., an imaginary structure whose axes are trajectories of bodies moving purely under inertia.

The question now arises to what extent this ideal system of reference is determined. Is there only one such system x, y, z, t , or are there perhaps infinitely many such systems? Newton's first law gives the answer at once, for it states that any two systems x, y, z, t and x', y', z', t' are equivalent if they differ only by a uniform translational motion. In mathematical form

$$(1) \quad \begin{aligned} x' &= x + \alpha_0 t \\ y' &= y + \beta_0 t \\ z' &= z + \gamma_0 t \\ t' &= t. \end{aligned}$$

We can generalize the transformation (1) by performing a rotation on the spatial system x, y, z about its origin, which amounts to replacing x, y, z in (1) by new space coordinates ξ, η, ζ such that

$$(2) \quad \xi^2 + \eta^2 + \zeta^2 = x^2 + y^2 + z^2.$$

This condition defines an arbitrary *orthogonal transformation*. With $\alpha_k, \beta_k, \gamma_k$, the direction cosines, it yields

$$(3) \quad \begin{array}{c|ccc} & x & y & z \\ \hline \xi & \alpha_1 & \alpha_2 & \alpha_3 \\ \eta & \beta_1 & \beta_2 & \beta_3 \\ \zeta & \gamma_1 & \gamma_2 & \gamma_3 \end{array}$$

This scheme can be read equally well from left to right as from top to bottom. Because of (2) the α , β , γ satisfy the well-known relations

$$(4) \quad \sum \alpha_k^2 = \sum \beta_k^2 = \sum \gamma_k^2 = 1, \quad \sum \alpha_k \beta_k = \dots = 0, \text{ etc.}$$

If now we replace the x, y, z in the right member of (1) by the ξ, η, ζ of (3), we obtain the generalized transformation scheme*

$$(5) \quad \begin{array}{c|cccc} & x & y & z & t \\ \hline x' & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_0 \\ y' & \beta_1 & \beta_2 & \beta_3 & \beta_0 \\ z' & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_0 \\ t' & 0 & 0 & 0 & 1 \end{array}$$

The fact that the primed system x', y', z', t' is just as good a reference frame for the purposes of classical mechanics as the unprimed system x, y, z, t is called the *principle of relativity of classical mechanics*. In what follows (5) will be called a *Galilean transformation*. It is a linear transformation in the four coordinates; it is orthogonal in the first three, and leaves the time coordinate invariant ($t' = t$). The last statement means that the principle of relativity of classical mechanics leaves intact the absolute character of time as postulated by Newton.

A new situation arises, however, in the field of electrodynamics, particularly in the electromagnetic theory of optical phenomena. Maxwell's equations, which form the basis of this field, require that the process of the propagation of light *in vacuo* with the velocity c be independent of the frame of reference from which this process is observed. The front of a spherical wave whose source is at the origin of coordinates is given by the equation

$$(6) \quad x^2 + y^2 + z^2 = c^2 t^2 \quad \text{or} \quad x'^2 + y'^2 + z'^2 = c^2 t'^2$$

respectively, depending on whether we are describing the wave front in the unprimed or the primed system. It is now convenient to change the names of the coordinates in the following manner:

$$(7) \quad x = x_1, y = x_2, z = x_3, i c t = x_4,$$

where i is the imaginary unit; we introduce a corresponding change of notation for the primed coordinates. Equations (6) then read

$$(8) \quad \sum_1^4 x_k^2 = 0, \quad \sum_1^4 x'_k{}^2 = 0,$$

* Note that this table can be read from left to right but no longer from top to bottom, since the transformation is no longer orthogonal.—TRANSLATOR.

and the fact that the propagation of light does not depend on the choice of reference frame demands that¹⁰

$$(9) \quad \sum_1^4 x_k'^2 = \sum_1^4 x_k^2.$$

Whereas Eq. (2) was an orthogonal transformation in three-dimensional space, we are dealing in (9) with an orthogonal transformation in four-dimensional space. True, the fourth coordinate is imaginary. This, however, will not affect the existence of equations analogous to (3), (4) and (5). The relation between the x_k and the x'_k arising from (5) is in general called a *Lorentz transformation*, after the great Dutch theoretical physicist Hendrik Antoon Lorentz. We write it in the form of the general scheme

$$(10) \quad \begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline x'_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ x'_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ x'_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ x'_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{array}$$

This table shows at once that the time coordinate (in the imaginary form x_4) is now involved in a change of reference system to the same extent as the space coordinates. As a necessary consequence of the invariance requirement (9) the absoluteness of time is now destroyed.

More instructive than the general Lorentz transformation is the special one which we obtain when we leave two space coordinates, say x_1 and x_2 , unchanged, and transform only x_3 and x_4 .

Then all the α_{ij} of the first and second rows of columns in (10) must vanish, except for

$$\alpha_{11} = \alpha_{22} = 1,$$

because $x'_1 = x_1$, $x'_2 = x_2$ (as read from left to right as well as from top to bottom). Furthermore we have the conditions analogous to (4),

$$(11) \quad \alpha_{33}^2 + \alpha_{34}^2 = \alpha_{33}^2 + \alpha_{43}^2 = \alpha_{43}^2 + \alpha_{44}^2 = \alpha_{34}^2 + \alpha_{44}^2 = 1,$$

and therefore

$$\alpha_{33}^2 = \alpha_{44}^2, \quad \alpha_{34}^2 = \alpha_{43}^2.$$

Letting $\delta = \pm 1$, we can write

$$(11a) \quad \alpha_{34} = \delta \alpha_{43}$$

¹⁰ For one of the Eqs. (8) must be the consequence of the other. In view of the linearity of the relation between them, one of the expressions (8) must be proportional to the other. Since the relation is a reciprocal one, the factor of proportionality must be unity.

and we must then put

$$(11b) \quad \alpha_{44} = -\delta\alpha_{33}$$

because of the other orthogonality condition $\alpha_{33}\alpha_{34} + \alpha_{43}\alpha_{44} = 0$. We now make use of (11a, b) to solve for the primed coordinates in terms of the unprimed. At the same time, with the help of (7), we go back to our original coordinates z, t, z', t' to obtain

$$(12) \quad \begin{aligned} z' &= \alpha_{33} \left(z + i\delta c \frac{\alpha_{43}}{\alpha_{33}} t \right), \\ t' &= -\delta \alpha_{33} \left(t + i \frac{\delta}{c} \frac{\alpha_{43}}{\alpha_{33}} z \right). \end{aligned}$$

The first of these equations shows that

$$(12a) \quad -i\delta c \frac{\alpha_{43}}{\alpha_{33}} = v$$

must be identified with the velocity with which the z' -axis moves parallel to the z -axis in the positive direction of the latter, as observed from the unprimed system. With the help of (12a) Eqs. (12) become

$$(13) \quad \begin{aligned} z' &= \alpha_{33} (z - vt), \\ t' &= -\delta \alpha_{33} \left(t - \frac{v}{c^2} z \right). \end{aligned}$$

Finally we must determine α_{33} . To this end we use Eq. (9) which, in the original coordinates, now simplifies to $z'^2 - c^2 t'^2 = z^2 - c^2 t^2$. Let us introduce here the values of z' and t' from (13). The factor of $2vzt$ vanishes on the left. Comparison of the factors of z^2 and t^2 on the left and right yields

$$\alpha_{33}^2 = \frac{1}{1 - v^2/c^2}.$$

In the limit $c \rightarrow \infty$ (13) must of course reduce to the Galilean transformation (1) with $\alpha_0 = \beta_0 = 0$ and $\gamma_0 = -v$. To this end we must put $\delta = -1$ and must choose the positive sign of α_{33} . We then obtain the characteristic two-dimensional Lorentz transformation

$$(14) \quad \begin{aligned} z' &= \frac{z - vt}{(1 - \beta^2)^{\frac{1}{2}}}, \\ t' &= \frac{t - \frac{v}{c^2} z}{(1 - \beta^2)^{\frac{1}{2}}}, \end{aligned}$$

where $\beta = \frac{v}{c}$, $(1 - \beta^2)^{\frac{1}{2}} > 0$.

The relativization of the time in (14) and the change of scale of the space coordinate z , as embodied in the denominator $(1 - \frac{v^2}{c^2})^{\frac{1}{2}}$, are, as we

have seen, a result of the fact that the velocity of light c is finite, a fact with which the principle of relativity of classical mechanics is incompatible.

If it be true that all electrodynamic effects are propagated with finite velocity c , it follows that for such effects the Galilean transformation must always be replaced by a Lorentz transformation, either in the general form (10), or the specialized form (14). We call this fact the *principle of relativity of electrodynamics*. It is evident, however, that mechanics too has to adapt itself to the fact of the finite propagation velocity of light. Now all velocities occurring in ordinary mechanics are quite small compared to c . This is the reason why for the purposes of mechanics we can, as a rule, neglect the change of scale of the space and time coordinates indicated by (14).

The wealth of physical facts embodied in the Lorentz transformation will be discussed in the third volume of this series. Here we shall only investigate the changes that we have to make in the concept of the fundamental quantity p , the momentum, as a result of our new relativity principle.

We have called p a vector. This means that the three components of p transform just like the coordinates themselves [i.e., the components of the radius vector $r = (x, y, z)$] in a change of the system of coordinates. We therefore say that p is covariant to r .

This is valid only from the viewpoint of the Galilean transformation, where the time is regarded as absolute. From the viewpoint of the Lorentz transformation the radius vector is a four-component quantity, a *four-vector*

$$(15) \quad \mathbf{x} = (x_1, x_2, x_3, x_4).$$

Our relativistic momentum will similarly have to be a four-vector, i.e., must be covariant to \mathbf{x} , if it is to have a meaning in relativity theory. We arrive at this four-vector in the following manner:

(a) (15) being a four-vector, the coordinate distance between two neighboring points

$$(16) \quad d\mathbf{x} = (dx_1, dx_2, dx_3, dx_4) = (dx_1, dx_2, dx_3, icdt)$$

is also a four-vector.

(b) The magnitude of this distance is certainly invariant under a Lorentz transformation. Apart from a factor ic , it is given by

$$(17) \quad d\tau = \left[dt^2 - \frac{1}{c^2} (dx_1^2 + dx_2^2 + dx_3^2) \right]^{\frac{1}{2}}$$

We follow Minkowski in calling $d\tau$ the element of *proper time*; in contrast to dt it is relativistically invariant. We shall factor out dt in (17) and introduce the ordinary velocity v of three dimensions, to obtain

$$(17a) \quad d\tau = dt \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = dt (1 - \beta^2)^{\frac{1}{2}}.$$

(c) Division of the four-vector (16) by the invariant (17a) yields another four-vector; we call it the four-vector velocity

$$(18) \quad \frac{1}{(1 - \beta^2)^{\frac{1}{2}}} \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, ic \right).$$

(d) Earlier we derived the momentum vector \mathbf{p} by multiplying the velocity three-vector by a mass m independent of the reference frame. We shall similarly deduce the momentum four-vector \mathbf{p} from the four-vector (18) by multiplication by a mass factor independent of the frame of reference. We shall call this mass factor the *rest mass* m_0 and obtain

$$(19) \quad \mathbf{p} = \frac{m_0}{(1 - \beta^2)^{\frac{1}{2}}} \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, ic \right).$$

It is proper to call the quantity in front of the parenthesis the moving mass (since it reduces to the rest mass for $\beta=0$), or simply the mass. We therefore assert that

$$(20) \quad m = \frac{m_0}{(1 - \beta^2)^{\frac{1}{2}}}.$$

This expression was first derived by Lorentz in 1904 under very special assumptions (deformable electron). The derivation from the principle of relativity makes such special assumptions unnecessary. Eq. (20) has been confirmed by many precision experiments with fast electrons. Together with optical experiments, notably that of Michelson and Morley, it forms the basis of the theory of relativity. Here we have proceeded in inverse order and deduced Eq. (20) from the principle of relativity in what appears to be a very formal procedure. This is not only logically admissible, but especially serviceable in view of the brevity of these introductory explanations. In § 4 we shall discuss what changes in the further application of Newton's laws of motion will have to be made as a result of the velocity-dependence of mass.

At this point we should, if only sketchily, bring to a conclusion the question of the permissible frames of reference; to this end we must pass from the *special theory of relativity* treated so far to the *general theory of relativity* (Einstein, 1915). In special relativity there are allowed reference systems which are obtained from one another by Lorentz transformations, and forbidden ones, such as, for example, those that are accelerated with respect to the former. In general relativity all possible frames of reference are admitted. Transformations between them need no longer be linear and orthogonal as in (10), but can instead be given by arbitrary functions $x'_k = f_k(x_1, x_2, x_3, x_4)$. Hence we are dealing with systems which are moving

and are being deformed with respect to each other in any way desired. As a result space and time lose any vestiges of the absolute character which they held in Newton's fundamental analysis. They become merely classification schemes for physical events. Euclidean geometry no longer suffices for this classification and must be replaced by the much more general metric geometry advanced by Riemann. The task then arises to give such a form to physical laws that they will remain valid in all frames of reference here considered, i.e., a form that remains invariant under arbitrary point transformations $x'_A = f_A(x_1 \cdots x_4)$ of four-dimensional space. The positive content of the general theory of relativity is precisely the possibility of this task. We cannot in this volume delve into the mathematically very involved form which the laws of mechanics take on in their invariant formulation. Suffice it to say that the general theory leads to a derivation and a more precise formulation of Newtonian gravitation.

We conclude with a remark about the name, theory of relativity. The positive achievement of the theory is not so much the complete relativization of space and time, but the proof that the laws of nature are independent of the choice of reference system, i.e., that events in nature are invariant under any change in the observer's view point. The names, "theory of the invariance of natural events," or, as occasionally proposed, "view-point theory," would be more appropriate than the customary name, general theory of relativity.

§ 3. Rectilinear Motion of a Mass Point

Let the motion of the particle take place along the x -axis. Only the x -components of any forces present will have any effect. Let X denote the resultant of these components.

We have $v = v = \frac{dx}{dt}$ and $p = m \frac{dx}{dt}$. Then

$$(1) \quad \dot{p} = X$$

and, with constant m ,

$$(2) \quad m \frac{d^2x}{dt^2} = X.$$

We wish to study the integration of this equation of motion for the three cases: X is given as a pure function of the time, [$X = X(t)$], of the position, [$X = X(x)$], or of the velocity, [$X = X(v)$].

(a) $X = X(t)$.

Immediate integration yields

$$(3) \quad v - v_0 = \frac{1}{m} \int_{t_0}^t X(t) dt = \frac{1}{m} Z(t).$$

Here $Z(t)$ is by definition the time integral of the force and equals the change in momentum during the time from t_0 to t .

A second integration leads to the equation of the trajectory,

$$(4) \quad x - x_0 = v_0(t - t_0) + \frac{1}{m} \int_{t_0}^t Z(t) dt.$$

(b) $X = X(x)$.

This is the typical case of the *force field* given as a function of position. Integration is achieved by use of the principle of the conservation of energy.

We multiply (2) on both sides by $\frac{dx}{dt}$,

$$(5) \quad m \frac{dx}{dt} \frac{d^2x}{dt^2} = X \frac{dx}{dt}.$$

The left member is now a complete differential,

$$\frac{d}{dt} \left\{ \frac{m}{2} \left(\frac{dx}{dt} \right)^2 \right\}.$$

In agreement with the general definition of (1.4) we write for the right member $dW = X dx$ and call dW the work done over the path dx . The equation thus arising says that *the change in kinetic energy equals the work done*.

For we define

$$(6) \quad T = E_{\text{kin}} = \frac{m}{2} v^2$$

as the *kinetic energy* or energy of motion of the mass point; the older name, live force (Leibniz), shows the ambiguity of the word force (he distinguished live force, *vis viva*, i.e., kinetic energy, and motor force, *vis motrix*, our present-day force; even Helmholtz, as late as 1847, entitled a treatise dealing with the conservation of energy "Concerning the Conservation of Force").

To the definition of the kinetic energy we add that of the potential energy V ,

$$(7) \quad dV = -dW = -X dx, \quad V = E_{\text{pot}} = - \int X dx.$$

In one-dimensional particle mechanics this definition suffices; in the case of two- or three-dimensional force fields the existence of V depends on the character of the fields (cf. § 6, Part 3). According to (7) V is determined only to within an additive constant.

With these definitions the integrated Eq. (5) yields the law of the conservation of energy,

$$(8) \quad T + V = \text{constant} = E.$$

Here E is the energy constant or *total energy*.

The principle of the conservation of energy possesses not only an exceedingly great physical importance, but also remarkable mathematical power. For it performs, as we have seen, not only the first integration of the equation of motion (hence its alternate name, "integral of energy"), but at once makes possible — at least in the present case (b) — a second integration as well. If we write (8) in the form

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{m} [E - V(x)],$$

we can solve for dt ,

$$dt = \left[\frac{m}{2(E - V)} \right]^{\frac{1}{2}} dx,$$

so that

$$(9) \quad t - t_0 = \left(\frac{m}{2}\right)^{\frac{1}{2}} \int_{x_0}^x \frac{dx}{(E - V)^{\frac{1}{2}}}.$$

Thus t is a known function of x , and therefore x can also be expressed in terms of t . (9) is then the completely integrated equation of motion.

(c) $X = X(v)$.

Now the equation of motion reads

$$m \frac{dv}{dt} = X(v)$$

which we rewrite as

$$dt = \frac{m dv}{X}$$

thereby at once obtaining

$$(10) \quad t - t_0 = m \int_{v_0}^v \frac{dv}{X} = F(v).$$

This also allows us to solve for v in terms of t , $v = f(t)$, so that

$$\frac{dx}{dt} = f(t),$$

from which we conclude that

$$x - x_0 = \int_{t_0}^t f(t) dt.$$

Examples

1. FREE FALL NEAR EARTH'S SURFACE (Falling Stone)

We take the positive x -direction as vertically upward. The force is constant,

$$(11) \quad X = -mg,$$

i.e., independent of t , x , and v . Here all three methods of integration (a), (b), (c) can be applied.

We shall carry out (a) and (b), and postulate explicitly that the "gravitational mass" and the "inertial mass" be equal,

$$(12) \quad m_{\text{inert}} = m_{\text{grav}}.$$

m_{inert} is the mass defined by the Second Law; m_{grav} is the mass occurring in the law of gravitation and hence also in our force equation (11).

Bessel recognized the necessity for testing Eq. (12) experimentally, by means of pendulum experiments.¹¹ A much more precise experimental proof was furnished by Eötvös with his torsion balance. Later on, Eq. (12) gave the first impulse to Einstein's theory of gravitation.

(a) $\ddot{x} = -g$. With suitable choice of the integration constants ($v=0$ and $x=h$ for $t=0$) we end up with

$$\dot{x} = -gt, \quad x = h - \frac{g}{2}t^2.$$

(b) Since $dW = -mgdx$, $V = mgx$ and $T + mgx = E$. If $v=0$ at $x=h$, we must have $E = mgh$; therefore

$$\frac{m}{2}v^2 + mgx = mgh.$$

From this we get for the particular value $x=0$ that $v^2 = 2gh$, or

$$(13) \quad v = (2gh)^{\frac{1}{2}}.$$

Inverting this equation we obtain

$$(13a) \quad h = \frac{v^2}{2g},$$

which is the height to which an arbitrary mass must be raised in order to attain, falling through this height in the gravitational field, a specified velocity v . The introduction of this height h instead of the velocity v is convenient, especially in certain engineering problems, such as the

¹¹ Incidentally we would like to direct the reader's attention to an interesting sentence occurring in Newton's *Mechanics*. At the beginning of this work, under Definition 1, Newton says: "Through very carefully performed experiments with pendula I have verified that mass and weight are proportional."

height to which water rises in a Pitot tube,¹² the pressure head in a centrifuge, etc. The height to which the water surface climbs in Newton's pail experiment is similarly given by (13a).

2. FREE FALL FROM A GREAT DISTANCE (Meteor)

Now the force of attraction is no longer constant. Instead we must use the law of gravitation

$$(14) \quad m \frac{d^2 r}{dt^2} = - \frac{mMG}{r^2},$$

where m is the mass of the meteor, M that of the earth, G the gravitational constant. Instead of the coordinate x we have introduced the distance r of the meteor from the center of the earth. Since the force is now a function of r , method of integration (b) should be used.

In particular, for the surface of the earth, with a the earth's radius, (14) yields

$$mg = \frac{mMG}{a^2},$$

so that mMG can be eliminated from (14),

$$\frac{d^2 r}{dt^2} = -g \frac{a^2}{r^2}.$$

With this notation (7) yields

$$dV = -dW = mga^2 \frac{dr}{r^3},$$

so that the potential energy, with zero level at infinity, becomes

$$(15) \quad V(r) = -mg \frac{a^2}{r}.$$

Eq. (8) therefore gives

$$\frac{m}{2} \left(\frac{dr}{dt} \right)^2 - \frac{mga^2}{r} = W = - \frac{mga^2}{R},$$

where R is some hypothetical initial distance from the earth's center at which the falling mass was in a state of rest. We thus obtain

$$(16) \quad \frac{dr}{dt} = a \left[2g \left(\frac{1}{r} - \frac{1}{R} \right) \right]^{\frac{1}{2}}$$

and, corresponding to (9),

$$(16a) \quad t = \frac{1}{a(2g)^{\frac{1}{2}}} \int \frac{dr}{\left(\frac{1}{r} - \frac{1}{R} \right)^{\frac{1}{2}}}.$$

¹² A hollow tube used in fluid flow to measure the dynamic pressure. It is often used on airplanes as an airspeed indicator. Cf. Glazebrook, *Dictionary of Applied Physics* V, p. 2.—TRANSLATOR.

We need not do the integration indicated in (16a) in detail, since only two special cases of (16) are of interest to us:

$$(a) \quad R = \infty, r = a.$$

The meteor reaches the earth with the velocity

$$\frac{dr}{dt} = (2ga)^{\frac{1}{2}},$$

i.e., a free fall in the earth's gravitational field from infinity results in the same velocity at the surface of the earth which would be achieved by a free fall from a height $h=a$ equal to the earth's radius under a constant acceleration due to gravity g [cf. Eq. (13)].

$$(b) \quad R = a + h, \quad h \ll a, \quad r = a.$$

Here we are concerned with a first order correction to the velocity of fall (13), taking into consideration the decreasing gravitational acceleration, but assuming that the meteor falls from not too great a height. From (16) we derive

$$\begin{aligned} \frac{dr}{dt} &= \left[2ga \left(1 - \frac{1}{1 + \frac{h}{a}} \right) \right]^{\frac{1}{2}} = (2ga)^{\frac{1}{2}} \left(\frac{h}{a} - \frac{h^2}{a^2} + \dots \right)^{\frac{1}{2}} \\ &= (2ga)^{\frac{1}{2}} \left(\frac{h}{a} \right)^{\frac{1}{2}} \left(1 - \frac{1}{2} \frac{h}{a} + \dots \right) = (2gh)^{\frac{1}{2}} \left(1 - \frac{1}{2} \frac{h}{a} + \dots \right). \end{aligned}$$

3. FREE FALL IN AIR

We shall assume that the air resistance is proportional to the square of the velocity. This assumption, introduced by Newton, agrees quite well with experience if the falling body is not too small and its velocity is neither comparable to that of sound, nor vanishingly small. The resultant force is then

$$X(v) = -mg + av^2,$$

where the signs indicate that the air resistance opposes the force of gravitation. Here method (c) of p. 18 applies, and the equation of motion becomes

$$(17) \quad \frac{dv}{dt} = -g + \frac{a}{m} v^2.$$

If we put $\frac{a}{mg} = b^2$, it goes over into

$$\frac{dv}{dt} = -g(1 - b^2 v^2).$$

From this we obtain the analogue of (10) with $t_0=0$,

$$-g dt = \frac{dv}{2} \left(\frac{1}{1-bv} + \frac{1}{1+bv} \right), \quad -gt = \frac{1}{2b} \cdot \ln \left(\frac{1+bv}{1-bv} \right),$$

so that

$$\frac{1+bv}{1-bv} = e^{-2bgt}$$

and

$$(18) \quad bv = \frac{e^{-2bgt} - 1}{e^{-2bgt} + 1} = -\frac{\sinh bgt}{\cosh bgt} = -\tanh bgt,$$

where \sinh , \cosh , and \tanh are the hyperbolic functions. $|bv|$ hence grows monotonically from 0 at $t=0$, and approaches the value 1 as $t \rightarrow \infty$. The limiting value of v itself is

$$|v| = \frac{1}{b} = \left(\frac{mg}{a} \right)^{\frac{1}{2}}.$$

This can also be read at once out of Eq. (17), since for the above limiting value $\frac{dv}{dt}$ becomes equal to zero.

We make use of Eq. (18) to obtain the first order correction due to air resistance which must be added to the formula derived for a free fall *in vacuo*. From the series expansion

$$\tanh \alpha = \frac{\sinh \alpha}{\cosh \alpha} = \frac{\alpha + \frac{\alpha^3}{6}}{1 + \frac{\alpha^2}{2}} = \alpha \left(1 - \frac{\alpha^2}{3} \right)$$

we obtain, according to (18), with $\alpha = bgt$,

$$v = -gt \left(1 - \frac{(bgt)^2}{3} \right).$$

4. HARMONIC OSCILLATIONS

Harmonic oscillations occur whenever a restoring force X proportional to the displacement x acts on a mass point m . We call the proportionality factor k , so that

$$X = -kx$$

and the equation of motion with constant m is

$$(19) \quad m \frac{d^2x}{dt^2} = -kx.$$

Since the force is a given function of the coordinate [case (b) of p. 17], we make use of the rule given there and apply the integral of energy.

We must therefore first determine the potential energy of the harmonic binding force. We have

$$dW = X dx = -\frac{k}{2} d(x^2),$$

so that, according to (7), with a suitable choice of the zero of V ,

$$V = - \int_0^x dW = \frac{k}{2} x^2.$$

The equation of energy is then

$$mv^2 + kx^2 = 2E.$$

As our initial conditions we may choose

$$(19a) \quad \text{at } t=0: \begin{cases} x=a \\ v=\dot{x}=0. \end{cases}$$

As a result $2E$ takes the value ka^2 , and

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 &= \frac{k}{m} (a^2 - x^2), \\ \left(\frac{k}{m}\right)^{\frac{1}{2}} dt &= \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}}. \end{aligned}$$

A quadrature, incorporating the initial conditions (19a), yields

$$(20) \quad \omega t = \sin^{-1}\left(\frac{x}{a} - \frac{\pi}{2}\right) \text{ with } \omega = \left(\frac{k}{m}\right)^{\frac{1}{2}}.$$

An inversion finally gives

$$(21) \quad x = a \sin\left(\omega t + \frac{\pi}{2}\right) = a \cos \omega t.$$

The physical meaning of the abbreviation ω is therefore clear. It is the circular frequency, i.e., the number of vibrations in 2π units of time; T being the period of oscillation, ν the frequency¹², we have the relation

$$(22) \quad \omega = \frac{2\pi}{T} = 2\pi\nu.$$

With the help of this abbreviation (19) can also be written

$$(23) \quad \ddot{x} + \omega^2 x = 0.$$

The equation of energy has the advantage that it always leads to the desired end, no matter how the force X depends on x . In our case, however, where X is linear in x , another much more elegant method exists. It is

¹² As opposed to ω , ν is the frequency, the number of vibrations in *unit* time.

based on the immediately plausible rule that a homogeneous linear differential equation with constant coefficients of any order (x being the dependent, t the independent variable) can always be solved by putting

$$(24) \quad x = C e^{\lambda t},$$

provided λ is chosen to be one of the roots of an algebraic equation obtained from our differential equation. This furnishes a *particular* solution. The *general* solution is obtained by a superposition of all such particular solutions in the form

$$(24a) \quad x = \sum C_j e^{\lambda_j t}.$$

The algebraic equation in λ is obtained by substitution of (24) in (23), and is here of second degree,

$$\lambda^2 + \omega^2 = 0 \text{ with the roots } \lambda = \pm i\omega.$$

The general solution is therefore

$$(24b) \quad x = C_1 e^{i\omega t} + C_2 e^{-i\omega t}.$$

Constants C_1, C_2 are determined by the boundary conditions (19a):

$$\dot{x} = 0 \quad , \quad C_1 i\omega - C_2 \omega i = 0: \quad C_1 = C_2.$$

$$x = a \quad , \quad a = C_1 + C_2 = 2C_1: \quad C_1 = \frac{a}{2}.$$

The final solution of the problem is, in agreement with (21),

$$x = a \cos \omega t.$$

We shall later (Chapter III, § 19) make extensive use of this method for damped, forced, coupled, etc., oscillations, provided these can be described by linear differential equations. The title, "harmonic oscillations," which we have given to this part, calls attention to the fact that the restoring force is linear in the coordinate, so that the resulting motion can be represented by a single constant frequency ω . The method fails in case the binding force is anharmonic, i.e., non-linear; in that event one has to resort to the less elegant method of the energy integral.

5. COLLISION OF TWO PARTICLES

Before the collision (cf. Fig. 1) let the masses m and M have velocities v_0 and V_0 respectively; after the collision they proceed with velocities v and V .



FIG. 1. Collision of two masses M and m ; velocities before collision v_0 and V_0 , after collision V and v .

No matter what the nature of the collision, whether elastic or inelastic, Newton's axiom "action=reaction" must be valid for the forces transmitted between m and M , and also for the time integral Z of these forces. Therefore, according to Eq. (3),

$$(25) \quad m(v - v_0) = Z = -M(V - V_0),$$

and hence also

$$(25a) \quad mv + MV = mv_0 + MV_0.$$

This equation states that the total momentum of the system is conserved.

Let us now introduce in (25a) the coordinate of the center of mass of the system,

$$(25b) \quad \xi = \frac{mx + MX}{m + M}.$$

We obtain

$$\dot{\xi} = \dot{\xi}_0.$$

This result says that the collision has no effect on the velocity of the center of mass.

Thus the center of mass of a shell fired *in vacuo* continues undisturbed in its parabolic path, even if at some point along the path the shell bursts into splinters each of which seems to follow an independent trajectory of its own.

So far we have two unknowns, v and V , and only one equation (25a). In order to find the complete solution of the collision problem a second relation is evidently necessary. We define an *elastic collision* as an interaction in which the kinetic energy as well as the momentum is conserved. We then require

$$(26) \quad \frac{m}{2}v^2 + \frac{M}{2}V^2 = \frac{m}{2}v_0^2 + \frac{M}{2}V_0^2,$$

or

$$m(v^2 - v_0^2) = M(V_0^2 - V^2).$$

But from (25)

$$m(v - v_0) = M(V_0 - V).$$

Division of these two equations yields

$$v + v_0 = V_0 + V,$$

or

$$(26a) \quad V - v = -(V_0 - v_0).$$

This equation states that the relative velocity of one mass with respect to the other after the collision is equal and opposite to that before the collision.

The combination of Eqs. (25a) and (26a),

$$mv + MV = mv_0 + MV_0$$

$$v - V = -v_0 + V_0$$

now completely determines the velocities after the collision,

$$(27) \quad \begin{aligned} v &= \frac{m-M}{m+M}v_0 + \frac{2M}{m+M}V_0 \\ V &= \frac{M-m}{m+M}V_0 + \frac{2m}{m+M}v_0. \end{aligned}$$

Notice that the determinant Δ of this "transformation" from initial values v_0, V_0 to final values v, V has the absolute value 1. For

$$\Delta = \begin{vmatrix} \frac{m-M}{m+M} & \frac{2M}{m+M} \\ \frac{2m}{m+M} & \frac{M-m}{m+M} \end{vmatrix} = -\left(\frac{M-m}{m+M}\right)^2 - \frac{4mM}{(m+M)^2} = -1.$$

This means that if we allow the initial velocities to have a certain range of values, the transformed surface element in v - V space has the same area as the initial surface element; the transformation is *area-preserving* (cf. Fig. 2a). This law is important in collision processes in the kinetic theory of gases and is related to Liouville's theorem (cf. Vol. V, this series).

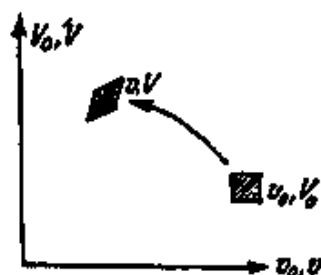


FIG. 2a. Velocity domains before and after collision. The mapping is area-preserving.

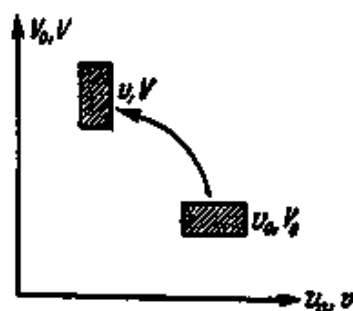


FIG. 2b. In the case of two equal masses, $m=M$, the mapping is not only area-preserving, but also angle-preserving.

Let us consider the case of two equal masses such as two billiard balls, $m=M$. Eqs. (27) become

$$(27a) \quad v = V_0, \quad V = v_0.$$

Now the transformation is not only area-preserving, but angle-preserving as well; cf. Fig. 2b, in which the transformed rectangle is obtained from the initial one by interchange of its sides. In particular, if, in a central collision (head-on collision), in a billiard game, one ball is initially at rest, then the other one transmits all its velocity to the former and thereby comes to rest [cf. (27a) with $V_0=0$].

If, on the other hand, one mass is very great compared to the other, $M \gg m$, the large mass retains practically all its original velocity after the collision, while the small mass follows the large one with a velocity equal to that of the large one minus the original relative velocity. For with $m \ll M$ the Eqs. (27) can be simplified to

$$(27b) \quad v = -v_0 + 2V_0 = V_0 - (v_0 - V_0), \quad V = V_0.$$

To complete the discussion of collisions we shall briefly go into inelastic collisions. In atomic physics one investigates inelastic collisions ("collisions of the second kind"), in which the colliding particle, say an electron, loses part of its energy in order to "excite" the atom with which it collides; such an excited atom has been raised from its ground state to a level of higher energy. Since in this type of process part of the initial energy is lost as far as the motion after collision is concerned, this motion can no longer be calculated using the formulas of elastic collisions. (Cf. problems I.1 to I.4.)

We shall here limit ourselves to the "completely inelastic collision," which is often considered in engineering problems. Such a collision is defined by the condition

$$v = V,$$

i.e., after the collision both masses m and M proceed at the same velocity, as if rigidly coupled together. The equation of momentum, as emphasized earlier, retains its validity under all circumstances; it becomes

$$(28) \quad (m+M)v = mv_0 + MV_0$$

and alone is sufficient to determine the sole unknown v . We would like to know the energy lost in this collision, which is

$$\frac{m}{2}v_0^2 + \frac{M}{2}V_0^2 - \frac{m+M}{2}v^2,$$

or, after a simple elimination of v with the help of (28),

$$(28a) \quad \frac{\mu}{2}(v_0 - V_0)^2.$$

The loss of energy equals the kinetic energy of a certain *reduced mass* μ which moves with the original relative velocity. μ is defined by

$$(28b) \quad \frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}, \text{ therefore } \mu = \frac{mM}{m+M}.$$

The theorem contained in Eqs. (28a, b) was first advanced by General Lazarus Carnot. (General Carnot was a mathematician and the organizer of universal military service during the French Revolution, as well as the father of Sadi Carnot, whose name has become famous in thermodynamics.)

§ 4. Variable Masses

The illustrations cited below will aid us in a critical evaluation of Newton's second law. We put this law in the form (1.3), "change of momentum equals force," rather than in the less general one (1.3a), "mass · acceleration equals force." We shall now learn how the rate of change of momentum is to be understood. We shall show that even in the case of variable mass the general form (1.3) may under certain circumstances reduce to (1.3a).

Let us consider a familiar example: a sprinkler wagon wets the asphalt on a hot summer day. The power of the motor is barely great enough to overcome the combined friction of the ground and wheels, of the air, and in the axle bearings. The vehicle therefore behaves as if under no forces. Let m be the mass of water in the tank at any instant + the constant mass of the empty vehicle. Let the amount of water squirted out per unit time be $\mu = -\dot{m}$, its exit velocity toward the rear, q as seen from the wagon, or $v - q$ as seen from the street, v being the speed of the vehicle.

If we were to use the formula (1.3) mechanically, we would obtain

$$(1) \quad \dot{\mathbf{p}} = \dot{\mathbf{p}} = \frac{d}{dt}(mv) = 0$$

from which would follow

$$(1a) \quad m\dot{v} = \mu v.$$

The acceleration of the wagon would then be independent of the exit velocity q . This is paradoxical, since one would expect the recoil (cf. gun) from the outgoing water jet to have some effect.

Actually we have not used the correct expression for the rate of change of momentum meant in (1.3), for it should consist not only of the member taken into account in (1), but also of a term giving the momentum contained in the water jets. This latter is $\mu(v - q)$ per unit time. Explicitly,

$$p_t = mv_t, \quad p_{t+dt} = (m + dm)(v + dv) + \mu dt(v - q)$$

so that the corrected rate of change of momentum becomes

$$(2) \quad \dot{\mathbf{p}} = \frac{d}{dt}(mv) + \mu(v - q) = 0,$$

or, remembering $\mu = -\dot{m}$ and simplifying,

$$(3) \quad m\dot{v} = \mu q.$$

From the viewpoint of (1.3a), the recoil of the water leaving the vehicle acts as an accelerating force on the latter, just as in the reaction water wheel used in rotary lawn sprinklers.

Instead of the sprinkler wagon we could have chosen as our example the interplanetary rocket, with which one might reach the moon. The rocket would be propelled by the expulsion of explosive gases. See problem I.5.

We generalize this result in two statements which are equivalent to Eqs. (2) and (3), respectively, of our illustrative example:

Either we take the viewpoint of (1.3), where we must then add to the change in momentum contained in the body in question the momentum convectively given off or added per unit time. The latter is to be calculated in the same frame of reference as the momentum of the body under investigation; the sign of \dot{m} takes care of the correct sign for this term. The equation of motion then becomes

$$(4) \quad \frac{d}{dt}(m\mathbf{v}) - \dot{m}\mathbf{v}' = \mathbf{F},$$

where \mathbf{v}' is the convective velocity. In our case we had $-\dot{m} = \mu$ and $|\mathbf{v}'| = |\mathbf{v}| - q$.

Or we take the viewpoint of (1.3a), in which case we must, however, add the recoil momentum gained or lost per unit time as a kind of external force. We then obtain the equation of motion in a form analogous to (3),

$$(5) \quad m\dot{\mathbf{v}} = \mathbf{F} + \dot{m}\mathbf{v}_{\text{rel}}.$$

\mathbf{v}_{rel} is the relative velocity of the convective momentum with respect to the body under observation, measured positive in the same sense as \mathbf{v} . In our example we had $|\mathbf{v}_{\text{rel}}| = -q$ and again $-\dot{m} = \mu$.

Two special cases deserve our attention:

(a) $\mathbf{v}' = 0$. The elements of mass gained or lost have zero velocity and therefore do not carry any momentum. In that case the equation of motion has the Newtonian form $\dot{\mathbf{p}} = \mathbf{F}$. Examples: water drop, chain, problems I.6 and I.7.

(b) $\mathbf{v}' = \mathbf{v}$ or, equivalently, $\mathbf{v}_{\text{rel}} = 0$. The equation of motion has the form, mass \cdot acceleration = force, in spite of the fact that the mass involved is variable. Example: rope hanging over the edge of a table, problem I.8.

In case (b) the Carnot energy loss, Eq. (3.28a), is zero; therefore the equation of energy applies in the usual form. In case (a) the form of the energy conservation law valid for a given problem is not obvious and must first be investigated.

We conclude these instructive remarks with the problem of the relativistic variation of mass. We shall talk specifically about the electron, even though equation (2.20) is of course valid not only for it, but for all masses. Here the variation of mass is a purely internal affair of the electron; there is no question of any momentum gained from or lost to the surroundings. As in case (a) the equation of motion is therefore $\dot{\mathbf{p}} = \mathbf{F}$, i.e., in view of (2.20),

$$(6) \quad \frac{d}{dt} \left(\frac{m_0 \mathbf{v}}{(1 - \beta^2)^{\frac{1}{2}}} \right) = \mathbf{F}.$$

Let us first consider the rectilinear motion of an electron; \mathbf{F} acts longitudinally, that is, in the direction of \mathbf{v} , so that $\mathbf{F} = F_{\text{long}}$ and $\mathbf{v} = v$.

We shall change Eq. (6) to the form "mass · acceleration = force," a customary procedure in the early part of the century, though unnecessarily complicated. To this end we carry out the differentiation on the left,

$$(6a) \quad \frac{m_0 \dot{v}}{(1 - \beta^2)^{\frac{1}{2}}} + m_0 v \frac{d}{dt} (1 - \beta^2)^{-\frac{1}{2}} = \frac{m_0}{(1 - \beta^2)^{\frac{1}{2}}} \left(\dot{v} + \frac{v \beta \dot{\beta}}{1 - \beta^2} \right).$$

Now $\beta = v/c$ so that

$$\dot{\beta} = \frac{\dot{v}}{c} \quad \text{and hence} \quad v \beta \dot{\beta} = \beta^2 v.$$

Consequently Eq. (6a) becomes

$$(6b) \quad \frac{m_0 \dot{v}}{(1 - \beta^2)^{\frac{1}{2}}} \left(1 + \frac{\beta^2}{1 - \beta^2} \right) = \frac{m_0}{(1 - \beta^2)^{\frac{1}{2}}} \dot{v} = F_{\text{long}}.$$

The *longitudinal mass* multiplying the acceleration \dot{v} is therefore

$$(7) \quad m_{\text{long}} = \frac{m_0}{(1 - \beta^2)^{\frac{3}{2}}}.$$

If, on the other hand, \mathbf{F} acts transversely, i.e., normal to the trajectory, only the direction, not the magnitude of the velocity is altered. In that case $\dot{\beta}$ is zero; (6) simply yields

$$\frac{m_0}{(1 - \beta^2)^{\frac{1}{2}}} \dot{v} = F_{\text{trans}}.$$

For this reason one introduced at the time a *transverse mass* different from the longitudinal mass and given by

$$(8) \quad m_{\text{trans}} = \frac{m_0}{(1 - \beta^2)^{\frac{1}{2}}}.$$

In view of these complications we emphasize that the above distinction between two kinds of masses becomes unnecessary if we use only the rational form (6) of the equation of motion.

Next we wish to determine the form of the equation of energy in relativity theory. Let us therefore multiply (6) by $\frac{dx}{dt} = v = \beta c$. In the right member we obtain

$$(9) \quad F \frac{dx}{dt} = \frac{dW}{dt} = \text{work done, or power.}$$

In the left member we have

$$m_0 c^2 \beta \frac{d}{dt} \left(\frac{\beta}{[1 - \beta^2]^{\frac{1}{2}}} \right) = m_0 c^2 \beta \dot{\beta} (1 - \beta^2)^{-\frac{1}{2}}.$$

We can at once convince ourselves that this is a total derivative in t , viz.,

$$(10) \quad m_0 c^2 \frac{d}{dt} \frac{1}{(1 - \beta^2)^{\frac{1}{2}}}.$$

Since (10) must be equal to (9), the rate of doing work, (10) must be the time rate of change of the kinetic energy T . We therefore have

$$T = m_0 c^2 \left(\frac{1}{[1 - \beta^2]^{\frac{1}{2}}} + \text{const.} \right).$$

We must put the constant equal to -1 , because T , by its nature, must vanish as β vanishes. Hence the *relativistic kinetic energy* is

$$(11) \quad T = m_0 c^2 \left(\frac{1}{[1 - \beta^2]^{\frac{1}{2}}} - 1 \right).$$

In view of (2.20) we can also write this as

$$(12) \quad T = c^2 (m - m_0).$$

In words: *the difference in energy between a moving electron and one at rest (which is nothing but the kinetic energy or "live force") equals the difference between the masses of the electron in motion and at rest, multiplied by c^2 .* Thus we have verified for the simplest case the law of the equivalence of mass and energy (law of the "inertia of energy"). This important law is fundamental to the whole field of atomic weight determinations and to nuclear physics and its applications to cosmology.

For the sake of completeness we point out that for small β , (11) can be expanded in a series which yields to a first approximation the elementary expression for T ,

$$T = m_0 c^2 \left(\frac{1}{2} \beta^2 + \frac{3}{8} \beta^4 + \dots \right) = \frac{m_0}{2} c^2 \beta^2 \left(1 + \frac{3}{4} \beta^2 + \dots \right) \rightarrow \frac{m_0}{2} v^2,$$

as is to be expected.

§ 5. Kinematics and Statics of a Single Mass Point in a Plane and in Space

Kinematics deals with the geometry of motions regardless of their physical realization. Statics¹⁴ is concerned with forces, their composition and equivalence, without regard to the motions caused by them.

(1) Plane Kinematics

We shall begin by writing down the formulas for the decomposition and composition of velocity and acceleration in Cartesian coordinates.

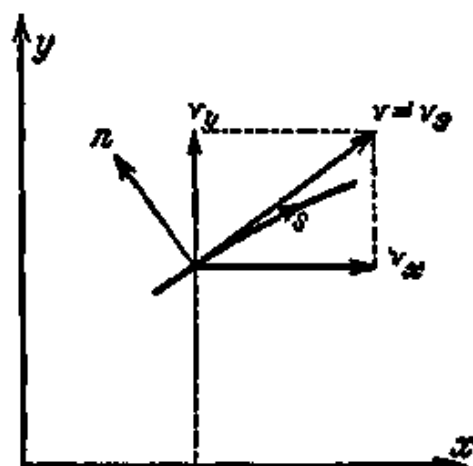


FIG. 3. Decomposition and composition of velocities in a plane; intrinsic coordinates s and n .

Velocity:

$$(1) \quad \mathbf{v} = (v_x, v_y) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (\dot{x}, \dot{y});$$

$$(2) \quad |\mathbf{v}| = (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} = v.$$

Acceleration:

$$(3) \quad \dot{\mathbf{v}} = (\dot{v}_x, \dot{v}_y) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right) = (\ddot{x}, \ddot{y});$$

$$(4) \quad |\dot{\mathbf{v}}| = (\ddot{x}^2 + \ddot{y}^2)^{\frac{1}{2}}.$$

Instead of decomposing velocity and acceleration in Cartesian coordinates we can also decompose them in terms of the *intrinsic coordinates* of the curve described by our mass point. Let s be the length of arc, subscript s denoting the path direction varying from point to point along the curve,

¹⁴ The name statics is actually not appropriate, for it refers only to equilibrium, whereas the content of statics applies to problems of motion as well as of equilibrium. The correct name would be dynamics. This term has been in historical usage for the study of motions caused by forces, and is therefore not available for the field which its name implies, i.e., the analysis of forces.

subscript n denoting the direction normal to s at any point of the curve. We then have

$$(5) \quad v_s = \pm v, v_n = 0.$$

This is trivial. The decomposition of $\dot{\mathbf{v}}$ into $\dot{\mathbf{v}}_s$ and $\dot{\mathbf{v}}_n$ is, however, significant. If we let α be the angle between the tangent to the path and the x -direction, we have

$$(6) \quad \dot{v}_s = \dot{v}_x \cos \alpha + \dot{v}_y \sin \alpha$$

for the tangential acceleration, and

$$(7) \quad \dot{v}_n = -\dot{v}_x \sin \alpha + \dot{v}_y \cos \alpha$$

for the normal acceleration.

Now

$$\cos \alpha = \frac{dx}{ds} = \frac{\dot{x}}{\dot{s}} = \frac{v_x}{v}, \quad \sin \alpha = \frac{dy}{ds} = \frac{\dot{y}}{\dot{s}} = \frac{v_y}{v},$$

so that

$$(8) \quad \begin{aligned} \dot{v}_s &= \frac{1}{v} (v_x \dot{v}_x + v_y \dot{v}_y) = \frac{1}{2v} \frac{d}{dt} (v_x^2 + v_y^2) \\ &= \frac{1}{2v} \frac{d}{dt} v^2 = \frac{dv}{dt} = |\dot{\mathbf{v}}|. \end{aligned}$$

This equation states that the tangential acceleration is the change in magnitude of the velocity, no matter what its change in direction may be.

Eq. (7), on the other hand, yields

$$(9) \quad \dot{v}_n = \frac{1}{v} (v_x \dot{v}_y - v_y \dot{v}_x) = \frac{1}{v} (\dot{x}\dot{y} - \dot{y}\dot{x}) = v^2 \cdot \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{v^3}{\rho},$$

where $\frac{1}{\rho}$ is the curvature of the path.¹⁵

The normal acceleration therefore does not depend on the change of velocity, but only on the velocity itself and on the shape of the trajectory. If, in particular, $\frac{dv}{dt} = 0$, the acceleration is normal to the velocity and hence to the path.

We shall now derive the same relations in a direct, differential-geometrical fashion by means of the *hodograph*¹⁶ introduced by Hamilton.

¹⁵ Cf., for example, Franklin, *Treatise on Advanced Calculus*, p. 295.—TRANSLATOR.

¹⁶ The name hodograph= path writer is misleading; it should really be called velocity writer, or better, polar diagram of velocity.

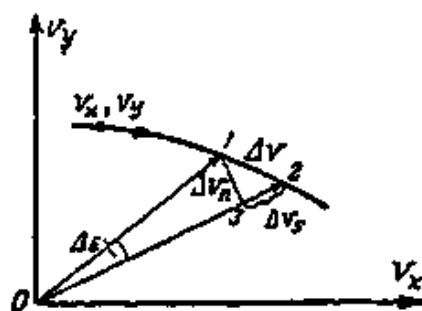


FIG. 4a. Hodograph of motion in a plane. Velocities v_1 and v_2 are laid off from the pole O in the polar diagram.

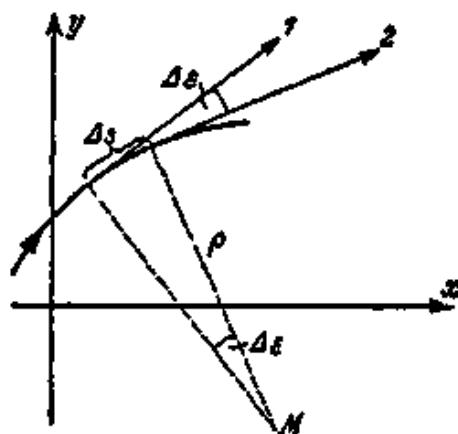


FIG. 4b. Trajectory and radius of curvature of motion in a plane.

The meaning of the hodograph becomes clear when we compare Figs. 4a and 4b. Fig. 4b shows the trajectory in the xy -plane. The velocities at two of its neighboring points, Δs apart, are indicated as tangents to the path; their included angle is $\Delta\epsilon$. The same angle $\Delta\epsilon$ also occurs at the center of curvature M . ρ being the radius of curvature,

$$(10) \quad \Delta s = \rho \Delta\epsilon.$$

The same two velocities are plotted in Fig. 4a from a common origin O , with directions preserved. Consider the two neighboring vectors $\vec{O1}$ and $\vec{O2}$ with $\Delta\epsilon$ as their included angle. Projection of 1 on $\vec{O2}$ gives point 3. $\Delta v = \vec{12}$ is decomposed into $\Delta v_n = \vec{32}$ and $\Delta v_s = \vec{13}$. We therefore obtain, in agreement with (8) and (9),

$$\begin{aligned} \dot{v}_s &= \frac{\vec{32}}{\Delta t} = \frac{v_2 - v_1}{\Delta t} = \frac{\Delta v}{\Delta t} = \frac{dv}{dt}, \\ \dot{v}_n &= \frac{\vec{13}}{\Delta t} = \frac{\Delta\epsilon \cdot v}{\Delta t} = \frac{\Delta\epsilon}{\Delta s} v^2 = \frac{v^2}{\rho}, \end{aligned}$$

the latter by recalling (10). Cf. problem I.9.

(2) The Concept of Moment in Plane Statics and Kinematics

The moment of a vector quantity \mathbf{E} about a given point of reference O is defined as the *vector product* of the radius vector \mathbf{r} from O to the point of application P of the vector \mathbf{E} by the vector \mathbf{E} itself, i.e.,

$$(11) \quad \mathbf{N} = \mathbf{r} \times \mathbf{E}.$$

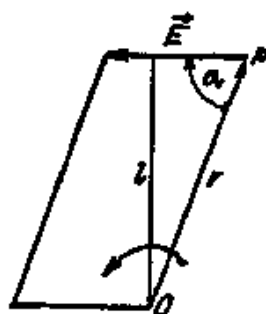


FIG. 5. Moment of an arbitrary vector quantity \vec{E} about an arbitrary point O .

N is therefore represented by the area of the parallelogram formed by \vec{r} and \vec{E} together with the sense of rotation of \vec{r} into \vec{E} as indicated by an arrow in Fig. 5. In magnitude,

$$(11a) \quad |N| = l|\vec{E}| = r|\vec{E}|\sin \alpha,$$

where l is the "lever arm" of \vec{E} about O . If we take for \vec{E} a force \vec{F} , we obtain the *moment of the force \vec{F} , or torque*

$$(12) \quad \vec{L} = \vec{r} \times \vec{F}.$$

The moment of a force \vec{F} is a fundamental concept of statics, whose discovery goes back to Archimedes himself. Let us denote the Cartesian components of \vec{F} by X and Y . Elementary vector algebra readily gives

$$(12a) \quad L_z = xY - yX.$$

The concept of moment is of importance in kinematics and kinetics as well. Let us still restrict ourselves to problems in a plane, and form

the moment of velocity $= \vec{r} \times \vec{v}$

the moment of acceleration $= \vec{r} \times \dot{\vec{v}}$

the moment of momentum $=$ angular momentum $= \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$

In Cartesian coordinates, with (12a) as a model, we have

$$(13) \quad \vec{r} \times \vec{v} = x\dot{y} - y\dot{x}, \quad \vec{r} \times \dot{\vec{v}} = x\ddot{y} - y\ddot{x}.$$

Between the moments of velocity and acceleration, there exists the relation

$$(14) \quad \vec{r} \times \dot{\vec{v}} = \frac{d}{dt}(\vec{r} \times \vec{v}).$$

It derives from the fact that $\frac{d\vec{r}}{dt} = \vec{v}$ and $\vec{v} \times \vec{v} = 0$, so that

$$(14a) \quad \frac{d}{dt}(\vec{r} \times \vec{v}) = \vec{r} \times \frac{d\vec{v}}{dt} + \vec{v} \times \vec{v} = \vec{r} \times \dot{\vec{v}}.$$

The customary proof by means of decomposition into coordinates runs exactly parallel to Eq. (14a):

$$(14b) \quad \frac{d}{dt}(xy - yx) = x\dot{y} + \dot{x}y - y\dot{x} - \dot{y}x = x\dot{y} - y\dot{x}.$$

If in Fig. 5 one thinks of the velocity \mathbf{v} of point P replacing the arbitrary vector \mathbf{E} , with P describing an arbitrary path, one can read off another simple relation, now between angular momentum and the so-called *areal velocity*. Indeed the infinitesimal element of area dS swept out by the radius vector \mathbf{r} with origin at O is equal to one half the area of the parallelogram $\mathbf{r} \times d\mathbf{s}$, so that the areal velocity

$$\frac{dS}{dt} = \frac{1}{2}(\mathbf{r} \times \mathbf{v}).$$

We therefore obtain the relation between areal velocity and angular momentum

$$(15) \quad \mathbf{r} \times \mathbf{p} = 2m \frac{dS}{dt}.$$

(3) Kinematics in Space

We decompose vectors along the three directions s (tangent), n (principal normal) and b (binormal) of the three-dimensional trajectory to obtain the following components:

$$\mathbf{v} = (v, 0, 0),$$

$$\dot{\mathbf{v}} = (\dot{v}, \frac{v^2}{\rho}, 0).$$

ρ is the radius of curvature introduced in (9) or (10), now constructed so as to lie in the osculating plane of the trajectory.

If we pass to the moments of velocity and acceleration, we keep the definitions $\mathbf{r} \times \mathbf{v}$ and $\mathbf{r} \times \dot{\mathbf{v}}$, but note that Fig. 5 must now be thought of as three-dimensional. In addition to magnitude and sense of rotation the parallelogram drawn there also has position in space. Because it is helpful in visualizing this point it has become customary to indicate the position by a normal to the plane of the parallelogram. By convention that side of the normal is chosen which points in the direction of advance of a right-handed screw rotated in the sense of rotation of the moment (from \mathbf{r} to \mathbf{v} or $\dot{\mathbf{v}}$ through an angle less than π). The vector picture of the moment then becomes an arrow pointing along this normal, its length being equal to the magnitude of the moment. In Fig. 5 one should, therefore, think of the moment as directed out of and perpendicular to the plane of the paper. We shall postpone a thorough investigation of this procedure and of the difference between axial and polar vectors to Chapter IV, § 23.

So far we have described the moment about an arbitrarily chosen point of reference O . In the following subsection we shall explain what is meant by the moment about a given axis.

(4) Statics in Space; Moment of Force About a Point and About an Axis

The moment of a force F about a reference point O is completely defined by

$$(16) \quad \mathbf{L} = \mathbf{r} \times \mathbf{F},$$

where \mathbf{r} is the radius vector from O to the point of application P of F ,

$$(16a) \quad \mathbf{r} = (x, y, z)$$

if O is taken as origin of coordinates. \mathbf{L} can be represented as a vector by the rule just given for moments (rule of the right-hand screw, with length of vector equal to $|\mathbf{L}|$). We now ask: what are the components of \mathbf{L} along the coordinate axes? We can define them as the projections of the moment vector on these three axes; for instance,

$$(17) \quad L_z = |\mathbf{L}| \cos(\mathbf{L}, z)$$

But $|\mathbf{L}|$ is the area of the parallelogram having sides \mathbf{r} and \mathbf{F} . The right member of (17) is therefore at the same time the projection of the area of the parallelogram on the x - y -plane. The latter has sides

$$\mathbf{r}_{\text{proj}} = (x, y); \quad \mathbf{F}_{\text{proj}} = (X, Y),$$

so that, with the help of (17), we obtain as in (12a)

$$(17a) \quad L_z = xY - yX,$$

and similarly

$$(17b) \quad L_x = yZ - zY, \quad L_y = zX - xZ.$$

The components L_x, L_y, L_z of \mathbf{L} can be called the moments of the force F about axes x, y, z . Cf. problem I.10.

What has been said of the coordinate axes also applies to any arbitrary axis a . The moment of a force F about an axis a is defined, just as in (17), by taking the moment about a point O located on a and projecting the corresponding moment vector on a . Or it can be formed as in (17a, b) by projecting the area of the moment about O on a plane perpendicular to a . A third method consists in finding the shortest distance from the point of application of the force to a , which we shall call the lever arm l . In this case F is decomposed into three components, F_a parallel to a , F_l in the

direction of l , and F_n in a direction perpendicular to both a and l . We then have

$$(18) \quad L_a(F) = L_a(F_a) + L_a(F_l) + L_a(F_n).$$

The first two terms on the right must vanish; for there can arise no moment of force about an axis a if the force is either parallel to a or if it intersects a .

There remains the third term which results from a force perpendicular to a acting with a lever arm l . Instead of (18) we have, therefore,

$$(18a) \quad L_a(F) = L_a(F_n) = F_n \cdot l.$$

At this point it may be well to say a few words about the different notations for the products of two vectors. The following list shows that unfortunately these notations vary widely, both in historical and national usage.

Name of product	This book	German Ed. SOMMERFELD	GIBBS	HEAVISIDE	ITALIANS	GRASSMANN
Scalar or inner ..	$A \cdot B$	$(\mathfrak{A} \mathfrak{B})$	$\mathfrak{A} \mathfrak{B}$	$\mathfrak{A} \mathfrak{B}$	$\mathfrak{A} \times \mathfrak{B}$	$\mathfrak{A} \mathfrak{B}$
Vector or outer ..	$A \times B$	$[\mathfrak{A} \mathfrak{B}]$	$\mathfrak{A} \times \mathfrak{B}$	$\nabla \mathfrak{A} \mathfrak{B}$	$\mathfrak{A} \wedge \mathfrak{B}$	$\mathfrak{A} \mathfrak{B}$ or $ \mathfrak{A} \mathfrak{B}$

Some explanatory remarks follow. The great thermodynamicist Willard Gibbs made a short summary of vector analysis, then still little known, for the use of his students. His notation is still followed (with slight variations) by many American and British authors. Heaviside's notation for the vector product, in which ∇ stands for vector, was thereupon generally abandoned. The Italian notation originated with Marcolongo. Hermann Grassmann, in his "Ausdehnungslehre" (Extension Analysis, 1844 and 1862), had developed a logical system of calculation with segments and points. According to him the simplest relation between two directed segments a and b , is the "planar magnitude" (*Plangrösse*), i.e., the parallelogram formed by a and b , which he therefore denotes by ab (though occasionally also by $[ab]$). The vertical line in Grassmann's notation for the vector product means "complement" (*Ergänzung*), that is, denotes passing to the vector arrow perpendicular to the planar magnitude.

§ 6. Dynamics (Kinetics) of the Freely Moving Mass Point; Kepler Problem; Concept of Potential Energy

(1) Kepler Problem with Fixed Sun

The simplest example we can think of in connection with a freely moving mass point is, at the same time, the most important for our picture

of the universe, namely, the motion of the planets. It is a two-dimensional problem, and motion takes place in the ecliptic if the planet in question is the earth. We assume the sun to be fixed in position and justify this by its large relative mass,

$$\text{Sun } 330000, \text{ Jupiter } 320, \text{ Earth } 1, \text{ Moon } \frac{1}{81}.$$

We shall deal with the problem including the sun's motion in Part 2 of this section. Let M be the mass of the sun, m that of the planet. The Newtonian attraction is

$$|F| = G \frac{mM}{r^2}, \quad G = \text{gravitational constant},$$

or, vectorially,

$$(1) \quad \mathbf{F} = -G \frac{mM}{r^2} \frac{\mathbf{r}}{r}.$$

It passes through the fixed point O at the center of the sun, which serves as origin for the radius vector \mathbf{r} .

It follows that $\mathbf{r} \times \mathbf{F} = 0$ and therefore, by the Second Law,

$$\mathbf{r} \times \dot{\mathbf{p}} = 0 \text{ and in view of (5.14), } \mathbf{r} \times \mathbf{p} = \text{const.}$$

The angular momentum about the sun is constant, therefore also the areal velocity of Eq. (5.15) is constant. This is the second Kepler law:

The radius vector from the sun to the planet sweeps over equal areas in equal times.

Let the constant areal velocity multiplied by two be called the "areal velocity constant" C ,

$$(2) \quad 2 \frac{dS}{dt} = C.$$

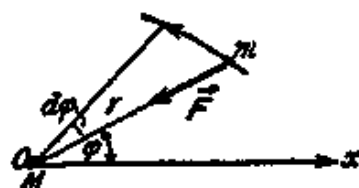


FIG. 6. Polar coordinates for the Kepler problem with sun as origin; area swept out by the radius vector.

We now introduce the polar angle ϕ , the *true anomaly*¹⁷ of the astronomers (cf. Fig. 6), and obtain

$$dS = \frac{1}{2} r^2 d\phi, \quad 2 \frac{dS}{dt} = r^2 \dot{\phi} = C$$

¹⁷ True anomaly is here defined as the angular distance of a planet from its aphelion, as seen from the sun.—TRANSLATOR.

so that

$$(3) \quad \dot{\phi} = \frac{C}{r^2}.$$

In order to derive the first Kepler law, the equation of the trajectory, we decompose the forces along Cartesian coordinates. After division by m the equation of motion becomes

$$(4) \quad \begin{aligned} \frac{dx}{dt} &= -\frac{GM}{r^2} \cos \phi \\ \frac{dy}{dt} &= -\frac{GM}{r^2} \sin \phi. \end{aligned}$$

If we multiply both sides of both equations by $\frac{1}{\dot{\phi}}$ and recall (3), we obtain

$$\begin{aligned} \frac{dx}{d\phi} &= -\frac{GM}{C} \cos \phi \\ \frac{dy}{d\phi} &= -\frac{GM}{C} \sin \phi. \end{aligned}$$

These can now be integrated. Let A and B be constants of integration. The result is

$$(5) \quad \begin{aligned} x &= -\frac{GM}{C} \sin \phi + A \\ y &= \frac{GM}{C} \cos \phi + B. \end{aligned}$$

This means that the hodograph of planetary motion is a circle,

$$(5a) \quad (x-A)^2 + (y-B)^2 = \left(\frac{GM}{C}\right)^2.$$

We shall return to this point in problem I.11. Let us transform the left members of (5) into polar coordinates,

$$x = r \cos \phi, \quad y = r \sin \phi,$$

so that

$$\begin{aligned} \dot{x} &= \dot{r} \cos \phi - r \dot{\phi} \sin \phi = -\frac{GM}{C} \sin \phi + A \\ \dot{y} &= \dot{r} \sin \phi + r \dot{\phi} \cos \phi = \frac{GM}{C} \cos \phi + B. \end{aligned}$$

We now eliminate \dot{r} by multiplication of the first equation by $-\sin \phi$, the second by $\cos \phi$, and subsequent addition. We obtain

$$r \dot{\phi} = \frac{GM}{C} - A \sin \phi + B \cos \phi$$

or, recalling (3),

$$(6) \quad \frac{1}{r} = \frac{GM}{O^2} - \frac{A}{O} \sin \phi + \frac{B}{O} \cos \phi.$$

This is the equation of a conic section in polar coordinates whose origin coincides with one of the foci of the conic section. We therefore obtain the first Kepler law: *The planet describes an ellipse with the sun at one focus.* In this connection we note that two equally possible trajectories, the hyperbola and the parabola, evidently do not apply to the planets, but only to comets. We shall not discuss them here, but refer the reader to problem I.12.

The derivation of the first Kepler law given here differs from that offered in most texts. The latter starts out with the equation of energy, which we shall now derive. We turn back to Eq. (4), where we replace $\cos \phi$ by $\frac{x}{r}$, $\sin \phi$ by $\frac{y}{r}$ in the right members. We then multiply the first of Eqs. (4) by \dot{x} , the second by \dot{y} , add the two, and get

$$\frac{d}{dt} \frac{1}{2} (\dot{x}^2 + \dot{y}^2) = -\frac{1}{2} \frac{GM}{r^3} \frac{d}{dt} (x^2 + y^2) = -\frac{GM}{r^3} \frac{dr}{dt}.$$

An integration with respect to t yields

$$(7) \quad \frac{1}{2} (\dot{x}^2 + \dot{y}^2) = \frac{GM}{r} + E.$$

The left member is the kinetic energy divided by m ; the first term on the right is, apart from sign, the potential energy divided by m (cf. Part 3 of this section); E is therefore the total energy divided by m . Our Eq. (7) has the same form as the equation of energy of one-dimensional motion, (3.8).

In order to pass from (7) to the equation of the path (6) in the simplest possible manner, we recall that in polar coordinates the square of the element of line is

$$dx^2 + dy^2 = dr^2 + r^2 d\phi^2.$$

We therefore have

$$\dot{x}^2 + \dot{y}^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2 = \left(\frac{d\phi}{dt}\right)^2 \left\{ \left(\frac{dr}{d\phi}\right)^2 + r^2 \right\},$$

or, in view of (3),

$$O^2 \left\{ \left(\frac{1}{r^2} \frac{dr}{d\phi} \right)^2 + \frac{1}{r^2} \right\}$$

If we put $s = \frac{1}{r}$, this becomes

$$O^2 \left\{ \left(\frac{ds}{d\phi} \right)^2 + s^2 \right\},$$

so that our equation of energy (7) is transformed into

$$\frac{1}{2}C^2 \left\{ \left(\frac{ds}{d\phi} \right)^2 + s^2 \right\} - GMs = E.$$

Differentiation with respect to ϕ gives

$$\frac{ds}{d\phi} \left\{ C^2 \left(\frac{d^2s}{d\phi^2} + s \right) - GM \right\} = 0.$$

Since $\frac{ds}{d\phi} \neq 0$, the bracket must vanish. Thus we obtain a linear homogeneous differential equation with constant coefficients of second order in s ,

$$\frac{d^2s}{d\phi^2} + s = \frac{GM}{C^2}.$$

The general solution of such an equation consists of a particular solution of the inhomogeneous equation plus the general solution of the homogeneous equation. Evidently

$$s = \text{constant} = \frac{GM}{C^2}$$

is a particular integral of the inhomogeneous equation. The general solution of the homogeneous equation is the sum of $\sin \phi$ and $\cos \phi$. We can now take A/C and B/C as our constants of integration and finally obtain

$$s = \frac{GM}{C^2} - \frac{A}{C} \sin \phi + \frac{B}{C} \cos \phi,$$

which is precisely the previously obtained Eq. (6).

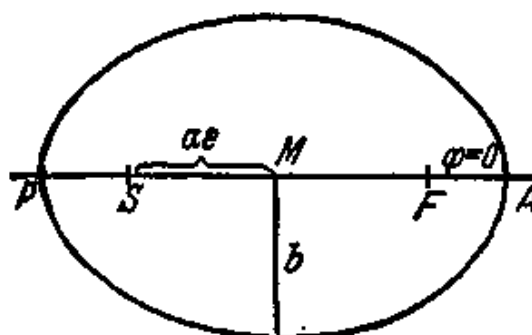


FIG. 7. Kepler ellipse with major and minor axes; perihelion, aphelion, eccentricity.

We shall now specialize this equation in such a way that the line $\phi=0$, which starts out from one focus, passes through the other focus as well, i.e., that it forms, together with the line $\phi=\pi$, the major axis of the ellipse (cf. Fig. 7). On this axis are located the points P, "perihelion" (closest

to the sun) and A , "aphelion" (farthest from the sun), at which r must be a minimum and a maximum respectively. We therefore impose the condition

$$\frac{dr}{d\phi} = 0 \text{ for } \phi = \begin{cases} 0 \\ \pi \end{cases}$$

which, from (6), requires $A=0$.

If, in addition, e is the eccentricity of the ellipse, Fig. 7 shows that at

$$\text{perihelion, } r=SP=a(1-e), \phi=\pi,$$

$$\text{aphelion, } r=SA=a(1+e), \phi=0.$$

According to (6) we then have at

$$\text{perihelion, } \frac{1}{a(1-e)} = \frac{GM}{C^2} - \frac{B}{C},$$

$$\text{aphelion, } \frac{1}{a(1+e)} = \frac{GM}{C^2} + \frac{B}{C}.$$

From these we obtain by addition and subtraction

$$(8) \quad \frac{GM}{C^2} = \frac{1}{a(1-e^2)}, \quad \frac{B}{C} = -\frac{e}{a(1-e^2)},$$

respectively.

We shall finally express the areal velocity constant C in terms of the period T . From (2) we immediately obtain

$$C = \frac{2S}{T} \text{ with } S = \pi ab = \pi a^2(1-e^2)^{\frac{1}{2}}$$

as the total area swept out by the radius vector. It follows that

$$(9) \quad C^2 = \frac{4\pi^2 a^4 (1-e^2)}{T^2}.$$

If we introduce this in the first of Eqs. (8), we have

$$(10) \quad \frac{T^2}{a^3} = \frac{4\pi^2}{GM}.$$

Since G and M are the same for all planetary trajectories, (10) is the expression of the third Kepler law: *the squares of the periodic times are proportional to the cubes of the major axes.*

Kepler greeted the discovery of this law with the enthusiastic statement¹⁸: "Finally I have brought to light and verified beyond all my hopes and

¹⁸ *Harmonices mundi*, 1619. The two first Kepler laws had been published in the *Astronomia Nova*, 1609.

expectations that the whole Nature of Harmonies permeates to the fullest extent, and in all its details, the motion of the heavenly bodies; not, it is true, in the manner in which I had earlier thought, but in a totally different, altogether complete way."

Actually the third Kepler law in the form (10) is not yet quite exact. It is valid only as long as one can neglect the planetary mass m in comparison with the mass M of the sun. We shall now drop this assumption, thereby passing to the two-body problem proper of astronomy. This problem is not significantly more difficult than the one-body problem treated so far.

(2) Kepler Problem Including Motion of the Sun

Let x_1, y_1 be the coordinates of the sun S ; x_2, y_2 those of the planet P .

According to Newton's third law the force on S must be equal and opposite to that on P , so that the complete equations of motion are

$$\begin{array}{ll} \text{for the sun,} & \text{for the planet,} \\ M \frac{d^2 x_1}{dt^2} = \frac{mMG}{r^3} \cos \phi; & m \frac{d^2 x_2}{dt^2} = -\frac{mMG}{r^3} \cos \phi; \\ M \frac{d^2 y_1}{dt^2} = \frac{mMG}{r^3} \sin \phi; & m \frac{d^2 y_2}{dt^2} = -\frac{mMG}{r^3} \sin \phi. \end{array}$$

We now introduce the relative position coordinates

$$(11a) \quad x_2 - x_1 = x, \quad y_2 - y_1 = y,$$

and furthermore the center of mass coordinates

$$(11b) \quad \frac{mx_2 + Mx_1}{m+M} = \xi, \quad \frac{my_2 + My_1}{m+M} = \eta.$$

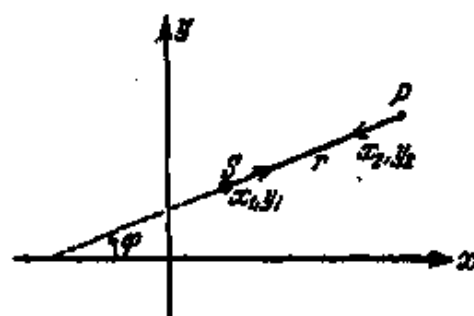


FIG. 8. Kepler problem with motion of sun taken into account.

Subtraction of the equations of motion gives

$$(12) \quad \begin{aligned} \frac{d^2 x}{dt^2} &= -\frac{(M+m)G}{r^3} \cos \phi, \\ \frac{d^2 y}{dt^2} &= -\frac{(M+m)G}{r^3} \sin \phi; \end{aligned}$$

whereas addition yields

$$(13) \quad \frac{d^2\xi}{dt^2} = 0, \quad \frac{d^2\eta}{dt^2} = 0.$$

Comparison of (12) with the earlier Eqs. (4) shows at once that the first two laws of Kepler remain intact, i.e., are valid for the relative motion as well. The third law takes the form

$$(14) \quad \frac{T^2}{a^3} = \frac{4\pi^2}{G(M+m)}.$$

The ratio T^2/a^3 is therefore no longer a universal constant, but in principle is somewhat different for every planet. Because of the preponderance of the sun's mass the deviations from (10) are, however, exceedingly slight.

Eqs. (13) further show that the mass center of sun and planet moves with constant velocity. If we do our calculations in terms of a coordinate system in which the mass center is fixed at the origin, this velocity must be put equal to 0; the same applies to the coordinates ξ, η of the center of mass themselves.

Eqs. (11b) are simplified accordingly. With their help and that of Eqs. (11a) the coordinates x_1, y_1 of the sun and the coordinates x_2, y_2 of the planet can now be expressed separately in terms of the relative position coordinates x, y :

$$x_1, y_1 = -\frac{m}{M+m}(x, y),$$

$$x_2, y_2 = \frac{M}{M+m}(x, y).$$

It follows that in the center of mass system the trajectories of the sun and planet are also ellipses; that of the planet is almost identical to the ellipse considered in Part I of this section. That of the sun is a very dwarfed ellipse, traversed in the same sense, but π out of phase with the planet's trajectory.

If we change the law of gravitation to

$$(15) \quad F = kr^n \cdot \frac{\mathbf{r}}{r}, \quad n \text{ arbitrary},$$

the second Kepler law will hold unchanged; the trajectories, however, become transcendental curves which are, in general, not closed. It is only in the case $n=1$ that we obtain ellipses just as in the case of gravitation, $n=-2$. (Cf. problem I.13).

(3) When Does a Force Field Have a Potential?

In one-dimensional motion we were able to define a potential energy V connected with a force X without any difficulty—see Eq. (3.7). As

mentioned at that time, this is possible for two- or three-dimensional motions only if certain conditions are met. If X , Y , Z are the Cartesian components of the force \mathbf{F} the definition of potential energy for the three-dimensional case analogous to (3.7) would be

$$(16) \quad V = - \int_{\text{sys}} (X \, dx + Y \, dy + Z \, dz).$$

If V is to be a quantity independent of the path of integration and dependent only on its endpoint (the choice of the initial point merely gives rise to an additive constant which remains arbitrary in any case), the expression

$$X \, dx + Y \, dy + Z \, dz$$

must be a *perfect differential*; i.e., X , Y , Z must be the derivatives with respect to x , y , z of a "field function." In our case the function is just $-V$, and we say that \mathbf{F} is "derivable from the potential V ." The well-known conditions for this are that

$$(17) \quad \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}, \quad \frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z}, \quad \frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x}.$$

It is only if these conditions are fulfilled that a field function $V(x, y, z)$ can be defined for each point x, y, z ; V is called the potential energy or simply the potential.

In the two-dimensional case, where $Z=0$ and X , Y are independent of z , the three Eqs. (17) are reduced to the first one of these.

Vector analysis (which has been relegated to Vol. II of this series since we need only vector algebra in this volume) shows that the conditions (17) have an invariant meaning, i.e. are independent of the choice of coordinates. In Vol. II these conditions will be summarized in the vector equation, $\text{curl } \mathbf{F} = 0$ (this is often expressed by saying that the vector field \mathbf{F} is *irrotational*).

Evidently one can without difficulty write down expressions for X , Y , Z in terms of x , y , z which do not satisfy conditions (17). On the other hand we see that these conditions are satisfied for the gravitational field

$$X = Y = 0, \quad Z = -mg$$

and lead to

$$(18) \quad V = mgz.$$

The same is true of the general gravitational fields based on Newton's law and the mathematically similar fields of electrostatics and magnetostatics. As a matter of fact fields that are irrotational and simultaneously time-independent ("potential fields") occupy a unique position in nature.

They will play a special role in the general developments of Chapters VI and VIII.

A mechanical system in which only forces derivable from potentials act is called a *conservative system* because its energy is conserved; otherwise we speak of non-conservative or *dissipative systems*.

MECHANICS OF SYSTEMS, PRINCIPLE OF VIRTUAL WORK, AND D'ALEMBERT'S PRINCIPLE

§ 7. *Degrees of Freedom and Virtual Displacements of a Mechanical System; Holonomic and Non-holonomic Constraints*

The single mass point has one degree of freedom if its motion is restricted to a straight line or a curve, two degrees of freedom if it is made to move in a plane or on a curved surface; the mass point moving freely in space has three degrees of freedom.

Two mass points connected by a weightless, rigid rod have five degrees of freedom; for the first point can be regarded as freely moving, in which case the second is restricted to the surface of a sphere described about the first, its radius equal to the length of the rod.

The number of degrees of freedom for n mass points which are coupled by r relations between their coordinates is

$$(1) \quad f = 3n - r.$$

If there is an infinity of mass points connected by infinitely many conditions such an enumeration is of course not feasible. The procedure to be used in that case will now be shown, the rigid body serving as example.

(a) *Freely Moving Rigid Body*

We single out a point of the rigid body. It has three degrees of freedom. A second point, at a constant distance from the first (definition of "rigid"), can move only on a spherical surface about the first point as center. This gives two more degrees of freedom. Finally a third point can describe a circle about the axis connecting the first two points, thus contributing one more degree of freedom. Once the motions of these three points have been specified, the paths of all other points of the rigid body are uniquely determined. It follows that

$$f = 3 + 2 + 1 = 6.$$

(b) *Top on a Plane Surface*

We assume that the bottom of the spinning top terminates in a point, and take this as the first point of our enumeration; it has two degrees of

freedom. A second point can move on a hemisphere about the first, and a third one on a circle about a line connecting the first two. Thus

$$f=2+2+1=5.$$

(c) *Top with Fixed Point*

Now the two degrees of freedom of the first point are lost, so that

$$f=2+1=3.$$

(d) *Rigid Body with Fixed Axis—Pendulum*

Here

$$f=1.$$

If the center of mass of the body does not lie on the axis we speak of a physical or *compound pendulum*. From this we obtain a mathematical or *simple pendulum* if the body shrinks to a point. The *spherical pendulum* — a mass point restricted in its motion to the surface of a sphere — has

$$f=2.$$

(e) *Infinitely Many Degrees of Freedom*

For a deformable solid body or a liquid

$$f=\infty.$$

In that case the equations of motion become partial differential equations. By contrast a system with a finite number of degrees of freedom n is determined by an equal number n of ordinary differential equations of second order.

(f) *Machine with One Degree of Freedom*

Such a machine consists of a series of nearly rigid bodies coupled to each other either by links or by means of guides of various types. The classical example of such a machine is the drive mechanism of a piston engine (Fig. 9). If the machine is provided with a centrifugal governor (also called Watt governor because it was first proposed by the inventor of the steam engine), it acquires a second degree of freedom.

In the aforementioned examples the number of degrees of freedom equals the number of independent coordinates which are necessary to determine the position of the system. The coordinates need not be Cartesian. In case of the drive mechanism we can equally well specify either the coordinate x determining the position of the piston or the angle ϕ giving the position of the crank pin on the shaft. In general we shall call the independent coordinates of a system of f degrees of freedom

$$(2) \quad q_1, q_2, \dots, q_f.$$

They can, within certain limits, be chosen arbitrarily. The r conditions among the coordinates referred to in Eq. (1) can be satisfied identically by suitable choice of the q , so that they drop out of the subsequent treatment of our system.

The mechanics of Hertz mentioned on p. 5 has the important merit of having called attention to *conditions of differential form*, to which the foregoing cannot be applied. Such a condition can be written as

$$(3) \quad \sum_{k=1}^f F_k(q_1 \dots q_f) dq_k = 0.$$

Here we assume that the F_k do not all have the form $\frac{\partial \Phi}{\partial q_k}$, so that (3) is not the total differential of some function $\Phi(q_1 \dots q_f)$, and we assume, moreover, that it cannot be converted into a total differential by means of an integrating factor.

In agreement with Hertz we shall call conditions of the form $\Phi(q_1 \dots q_f) = \text{const.}$ *holonomic* (*holos* in Greek = *integer* in Latin = whole = integrable); conditions of the form (3) which cannot formally be integrated will be called *non-holonomic*.¹ The simplest example of a non-holonomic condition is furnished by a sharp-edged wheel rolling on a horizontal plane, cf. problem II.1 (the sleigh and the flexible coupling mechanism of a bicycle also belong in this category). Such a wheel is restricted to move always in the direction it may have at any given instant. Nevertheless it is able to reach all points of its supporting plane, even if at times only by pivoting about its sharp point of contact. It therefore possesses more degrees of freedom in finite than in infinitesimal motion. In general, if a system subject to r non-holonomic conditions has f degrees of freedom in finite motion, it has only $f-r$ degrees of freedom in infinitesimal motion. This point will be investigated in problem II.1.

The foregoing distinction is important for the concept of virtual displacements. A *virtual displacement* is an arbitrary, instantaneous, infinitesimal change of the position of the system compatible with the conditions of constraint. Whereas we shall denote real displacements due to given forces under given conditions by

$$dq_1, dq_2 \dots dq_f,$$

the symbols

$$\delta q_1, \delta q_2 \dots \delta q_f$$

will be used to denote virtual displacements. The δq have nothing to do

¹ A. Voss made a general study of such conditions in 1884, long before Hertz; cf. *Math. Ann.* 25.

with the actual motion. They are introduced, so to speak, as test quantities, whose function it is to make the system reveal something about its internal connections and about the forces acting on it.

For purely holonomic constraints the δq are independent of each other, one δq corresponding to each degree of freedom. A larger number of δq must be introduced for non-holonomic constraints; in that case the δq are related by differential conditions of the form (3), or, for virtual displacements,

$$(4) \quad \sum_{k=1}^f F_k(q_1 \dots q_f) \delta q_k = 0.$$

Here f is the number of degrees of freedom for finite motion. As previously emphasized, this number is greater than that for infinitesimal motion.

§ 8. The Principle of Virtual Work

Let us consider a mechanical system in equilibrium under applied forces. The forces may have any desired direction, may act on various parts of the system, and need not have the positions required for the equilibrium of a simple rigid body. Whether the forces lead to the equilibrium of the system under investigation depends as much on the system as on the forces.

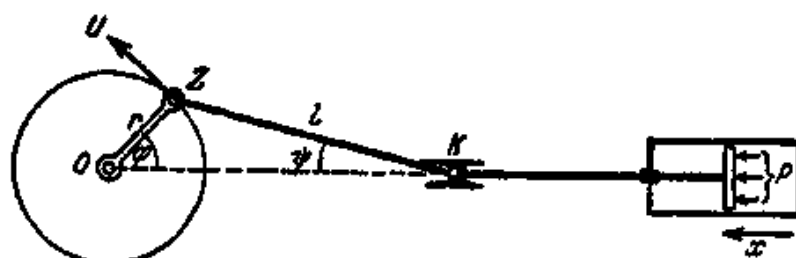


FIG. 9. Schematic diagram of the drive mechanism of a piston engine.

In the spirit of elementary particle mechanics we would ask for the *reactions* which are exerted by one part of the system on another due to the applied forces. This procedure would, for instance, be used by a mechanical engineer in the analysis of the crank mechanism (Fig. 9). The steam pressure P acting on the piston is transmitted by the piston rod to the crosshead K , whence it is passed on as longitudinal compression to the connecting rod of length l . The connecting rod acts on the crank pin Z with a thrust which has the direction of the rod. In order that the system be in equilibrium only that part U of the thrust which is perpendicular to the crank, therefore tangential to the crank circle, need be opposed by an equal applied force. The component in the direction of the crank,

i.e., toward the center of the crank shaft, is absorbed by the rigidly fixed shaft bearing, O . It only puts a stress on the bearing and is irrelevant to the question of the equilibrium of the system.

It is therefore the reactions within the system which make equilibrium possible. To investigate them individually is possible in simple cases, but tedious in general. We can, however, assert without knowing them in detail that *they do no work on the system*. In our case the guide pressure at the guide rails is perpendicular to the motion of the crosshead, and that part of the force acting on the crank pin which is transmitted to the crank shaft acts through the fixed point O of the crank shaft bearing. We establish this assertion in the general case by giving the system a tentative virtual displacement from its position of equilibrium. The "virtual work" of the reactions in such a displacement is found to be zero.

Let us verify the principle in detail on the simple rigid body. We must imagine that every point i is related to every point k of the body by means of reactions R_{ki} and R_{ik} acting on i and k respectively. If we single out two such points, we have the system of two mass points mentioned at the beginning of § 7, the two masses being connected by a weightless, rigid rod. The reactions acting in this rod must satisfy Newton's third law,

$$(1) \quad R_{ik} = -R_{ki}.$$

Just as in § 7, in the enumeration of the degrees of freedom, we shall now decompose the virtual displacement into a translation δs_i common to both points and a rotation δs_n of point k about the already displaced point i , this rotation being a motion normal to the rod. Then

$$\delta s_k = \delta s_i + \delta s_n$$

For the virtual work of translation we therefore obtain, in view of (1),

$$\delta W_{tr} = R_{ik} \cdot \delta s_i + R_{ki} \cdot \delta s_i = 0;$$

for that of rotation, for which i remains fixed and k is displaced normal to the rod,

$$\delta W_{rot} = R_{ik} \cdot \delta s_n = 0.$$

This example illustrates that Newton's law of action and reaction is the salient point in the transition from particle mechanics to the mechanics of systems.

We shall now expand what we have learned with the help of the foregoing examples into a general postulate: *in any mechanical system the virtual work of the reactions equals zero*. Far be it from us to want to give

a general proof of this postulate.² Rather we regard it practically as definition of a "mechanical system."

It is now only a small step to the general formulation of the principle of virtual work. We argue as follows: every physically given force acting on a system in equilibrium is in equilibrium with the reactions induced at its point of application; the work done by such an applied force plus that done by its reactions in any virtual displacement of the point of application is therefore zero. The same is true of the sum of all applied forces and the sum of all the reactions induced by them. Now the reactions, taken by themselves, do no virtual work (by the previous paragraph). *Therefore the virtual work done by the applied forces keeping a system in equilibrium must equal zero as well.* The tedious investigation of the reactions is thereby eliminated.

This is the principle of virtual work, often called *Prinzip der virtuellen Verrückungen oder Verschiebungen* (principle of virtual displacements) in the German literature. This name is not as fortunate as the one used in English-speaking countries, which was taken over from the Italian *principio dei lavori virtuali*. The term, principle of virtual velocities, which is often used in the mathematical literature and was first proposed by Jean Bernoulli, seems unsuited to us.

Historically the principle was already sketched by Galileo. It was further developed by Stevin, Jacques and Jean Bernoulli and d'Alembert. It achieved its dominating position as the most general equilibrium principle only with the "Mécanique analytique" of Lagrange.

Whether the constraints of the system are of the holonomic or the non-holonomic variety affects the application of the principle of virtual work but little. Indeed, a condition of the form (7.4) can be introduced in the expression for the virtual work by elimination of one of the δq , regardless of whether this condition is integrable or not.

Instead of forces of reaction we can use the more descriptive term, forces of geometric origin. For they are given by the geometric relations between the various parts of the system, or, as in the case of the rigid body, between its individual mass points.

Antonymous to forces of geometric origin are the "forces of physical origin" or *applied forces*. The commonly used term "external forces" is less clear and will not be used here in this sense. Applied forces are caused by physical effects, such as gravity, steam pressure, cable tensions acting on the system from the outside, etc. They betray their physical origin by the fact that their mathematical expressions contain specific

² Lagrange attempts this in the introduction of his *Mécanique Analytique* (cf. p. 1) by means of certain block and tackle constructions.

constants (gravitational constant, readings of the scale of a manometer or barometer, etc.) which can be determined only experimentally. In § 14 we shall talk about the force of friction, which must sometimes be counted among the forces of reaction, sometimes among the applied forces. It is a force of reaction if it occurs as static friction; an applied force if it occurs as sliding or kinetic friction. Static friction is automatically eliminated by the principle of virtual work; kinetic friction must be introduced as an applied force. An external indication of this is the occurrence of the experimental constant μ in the law of sliding friction (14.4).

§ 9. Illustrations of the Principle of Virtual Work

(1) The Lever (Archimedes)

The lever possesses one degree of freedom, $f=1$, therefore only one displacement δq which corresponds to the virtual angular displacement $\delta\phi$.

Equilibrium exists if, and only if, the virtual work done in a rotation $\delta\phi$ of the lever is zero. Let δs_A , δs_B be the virtual displacements of the points of application P and Q of the forces A and B respectively. We then demand that

$$A \delta s_A + B \delta s_B = 0.$$

But from Fig. 10a $\delta s_A = a \delta\phi$, $\delta s_B = -b \delta\phi$. Therefore

$$(Aa - Bb) \delta\phi = 0$$

and consequently

$$Aa = Bb.$$

The moments of the forces about the fulcrum O are equal, i.e., their algebraic sum is zero.

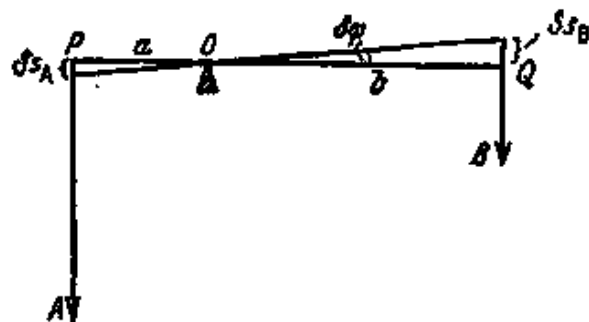


FIG. 10a. Lever with arms a and b under vertical loads A and B .

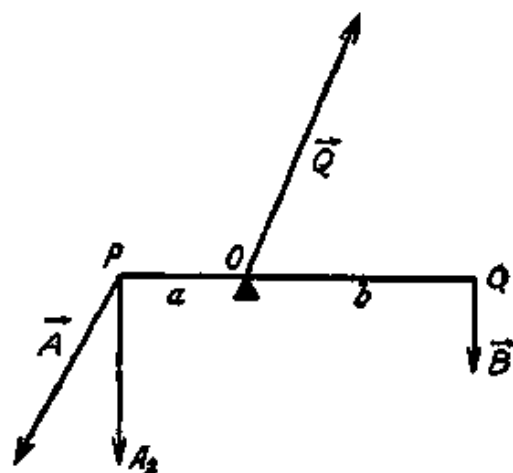


FIG. 10b. Lever under oblique load, showing the reaction of the fulcrum on the beam.

If, as in Fig. 10b, the force A is not perpendicular to the lever arm, we can decompose it into a component A_1 in the direction of the arm, and A_2 perpendicular to it. With point O fixed, A_1 has no effect, so that we have

$$A_2 a = |B| b.$$

In order to obtain the load at O , we must introduce an opposing force acting on the arm; in Fig. 10a it is directed vertically upward and has magnitude $Q = A + B$; the load on the fulcrum is equal and opposite to this force Q . In the case of Fig. 10b we have the vector equation $Q = A + B$; here, too, the force on O is the opposite (i.e., "equilibrant") of Q . In posing these questions, we actually transgress the limits of the principle of virtual work. The fixed position of the pivot O is characteristic of the mechanical system of the lever. Its virtual displacement, and the virtual work done on it, are therefore zero. In order to obtain Q or Q by means of our principle we should have to consider an altogether different mechanical system: we should have to provide O with two degrees of freedom and ask for the condition of equilibrium when we add a virtual translation of the whole lever parallel to itself to the rotation so far considered.

(2) Inverse of the Lever: Cyclist, Bridge

Consider the bicycle of Fig. 11a. The earth opposes the weight in the two points R (rear wheel) and F (front wheel). The rear wheel is exposed to the greater pressure, since the weight Q of bicycle and rider lies closer to R than to F . Accordingly a cyclist pumps his rear wheel to a higher

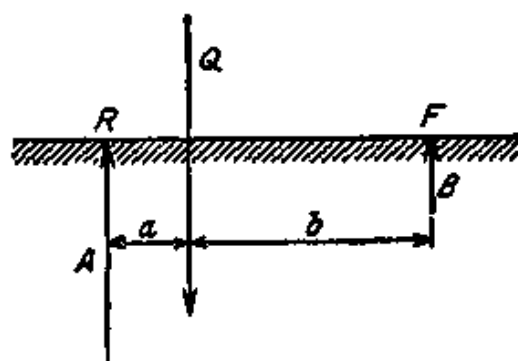


FIG. 11a. Distribution of weight on front and rear wheels of a bicycle with rider.

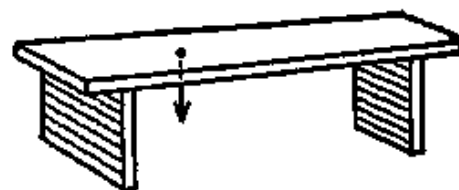


FIG. 11b. Distribution of load on the two supports of a schematic bridge.

pressure than his front wheel. The load on the rear wheel is $A = \frac{b}{a+b} Q$, that on the front wheel, $B = \frac{a}{a+b} Q$.

The same situation obtains with a bridge loaded off center (Fig. 11b).

(3) The Block and Tackle (also known to the Greeks)

Let n be the number of pulleys at both the upper and the lower end of the tackle. Q is the load to be lifted, P the force required at the loose end of the rope. In a virtual displacement of the system let

P move a distance δp ,

Q move a distance δq ,

the positive direction of motion being indicated by the arrows of Fig. 12. Equilibrium exists if

$$(1) \quad P \delta p - Q \delta q = 0.$$

If now Q is lifted an amount δq , the $2n$ rope lengths between the upper and lower pulleys are shortened by δq each, the total shortening therefore being $2n\delta q$. The loosely hanging rope at P must lengthen by precisely the same amount. Thus

$$\delta p = 2n\delta q$$

and, in view of (1),

$$(Q - 2nP) \delta q = 0.$$

We then obtain

$$(2) \quad P = \frac{Q}{2n}.$$

We have here treated the block and tackle as an "ideal" mechanical system, i.e., we have neglected the friction between ropes and pulleys and the friction in the pulley bearings.

This simple example can of course also be treated by the elementary method of rope tension, which in this case affords perhaps a more concrete picture of the interplay of forces.

Let S be the tension in the rope, taken over its total cross-section. If we neglect all frictional effects, the tension must be the same at every point of the rope; no matter where the rope is cut, one encounters the same tension S , which in both severed ends acts away from the point of severance. Let us cut the rope once on the left side, above P . The severed piece, in which P

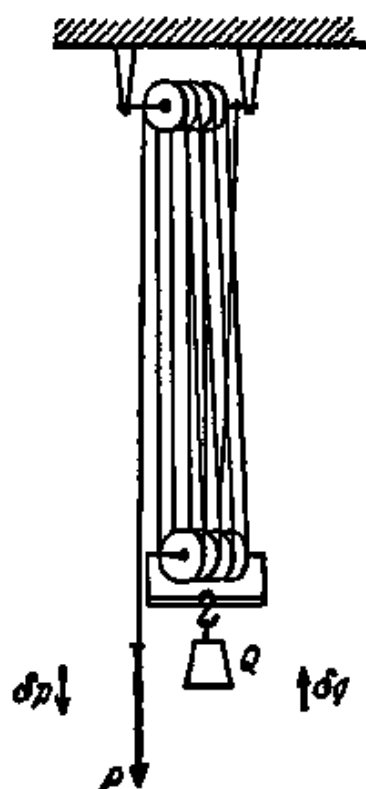


FIG. 12. Block and tackle. Virtual displacements of load and force.

acts downward and S upward, gives

$$P=S.$$

Next we cut all the ropes in the right part of the figure, thereby exposing $2n$ cross-sections on each side of the cut. The equilibrium of the forces acting on the severed lower right part demands

$$Q=2nS.$$

We therefore have again

$$P=\frac{Q}{2n}.$$

In addition, a consideration of the upper part of the system yields the loading of the beam from which the block is suspended. Evidently it amounts to $P+Q$.

(4) The Drive Mechanism of a Piston Engine

As in Fig. 9, P is the total force due to the steam pressure exerted on the piston, so that the virtual work done on the piston is $P\delta x$. Let Q be the equilibrant of the peripheral force U on the crank, i.e., the force causing P to be in equilibrium. The virtual work done by Q is $-Qr\delta\phi$. Our principle requires

$$(3) \quad Qr\delta\phi = P\delta x, \quad Q = P \frac{\delta x}{r\delta\phi}.$$

The calculation of Q therefore reduces to the purely kinematic task of determining the relation between δx and $\delta\phi$.

According to Fig. 9 (projection on the x -direction),

$$(4) \quad r \cos \phi + l \cos \psi = \text{const} - x,$$

so that, differentiating,

$$(4a) \quad r \sin \phi \delta\phi + l \sin \psi \delta\psi = \delta x.$$

The triangle OZK gives

$$(4b) \quad \sin \psi = \frac{r}{l} \sin \phi, \quad \delta\psi = \frac{r \cos \phi}{l \cos \psi} \delta\phi = \frac{r}{l} \frac{\cos \phi}{\left[1 - \left(\frac{r}{l}\right)^2 \sin^2 \phi\right]^{\frac{1}{2}}} \delta\phi$$

If we introduce this in (4a), we obtain

$$(4c) \quad r \sin \phi \delta\phi \left(1 + \frac{r}{l} \frac{\cos \phi}{\left[1 - \left(\frac{r}{l}\right)^2 \sin^2 \phi\right]^{\frac{1}{2}}}\right) = \delta x.$$

This relation furnishes the kinematic quantity $\frac{\delta x}{r \delta \phi}$. Substitution in (3) now gives

$$(5) \quad Q = P \sin \phi \left(1 + \frac{r}{l} \frac{\cos \phi}{\left[1 - \left(\frac{r}{l} \right)^2 \sin^2 \phi \right]^{\frac{1}{2}}} \right).$$

Thus the peripheral force $U = Q$ transmitted by the crank pin Z is determined for every crank position ϕ . Its precise knowledge is essential for an evaluation of the amount of cyclic fluctuation of the machine, and hence for the determination of the flywheel required. Since r/l is a small proper fraction, (5) can be expanded into a rapidly converging series in r/l . Cf. also problem II.2.

Finally, for the sake of a later application we shall calculate the piston position x as a power series in r/l . According to (4) and (4b) we have

$$(6) \quad x + r \left(\cos \phi - \frac{1}{2} \frac{r}{l} \sin^2 \phi + \dots \right) = \text{const.}$$

(5) Moment of a Force About an Axis and Work in a Virtual Rotation

Let a point P be at a distance l from an axis a . Let a force \mathbf{F} of arbitrary direction act at P . In a virtual rotation $\delta \phi$ about the axis a , P is displaced by

$$\delta s_P = l \delta \phi.$$

What is the work δW done by \mathbf{F} in this displacement?

We decompose \mathbf{F} into the mutually perpendicular components F_a , F_t , F_n , just as for Eq. (5.18). The work done depends only on F_n , for

$$\delta W = F_n \delta s_P = F_n l \delta \phi.$$

A comparison with (5.18a) will allow us to make a general statement:

The moment of a force about an axis can be regarded as the virtual work of the force in a rotation $\delta \phi$ of its point of application about the axis, divided by $\delta \phi$,

$$(7) \quad L_a(\mathbf{F}) = \frac{\delta W}{\delta \phi} = l F_n.$$

The concept of moment, basic to statics, is thereby brought into relation with the concept of virtual work basic to all questions of equilibrium.

Let us remark in this connection that the dimensions of moment (force-lever arm) are the same as those of work (force · distance). This is in agreement with (7) if, as is customary, we regard the angle measured in radians as dimensionless,

§ 10. *D'Alembert's Principle ; Introduction of Inertial Forces*

As we have seen, all bodies have the tendency to remain in a state of rest or of uniform rectilinear motion. We can think of this tendency as a resistance to changes in the motion, an inertial resistance, or, for brevity, as an *inertial force*. The definition of inertial force F^* for the single mass point is therefore

$$(1) \quad F^* \equiv -\dot{\mathbf{p}}$$

and the fundamental law $\dot{\mathbf{p}} = \mathbf{F}$ takes on the form

$$(2) \quad F^* + F = 0.$$

The inertial force is in vectorial equilibrium with the applied force.

While F is a force given by the physical situation, F^* is a fictitious force. We introduce it in order to reduce problems of motion to problems involving equilibrium, a procedure that is often convenient.

Inertial forces are familiar to us from everyday life. When we set the heavy revolving door of an hotel in motion, it is not the force of gravity or friction, but the inertia of the door that has to be overcome. A similar example is that of the sliding doors of street cars and trolleys.* On the forward platform the door opens in the direction of travel. When the car brakes, the door tends to move forward and can therefore be opened easily. When the car accelerates after a stop, the open door seeks to retain its position of rest; it therefore tends to move to the rear and can be closed without effort. It is easier to get on and off at the front platform than at the rear, where the door opens in the reverse manner.

The best-known form of an inertial force is the *centrifugal force*, which is noticeable in any curved motion. It, too, is a fictitious force. It corresponds to the acceleration $\dot{\mathbf{v}}_n$ normal to the curve, which is a *centripetal* acceleration, i.e., directed toward the center of curvature. According to (5.9) the centrifugal force is given by

$$(3) \quad \mathbf{C} = -m\dot{\mathbf{v}}_n, \quad |\mathbf{C}| = m|\dot{\mathbf{v}}_n| = m\frac{v^2}{\rho},$$

where the minus sign refers to the outward direction.

The *Coriolis force* (cf. § 28) and the various gyroscopic effects (cf. § 27) also come under the heading of inertial forces.

Incidentally the operation of railroads furnishes a very vivid example of the fact that the "fictitious" centrifugal force has a very real existence.

* The translator does not guarantee that the following is applicable to trolleys in the United States. It applies at least in part to the streetcars of San Francisco, which belong, however, to a breed rapidly approaching extinction.

On a curve the rail bed is banked in such a way that the outer rail is higher than the inner. The difference in height is always such that for some mean velocity of the train the resultant of gravity and centrifugal force is perpendicular to the rail bed. This procedure eliminates not only the danger of overturning about the outer rail, but also a harmful unequal loading of the rails.

Strangely enough, the great Heinrich Hertz raises objections to the introduction of the centrifugal force in the unusually beautiful and beautifully written introduction to his "Mechanics" (Collected Works, Vol. III, p. 6):

"We swing a stone attached to a string in a circle; we thereby consciously exert a force on the stone; this force constantly deviates the stone from a straight path, and if we alter this force, the mass of the stone or the length of the string, we discover that indeed the motion of the stone occurs at all times in agreement with Newton's second law. Now the third law demands a force opposing that which is exerted by our hand on the stone. If we ask for this force, we obtain the answer familiar to everybody, that the stone reacts on the hand by virtue of the centrifugal force, and that this centrifugal force is indeed equal and opposite to the force exerted by us on the stone. Is this mode of expression admissible? Is that which we now call centrifugal force anything but the inertia of the stone?"

We answer this question with a flat no; indeed the centrifugal force, by virtue of our definition (3), is identical to the inertia of the stone. But the force opposing that which we exert on the stone, i.e., really on the string, is the pull which the string exerts on our hand. Hertz further remarks that "we are forced to the conclusion that the classification of the centrifugal force as a force is not suitable; its name, just like that of live force, is to be regarded as a heritage passed down from former times; and from the point of view of usefulness the retention of this name is easier to excuse than to justify." In regard to this we would like to say that the name centrifugal force needs no justification, for it rests, like the more general term, inertial force, on a clear definition.

Incidentally, it is precisely this alleged lack of clarity of the force concept which induced Hertz, in an interesting but not very fruitful attempt, to construct his mechanics entirely without the notion of force (cf. § 1, p. 5).

We now come to the achievement of d'Alembert (mathematician, philosopher, astronomer, physicist, encyclopedist; "Traité de Dynamique," 1758).

If a mass point k , part of an arbitrary mechanical system, is acted on by an applied force F , Eq. (2) must be changed to read

$$(4) \quad \mathbf{F}_k^* + \mathbf{F}_k + \sum_i \mathbf{R}_{ik} = 0.$$

Here \mathbf{R}_{ik} is the reaction which the mass point i connected with k exerts on k . According to our general postulate of p. 52, the \mathbf{R}_{ik} , taken together, do no work in an arbitrary virtual displacement compatible with the (here internal) constraints. It follows that the virtual work of the sum of all the $\mathbf{F}^* + \mathbf{F}$ is zero as well,

$$(5) \quad \sum_k (\mathbf{F}_k^* + \mathbf{F}_k) \cdot \delta \mathbf{s}_k = 0.$$

Recalling now the principle of virtual work, we can express Eq. (5) by saying that *the inertial forces of a system are in equilibrium with the forces applied to the system*. A knowledge of the reactions is not required.

This is *d'Alembert's principle* in its simplest and most natural form. In order to obtain another interesting formulation of the principle, let us look at the quantity

$$\mathbf{F}_k + \mathbf{F}_k^* = \mathbf{F}_k - \dot{\mathbf{p}}_k.$$

It is that part of the force \mathbf{F}_k that cannot be converted into motion of the point k . We can call this part the "lost force" and can therefore re-frame (5) by stating that *the lost forces of a system are in equilibrium*.

A formulation of d'Alembert's principle widely used in textbooks is that expressed in Cartesian coordinates. We call the components of \mathbf{F}_k , X_k, Y_k, Z_k and those of $\delta \mathbf{s}_k$, $\delta x_k, \delta y_k, \delta z_k$. Furthermore, we stipulate that the masses m_k involved are constant; for a system consisting of n mass points we can then replace (5) by

$$(6) \quad \sum_{k=1}^n \{ (X_k - m_k \ddot{x}_k) \delta x_k + (Y_k - m_k \ddot{y}_k) \delta y_k + (Z_k - m_k \ddot{z}_k) \delta z_k \} = 0.$$

It is here required that the $\delta x_k, \delta y_k, \delta z_k$ be compatible with the constraints of the system. Let us at once consider the general case of non-holonomic constraints. There relations of type (7.4) exist; if we replace the general coordinates q of (7.4) by Cartesian coordinates, these relations become

$$(6a) \quad \sum_{\mu=1}^n [F_{\mu}(x_1 \dots x_n) \delta x_{\mu} + G_{\mu}(x_1 \dots x_n) \delta y_{\mu} + H_{\mu}(x_1 \dots x_n) \delta z_{\mu}] = 0.$$

If f is the number of degrees of freedom for infinitesimal motion, there must be $3n - f$ such relations for the $\delta x, \delta y, \delta z$ (cf. p. 50). In the case of holonomic constraints the $F_{\mu}, G_{\mu}, H_{\mu}$ are derivatives of one and the same function with respect to $x_{\mu}, y_{\mu}, z_{\mu}$.

Let the reader be warned emphatically not to look for the true content of d'Alembert's principle in the clumsy formulation (6), (6a). Equation (5) or the statement of equilibrium equivalent to it is not only more readily useful, but also, by virtue of its invariant form, more natural.

§ 11. Application of d'Alembert's Principle to the Simplest Problems

(1) Rotation of a Rigid Body About a Fixed Axis

Here we are dealing with a single degree of freedom, viz., the angle of rotation ϕ . We let $\dot{\phi} = \omega$ be the angular velocity, $\ddot{\phi} = \dot{\omega}$ the angular acceleration. For the present we are not interested in the axle bearings.

We suppose that arbitrary applied forces F act on the body. According to § 9, Eq. (7), their virtual work is given by the sum of their moments about the axis of rotation, i.e., by

$$(1) \quad \delta W = L \cdot \delta\phi = L_a \delta\phi$$

where L_a is the sum of the moments of the F about the axis of rotation a . We also wish to know the work done by the inertial forces F^* . For this purpose we subdivide the body into mass elements dm . In view of (10.3) the inertial force acting on dm directed normal to the path is the centrifugal force $dm \frac{v^2}{r} = dm \omega v$. (In circular motion the radius of curvature ρ is of course equal to the distance r from the rotation axis, the velocity v of each element of mass therefore becomes $r\omega$, and its acceleration \dot{v} along the path is $r\dot{\omega}$). But the centrifugal force does no work. Along the path direction, on the other hand, the inertial force is

$$-dm\dot{v} = -dmr\dot{\omega}.$$

The total virtual work of the inertial forces is therefore

$$(2) \quad \sum (-dm\dot{v})\delta s = \sum -dmr\dot{\omega}r\delta\phi = -\delta\phi\dot{\omega} \int r^2 dm = -\delta\phi\dot{\omega}I,$$

where

$$(3) \quad I = \int r^2 dm$$

is the *moment of inertia* of the body. The dimensions of I are ML^2 , therefore g cm^2 in the absolute system, g cm sec^2 in the gravitational system.

By virtue of (1) and (2) d'Alembert's principle takes the form

$$\delta\phi(L_a - I\dot{\omega}) = 0$$

so that we obtain the basic equation of rotational motion

$$(4) \quad I\dot{\omega} = L_a.$$

Let us compare this equation with the basic equation of translational motion of one degree of freedom, say in the x -direction,

$$m\ddot{x} = F_x.$$

We see that in rotational motion I takes the place of m .

The same substitution holds in the expression for the kinetic energy. The kinetic energy of rotation of a rigid body is

$$(5) \quad E_{\text{kin}} = T = \int \frac{dm}{2} v^2 = \int \frac{dm}{2} r^2 \omega^2 = \frac{\omega^2}{2} \int r^2 dm = \frac{\omega^2}{2} I$$

and therefore corresponds exactly to the elementary expression of particle mechanics,

$$(5a) \quad E_{\text{kin}} = T = \frac{\dot{x}^2}{2} m.$$

In the case of a rigid body with fixed axis, I is time-independent; in mechanisms with flexible joints and in living beings it is, however, variable in a characteristic manner. In § 13 we shall see that all athletic activities, in particular apparatus gymnastics, are based primarily on the ability of the human body to change its moment of inertia.

An investigation of the manner in which the moment of inertia of a rigid body depends on the position of the axis of rotation will be deferred to § 22.

Finally we shall turn to the connection of the kinetic energy with the basic equation of motion. Just as, in the case of constant mass, we can obtain the equation of motion $m\ddot{x} = F_x$ from the law of kinetic energy in particle mechanics, i.e.,

$$\frac{dT}{dt} = \frac{dW}{dt} \quad \text{with} \quad dW = F_x dx,$$

we obtain, in the case of constant I , the equation of motion (4) for rotation. We need merely make use of (5) in

$$\frac{dT}{dt} = \frac{dW}{dt} \quad \text{with} \quad dW = L_a d\phi \quad [\text{Eq. (9.7)}].$$

The moment of inertia occurs also in the expression for the *moment of momentum* or *angular momentum* of the rotating body. If we let M be the angular momentum of the body, we evidently have

$$(6) \quad M = \sum dm \, vr = \omega \sum dm \, r^2 = \omega I.$$

(2) Coupling of Rotational and Translational Motion

Think of the coal basket in a mine, or of an elevator. The cable carrying the elevator is wound around a drum and driven by a force P . Let r be the drum radius. The two virtual displacements that take place (cf. Fig. 13) are related by

$$(7) \quad \delta z = r \delta \phi.$$

d'Alembert's principle requires

$$(7a) \quad (-Q - M\ddot{z})\delta z + (rP - I\dot{\omega})\delta \phi = 0.$$

It is convenient to "reduce" the mass of the drum, so to speak, to the periphery of the drum, i.e., to replace I by a "reduced mass" defined by

$$(8) \quad I = M_{\text{red}} r^2.$$

By virtue of (7), Eq. (7a) can then be rewritten in the form

$$(P - Q - M\ddot{z} - M_{\text{red}} r \dot{\omega}) \delta z = 0.$$

Since $r\omega = \dot{z}$, $r\dot{\omega} = \ddot{z}$, we then obtain the equation of motion

$$(9) \quad (M + M_{\text{red}})\ddot{z} = P - Q.$$

The inertia of the drum therefore adds a term M_{red} to the mass of the elevator.

(3) Sphere Rolling on Inclined Plane

Here again we are dealing with the coupling of translation (motion down the incline) and rotation (about an axis through the center of the sphere perpendicular to the plane of the paper in Fig. 14). The component of gravity effective in this case is $P = Mg \sin \alpha$; the static friction F indicated on the diagram does not enter d'Alembert's principle, since it acts at the point of contact which is instantaneously at rest. The condition for pure rolling motion is

$$(10) \quad \dot{z} = r\omega, \text{ or, written for virtual motion, } \delta z = r \delta \phi.$$

With d'Alembert we now require that

$$(11) \quad \delta z (Mg \sin \alpha - M\ddot{z}) + \delta \phi (-I\dot{\omega}) = 0.$$

The calculation of I is a problem of integral calculus. We shall state without proof that the moment of inertia of a homogeneous ellipsoid of

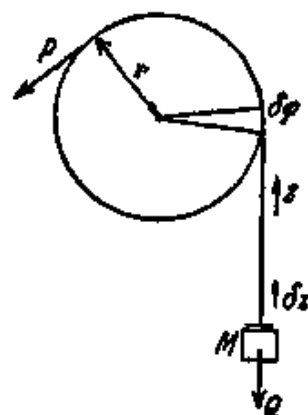


FIG. 13. Coupling of translational and rotational motion (elevator, coal basket).

semi-axes a , b , c about axis c (and correspondingly about a and b) is

$$(12) \quad I_c = \frac{M}{5} (a^2 + b^2).$$

As a special case we obtain for the moment of inertia of a sphere

$$(12a) \quad I = \frac{2}{5} Mr^2.$$

As in (8) we introduce a mass reduced to the distance r , which by virtue of (12a) becomes

$$(12b) \quad M_{\text{red}} = \frac{2}{5} M.$$

If we substitute this in (11) and also take (10) into account, we easily obtain

$$(13) \quad \ddot{z} = \frac{5}{7} g \sin \alpha.$$

The factor $\frac{5}{7}$ shows how the "fall" on an inclined plane is delayed by the angular acceleration of the sphere and the increased inertia due to it.

Whereas from (3.13) the final velocity in a free fall was found to be

$$v = (2gh)^{\frac{1}{2}}, \quad h = \text{height of fall},$$

equation (13) now gives the final velocity

$$v = (2 \cdot \frac{5}{7} gh)^{\frac{1}{2}}.$$

The difference is due to the fact that now the gravitational potential energy is converted not only into kinetic energy of descent, but also into rotational energy of the rolling sphere.

(4) Mass Guided Along Prescribed Trajectory

If we assume the guide ways to be frictionless, d'Alembert's principle applied to the one degree of freedom here present (displacement along the guide) simply says that

$$\delta s (F_s^* + F_s) = 0,$$

i.e., according to (5.8),

$$(14) \quad m\dot{v}_s = m|\dot{v}| = F_s$$

with arbitrary direction of the applied force F . The component F_s of F

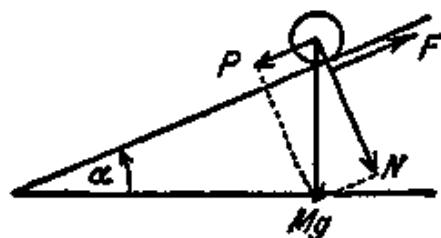


FIG. 14. Sphere on inclined plane. The static friction F causes pure rolling, but does not enter d'Alembert's principle.

normal to the guide, which we may take positive in the centripetal direction, must then add to the reaction R_n (counted positive in the same direction) to give the equilibrant of the centrifugal force C ; i.e.,

$$(15) \quad R_n + F_n = C = m \frac{v^2}{\rho}.$$

In general, especially if the guiding action is achieved by a material device such as a rail, we are compelled to take into account also a tangential component R_s of the reaction, the *friction*. If we count the friction positive in the negative direction of δs , Eq. (14) is therefore enlarged to

$$(16) \quad m\dot{v} = F_s - R_s.$$

Whereas R_n is determined by Eq. (15), R_s in (16), on the other hand, remains "statically and dynamically indeterminate" and can be determined only from experiment. In § 14 we shall discuss how such experiments are carried out.

§ 12. Lagrange's Equations of the First Kind

Let us consider a system of discrete mass points m_1, m_2, \dots, m_n , connected with each other by r holonomic conditions

$$(1) \quad F_1 = 0, F_2 = 0, \dots, F_r = 0.$$

The number of degrees of freedom is then $f = 3n - r$. We operate in Cartesian coordinates and make use of the formulation (10.6) of d'Alembert's principle. In order to write the clumsy sums occurring there in a more convenient way, we number the coordinates $x_1, y_1, z_1, \dots, x_n, y_n, z_n$ consecutively as

$$x_1, x_2, x_3, x_4, \dots, x_{3n-1}, x_{3n},$$

and likewise the components of force X, Y, Z . The mass belonging to x_k , X_k will be denoted by m_k ; evidently the m_k will be equal in groups of three. Eq. (10.6) now becomes

$$(2) \quad \sum_{k=1}^{3n} (X_k - m_k \ddot{x}_k) \delta x_k = 0.$$

By virtue of the r conditions of constraint (1), the δx_k are subject to the restrictions

$$(3) \quad \delta F_i = 0, \quad i = 1, 2, \dots, r,$$

which can also be written

$$(4) \quad \sum_{k=1}^{3n} \frac{\partial F_i}{\partial x_k} \delta x_k = 0, \quad i = 1, 2, \dots, r.$$

Let us multiply each of the δF_i by an arbitrary numerical factor λ_i (Lagrange multiplier) and add it to the d'Alembert equation (2), giving

$$(5) \quad \sum_{k=1}^{3n} \left(X_k - m_k \ddot{x}_k + \sum_{i=1}^r \lambda_i \frac{\partial F_i}{\partial x_k} \right) \delta x_k = 0.$$

Only f of the $3n$ displacements δx are independent of each other. The remaining r are functions of these independent ones. Let these r displacements be given by the quantities $\delta x_1, \delta x_2, \dots, \delta x_r$. Now we have precisely r quantities $\lambda_1, \lambda_2, \dots, \lambda_r$ over which we can dispose freely. We choose them so as to make

$$(6) \quad X_k - m_k \ddot{x}_k + \sum_{i=1}^r \lambda_i \frac{\partial F_i}{\partial x_k} = 0; \quad k=1, 2, \dots, r.$$

Eq. (5), with the numbers λ_i now determined, reduces to

$$(7) \quad \sum_{k=r+1}^{3n} \left(X_k - m_k \ddot{x}_k + \sum_{i=1}^r \lambda_i \frac{\partial F_i}{\partial x_k} \right) \delta x_k = 0$$

where the δx_k are completely independent, there being indeed $f=3n-r$ of these. If, for example, we choose

$$(8) \quad \delta x_{r+v} \neq 0; \delta x_{r+1} = \delta x_{r+2} = \dots = \delta x_{r+v-1} = \delta x_{r+v+1} = \dots = \delta x_{3n} = 0,$$

we see that the factor of δx_{r+v} must vanish. Letting v run through all values $1, 2, \dots, f$, we conclude that all expressions in parentheses have to be $=0$,

$$X_k - m_k \ddot{x}_k + \sum_{i=1}^r \lambda_i \frac{\partial F_i}{\partial x_k} = 0; \quad k=r+1, r+2, \dots, 3n.$$

Together with the Eqs. (6) these form $3n$ differential equations

$$(9) \quad m_k \ddot{x}_k = X_k + \sum_{i=1}^{i=r} \lambda_i \frac{\partial F_i}{\partial x_k}; \quad k=1, 2, \dots, 3n,$$

which are called the *Lagrange equations of the first kind*. Of course the m_k are equal in groups of three; thus $m_1=m_2=m_3$, since we are dealing with the same mass point m_1 having the three coordinates $x_1=x_1, x_2=y_1, x_3=z_1$.

So far we have assumed that the conditions (1) are holonomic; we can easily convince ourselves that all of the preceding can be carried over to the case of non-holonomic constraints with only slight modification. The only difference is that the factors $\frac{\partial F_i}{\partial x_k}$ in (4) must be replaced by general

functions of the coordinates, F_{ik} , which cannot be written in the form of partial derivatives of a function. If we make this replacement in Eqs. (9), we at once obtain Lagrange's equations of the first kind for non-holonomic systems,

$$(9a) \quad m_k \ddot{x}_k = X_k + \sum_{i=1}^{i=r} \lambda_i F_{ik}.$$

Let us make a more interesting generalization by assuming that the conditions (1) vary with time. Then the F_i depend explicitly not only on the x_k but also on t . We must now demand that in forming (4) the time be held constant, a stipulation which is not only permissible but also plausible since our virtual displacement has nothing to do with the passage of time. The derivation of (9) is not affected by this requirement. But we obtain an important consequence regarding the form of the equation of energy.

If we want to derive this equation in the case of time-independent constraints, we proceed as follows: we multiply (9) by dx_k and sum over k . On the left we obtain

$$(9b) \quad dt \sum m_k \dot{x}_k \ddot{x}_k = dt \frac{d}{dt} \sum \frac{m_k}{2} \dot{x}_k^2 = dt \frac{dT}{dt} = dT.$$

The first term of the right member gives the work done by the applied forces in time dt ,

$$(9c) \quad \sum dx_k X_k = dW.$$

The second term on the right vanishes. For

$$(9d) \quad \sum_{i=1}^r \lambda_i \sum_{k=1}^{3n} \frac{\partial F_i}{\partial x_k} dx_k = \sum_{i=1}^r \lambda_i dF_i = 0$$

by virtue of the fact that the F_i depend only on the x_k , so that $F_i = 0$ implies

$$(9e) \quad dF_i = \sum \frac{\partial F_i}{\partial x_k} dx_k = 0.$$

From (9b, c) we then have

$$(10) \quad dT = dW.$$

This is no longer so if the F_i also depend on t . Then the zero in (9d, e) is to be replaced by

$$-\sum_{i=1}^r \lambda_i \frac{\partial F_i}{\partial t} dt \quad \text{and} \quad -\frac{\partial F_i}{\partial t} dt$$

respectively. For time-dependent constraints the equation of energy is then

$$(10a) \quad dT = dW - dt \sum_{i=1}^r \lambda_i \frac{\partial F_i}{\partial t}.$$

This means that *time-dependent constraints do work on the system*.

To make this principle more concrete, let us think of a tennis racquet. If the racquet is kept fixed, it reflects the ball with unchanged energy. If instead it yields backward or swings toward the ball, it takes energy away from or gives energy to the ball.

In non-holonomic systems an explicit dependence on t of the F_{ik} occurring in (9a) would be compatible with an equation of energy of the form (10). If, however, the non-holonomic conditions had the form

$$\sum F_{ik} dx_k + G_i dt = 0$$

instead of (7.4), it would be necessary to add members in G_i to (10) which would then take on a form analogous to (10a), i.e.,

$$(10b) \quad dT = dW - dt \sum_{i=1}^r \lambda_i G_i.$$

We shall learn from the example of the spherical pendulum in the following chapter that the λ_i can be regarded as the reactions of the system against the constraint exerted by the holonomic or non-holonomic conditions. There we shall also see that the determination of the λ cannot be effected by means of r Lagrange equations arbitrarily singled out, even though this was a permissible assumption for purposes of our derivation. Instead the λ must be determined from all $3n$ of the Lagrange equations taken together. It should be emphasized that the method of Lagrange multipliers plays an important role not only in the Lagrange equations of the first kind, but also (cf. Ch. VI, § 34) in types of equations of a much more general nature. Apart from their use in mechanics, the Lagrange multipliers are encountered in the elementary theory of maxima and minima.

§ 13. Equations of Momentum and of Angular Momentum

We derive these equations for a system of discrete mass points which can be translated and rotated as a whole in space. Through a limiting process they can, however, be applied equally well to a freely moving rigid body or to an arbitrary mechanical system whose motion is not restricted by external constraints.

We divide the forces acting into *external* and *internal forces*. This classification says nothing about the origin of the forces and is therefore by no means identical with the classification of p. 53 into applied forces and forces of reaction. Our present distinction is strictly based on the criterion of whether the law of action and reaction is or is not satisfied within the system itself. In the first case we speak of internal forces, in the second of external forces. The internal forces of the solar system, for instance, are applied forces because they are gravitational, whereas the external force which drives a railroad train forward is a force of reaction (as we shall see on p. 84), viz., the static friction at the rolling wheels.

We call F_k the external force acting at the point k ; the internal forces will be called F_{ik} to remind us that they act between two points contained in the system and therefore within the system satisfy Newton's third law,

$$(1) \quad F_{ik} = -F_{ki}.$$

(1) Equation of Momentum

Let us now make use of d'Alembert's principle in the form (10.5). We replace F_k by $F_k + \sum_i F_{ik}$, F_k^* by $-\dot{p}_k$ in agreement with definition, and make all the δs_k equal to each other. We therefore impart the same virtual displacement to all the mass points of the system. The F_{ik} drop out because of (1) once we sum over i and k , and we are left with

$$(2) \quad \delta s \cdot \left(\sum_k F_k - \sum_k \dot{p}_k \right) = 0.$$

Let us indicate the summation over k by means of a bar. From (2) we conclude that

$$(3) \quad \dot{\bar{p}} = \bar{F}.$$

\bar{p} is the total momentum of the system, equal to the vector sum of the individual momenta. We define the center of mass velocity V by

$$MV = \bar{m}V = \bar{p}, \quad M = \bar{m}$$

and have, in lieu of (3),

$$(3a) \quad M\dot{V} = \bar{F}.$$

We now choose an arbitrary but fixed point of reference O . We measure the distance r_k of the points of the system from O and define the position R of the center of mass with respect to O by the equation

$$(3b) \quad MR = \bar{mr}.$$

The content of equations (3a, b) can be summed thus: *the center of mass of a freely moving mechanical system moves like a single mass point, having a mass M equal to the total mass of the system, and acted on by the resultant $\bar{\mathbf{F}}$ of all the external forces acting on the system.*

(2) Equation of Angular Momentum

Suppose we impart to the system a virtual rotation $\delta\phi$ about an arbitrary axis passing through a point O . The displacements $\delta\mathbf{s}_k$ of the various points m_k of the system are then unequal; for

$$(4) \quad \delta\mathbf{s}_k = \delta\phi \times \mathbf{r}_k.$$

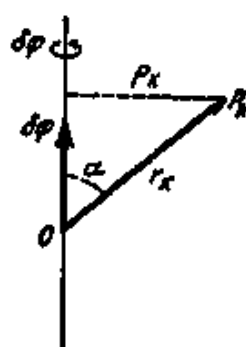


FIG. 15. The virtual displacement $\delta\mathbf{s}$ resulting from a virtual rotation $\delta\phi$.

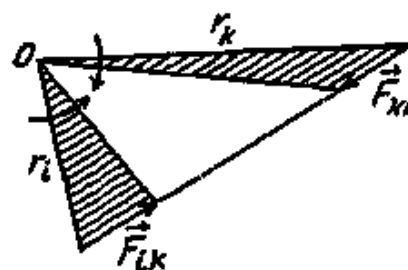


FIG. 16. The moments of internal forces cancel in pairs.

To prove this, let us look at Fig. 15. $\delta\phi$ is there drawn as a vector along the axis of rotation and at the same time as a curved arrow about this axis in agreement with the rule of the right-handed screw. By virtue of the definition of the vector product the magnitude δs_k of $\delta\mathbf{s}_k$ is

$$\delta s_k = \delta\phi |\mathbf{r}_k| \sin\alpha = \delta\phi \rho_k,$$

as must be the case for the rotation in question. The direction and sense of $\delta\mathbf{s}_k$ are likewise correctly given by (4). $\delta\mathbf{s}_k$ is directed normal to the plane of the drawing, into the paper.

We introduce (4) in (10.5), while replacing \mathbf{F}^* and \mathbf{F} as in subsec. 1, and immediately obtain

$$(5) \quad \sum_k \left\{ (\mathbf{F}_k + \sum_i \mathbf{F}_{ik} - \dot{\mathbf{p}}_k) \cdot (\delta\phi \times \mathbf{r}_k) \right\} = 0.$$

Next we use a rule of elementary vector algebra,

$$(6) \quad \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$$

which says that the parallelepiped formed by any three vectors \mathbf{A} , \mathbf{B} , \mathbf{C} has a volume which is independent of cyclic permutation of the labels of its three edges.

Instead of (5) we can therefore write

$$(7) \quad \delta\phi \cdot \left\{ \sum_k (\mathbf{r}_k \times \mathbf{F}_k) + \sum_i \sum_k (\mathbf{r}_k \times \mathbf{F}_{ik}) - \sum_k (\mathbf{r}_k \times \dot{\mathbf{p}}_k) \right\} = 0.$$

In this fashion the connection between $\delta\phi$ and \mathbf{r} is severed, so that, with $\delta\phi$ arbitrary, the factor in brackets $\{ \}$ must itself vanish. In order to write this factor more simply we introduce the following notation:

$$(7a) \quad \mathbf{L}_k = \mathbf{r}_k \times \mathbf{F}_k \text{ as in (5.12), } \bar{\mathbf{L}} = \sum \mathbf{L}_k;$$

$$(7b) \quad \mathbf{M}_k = \mathbf{r}_k \times \mathbf{p}_k, \quad \mathbf{r}_k \times \dot{\mathbf{p}}_k = \frac{d}{dt}(\mathbf{r}_k \times \mathbf{p}_k) = \dot{\mathbf{M}}_k \text{ as in (5.14);}$$

$$(7c) \quad \bar{\mathbf{M}} = \sum \mathbf{M}_k, \quad \dot{\bar{\mathbf{M}}} = \sum \dot{\mathbf{M}}_k.$$

$\bar{\mathbf{L}}$ is therefore the vector sum of all moments of the external forces about the common point of reference O , $\bar{\mathbf{M}}$ is the vector sum of the angular momenta of all the mass points of the system about the same point of reference, or, more briefly, the total angular momentum of the system about O .

Moreover we show with the help of Fig. 16 that in the double sum of Eq. (7) all the terms cancel in pairs, viz., that

$$(8) \quad \mathbf{r}_k \times \mathbf{F}_{ik} + \mathbf{r}_i \times \mathbf{F}_{ki} = 0.$$

We see that in this expression the Third Law, Eq. (1), acts essentially as the definition of internal force.

From (8) it follows that the double sum in (7) vanishes. Recalling (7a, b, c) we therefore conclude from (7) that

$$(9) \quad \dot{\bar{\mathbf{M}}} = \bar{\mathbf{L}}.$$

This equation is the exact counterpart of Eq. (3). It states that *the time rate of change of the total angular momentum of the system is equal to the resultant moment of the external forces*, just as Eq. (3) stated that *the time rate of change of the total momentum of the system is equal to the resultant of all the external forces*.

These two laws will be called the equations (or principles) of angular momentum and (linear) momentum respectively.

Formerly it was the custom in the German literature to call basic equation (9) the principle of areas (Flächensatz). This name had its origin in the Kepler problem. There we found that in the case of *one* planet the areal velocity was proportional to the angular momentum, and the direction of the angular momentum was normal to the orbital plane of the planet. This is no longer the case for the planetary many-body problem, where we have instead

$$(10) \quad \bar{\mathbf{M}} = \sum 2m_k \frac{d\mathbf{A}_k}{dt},$$

so that not only the different planetary masses occur as factors, but the individual areal velocities corresponding to the planets must be added vectorially. The areal velocity thus arising for a complete planetary system is defined, as is well known, by the *invariable plane* (plane normal to $\bar{\mathbf{M}}$). It is invariable because in a planetary system external forces are absent, so that $\bar{\mathbf{L}}=0$ and, according to (9),

$$(10a) \qquad \bar{\mathbf{M}} = \text{const.}$$

In general for $\bar{\mathbf{L}}=0$ we obtain the special *principle of the conservation of angular momentum*. The notion of areal velocity is even more difficult to visualize, hence less useful, for a system of infinitely many particles such as a rigid body, so that the term "Flächensatz" should be abandoned for general use.

(3) Proof Using the Coordinate Method

We shall now sketch the proof of our principles by an alternate method, that of decomposition into Cartesian coordinates, because the use of these coordinates is so widespread and has been so greatly favored by older texts that we wish to defer to usage in some measure.

We begin with the equations

$$(11) \qquad \begin{aligned} m_k \ddot{x}_k &= X_k + \sum_i X_{ik} \\ m_k \ddot{y}_k &= Y_k + \sum_i Y_{ik} \end{aligned}$$

which are written in easily understandable form. Summation of the first of these equations over k , with $X_{ik} = -X_{ki}$, at once yields the x -component of the equation of momentum,

$$(12) \qquad \frac{d^2}{dt^2} \sum_k m_k x_k = \sum_k X_k.$$

Multiplication of the first equation by $-y_k$, the second by x_k , yields as their sum

$$(13) \qquad \sum_k m_k (x_k \ddot{y}_k - y_k \ddot{x}_k) = \sum_k (x_k Y_k - y_k X_k) + \dots$$

We group together in pairs ik and ki the terms...not written down, thereby bringing out the direction of the internal forces, $i \rightarrow k$ and $k \rightarrow i$. We then obtain

$$\begin{aligned} x_k Y_{ki} - y_k X_{ki} + x_i Y_{ik} - y_i X_{ik} \\ = \frac{F_{ik}}{r_{ik}} [x_k (y_i - y_k) - y_k (x_i - x_k) + x_i (y_k - y_i) - y_i (x_k - x_i)]. \end{aligned}$$

Simplification shows this to equal zero, in agreement with Fig. 16. With the help of (5.17a) the right member of (13) reduces to

$$\sum_k L_{kz} = L_z.$$

The left member of (13) is, in view of (5.14b),

$$(13a) \quad \frac{d}{dt} \sum_k m_k (x_k \dot{y}_k - y_k \dot{x}_k) = \sum_k \dot{M}_{kz} = \dot{\bar{M}}_z.$$

Equation (13) is then identical with the z -component of our equation of angular momentum (9).

(4) Examples

There exists a profound difference between the principles of linear and of angular momentum which we shall explain with the help of the special case in which no external forces act on the system.

According to Eq. (3a) in this case the velocity of the mass center remains constant; for the total mass M occurring as factor is constant, even for a system with internal motion. If, then, the mass center is initially at rest, it remains at rest. *Internal forces are unable to impart motion to the center of mass, even in a mechanism with flexible joints or in a living body.* In order to move one's center of mass, one must be able to push against a support; therefore an *external force* is necessary.

It is evident that in the absence of external forces $\mathbf{L}=0$, so that (9) yields

$$(14) \quad \bar{\mathbf{M}} = \text{const.}$$

If the moment of momentum is initially zero, it remains zero, even for a system with internal motion. From this it does not follow, however, that the angular position of the system is conserved permanently. Rather, this angular position can be varied *ad libitum* with the help of internal forces alone, and without a push against some outside object.

An example of this is the cat, which always manages to fall on its feet. It achieves this by suitable rotation of the anterior extremities coupled with opposite rotation of the posterior ones. This action is illustrated by the rapid exposure photographs published in the "Comptes Rendus of the Paris Academy," 1894, p. 714.

The essential points of this process can conveniently be followed by means of an experiment with a turning stool. Such a stool consists of a horizontal disc which revolves with as little friction as possible about

a vertical axis. The victim of the experiment is seated on the disc, initially at rest:

$$M_0 = 0.$$

He lifts his right arm forward and describes with it a backward rotation. The "area swept out" in this process must be compensated by a counter-rotation of the remainder of the body including the disc of the stool. More precisely, the moment of momentum M_1 of the moving arm induces a moment of momentum of torso and disc M_2 such that

$$M_2 = -M_1.$$

The experimental subject now lowers his arm; this causes no change in M . Now the initial position of the body is restored, and the process can be repeated. With each repetition the same counter-rotation M_2 takes place. After n repetitions the subject notices that he is facing in a direction opposite to the initial one. In contrast to the position of the center of mass, the angular position is not fixed by the initial state of rest.

One can strengthen the effect by making the subject hold a heavy weight in the right hand. The "area swept out" is thereby, so to speak, multiplied, so that the counter-rotation is also visibly increased.

Let us perform two more experiments: the subject stands on the stool with lowered arms and is given an angular momentum M_0 ; he now raises his arms (with weights in his hands if desired) sideways; the rotation suddenly decreases. Instead, we can set the person spinning with outstretched arms; he next lowers his arms and usually falls off the stool because the rotation, especially when weights are used, is suddenly increased considerably.

In both foregoing cases

$$M_2 = M_1 \text{ and therefore } I_0\omega_0 = I_1\omega_1 \text{ from Eq. (11.6).}$$

In the first case, however, we have

$$I_0 \ll I_1 \text{ and hence } \omega_1 \ll \omega_0,$$

whereas in the second case

$$I_0 \gg I_1 \text{ so that } \omega_1 \gg \omega_0.$$

The changeability of the moment of inertia under conservation of angular momentum is used extensively in all athletic feats, especially in exercises on the horizontal bar. Consider, for example, the "forward upswing." In the initial act of acquiring swing the body is stretched, its moment of inertia great, and its angular velocity about the bar moderate. As he swings forward, shortly before reaching the highest point, the performer pulls in his legs, reduces his moment of inertia about the bar and

his angular velocity becomes high. His mass center swings over the bar and the performer achieves an upright position on the bar. Notice that the reactions produced by the grasp of the hands on the bar do not influence the angular momentum to any noticeable degree since the bar is so thin that the forces of reaction have a vanishingly small lever arm.

The same principles are used in the "circles," (backward hip circle, knee circle, etc.). Gymnastics, ice skating and skiing are, in a way, practical lessons in experimental and theoretical mechanics.

(5) Mass Balancing of Marine Engines

Let us finally consider an illustration on a large scale, the Schlick method for balancing the reciprocating masses of marine engines.

In the transition period leading to the modern express steamers, toward the end of the last century, the shipbuilding industry went through a crisis. For technical reasons the speed of revolution of the propellor shaft is fixed at approximately 100 per min. The inertial effects of the piston engines, which have to be absorbed by the ship's body, change in this same rhythm. As the length of ships was increased more and more, the "proper frequency" of the vessel was continually depressed, so that this frequency came dangerously near to the rhythm of the inertial effects. Let us anticipate by using the word "resonance," a phenomenon with which we shall deal at great length in the next chapter. The word originated in acoustics, where resonance phenomena are most immediate and where they were studied first.

For lack of space the steam cylinders of fast steamers have to be arranged vertically. Let us assume, to make things specific, that we are dealing with four pistons (cf. Fig. 17), which are all connected to the same crank shaft oriented lengthwise, along the z -direction in our diagram. We shall see that for a smaller number of pistons a mass balance even to first order (to which we shall restrict ourselves here) is impossible. With the choice of coordinates of Fig. 17, the inertial forces are directed along the z -axis; they give rise to moments only about the y -axis. The inertial effects must be absorbed by the reactions of the body of the ship, in which they induce rhythmic countervibrations.

This is beautifully illustrated by the models which Consul Otto Schlick donated to the German Museum in Munich at the time of his invention. The ship's hull is here idealized as an elongated beam; it is suspended by spiral springs which represent the buoyancy of the water and enable the ship to oscillate. When the engine models carried by the beam are set in motion, the beam starts oscillating with slight amplitude. If the speed of revolution of the engines is increased, the vibrations of the beam grow larger the more the rotation frequency approaches the fundamental proper

frequency of the beam (cf. Fig. 18). Great amplitudes of oscillation would have disastrous effects on the safety of the ship—and also on the wellbeing of the passengers. The idea of mass balancing is to bring about a cancellation of the inertial forces and torques of the reciprocating masses of the marine engine in order to protect the ship's body from their harmful effects.

If we pass at once from accelerations to position coordinates, the balancing of the inertial forces, which are all in the x -direction, demands that

$$(15) \quad \sum M_k x_k = 0.$$

The masses M_k include not only those of the pistons and piston rods, but to first approximation also those of the connecting rods and portions of the eccentric parts of the crank shaft.

Just as important is the balancing of the moments of the inertial forces. It is mentioned above and made plausible by Fig. 17 that only the moments about the y -axis play any role here. Again we immediately pass from the accelerations to the position coordinates, which is permissible since the lever arms, i.e., the a of our Fig. 17, are constant. We then require

$$(16) \quad \sum M_k a_k x_k = 0.$$

We now express the piston coordinates x_k in terms of the crank pin

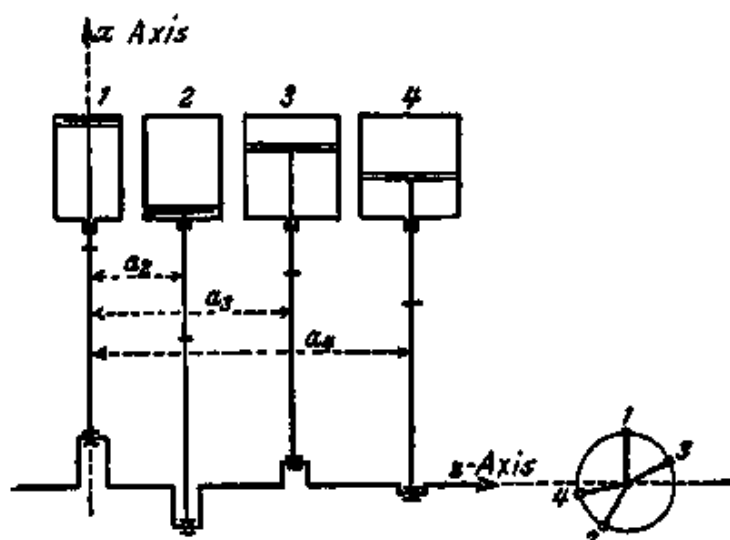


FIG. 17. Schlick mass balance of a vertically arranged four-cylinder piston engine. Diagram at lower right shows the position of the four crank pins relative to each other.



FIG. 18. Proper frequency of a freely vibrating beam as a model for the fundamental frequency of a ship.

coordinates ϕ_k . From Fig. 9 and Eq. (9.6) we have, to a *first approximation*,

$$(17) \quad x_k + r_k \cos \phi_k = \text{const.}$$

First approximation⁴ here means that we pass to the limit of an infinitely long connecting rod, or $r/l \rightarrow 0$. We shall not go into the calculation to second order where the first power of r/l is retained, as in Eqs. (9.5) and (9.6). Since all the pistons work on the same shaft, the ϕ_k are equal to each other apart from a phase shift α_k constant in time;

$$(18) \quad \phi_k = \phi_1 + \alpha_k,$$

where $\alpha_1 = 0$ and $\alpha_2, \alpha_3, \alpha_4$ can be chosen at will. By virtue of (17) and (18), the variable part of the conditions (15) and (16), which alone concerns us, gives

$$(19) \quad \sum M_k r_k \cos(\phi_1 + \alpha_k) = 0, \quad \sum M_k r_k a_k \cos(\phi_1 + \alpha_k) = 0.$$

If we expand the trigonometric functions, we see that with ϕ_1 arbitrary the factors of $\cos \phi_1$ and $\sin \phi_1$ must vanish separately. We then obtain four equations between the parameters α_k and a_k .

$$(20) \quad \begin{aligned} \sum M_k r_k \cos \alpha_k &= 0, & \sum M_k r_k \sin \alpha_k &= 0, \\ \sum M_k r_k a_k \cos \alpha_k &= 0, & \sum M_k r_k a_k \sin \alpha_k &= 0. \end{aligned}$$

The M_k and r_k are fixed by construction. The quantities at our disposal are the three phase displacements $\alpha_2, \alpha_3, \alpha_4$, and the two lever arm ratios $a_3: a_4$ [the absolute magnitudes of the a do not enter in Eq. (20)], altogether therefore five parameters; they allow a certain freedom of choice in fulfilling conditions (20). This freedom in turn makes it possible to avoid solutions which are technically objectionable. The preceding shows that the mass balancing can be carried through to first order in four-cylinder engines; it also shows that for lack of enough parameters it cannot be effected in engines with a smaller number of cylinders, as asserted above. The external characteristic of the Schlick mass balancing method is that the pistons of a four-cylinder engine are not equidistant and that their crank pins are not arranged at equal angles to each other. The latter feature is illustrated in the lower right-hand corner of Fig. 17.

The Schlick method proved its worth in the first modern steamers of the Hamburg-America Line; it eliminated the danger of resonance. It is true, however, that it had only a transient importance in the practices of ship-building, since piston engines were soon to be displaced by turbines, where there are no reciprocating masses. Even nowadays, however, mass balancing is important in automobile and airplane engines as well as in the Diesel engines of submarines.

⁴ This first approximation defines the mass balancing to first order (i.e., the "balancing for primary forces and primary couples," as it is called). Since we want to restrict ourselves to the latter, we need not carry out the second approximation.

(6) General Rule on the Number of Integrations Feasible in a Closed System

A mechanical system is called closed if no external forces, but only internal ones, act on it.⁵ In that case the equations of linear and angular momentum become principles of conservation. The conservation of momentum introduces $2 \cdot 3$ constants, that of angular momentum 3 constants of integration.⁶ The equation of energy yields one additional constant. We therefore have a total number of

$$(21) \qquad 2 \cdot 3 + 3 + 1 = 10$$

integrals of the equations of motion.

So much for the three-dimensional case. In the case of two dimensions, such as the two-body problem of astronomy, we have only one component of angular momentum (directed perpendicular to the plane containing the trajectories of the two bodies), so that we obtain, together with the integral of energy,

$$(22) \qquad 2 \cdot 2 + 1 + 1 = 6$$

generally feasible integrals.

In the one-dimensional case this number evidently reduces to

$$(23) \qquad 2 \cdot 1 + 0 + 1 = 3.$$

The general expression for n dimensions is

$$(24) \qquad n + 1 + \frac{1}{2}n(n+1).$$

The best method of clarifying this expression is to appeal to the concepts of relativity: we put $n=3$ and add the time as the fourth coordinate. We must then form the four-vector momentum which is obtained from Eq. (2.19) by summing over all the particles of the system. The basic equations of relativistic mechanics now tell us that for a closed system this four-vector remains constant; incidentally its time component is, apart from a factor $-ic$ and an additive constant, equal to the kinetic energy. The four integrals thus obtained (conservation of momentum and energy) are represented in (24) by the term $n+1$. The second term of the expression is the result of the combination of two axes at a time in the formation of moments. Evidently the combination of two space axes yields the equations of angular momentum in the ordinary sense. The combination of the time axis with one of the space axes, on the other hand, gives the second

⁵ Every system becomes closed, of course, if one makes it large enough, i.e., if one includes the sources of the external forces in the system.

⁶ The $2 \cdot 3$ constants arising from the equation of the straight line described by the center of mass, and the three areal velocity constants.

integrals of motion of the mass center which express the rectilinearity of this motion. For according to (2.19), if we indicate summation over all mass points by a bar as on p. 70 and replace $(1-\beta^2)^{\frac{1}{2}}$ by unity from the start, we calculate

$$x_k p_k - x_k p_k = ic(\overline{m_k x_k} - t \overline{m_k \dot{x}_k}), \quad k=1, 2, 3.$$

From the principle of conservation of angular momenta this quantity must be equal to a constant, which we may call icA_k . In three-dimensional vector notation and with the symbols of (3a, b) we then have

$$(25) \quad \mathbf{R} - t\mathbf{V} = \mathbf{A}.$$

With \mathbf{A} and \mathbf{V} constant this means that indeed the mass center moves in a straight line with constant speed. The foregoing should be sufficient explanation for the origin of (24); the use of the four-dimensional space-time symmetry has lent additional clarity to it.

We wish finally to make a remark concerning the enumeration of (21) and (22) pertaining to the field of astronomy. The famous three-body problem would need for its complete integration, i.e., for a determination of its $3 \cdot 3$ coordinates and $3 \cdot 3$ components of velocity,

$$(26) \quad 2 \cdot 3 \cdot 3 = 18$$

first integrals. Each of these, as exemplified by Eq. (25), would give one relation between the position and velocity coordinates involving one constant of integration. But a comparison of (26) with (21) shows that we are lacking eight integrals for the complete integration; above and beyond this the unrelenting efforts of the greatest mathematicians from Lagrange to Poincaré have shown that the missing integrals cannot be obtained in algebraic form; a conclusive proof of this was given by H. Bruns.

A similar enumeration for the two-body problem, plane by its very nature, requires only

$$2 \cdot 2 \cdot 2 = 8$$

instead of $2 \cdot 3 \cdot 3 = 18$ constants of integration for its complete integration. Thus only two constants are required beyond those which according to (22) are in all cases available for a two-dimensional problem. As a matter of fact these two integrals with their corresponding arbitrary constants can be found here, as shown by the transition from Eqs. (6.4) to (6.5). Hence the two-body problem can be solved exactly; the three-body problem is in general insoluble, i.e., it can be solved only by analytical approximation methods. It is only under very special assumptions about the type of motion that we shall be able to find a solution in closed form for the latter problem in § 32.

§ 14. The Laws of Friction

As already emphasized in § 11, subsec. 4, the guiding of a mass on a prescribed path introduces a component of reaction along the path direction which cannot be obtained from general principles of mechanics, but must be determined experimentally. Apart from some preliminary work of other investigators this determination was carried out for the first time in 1785 in the famous, and for those times very accurate, experiments of Ch. A. Coulomb, whose name, we recall, is permanently linked with the basic laws of electrostatics and magnetostatics.

With Coulomb we distinguish

- (a) Static friction
- (b) Kinetic or sliding friction.

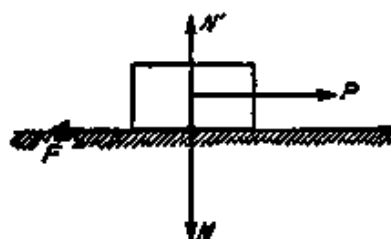


FIG. 19. Static friction on plane support.

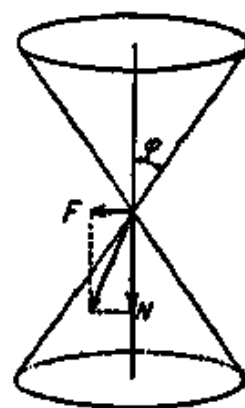


FIG. 20. Construction of the angle of friction and the cone of friction.

(1) Static Friction

Consider a body resting on a horizontal support. If we exert a gradually increasing pull P on the body parallel to the support, no motion will occur at first. We must therefore assume that a force of friction F balances the pull P . If, however, P exceeds a very definite limit, acceleration takes place.

This limit F_{\max} is, according to Coulomb (and his predecessors), proportional to the normal pressure N , which in the case of rest on a horizontal support is simply equal to the weight G of the body. We have

$$(1) \quad F_{\max} = \mu_0 N.$$

μ_0 is the *coefficient of static friction*; it depends on the nature and the state of the surfaces of the two materials in contact. If the two materials are the same, μ_0 is especially great (interpenetration).

By means of

$$(2) \quad \mu_0 = \tan \phi$$

one can introduce an angle ϕ which can be thought of as the vertex angle of a "cone of friction." As long as the resultant of the two forces F and N

falls inside this cone, no motion takes place, cf. Fig. 20. Motion occurs when their resultant lies in the surface of the cone or outside it.

The significance of the angle of friction is illustrated by experiments with the inclined plane (Fig. 21) which go back to Galileo. We write down without further explanation

$$N = G \cos \alpha, \quad P = G \sin \alpha = -F.$$

From

$$F < F_{\max} = \mu_0 N = N \tan \phi$$

we therefore obtain as the condition of rest

$$G \sin \alpha < \tan \phi \cos \alpha \cdot G$$

so that

$$\tan \alpha < \tan \phi$$

or

$$\alpha < \phi.$$

The body remains in a state of rest on the inclined plane as long as $\alpha < \phi$. The angle of friction ϕ is therefore that inclination of a plane at which sliding will set in.

The following is a less trivial example. An oblique arm is attached to a vertical axle at an angle $\frac{\pi}{2} - \alpha$. This arm carries a movable sleeve or bead (cf. Fig. 22). When the axle does not rotate the bead is at rest or in motion depending on whether $\alpha < \phi$ or $\alpha > \phi$. If now the axle is set rotating, the centrifugal force $mr\omega^2$ is added vectorially to the force of gravity mg . The normal force N resulting from these two and the pull P along the guiding rod are, from the diagram,

$$N = m(g \cos \alpha + r\omega^2 \sin \alpha), \quad P = \pm m(g \sin \alpha - r\omega^2 \cos \alpha).$$

The double sign in front of P means that we count the pull positive downward as well as upward, so that we can take into consideration a downward as well as an upward sliding of the bead.



FIG. 21. Equilibrium on an inclined plane.

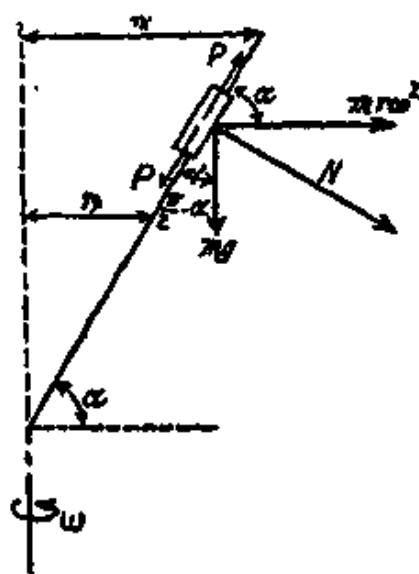


FIG. 22. Movable sleeve or bead on an oblique rotating rod. Equilibrium under friction.

From (1) and (2) the bead is in equilibrium if

$$\pm(g \sin \alpha - r\omega^2 \cos \alpha) < \tan \phi (g \cos \alpha + r\omega^2 \sin \alpha).$$

We now replace the $<$ sign by an $=$ sign, thereby obtaining the condition for "just sliding," i.e., the limit of equilibrium. By trigonometric transformation we carry out a separate calculation for the two cases \pm .

$$+ \text{ sign, downward sliding: } g \sin (\alpha + \phi) = r_2 \omega^2 \cos (\alpha + \phi),$$

$$- \text{ sign, upward sliding: } g \sin (\alpha - \phi) = r_1 \omega^2 \cos (\alpha - \phi),$$

or, collected together,

$$\left. \begin{matrix} r_1 \\ r_2 \end{matrix} \right\} = \frac{g}{\omega^2} \tan (\alpha \mp \phi).$$

The force of friction hence results in a finite interval

$$r_1 < r < r_2$$

of r in which the bead is in equilibrium.

For $\alpha > \phi$ (the bead slides down as $\omega \rightarrow 0$) both r are positive; the smaller ω , the greater the interval between them. With $\alpha < \phi$ (the bead is in equilibrium under static friction for $\omega \rightarrow 0$) $r_1 = 0$ (even negative according to the equation) and only r_2 is positive; with increasing ω , r_2 approaches zero as well.

(2) Sliding Friction

Here the law of friction

$$(4) \quad F = \mu N$$

applies.

The coefficient of sliding friction μ is roughly independent⁷ of the velocity, and, like μ_0 , a constant depending on the nature of the materials and conditions of the surfaces. It is universally true that

$$(5) \quad \mu < \mu_0.$$

If the path along which the body slides is rectilinear, N equals the force of gravity (or its component perpendicular to the path); if the path is curved, we must, according to Eq. (II.15), add the effect of the centrifugal force.

We illustrate Eq. (5) by means of an extremely primitive experiment which is, however, very surprising in its result. Let us put a smooth cane

⁷ Experience in railroad operation (sliding friction between wheel and brakehoe) indicates that for high velocities v the factor μ decreases monotonically with increasing v .

or walking stick over the forefingers of the right and left hand, held some distance apart. From Fig. 11a the distribution of forces is

$$A = \frac{b}{a+b}G, \quad B = \frac{a}{a+b}G.$$

We now let the two fingers approach each other. Sliding occurs alternately on the right and left fingers, until the fingers meet. Where on the stick do they meet?

Let $A > B$ initially. The sliding therefore begins at B . B remains in motion not only until $a=b$, but slides to the point $b_1 < a$ where the sliding friction of B equals the static friction of A . In general we have

$$F_{B,s} = \mu a \frac{G}{a+b}, \quad F_{A,s} = \mu_0 b \frac{G}{a+b}.$$

Putting these two expressions equal for $b=b_1$, we obtain

$$\mu a = \mu_0 b_1, \quad \frac{a}{b_1} = \frac{\mu_0}{\mu} > 1.$$

At this instant the stick must begin to move over A . At once the friction $F_{A,s}$ falls to $F_{A,s} < F_{A,s}$ so that in b_1 the friction $F_{B,s}$ exceeds that in A ; i.e., B comes to rest, and $F_{B,s}$ changes to $F_{B,s}$.

This process is now repeated at each turning point. A and B thereby approach in geometric progression (since the quotient $\frac{\mu_0}{\mu}$ occurs each time) the center of mass of the stick for which $a=b=0$. In the final state the stick balances in equilibrium over the juxtaposed fingers.

We now return to static friction, which plays a decisive role in pure rolling motion. Paradoxical as it may sound, it is the static friction which drives a railroad train forward. (The same is true of an automobile; a pedestrian on slippery ground likewise propels himself only by means of static friction.) The steam pressure is an internal force, and as such could never set the mass center of the locomotive in motion. To do this an external force is needed. This external force is the reaction between rail and wheel, i.e., just the static friction.

Consider one of the driven wheels of the locomotive (Fig. 23). By means of the connecting rod the engine transmits a torque L to the wheel; its primary action would be to impart a rotational acceleration to the

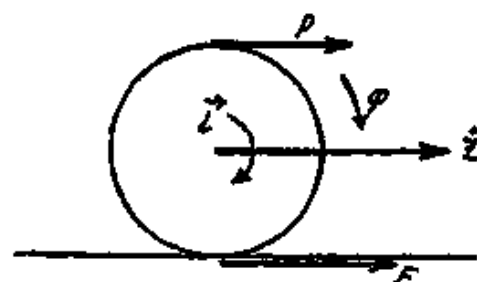


FIG. 23. Reaction between wheel and rail in a locomotive. For the case of pure rolling the static friction provides the driving force of the train.

wheel. This is incompatible with the condition of pure rolling, Eq. (11.10),

$$(6) \quad \dot{z} = r\omega.$$

Let M be the mass of the train per actuated wheel, R the resistance to motion (air resistance, frictional losses in the axle bearings, etc.), I the moment of inertia of the wheel, and F the force of static friction. The equations of motion become

$$(7) \quad \begin{aligned} M\ddot{z} &= F - R; \\ I\ddot{\phi} &= L - Fr. \end{aligned}$$

The static friction F cannot be determined *a priori*; it can, however, be obtained from the foregoing equations as follows. Let us at first eliminate F from the equations

$$(8) \quad \begin{aligned} M\ddot{z} &= F - R; \\ M_{\text{red}}\ddot{z} &= P - F. \end{aligned}$$

equivalent to (7). P is the peripheral force corresponding to the torque L , and M_{red} , as in (11.8), is the reduced mass corresponding to the moment of inertia I , i.e.,

$$L = Pr, \quad I = M_{\text{red}}r^2.$$

From (8) one obtains

$$(9) \quad (M + M_{\text{red}})\ddot{z} = P - R$$

and, by virtue of the first Eq. (8),

$$(10) \quad F = R + \frac{M}{M + M_{\text{red}}}(P - R) = \frac{MP + M_{\text{red}}R}{M + M_{\text{red}}}.$$

D'Alembert's principle could have furnished equation (9) directly. The first Eq. (8) contains the quantitative proof of our assertion that the static friction F is the driving force in the operation of a train. For in the case of uniform motion it gives

$$R = F.$$

As the second Eq. (8) shows, the peripheral force P resulting from the steam pressure has merely the function of calling into play the static friction at the rails.

Another evidence of this is the fact that as trains have become faster or the freight per train greater, locomotives have become constantly heavier. This circumstance points directly to Coulomb's law of friction, Eq. (1), which states that the limit of static friction available is proportional to

the normal pressure N . The well-known fact that static friction fails and sliding occurs when the rails are too smooth (due to ice, or, for instance, to lubrication from run-over migrating caterpillars) points to the other factor μ_0 in Eq. (1), which, as emphasized, depends on the state of the surface of the rails. When the rails are too smooth, the factor μ_0 must be artificially increased; the sander serves this end.

CHAPTER III

OSCILLATION PROBLEMS

The investigations that are to follow will teach us nothing new about the principles of mechanics. So great, however, is the significance of oscillation processes for physics and engineering that their separate systematic treatment is deemed essential.

§ 15. The Simple Pendulum

The oscillating body is a particle of mass m which is attached to a fixed point O by means of a weightless rigid rod of length l ; l is called the length of the pendulum. We may neglect friction at the point of suspension and air resistance, so that the only force acting is that of gravity, with a component $-mg \sin \phi$ in the direction of increasing ϕ (cf. Fig. 24). The general equation (11.14) for the guided motion along an arbitrary path gives us, with $v = l\dot{\phi}$ (circular path), the exact equation

$$(1) \quad ml \frac{d^2\phi}{dt^2} = -mg \sin \phi.$$

For sufficiently small oscillations, $\phi \ll 1$, we can put $\sin \phi = \phi$. With the abbreviation

$$(2) \quad \frac{g}{l} = \omega^2$$

we then obtain the linear pendulum equation

$$(3) \quad \frac{d^2\phi}{dt^2} + \omega^2\phi = 0.$$

This is the differential equation of "harmonic oscillations" as treated in § 3 (4). Apart from the designation for the dependent variable it is identical with Eq. (3.23). The circular frequency ω defined in (3.22) is now given by Eq. (2) above. We therefore have

$$(4) \quad \omega = \frac{2\pi}{T} = \left(\frac{g}{l}\right)^{\frac{1}{2}}, \quad T = 2\pi \left(\frac{l}{g}\right)^{\frac{1}{2}}.$$

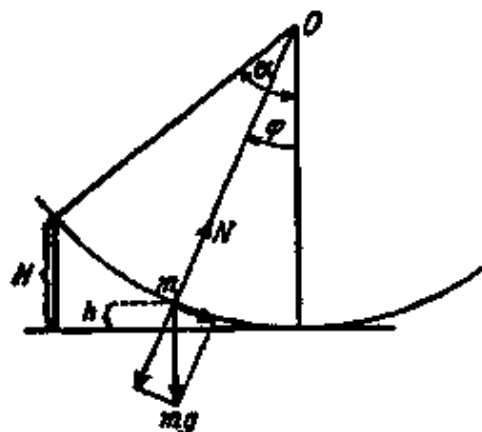


FIG. 24. Simple pendulum. Component of gravity along the direction of motion.

Notice that T is independent of the mass m , which dropped out already in (1). Thus different masses have the same period if the pendulum length l is the same. T is the full period, covering a complete swing to and fro. Sometimes one half of this time is designated as the period of oscillation. Thus one speaks of a "seconds pendulum" for which $\frac{1}{2}T$ equals one second. Its length is calculated from (4) to be

$$l = \frac{g}{\pi^2} \cong 1 \text{ meter.}$$

To the extent to which Eq. (3) is valid the period of oscillation is independent also of the amplitude of swing; i.e., small pendulum oscillations are *isochronous*.

The general solution of (3) has the form

$$\phi = a \sin \omega t + b \cos \omega t.$$

If we specify that $\phi = 0$ at $t = 0$ and $\phi = \alpha$ at $t = \frac{T}{4}$, we must put $b = 0$ and $a = \alpha$, so that

$$(5) \quad \phi = \alpha \sin \omega t.$$

α is therefore the amplitude of ϕ , i.e., the maximum displacement of the particle measured in units of angle (radians).

For finite deflections the isochronism is destroyed because of the non-linearity of Eq. (1) which applies in that case. In order to integrate (1) we multiply it on the left and right by $\frac{d\phi}{dt}$; this amounts to passing from the equation of motion to the equation of energy. An integration yields

$$(6) \quad \left(\frac{d\phi}{dt}\right)^2 = 2\omega^2 \cos \phi + C.$$

C is determined by the condition that $\frac{d\phi}{dt} = 0$ for $\phi = \alpha$, i.e.,

$$C = -2\omega^2 \cos \alpha.$$

Alternately we can proceed directly from the equation of energy. With the meaning of H indicated in Fig. 24 we obtain

$$(6a) \quad \frac{m}{2} l^2 \left(\frac{d\phi}{dt}\right)^2 + mgh = mgH$$

where $\begin{cases} h = l(1 - \cos \phi) \\ H = l(1 - \cos \alpha), \end{cases}$

which is evidently identical to (6).

Consider now the equality

$$\cos \phi - \cos \alpha = 2 \left(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\phi}{2} \right);$$

we substitute it in (6) to obtain

$$(7) \quad \frac{d\left(\frac{\phi}{2}\right)}{\left(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\phi}{2}\right)^{\frac{1}{2}}} = \omega dt$$

or

$$(8) \quad \int_0^{\frac{\phi}{2}} \frac{d\left(\frac{\phi}{2}\right)}{\left(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\phi}{2}\right)^{\frac{1}{2}}} = \omega t.$$

We have thus arrived at an *elliptic integral of the first kind*. In order to explain this name we shall have to speak in passing of the "rectification of the ellipse," i.e., the measurement of the length of an arc of an ellipse. Let us use the parametric form of the equation of an ellipse,

$$x = a \sin v$$

$$y = b \cos v$$

from which we calculate

$$ds^2 = dx^2 + dy^2 = (a^2 \cos^2 v + b^2 \sin^2 v) dv^2,$$

$$ds = [a^2 - (a^2 - b^2) \sin^2 v]^{\frac{1}{2}} dv.$$

We now put

$$k^2 = + \frac{a^2 - b^2}{a^2} \quad (< 1 \text{ for } a > b),$$

and obtain for the length of the arc of the ellipse between the endpoint $v=0$ of the minor axis and an arbitrary point v of the ellipse

$$(9) \quad s = a \int_0^v (1 - k^2 \sin^2 v)^{\frac{1}{2}} dv.$$

This is an "elliptic integral of the second kind."

The elliptic integral of the first kind is the simpler of the two from the viewpoint of function theory. In the "Legendre standard form" it is

$$\int_0^v \frac{dv}{(1 - k^2 \sin^2 v)^{\frac{1}{2}}}.$$

We shall put our integral (8) in this form by means of the transformation

$$\sin \frac{\phi}{2} = \sin \frac{\alpha}{2} \cdot \sin v.$$

$$(10) \quad \left(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\phi}{2} \right)^{\frac{1}{2}} = \sin \frac{\alpha}{2} \cos v,$$

$$\frac{d\frac{\phi}{2}}{\left(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\phi}{2} \right)^{\frac{1}{2}}} = \frac{dv}{\cos \frac{\phi}{2}} = \frac{dv}{(1 - k^2 \sin^2 v)^{\frac{1}{2}}},$$

where the "modulus" k stands for

$$(11) \quad k = \sin \frac{1}{2} \alpha.$$

If we wish to calculate the period T , we must put in Eq. (8)

$$t = \frac{T}{4} \quad \text{and} \quad \phi = \alpha,$$

so that, according to (10), $v = \frac{\pi}{2}$. This yields the so-called "complete integral of the first kind," which is designated by the letter K ,

$$(12) \quad K = \int_0^{\frac{\pi}{2}} \frac{dv}{(1 - k^2 \sin^2 v)^{\frac{1}{2}}}.$$

ω being defined by (2), we then obtain from (8) the period

$$(13) \quad T = 4K \left(\frac{l}{g} \right)^{\frac{1}{2}}.$$

From (12) we can read off directly that

$K = \frac{\pi}{2}$ as $k \rightarrow 0$, i.e., according to (11), for sufficiently small amplitudes α ;
 $K = \infty$ as $k \rightarrow 1$, i.e., according to (11), for $\alpha = \pi$, 180° swing to upright position.

In the first case we obtain our former expression (4), as would be expected. In the latter case the deviation from this expression reaches an extreme.

In general a binomial expansion and term-by-term integration of (12) leads to

$$K = \frac{\pi}{2} \left(1 + \frac{k^2}{4} + \frac{9k^4}{64} + \dots \right).$$

The corresponding expression for T is

$$(14) \quad T = 2\pi \left(\frac{l}{g} \right)^{\frac{1}{2}} \left(1 + \frac{1}{4} \sin^2 \frac{\alpha}{2} + \frac{9}{64} \sin^4 \frac{\alpha}{2} + \dots \right),$$

which gives the deviation from isochronism for finite deflections in quantitative fashion.

Astronomical clocks have simply-constructed pendulums with $\alpha \leq 1\frac{1}{2}^\circ$. For them the first correction term in the parenthesis of (14) amounts to approximately 1 part in 20,000.

§ 16. The Compound Pendulum

This problem is essentially that of rotation of a rigid body about a fixed axis, treated already in § 11, subsec. 1, from which it differs only in that the external forces are now specified to be gravitational. Let s be the distance of the center of gravity G from the fixed axis O [we use the term "center of gravity" deliberately here, though, from (3.12), it coincides with the center of mass]; moreover, let ϕ be the angle which the line OG makes with the vertical. The total moment L of the gravitational forces acting on the individual elements of mass dm is evidently

$$(1) \quad L = -mgs \sin \phi,$$

where m is the total mass; from (11.4) the equation of motion is then

$$(2) \quad I\ddot{\phi} = -mgs \sin \phi.$$

A comparison with the equation of motion (15.1) of the simple pendulum shows that the length l of the equivalent simple pendulum, i.e., the simple pendulum having the same period of oscillation as our compound one, is

$$(3) \quad l = \frac{I}{ms}.$$

Let us replace I by the so-called *radius of gyration* a , defined by

$$(4) \quad I = ma^2.$$

The radius of gyration is therefore that distance from the point of suspension O of the pendulum at which we must concentrate the total mass m in order to obtain the moment of inertia I of the actual mass distribution. Note: in (11.8) we introduced a "reduced mass" for the distance r at which the initially unknown mass M_{red} was to be placed; here, *per contra*, the mass m is given and we are looking for the distance a at which this mass is to be located.

Comparison of (3) and (4) shows that a is the geometric mean of s and l ,

$$(5) \quad a^2 = ls.$$

Let us now lay off the equivalent pendulum length l from O along the center line OG of the pendulum. The point P thus obtained is called the *center of oscillation* (Huygens). Fig. 25 shows the relative positions of O , G and P and allows us to form a picture of the relation between s , a and l .

FIG. 25. Point of suspension O , center of gravity G , and center of oscillation P of a compound pendulum. The radius of gyration a is the geometric mean of the equivalent pendulum length l and distance from center of gravity s .



We now claim that the roles of O and P are interchangeable. So far O has been our point of suspension, P the center of oscillation. We shall now take P as the point of suspension and show that O becomes the center of oscillation. This is the idea underlying the *reversible pendulum*.

The scheme below tabulates the symbols so far used and completes the list for purposes of what is to follow.

Point of Suspension	Center of Oscillation	Equivalent Pend. Length	Moment of Inertia	Radius of Gyration	Distance of Mass Center
O	P	l	I	a	s
P	O'	l_P	I_P	a_P	$l-s$

Our assertion is that

$$l_P = l, \text{ i.e., } O' = O.$$

Proof: let us calculate l_P from equations (3) and (4) rewritten in terms of the corresponding new symbols. We have

$$(6) \quad l_P = \frac{I_P}{m(l-s)} = \frac{a_P^2}{l-s}.$$

Now according to Eq. (10) of the supplement to this section

$$(6a) \quad a_P^2 = l(l-s)$$

so that indeed the last member of (6) equals l .

The pendulum is used in the determination of the gravitational acceleration g at different points on or below the surface of the earth. Since in practice no simple pendulum is available and since in a compound pendulum the moment of inertia I cannot be calculated accurately (not only because of the complicated shape of the bob, but also because of possible internal inhomogeneities), one is forced to resort to the experimental method of the reversible pendulum for the determination of the equivalent pendulum length. We have to imagine that the pendulum of Fig. 25 is provided with two knife-edges for its points of support, one at O and one at P , the latter with its edge facing up, and both with their triangular cross-sections in the plane of the drawing. The knife-edge at P can be moved up and down by means of a micrometer screw. Given a sufficiently long period of observation the number of oscillations can be counted with very great accuracy, so that the equality or inequality of the periodic times for oscillations about O and P can be determined exceedingly precisely, and, if necessary, corrected by means of the micrometer screw.

The principle of the reversible pendulum is a first illustration of a type of very general reciprocity relation which recurs in all branches of physics.

Another example of such a relation is the interchangeability of source point and field point ("Aufpunkt") in acoustics and electrodynamics.

SUPPLEMENT: A RULE CONCERNING MOMENTS OF INERTIA

We have in mind the rule of parallel axes, which states that the moment of inertia of a body of mass m about an axis through an arbitrary point O is equal to the sum of its moment of inertia about the parallel axis through the center of mass G and ms^2 , where s is the distance between G and the axis through O .

If y is the direction of the axis in question and x the direction from O to G , the distance r from the axis through O of some element of mass dm must be

$$r^2 = x^2 + z^2.$$

Here x is measured from O . If, instead, x is measured from G , and if, as in Fig. 25, $OG=s$, we have

$$r^2 = (x+s)^2 + z^2 = x^2 + z^2 + 2xs + s^2.$$

If we sum over all dm , it follows that

$$(7) \quad I = I_G + 2s \int x dm + ms^2.$$

The middle term vanishes [cf., for instance, Eq. (13.3b)] provided the plane $x=0$ passes through the center of mass. If this is the case,

$$(8) \quad I = I_G + ms^2,$$

as asserted above.

Accordingly we have from Fig. 25 that

$$(8a) \quad I_P = I_G + m(l-s)^2.$$

But from (8) and (8a)

$$I_P - I = ml^2 - 2m ls$$

which, in view of (4), can be written

$$(9) \quad a_P^2 - a^2 = l^2 - 2ls$$

or, by virtue of (5),

$$(10) \quad a_P^2 = l^2 - ls = l(l-s).$$

This is the relation that was used in (6a).

§ 17. The Cycloidal Pendulum

This pendulum was invented by Christian Huygens¹, the most ingenious watchmaker of all time. Its purpose is to eliminate the lack of isochronism of the ordinary simple pendulum. This is achieved by making the mass point move on a cycloidal instead of a circular arc. Later on we shall see how this motion can be realized in practice.

The parametric representation of a common cycloid is

$$(1) \quad \begin{aligned} x &= a(\phi - \sin \phi), \\ y &= a(1 - \cos \phi). \end{aligned}$$

The parameter ϕ is the angle through which a wheel of radius a rolling on the horizontal x -axis has turned from its initial position. The common cycloid is generated by a point on the periphery of the wheel (Fig. 26).

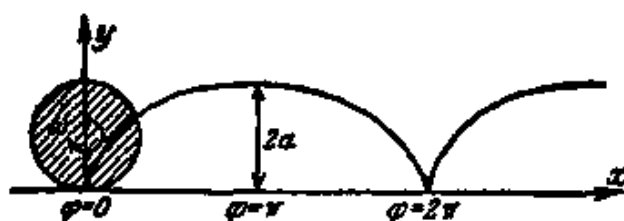


FIG. 26. Generation of common cycloid by point on the periphery of a rolling wheel. Definition of angle of rotation ϕ .

For our pendulum we need a cycloid that has its cusps at the top rather than on the bottom (cf. Fig. 27 on p. 96); this is generated by having our wheel roll on the underside of the x -axis. The x of such a curve is that given in (1) while its y is obtained by subtracting the y given in (1) from $2a$,

$$(2) \quad \begin{aligned} x &= a(\phi - \sin \phi), \\ y &= a(1 + \cos \phi). \end{aligned}$$

The component of gravity mg along the tangent of the trajectory (in our case the cycloid) is

$$F_s = -mg \cos(y, s) = -mg \frac{dy}{ds}.$$

The general relation (11.14) therefore yields

$$(3) \quad m\dot{v} = -mg \frac{dy}{ds},$$

where, just as in the case of the circular pendulum, the mass m cancels on

¹ *Horologium Oscillatorium*, Paris (1673). Collected Works, Vol. 18, The Hague (1934).

the left and right. Differentiation of (2) gives

$$\begin{aligned} dx &= a(1 - \cos \phi) d\phi, & dy &= -a \sin \phi d\phi. \\ ds^2 &= a^2(2 - 2 \cos \phi) d\phi^2, & ds &= 2a \sin \frac{\phi}{2} d\phi. \end{aligned}$$

Thus in our case

$$(4) \quad v = \frac{ds}{dt} = 2a \sin \frac{\phi}{2} \frac{d\phi}{dt} = -4a \frac{d}{dt} \cos \frac{\phi}{2}$$

and

$$(5) \quad \frac{dy}{ds} = -\frac{1}{2} \frac{\sin \phi}{\sin \phi/2} = -\cos \frac{\phi}{2}.$$

If we replace (4) and (5) in (3), we obtain

$$(6) \quad \frac{d^2}{dt^2} \cos \frac{\phi}{2} = -\frac{g}{4a} \cos \frac{\phi}{2}.$$

This equation differs from Eq. (15.3) of the simple pendulum only in that the dependent variable is now called $\cos \frac{\phi}{2}$ rather than ϕ . This is of course of no consequence for the integration of (6). The earlier Eq. (15.4) therefore holds unchanged, viz.

$$(7) \quad T = 2\pi \left(\frac{l}{g} \right)^{\frac{1}{2}} \text{ with } l = 4a,$$

the latter because in (6) $4a$ took the place of our former l .

Eq. (15.3) described only the small displacements of a simple pendulum and was obtained from the exact relation (15.1) by an approximation; our present equation (6) and Eq. (7) resulting from an integration thereof are, on the other hand, exact for oscillations of arbitrary amplitude. The cycloidal pendulum is then rigorously isochronous; its periodic time is completely independent of the amplitude of oscillation.²

As regards the method used, we notice that in (6) the motion of our particle was represented not by its Cartesian coordinates or by some parameter bearing an immediate relation to the cycloidal curve, but by one half the angle of rotation ϕ of the wheel generating the cycloid. We see that

² The cycloid can also be called *tautochrone* (oscillations on a cycloid are "isochronous to each other"); it is also called *brachistochrone* (because it answers the question, "on what curve must a mass acted on by constant gravitational force slide in order to traverse the distance between two given endpoints in the least possible time?"). It turns out that the mass takes less time on a cycloid than on a straight line or any other curve joining the same points). The brachistochrone problem is all the more notable because it was for it that the first principles of the Calculus of Variations were developed.

this parameter, although only indirectly connected with the cycloid, provides the simplest method of approach to the problem. Its introduction gives us a foretaste of the general Lagrange method of Chapter VI, which enables us to introduce arbitrary parameters as dependent variables in the equations of motion.

Just as remarkable as Huygens' discovery of the isochronism of the cycloidal pendulum is the way in which he actually achieved the frictionless motion of the bob on the cycloid. He availed himself of the rule that the evolute of a cycloid is another cycloid equal to the generating one. If, therefore, we tie a string of length $l=4a$ to the point O of Fig. 27 in which the two upper cycloid arcs form a cusp, and if this string be pulled taut so that it rests against the right part of the cycloid (or the left part if deflected to the left), the endpoint P of the string describes the lower cycloidal arc. The guiding of the bob along the lower cycloid effected in this manner is almost as frictionless as the guiding of the simple pendulum along a circular arc.

Actually Huygens' idea has been abandoned in the practice of pendulum clock construction; according to investigations of Bessel among others it is sufficient to install a spring — usually a short elastic lamina — at the upper end of the pendulum. If the length of the lamina and the mass of the bob are suitably chosen, a sufficient degree of isochronism is achieved.

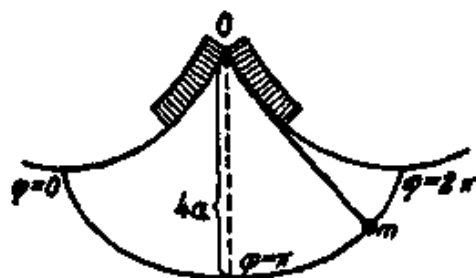


FIG. 27. Huygens' isochronous cycloidal pendulum.

§ 18. The Spherical Pendulum

We require the pendulum to be suspended in such a fashion that the mass point m is able to move freely on the surface of a sphere of radius l (the length of the pendulum). It is then subject to the condition of constraint

$$(1) \quad F = \frac{1}{2}(x^2 + y^2 + z^2 - l^2) = 0,$$

where the factor $\frac{1}{2}$ has been added for convenience's sake.

Here r , the number of conditions of constraint, equals 1, and $X_1 = X_2 = 0$, $X_3 = -mg$, so that the Lagrange equations of the first kind (12.9) take the form

$$(2) \quad \begin{aligned} m\ddot{x} &= \lambda x, \\ m\ddot{y} &= \lambda y, \\ m\ddot{z} &= -mg + \lambda z. \end{aligned}$$

In view of Eqs. (13.13) and (13.13a), elimination of λ from the first two equations (2) yields the constancy of angular momentum about the z -axis, or, what amounts to the same thing, the conservation of the areal velocity

$$(3) \quad x \frac{dy}{dt} - y \frac{dx}{dt} = 2 \frac{dS}{dt} = C \quad (S = \text{area swept out}).$$

If, on the other hand, we multiply the Lagrange equations (2) by \dot{x} , \dot{y} , \dot{z} , we obtain the equation of energy, for condition (1) is independent of t (cf. p. 68). Addition yields

$$(4) \quad m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = -mg\dot{z} + \lambda(x\dot{x} + y\dot{y} + z\dot{z}).$$

But from (1)

$$\frac{dF}{dt} = x\dot{x} + y\dot{y} + z\dot{z} = 0.$$

On the other hand we evidently have

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{1}{2} \frac{d}{dt} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} \frac{dv^2}{dt}.$$

Integration of (4) with respect to t then gives

$$(5) \quad \frac{m}{2} v^2 = -mgz + \text{const.},$$

which we shall write in the form

$$(5a) \quad T + V = E \quad \text{with} \quad V = mgz.$$

Let us finally multiply the Lagrange equations by x , y , z , respectively. With the aid of (1) this allows us to calculate λ ,

$$\lambda^2 - mgz = m(x\ddot{x} + y\ddot{y} + z\ddot{z})$$

or

$$(6) \quad \lambda = mg \frac{z}{l} + m \left(\frac{x}{l} \ddot{x} + \frac{y}{l} \ddot{y} + \frac{z}{l} \ddot{z} \right).$$

Now the normal to the surface of the sphere at the point x, y, z has direction cosines $\frac{x}{l}, \frac{y}{l}, \frac{z}{l}$, so that apart from sign the second term on the right is the inertial force F_n^* normal to the spherical surface; similarly the first term on the right is, apart from sign, the component F_n of gravity in the same direction. According to d'Alembert the sum of these two must be equilibrated by the reaction R_n of the surface of the sphere, or, physically speaking, by the tension in the pendulum suspension. The meaning of Eq. (6) can hence be summed concisely by the equation

$$(7) \quad \lambda = -(F_n + F_n^*) = R_n.$$

We notice that within a factor l , λ is the constraint which is exerted on the motion by virtue of condition (1), this constraint acting in a direction normal to the motion. Corresponding statements hold in more general cases where several conditions of constraint and therefore several Lagrange multipliers are present.

In order to carry out a second integration of (5) we shall pass to spherical coordinates given by

$$\begin{aligned}x &= l \cos \phi \sin \theta \\y &= l \sin \phi \sin \theta \\z &= l \cos \theta.\end{aligned}$$

We form

$$\begin{aligned}\dot{x} &= l \dot{\theta} \cos \phi \cos \theta - l \dot{\phi} \sin \phi \sin \theta, \\ \dot{y} &= l \dot{\theta} \sin \phi \cos \theta + l \dot{\phi} \cos \phi \sin \theta, \\ \dot{z} &= -l \dot{\theta} \sin \theta.\end{aligned}$$

The equation of conservation of angular momentum (3) becomes

$$(8) \quad 2 \frac{dS}{dt} = x\dot{y} - y\dot{x} = l^2 \sin^2 \theta \cdot \dot{\phi} = C$$

and the equation of energy (5a),

$$(9) \quad \frac{ml^2}{2} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + mgl \cos \theta = E.$$

A further change of variables

$$u = \cos \theta, \quad \dot{\theta} = -\frac{1}{(1-u^2)^{1/2}} \frac{du}{dt}$$

transforms (8) into

$$(10) \quad \dot{\phi} = \frac{C}{l^2(1-u^2)}$$

and (9) into

$$(11) \quad \left(\frac{du}{dt}\right)^2 = U(u) = \frac{2}{ml^2} (E - mgl u) (1 - u^2) - \frac{C^2}{l^4}.$$

This relation between t and u allows us to find t as a function of u ,

$$(12) \quad t = \int \frac{du}{U^{1/2}}.$$

Eq. (10) can now likewise be written in integrated form, for from (10) and (11)

$$\frac{d\phi}{du} = \dot{\phi} \cdot \frac{dt}{du} = \frac{C}{l^2(1-u^2)} \frac{1}{U^{1/2}},$$

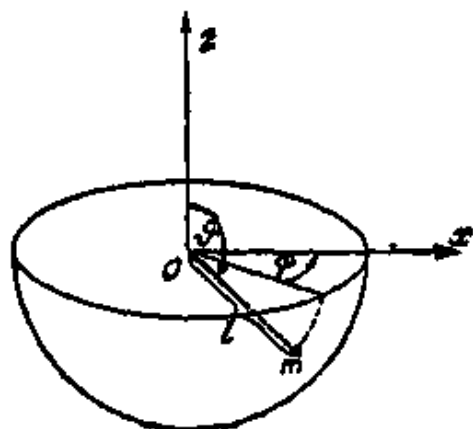


FIG. 28. Spherical pendulum treated as mass point m moving under gravity on the surface of a sphere of radius l .

so that one obtains

$$(13) \quad \phi = \frac{C}{I^2} \int \frac{du}{1-u^2} \cdot \frac{1}{U^{\frac{1}{2}}}.$$

U is a function of third degree in $u = \cos \theta$. $U^{\frac{1}{2}}$ is real only for $U > 0$. If then the constants of the equation correspond to a real physical problem, there must be two values $u = u_2 < u = u_1$ in the interval

$$-1 < u < +1$$

between which U is positive (cf. Fig. 29).

$u_1 = \cos \theta_1$ and $u_2 = \cos \theta_2$ are the two latitudes between which the mass point oscillates back and forth. If the integration of (12) or (13) reaches one of these limits of u , not only the direction of integration but also $U^{\frac{1}{2}}$ must change sign, in order that the integrals remain real and positive. Between two successive turning points one quarter of the full period of oscillation elapses, i.e.,

$$(14) \quad \frac{T}{4} = \int_{u_2}^{u_1} \frac{du}{U^{\frac{1}{2}}}.$$

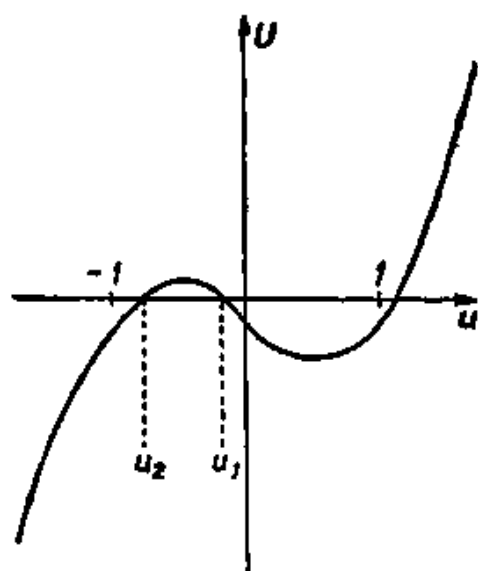


FIG. 29. Curve of third degree $U(u)$ and its intersections $u = u_1$ and $u = u_2$ with the abscissa. $u_2 < u_1 < 0$ means that the trajectory is located in the lower hemisphere.

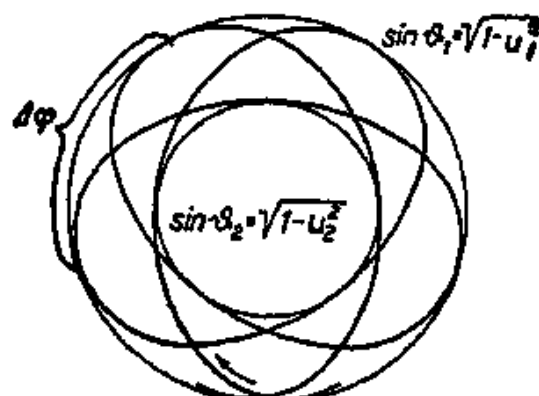


FIG. 30. "Bird's-eye" view of the path of the spherical pendulum. Angle of precession $\Delta\phi$. A passage from θ_1 over θ_2 back to θ_1 corresponds to a half-period, $\Delta\phi$ therefore to a full cycle.

Note that now the oscillation is no longer periodic in space as in the case of the pendulum moving in a plane, but is modified by a slow *precession*. The angle of precession $\Delta\phi$ by which the mass advances (or recedes) in a full period T is calculated from (13) to be

$$(15) \quad 2\pi + \Delta\phi = \frac{4C}{I^2} \int_{u_2}^{u_1} \frac{du}{(1-u^2)U^{\frac{1}{2}}}.$$

This precession is illustrated in Fig. 30, which is taken from A. G. Webster, "Dynamics of Particles," Leipzig, Teubner (1912), p. 51.

The integral (12) is an elliptic integral of the first kind, just like the integral (15.8) for the simple pendulum. This is the generic name applied to all integrals whose integrand contains the square root of a polynomial of the third or fourth degree in the variable of integration in the denominator. That Eq. (15.8) falls in this class can be seen by introducing the transformation $u = \sin \frac{\phi}{2}$, so that u becomes our variable of integration; if, moreover, we put $a = \sin \frac{\alpha}{2}$, (15.8) goes over into

$$\int \frac{du}{[(a^2 - u^2)(1 - u^2)]^{\frac{1}{2}}}$$

In particular, expression (14) for T is, just like (15.12), a complete integral of the first kind. On the other hand, integral (13), which has the two factors $(1 \pm u)$ in addition to $U^{\frac{1}{2}}$ in the denominator, is an "elliptic integral of the third kind," and (15) is a "complete elliptic integral of the third kind."

Problem III.1 shows that for infinitesimal oscillations the equation expressing the motion of the spherical pendulum becomes elementary and the angle of precession $\Delta\phi \rightarrow 0$.

§ 19. Various Types of Oscillations

Free and Forced, Damped and Undamped Oscillations

Free, undamped oscillations were treated in § 3, subsec. 4; we called them harmonic oscillations. At this point we shall consider, first of all,

Undamped, Forced Oscillations

We shall take as their differential equation

$$(1) \quad m\ddot{x} + kx = c \sin \omega t,$$

where $\omega = \frac{2\pi}{T}$ is the circular frequency of the driving force.

We have here made the differential equation linear in the dependent variable x , which is permissible, at any rate, for small oscillations (cf. simple pendulum). The same remark applies to the remaining examples in this and the following section.

The restoring force is $-kx$ as in (3.19); c of Eq. (1) is the amplitude of the driving force causing our particle to oscillate.

By virtue of the addition of the right member, (1) is an inhomogeneous linear differential equation. The left side, when set equal to zero, gives the associated homogeneous differential equation, as previously mentioned in connection with Eq. (3.23).

A particular solution of the inhomogeneous differential equation is given by

$$x = C \sin \omega t,$$

where C must satisfy the equation

$$C(k - m\omega^2) = c.$$

If, with (3.20) as model, we put

$$(2) \quad \omega_0 = \left(\frac{k}{m}\right)^{\frac{1}{2}},$$

we obtain

$$(3) \quad C = \frac{c/m}{\omega_0^2 - \omega^2}.$$

The general solution of (1) is formed from this particular solution and the general solution of the associated homogeneous equation:

$$(4) \quad x = C \sin \omega t + A \cos \omega_0 t + B \sin \omega_0 t.$$

The amplitude C of the first term grows with increasing ω to become infinite for $\omega = \omega_0$; thereupon it jumps to negative infinity, and decreases slowly in absolute value toward 0 as $\omega \rightarrow \infty$.

Actually, when C becomes negative the amplitude does not change sign, for amplitudes are positive by definition. We therefore continue to define the amplitude by $|C|$ and put the change of sign that takes place into the sine factor, where it appears as a phase change of $\delta = \pm\pi$.

The foregoing is illustrated in Figs. 31a, b, where $|C|$ and δ have been plotted as functions of ω .

In Fig. 31b we cannot *a priori* decide whether the phase leads or lags for $\omega > \omega_0$, i.e., whether we are to take $\delta = +\pi$ or $\delta = -\pi$. We shall, however, anticipate and consider undamped vibrations as a limiting case of damped vibrations (see below); this leads us to decide in favor of $-\pi$, so that the first term of (4) can be written in detail

$$(4a) \quad x = \frac{c/m}{\omega^2 - \omega_0^2} \sin(\omega t - \pi) \quad (\omega > \omega_0).$$

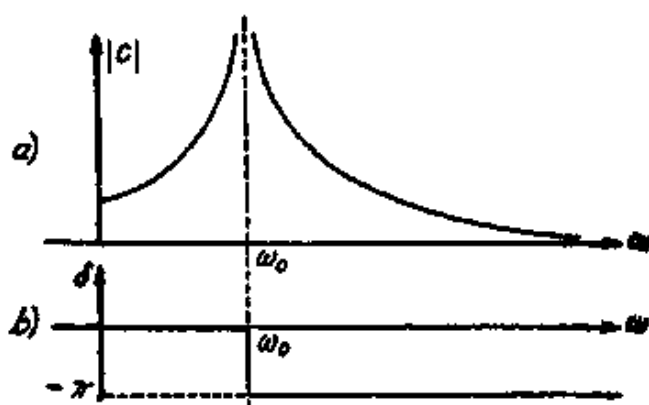


FIG. 31. Amplitude and phase of undamped forced oscillations.

The fact that the amplitude becomes infinite for $\omega = \omega_0$ illustrates the phenomenon of *resonance* between free and forced oscillations, a phenomenon that plays an important role in all of physics. The denominator of (3) and (4a) whose vanishing causes this infinite amplitude is called a "resonance denominator." It is intuitively clear that the closer the proper frequency of the oscillating system is to that of the driving force, the better the system will follow this force.

Incidentally we must keep in mind that we are guilty of gross extrapolation when we deduce infinite amplitudes at resonance, for in almost all cases our linear differential equation holds only for infinitesimal oscillations.

So far we have directed all our attention to the first term of the right member of Eq. (4). The other two terms are determined by the initial conditions. Let us take

$$x=0, \dot{x}=0 \text{ at } t=0,$$

so that, from (4),

$$A=0, \omega C + \omega_0 B = 0, \text{ hence } B = -\frac{\omega}{\omega_0} C.$$

It follows that

$$(5) \quad x = C \left(\sin \omega t - \frac{\omega}{\omega_0} \sin \omega_0 t \right).$$

Let us make the content of this equation clearer by considering the special case of near resonance of the two frequencies ω and ω_0 .

We put

$$\omega = \omega_0 + \Delta\omega$$

and expand

$$\sin \omega t - \frac{\omega}{\omega_0} \sin \omega_0 t = \sin \omega_0 t + t \Delta\omega \cos \omega_0 t - \sin \omega_0 t - \frac{\Delta\omega}{\omega_0} \sin \omega_0 t.$$

Eq. (5) then yields

$$x = C \Delta\omega \left(t \cos \omega_0 t - \frac{1}{\omega_0} \sin \omega_0 t \right).$$

and, by virtue of (3), in the limit $\Delta\omega=0$,

$$(6) \quad x = \frac{0}{2m\omega_0^3} (\sin \omega_0 t - \omega_0 t \cos \omega_0 t).$$

This type of oscillation, illustrated in Fig. 32, is no longer periodic as was that of free oscillations; indeed t appears in (6) as a secular term (i.e., no longer solely in the argument of a trigonometric function). For

$t \rightarrow \infty$ the amplitude approaches the value $C = \infty$ as indicated in Fig. 31 for the case $\omega = \omega_0$.

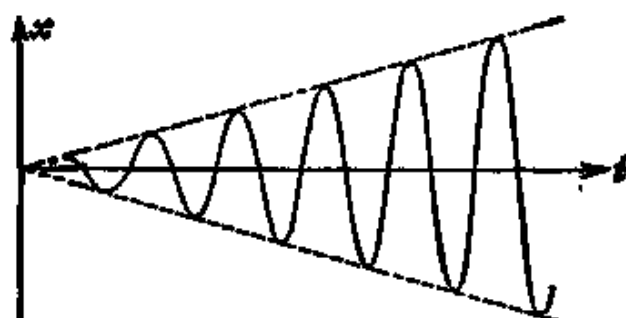


FIG. 32. Resonance of free and forced oscillations. Secular increase of amplitude.

Free, Damped Oscillations

These have the differential equation

$$(7) \quad m\ddot{x} + kx = -w\dot{x}.$$

The frictional term on the right has been put proportional to the velocity, an assumption which finds its justification in the hydrodynamics of slow, laminar (=non-turbulent) flow (e.g., air friction).

Eq. (7) is a homogeneous linear differential equation. As before we put

$$(7a) \quad \frac{k}{m} = \omega_0^2, \quad \omega_0 = \text{undamped proper frequency.}$$

Let us also make the convenient change of symbols

$$(7b) \quad \frac{w}{m} = 2\rho, \quad \rho > 0.$$

Eq. (7) then takes the form

$$(8) \quad \ddot{x} + 2\rho\dot{x} + \omega_0^2 x = 0.$$

The method described under Eq. (3.23) now proves its full worth. As there, we substitute

$$(8a) \quad x = Ce^{\lambda t}$$

in (8) and thus obtain the *characteristic equation* in λ ,

$$\lambda^2 + 2\rho\lambda + \omega_0^2 = 0$$

with the two roots

$$\lambda = -\rho \pm (-\omega_0^2 + \rho^2)^{\frac{1}{2}} = \begin{cases} \lambda_1 \\ \lambda_2 \end{cases}.$$

Expression (8a) must therefore be generalized to

$$(8b) \quad x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

We now distinguish two cases:

$$1. \quad \rho < \omega_0, \quad 2. \quad \rho > \omega_0.$$

The first case is that usually prevailing in practice. The motion is a periodic oscillation with decaying amplitude. The second case is that of strong or "aperiodic" damping. In both cases we shall specialize the motion by imposing the condition, $x=0$ at $t=0$, which, according to (8b), leads to $C_2 = -C_1$.

$$1. \quad \rho < \omega_0. \quad \lambda = -\rho \pm i(\omega_0^2 - \rho^2)^{\frac{1}{2}},$$

$$x = 2C_1' e^{-\rho t} \sin (\omega_0^2 - \rho^2)^{\frac{1}{2}} t.$$

For small ρ the periodic time

$$T = \frac{2\pi}{(\omega_0^2 - \rho^2)^{\frac{1}{2}}}$$

differs little from that of the undamped oscillation. $e^{-\rho t}$ is the damping factor, ρT the *logarithmic decrement*.

$$2. \quad \rho > \omega_0. \quad \lambda_1 \text{ and } \lambda_2 \text{ are real and we obtain}$$

$$x = 2C_1 e^{-\rho t} \sinh(\rho^2 - \omega_0^2)^{\frac{1}{2}} t$$

where \sinh is the hyperbolic sine.

We shall finally deal with a type of oscillation including all those so far considered, namely that of

Damped, Forced Oscillations

We may write their differential equation in the form

$$m\ddot{x} + w\dot{x} + kx = c \sin \omega t$$

or, with the abbreviations defined in (7a, b),

$$(9) \quad \ddot{x} + 2\rho\dot{x} + \omega_0^2 x = \frac{c}{2m\omega} (e^{i\omega t} - e^{-i\omega t}).$$

To the general integral (8b) of the homogeneous equation we must now add a particular solution which we shall write in the form

$$x = |O| \sin (\omega t + \delta) = \frac{|O|}{2i} (e^{i(\omega t + \delta)} - e^{-i(\omega t + \delta)}).$$

Let us introduce this in (9). A comparison of the factors of $e^{\pm i\omega t}$ left and right yields

$$|C| (-\omega^2 + 2i\rho\omega + \omega_0^2) e^{i\delta} = \frac{c}{m},$$

$$|C| (-\omega^2 - 2i\rho\omega + \omega_0^2) e^{-i\delta} = \frac{c}{m}.$$

Multiplication and division of these two relations yields

$$|C|^2 = \left(\frac{c}{m}\right)^2 \frac{1}{(\omega_0^2 - \omega^2)^2 + 4\rho^2\omega^2}$$

$$e^{2i\delta} = \frac{\omega_0^2 - \omega^2 - 2i\rho\omega}{\omega_0^2 - \omega^2 + 2i\rho\omega},$$

respectively. Accordingly

$$(10) \quad |C| = \frac{c}{m} \frac{1}{[(\omega_0^2 - \omega^2)^2 + 4\rho^2\omega^2]^{\frac{1}{2}}},$$

$$(11) \quad \tan \delta = \frac{1 e^{2i\delta} - 1}{i e^{2i\delta} + 1} = - \frac{2\rho\omega}{\omega_0^2 - \omega^2}.$$

Compare the plot of these two functions of ω in Fig. 33 with Figs. 31a, b.

Fig. 33 shows that our formerly infinite resonance maximum has been depressed to a finite value as a result of the damping (note, by the way, that the maximum value no longer occurs at the exact point $\omega = \omega_0$, but rather at a somewhat smaller ω ; cf. problem III.2).

Fig. 33 also demonstrates that with increasing ω , δ goes from the value 0 at $\omega = 0$ to negative values; for $\omega = \omega_0$ it exactly equals $-\frac{1}{2}\pi$, and it approaches $-\pi$ as $\omega \rightarrow \infty$. Thus we have justified the arbitrary choice between $\pm\pi$, made earlier (in Fig. 31), when we were dealing with the undamped case. As a matter of fact we see now that the phase of the oscillation always *lags* behind that of the driving force. For further examples of forced vibrations see problems III.3 and III.4.

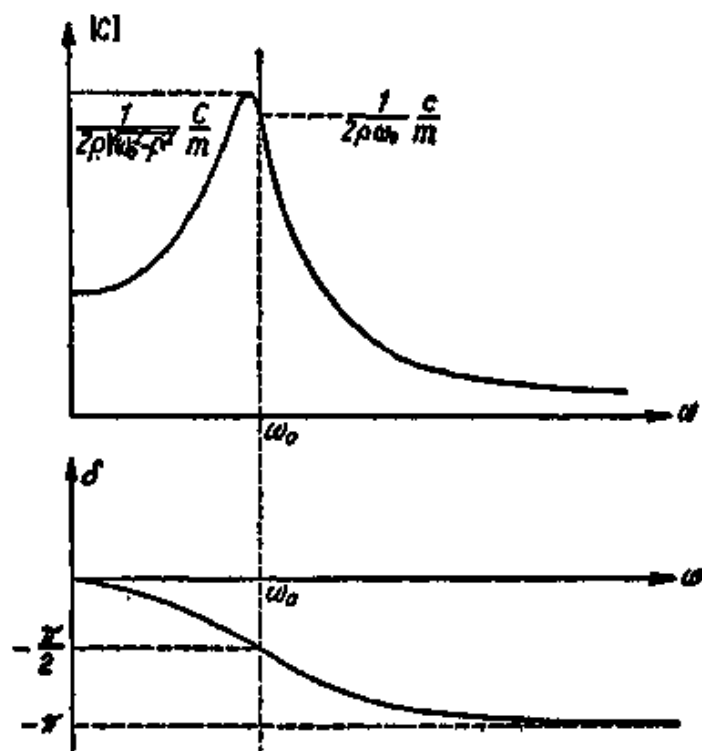


FIG. 33. Amplitude and phase of damped forced oscillations.

§ 20. Sympathetic Oscillations

The types of oscillation so far considered have concerned one mass point. We shall now deal with types of oscillation involving two masses capable of oscillation, these two masses being weakly coupled to each other. Sympathetic oscillations have for many years been important in electric measurements. There one speaks of a primary and a secondary circuit, the latter usually being "inductively" coupled to the former. The primary circuit is made to oscillate ("is excited"), whereupon the secondary circuit does likewise, and especially strongly so if resonance prevails. Indeed the "doubly tuned coupling stage" widely used in radio consists of a primary circuit and a secondary one tuned to the former. Here we shall of course restrict ourselves to coupled *mechanical* oscillations, which have often been used as models for electrical ones.

A particularly instructive example of sympathetic oscillations is furnished by the so-called "coupled pendulums." In the case of resonance these are two equally long and equally heavy pendulums. We may picture them most simply as oscillating in the same plane; their coupling may be effected by means of a helical spring as indicated in Fig. 35. If the spring offers but slight resistance to the relative motion of the two pendulums we speak of weak coupling; in the case of greater spring tension we speak of strong coupling. We assume that the coupling of our pendulums is *weak*. If the pendulums are not exactly equal in length or in weight, we shall say that they are "out of tune," or "detuned."

We shall first describe the phenomena which are observed in the case of resonance.

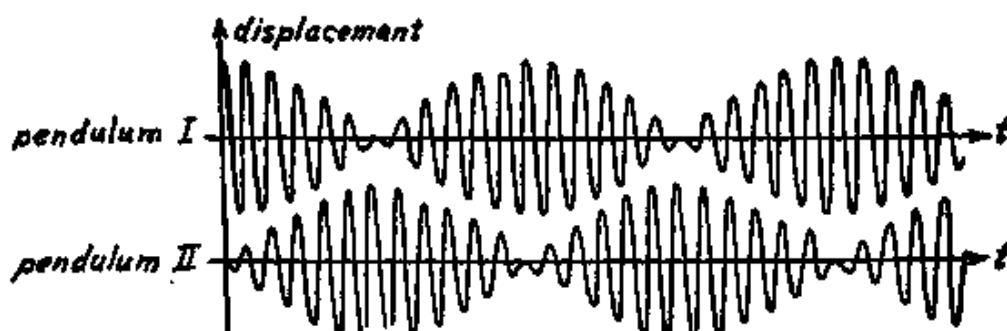


FIG. 34. Coupled pendulums in the case of resonance.

Let the first pendulum be excited, the second one being initially at rest. In Fig. 34 we have drawn a picture of the resulting oscillations.

The oscillations of each pendulum are *modulated*. The energy alternates between one pendulum and the other. When one pendulum oscillates with maximum amplitude, the other one is at rest.

If, instead (cf. Fig. 35) both pendulums are set in motion simultaneously and with equal strength, either in the same direction (Fig. 35, left), or in opposite directions (Fig. 35, right), no energy is exchanged. These two oscillatory modes are called the *normal modes of oscillation* of our coupled system of two degrees of freedom. We have the general rule that *an oscillatory system of n degrees of freedom has n normal modes of oscillation*.

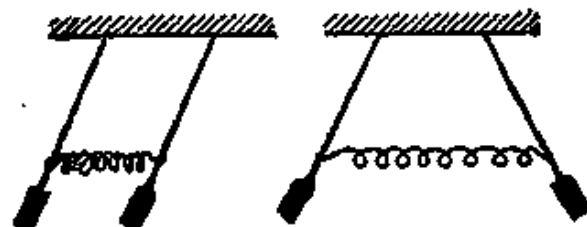


FIG. 35. The two normal modes of oscillation of coupled pendulums in resonance.

If, on the other hand, the pendulums are detuned, an energy exchange still takes place to be sure, but this exchange is of such a nature that the initially excited pendulum has a minimum amplitude different from zero. Only the pendulum initially at rest again reaches the state of rest in the course of the motion. Thus the "sympathy" of the two pendulums is upset by imperfect tuning.

We shall now sketch the theory for *complete resonance*, making the simplest possible assumptions: we neglect all damping, and approximate the circular trajectories of the bobs by the tangents at their lowest points, which is permissible for sufficiently small displacements. Let x_1 be the amplitude of oscillation of pendulum I, x_2 that of pendulum II; call k the "coupling coefficient," i.e., the spring tension caused by an elongation of unit length, divided by the mass of one of the pendulums. The simultaneous differential equations of the problem are

$$(1) \quad \begin{aligned} \ddot{x}_1 + \omega_0^2 x_1 &= -k(x_1 - x_2) \\ \ddot{x}_2 + \omega_0^2 x_2 &= -k(x_2 - x_1). \end{aligned}$$

If we introduce in (1)

$$(2) \quad z_1 = x_1 - x_2, \quad z_2 = x_1 + x_2,$$

subtraction and addition yield the two equations for the normal modes,

$$(3) \quad \begin{aligned} \ddot{z}_1 + \omega_0^2 z_1 &= -2kz_1 \quad \text{or} \quad \ddot{z}_1 + (\omega_0^2 + 2k)z_1 = 0, \\ \ddot{z}_2 + \omega_0^2 z_2 &= 0 \end{aligned}$$

respectively, with the corresponding frequencies

$$(4) \quad \begin{aligned} \text{for } z_1: \quad \omega &= (\omega_0^2 + 2k)^{\frac{1}{2}} \approx \omega_0 + \frac{k}{\omega_0}; \\ \text{for } z_2: \quad \omega' &= \omega_0. \end{aligned}$$

The general solutions of Eqs. (3) are

$$(5) \quad \begin{aligned} z_1 &= a_1 \cos \omega t + b_1 \sin \omega t; \\ z_2 &= a_2 \cos \omega' t + b_2 \sin \omega' t. \end{aligned}$$

At the moment of excitation $t=0$ let

$$(6) \quad x_2 = \dot{x}_2 = 0, \quad \dot{x}_1 = 0, \quad x_1 = C,$$

giving

$$(7) \quad \dot{z}_1 = \dot{z}_2 = 0, \quad z_1 = z_2 = C.$$

It follows that

$$(8) \quad b_1 = b_2 = 0, \quad a_1 = a_2 = C,$$

so that

$$z_1 = C \cos \omega t, \quad z_2 = C \cos \omega' t.$$

Finally

$$(9) \quad \begin{aligned} x_1 &= \frac{z_1 + z_2}{2} = C \cos \frac{\omega' - \omega}{2} t \cdot \cos \frac{\omega' + \omega}{2} t \\ x_2 &= \frac{z_2 - z_1}{2} = -C \sin \frac{\omega' - \omega}{2} t \cdot \sin \frac{\omega' + \omega}{2} t. \end{aligned}$$

According to (4) $\frac{\omega - \omega'}{2} \cong \frac{k}{2\omega_0} \ll 1$ in the case of weak coupling. The first factors of the right members of (9) therefore vary slowly with time; it is this circumstance which determines the *beats* in the oscillation illustrated in Fig. 34.

The theory is not quite so simple if the two pendulums are out of tune, i.e., if $l_1 \neq l_2$ or/and $m_1 \neq m_2$. Letting c be the tension of the spring due to unit elongation, we now put

$$\omega_1^2 = \frac{g}{l_1}, \quad \omega_2^2 = \frac{g}{l_2}, \quad k_1 = \frac{c}{m_1}, \quad k_2 = \frac{c}{m_2}$$

and instead of (1) we obtain the initial equations

$$(10) \quad \begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 &= -k_1 (x_1 - x_2) \\ \ddot{x}_2 + \omega_2^2 x_2 &= -k_2 (x_2 - x_1). \end{aligned}$$

Here again there are two normal modes, which can be obtained by an extension of the method set forth in (3.24). [In Eq. (1) we were able to use a more convenient method especially suited to that case; this method is not applicable in general.] We substitute

$$(11) \quad x_1 = A e^{i\lambda t}, \quad x_2 = B e^{i\lambda t}$$

and obtain from (10) the two characteristic equations

$$(12) \quad \begin{aligned} A(\omega_1^2 - \lambda^2 + k_1) &= k_1 B \\ B(\omega_2^2 - \lambda^2 + k_2) &= k_2 A. \end{aligned}$$

The so-called *secular equation*³ obtained from (12) is quadratic in λ^2 , since

$$(13) \quad \frac{B}{A} = \frac{\omega_1^2 - \lambda^2 + k_1}{k_1} = \frac{k_2}{\omega_2^2 - \lambda^2 + k_2}$$

so that

$$(14) \quad \{\lambda^2 - (\omega_1^2 + k_1)\} \{\lambda^2 - (\omega_2^2 + k_2)\} = k_1 k_2.$$

For small k_1, k_2 (14) has the two approximate roots

$$(15) \quad \lambda^2 = \begin{cases} \omega_1^2 + k_1 + \frac{k_1 k_2}{\omega_1^2 - \omega_2^2} \\ \omega_2^2 + k_2 + \frac{k_1 k_2}{\omega_2^2 - \omega_1^2}. \end{cases}$$

We designate these two roots of the secular equation by ω^2 and ω'^2 ; furthermore we generalize the tentative solution (11) in the same manner as was done in (3.24b), using the principle of superposition of solutions of linear differential equations. Written in real form the general solution is then

$$(16) \quad \begin{aligned} x_1 &= a \cos \omega t + b \sin \omega t + a' \cos \omega' t + b' \sin \omega' t \\ x_2 &= \gamma a \cos \omega t + \gamma b \sin \omega t + \gamma' a' \cos \omega' t + \gamma' b' \sin \omega' t. \end{aligned}$$

Here γ and γ' are the specific values of B/A which arise from (13) for $\lambda^2 = \omega^2$ and $\lambda^2 = \omega'^2$ respectively.

Let us once again take as the condition of excitation at $t=0$

$$x_2=0, \dot{x}_2=0, \dot{x}_1=0, x_1=C.$$

This yields

$$(17) \quad \begin{aligned} \gamma a + \gamma' a' &= 0, \quad \gamma \omega b + \gamma' \omega' b' = 0, \\ \omega b + \omega' b' &= 0, \quad a + a' = C, \end{aligned}$$

from which

$$b = b' = 0$$

and

$$a = \frac{\gamma}{\gamma - \gamma'} C, \quad a' = \frac{\gamma}{\gamma - \gamma'} C.$$

³ The word originated in the perturbation theory of celestial mechanics.

If we substitute these values in (16), we have

$$(18) \quad \begin{aligned} x_1 &= \frac{C}{\gamma' - \gamma} (\gamma' \cos \omega t - \gamma \cos \omega' t) \\ x_2 &= \frac{C}{\gamma' - \gamma} \gamma \gamma' (\cos \omega t - \cos \omega' t). \end{aligned}$$

In the equation for x_2 we can perform the trigonometric transformation used in (9) to obtain

$$(19) \quad x_2 = \frac{2\gamma\gamma'}{\gamma' - \gamma} C \sin \frac{\omega' - \omega}{2} t \cdot \sin \frac{\omega' + \omega}{2} t$$

We see that the second pendulum still comes to rest at the times

$$\frac{\omega' - \omega}{2} t = n\pi;$$

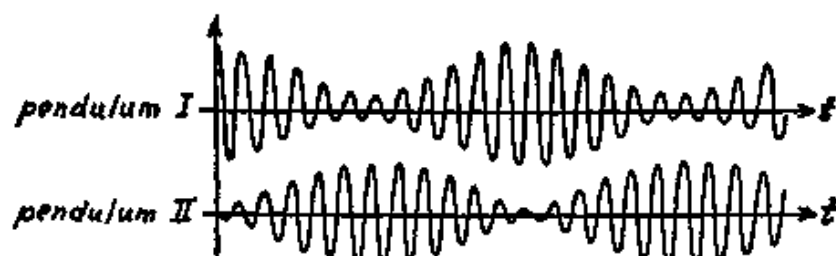


FIG. 36. Oscillograph of two slightly detuned coupled pendulums.

not so the first pendulum, which [cf. the first Eq. (18) and Fig. 36] retains a finite amplitude when that of x_2 is at a maximum. Imperfect tuning results in an incomplete transfer of energy.

If we desire to apply the foregoing theory to electrical phenomena, we must extend it to include damping of the pendulums; damping has its electrical analogue in the Ohmic resistance (our acceleration term corresponds to the self-induction, our restoring force to capacitive effects); moreover the analysis of electrical oscillations in coupled circuits demands that we introduce "acceleration and velocity coupling" in addition to the "position coupling" [k multiplied by $\pm(x_2 - x_1)$] which was the only type of coupling taken into account in our mechanical problem.

In problem III.5 we shall investigate the motion of an experimentally convenient arrangement, in which the pendulums are suspended bifilarly from a flexible wire and oscillate not in the plane of their positions of rest, but perpendicularly to it.

An interesting arrangement, in which both coupled pendulums are, so to speak, realized in the same body, is that of an oscillating helical spring.⁴

⁴ For details the reader is referred to the *Wüllner-Festschrift*, Teubner (1905): *Lissajous Figures and Resonance Effects of Oscillating Helical Springs; Their Use in the Determination of the Poisson Ratio*.

Such a spring (cf. Fig. 37) is capable not only of an oscillation (y) along its axis but also of a rotary oscillation (x) about this axis. For finite displacements the coupling between these two motions is produced by the spring itself. For if the spring is pulled vertically downward, a lateral force is experienced; the spring seeks to withdraw along the wire-direction in order to uncoil itself. If, on the other hand, the spring is coiled up, it will seek to shorten itself along the y -axis. In other words, if one excites an oscillation in the y -direction, an x -oscillation is induced, and conversely. (Note: as far as the elastic stress on the material is concerned, the y -oscillation is one of torsion, the x -oscillation one of deflection. For details about this consult Vol. II this series.)

By means of the adjustable mass Z , one can bring the vertical and horizontal oscillations into accurate or approximate resonance. If then one of the two vibrations is excited, an exchange of amplitudes of the type of Fig. 34 or Fig. 36 takes place.

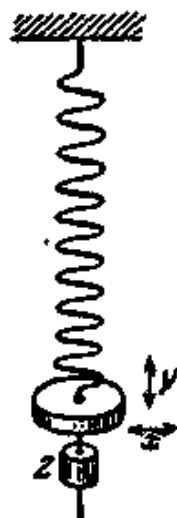


FIG. 37. Torsional and deflection oscillations of a helical spring.

§ 21. The Double Pendulum

As at the beginning of the previous section, we shall first describe the empirical phenomena involved.

From a heavy pendulum (a chandelier, for instance) we suspend a light pendulum of about the same period of oscillation. Let us impart a sharp impulse to the heavy bob; the light bob will be set in vigorous motion, which suddenly subsides and stays at zero for a short time. At this instant one perceives that the heavy bob, which had previously remained practically at rest, now starts oscillating with noticeable amplitude. This oscillation soon ceases, however, whereupon in its turn the light pendulum again begins to move with considerable vigor, and so forth.

As mentioned, we demand that the masses of the two bobs, M and m , be very unequal, but that the equivalent lengths L , l be approximately the same. We let

$$\frac{m}{M} = \mu \ll 1.$$

We shall treat the displacements, X of the heavy pendulum, x of the light one, as small quantities, so that once again we can approximate arcs of circles by their tangents. Consequently we must also keep the angles ϕ and ψ (cf. Fig. 38, where ψ belongs to the relative

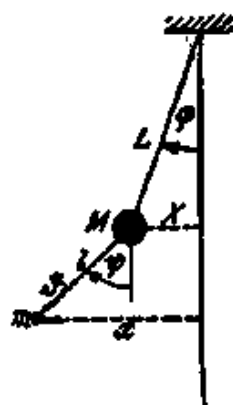


FIG. 38. Schematic arrangement of a double pendulum.

displacement $x - X$ small. We can therefore put

$$(1) \quad \begin{aligned} \sin \phi = \phi = \frac{x}{L}, \quad \sin \psi = \psi = \frac{x-X}{l} \quad \text{and} \quad \sin(\psi - \phi) = \psi - \phi = \frac{x-X}{l} - \frac{x}{L}. \\ \cos \phi = \cos \psi = \cos(\phi - \psi) = 1. \end{aligned}$$

The upper pendulum is acted upon not only by the force of gravity, but also by the lower pendulum; the string tension^{*} $S \approx mg \cos \psi$ contributes a component tangential to the motion of M of amount $-mg \cos \psi \sin(\phi - \psi)$. Thus we arrive at the equations of motion

$$(2) \quad \begin{aligned} M\ddot{X} &= -M\frac{g}{L}X + mg\left(\frac{x-X}{l} - \frac{x}{L}\right) \\ m\ddot{x} &= -m\frac{g}{l}(x-X) \end{aligned}$$

or, in more convenient form,

$$(3) \quad \begin{aligned} \ddot{X} + \left(\frac{g}{L} + \mu\frac{g}{l} + \mu\frac{g}{L}\right)X &= \mu\frac{g}{l}x, \\ \ddot{x} + \frac{g}{l}x &= \frac{g}{l}X. \end{aligned}$$

From now on we shall put $L = l$ and introduce the abbreviation

$$(4) \quad \omega_0^2 = \frac{g}{l}.$$

Our Eqs. (3) then become

$$(5) \quad \begin{aligned} \ddot{X} + \omega_0^2(1 + 2\mu)X &= \mu\omega_0^2 x, \\ \ddot{x} + \omega_0^2 x &= \omega_0^2 X. \end{aligned}$$

These equations of motion state that the upper pendulum is μ times more weakly coupled to the lower one than vice versa.

To integrate (5) we use a substitution similar to (20.11),

$$(6) \quad x = Ae^{i\lambda t}; \quad X = Be^{i\lambda t}.$$

From (5) we have as a result

$$(7) \quad \begin{aligned} A(\omega_0^2 - \lambda^2) &= B\omega_0^2 \\ B[\omega_0^2(1 + 2\mu) - \lambda^2] &= A\mu\omega_0^2. \end{aligned}$$

^{*} In the present elementary treatment we have to introduce this tension S as a descriptive auxiliary quantity; later, when we analyse the same problem by means of the general Lagrange method, this procedure will become superfluous. In order to determine S we reason as follows: the tension in the suspension of the light bob is in equilibrium with gravity and inertial force (centrifugal force); the latter is a small quantity of second order and can therefore be neglected. We then have $S = mg \cos \psi$ as stated above.

If we put the two values of B/A obtained from these two equations equal, we arrive at the quadratic equation in λ^2 ,

$$(8) \quad (\lambda^2 - \omega_0^2)^2 + 2\mu\omega_0^2(\omega_0^2 - \lambda^2) = \mu\omega_0^4.$$

Let its two roots be called $\lambda^2 = \omega^2$ and $\lambda^2 = \omega'^2$. Omission of higher powers of μ easily yields their approximate values

$$(9) \quad \left. \begin{matrix} \omega \\ \omega' \end{matrix} \right\} = \omega_0 \left(1 \pm \frac{1}{2}\mu^{\frac{1}{2}} \right).$$

Written in real form, the general solution of (5) is then

$$(10) \quad \begin{aligned} x &= a \cos \omega t + b \sin \omega t + a' \cos \omega' t + b' \sin \omega' t, \\ X &= \gamma a \cos \omega t + \gamma b \sin \omega t + \gamma' a' \cos \omega' t + \gamma' b' \sin \omega' t. \end{aligned}$$

As in § 20, γ and γ' are here the values of B/A which result from (7) for $\lambda^2 = \omega^2$ and $\lambda^2 = \omega'^2$ respectively, viz.,

$$(11) \quad \gamma = -\mu^{\frac{1}{2}}, \quad \gamma' = +\mu^{\frac{1}{2}} \quad \text{and hence} \quad \gamma' - \gamma = 2\mu^{\frac{1}{2}}.$$

Let the excitation of the system at $t=0$ be given by

$$(12) \quad x=0, \quad \dot{x}=0, \quad X=0, \quad \dot{X}=0.$$

It follows that

$$\left. \begin{aligned} a + a' &= 0 \\ \gamma a + \gamma' a' &= 0 \end{aligned} \right\} a = a' = 0.$$

$$\left. \begin{aligned} \omega b + \omega' b' &= 0 \\ \gamma \omega b + \gamma' \omega' b' &= 0 \end{aligned} \right\} b = \frac{0}{\omega(\gamma - \gamma')}; \quad b' = \frac{0}{\omega'(\gamma' - \gamma)}.$$

Thus we obtain the final solutions

$$(13) \quad \begin{aligned} x &= \frac{0}{\gamma - \gamma'} \left(\frac{\sin \omega t}{\omega} - \frac{\sin \omega' t}{\omega'} \right) \\ X &= \frac{0}{\gamma - \gamma'} \left(\frac{\gamma}{\omega} \sin \omega t - \frac{\gamma'}{\omega'} \sin \omega' t \right). \end{aligned}$$

Let us pass from these to the velocities \dot{x} and \dot{X} , taking (11) into account. We end up with

$$(14) \quad \begin{aligned} \dot{x} &= \frac{0}{2\mu^{\frac{1}{2}}} (\cos \omega' t - \cos \omega t), \\ \dot{X} &= \frac{0}{2} (\cos \omega' t + \cos \omega t). \end{aligned}$$

Given the same phase, the velocity of the heavy upper bob is hence $\mu^{\frac{1}{2}}$ times smaller than that of the light lower one; notice also that (14)

satisfies our initial conditions (12). The same can be said about the displacements themselves. Like the velocities, these are subject to beats because of the closeness of the values of ω and ω' . This modulation can be shown explicitly by writing the equations (13) and (14) in a form resembling Eq. (20.9).

We conclude the chapter with a problem which also pertains to the class of coupled oscillations and leads to oscillations very similar to the ones treated above. We shall, however, avail ourselves of a simpler mathematical method resembling that of the forced⁶ undamped oscillations of § 19, so that we have to cope with the integration of only one differential equation rather than with that of a system of two simultaneous ones.

Let us suspend our pocket watch from a smooth nail, in such a way that the watch hangs completely free and friction is reduced to a minimum. By means of gentle contact with our fingers or a piece of cloth we bring the watch into a state of complete rest. When released, the timepiece at once begins to move, performing increasing oscillations about the vertical rest position. These oscillations reach a maximum, then gradually decrease once more to zero, after which the process repeats itself.

In these oscillations of the watch we are evidently confronted with a motion reacting against the rhythm of the balance wheel, i.e., a manifestation of the principle of conservation of angular momentum. The fluctuation of the oscillation amplitude, on the other hand, is caused by interference between the free pendulum oscillations of the watch in the gravitational field and the forced oscillations excited by the balance wheel.

We shall follow § 13, subsec. 2 in our notation. Accordingly we let M be the angular momentum of the total motion of the system. We decompose it into that of the pendulum motion (p) and that of the balance wheel oscillations (δ),

$$(15) \quad M = M_p + M_\delta.$$

M_p is calculated about the point of suspension O (nail), M_δ about the center B of the balance wheel. The latter is permitted because a *pure* angular momentum (i.e., one caused by a motion in which the center of mass of the system remains fixed) can, just like a force couple (cf. p. 128), be shifted

⁶ We can say quite generally that the excitation of forced oscillations in a system by means of an external force is equivalent to coupling with a second system on which the first one does not react. In the case about to be described it is certainly true that the reaction of the pendulum oscillations on the balance wheel is vanishingly small.

at will in its plane⁷; indeed, due to the symmetry of the balance wheel about B , the inertial action of the balance consists of a pure moment of momentum. Let ω be the circular frequency of the balance wheel; it is determined by the stiffness of the balance spring. Let ω_0 be the undisturbed, i.e., proper circular frequency of the pendulum oscillations. According to (11.6) and (16.4) we put

$$(16) \quad M_p = I\dot{\phi}, \quad I = m_p a^2;$$

m_p is the total mass of the watch, a its radius of gyration measured from O . We postulate a sinusoidal balance wheel oscillation which we shall therefore describe by $\phi_b = \alpha \sin \omega t$, B being the vertex of angle ϕ_b . The angular momentum of the balance wheel is then

$$(17) \quad M_b = m_b \omega b^2 \alpha \cos \omega t,$$

where m_b is the mass of the balance wheel, b its radius of gyration measured from B .

As in the case of the compound pendulum [Eq. (16.1)] the moment of the external force is

$$(18) \quad L = -m_p g s \phi,$$

where we have, as usual, made the approximation for small ϕ . Here s is the distance of the center of gravity of the watch from O , and ϕ the angle formed at O by the vertical and a line through the center of gravity. We now apply (13.9), use therein the values given by (15), (16), (17), and (18), and obtain the equation of motion

$$(19) \quad \ddot{\phi} + \frac{g s}{a^2} \phi = \frac{m_b}{m_p} \left(\frac{b}{a}\right)^2 \alpha \omega^2 \sin \omega t$$

for our system.

This equation represents the type of oscillation which was treated in §19 as undamped forced oscillation. Again we put

$$\frac{g s}{a^2} = \omega_0^2$$

⁷ This is a direct consequence of the fact that the angular momentum of a system about a given axis can be decomposed into the sum of the angular momentum of the system about a parallel axis through its mass center and the angular momentum of the mass center (containing the total mass of the system) about the given axis. In our case the latter term vanishes since the angular momentum of the mass center of the balance wheel due to the oscillation of the watch as a whole was included in M_{pend} .

where ω_0 is, we recall, the proper frequency of the pendulum motion; let us moreover abbreviate

$$c = \frac{m_b}{m_p} \left(\frac{b}{a} \right)^2 \alpha \omega^2 \ll 1.$$

Equation (19) becomes

$$(20) \quad \ddot{\phi} + \omega_0^2 \phi = c \sin \omega t.$$

The solution satisfying the initial conditions $\phi=0$, $\dot{\phi}=0$ at $t=0$ is

$$(21) \quad \phi = \frac{c}{\omega_0^2 - \omega^2} \left(\sin \omega t - \frac{\omega}{\omega_0} \sin \omega_0 t \right).$$

The constant c is so small (factor m_b/m_p) that the oscillation is of visible magnitude only when the relation $\omega_0 = \omega$ is approximated, i.e., when approximate resonance exists between the external pendulum oscillations and the internal oscillations of the balance wheel. Surprisingly it turns out that this resonance is more or less well realized in pocket watches of not too small a size (ladies' watches are unsuitable for our purposes).

Eq. (21) further shows that amplitude modulation goes hand in hand with the approach to resonance $\omega_0 \rightarrow \omega$. The period T of the beats is determined by the requirement

$$(22) \quad \omega T = \omega_0 T \pm 2\pi,$$

and has therefore the value

$$(22a) \quad T = \frac{2\pi}{|\omega - \omega_0|}.$$

It can be determined very accurately by counting the number of pendulum oscillations between two nodes of the beats, and furnishes therefore a convenient and precise measure of the degree of resonance. We can refer back to Fig. 32 which, as pointed out, represents the same differential equation as (20); we must, however, bear in mind that in the diagram we postulated complete resonance, i.e., $T = \infty$.

If one leaves the watch to itself for some time, one observes that the beats have ceased. The reason for this is evidently friction (at the point of suspension and in the air), which we have so far neglected. This friction damps the contribution of the free pendulum oscillations to the motion of the watch, leaving only the forced oscillations due to the motion of the balance wheel, the latter contribution (cf. Fig. 33, for instance) being somewhat reduced in amplitude due to friction. We can reason as follows: initially the forced oscillation is present in its full amount, and the free pendulum oscillation is excited to such a degree that at $t=0$ it just cancels the forced one—in agreement with the initial conditions $\phi = \dot{\phi} = 0$.

Indeed the initially motionless state of the watch can be interpreted as being caused by an impulse exactly cancelling the balance wheel oscillation. The effect of this impulse is gradually used up by friction, so that only the forced oscillation due to the balance wheel remains.

The example of the watch appeared in the literature for the first time in the "Elektrotechnische Zeitschrift" of the year 1904, in connection with the phenomenon of "hunting" of synchronous machinery, then timely and surprising. Two synchronous alternators feeding the same power line and connected in parallel show undesirable fluctuations in their motions and their currents when resonance occurs. They provide a greatly magnified picture of the beats of our watch and of the coupling and resonance phenomena occurring in the coupled oscillations that we have just analyzed.

CHAPTER IV

THE RIGID BODY

§ 22. *Kinematics of Rigid Bodies*

At the beginning of § 7 we saw that a rigid body is endowed with six degrees of freedom; these we shall subdivide into three of translation and three of rotation.

Let us consider the body in two different positions, the "initial position" and the "final position." We pick out an arbitrary point of the body as "point of reference" O , and describe a sphere of reference (say of unit radius) about it. On this sphere we mark two points A and B . Once we have guided the three points OAB from their initial positions to their final ones, all other points of the rigid body have similarly reached their destinations.

First we take the point O from its initial position O_1 to its final position O_2 . Let this be achieved by means of a parallel displacement or *translation* in which each point of the body is subjected to the same rectilinear displacement $O_1 \rightarrow O_2$. We have thus described the three degrees of freedom of translation.

The sphere K_1 described about O_1 is now in coincidence with the corresponding sphere K_2 described about O_2 . In general this is not true of the position of the points A, B , which we designate by A_1, B_1 on K_1 and A_2, B_2 on K_2 . We shall show that there is one definite *rotation* about the point $O_1 = O_2$ which will take points A_1, B_1 over into A_2, B_2 . Axis and angle of this rotation define the three degrees of freedom of rotation to be added to those of translation.

In order to construct the axis of rotation, i.e., the point Ω at which the axis cuts the unit sphere, we connect A_1 to A_2 and B_1 to B_2 by means of arcs of great circles. At the centers A' and B' of these arcs we erect their perpendicular bisectors whose intersection is the point Ω in question. The angle of rotation, which we shall also call Ω , is

$$(1) \quad \Omega = \sphericalangle A_1 \Omega A_2 = \sphericalangle B_1 \Omega B_2.$$

The equality of these two angles results from the congruence of the shaded spherical triangles $A_1 \Omega B_1$ and $A_2 \Omega B_2$ of Fig. 39, whose three corresponding sides are equal to each other. It follows that the two angles designated by γ in Fig. 39 are equal. If we subtract one or the other of these angles

from the total angle $A_1\Omega B_2$ we obtain the right or middle member of Eq. (1). This equation evidently states that the same rotation Ω not only takes point A_1 over into A_2 , but also point B_1 into B_2 .

So far the magnitude and direction of the translation are still arbitrary¹ within wide limits, for we have free choice over the reference point O . The magnitude and axis of the rotation, on the other hand, are *independent* of the choice of the reference point. For let us substitute for O a new reference point O' . The difference between the translations associated with O' and O for a given total displacement of the rigid body is again a translation. This latter translation, however, does not affect the positions of the points A, B on the spheres K_1 and K_2 . It follows that the construction of Fig. 39 carries over unchanged to the present case and yields not only the same angle of rotation Ω as previously, but also an axis of rotation passing through the point of reference O' and parallel to our former axis.

Of much greater importance than finite displacements of the rigid body are its infinitesimal displacements which succeed each other continuously to result in a finite motion. We shall therefore assume that now the magnitude O_1O_2 of the translation and the angle Ω of rotation are arbitrarily small. Let us divide them by the correspondingly small interval of time Δt . We then obtain the velocity \mathbf{u} of translation and the angular velocity ω of rotation,

$$(2) \quad \mathbf{u} = \frac{\overrightarrow{O_1O_2}}{\Delta t}, \quad \omega = \frac{\Omega}{\Delta t}.$$

As before, the angular velocity is independent of the choice of reference point O , whereas \mathbf{u} depends on this choice. The heavy type indicates that

¹ In the supplement to § 23 we shall see that we can, in particular, make the direction of the translation parallel to the axis of rotation. We then speak of a "screw displacement."

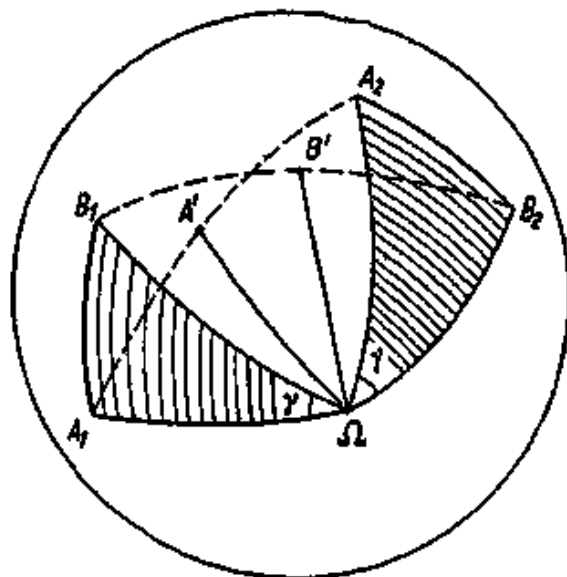


FIG. 39. Construction of the point Ω determining the axis of rotation for a rigid body revolving about a fixed point O . This diagram also suggests how the resultant of two finite rotations can be found.

ω is to be regarded as a vector which expresses not only the magnitude, but also the axial direction of rotation of the angular velocity.

We can easily show that ω does indeed possess vector character. In Fig. 15 and Eq. (13.4), while discussing virtual rotations, we derived the relation

$$(3) \quad \delta s = \delta \phi \times r.$$

If we now pass from the virtual rotation $\delta \phi$ to the angular velocity $\omega = \frac{d\phi}{dt}$ and from the virtual displacement δs caused by the rotation to the velocity $w = \frac{ds}{dt}$, we obtain from (3) that

$$(4) \quad w = \omega \times r.$$

As in Fig. 15, r is here the radius vector from the point of reference O located on the axis of rotation to the point P whose velocity w is to be determined.

Consider now the total effect of two successive infinitesimal rotations $\omega_1 dt$ and $\omega_2 dt$ on the motion of the point P of the rigid body, reference point O being common to both axes ω_1 and ω_2 . We have

$$(4a) \quad w_1 = \omega_1 \times r, \quad w_2 = \omega_2 \times r, \quad w_1 + w_2 = (\omega_1 + \omega_2) \times r.$$

In the last of these equations the left member is the velocity w , resulting from w_1 and w_2 . A comparison with (4) shows that

$$(5) \quad \omega_r = \omega_1 + \omega_2$$

is likewise the resultant angular velocity, equivalent to the two rotations $\omega_1 dt$ and $\omega_2 dt$ in its effect on the rigid body. We conclude that *angular velocities add like vectors*. As in the case of vectors their order in addition is immaterial, i.e., *their addition is commutative*, for

$$(6) \quad \omega_1 + \omega_2 = \omega_2 + \omega_1.$$

Neither of these two laws is valid for finite rotations. Their composition does not follow the simple rules of vector algebra, but those of the algebra of *quaternions* invented by Hamilton. Moreover the effect of two finite rotations depends on their order; two such rotations do not commute.

At this point it is convenient to discuss the difference between polar and axial vectors.

Examples of *polar vectors* are velocity, acceleration, force, radius vector, etc. They can be represented by directed segments provided with an arrow head. In a rotation of the system of coordinates their rectangular components transform like the coordinates themselves, i.e., according to

the scheme of orthogonal transformations with determinant $+1$. In an inversion of the coordinate system through the origin, in which x, y, z are replaced by $-x, -y, -z$ respectively so that the transformation has determinant -1 , the components of polar vectors change sign.

Angular velocity, angular acceleration, torque and angular momentum are examples of *axial vectors*. In accordance with their nature they are represented by an axis on which sense and magnitude of rotation are indicated (e.g., by a curved arrow and a number). If, instead, we represent them by means of an arrow of corresponding magnitude laid off on the axis, we must make some arbitrary agreement about the direction of this arrow, such as the rule of the right-handed screw. In a pure rotation of the coordinate system the rectangular components of axial vectors transform like the components of their associated arrows, i.e., orthogonally; in an inversion of the coordinates through the origin, however, these rectangular components do not change sign. In such a transformation the rule of the right-handed screw must be replaced by that of the left-handed screw, in agreement with the fact that an inversion through the origin takes a right-handed coordinate system over into a left-handed one.

The vector product of two polar vectors is an axial vector (e.g., the moment of a force). The vector product of an axial and a polar vector is a polar vector [e.g., the velocity \mathbf{w} in Eq. (4)]. The reader may easily convince himself of this by checking the behavior of these products under inversion of coordinates.²

After this digression we return to the kinematics of the rigid body. The motion of each one of its points is composed of the velocity \mathbf{u} of Eq. (2) connected with translation and the velocity \mathbf{w} in Eq. (4) connected with rotation. The velocity \mathbf{v} of an arbitrary point of the rigid body is hence given by

$$(7) \quad \mathbf{v} = \mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}.$$

The choice of the reference point O is completely up to us; for it we have

$$(7a) \quad \mathbf{v} = \mathbf{u}.$$

For many purposes it is advantageous to put O at the mass center G . This becomes evident if, for instance, we wish to calculate the kinetic

² From now on we shall simply talk about the torque \mathbf{L} , the angular velocity $\boldsymbol{\omega}$, where the reader should bear in mind that we mean by this the axial vectors representing the torque and the angular velocity respectively. When, on the other hand, we speak of the plane of the torque and the plane of the angular velocity, we mean, of course, the planes perpendicular to the axial vectors \mathbf{L} and $\boldsymbol{\omega}$ respectively.

energy of the body,

$$(8) \quad T = \int \frac{dm}{2} v^2.$$

To this end we form, with the help of (7),

$$(8a) \quad v^2 = u^2 + (\omega \times \mathbf{r})^2 + 2\mathbf{u} \cdot (\omega \times \mathbf{r})$$

and accordingly break up T into three parts,

$$(9) \quad T = T_{\text{transl}} + T_{\text{rot}} + T_m,$$

where T_m is a "mixed" energy which is determined by the translation and the rotation combined.

Since \mathbf{u} has the same value for all points dm , we evidently have

$$(10) \quad T_{\text{transl}} = \frac{u^2}{2} \int dm = \frac{m}{2} u^2.$$

In order to calculate T_m we perform the transformation

$$(11) \quad T_m = \int \mathbf{u} \cdot \omega \times \mathbf{r} \, dm = \mathbf{u} \cdot \omega \times \int \mathbf{r} \, dm = m \mathbf{u} \cdot \omega \times \mathbf{R},$$

where \mathbf{R} is the directed segment from O to the mass center G ,

$$(11a) \quad \mathbf{R} = \frac{1}{m} \int \mathbf{r} \, dm$$

as in Eq. (13.3b). If now we let O coincide with G , we have $\mathbf{R} = 0$ and, from (11),

$$(11b) \quad T_m = 0.$$

The kinetic energy T then becomes simply the sum of T_{transl} and T_{rot} . Notice, in passing, that if the body rotates about a fixed point and if one chooses this fixed point as reference point O , not only T_m , but also T_{transl} vanishes (in both cases because $\mathbf{u} = 0$), so that

$$(11c) \quad T = T_{\text{rot}}.$$

We shall now focus our attention on the rotational contribution to the kinetic energy. If we square the components of $\omega \times \mathbf{r}$, we obtain from the middle term of the right member of (8a)

$$(12) \quad \begin{aligned} 2T_{\text{rot}} = & \omega_x^2 \int (y^2 + z^2) \, dm + \omega_y^2 \int (z^2 + x^2) \, dm + \omega_z^2 \int (x^2 + y^2) \, dm \\ & - 2\omega_x \omega_y \int yz \, dm - 2\omega_x \omega_z \int zx \, dm - 2\omega_y \omega_z \int xy \, dm. \end{aligned}$$

With the notation

$$(12a) \quad \begin{aligned} I_{xx} &= \int (y^2 + z^2) dm \dots \\ I_{xy} &= \int xy dm \dots \end{aligned}$$

this yields

$$(12b) \quad 2T_{\text{rot}} = I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 - 2I_{yz}\omega_y\omega_z - 2I_{zx}\omega_z\omega_x - 2I_{xy}\omega_x\omega_y.$$

According to the definition introduced in (11.3), I_{xx} is the moment of inertia of the mass distribution about the x -axis; a corresponding statement holds for I_{yy} and I_{zz} . We shall call I_{xy} , I_{yz} , I_{zx} the *products of inertia* (the name "centrifugal moments" is sometimes used synonymously). We can also abbreviate I_{xx} , . . . without ambiguity to I_x , . . .

In accordance with (11.5) we put the left member of (12) equal to $I\omega^2$ and with the abbreviations

$$(13) \quad \frac{\omega_x}{\omega} = \alpha, \quad \frac{\omega_y}{\omega} = \beta, \quad \frac{\omega_z}{\omega} = \gamma$$

obtain

$$(13a) \quad I = I_{xx}\alpha^2 + I_{yy}\beta^2 + I_{zz}\gamma^2 - 2I_{yz}\beta\gamma - 2I_{zx}\gamma\alpha - 2I_{xy}\alpha\beta.$$

α , β , γ are the direction cosines of the vector ω whose axis is arbitrarily located in the rigid body. It follows from (13a) that the moment of inertia about any axis is completely determined once the six magnitudes I_{ik} are given.

A sextet of magnitudes of the type of our I_{ik} is called a *tensor*, or, more precisely, a *symmetrical tensor*. The name originated in the theory of elasticity where stress and strain tensors play a central role. In general a tensor is very aptly written as a square scheme, which in our case would be

$$(13b) \quad I_{ik} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

where $I_{xy} = I_{yx}$, . . .

From an elementary viewpoint the mathematics of tensors is less concrete and easily intelligible than that of vectors. Whereas a vector is represented by a line segment, we must resort to a surface of second degree for the geometrical representation of a tensor. In our case this "tensor surface" is obtained as follows: we put

$$(14) \quad \alpha = \frac{\xi}{\rho}, \quad \beta = \frac{\eta}{\rho}, \quad \gamma = \frac{\zeta}{\rho},$$

where ξ , η , ζ are interpreted as Cartesian coordinates, hence $\rho = (\xi^2 + \eta^2 + \zeta^2)^{\frac{1}{2}}$

as radius vector from the point O . We now set ρ equal to $I^{-\frac{1}{2}}$, so that along every axis through O we lay off, not I , but rather the reciprocal of $I^{\frac{1}{2}}$ (else we should not obtain a surface of *second degree*). In this manner we obtain from (13a)

$$(15) \quad 1 = I_{xx}\xi^2 + I_{yy}\eta^2 + I_{zz}\zeta^2 - 2I_{yz}\eta\zeta - 2I_{xz}\xi\zeta - 2I_{xy}\xi\eta.$$

Apart from possible degeneracies this is the equation of an ellipsoid, since for a finite mass distribution I is, in general, greater than zero. The surface represented by (15) is called the *momental ellipsoid*.

If one transforms the coordinates so that they coincide with the principal axes of the ellipsoid, one obtains an equation of the form

$$(15a) \quad 1 = I_1\xi_1^2 + I_2\xi_2^2 + I_3\xi_3^2,$$

where I_1, I_2, I_3 are the three *principal moments of inertia*. The products of inertia vanish for the principal axes, which can be regarded as a definition of the latter. The tensor scheme (13b) reduces to diagonal form. When the tensor is described in a system of coordinates different from that of the principal axes one must mentally add the three direction parameters of the principal axes; thus we are again lead to the six magnitudes characterizing a symmetrical tensor.

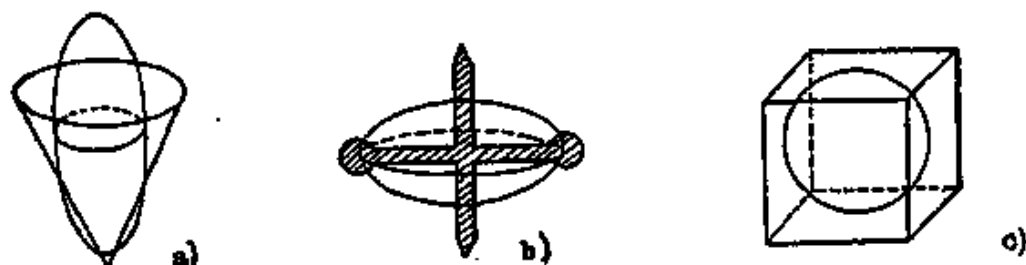


FIG. 40a—c. (a) Momental ellipsoid of the toy top. (b) Momental ellipsoid of the flywheel top. (c) An example of a spherical top.

Every plane of symmetry of the mass distribution is of course also a plane of symmetry of the momental ellipsoid. A mass distribution with rotational symmetry has a momental ellipsoid of revolution, i.e., in addition to the principal axis along the "axis of figure" it possesses infinitely many other "equatorial" principal axes. As examples we may mention two types of tops; one is of the conical type used as toy, the other has the shape of a flywheel and is usually employed for demonstration purposes (Figs. 40a and b). In the first type the moment of inertia about the axis of the body is a minimum, so that the corresponding principal axis is longer than the equatorial ones (by virtue of the relation $\rho = I^{-\frac{1}{2}}$); we have a *prolate spheroid*. In the second case the moment of inertia about the axis of figure is a maximum, hence the corresponding principal axis is, for the

same reason, smaller than the equatorial ones; the result is an *oblate spheroid*.

Incidentally a momental ellipsoid becomes one of revolution not only for mass distributions with rotational symmetry, but also whenever more than two planes of symmetry pass through an axis, as for example in the case of a square or hexagonal prism.

Similarly the ellipsoid degenerates into a sphere not only in the case of a spherically symmetrical distribution, but also in cases such as that of a cubical distribution, for instance, because here there exist more planes of symmetry than are compatible with the ellipsoidal shape of the tensor surface. In such a case we speak of a "spherical top." In a spherical top (cf. Fig. 40c) any axis is a principal axis.

§ 23. Statics of Rigid Bodies

This subject forms the theoretical basis for the whole field of structural mechanics dealing with such topics as the construction of bridges, trusses, arches, etc., and for this reason it is treated with the greatest detail in the texts of mechanical engineering, both analytically and graphically. Here we shall restrict ourselves to the general features of the subject.

(1) The Conditions of Equilibrium

These, like all questions of equilibrium, are governed by the principle of virtual work. Since this principle can be regarded as the special case of d'Alembert's principle in which the inertial forces vanish, our present analysis can be directly modeled after that of the principles of linear and angular momentum of § 13. Indeed the virtual displacements (translation and rotation) used there are evidently compatible with the internal connections of the rigid body and correspond to the two component parts of the general motion of a rigid body considered in the preceding section.

By deleting the inertial forces in Eqs. (13.3) and (13.9), we obtain the general conditions of equilibrium of a rigid body,

$$(1) \quad \sum \mathbf{F}_k = 0, \quad \sum \mathbf{L}_k = 0.$$

The \mathbf{F}_k are external forces acting at arbitrary points P_k of the rigid body. The first Eq. (1) asks us to lay off the force vectors end to end in arbitrary order and with no regard to their points of application, and to examine the resulting *force polygon*. According to Eq. (1) *for equilibrium the polygon of forces must be closed*.

The \mathbf{L}_k are the moments of the \mathbf{F}_k about a reference point O whose choice is arbitrary but which must be the same for all the \mathbf{F}_k . The second Eq. (1) asks us to replace these \mathbf{L}_k by their (axial) vector representations

(cf. p. 37) and to examine the polygon of torques arising when all these vectors are added vectorially. According to the second Eq. (1) *the torque polygon must also be closed for equilibrium.*

In analogy to Eqs. (13.12) and (13.13) we can pass from the two vector equations (1) to the following six component equations:

$$(2) \quad \begin{aligned} \sum X_k &= \sum Y_k = \sum Z_k = 0 \\ \sum (y_k Z_k - z_k Y_k) &= \sum (z_k X_k - x_k Z_k) = \sum (x_k Y_k - y_k X_k) = 0. \end{aligned}$$

These represent the projections of the vector equations (1) on the coordinate axes; the x_k, y_k, z_k are the coordinates of the points of application, measured from O as origin.

(2) Equipollence; the Reduction of Force Systems

If the external forces (or torques) are not in equilibrium, we can ask whether there exists a single force (or single torque) of such properties that under its action alone the rigid body moves in the same way as it would under the action of the given system of forces (or torques).

Posing this question is, among other things, useful (even though in general not sufficient) for the determination of the forces which are exerted on a rigid body by its supports if the rigid body is acted on by a system of forces which themselves are not sufficient to bring about a state of equilibrium.

We obtain the answer by drawing the closing segment in the now "open" polygon F_1, F_2, \dots, F_n , once in the direction in which the polygon is traced (F_{n+1}) and once (cf. Fig. 41) in the opposite direction (F_r , resultant force). Nothing is changed thereby. We have now a closed force polygon F_1, \dots, F_{n+1} , and a single force F_r , which, taken together, are *equipollent* to the "open" polygon of forces F_1, \dots, F_n . The forces F_1, \dots, F_{n+1} are, however, in equilibrium and can therefore be left out, so that the single force F_r is equipollent to the given system of forces F_1, \dots, F_n . Mathematically,

$$(3) \quad F_r = \sum_{k=1}^n F_k.$$

The same process of reasoning can be carried out with an "open" torque polygon. One thereby obtains a resultant force moment L_r which

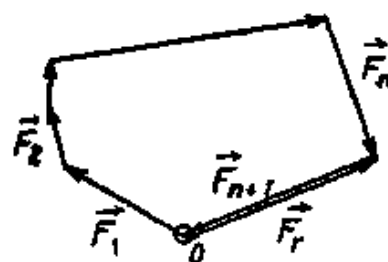


FIG. 41. Construction of the resultant force for an "open" polygon of forces.

is equipollent to the given system of moments L_1, L_2, \dots, L_n , i.e.,

$$(4) \quad L_r = \sum_{k=1}^n L_k$$

Let us mention in passing that there is nothing to stop us from making the single force F_r act at the same point O which serves as reference point in the calculation of moments L_k . This choice is indicated in Fig. 41.

(3) Change of Reference Point

Eq. (3) immediately shows that F_r is independent of the choice of reference point O . If F_r' is the resultant single force associated with a different reference point O' , we therefore have

$$(5) \quad F_r' = F_r.$$

From Eq. (4), on the other hand, we have with corresponding meaning of L_r'

$$(6) \quad L_r' = \sum_{k=1}^n L_k' \text{ with } L_k' = r_k' \times F_k,$$

where r_k' is the radius vector from O' to the point of application P_k of F_k . Let a be the vector distance from O' to O . Then

$$(6a) \quad r_k' = a + r_k, \quad L_k' = a \times F_k + r_k \times F_k = a \times F_k + L_k.$$

Therefore

$$(6b) \quad L_r' = \sum_{k=1}^n a \times F_k + \sum_{k=1}^n L_k = a \times \sum_{k=1}^n F_k + L_r.$$

But in view of (3)

$$a \times \sum_{k=1}^n F_k = a \times F_r.$$

Thus we have

$$(7) \quad L_r' = L_r + a \times F_r.$$

(4) Comparison of Kinematics and Statics

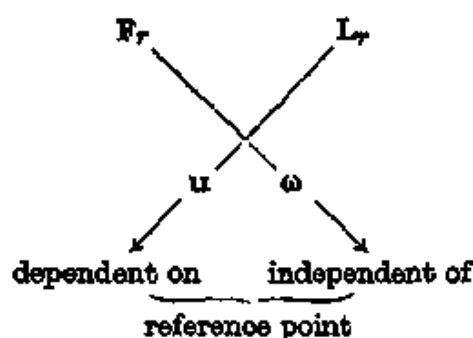
As remarked in connection with Eq. (22.2), in kinematics ω is independent of the choice of reference point, whereas u depends on that choice. We write

$$(8) \quad \omega' = \omega$$

and, from (22.7), with $\mathbf{v}=\mathbf{u}'$ and $\mathbf{r}=\mathbf{a}$,

$$(9) \quad \mathbf{u}' = \mathbf{u} + \boldsymbol{\omega} \times \mathbf{a}.$$

This equation has the same structure as the preceding Eq. (7) provided we disregard the sequence of the factors in the corresponding vector products. If we also take into consideration Eqs. (5) and (8), we arrive at a remarkable reciprocity between statics and kinematics which can be expressed by the scheme below:



This crosswise reciprocity holds as well between the concepts of force couple and rotational couple which we shall now take up.

The *force couple* (or "couple," for short) is a basic element in elementary statics. As is well known, a couple consists of two parallel and opposite forces of equal magnitude, $\pm \mathbf{F}$, whose lines of action are a finite distance, say l , apart. If we carry out the reduction of such a couple in the sense of subsec. 2, we obtain

$$(10) \quad \mathbf{F}_r = 0, \quad \mathbf{L}_r = \vec{L}, \quad |\vec{L}| = |\mathbf{F}| l,$$

where one should think of the vector \vec{L} as directed normal to the plane of the two forces. Whereas, however, the former \mathbf{L}_r was, so to speak, attached to the reference point O , our present \vec{L} is the same for all reference points and completely free to move in space; i.e., two given couples can be added vectorially to yield a third couple; two couples of equal and opposite moments located in parallel planes cancel, etc.

Let us follow up the crosswise reciprocity indicated by our scheme by defining a rotational couple. By a *rotational couple* we understand two equal and opposite rotational velocities $\pm \boldsymbol{\omega}$, whose axes are parallel to each other and a distance l apart. According to the rule of addition (22.5), the reduction of a rotational couple yields a resultant rotational velocity $\boldsymbol{\omega}_r = 0$. Our rotational couple generates then a pure translation perpendicular to the plane of the two axes of rotation. The magnitude of the velocity of translation is easily found to be $|\vec{u}| = \omega l$. The

analogy to Eqs. (10) in the sense of our reciprocity scheme is therefore complete. Whereas our former \mathbf{u} depended on the choice of the reference point O , the $\vec{\mathbf{u}}$ equivalent to a rotational couple is independent of O , and can be translated parallel to itself in space in any manner whatever. From this it follows that two arbitrarily located rotational couples add vectorially just like their velocities of translation $\vec{\mathbf{u}}$; two rotational couples of equal and opposite moment $\pm \omega l$ located in parallel planes cancel, etc.

SUPPLEMENT: WRENCHES AND SCREW DISPLACEMENTS

From (7) we see that \mathbf{L}_r depends on the reference point. We are therefore tempted to choose this point in such a way that \mathbf{L}_r and \mathbf{F}_r become parallel. We then obtain an especially simple picture of our system of forces called a *wrench*, i.e., a single force and a moment acting about this force or, equivalently, a couple located in a plane perpendicular to the force. If our initial reference point is O , the position of O' required for a wrench is obtained as follows: in Eq. (7) we decompose \mathbf{L}_r into \mathbf{L}_p parallel to \mathbf{F}_r and \mathbf{L}_n perpendicular to it and determine \mathbf{a} from the equation

$$(11) \quad \mathbf{L}_n = -\mathbf{a} \times \mathbf{F}_r.$$

From (5) and (7) we then have for the reference point O' ,

$$\mathbf{F}_r' = \mathbf{F}_r, \quad \mathbf{L}_r' = \mathbf{L}_p \parallel \mathbf{F}_r$$

as demanded by the definition of a wrench. Eq. (11) states that for this purpose the reference point O must be displaced a certain distance

$$\mathbf{a} = -\frac{|\mathbf{L}_n|}{|\mathbf{F}_r|}$$

normal to \mathbf{F}_r and \mathbf{L}_n .

A line of reasoning exactly reciprocal in the sense of the preceding discussion leads to the *screw displacement*. With Eq. (9) as starting point we decompose \mathbf{u} into \mathbf{u}_p parallel to ω and \mathbf{u}_n perpendicular to it. The displacement \mathbf{a} of the reference point required for a screw is determined by the equation

$$(12) \quad \mathbf{u}_n = -\omega \times \mathbf{a}.$$

From (8) and (9) we then obtain for the reference point O' ,

$$(13) \quad \omega' = \omega, \quad \mathbf{u}' = \mathbf{u}_p \parallel \omega,$$

which in fact represents a screw displacement. Eq. (12) states that the reference point O must here be displaced by a certain distance normal to ω and \mathbf{u}_n .

Attractive as the concept of the wrench and the screw displacement may be, it is of no great practical value in the treatment of specific problems involving rotation. For this reason mention of them has been relegated to a supplement.

§ 24. *Linear and Angular Momentum of a Rigid Body. Their Connection with Linear and Angular Velocity*

Let us imagine that a momentum of translation (linear momentum, impulsive force) and a momentum of rotation (moment of momentum, impulsive torque) have been imparted to a rigid body. Let the first one of these be designated by the letter \mathbf{p} , the latter by \mathbf{M} .

\mathbf{p} is calculated as a sum over all linear momenta $d\mathbf{p} = \mathbf{v} dm$, i.e.,

$$(1) \quad \mathbf{p} = \int d\mathbf{p} = \int \mathbf{v} dm.$$

With the help of Eq. (22.7) we get

$$\mathbf{p} = \mathbf{u} \int dm + \boldsymbol{\omega} \times \int \mathbf{r} dm$$

or, with introduction of the radius vector \mathbf{R} from O to the center of mass, cf. (22.11a),

$$(2) \quad \mathbf{p} = m\mathbf{u} + m\boldsymbol{\omega} \times \mathbf{R}.$$

In particular, if we choose $O = G$, we have $\mathbf{R} = 0$ and

$$(3) \quad \mathbf{p} = m\mathbf{u}.$$

The angular momentum \mathbf{M} of the rigid body, on the other hand, is composed of the moments of all the elements of linear momentum taken about the common reference point O . We therefore have

$$(4) \quad \mathbf{M} = \int \mathbf{r} \times d\mathbf{p} = \int dm (\mathbf{r} \times \mathbf{v}),$$

from which, because of (22.7) and (22.11a),

$$(5) \quad \mathbf{M} = \int dm (\mathbf{r} \times \mathbf{u}) + \int dm \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = m\mathbf{R} \times \mathbf{u} + \int dm \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

The first term on the right vanishes for $O = G$ as well as for $\mathbf{u} = 0$, so that in both these cases

$$(6) \quad \mathbf{M} = \int dm \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

In order to evaluate this integral we remind the reader of the vector rule for the triple cross-product, valid for any three vectors \mathbf{A} , \mathbf{B} , \mathbf{C} ,

$$(7) \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

It follows that

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega} r^2 - \mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{r})$$

and therefore, taking the x -component as example,

$$\begin{aligned} (8) \quad M_x &= \int [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]_x dm \\ &= \omega_x \int (x^2 + y^2 + z^2) dm - \omega_y \int x y dm - \omega_z \int x z dm. \end{aligned}$$

By introducing the moments and products of inertia from (22.12a), we can then write (6) in the form

$$\begin{aligned} (9) \quad M_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ M_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ M_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z. \end{aligned}$$

We have thus arrived at a linear relation between the dynamic vector \mathbf{M} and the kinematic vector $\boldsymbol{\omega}$; this relation is achieved by means of the tensor I of Eq. (22.13b). We therefore say that \mathbf{M} is a "linear vector function" of $\boldsymbol{\omega}$. Such linear vector functions play an important role in all aspects of the tensor calculus, especially in the theory of elasticity (cf. Vol. II, this series).

Eqs. (9) can be put into instructive form if we make use of expression (22.12b) for the kinetic energy of rotation. For then we simply have

$$(10) \quad M_i = \frac{\partial T_{\text{rot}}}{\partial \omega_i}, \quad i = x, y, z.$$

Notice, moreover, that this expression is valid not only for the case $O=G$ or $\mathbf{u}=0$ presupposed in (9), but also for $\mathbf{u} \neq 0$ and arbitrary position of O . For in the more general case one need only complete expression (22.12b) for T_{rot} by adding expression (22.11) for T_m , so that the term

$$\frac{\partial T_m}{\partial \omega_i} = m(\mathbf{R} \times \mathbf{u})_i$$

will be added on the right of Eq. (10). But this is the same term which appears on the right side of Eq. (5) for \mathbf{M} whenever O and G do not coincide. Since, finally, the total kinetic energy T differs from $T_{\text{rot}} + T_m$ only by the term T_{transl} independent of $\boldsymbol{\omega}$ [cf. (22.9) and (22.10)], we can generalize (10) in the form

$$(10a) \quad M_i = \frac{\partial T}{\partial \omega_i}, \quad i = x, y, z$$

valid for arbitrary position of O .

What has been said of the angular momentum \mathbf{M} is also valid for the

linear momentum \mathbf{p} . Here we consider at once the general case $O \neq G$ and from Eqs. (22.9), (22.10) and (22.11) form

$$\frac{\partial T}{\partial u_i} = m u_i + m(\boldsymbol{\omega} \times \mathbf{R})_i$$

which is in agreement with Eq. (2) for \mathbf{p} . The equation complementary to (10a) is therefore

$$(11) \quad p_i = \frac{\partial T}{\partial u_i}, \quad i = x, y, z.$$

Eqs. (10a) and (11) are special cases of a much more general relationship connecting momentum and velocity coordinates of an arbitrary mechanical system. The proof of this must be postponed to Chapter VI, § 36. Here we shall only concern ourselves with the geometrical meaning of Eq. (10), which leads us to the celebrated geometrical construction of Poinsot. The *Poinsot method* tells us how to find the position of the axis of angular momentum \mathbf{M} with reference to a given axis of rotation. The same can be said of this method as of the foregoing equations, namely, that it is not restricted to the case of the rigid body, but is applicable whenever one deals with a symmetric tensor; one represents this tensor by means of a tensor surface of second degree, and asks for the linear vector function given by means of this tensor.

The Poinsot construction runs as follows: from the center O of the momental ellipsoid we lay off the angular velocity vector $\boldsymbol{\omega}$ and construct the tangent plane to the ellipsoid at the point where $\boldsymbol{\omega}$ intersects it. The perpendicular from O to this tangent plane gives the direction of \mathbf{M} . As proof we need merely recall that for an arbitrary surface $f(\xi, \eta, \zeta) = \text{const.}$, the direction cosines of the normal to the tangent plane are proportional to

$$(12) \quad \frac{\partial f}{\partial \xi}, \frac{\partial f}{\partial \eta}, \frac{\partial f}{\partial \zeta}.$$

In our case $f(\xi, \eta, \zeta) = \text{const.}$ is the equation (22.15) of the momental ellipsoid and its derivatives with respect to ξ, η, ζ are indeed proportional to the components of \mathbf{M} of Eq. (9).

We may also interpret the Poinsot construction as the direct geometrical expression of our Eq. (10), for the momental ellipsoid is essentially identical to the surface $T_{\text{rot}} = \text{const.}$

Our Figs. 42a, b represent the case of the symmetrical momental ellipsoid, where $\boldsymbol{\omega}$, \mathbf{M} and the axis of symmetry ("axis of figure") \mathbf{f} are coplanar; so that the tangent plane can be represented as the tangent to the cross-section of the ellipse in this plane. In the prolate ellipsoid of

revolution (i.e., spheroid), Fig. 42b, \mathbf{M} and \mathbf{f} , the axis, lie on opposite sides of $\boldsymbol{\omega}$; in the oblate spheroid of Fig. 42a, \mathbf{M} lies between \mathbf{f} and $\boldsymbol{\omega}$. The case of the ellipsoid with three unequal axes presents a more difficult graphical problem.

In conclusion we emphasize that the relations discussed in this section are basically nothing but expressions of the Newtonian definition, "the quantity of motion is the measure of the same, arising from the velocity and the quantity of matter conjointly," extended to the rigid body. The reason why our present relations are so much more involved than the one between momentum and velocity of a single particle is that in particle mechanics the "quantity of matter," i.e., the mass, is a *scalar*, whereas in the case of the rigid body the moment of inertia that takes its place is a *tensor*.

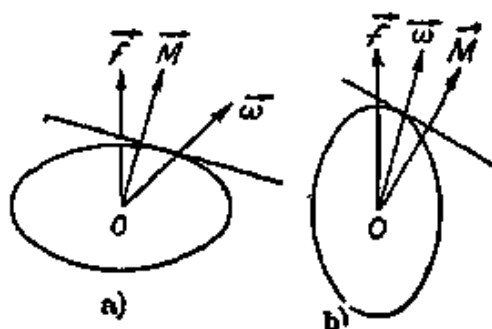


FIG. 42. Poincaré construction giving the relative position of angular velocity $\boldsymbol{\omega}$ and angular momentum \mathbf{M} for the two cases where the momental ellipsoid degenerates into a) an oblate spheroid, b) a prolate spheroid.

§ 25. Dynamics of a Rigid Body. Survey of its Forms of Motion

Let us first consider the rigid body moving freely in space. As reference point we choose its center of mass, and reduce all the forces acting on the body to forces acting on this point, in agreement with the prescription of § 23. We need then only deal with a single resultant force \mathbf{F} and a resultant torque \mathbf{L} . The equations of motion are the equations of momentum and of moment of momentum of § 13; they read

$$(1) \quad \dot{\mathbf{p}} = \mathbf{F},$$

$$(2) \quad \dot{\mathbf{M}} = \mathbf{L}.$$

Since the rigid body possesses but six degrees of freedom, these two vector equations suffice for the complete description of its state of motion.

Eqs. (1) and (2) can be treated separately whenever \mathbf{F} is independent of the angular velocity and \mathbf{L} independent of the translational velocity. In ballistics, for instance, this is not the case. If it is the case, (1) becomes a problem of pure particle mechanics, (2) a problem of rotation about a fixed point or, as we shall say for brevity, a "problem of the spinning top."

At this point we shall be interested principally in the latter. With the choice of reference point made above, we can disregard the force of

gravity since it has no moment about the mass center. If, furthermore, we neglect air resistance, friction and the like, we are confronted with the problem of the *spinning top under no forces*. Thus the gyroscope in a Cardan suspension (cf. Fig. 47) is a top under no forces provided we can neglect the mass of the gimbals in comparison with that of the flywheel, which is approximately valid in the usual constructions. Otherwise we would be confronted with a considerably more involved mathematical problem.

We shall also deal with rotation about a fixed point other than the mass center. As remarked on p. 122, it is then advisable to take this fixed point as reference point O and introduce the gravitational moment L acting about it. In that case we speak of a *heavy top*. Subsecs. 4 and 5 are devoted to its discussion.

We shall postpone the complete analytical treatment of the top under no forces until the following section, where we shall become acquainted with the tool provided by *Euler's equations*. The complete treatment of the heavy top — to the extent to which it can be carried through at all — must be postponed even further, namely to § 35. There we shall have at our command the yet more powerful method of the *generalized Lagrange equations*.

For the top under no forces Eq. (2) yields $\dot{M}=0$. This can be integrated at once to yield

$$(3) \quad M = \text{const.}$$

The angular momentum of a top under no forces is constant in magnitude and spatial direction. This statement completely parallels Galileo's law of inertia, but in general does not lead to an expression for velocity and position in space which is as simple as in the other case.

(1) The Spherical Top Under No Forces

Only in the case of a spherical momental ellipsoid do we have $M = I\omega$, from which $M = \text{const.}$ leads to $\omega = \text{constant}$. The axis of rotation is in permanent coincidence with the fixed axis of angular momentum. Each point of the body, no matter what the external shape of the same (cf. Fig. 40c, for instance), describes a circle about this axis with constant velocity.

(2) The Symmetrical Top Under No Forces

Here a simple rotational motion occurs only if the direction of M coincides with one of the principal axes, that is, either with the axis of the body or an equatorial axis. The *general* form of motion of the symmetrical top under no forces is the so-called *regular precession*.

We explain this form of motion with the aid of Fig. 43. We have drawn the axis of the angular momentum, which is fixed in space, vertically upward; let M be the point at which it intersects a unit sphere described about the center of the momental ellipsoid. Call R and F the points of intersection of this sphere and the axes of rotation and of symmetry at an arbitrary instant. Since by the Poinsot method these three axes lie in a meridian plane through F , the three points M , R and F are located on a great circle passing through the fixed point M ; in the case of a momental oblate spheroid, which we shall postulate for definiteness, M is situated between F and R . At any instant the motion consists of a rotation about OR . In this process F advances normally to the arc of the great circle just mentioned. The angular distance between F and M is not changed thereby; thus we can draw the instantaneous path of F as a short arc of a circle of latitude about M (arrow at the left in Fig. 43). Now R too must change its position — it must move to the great circle defined by M and the new position of F . In this motion the angular distance between M and R is conserved, since it is determined by the Poinsot construction. Thus R , too, advances on the arc of a circle of latitude about M (arrow at right of Fig. 43). The relative position of points F , M , and R is now the same as initially, so that our process of reasoning can be repeated. It follows that *axes of symmetry and rotation each describe a circular cone about the spatially fixed angular momentum, each cone being traced with constant angular velocity;*

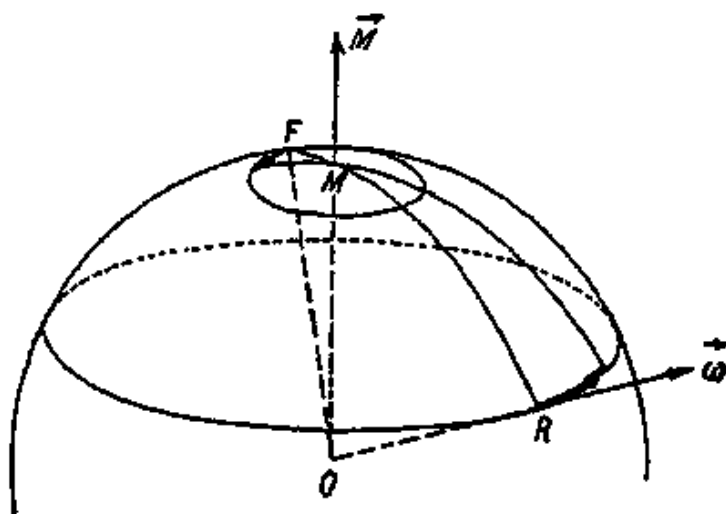


FIG. 43. Regular precession of the symmetrical top under no forces.

the latter because the angular velocity is completely determined by the magnitude of M and its position with respect to the momental ellipsoid. Thus the character of the regular precession has been fully described.

The same applies of course to a momental prolate spheroid, with the only difference (cf. Fig. 42b) that R would now be located between F and M .

(3) The Unsymmetrical Top Under No Forces

The form of motion of the symmetrical top just derived could have been described with more brevity, but less clarity of detail, as follows:

through the terminus of the angular momentum vector M we pass the "invariable plane" ϵ (cf. p. 73) normal to M . About the origin of M we construct the ellipsoid of twice the kinetic energy ("Poinset ellipsoid") which is similar to the momental ellipsoid. The Poinset ellipsoid is tangent to ϵ^3 , and the point of tangency is the terminus of the angular velocity vector ω . The instantaneous motion of the top consists of a rotation of this ellipsoid about ω . In this process the ellipsoid rolls without slipping on the plane ϵ^4 . If the Poinset ellipsoid is one of revolution, the curve of the point of tangency becomes a circle about M ; the cones described by ω ("space cone") and the axis of figure therefore become circular cones. Thus we have again the regular precession of the top.

The same construction now leads at once to the Poinset picture of the force-free motion of a general ("unsymmetrical") top of three distinct principal moments of inertia. Again we let the Poinset ellipsoid roll on the invariable plane ϵ (cf. footnote 3 below). Now the curve of contact is no longer a circle, but a transcendental curve which in general does not close on itself. Similarly the cones which describe the motion of the axis of rotation and of the body "axis" in space are now transcendental cones. The analysis of the unsymmetrical top, even when under no forces, leads to elliptic integrals [cf. § 26, (3)], while that of the symmetrical top under no forces requires only elementary functions. Of course even for the unsymmetrical top a pure rotation about one of the three principal axes is a steady rotation whose representation is elementary.

(4) The Heavy Symmetrical Top

Here we shall not treat the spherical top separately, since its motion is hardly simpler than that of the symmetrical top.

For the heavy symmetrical top the fixed point O (point of support in the socket) no longer coincides with the center of mass G (located on the axis of symmetry); call s the distance OG . The magnitude of the gravitational torque is then

$$(4) \quad |L| = mgs \sin \theta,$$

where θ is the angle between the vertical and the axis of figure. L is normal to the vertical and to the axis of symmetry or, in other words, it lies along the line of intersection of the horizontal plane with the equatorial plane of the momental ellipsoid. This line of intersection is called the *line of nodes*,

³ This follows from the Poinset construction of p. 132 and from Eq. (26.17a) soon to be encountered.

⁴ Rolling without sliding is equivalent in meaning to the equality of the rate of change of the angular velocity vector ω as observed from space and from the body. In this connection refer to Eq. (26.8a), where this equality is proved.

a term borrowed from astronomy. For a more precise definition of signs refer to p. 141.

Our general Eq. (2) can no longer be integrated immediately as in the case of a top under no forces; rather, the angular momentum is subject to continuous change given by the law

$$(5) \quad d\mathbf{M} = \mathbf{L} dt.$$

Thus the infinitesimal vector $\mathbf{L} dt$ adds to the vector \mathbf{M} at any given instant t to give the angular momentum at $t+dt$. The terminus of \mathbf{M} advances in the direction of the instantaneous line of nodes, i.e., normal to the vertical and the axis of symmetry. From this it follows that the projections of \mathbf{M} on the vertical as well as on this axis must be constant. Let us call the two constants

$$(6) \quad M' = M_{\text{vert}} \text{ and } M'' = M_{\text{sg}}.$$

The two quantities M' and M'' , which can be prescribed arbitrarily, are two constants of integration of the equations of motion.

A third constant is that of total energy E . Corresponding to Eq. (6.18) we have the gravitational potential energy

$$(6a) \quad V = mgs \cos \theta$$

so that

$$(7) \quad T + mgs \cos \theta = E.$$

In order to pass to an analytical description of the motion we must express T and the projections of \mathbf{M} mentioned in (6) in terms of suitable position parameters of the top (the Eulerian angles); this will be carried out in § 35. The calculation of the motion is there shown to lead to elliptic integrals.

The *regular precession* is now no longer the general form of motion as in the case of the top under no forces, but only results for specially chosen values of M' , M'' and E . The precessional motion usually observed with a heavy top excited in the customary way appears to be — but is not — regular; it can be called *pseudo-regular precession*. Finally a *pure rotation* about the vertically oriented axis of figure is also a possible (stable or unstable) form of motion, no matter what the magnitude of ω .

So far we have considered only the equation of angular momentum (2). We must throw a quick glance at the equation of linear momentum (1). Its right member consists of the force \mathbf{F} acting at the fixed point O , which is composed of the force of gravity $m\mathbf{g}$ acting vertically downward, and the

reaction of the support, F_{sup} . The change in momentum in the left member is

$$\dot{\mathbf{p}} = m \frac{d}{dt} (\boldsymbol{\omega} \times \mathbf{R}) = m \dot{\mathbf{V}}$$

from Eq. (24.2) with $\mathbf{u}=0$, where \mathbf{V} is the velocity of the center of mass. Eq. (1) then makes the simple statement that

$$F_{\text{sup}} = m(\dot{V} - g).$$

In other words, the law of linear momentum demands that at any given instant the support furnish a force equal to the mass of the top \times the acceleration of the mass center diminished by the gravitational acceleration.

(5) The Heavy Unsymmetrical Top

In spite of the efforts of many great mathematicians, all attempts to integrate the differential equations of this problem in the most general form have failed so far. Among the integrals of angular momentum (6) the first remains in effect, to be sure, because even here the gravitational torque acts about a horizontal axis so that the terminus of the vector \mathbf{M} remains in a horizontal plane fixed in space. The second integral (6) is, however, invalidated, because it is based on the symmetry of the momental ellipsoid. Of course the energy integral (7) is valid also for a general momental ellipsoid.

The soluble special cases of the problem postulate either a particular mass distribution or a particular form of motion.

The best-known case is that of Kowalewski. The momental ellipsoid is here assumed to be symmetrical; the center of mass no longer lies on the axis of the body, but in the equatorial plane defined as the plane perpendicular to the axis and passing through the fixed point; in addition it is required that the moment of inertia about the axis of the body be one half the equatorial one. In that case the form of motion need not be restricted.

The case of Staudé concerns the question as to which axes can serve as axes of steady rotation when directed vertically. It turns out that these axes lie in the body on a cone of second degree which, in addition to the three principal axes, contains also the axis through the center of mass. To each axis belongs a (to within a sign) definite angular velocity. Neither mass distribution nor position of the center of mass need be specialized in this problem.

The case of Hesse, finally, is concerned with the analogue to the simple motion of a pendulum (spherical pendulum or, in particular, ordinary pendulum). For such motion the mass center must lie on a certain axis in the momental ellipsoid, and the initial excitation must be of proper form,

just as in the case of the symmetrical top, whose mass center describes a pure pendulum motion only if the initial angular momentum has no component along the axis of symmetry.

§ 26. Euler's Equations.

Quantitative Treatment of the Top Under No Forces

(1) Euler's Equations of Motion

We distinguish between a reference system x, y, z fixed in space and a second one, X, Y, Z fixed in the body. In the (x, y, z) -system the angular momentum for motion under no forces has an invariable position: $\mathbf{M} = \text{constant}$ [Eq. (25.3)]; seen from the body, the position of \mathbf{M} varies continuously. We want to study the law of this variation.

Let us therefore focus our attention on a point P fixed in the body, and a point Q fixed in space, the two points being momentarily in coincidence. Let \mathbf{v} be the velocity of P in space, \mathbf{V} that of Q in the body. According to the kinematic Eq. (22.4), $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. As seen from the body, Q moves with equal but opposite velocity to that of P as seen from space, so that

$$\mathbf{V} = -\boldsymbol{\omega} \times \mathbf{r} = \mathbf{r} \times \boldsymbol{\omega}.$$

In tabular form we have

	Seen from Space	Seen from Body
P	$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$	$\mathbf{V} = 0$
Q	$\mathbf{v} = 0$	$\mathbf{V} = \mathbf{r} \times \boldsymbol{\omega}$

For point Q we choose the spatially fixed terminus of the vector \mathbf{M} and hence write

$$\mathbf{r} = \mathbf{M}, \quad \mathbf{V} = \frac{d\mathbf{M}}{dt}.$$

Thus $\frac{d\mathbf{M}}{dt}$ means "change in the body" (we called the change in space $\dot{\mathbf{M}}$; it is equal to zero here).

From the second line of our table we then read off

$$(1) \quad \frac{d\mathbf{M}}{dt} = \mathbf{M} \times \boldsymbol{\omega}.$$

This completes the derivation of *Euler's equations for a rotating body under no forces*.

We shall rewrite them in terms of their components in the (X, Y, Z) -system. We shall call $\omega_1, \omega_2, \omega_3$ the components of ω , and M_1, M_2, M_3 those of M . Eq. (1) yields

$$\begin{aligned} \frac{dM_1}{dt} &= M_2 \omega_3 - M_3 \omega_2, \\ (2) \quad \frac{dM_2}{dt} &= M_3 \omega_1 - M_1 \omega_3, \\ \frac{dM_3}{dt} &= M_1 \omega_2 - M_2 \omega_1. \end{aligned}$$

The system of the X, Y, Z is so far completely arbitrary. If, now, we take the directions X, Y, Z along the principal moments of inertia of Eq. (22.15a) and call these I_1, I_2, I_3 , we obtain, by virtue of the general relation (24.9),

$$(3) \quad M_1 = I_1 \omega_1, \quad M_2 = I_2 \omega_2, \quad M_3 = I_3 \omega_3;$$

and (2) takes the simple form

$$\begin{aligned} I_1 \frac{d\omega_1}{dt} &= (I_2 - I_3) \omega_2 \omega_3, \\ (4) \quad I_2 \frac{d\omega_2}{dt} &= (I_3 - I_1) \omega_3 \omega_1, \\ I_3 \frac{d\omega_3}{dt} &= (I_1 - I_2) \omega_1 \omega_2. \end{aligned}$$

It is these remarkably symmetrical and elegant equations one usually thinks of when one speaks of Euler's equations.

Let us now extend them to include the case that an external torque L is in effect. In that case the terminus of M is no longer fixed in space, but, according to (25.2), has the velocity $v = L$.

As seen from the body, our point Q now moves with a velocity composed of $v = L$ and $V = r \times \omega$. It follows that Eq. (1) must be changed to

$$(5) \quad \frac{dM}{dt} = M \times \omega + L$$

and the components of L with respect to X, Y, Z must be added to the right members of (2) and (4). This yields *Euler's equations of motion for a rigid body with a fixed point*.

We shall write these equations explicitly only for the case of the heavy symmetrical top, where L acts about the line of nodes and, from (25.4), has the magnitude

$$|L| = mgs \sin \theta.$$

In order to dispel all ambiguities contained in the meaning of the words vertical, axis of symmetry, line of nodes, we agree that

the positive side of the spatially fixed z -axis points up and defines the vertical;

the positive side of the Z -axis passes through the mass center and defines the axis of symmetry; it makes an angle θ with the vertical;

the line of nodes is the semi-infinite line normal to the positive z - and Z -axes and in the direction of advance of a right-handed screw as θ increases.

We further specify that the distance s is to be a positive quantity. Call ϕ the angle which the line of nodes makes with the positive X -axis. The components of L with respect to X, Y, Z are then given by

$$(5a) \quad mgs \sin \theta \cos \phi, \quad -mgs \sin \theta \sin \phi, \quad 0$$

respectively, and with $I_1 = I_2$ equations (4) go over into

$$(6) \quad \begin{aligned} I_1 \frac{d\omega_1}{dt} &= (I_1 - I_3) \omega_2 \omega_3 + mgs \sin \theta \cos \phi \\ I_1 \frac{d\omega_2}{dt} &= (I_3 - I_1) \omega_3 \omega_1 - mgs \sin \theta \sin \phi \\ I_3 \frac{d\omega_3}{dt} &= 0. \end{aligned}$$

The last equation shows that for the heavy symmetrical top (and therefore *a fortiori* for one under no forces) we have

$$(7) \quad I_3 \omega_3 = M_3 = \text{const.},$$

which we already knew. We see at the same time that Euler's equations are not suited for a further integration for the heavy top, since as yet we are ignorant of the relation between the ω_1, ω_2 and the θ, ϕ .

As far as the $\omega_1, \omega_2, \omega_3$ are concerned, we wish to emphasize very strongly that they are not velocities in the ordinary sense, i.e., not derivatives with respect to time of spatial measurements of some sort. Indeed, in view of the expression defined on p. 50, we can aptly designate them as "non-holonomic velocity components."

We shall finally write (5) in a somewhat different form. Since \mathbf{v} is the velocity as seen from space, we can generalize our expression by substituting $\mathbf{v} = \dot{\mathbf{M}}$ for $\mathbf{v} = \mathbf{L}$. We thus obtain

$$(8) \quad \dot{\mathbf{M}} = \frac{d\mathbf{M}}{dt} + \boldsymbol{\omega} \times \mathbf{M},$$

an equation which, by the analysis of p. 139, is valid for all (axial or polar)

vectors. If, specifically, we apply it to the angular velocity vector ω , it simply yields

$$(8a) \quad \dot{\omega} = \frac{d\omega}{dt}.$$

For the angular velocity vector ω and only for this vector the spatial change is equal to the change as judged from the body. It is this rule to which we referred in the footnote on p. 136.

(2) Regular Precession of the Symmetrical Top Under No Forces and Euler's Theory of Polar Fluctuations

We need not say any more about the spherical top. Its general motion is a pure rotation about an axis fixed in the body. This follows at once from Eqs. (4) if we put $I_1 = I_2 = I_3$. As we know from § 25, subsec. 1, this axis is at the same time fixed in space and coincides with the angular momentum direction.

Let us now turn to the symmetrical top, $I_1 = I_2 \neq I_3$. The third Eq. (4) yields

$$\omega_3 = \text{const.}$$

as we already know from Eq. (7). The first two equations are

$$(9) \quad \begin{aligned} I_1 \frac{d\omega_1}{dt} &= (I_1 - I_3) \omega_2 \omega_3 \\ I_1 \frac{d\omega_2}{dt} &= (I_3 - I_1) \omega_1 \omega_3 \end{aligned}$$

It is convenient to consolidate them into one by introducing a complex variable. Multiply the second equation by i and add to the first to obtain

$$(10) \quad I_1 \frac{ds}{dt} = i(I_3 - I_1) s \omega_3, \quad s = \omega_1 + i\omega_2.$$

Let us abbreviate this by putting

$$(11) \quad \alpha = \frac{I_3 - I_1}{I_1} \omega_3,$$

so that an integration of (10) gives

$$(12) \quad s = s_0 e^{i\alpha t}, \quad s_0 = \text{constant of integration.}$$

s is the projection of the angular velocity vector ω on the equatorial plane of the top, if we use this plane as the complex plane of s . Eq. (12) states that this projection describes a circle of radius s_0 with the constant angular velocity α . At the same time the total angular velocity vector

ω describes a circular cone about the axis of figure. The vertex angle β of the cone is given by

$$(12a) \quad \tan \beta = \frac{(\omega_1^2 + \omega_2^2)^{\frac{1}{2}}}{\omega_3} = \frac{|\omega_2|}{\omega_3}.$$

This is the picture of the regular precession which is seen by an observer located on the top. (To an observer fixed in space the axis of the top rotates of course about the instantaneous axis of rotation which, as we saw earlier describes in its turn a circular cone about the spatially fixed angular momentum vector M .) Since it is our intention to apply the foregoing to the earth, the viewpoint of the observer located on the top rather than that of the one fixed in space will be useful, as it corresponds to the viewpoint of a human being located on the earth.

The earth is a top whose momental ellipsoid is an oblate spheroid. We call the *geometric North Pole* the point at which the axis of symmetry pierces the surface of the earth; it is, in general, distinct from the *celestial North Pole* which is the point at which the angular velocity vector cuts through the earth's surface. According to the Euler theory reproduced above, the celestial North Pole describes a circle about the geometric North Pole, a phenomenon called *Eulerian motion*. Inasmuch as it is the path of the rotational pole, this circle is also referred to as the *polhode*.

A suitable measure of the flattening of the earth is the so-called ellipticity

$$(13) \quad \frac{I_2 - I_1}{I_1} \sim \frac{1}{300}.$$

The angular velocity of the earth is determined by the length of the day; we have

$$(14) \quad \omega_3 \sim \omega = \frac{2\pi}{\text{day}}$$

from which, according to (11),

$$(15) \quad \alpha = \frac{I_2 - I_1}{I_1} \omega_3 = \frac{2\pi}{300} \text{ day}^{-1}.$$

Thus Euler's period for the precession amounts to

$$(16) \quad \frac{2\pi}{\alpha} = 300 \text{ days} = 10 \text{ months}.$$

We are accustomed to think of the axis of rotation of the earth as fixed in the globe and passing through the geometrical poles. This is not rigorously true. Every movement of mass on the earth along a longitude

must change the position of the axis of rotation⁵, and every movement of mass along a circle of latitude must change the angular velocity, that is, the length of the day; both changes are a result of the law of conservation of angular momentum. Let us imagine that this movement has ceased and that the celestial pole is deviated from the geometric one. In that case the axis of rotation would, by virtue of the Eulerian motion, commence a circular motion about the geometric pole.

Let us now compare our theoretical results with the observations of polar fluctuations, which have been gathered by international cooperation. In Fig. 44 we have sketched the polhode obtained between the years 1895 and 1900.

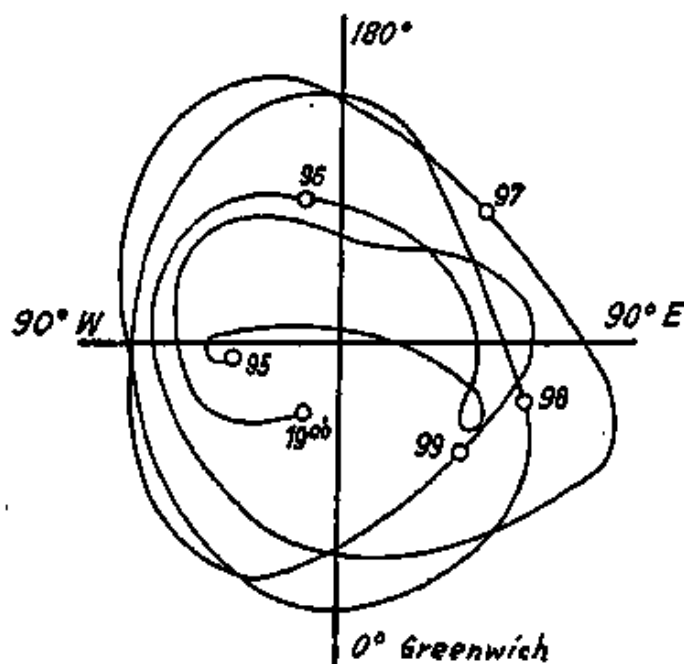


FIG. 44. Polar fluctuations between the years 1895 and 1900. Confirmation of Chandler's period.

The average deviation of the celestial pole, i.e., the mean radius of Euler's circle, amounts to about $\frac{1}{2}''$ of arc or 4 meters on the earth's surface,

according to observations between these years. But instead of a period of 10 months we have, according to Fig. 44, $3\frac{1}{2}$ complete revolutions for the four years 1896–1900, which corresponds to a period of 14 months.

The fourteen-month period is called Chandler's period after its discoverer. Its explanation lies in the elastic deformations that the earth suffers as a result of the changed centrifugal effect caused by polar fluctuations. The modulus of elasticity of the earth compares in magnitude to that of steel.

The observed polhode, as drawn in Fig. 44, can now be explained as a superposition of 1) fluctuations occurring with Chandler's period, 2) annual fluctuations evidently of meteorological origin, and 3) deviations at irregular intervals which may point to isolated and unrelated mass transports. No trace remains of Euler's ten-month period which was derived by assuming the earth to be an ideal rigid body.

⁵ The terrestrial mass transport most important for this effect seems to be the yearly migration of the air pressure maximum from the continent of Asia to the Pacific Ocean and back.

In agreement with usage in gyroscopic theory we have here described the motion of the earth's axis first investigated by Euler as a "precession under no forces." We have thus usurped a word having an entirely different meaning in astronomical usage. There, "precession" denotes a slow rotation of the earth's axis about the normal to the ecliptic which causes an advance of the equinoctial points of $50''$ per year. This *precession of the equinoxes* has a period of $\frac{360^\circ}{50''} = 26,000$ years. Instead of "precession of the equinoxes" we could also speak of an "advance of the line of nodes" (line of inter-section of the plane of the ecliptic with the equatorial plane of the earth); as mentioned earlier, our designation, "line of nodes," was borrowed from astronomy.

The precession of the equinoxes is not a free one, but rather a motion forced on the global top by the joint effect of the attractions of sun and moon.

We shall clarify this effect by means of Fig. 45, where we have, at least qualitatively, anticipated the theory of the heavy symmetrical top.

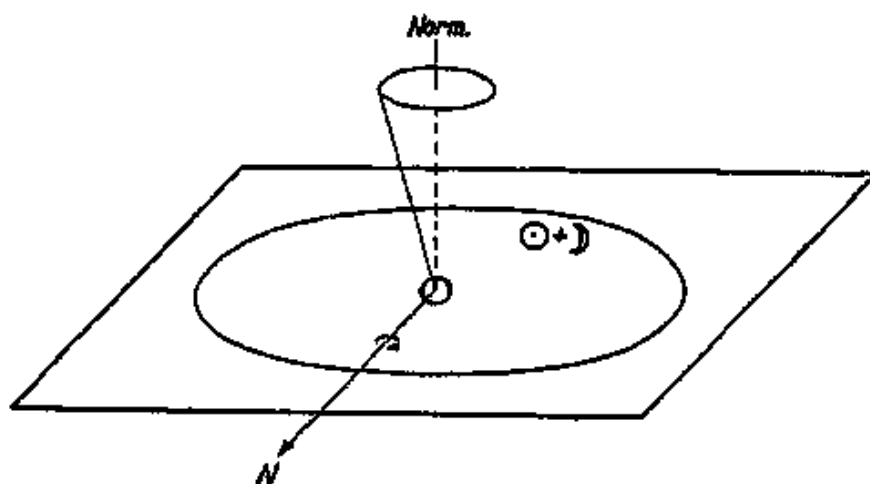


FIG. 45. Precession of the earth's axis, called "precession of the equinoxes."

The diagram shows the plane of the ecliptic on which a circle is drawn. One should think of the circumference of this circle as being uniformly "smeared" with the masses of sun \odot and moon \circ (actually we should draw two circles, one for the sun and one for the moon⁶; we have fused these two circles into one). The uniform mass distribution represents a time average over the instantaneous positions relative to the earth of sun and moon during their revolutions (in the sense of a Gaussian perturbation method). We justify the taking of this time average by the experimental fact that the periods of sun and moon are very small compared with the precession period mentioned above, so that this precession can in no way

⁶ As a matter of fact the moon is so close to the earth that its effect is about twice as great as that of the sun.

depend on the instantaneous positions of sun and moon. At the center of the $\odot + \rangle$ circle we see a cross-section of the earth with its two protuberances at the equator. Only these latter have a part in the phenomenon in question; for the attraction of the $\odot + \rangle$ ring tends to pull the two protuberances into the plane of the ecliptic, an effect which is intuitively almost obvious. We therefore have a torque about the line of nodes N in the sense of the arrow drawn about N . Now this torque is of the same type as the gravitational torque acting on a top whose mass center lies below the fixed point of support. The result is therefore similar to that in the case of the top. Rather than yield to the torque the axis of figure "escapes" in a perpendicular direction and describes a cone of precession about the vertical, here the normal to the ecliptic.

To be sure, the *regular* precession is only a special form of motion of the heavy top (cf. p. 137); under the present circumstances one would therefore expect the more general *pseudo-regular* precession consisting of a regular precession on which small "nutations" are superposed. Now these small nutations are nothing but the conical oscillations of the axis of figure occurring under no forces, hence, in our case, the polar fluctuations that take place with the period of Euler (or that of Chandler, obtained from the former by global deformation). The pseudo-regular precession to be expected is thus obtained from the precession of the equinoxes by addition of the Eulerian nutations occurring in the absence of forces.

Here we must once more apologize for the ambiguous use of a term. In astronomy one understands by nutation not a *free* fluctuation of the earth's axis, but one *forced* on it by the motion of the moon. Contrary to our preceding assumption in Fig. 45, the orbital plane of the moon does not coincide with that of the ecliptic, but is tilted at an angle of 5° with respect to it. Under joint action of sun and earth its normal too describes a cone of precession about the normal to the ecliptic. This precession is tantamount to a *recession of the lunar nodes* (intersection of moon's orbit with ecliptic) which, however, occurs at a much livelier rate than the advance of the line of nodes of the earth, viz., in $18\frac{1}{2}$ years. It is understandable that the earth's axis is in its turn implicated in this precession; the recession of the lunar nodes results in the *astronomical nutation of the earth's axis*, which takes place with the same period.

(3) Motion of an Unsymmetrical Top Under No Forces. Examination of its Permanent Rotations as to Stability

We turn to the integration of Eqs. (4) in the case $I_1 \neq I_2 \neq I_3$. Multiplication of these equations by ω_1 , ω_2 , ω_3 and addition yields

$$I_1 \omega_1 \frac{d\omega_1}{dt} + I_2 \omega_2 \frac{d\omega_2}{dt} + I_3 \omega_3 \frac{d\omega_3}{dt} = 0$$

or, integrated,

$$(17) \quad \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) = \text{const.} = E.$$

E is the energy constant, and the left member is the kinetic energy, in agreement with Eq. (22.12b) specialized to principal axes. Instead of (17) one can evidently also write

$$(17a) \quad E_{\text{kin}} = \frac{1}{2} \mathbf{M} \cdot \boldsymbol{\omega}.$$

We can instead multiply Eqs. (4) by $I_1 \omega_1$, $I_2 \omega_2$, $I_3 \omega_3$; addition once more yields zero on the right. The result of the integration can be written

$$(18) \quad (I_1 \omega_1)^2 + (I_2 \omega_2)^2 + (I_3 \omega_3)^2 = \text{const.} = |\mathbf{M}|^2.$$

On the left we have the sum of the squares of the angular momentum components. This sum, as we know, remains invariant in the absence of forces, even if the components themselves vary in the course of the motion.

In (17) and (18) we have two linear homogeneous equations for ω_1^2 , ω_2^2 , ω_3^2 , from which we can, for instance, solve for ω_2^2 and ω_3^2 in terms of ω_1^2 :

$$(19) \quad \begin{aligned} \omega_2^2 &= \beta_1 - \beta_2 \omega_1^2, & \beta_1 &= \frac{2EI_2 - |\mathbf{M}|^2}{I_2(I_2 - I_1)}, & \beta_2 &= \frac{I_1(I_3 - I_1)}{I_2(I_2 - I_1)}, \\ \omega_3^2 &= \gamma_1 - \gamma_2 \omega_1^2, & \gamma_1 &= \frac{2EI_3 - |\mathbf{M}|^2}{I_3(I_3 - I_1)}, & \gamma_2 &= \frac{I_1(I_3 - I_1)}{I_3(I_3 - I_1)}. \end{aligned}$$

If we replace these values of ω_2 and ω_3 in the first Eq. (4), we have

$$(20) \quad \frac{d\omega_1}{[(\beta_1 - \beta_2 \omega_1^2)(\gamma_1 - \gamma_2 \omega_1^2)]^{\frac{1}{2}}} = \frac{I_2 - I_3}{I_1} dt.$$

t is therefore an elliptic integral of the first kind in ω_1 (cf. p. 100); function theory allows us to state conversely that ω_1 is an elliptic function of the time. The same holds of course for ω_2 and ω_3 .

We furthermore deduce from Eqs. (17) and (18) that the polhode cone or *body cone* is no longer a circular cone as in the case of the symmetrical top, but a cone of fourth degree.

We shall finally consider the rotations of the unsymmetrical top about one of its three principal axes which, as we know [cf. § 25, toward end of (3)], are steady rotations. Let us, for definiteness, put

$$A > B > C.$$

We shall show that the rotations about the axes of the *greatest* and *smallest* principal moment of inertia are stable, those about the axis of the *intermediate*

principal moment are unstable. We choose Eqs. (17) and (18) as starting point. It will be convenient in connection with the diagrams below to rewrite these in terms of the angular momentum components M_1, M_2, M_3 ,

$$(21a) \quad \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} = \text{const.},$$

$$(21b) \quad M_1^2 + M_2^2 + M_3^2 = \text{const.} = |M|^2.$$

Eq. (21b) describes a sphere of radius $|M|$, (21a) an ellipsoid with three distinct axes (a "non-degenerate" ellipsoid).

Case 1. Rotation about the longest axis of the ellipsoid (21a). In a pure rotation the sphere is tangent to the ellipsoid from the *outside* at point A , Fig. 46a. A small jolt will in general alter both the sphere and the ellipsoid. The point of tangency A will change to a small curve of intersection which remains, however, in the neighborhood of A . A narrow body cone is the result; the original rotation proves to be stable.

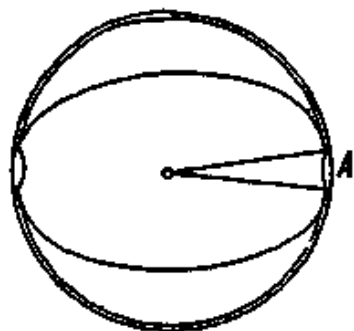


FIG. 46a. Stable rotation of unsymmetrical top about the longest axis of the momental ellipsoid.

The same is true in case 3, rotation about the shortest axis of the ellipsoid (21a). The sphere now lies inside the ellipsoid and is hence tangent to it from the *inside*. A small jolt will again cause the point of tangency to transform into a neighboring curve; again the original rotation is stable.

Case 2. Rotation about the intermediate axis. The sphere *intersects* the ellipsoid in a curve of the fourth degree; its singular point B (foremost point of Fig. 46b) represents the original rotation. If the top is given a small impulse, the curve of intersection splits into two branches. The axis of rotation wanders off along one of these branches and moves further and further from its initial position in the body. The rotation is unstable.

It is instructive to prove this analytically; one proceeds from the differential equations (4). One can show (problem IV.2) that the lateral components generated by a small perturbation of the original rotation satisfy two simultaneous differential equations of first order. These have solutions of trigonometric character in cases 1 and 3, exponential character in case 2 (method of infinitesimal oscillations as stability criterion).

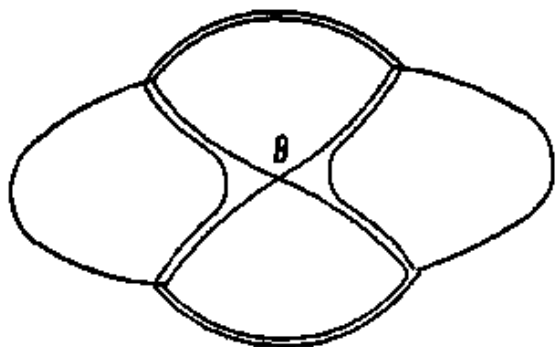


FIG. 46b. Unstable rotation of unsymmetrical top about the intermediate axis of the momental ellipsoid.

Let us perform the following experiment with a (full) matchbox: we hold the box between thumb and forefinger at opposite ends of its shortest edge and flip it into the air, thus imparting to it considerable angular momentum about this shortest edge. We notice that if the box originally shows its label, it will continue to do so throughout the motion. The same phenomenon occurs, though less clearly, if we hold the box at opposite ends of its longest edge, and flip it as before. If, instead, we hold it at opposite ends of the intermediate edge, with the striking surface showing, and repeat the procedure, we shall not see this surface throughout the motion, but rather a distinct change of colors.

Another striking example of instability of a state of motion is the following: occasionally one finds smooth-worn, flat pebbles in nature which, if spun about their vertical axis on a flat support, show stability of motion only for one sense of rotation; if made to spin in the opposite sense, they will start to wobble more and more violently, and finally end up by spinning in the stable direction opposite to their original angular momentum. The same can often be observed with small pocketknives (penknives) set to stand edgewise with blade folded in, when one gives them a gentle impulse.

We can perform a geometrically well-defined, instructive experiment in this connection. Let us take the wooden model of a non-degenerate flat ellipsoid of principal axes a , b , c (a and b much larger than c) and equip it with a heavy metal strip which, in its original position, hugs the upper surface of the ellipsoid in its (ac) -section. The strip can be rotated about the short c -axis, but is clamped down during each experiment. In the position ac the strip does not disturb the symmetry of the mass distribution. Both senses of spin about c are therefore equally stable. Let us now turn the strip by a small angle from this position. The two principal axes of inertia a and b are then each displaced by a small angle γ ; the symmetry of the lower surface facing the plane support is determined by the two principal radii of curvature in the planes ac and bc ; thus the symmetry of this surface remains unchanged. The direction of spin in the sense of the acute angle γ is now geometrically "distinguishable" from that in the opposite sense. Indeed the former is stable, the latter unstable since it is accompanied by rolling motions which increase with time.

A more elegant, though less easily achievable, form of the experiment is the following (G. T. Walker demonstrated it to us at Trinity College, Cambridge, in 1899): the non-degenerate ellipsoid is made of brass sheet; a certain circular region about the point of support has been stamped out and can be moved with respect to the remaining ellipsoidal shell. By a small angular displacement of this circular plug the curvature relations of the lower surface near the point of support are altered with respect to the

inertial distribution of the shell, which remains sensibly unchanged. This alteration is so slight that it passes unnoticed when the ellipsoid is examined. Nevertheless one sense of spin is again preferred to the other.

These experiments with the non-degenerate ellipsoid, while enlightening in themselves, also furnish an adequate substitute for the analytical theory of the phenomenon. Such a theory would have to investigate the rolling oscillations which might accompany the spin in one direction or the other when a small perturbation is superposed on this spin; it would show that the characteristic equation for the frequency of these oscillations has only real roots in the one case, some complex roots in the other. In the first case one would decide that the spin was stable, in the second, that it was unstable, i.e., subject to secular increase of the perturbation. Equations for this treatment are set up in the treatise of Routh (Advanced Part, Art. 241 and ff.) cited in § 42.

§ 27. *Demonstration Experiments Illustrating the Theory of the Spinning Top; Practical Applications*

We begin by describing the well-known device known as *Cardan's suspension*, which affords an unusually effective means of demonstrating the properties of tops and gyroscopes.

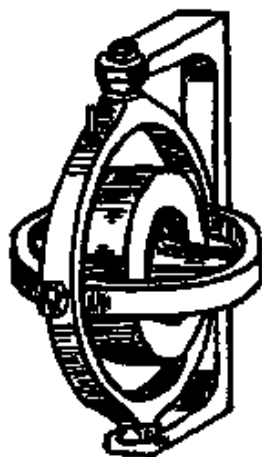


FIG. 47. Gyroscope in Cardan's suspension. Axis of rotation of outer ring = vertical, axis of rotation of inner ring = horizontal perpendicular to paper, axis of rotation of gyroscope = horizontal in plane of paper.

The suspension consists of an outer and an inner ring. The outer ring has a vertical axle borne by the outer frame or cage; the inner ring has an horizontal axle with bearings in the outer ring. The flywheel-shaped top revolves with its axis perpendicular to the axis of rotation of the inner ring. Fig. 47 shows the flywheel axle pointing normal to that of the outer ring, which causes the inner ring to lie in a horizontal plane. We shall designate this arrangement of the apparatus as its normal position.

On the axle of the flywheel provision is made for a means by which angular momentum can be imparted to the wheel while in its normal position, with the gimbals at rest. This angular momentum must be so great that all phenomena are essentially dominated by it and the effect of the mass of the gimbals becomes negligible.

In the following experiments a considerable angular momentum and the initial normal position are presupposed.

1. We exert a slight pressure downward on the *inner ring*. This ring does not give way; instead it is the *outer ring* that turns. Thus the axis of the flywheel moves backward or forward in an horizontal plane, depending on the position of the point at which the pressure is exerted. Instead of pressing on the inner ring we can load it unilaterally by means of a small weight. As long as the angular momentum remains sufficiently great, the top then describes a *regular precession* with horizontal axis.

2. We press on the *outer ring*. It remains motionless, whereas the *inner ring* turns upward or downward from its horizontal position depending on the sense in which pressure is exerted on the outer one. We can even deliver a vigorous blow to the outer ring without its yielding noticeably. All one perceives in that case is a rapid conical oscillation of the axis of the top about an axis close to that of the normal position.

3. If the pressure on the outer ring continues so that, with continual rotation of the inner ring, the axis of the top approaches the vertical, we notice that the resistance of the outer ring weakens more and more. One can then without effort set the outer ring spinning rapidly, but only in that sense which corresponds to the direction of the pressure originally exerted on the ring. If one attempts to rotate the outer ring in the opposite sense, the flywheel "rebels"; its axis suddenly tends in the opposite direction, thus causing the inner ring to flip through an angle of 180° . Now we can turn the outer ring without effort in this opposite direction, but another flipping of the top occurs if we return to the original sense of rotation.

4. This is the *tendency of the spins to align parallel to each other* which was emphasized by Foucault. The axis of the top is stable in the vertical position as long as its spin is *homologous* (=in the same sense) to that of the outer ring. If the spins are anti-parallel, this position is, on the contrary, unstable to a high degree and the axis comes to rest only when the opposite direction has been attained; in this latter direction homologous parallelism of the two spin axes prevails again. If we exert pressure on alternate sides of the outer ring in the proper rhythm, we can cause the top to revolve continually about the axis of the inner ring.

5. If we tie the inner ring to the outer one, so that the movability of the inner ring is destroyed, the resistance of the top to motion is destroyed as well. Seemingly without a will of its own, the top then obeys all pressures exerted on the outer ring, just as if it did not have any spin. Thus, typical gyroscopic effects occur only in the case of the top with three degrees of freedom, and are completely absent in that of two degrees. One can, however, reconstitute the missing degree of freedom by clamping the top to the rotating surface of the turning stool described on p. 74; this must be done in such a way that the axis of the outer ring, which has so far been

kept vertical, is tilted with respect to the axis of the stool (which remains vertical) at not too small an angle. Then the axis of the top with two degrees of freedom tends to align itself with the axis of the rotating support, just as a compass needle turns towards the North Pole, i.e., in the sense of the homologous parallelism described above. Thus the single ring containing the top will come to lie in a vertical plane, with one or the other of the axle pins of the top uppermost, depending on the sense in which the stool is rotated.

The explanation of all these phenomena is contained in the fundamental principle (25.5),

$$(1) \qquad dM = L dt.$$

1. If we press on the inner ring, L is horizontal and coincides with the axis of rotation of the inner ring. The angular momentum M is directed toward the left or right of Fig. 47 and is hence deflected laterally by L . If then we are allowed to assume that the axis of the top, originally in coincidence with the angular momentum, tends to remain in coincidence by following it, we have explained the lateral deflection of the axis of figure, that is, the rotation of the outer ring. That the assumption made here is actually valid for sufficiently rapid spin of the top will be justified in § 35 (cf. the discussion about the pseudo-regular precession in that section).

2. If we exert pressure on the outer ring, L is directed vertically. The angular momentum, originally directed horizontally to the right or left, is deflected upward or downward. Under the same assumption as in 1, we therefore obtain a rotation of the inner ring. If we impart a very strong blow to the outer ring, our assumption regarding the coincidence of angular momentum and axis of the top is only approximately satisfied; we then obtain the small conical oscillations mentioned earlier, which betray a small dislocation of the two axes.

3 and 4. By the same token we see that if the axis of angular momentum is almost vertical and if we rotate the outer ring in a sense homologous to that of the spin of the top, the axis of angular momentum becomes more nearly vertical. Gimbals and flywheel then rotate as a whole about the vertical. The resistance of the outer ring vanishes. If we rotate the outer ring in the non-homologous or anti-parallel sense, a small deviation of the axis of angular momentum from the vertical suffices to make the former recede further and further from the vertical; the almost-vertical position of the top proves to be unstable with respect to such a non-homologous rotation.

5. If we tie inner and outer ring together, the axis of angular momentum can no longer move in a vertical plane when a vertical torque L is imposed thereon by a rotation of the outer wheel. The torque is therefore transmitted to the whole system. This is possible because the horizontal change in direction that the vector M suffers can be compensated by the bearings of the outer ring, since inner and outer ring are now rigidly connected. Not so on the turning stool, where the angular momentum can follow the imposed L at least to some extent, which explains why the axis of the top tends to point in the direction of the axis of the stool.

We shall now discuss some practical applications. Let it be remarked in advance that details on many points of the discussion can be found in the older literature from which much of the following is borrowed.

(1) The Gyrostabilizer and Related Topics

Around the year 1870 Henry Bessemer, whose name is renowned in metallurgy, built a drawing room cabin destined for navigation on the English Channel. The cabin was suspended so that it could move about a fore-aft axis of the ship and was to be stabilized against the ship's roll by means of a flywheel. The axis of the flywheel was, however, rigidly fixed in the cabin, and therefore lacked the required third degree of freedom (cf. above under 5). As a result the construction was a failure soon to be abandoned.

It was O. Schlick, mentioned in connection with the mass balancing of piston engines (cf. p. 76), who successfully worked out the present problem. His method was applied to several steamers, including the "Silvana" of the Hamburg-America Line, and the Italian "Conte di Savoia" (considerable literature on the latter exists in American publications). In the "Silvana" the flywheel had a weight of 5,100 kg., a diameter of 1.6 m., and made 1,800 r.p.m. (a peripheral velocity of 150 m. per sec.). It was fixed in a cage which could, like a pendulum, swing about an axis in the port-starboard direction, so that the axis of symmetry of the flywheel oscillated in the vertical fore-aft plane of the ship. This cage corresponds to the inner ring of our demonstration top, the ship's hull itself to the outer one. The vertical of Fig. 47 is replaced by the long axis of the ship; instead of the former rotations about the vertical there is now the rolling of the vessel. The required three degrees of freedom then consist in the rolling of the ship, the oscillations of the cage, and the spin of the flywheel. When the vessel rolls, the axis of the flywheel, vertical in its normal position, alternately swings fore and aft in its cage, so that the energy contained in the rolling is converted to energy of motion and position of the cage. The rolling of the ship and the swinging of the cage are now coupled to each

other; if, in particular, their corresponding proper oscillations are in resonance, conditions resembling those of coupled pendulums obtain. To be sure, no damping of the ship's oscillation has so far been achieved. But it is now possible to absorb the oscillation energy of the cage and thus the energy of roll of the vessel by a braking device acting at the axle of the cage, just as the velocity of a car is reduced by a brakeshoe tangent to the wheel. Of course the braking action at the cage must not be so strong as to prevent the deflection of the flywheel axis altogether; for then we should again be confronted with the ineffective top of two degrees of freedom. Graphs of the rolling motion, similar to seismograms in an earthquake, show that there exists an optimum or "best compromise" value of braking action; in the "Silvana" the amplitude of roll was reduced to $\frac{1}{10}$ to $\frac{1}{20}$ of its original value almost as soon as the flywheel was put in action; the amplitude of oscillation of the frame hovered around 30° to 40° under these circumstances.

Nevertheless the gyrostabilizer has not been applied extensively. This is partly due to the danger inherent in the construction — a rapidly rotating, massive flywheel is an unpleasant passenger —, partly to the invention of an even more successful competitor, the Frahm stabilization tank, a device based on an entirely different principle.

A problem connected with the foregoing is that of stabilizing by gyroscopic action a turntable on board a ship. We do not know to what extent this problem has been solved for practical use; for obvious reasons work on it has been going on in all countries.

(2) The Gyrocompass

This is the finest and most nearly perfect gyroscopic device. Its conception goes back to Foucault. After Foucault had demonstrated the rotation of the earth by means of his pendulum experiments (cf. Ch. V, § 31), he made plans to achieve the same end by means of spinning tops. Of his several attempts we mention only the gyrocompass which was to replace the magnetic compass. The Foucault gyrocompass consists of a spinning top of two degrees of freedom constrained to the horizontal plane, which points, not to the magnetic North Pole, but to the actual celestial North Pole, the axis of rotation of the earth. Actually we dealt with this arrangement already in the fifth of our demonstration experiments, where we put the top with fixed inner ring on the turning stool. The rotating earth now takes the place of the turning platform of the stool. The only difference between the two cases lies in the fact that we were able to impart an arbitrarily large angular velocity to the rotating platform, resulting in a very strong orientation effect on the top, whereas the angular velocity of

the earth is very small, so that the alignment of the Foucault gyroscope takes a considerable time. In the earlier arrangement we mentioned that the angle between the axes of rotation of the outer ring and the stool should not be too small. In the present case this angle is the complement of the geographic latitude, the "co-latitude" at the point of observation. At the two poles of the earth, where this angle is zero, the orientation power of the gyrocompass vanishes. In general it is proportional to the angular velocity of the earth, the angular momentum of the top and the sine of the co-latitude.

Foucault's experiments lead only to rough indications of the effect. Its full realization was achieved by Hermann Anschütz-Kaempfe, by means of successive improvements in construction. His original goal was to reach the North Pole by means of a submarine passing under the drift ice. Since the readings of a magnetic compass become very unreliable near the North Pole, failing altogether inside a submarine, he had the idea of making the top serve as his direction-finder. It is true that in the pursuit of this idea through several decades he did not reach the North Pole; but his experimentation lead to an ideal instrument which has become indispensable in navigation.

The Anschütz gyroscope, unlike that of Foucault, is not constrained to a horizontal plane, but is merely pulled back into this plane by its weight, like a pendulum. Originally it was arranged so as to swim in a bath of mercury. Later constructions made use of two or three tops whose effects strengthened and corrected each other. The angular momentum of the spinning tops is kept constant by electric drive. In the latest Anschütz construction the whole system is enclosed in a sphere which floats with almost no friction in a second sphere of only slightly larger radius. Since the gyroscope is taken along on trips during which it may not be touched for several months, provision must be made for a particularly ingenious automatic lubrication method.

Measures to eliminate the harmful effects of the ship's own motion are of special importance. When the ship travels in a curve or changes its speed, the gyrocompass, with its ability to oscillate about the horizontal plane, is sensitive to the corresponding inertial forces. These exert pressures on the axis of spin, causing it to deflect from its undisturbed position, with the result that erroneous readings are obtained. One can show that the motion of the vessel becomes harmless if the free oscillation of the compass needle about the meridian has the period

$$T = 2\pi \left(\frac{I}{g} \right)^{\frac{1}{2}} = (8\pi)^{\frac{1}{2}} 10^3 \text{ sec} = 84.4 \text{ min.}$$

which is the same as that of a pendulum of length equal to the earth's radius

$$l = \frac{2}{\pi} \cdot 10^7 \text{ m.}$$

(Law of Schuler, completed by Glitscher⁷).

A further beautiful application of the gyroscope concerns the automatic steering mechanism of large steamers. If a ship is to retain its course in spite of the motion of waves and ocean currents, the uninterrupted attention of the helmsman and the corresponding corrective action of the steering mechanism are required. This corrective action is, however, always too late by a certain amount of time, therefore causing losses in mileage and time. The gyrocompass is, on the contrary, a sense organ which "feels" much more accurately and swiftly than man, and takes instantaneous countermeasures. As a result of these countermeasures the line of travel becomes almost rigorously rectilinear (actually loxodromic, i.e., a rhumb line), which results in a considerable saving in energy. For this reason every passenger ship of good size is now equipped with such an automatic steering mechanism.

(3) Gyroscopic Effects in Railroad Wheels and Bicycles

A set of rolling wheels of a railroad car is a spinning top whose angular momentum can become considerable for fast trains. When the wheels go around a curve, the angular momentum must, at any instant, be deflected to a position determined by the normal to the curve. For this, according to Eq. (1), a torque is required whose axis lies along the direction of travel. Since such a torque (often called "gyroscopic couple") is not present, the "gyroscopic effect" will result in a countertorque which presses the set of wheels against the outer rail and pulls it off the inner one. This countertorque adds to the moment of the centrifugal force about the direction of travel. The latter effect is compensated, as we know, by adequate banking of the roadbed. Both moments have the form

$$mv\omega$$

where v is the velocity of travel, and ω the angular velocity of the train in the curve; m is, in the present case, the mass of the set of wheels reduced to the wheel periphery, whereas in the centrifugal effect m is the total mass of the car carried by the wheels. Our gyroscopic couple and its equal and opposite countertorque are therefore extremely small compared with the moment of the centrifugal force; one could compensate for it by lifting the outer rail a very slight additional amount.

⁷ Cf. *Wissensch. Veröffentl. aus den Siemenswerken*, 19, 57 (1940).

More serious effects may result from any vertical irregularities in the rails, such as, for instance, a "hump" on one of the rails (to this category also belong the increasing and decreasing elevation of one of the rails at the beginning and end of a banked curve). Such a hump causes a deviation of the angular momentum in a vertical direction, and hence a countertorque which seeks to twist the set of wheels out of the rail-bed by pressing, say, the front wheel of the set against the rail, and pushing the last wheel of the set away from the rail. The play allowed by the rails will thus cause the flanges of the wheels to bite now into one rail, now into the other. This has indeed been observed on test runs with fast electric trains. In order to control the condition and exact position of the rails at all times, the German Reichsbahn uses test cars equipped with gyroscopic instruments, the latter manufactured by the Anschütz company.

A bicycle is a doubly non-holonomic system; for, like the wheel in problem II.1, it has five degrees of freedom in finite motion, but only three such degrees in infinitesimal motion (rotation of the rear wheel in its instantaneous plane, to which the rotation of the front wheel is coupled by the condition of pure rolling; rotation about the handle bar axis; and common rotation of front and rear wheel about the line connecting their points of contact with the ground), as long as we do not consider the degrees of freedom of the cyclist himself. It is well-known that given sufficient velocity the stability of this system relies on the fact that either by means of rotations of the handle bar, or by means of unconsciously released motions of the body, the cyclist calls forth suitable centrifugal effects. That the gyroscopic effects of the wheels are very small compared with these can be seen from the construction of the wheel; if one wanted to strengthen the gyroscopic effects, one should provide the wheels with heavy rims and tires instead of making them as light as possible. It can nevertheless be shown^a that these weak effects contribute their share to the stability of the system. This is the case because, just as in the automatic steering mechanism of ships, they react more quickly against a sinking of the center of gravity than do the centrifugal effects. In the small oscillations which one has to consider in testing the stability of the motion, the gyroscopic action lags the oscillations of the center of gravity by a quarter period, whereas the centrifugal action lags them by a half period.

^a Cf. F. Klein and A. Sommerfeld, *Theorie des Kreisels*, Vol. IV, p. 880 and ff. In order to carry out the stability considerations we must of course exclude all participating action on the part of the cyclist. Not only must he be assumed to ride without hands, but also with motionless body; he should act only by means of his weight. This work also offers detailed material on other applications and on the mathematical foundations of the theory of the spinning top.

SUPPLEMENT: THE MECHANICS OF BILLIARDS

The beautiful game of billiards opens up a rich field for applications of the dynamics of rigid bodies. One of the illustrious names in the history of mechanics, that of Coriolis*, is connected with it.

The following explanations have as their main object the clarification of some problems which we shall pose on the subject. In these problems not only the dynamics of the rolling and sliding ball, but also the theory of friction on the billiard cloth will come into its own.

(a) *High and Low Shots*

The experienced player almost always gives the ball a "side" or "English." For the time being we shall, however, consider only shots without English, in which the cue therefore hits the ball in its vertical median plane, and in a horizontal direction. We distinguish high and low shots.

We speak of a *high shot* if the point of impact between cue and ball lies above $\frac{7}{8}a$ (a =radius of the cue ball), as measured from the plane of the table; of a *low shot* if the ball is hit at a height less than $\frac{7}{8}a$ (cf. problem IV.3 in connection with this and the following). Only if the ball is hit at exactly this height does pure rolling take place from the very start. By virtue of the moment of inertia of a sphere given on p. 65, the rotation transmitted to the ball is then of such magnitude that the peripheral velocity corresponding to it is just equal and opposite at the point of support to the forward motion of the ball, so that the condition (II.10) of pure rolling is fulfilled.

For high shots the peripheral velocity at the point of contact generated by the rotation is opposite to that of the center of mass of the ball and exceeds the latter. The friction at the cloth opposes the excess velocity (peripheral velocity—forward velocity), thus augmenting the original velocity of the mass center: for high shots, friction acts on the ball in the direction of the shot. The final velocity under pure rolling, which sets in once the friction has consumed the excess velocity, is greater than the initial one. Balls that are hit high run for a long time and in general betray the experienced player.

For low shots the peripheral velocity at the point of contact is opposite to that of the center of mass, but outweighed by it; for even lower shots it is directed forward. In both cases friction acts in a direction opposite to that of the original impact. The final velocity under pure rolling is smaller than the initial one.

* G. Coriolis, *Théorie mathématique des effets du jeu de billard*. Paris, 1835.

As for the *impulse* Z (dimensions dyne-sec), it is of course to be interpreted as the time integral of a very great force F in the direction of the cue over the very short time of duration τ ,

$$Z = \int_0^{\tau} F dt.$$

The *impulsive torque* about the center of the ball is accordingly given by

$$Zl = \int_0^{\tau} Fl dt$$

where l is the distance of the center from the axis of the cue. The impulsive torque vector is directed perpendicularly to the plane passing through center and cue axis. For the shots without English so far considered, it is directed horizontally and is normal to the median plane mentioned above.

(b) *Follow Shots and Draw Shots*

If the ball, after being struck high, meets one of the other two balls in central impact, it transfers all its forward motion to the latter because of the equality of the two masses involved [cf. Eq. (3.27a)]; but it retains its rotational motion if we neglect the friction between the two balls during the short time of contact. The instant after the impact the center of the striking ball is therefore momentarily at rest, while its lowest point glides over the billiard cloth. The friction thus arising is constant in time and acts on the ball in the sense of the original forward motion, while its moment about the center simultaneously slows down the existing rotation. Thus the ball is accelerated from the state of rest, while its rotation decreases accordingly. The acceleration ceases as soon as the peripheral velocity at the cloth has become equal to the forward velocity of the center, whereupon pure rolling sets in. Once this stage is reached, the ball rolls on with constant final velocity (we shall neglect the very slow effect of the rolling friction). This is the theory of the *follow shot*.

The ball which is hit low similarly transfers its center of mass velocity to the struck ball and is momentarily at rest. We shall assume that the ball was hit very low, at any rate below the center, so that the peripheral velocity at the point of contact remaining after collision is directed forward. The friction now acts backward. The ball begins to move with constant backward acceleration, while at the same time its rotational velocity decreases, until pure rolling sets in. This is the theory of the *draw shot*.

Since sliding friction is independent of the velocity, the variation with time of the center of mass velocity v as well as of the peripheral velocity $u = a\omega$ is a linear one. The exercises so far considered can therefore be treated more conveniently by graphical than by mathematical methods.

To do the former we may construct a diagram in which we plot the instantaneous values of v and u as ordinates against the time (problem IV.3).

(c) *Trajectories with "English" Under Horizontal Impact*

If the ball is not hit in the vertical median plane, but to either side of it, we speak of "*right English*" and "*left English*." As long as the cue is advanced horizontally against the ball, the trajectory remains a straight line in the direction of the initial impact.

The plane of the impulsive torque is now inclined to the vertical median plane, in high shots either to the right for right English, or to the left for left English, this inclination being such that the normal to the plane of the impulsive torque (this normal is parallel to the axial vector torque) is contained in the vertical plane through the center of the ball normal to the median plane. We can decompose the torque into a vertical component and a horizontal one at right angles to the direction of the impact. The first component causes a spin about the vertical diameter of the ball and generates a small "boring friction" at the cloth which has, however, no effect on the path of the ball. The lateral component on the other hand acts in the same way that it did in the shots considered under 1 and 2, so that the phenomena there observed apply without change to shots with English. In particular the trajectory remains rectilinear.

The spin about the vertical diameter makes itself felt in the collision of the ball with a cushion or with a second ball. In the first case friction at the cushion occurs which deviates the ball to the left for right English and to the right for left English as seen by the player. The angle of reflection, which, for shots without English, is equal to the angle of incidence, is thereby altered; as a matter of fact the actual reflected path is generated from the equiangular reflected path by a rotation of the latter in the sense of the vertical spin imparted to the ball. This phenomenon is familiar to every billiards player. Together with the frictional force at the cushion there appears a frictional torque about the vertical which weakens the spin about the vertical diameter. The original English therefore gradually disappears after several impacts, a fact which is likewise known to every player. In a collision of ball against ball the effect of the English is similar, acting in the same sense as in a ball-cushion impact.

(d) *Parabolic Path Due to Shot with Vertical Component*

The plane of the impulsive torque is now not only inclined as under (c) but also tilted forward as seen by the player. The vector torque has therefore not only components along the vertical and lateral directions, but also a component in the direction of motion. Thus the point of contact has a component of sliding velocity perpendicular to the initial motion.

The friction, which is opposed to the resultant velocity of the point of contact, therefore makes an angle different from zero with the initial motion. If we convince ourselves (cf. problem IV.4) that this angle formed with the original motion remains constant during the motion, and if we remember that the magnitude of the friction likewise remains constant, we conclude that the path of the ball is a parabola in the horizontal plane, since it is under the influence of a single force of constant magnitude and direction (principle of J. A. Euler, son of the great Leonhard).

Shots of this type are very surprising to a player who does not have full knowledge of the laws of friction and the vectorial decomposition of angular momentum. They are especially useful when the two balls to be hit are at the two opposite ends of the short side of the table. In that case the vertical component of the impulse must be very strong, i.e., the cue must be guided at a small angle to the vertical.

CHAPTER V

RELATIVE MOTION

The interest in the subject matter of this chapter derives mainly from the fact that we make all our observations on the rotating earth, which is not an allowable frame of reference, either in the sense of classical mechanics or in the sense of the special theory of relativity. In general relativity, on the other hand, all systems of reference are permitted (cf. p. 15), so that a separate theory of relative motion becomes meaningless.

In this chapter we shall adopt the viewpoint that in every theoretically admitted reference system the mechanics of Newton holds rigorously. We shall then ask for the deviations from Newtonian mechanics that result from the motion of the reference system to which, for practical reasons, we are chained.

§ 28. *Derivation of the Coriolis Force in a Special Case*

Let a mass point move along a meridian of the terrestrial globe, of radius a , with the constant angular velocity μ , while at the same time the earth rotates about its axis with constant angular velocity ω . As usual, we call θ the colatitude, ϕ the (celestial !) longitude. Apart from arbitrary initial values the motion of our mass point is then given by

$$(1) \quad \theta = \mu t, \quad \phi = \omega t.$$

From the Cartesian coordinates of the point,

$$(2) \quad \begin{aligned} x &= a \sin \theta \cos \phi \\ y &= a \sin \theta \sin \phi \\ z &= a \cos \theta, \end{aligned}$$

we obtain by differentiation with respect to t ,

$$(3) \quad \begin{aligned} \dot{x} &= a\mu \cos \theta \cos \phi - a\omega \sin \theta \sin \phi \\ \dot{y} &= a\mu \cos \theta \sin \phi + a\omega \sin \theta \cos \phi \\ \dot{z} &= -a\mu \sin \theta. \end{aligned}$$

$$(4) \quad \begin{aligned} \ddot{x} &= -a\mu^2 \sin \theta \cos \phi - a\omega^2 \sin \theta \cos \phi - 2a\mu\omega \cos \theta \sin \phi \\ \ddot{y} &= -a\mu^2 \sin \theta \sin \phi - a\omega^2 \sin \theta \sin \phi + 2a\mu\omega \cos \theta \cos \phi \\ \ddot{z} &= -a\mu^2 \cos \theta. \end{aligned}$$

In the triplet of equations (4) the first terms on the right represent the usual centripetal acceleration which is associated with the motion along the meridian if the latter is at rest in space. The second terms give the centripetal acceleration resulting from the motion of a fixed point of the meridian in a circle of latitude (due to the earth's rotation about its axis). The third terms, however, constitute something new, for they represent the kinematic interplay of both motions. If we multiply (4) by $-m$, we obtain the inertial force F^* of our mass point in the compound rotation. In vector form it is

$$(5) \quad F^* = C_1 + C_2 + F_c.$$

The symbols C_1 and C_2 refer, as in (10.3), to "ordinary centrifugal forces." C_1 is directed radially outward from the earth's center and has the magnitude

$$|C_1| = m a \mu^2 = m \frac{v_1^2}{a}, \quad v_1 = a \mu.$$

C_2 is directed outward normal to the earth's axis, and has the magnitude

$$|C_2| = m a \omega^2 \sin \theta = m \frac{v_2^2}{a \sin \theta}, \quad v_2 = a \omega \sin \theta.$$

We can call the third constituent F_c the "composite centrifugal force" (force centrifuge composée) or *Coriolis force*. Its complete vector expression [cf. Eq. (29.4a)] is given by

$$(6) \quad F_c = 2m \mathbf{v}_{\text{rel}} \times \boldsymbol{\omega}.$$

We have here written \mathbf{v}_{rel} instead of the vector \mathbf{v}_1 corresponding to the preceding v_1 ; by this we wish to indicate that quite generally it is the velocity relative to the rotating reference system that gives rise to F_c .

According to (6) the magnitude of F_c is

$$(6a) \quad |F_c| = 2m v_{\text{rel}} \omega \sin(\mathbf{v}_{\text{rel}}, \boldsymbol{\omega}),$$

so that, in our case,

$$(6b) \quad |F_c| = 2m v_{\text{rel}} \omega \cos \theta.$$

$\cos \theta$ is of course just the sine of the geographic latitude. As for direction, F_c is perpendicular to both \mathbf{v}_{rel} and $\boldsymbol{\omega}$, or, equivalently, to C_1 and C_2 . The sense of F_c is given by the direction of advance of a right-handed screw turning from \mathbf{v}_{rel} to $\boldsymbol{\omega}$. This is illustrated in Fig. 48 for a particle moving from south to north. Two positions, one in the southern and one in the northern hemisphere, are shown. In the former, corresponding to the sense of the right-handed screw $\mathbf{v}_{\text{rel}} \rightarrow \boldsymbol{\omega}$, F_c acts from east to west; in the latter, from west to east.

Instead of a single particle we can also consider a continuous sequence of such particles, hence a river flowing along the meridian. Fig. 48 then tells us that the inertial force of the water moving from south to north presses against the *right bank in the northern hemisphere, against the left bank in the southern hemisphere*. The change of sign in the pressure is evidently connected with the sine of the geographic latitude occurring in (6b). This rule is valid not only for south-north flow, but, as will be shown in the next section, for any direction of \mathbf{v}_{rel} , and therefore, in particular, also for the north-south direction of flow. This is intuitively obvious in our example. The west-east velocity of the water deriving from the earth's rotation depends on its distance from the axis of rotation, hence on the geographic latitude. If the stream moves from south to north, the water in the northern hemisphere has an excess of west-east momentum imported from more southerly latitudes; this excess manifests itself as a pressure eastward, that is, against the right bank. But similar reasoning must hold in the case of north-south motion. In that case the water imports a deficiency of west-east motion from the northern latitudes.

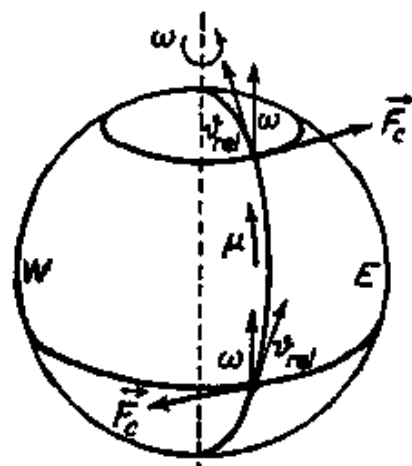


FIG. 48. Special derivation of Coriolis force: a mass point moves along a meridian of the rotating earth with constant velocity \mathbf{v}_{rel} corresponding to the constant angular velocity μ as seen from the earth's center.

Let us mentally add the deficient amount in the sense of Fig. 41, once with + sign, once with - sign. The part added with - sign has an east-west direction, and therefore exerts a pressure westward, i.e., again on the right bank. The same process of reasoning shows that in the southern hemisphere the river exerts excess pressure on its left bank, for south-north as well as north-south motion of the water.

Geographers have proved by numerous examples that the pressure against the right bank in the northern hemisphere manifests itself in a stronger erosion of the right embankment (Baer law of river displacements); in addition the water stands slightly but measurably higher at the right shore of the river.

Of much greater significance are the effects of the Coriolis force on ocean currents (deviation of the Gulf Stream and tidal currents of the northern hemisphere to the right).

It is, however, in the atmosphere that its effects are most pronounced. The well-known law of Buys-Ballot states that the wind does not blow in the direction of the pressure gradient, but is deviated considerably,

to the right in the northern, to the left in the southern hemisphere; it is only at the equator that it follows the pressure gradient exactly.

All these phenomena are immediate results of Newton's first law and in the last analysis derive from the fact that in mechanics the rotating earth is not an admissible reference frame.

In this section we have calculated the Coriolis force with the help of spherical polar coordinates. In problem V.1 we shall derive it in cylindrical coordinates.

§ 29. *The General Differential Equations of Relative Motion*

We replace the earth by an arbitrary rigid body B , which rotates with instantaneous angular velocity ω about a fixed point O . Let P be a particle which moves with arbitrarily varying velocity relative to B . Its velocity with respect to space is then composed of this relative velocity and the velocity in space of a point of the body instantaneously in coincidence with P . According to (22.4) the latter is given by $\omega \times \mathbf{r}$. As in (22.4) we shall designate by \mathbf{w} the velocity of P with respect to space; furthermore we shall call \mathbf{v} (instead of \mathbf{v}_{rel}) the relative velocity of P with respect to B . We then have

$$(1) \quad \mathbf{w} = \mathbf{v} + \omega \times \mathbf{r}.$$

Let us agree that temporal changes be designated by an overhead dot if observed from space, by $\frac{d}{dt}$ if observed from the body B . We can then write

$$(2a) \quad \mathbf{w} = \dot{\mathbf{r}},$$

$$(2b) \quad \mathbf{v} = \frac{d\mathbf{r}}{dt},$$

$$(2c) \quad \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} + \omega \times \mathbf{r}.$$

The acceleration of our point P in space is given from (1) as

$$(3) \quad \dot{\mathbf{w}} = \dot{\mathbf{v}} + \omega \times \dot{\mathbf{r}} + \dot{\omega} \times \mathbf{r}.$$

In the middle term of the right member we substitute the value of $\dot{\mathbf{r}}$ from (2a) and (1) to obtain

$$(3a) \quad \omega \times \dot{\mathbf{r}} = \omega \times \mathbf{v} + \omega \times (\omega \times \mathbf{r}).$$

We shall transform the first term on the right of (3) by replacing the arbitrary vector \mathbf{r} in (2c) by \mathbf{v} . This yields

$$(3b) \quad \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} + \boldsymbol{\omega} \times \mathbf{v}.$$

Substitution of (3a) and (3b) into (3) gives

$$(4) \quad \dot{\mathbf{w}} = \frac{d\mathbf{v}}{dt} + 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}.$$

We notice that according to (26.8a) we can write either $\dot{\boldsymbol{\omega}}$ or $\frac{d\boldsymbol{\omega}}{dt}$ in the last term of Eq. (4).

From (4) we proceed to the inertial force acting on our particle by multiplying both sides by $-m$. At the left we then have the inertial force \mathbf{F}^* in space; the first term on the right is the inertial force observed in the non-inertial reference system B , which we shall call $\mathbf{F}_{\text{rel}}^*$. The second term on the right gives the expression for the Coriolis force which we met in (28.6), viz.,

$$(4a) \quad -2m \boldsymbol{\omega} \times \mathbf{v} = +2m \mathbf{v} \times \boldsymbol{\omega} = \mathbf{F}_c.$$

Our present treatment therefore supplements that of the preceding section by furnishing a general derivation of the Coriolis force. In the next to last term of Eq. (4) one easily recognizes (after multiplication by $-m$) the ordinary centrifugal force \mathbf{C} , which appears to act on our particle by virtue of the rotation of the reference system B , and which was designated by \mathbf{C}_2 in Eq. (28.5).

From (4) we therefore have, collecting all the terms,

$$(5) \quad \mathbf{F}^* = \mathbf{F}_{\text{rel}}^* + \mathbf{C} + \mathbf{F}_c + m\mathbf{r} \times \dot{\boldsymbol{\omega}}.$$

Here we replace $\mathbf{F}_{\text{rel}}^*$ by its value from the definition

$$\mathbf{F}_{\text{rel}}^* = -m \frac{d\mathbf{v}}{dt}$$

and recall that due to the equilibrium of external and inertial forces in the system fixed in space we must have

$$\mathbf{F} + \mathbf{F}^* = 0.$$

Thus we obtain the *general differential equation of relative motion*

$$(6) \quad m \frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{C} + \mathbf{F}_c + m\mathbf{r} \times \dot{\boldsymbol{\omega}}.$$

We see that in the system B there appear, in addition to the actual external force, the *fictitious forces* \mathbf{C} and \mathbf{F}_c ; from the standpoint of an observer moving with B , they act in the same manner as the external

force \mathbf{F} ; actually, they result solely from the inertia of the particle m fixed in, or moving relative to, a non-Newtonian reference frame. The last term on the right of (6) is of similar origin; it stems from a possible acceleration or change in direction of the rotation. Applied to the earth it corresponds to the polar fluctuations and can certainly be neglected as vanishingly small. The differential equation (6) will be used in the three following sections and in problems V.1 and V.2.

§ 30. Free Fall on the Rotating Earth; Nature of the Gyroscopic Terms

Whenever we try to measure the effect of gravity, it is not just the gravitational attraction itself, but the resultant of the earth's attraction \mathbf{F} and the centrifugal force \mathbf{C} that is observed. The flattening of the geoid, i.e., of the mean terrestrial surface, is itself determined by this resultant, and, in fact, in such a manner that the geoid is everywhere normal to it. If we put

$$(1) \quad \mathbf{F} + \mathbf{C} = -m \mathbf{g},$$

the gravitational acceleration \mathbf{g} is a vector which has the magnitude g , but a direction along the normal to the geoid, rather than along the produced radius of the earth.

From (29.6) we obtain, in view of (1) and (28.6) and with neglect of the term in $\dot{\omega}$,

$$(2) \quad \frac{d\mathbf{v}}{dt} = -\mathbf{g} + 2\mathbf{v} \times \boldsymbol{\omega}.$$

Let us now resolve this vector equation into coordinate equations by introducing an orthogonal system ξ, η, ζ , fixed in the earth, and defined as follows (cf. Fig. 49):

- ξ = north-south direction on the earth,
 (3) η = west-east direction on the earth,
 ζ = point of observation \rightarrow zenith =
 normal to geoid.

We then have, in component form,

$$(4) \quad \begin{aligned} \mathbf{v} &= \left(\frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt} \right); \\ \mathbf{g} &= (0, 0, g); \\ \boldsymbol{\omega} &= (-\omega \cos \phi, 0, \omega \sin \phi); \end{aligned}$$

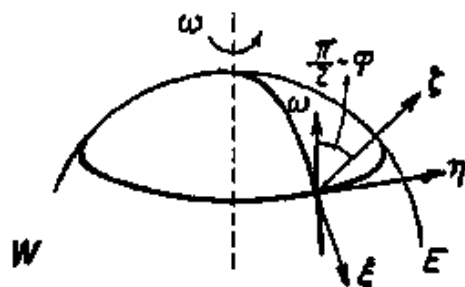


FIG. 49. Free fall on rotating earth. System of coordinates: ξ along a meridian, η along a circle of latitude, ζ along the normal to the geoid.

ϕ being the geographic latitude as in Fig. 49. It then follows from (2) that

$$\begin{aligned}
 \frac{d^2\xi}{dt^2} &= 2\omega \sin \phi \frac{d\eta}{dt} \\
 (5) \quad \frac{d^2\eta}{dt^2} &= -2\omega \sin \phi \frac{d\xi}{dt} - 2\omega \cos \phi \frac{d\zeta}{dt} \\
 \frac{d^2\zeta}{dt^2} + g &= 2\omega \cos \phi \frac{d\eta}{dt}
 \end{aligned}$$

Before proceeding to integrate (5) we wish to examine the general character of these equations. They are distinguished by the fact that the array of coefficients of the right-hand side is antisymmetric. Let us introduce abbreviations

$$(6) \quad \alpha = 2\omega \sin \phi, \quad \beta = 0, \quad \gamma = -2\omega \cos \phi.$$

The array then is clearly antisymmetric about the diagonal, as shown below:

$$(7) \quad \begin{array}{c|ccc} & \frac{d\xi}{dt} & \frac{d\eta}{dt} & \frac{d\zeta}{dt} \\ \hline \frac{d^2\xi}{dt^2} & 0 & \alpha & \beta \\ \hline \frac{d^2\eta}{dt^2} & -\alpha & 0 & \gamma \\ \hline g + \frac{d^2\zeta}{dt^2} & -\beta & -\gamma & 0 \end{array}$$

This *antisymmetric* character indicates *conservation of energy*. If diagonal terms were present or if, speaking more generally, the array of coefficients had a *symmetric part*, we should have *dissipation of energy*.

For let us multiply Eqs. (5) row by row by $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$, and add. All the coefficients of α , β and γ on the right vanish, and we are left with

$$\frac{1}{2} \frac{d}{dt} \left[\left(\frac{d\xi}{dt} \right)^2 + \left(\frac{d\eta}{dt} \right)^2 + \left(\frac{d\zeta}{dt} \right)^2 \right] + g \frac{d\zeta}{dt} = 0,$$

that is,

$$(8) \quad T + V = \text{const.}$$

Here T and V are the kinetic and potential energy of relative motion (where we have put the mass=1). This conservative character of our array of coefficients can be made evident even without calculation; for by virtue

of the factor $\mathbf{v} \times \boldsymbol{\omega}$, \mathbf{F}_g is perpendicular to the motion and therefore does no work, in analogy with magnetic forces in electrodynamics.

If, on the other hand, the array of coefficients had a symmetric contribution, we should have

$$(9) \quad \frac{d}{dt}(T+V) < 0$$

where the $<$ sign results from the assumption that the signs of the coefficients satisfy the physically necessary conditions corresponding to a damping of the motion. It is seen that (9) results not in conservation, but rather, as asserted, in dissipation of energy. An example (only one-dimensional, it must be admitted) of the dissipative character of an even array of coefficients is furnished by Eqs. (8) and (9) in the treatment of damped oscillations of Chapter II, § 19.

With Lord Kelvin we call the terms of an antisymmetric array of coefficients *gyroscopic terms*. The name suggests that they indicate an internal gyration of the system (in our case the earth) which has not been taken into account explicitly in setting up the problem, but has instead been incorporated in the choice of coordinates (in our case the ξ, η, ζ). Such gyroscopic terms play an important role in general laws concerning the stability of equilibria and motions.

We shall now proceed with the integration of Eqs. (5). Let us postulate a free fall from height h without initial velocity. We therefore require at $t=0$:

$$(10) \quad \begin{aligned} \xi = \eta = 0, \quad \zeta = h \\ \frac{d\xi}{dt} = \frac{d\eta}{dt} = \frac{d\zeta}{dt} = 0. \end{aligned}$$

From the first and third Eq. (5) we then have

$$(11) \quad \frac{d\xi}{dt} = 2\omega\eta \sin \phi, \quad \frac{d\zeta}{dt} + gt = 2\omega\eta \cos \phi.$$

Replacing these in the second Eq. (5), we obtain

$$(12) \quad \frac{d^2\eta}{dt^2} + 4\omega^2\eta = Ct, \quad C = 2\omega g \cos \phi.$$

The integral of this equation is found by the general rule laid down in connection with Eq. (19.4), viz., "particular solution of the inhomogeneous equation + general solution of the homogeneous equation." In the present case this leads to

$$\eta = \frac{C}{4\omega^2}t + A \sin 2\omega t + B \cos 2\omega t.$$

Conditions (10) require that we put

$$(13) \quad B=0, \quad 2\omega A = -\frac{C}{4\omega^2}, \text{ i.e.,} \\ \eta = \frac{C}{4\omega^2} \left(t - \frac{\sin 2\omega t}{2\omega} \right) = \frac{g \cos \phi}{2\omega} \left(t - \frac{\sin 2\omega t}{2\omega} \right).$$

According to the meaning of η , cf. (3), this is the *eastward deflection*.

ξ is the *southward deflection*. From (11) and (13) it satisfies

$$\frac{d\xi}{dt} = g \sin \phi \cos \phi \left(t - \frac{\sin 2\omega t}{2\omega} \right)$$

whose solution, with due regard for (10), is

$$(14) \quad \xi = g \sin \phi \cos \phi \left(\frac{t^2}{2} - \frac{1 - \cos 2\omega t}{4\omega^2} \right).$$

With the help of (13) and (10) we finally obtain from the second Eq. (11) the motion along the vertical,

$$(15) \quad \zeta = h - \frac{gt^2}{2} + g \cos^2 \phi \left(\frac{t^2}{2} - \frac{1 - \cos 2\omega t}{4\omega^2} \right).$$

ωt is a very small number of order of magnitude (time of fall) \div (1 day). We can therefore develop the solutions in powers of ωt . In lieu of (13), (14), and (15) we then obtain

$$\eta = \frac{gt^2}{3} \cos \phi \omega t, \quad \xi = \frac{gt^2}{8} \sin \phi \cos \phi (\omega t)^2 \\ \zeta = h - \frac{gt^2}{2} \left(1 - \frac{\cos^2 \phi}{3} (\omega t)^2 \right).$$

The eastward deflection is accordingly of the first, the southward deflection of the second order in ωt . The deviation from the law of freely falling bodies along the vertical caused by the earth's rotation is likewise only of the second order in ωt . The eastward deflection has been observed in several instances and found to be in agreement with theory; under favorable circumstances (deep mine shaft) it amounts to several centimeters.

Evidently these (observable or unobservable) deflections are due to the fact that the initial conditions (10) which lie at the very basis of both theory and experiment prescribe *rest with respect to the earth*. They hence imply a certain velocity in space, which is of the magnitude (earth's angular velocity) \cdot (distance from axis of earth). This velocity is somewhat different from the velocity with which the earth's surface moves away under the falling body. It is then clear that the body does not hit the earth at the exact vertical projection of its initial position,

§ 31. Foucault's Pendulum

Once more Eqs. (30.5) are in force, but with the added condition that the mass point have the constant distance l from the point of suspension of the pendulum. We write this condition in a form similar to that used for the spherical pendulum (18.1), i.e.,

$$(1) \quad F = \frac{m}{2}(\xi^2 + \eta^2 + \zeta^2 - l^2) = 0$$

and introduce the Lagrange multiplier associated with it. Eqs. (30.5) then read

$$\begin{aligned} \frac{d^2\xi}{dt^2} &= 2\omega \sin \phi \frac{d\eta}{dt} + \lambda \xi \\ (2) \quad \frac{d^2\eta}{dt^2} &= -2\omega \sin \phi \frac{d\xi}{dt} - 2\omega \cos \phi \frac{d\zeta}{dt} + \lambda \eta \\ \frac{d^2\zeta}{dt^2} + g &= 2\omega \cos \phi \frac{d\eta}{dt} + \lambda \zeta. \end{aligned}$$

We shall of course restrict ourselves to small oscillations. We therefore regard $\frac{\xi}{l}$ and $\frac{\eta}{l}$ as small quantities of the first order; from (1) it follows that $\frac{\zeta^2}{l^2} = 1$ up to quantities of the second order. More precisely, for points in the neighborhood of the rest position we can write

$$\zeta = -l(1 + \text{quantities of second order}),$$

since ζ is of course directed vertically upward. The third Eq. (12) then shows that up to quantities of the first order

$$(3) \quad g = -\lambda l, \text{ hence } \lambda = -\frac{g}{l}.$$

Once more we write down the first two Eqs. (2), neglecting the term in $\frac{d\zeta}{dt}$ because small of second order, and using the abbreviation

$$(4) \quad u = \omega \sin \phi,$$

to obtain

$$\begin{aligned} \frac{d^2\xi}{dt^2} - 2u \frac{d\eta}{dt} + \frac{g}{l}\xi &= 0 \\ (5) \quad \frac{d^2\eta}{dt^2} + 2u \frac{d\xi}{dt} + \frac{g}{l}\eta &= 0. \end{aligned}$$

It is convenient to consolidate them in complex form by multiplying the

second Eq. (5) by i , adding it to the first one, and, as on p. 142, Eq. (26.10), introducing the new variable

$$(6) \quad s = \xi + i\eta.$$

We obtain

$$(7) \quad \frac{d^2 s}{dt^2} + 2iu \frac{ds}{dt} + \frac{g}{l}s = 0,$$

which is a homogeneous linear differential equation of second order with constant coefficients. Note that it is the gyroscopic character of the middle terms of Eqs. (5) which made step (5) \rightarrow (7) possible.

Eq. (7) is solved by putting

$$s = A e^{i\alpha t}.$$

Substitution in (7) gives

$$\alpha^2 + 2u\alpha - \frac{g}{l} = 0,$$

a quadratic equation in α with the roots

$$(8) \quad \alpha_1 = -u + \left(u^2 + \frac{g}{l}\right)^{\frac{1}{2}} \quad \text{and} \quad \alpha_2 = -u - \left(u^2 + \frac{g}{l}\right)^{\frac{1}{2}}.$$

It follows that the general solution of (7) is

$$(9) \quad s = A_1 e^{i\alpha_1 t} + A_2 e^{i\alpha_2 t}.$$

The constants A_1 and A_2 are determined from the initial conditions. In agreement with the experimental arrangement we shall stipulate that these be

$$(10) \quad \xi = a, \quad \eta = 0, \quad \frac{d\xi}{dt} = \frac{d\eta}{dt} = 0 \quad \text{at } t = 0.$$

We therefore imagine that the bob is pulled by an amount a out of its plumb line position along the positive ξ -axis, i.e. (cf. Fig. 50), southward along the meridian, and then released without impulse. From (10) the initial values of our complex variables are

$$(10a) \quad s = a, \quad \frac{ds}{dt} = 0 \quad \text{at } t = 0.$$

Eq. (9) then gives

$$(11) \quad A_1 + A_2 = a,$$

$$(11a) \quad A_1 \alpha_1 + A_2 \alpha_2 = 0,$$

$$(11b) \quad A_1 = \frac{a}{2} \left[1 + \frac{u}{\left(u^2 + \frac{g}{l}\right)^{\frac{1}{2}}} \right], \quad A_2 = \frac{a}{2} \left[1 - \frac{u}{\left(u^2 + \frac{g}{l}\right)^{\frac{1}{2}}} \right].$$

Next we calculate the expression for $\frac{ds}{dt}$; it is somewhat less involved than that for s itself. Recalling (11a) we have

$$\frac{ds}{dt} = i\alpha_1 A_1 e^{-iut} \left[e^{i(u^2 + \frac{g}{l})^{\frac{1}{2}} t} - e^{-i(u^2 + \frac{g}{l})^{\frac{1}{2}} t} \right],$$

from which, according to (8) and (11b),

$$(12) \quad \frac{ds}{dt} = -\alpha \frac{g}{l} \frac{1}{(u^2 + \frac{g}{l})^{\frac{1}{2}}} e^{-iut} \sin \left((u^2 + \frac{g}{l})^{\frac{1}{2}} t \right).$$

We arrive at the following conclusions: whenever the sine factor vanishes, we have

$$\frac{ds}{dt} = 0 \quad \text{and hence} \quad \frac{d\xi}{dt} = \frac{d\eta}{ds} = 0.$$

This represents the occurrence of a turning point or cusp in the trajectory of the bob. According to our initial conditions (10), the first one of these occurs at $t=0$. If we put

$$(13) \quad T = \frac{2\pi}{(u^2 + \frac{g}{l})^{\frac{1}{2}}},$$

succeeding cusps occur at

$$t = \frac{T}{2}, t = T, t = \frac{3T}{2}, \dots$$

$t=T$ is the duration of a complete to-and-fro motion. Putting $u=0$ (that is, $\omega=0$) makes Eq. (13) agree with the period of oscillation of a simple pendulum without terrestrial rotation — as would be expected.

In order to see where the bob of our Foucault pendulum is located at $t=T$, we make use of (13) and (11) to obtain from (9)

$$s_{t=T} = A_1 e^{-i u T + 2\pi i} + A_2 e^{-i u T - 2\pi i} = (A_1 + A_2) e^{-i u T} = a e^{-i u T}.$$

The bob therefore has the same distance a from its rest position as it had at the outset of the motion, but its azimuth no longer coincides with the southward meridian, as initially, but has acquired a lag with respect to this direction given by the angle

$$uT = 2\pi \frac{u}{(u^2 + \frac{g}{l})^{\frac{1}{2}}} \cong 2\pi \left(\frac{l}{g} \right)^{\frac{1}{2}} \omega \sin \phi.$$



FIG. 50.
Foucault's pendulum. Bird's-eye view of the trajectory of the bob. Initial displacement to the south; westward deflection in a complete oscillation.

The bob is thus deflected westward, cf. Fig. 50. We can explain this by saying that for zero rotation of the earth the pendulum bob would pursue a straight south-north-south course. In our case, however, the Coriolis force, through its "pressure on the right bank," displaces the trajectory by an angle $\frac{1}{2}\omega T$ eastward while the bob is swinging out, $\frac{1}{2}\omega T$ westward while it is moving back.

Foucault's experiments of 1851 and those of his countless successors yielded only qualitative results; a quantitative investigation of all sources of error was carried out by H. Kamerlingh Onnes, later leading authority in the field of low temperatures and discoverer of superconductivity, in his Groningen thesis of 1879.

§ 32. Lagrange's Case of the Three-Body Problem

We cannot resist the temptation to conclude our analysis of relative motion with the proof of a famous principle enounced by Lagrange (Paris Academy, 1772): *The three-body problem can be solved in closed and elementary form if one assumes that the triangle formed by the three celestial bodies always remains similar to itself.* The masses of the three bodies are completely arbitrary.

The proof of this principle will show that

1. The plane through the three mass points is fixed in space.
2. The resultant of the Newtonian forces on each of the three points passes through their common mass center.
3. The triangle formed by them is equilateral.
4. The three points describe conic sections similar to each other, with the common mass center at one focus.

The proof given by Lagrange is rather involved. It can be simplified if, with Laplace, we assume from the start the first conclusion above. Carathéodory¹ has, however, shown that even without this assumption an elementary proof is possible. His starting point is our vector equation (29.4) resolved in orthogonal coordinates. We shall follow his proof with minor modifications.

We consider the plane S which passes through the three points P_1, P_2, P_3 (masses m_1, m_2, m_3) and therefore also through their center of mass O . Without spoiling the generality of the problem we can define the latter as being at rest. S therefore rotates about the fixed point O ; this rotation includes a component causing S to turn into itself about its normal at O . Call ω the total angular velocity. We imagine ourselves to be located on a frame fixed in S from which we observe the motion of the points P_k , in a way similar to that in which we observed the motion of Foucault's pendulum

¹ *Sitz. Bayr. Akad. Wiss.* 257 (1933).

from the earth. From O we measure the radius vectors \mathbf{r}_k to the points P_k ; \mathbf{v}_k and $\frac{d\mathbf{v}_k}{dt}$ are their velocities and accelerations as observed from S . Making use of the vector rule (24.7), we write down the differential equations (29.4) of the motion in the form

$$(1) \quad \frac{d\mathbf{v}_k}{dt} + 2\boldsymbol{\omega} \times \mathbf{v}_k + \boldsymbol{\omega}(\mathbf{r}_k \cdot \boldsymbol{\omega}) - \mathbf{r}_k \omega^2 + \dot{\boldsymbol{\omega}} \times \mathbf{r}_k = \frac{\mathbf{F}_k}{m_k}.$$

\mathbf{F}_k is the vector sum of the Newtonian gravitational forces acting at m_k . Thus, for example,

$$(2) \quad \frac{\mathbf{F}_1}{m_1} = \frac{Gm_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} + \frac{Gm_3}{|\mathbf{r}_3 - \mathbf{r}_1|^3} \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|}.$$

We fix a Cartesian coordinate system in S , with origin at O , and x, y , arbitrarily oriented, in the plane of S ; at O we erect the z -axis perpendicular to S . In Eulerian fashion we resolve $\boldsymbol{\omega}$ along these axes,

$$(3) \quad \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3).$$

Let the component ω_3 (rotation of S into itself) be determined by considering the direction of one of the vectors $\overrightarrow{OP_k}$ fixed in S . But we assumed that the triangle $P_1P_2P_3$ was to remain similar to itself; it follows that each of the other two vectors $\overrightarrow{OP_k}$ as well has a direction fixed in S . We can then write

$$(4) \quad \mathbf{r}_k = \lambda(t) (a_k, b_k, 0),$$

where a_k, b_k are the Cartesian components of P_k at some given initial time. The function $\lambda(t)$ determines the common change in scale of the vectors $\overrightarrow{OP_k}$ and hence also that of the triangle $P_1P_2P_3$: with $\dot{\lambda}$ and $\ddot{\lambda}$ the derivatives of λ , we obtain from (4) that

$$(4a) \quad \begin{aligned} \mathbf{v}_k &= \dot{\lambda}(t) (a_k, b_k, 0), \\ \frac{d\mathbf{v}_k}{dt} &= \ddot{\lambda}(t) (a_k, b_k, 0). \end{aligned}$$

It further follows that the resultant force \mathbf{F}_k of Eq. (1) has a vanishing z -component, and x - and y -components inversely proportional to λ^2 . We shall write this force in abbreviated form as

$$(5) \quad \frac{\mathbf{F}_k}{m_k} = \frac{1}{\lambda^2(t)} (L_k, M_k, 0).$$

Next we write down the z -component of Eq. (1) perpendicular to S ,

$$2\dot{\lambda}(\omega_1 b_k - \omega_2 a_k) + \lambda \omega_3 (a_k \omega_1 + b_k \omega_2) + \lambda (\dot{\omega}_1 b_k - \dot{\omega}_2 a_k) = 0,$$

or, factoring out a_k, b_k ,

$$(6) \quad \{-2\dot{\lambda}\omega_2 + \lambda(\omega_3\omega_1 - \dot{\omega}_2)\}a_k + \{2\dot{\lambda}\omega_1 + \lambda(\omega_3\omega_2 + \dot{\omega}_1)\}b_k = 0.$$

The two brackets $\{ \}$ are functions of t independent of k . Calling them $f(t)$ and $g(t)$, we obtain

$$(6a) \quad \frac{f(t)}{g(t)} = -\frac{b_k}{a_k}.$$

We have, however, assumed that the points P_k form a triangle, i.e., are not collinear. The three ratios b/a must therefore be unequal. In that case we can satisfy (6) only by putting $f=g=0$. Explicitly

$$(7) \quad \begin{aligned} 2\dot{\lambda}\omega_1 &= -\lambda(\omega_3\omega_2 + \dot{\omega}_1), \\ 2\dot{\lambda}\omega_2 &= \lambda(\omega_3\omega_1 - \dot{\omega}_2). \end{aligned}$$

Multiplication by ω_1 and ω_2 respectively, followed by addition, yields

$$\frac{2\dot{\lambda}}{\lambda} = -\frac{\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2}{\omega_1^2 + \omega_2^2}$$

and, by quadrature,

$$(8) \quad \omega_1^2 + \omega_2^2 = \frac{C}{\lambda^2}, \quad C = \text{constant of integration}.$$

We proceed to write the x - and y -components of the differential Eq. (1). They are

$$\begin{aligned} \ddot{\lambda}a_k - 2\omega_3\dot{\lambda}b_k + \omega_1\lambda(a_k\omega_1 + b_k\omega_2) - \lambda a_k(\omega_1^2 + \omega_2^2 + \omega_3^2) - \dot{\omega}_3\lambda b_k &= \frac{L_k}{\lambda^2}, \\ \ddot{\lambda}b_k + 2\omega_3\dot{\lambda}a_k + \omega_2\lambda(a_k\omega_1 + b_k\omega_2) - \lambda b_k(\omega_1^2 + \omega_2^2 + \omega_3^2) + \dot{\omega}_3\lambda a_k &= \frac{M_k}{\lambda^2}, \end{aligned}$$

or, arranged in factored form,

$$(9) \quad \begin{aligned} \{\ddot{\lambda} - \lambda(\omega_2^2 + \omega_3^2)\}a_k - \{2\omega_3\dot{\lambda} + \lambda(-\omega_1\omega_2 + \dot{\omega}_3)\}b_k &= \frac{L_k}{\lambda^2}, \\ \{2\omega_3\dot{\lambda} + \lambda(\omega_1\omega_2 + \dot{\omega}_3)\}a_k + \{\ddot{\lambda} - \lambda(\omega_1^2 + \omega_3^2)\}b_k &= \frac{M_k}{\lambda^2}. \end{aligned}$$

The brackets $\{ \}$ of the first equation and similarly those of the second, when multiplied by λ^2 , must therefore each satisfy three linear equations with constant coefficients (independent of t). This is possible only if they are themselves constant. It follows that the difference of the first and fourth brackets and that of the third and second brackets each equal a constant divided by λ^2 . We then have

$$(10) \quad \omega_1^2 - \omega_2^2 = \frac{A}{\lambda^2}, \quad 2\omega_1\omega_2 = \frac{B}{\lambda^2}.$$

A suitable consolidation gives

$$(\omega_1 \pm i\omega_2)^2 = \frac{A \pm iB}{\lambda^2}$$

from which the absolute magnitude

$$(11) \quad \omega_1^2 + \omega_2^2 = \frac{D}{\lambda^2}, \quad D = (A^2 + B^2)^{\frac{1}{2}}$$

is obtained. A comparison with (8) would lead to

$$(11a) \quad \lambda = \frac{C}{D} = \text{const.}$$

unless both C and D were to vanish. Now according to (10) $\lambda = \text{const.}$ would make both ω_1 and ω_2 constant, so that, from (7), ω_3 would have to be zero. By suitable choice of the coordinates x, y one could even make $\omega_2 = 0$; the first Eq. (9) would then yield $L_k = 0$. In that case the three points P_k would have to be collinear, contrary to our hypothesis.

We must therefore put $C = D = 0$ and obtain from either (8) or (11) that

$$(12) \quad \omega_1 = \omega_2 = 0.$$

This proves statement 1, that *the plane S rotates with angular velocity ω_3 into itself; its normal is fixed in space.*

If we apply the equation of angular momentum to our system we see that the motion of the points m_k within the plane S cannot contribute to the areal velocity constant. This constant is hence directly determined by the angular velocity ω_3 of S . We must have

$$\text{const.} = \omega_3 \sum m_k |\mathbf{r}_k|^2 = \omega_3 \lambda^2 \sum m_k (a_k^2 + b_k^2).$$

For this we can write

$$(12a) \quad \lambda^2 \omega_3 = \gamma \quad (\gamma = \text{constant});$$

it follows that

$$(12b) \quad 2\dot{\lambda}\omega_3 + \lambda^2 \dot{\omega}_3 = 0.$$

By virtue of (12) and (12a, b) Eqs. (9) simplify to

$$(13) \quad \lambda^2 \ddot{\lambda} - \frac{\gamma^2}{\lambda} = \frac{L_k}{a_k} = \frac{M_k}{b_k}.$$

The requirement $\frac{L_1}{a_1} = \frac{M_2}{b_1}$ contained in them says that the moment of F_1 about O vanishes, for

$$(14) \quad |\mathbf{r}_1 \times \mathbf{F}_1| = \frac{1}{\lambda^2} (a_1 M_1 - b_1 L_1) = 0,$$

so that F_1 passes through the mass center O . The same holds for F_2 and F_3 . This is our assertion 2 which states that *the resultant of the forces acting at P_k passes through the mass center of the particles m_k .*

We can make use of (2) to write (14) more explicitly. We have at once

$$(15) \quad \frac{\mathbf{r}_1 \times \mathbf{F}_1}{m_1 G} = \frac{m_2 \mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + \frac{m_3 \mathbf{r}_1 \times \mathbf{r}_3}{|\mathbf{r}_3 - \mathbf{r}_1|^3} = 0.$$

But from the definition of the mass center,

$$(16) \quad m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3 = 0,$$

and therefore

$$m_2 \mathbf{r}_1 \times \mathbf{r}_2 + m_3 \mathbf{r}_1 \times \mathbf{r}_3 = 0.$$

Substitution into (15) yields

$$m_2 \mathbf{r}_1 \times \mathbf{r}_2 \left(\frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} - \frac{1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} \right) = 0,$$

that is,

$$(17) \quad |\mathbf{r}_2 - \mathbf{r}_1| = |\mathbf{r}_3 - \mathbf{r}_1|.$$

Similarly we find

$$(17a) \quad |\mathbf{r}_3 - \mathbf{r}_2| = |\mathbf{r}_1 - \mathbf{r}_2|, \text{ etc.}$$

We have thus arrived at statement 3: *the triangle is equilateral.*

The quotients $\frac{L_k}{a_k}$ and $\frac{M_k}{b_k}$ occurring in (13) can each be determined. To this end, let us call λ the side of the triangle, where

$$\lambda^2 = (a_2 - a_1)^2 + (b_2 - b_1)^2 = (a_3 - a_2)^2 + (b_3 - b_2)^2 = \dots$$

According to (2) and (5) we then have

$$\frac{L_1}{a_1} = \frac{G}{\lambda^3} \{ m_2 (a_2 - a_1) + m_3 (a_3 - a_1) \}$$

and, in view of (16),

$$(18) \quad \frac{L_1}{a_1} = \frac{G}{\lambda^3} \{ -m_1 - m_2 - m_3 \}.$$

The right member of this equation is symmetric in the m_k and the coordinates a_k, b_k ; it therefore represents the value not only of $\frac{L_1}{a_1}$, but of $\frac{L_k}{a_k}$ and also of $\frac{M_k}{b_k}$. Substitution of this value in (13) yields

$$(19) \quad \lambda^2 \ddot{\lambda} - \frac{\gamma^2}{\lambda} = -\frac{G}{\lambda^3} (m_1 + m_2 + m_3).$$

This differential equation in λ describes the motion in time, i.e., the rhythm with which our equilateral triangle alternately expands and contracts.

There is, however, a simpler way to gain insight into this secular motion and at the same time into the form of the trajectories; we abandon the plane S and observe the motion from a plane S' coinciding with S , but fixed in space. In S' the only force acting on the mass point m_k is the resultant force F_k directed toward the mass center which is at rest; the fictitious forces (Coriolis, centrifugal, etc.) occurring in (1) drop out. From (5) and (18) the magnitude of F_k is

$$(20) \quad |F_k| = \frac{m_k}{\lambda^3} (L_k^2 + M_k^2)^{\frac{1}{2}} = - \frac{m_k G}{\lambda^3 s^3} (m_1 + m_2 + m_3) \frac{(a_k^2 + b_k^2)^{\frac{1}{2}}}{s}.$$

The only quantity in the right member that varies in time is λ^3 . With the help of (4) it can be expressed in terms of $|r_k|$,

$$\lambda^3 = \frac{|r_k|^3}{a_k^2 + b_k^2}.$$

Let us replace λ by this value in (20), define a new mass

$$(20a) \quad m'_k = m_k \frac{(a_k^2 + b_k^2)^{\frac{1}{2}}}{s^3}$$

and the total mass $M = m_1 + m_2 + m_3$. Instead of (20) we then obtain

$$|F_k| = - \frac{m'_k M G}{|r_k|^3}.$$

Each of our three mass points hence moves in space independently of the others, as if endowed with a mass m'_k , and attracted to a mass M at rest in O in a Newtonian manner. *It therefore describes a conic section with one focus at O .*

In order to be able to say something about the magnitude and mutual position of the three conic sections we must take into account the initial conditions implicit in the state of motion we have postulated. Let us for example consider the instant at which $\lambda = \lambda_{\text{extr}}$ when the distance

$$(21) \quad \lambda_{\text{extr}} (a_k^2 + b_k^2)^{\frac{1}{2}}$$

of all the m_k from O is an extremum. According to (4) the radial velocity in S is then equal to zero; the velocity in S' , i.e., in space, is given by the component ω_s of angular velocity multiplied by the distance (21); the factor $(a_k^2 + b_k^2)^{\frac{1}{2}}$ occurring in this distance is thus a *measure of the similarity* not only of the initial velocities and initial distances from the common mass center, but at the same time of the size of the three conic sections resulting

from these initial values. *With this, statement 4 is established.* The positions of the three conic sections are distinguished by the angles which the three radius vectors $\overrightarrow{OP_i}$ form with each other.

In the special case $m_1 = m_2 = m_3$, where the mass center coincides with the intersection of the medians of the equilateral triangle, the conic sections are congruent and displaced by 120° with respect to each other.

In addition to this motion in conic sections there is, according to Lagrange, a class of motions expressible in elementary form in which the three bodies are located on a rotating straight line. However we do not want to go into this here.

Let us finally point out that from the specialized three-body problem of Lagrange one can pass to a correspondingly specialized n -body problem. In the case of n equal masses and suitable initial velocities one then obtains n congruent Kepler ellipses, which are displaced by an angle $\frac{2\pi}{n}$ with respect to each other and traversed in the same rhythm. At one time this mode of motion was temporarily advanced for electrons to explain the L -spectra of X -rays [*Physikal. Zeits.* 19, 297 (1918)].

INTEGRAL VARIATIONAL PRINCIPLES OF MECHANICS AND LAGRANGE'S EQUATIONS FOR GENERALIZED COORDINATES

§ 33. *Hamilton's Principle*

We have already met a variational principle of mechanics, that of d'Alembert. This principle compares the state of a system at any given arbitrary instant with a neighboring state obtained from it by a virtual displacement. The principles which we are about to consider are *integral principles*. They differ from the former in that we shall be concerned with the successive states of the system during a finite interval of time, or, what amounts to the same thing, over a finite section of the trajectory. These states are then compared with certain corresponding virtual neighboring states.

The different integral principles with their various names are distinguished by the way in which the correspondence between the original states and their neighboring or varied states is established. They all have this in common: the quantity to be varied has the dimensions of *action*. They can therefore all be collected under the name "principles of Least Action¹."

While power, as we already know, is a quantity of dimensions Energy \times Time⁻¹, action has dimensions Energy \times Time. An example of this is the elementary quantum of action, or Planck's constant, which we shall encounter in § 45, i.e., the quantity

$$h = 6.624 \cdot 10^{-27} \text{ erg sec.}$$

We shall first deal with *Hamilton's principle*. It differs from that of Maupertuis, to be treated in § 37 (though historically the latter came first), in that here *the time is not varied*. This means that the system arrives at any given point of the actual trajectory, of coordinates x_k , at the same time as at the corresponding point of the varied trajectory, of coordinates $x_k + \delta x_k$. The following statement sums up this property of Hamilton's principle:

$$(1) \quad \delta t = 0.$$

¹ In English-speaking countries this usage is not common. We shall hence at once distinguish Hamilton's principle from the principle of least action (sometimes called the Principle of Maupertuis).—TRANSLATOR.

We must remark at this point that when we speak of the trajectory or path of the system, we do not mean the trajectory of a point of the system in a space of three dimensions, but rather a curve in a space of many dimensions, characteristic of the motion of the system as a whole. Thus, in the case of f degrees of freedom, this curve lies in the f -dimensional space of the coordinates $q_1 \dots q_f$ (cf. p. 48).

In addition to the condition (1) we demand that another restriction be imposed on the variations in Hamilton's principle; the end points O and P of the section of the trajectory under consideration and of its varied neighboring trajectory must coincide in space. Hence we have, for any coordinate x ,

$$(2) \quad \delta x = 0 \text{ at } t = t_0 \text{ and at } t = t_1.$$

The adjoining figure has been drawn to aid in visualizing symbolically, in three dimensions, the relation of the actual path (solid) to the virtual one (dashed). The displacement δq , resulting from the variations of the coordinates δx , is to be completely arbitrary except at the two end points, with the restriction that δq be continuous and differentiable in t . There is a one-to-one correspondence between any point on the real path and one on the varied path, obtained from the former by a displacement δq , and two such points belong to the same time t .

We shall now derive Hamilton's principle. We start out with the form (10.6) of d'Alembert's principle,

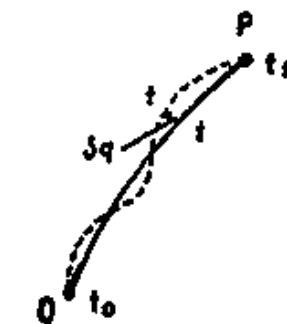


FIG. 51. Variation of the "trajectory" in Hamilton's principle. The time is not varied.

$$(3) \quad \sum_{k=1}^n \{ (m_k \ddot{x}_k - X_k) \delta x_k + (m_k \ddot{y}_k - Y_k) \delta y_k + (m_k \ddot{z}_k - Z_k) \delta z_k \} = 0.$$

We therefore consider a system of n discrete mass points which may, however, be coupled by either holonomic or non-holonomic forces of constraint of unspecified nature. It follows that the δx_k , δy_k , δz_k , which must of course satisfy these constraints, are not independent of each other; in the holonomic case of f degrees of freedom only f can be chosen arbitrarily. In the non-holonomic case they are related by differential conditions.

We shall at first take up a purely formal transformation of relation (3), by writing

$$(4) \quad \ddot{x}_k \delta x_k = \frac{d}{dt} (\dot{x}_k \delta x_k) - \dot{x}_k \frac{d}{dt} (\delta x_k),$$

where we shall at once ask ourselves what the meaning of an expression

such as $\frac{d}{dt}(\delta x_k)$ is. For this purpose we compare not only the actual path of the x_k with the virtual path of the $x_k + \delta x_k$, but also the velocity \dot{x}_k along the actual path with the velocity $\dot{x}_k + \delta \dot{x}_k$ along the virtual path *at the same instant* t . The latter velocity is defined by the identity

$$\frac{d}{dt}(x_k + \delta x_k) = \dot{x}_k + \frac{d}{dt}(\delta x_k).$$

We equate these two ways of writing the varied velocity and obtain

$$(5) \quad \frac{d}{dt}(\delta x_k) = \delta \dot{x}_k.$$

Let us introduce this result into (4),

$$(6) \quad \ddot{x}_k \delta x_k = \frac{d}{dt}(\dot{x}_k \delta x_k) - \dot{x}_k \delta \dot{x}_k = \frac{d}{dt}(\dot{x}_k \delta x_k) - \frac{1}{2} \delta (\dot{x}_k^2).$$

Similar equations hold of course for the coordinates y_k and z_k . Hence (3) can now be written in the form

$$(7) \quad \begin{aligned} \frac{d}{dt} \sum m_k (\dot{x}_k \delta x_k + \dot{y}_k \delta y_k + \dot{z}_k \delta z_k) = \\ \sum \frac{m_k}{2} \delta (\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2) + \sum (X_k \delta x_k + Y_k \delta y_k + Z_k \delta z_k). \end{aligned}$$

The second term on the right is nothing else but the virtual work δW , that is, the work done by the external forces in our virtual displacement. On the other hand, the first term on the right is the variation of the kinetic energy T given by

$$T = \sum \frac{m_k}{2} (\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2)$$

which occurs when we pass from the real to the virtual trajectory. Eq. (7) can therefore be simplified to

$$(8) \quad \frac{d}{dt} \sum m_k (\dot{x}_k \delta x_k + \dot{y}_k \delta y_k + \dot{z}_k \delta z_k) = \delta T + \delta W.$$

Before deriving some further conclusions from this, we shall digress for a moment to make some remarks about the relation (5). Let us write it down once more, in the form

$$(9) \quad \frac{d}{dt} \delta x = \delta \frac{dx}{dt}.$$

If we recall that t is not varied, and that $\delta t = 0$ implies $\delta dt = 0$, we can replace (9) by

$$(9a) \quad \frac{d \delta x}{dt} = \frac{\delta dx}{dt} \quad \text{or also} \quad d \delta x = \delta dx.$$

Eq. (9a), especially in the second form $d\delta = \delta d$, plays a fruitful if somewhat mysterious role in the older calculus of variations of the Euler type. We note that (9a) really says the same thing as the somewhat trivial Eq. (5) relating the time derivative of the virtual displacement to the virtual variation of velocity, except that (9a) contains the two assumptions that the time is not subject to variation and that the virtual displacement is continuous.

We now return to Eq. (8) and integrate it over t from t_0 to t_1 . The left-hand side vanishes because of (2) and we are left with

$$(10) \quad \int_{t_0}^{t_1} (\delta T + \delta W) dt = 0.$$

Owing to the type of variation embodied in Hamilton's principle, this can also be rewritten as

$$(11) \quad \delta \int_{t_0}^{t_1} T dt + \int_{t_0}^{t_1} \delta W dt = 0.$$

It would, however, be erroneous to replace the latter integral by $\delta \int W dt$; for while it is true that the virtual work δW and the amount of work done in dt , i.e., dW , have a well-defined meaning, this is not so for the work W itself. W is, in general, not a "state variable." It is a state variable only if dW is a perfect differential, that is, if the external forces satisfy those conditions which guarantee the existence of a potential energy V [cf. § 6, (3)]. In that case we can replace

$$\int \delta W dt \quad \text{by} \quad - \int \delta V dt = - \delta \int V dt$$

in Eq. (11), which then takes the classically simple form

$$(12) \quad \delta \int_{t_0}^{t_1} (T - V) dt = 0.$$

This is the equation one usually thinks of when one speaks of *Hamilton's principle*. It is valid, according to the statements of p. 46, for *conservative systems*. We can call equation (11) *Hamilton's principle generalized* to include non-conservative systems.

We now claim that Eqs. (12) or (11), respectively, contain the sum-total of mechanics, just as does d'Alembert's principle. This emphasizes the special significance of the energy-like expression $T - V$. In mechanics it is called the *Lagrangian function* (or *Lagrangian*, for short) and takes Eq. (12) into

$$(13) \quad \delta \int_{t_0}^{t_1} L dt = 0 \quad \text{where} \quad L = T - V.$$

In words, *the time integral of the Lagrangian is an extremum*. Helmholtz relied heavily on the variational principle in Hamiltonian form in his last works; he extended it to electrodynamics, and called L the *kinetic potential*. The name "free energy," as opposed to the "total energy" $T+V$, would be equally justified in view of its wide use in thermodynamics.

Hamilton's principle derives special value from the fact that it is totally independent of the choice of coordinates. In fact, T and V (as well as δW) are quantities of immediate physical significance, which can be expressed in any desired set of coordinates. We shall make use of this property in the following section.

Hertz was of the opinion that Hamilton's principle was valid only for holonomic systems. This error was corrected by O. Hölder (Göttinger Nachr. 1896).

Hamilton's principle goes counter to our need for cause-and-effect relationships, as do all the other variational principles involving action integrals. For here the sequence of events is determined not by the present state of the system, but is instead derived under equal consideration of both its past and future states. It seems then that the variation principles are *not causal*, but rather *teleological*. We shall come back to this point in § 37, where we shall deal with the historical origin of the principles. There we shall also briefly touch on the conversion of Hamilton's principle into forms useful in fields of physics other than mechanics.

§ 34. Lagrange's Equations for Generalized Coordinates

Let us consider an arbitrary mechanical system. We shall for the present assume that its parts are coupled by holonomic conditions only. The number of degrees of freedom of the system is f . We can then introduce f independent coordinates which determine the position of the system at any given instant. We shall call them, as on p. 49,

$$(1) \quad q_1, q_2, \dots, q_f.$$

These are our position coordinates. To them we add the "velocity coordinates"

$$(1a) \quad \dot{q}_1, \dot{q}_2, \dots, \dot{q}_f.$$

The q_k and \dot{q}_k together completely specify the state of the system at any instant.

Let us be more explicit: let the system be described for the moment by $n > f$ coordinates x_1, \dots, x_n , which need not necessarily be Cartesian. Let $n-f$ conditions hold between them, of the form

$$(2) \quad F_k(x_1, x_2, \dots, x_n) = 0, \quad k=f+1, f+2, \dots, n$$

We can then define q_k as some function F_k of x_1, \dots, x_n ,

$$(2a) \quad F_k(x_1, x_2, \dots, x_n) = q_k, \quad k=1, 2, \dots, f.$$

Let us denote the partial derivatives of F_k with respect to x_i by F_{ik} ; a differentiation of (2) and (2a) with respect to t then gives

$$(2b) \quad \sum_{i=1}^n F_{ik}(x_1, \dots, x_n) \dot{x}_i = \begin{cases} \dot{q}_k, & k=1, 2, \dots, f \\ 0, & k=f+1, \dots, n. \end{cases}$$

We can calculate from this the \dot{x}_i as linear functions of the \dot{q}_k , with coefficients that depend on the x_1, \dots, x_n , or, by virtue of (2) and (2a), on the q_1, \dots, q_f . The kinetic energy T , a homogeneous quadratic function of the \dot{x}_i , just as it would be were it expressed in Cartesian coordinates to begin with, again becomes a homogeneous quadratic function of the \dot{q}_k with coefficients that depend on the q_k . For the present we shall postulate that the potential energy V is a function of the q_k only, without, in principle, excluding the possibility of later making V a function of the \dot{q}_k as well. In this connection we may now complete the definition (33.13) of L by stating that

L is to be regarded as a function of the q_k and \dot{q}_k .

For the time being we shall exclude an explicit dependence of L on t .

It is in this sense that we now write down the variation of L , i.e., the difference between the values of L in the virtual varied state $q_k + \delta q_k$, $\dot{q}_k + \delta \dot{q}_k$ and in the original state q_k , \dot{q}_k :

$$(3) \quad \delta L = \sum_k \frac{\partial L}{\partial q_k} \delta q_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k.$$

This variation is now introduced into Hamilton's principle

$$(3a) \quad \int_{t_1}^{t_2} \delta L dt = 0.$$

This form differs from that in (33.13) in that we have written the variation under the integral sign, whereas we had previously put it in front. The two forms are, of course, equivalent by virtue of rule (33.1), which says that t and dt are not varied. In any case, Eq. (3a) corresponds to the formulation (33.10) in which we first encountered the principle.

We now carry out the integration over the time indicated by (3a) on the general term of the second sum of (3). For this purpose we alter the form of this term by a *partial integration*, a procedure which has been

characteristic of the whole calculus of variations ever since Euler²:

$$(4) \quad \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} \delta q_k dt = \frac{\partial L}{\partial \dot{q}_k} \delta q_k \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \delta q_k dt.$$

In the last member of this double equality the first term vanishes because of the conditions laid down in (33.2). The complete expression (3) for δL therefore yields

$$(4a) \quad \int_{t_1}^{t_2} \delta L dt = - \int_{t_1}^{t_2} \sum_k \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} \right) \delta q_k dt = 0.$$

Now the δq_k are independent of each other. We can therefore make all but one of them zero. This one we can also make zero everywhere along the "trajectory" of Fig. 51 except in the neighborhood of a single point, or, what amounts to the same thing, during a time interval Δt at an arbitrary time t . In order to satisfy (4a) we now require that

$$(5) \quad \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} \right) \int_{\Delta t} \delta q_k dt = 0.$$

But Δt is finite, and δq_k does not vanish during the interval Δt . Hence we must have, for any time t and any index k ,

$$(6) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0.$$

These are Lagrange's equations for generalized coordinates, or, as they are also called, *Lagrange's equations of the second kind*, specialized to the case so far considered in which the forces acting on the system have a potential and the internal constraints of the system are holonomic.

If one or the other of these assumptions is dropped, we arrive at an extended form of these equations; let us hence consider two cases.

The first case is that in which the forces are not derivable from a potential. In that case the form (33.11) of Hamilton's principle will have to be our starting point. We think of the virtual work δW of the external forces as expressed in terms of the virtual displacements δq_k , and are led to write

$$(7) \quad \delta W = \sum Q_k \delta q_k.$$

² In general we use the term "Euler's equation" of a given variational problem to designate an equation of type (6), and the derivation of (6) from (4) and (5) is typical of the derivation of Euler's equation in any such problem. We can therefore say that Lagrange's equations are the Euler equations for the variational problem characterized by the function L .

We shall call the coefficients Q_k here introduced the *generalized components of force* associated with the coordinates q_k . This is a formal extension of the force concept, which is of course admissible as a mathematical definition. Furthermore it is quite useful. Thus we can now restate the definition of the moment of a force about an axis given in (9.7) as follows: the moment of a force is the generalized force associated with the corresponding angle of rotation. It is clear that the quantities Q_k defined in (7) no longer possess vector character, nor need they in general have the dimensions of dynes any longer. From (7) it is seen that their dimensions depend instead on the dimension of the associated q_k . Thus moments of force must, as we already know, have dimensions of work, hence ergs, for the associated δq_k are angles and therefore dimensionless.

If we now introduce (7) in (33.11) and carry out the transformations indicated by Eqs. (4) and (5), we clearly obtain, in place of (6),

$$(8) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k.$$

We can write this in a somewhat more general form as

$$(8a) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k.$$

This is more general because now we can take into account the case where some of the forces acting are derivable from potentials, others not. We need only write the Q_k corresponding to the latter type of forces on the right side of (8a). The potential energy of the former, on the other hand, can be combined with the kinetic energy T to form the Lagrangian L of (8a).

Eqs. (8a) are then the *Lagrange equations for forces some of which are not derivable from potentials*.

If now we drop the second of the previously stated assumptions, i.e., postulate that the constraints of the system are in part non-holonomic, the introduction of the coordinates q_k is made invalid. For by definition non-holonomic conditions cannot be put in the form (2) and hence cannot be eliminated by proper choice of the q . We are then forced to introduce an excessive number of q , that is, a number greater than the number of degrees of freedom for infinitesimal motion. The latter is $f-r$ where f is the number of degrees of freedom for finite motion and r the number of non-holonomic conditions. These can be written as virtual conditions in a form similar to Eq. (7.4),

$$(9) \quad \sum_{k=1}^f F_{k\mu}(q_1, \dots, q_f) \delta q_k = 0, \quad \mu = 1, 2, \dots, r.$$

They imply a restriction on allowable variations δq_k . One takes this restriction into account by multiplying each of the Eqs. (9) by a Lagrangian multiplier λ_μ and then adding it under the integral of (8.13). One obtains, with the F in somewhat abbreviated notation,

$$\int_{t_0}^{t_1} (\delta L + \sum_{\mu=1}^r \lambda_\mu F_{k\mu} \delta q_k) dt = 0.$$

The Eulerian transformation proceeds as in (4), where instead of (4a) we obtain

$$(10) \quad \int_{t_0}^{t_1} \sum_k \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} - \sum_{\mu=1}^r \lambda_\mu F_{k\mu} \right) \delta q_k dt.$$

Here the δq_k are no longer independent of each other, but are connected through relations (9). One can, however, argue as on p. 67: of the bracketed () coefficients of δq_k in (10), r can be made to vanish by a suitable choice of the λ_μ . In the remaining sum over k , only $f-r$ of the q_k , all independent of each other, are left. The same line of reasoning as after (5) now forces us to the conclusion that the remaining brackets must vanish, too. We then obtain the complete system of f equations,

$$(11) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \sum_{\mu=1}^r \lambda_\mu F_{k\mu}.$$

We can designate these as *Lagrange's equations of the mixed type*, since they fall halfway between Lagrange's equations of the first and second kind.

We may mention that this mixed type occurs not only when we are *unable* to eliminate some of the conditions (case of non-holonomic constraints), but also whenever we *do not wish* to eliminate them. For it can happen that we are interested in the force of constraint that a holonomic condition exerts on the system. This force, as it turns out, is represented by the λ_μ associated with the condition in question [just as in Eq. (18.7) dealing with the spherical pendulum], and can be obtained by integration of Eq. (11).

Evidently we can finally combine the types (11) and (8a), for the case that we simultaneously drop both assumptions stated after (6).

Instead of doing this, we shall lastly concern ourselves with the following question: how and under what assumptions can the principle of the conservation of energy be derived from Lagrange's equations (6)?

As already emphasized, above Eq. (3), L is a function of the q_k and the \dot{q}_k ; we further require, as earlier, that L not contain t explicitly. In that

case Eq. (8) is valid not only for the virtual changes δq , $\delta \dot{q}$, but also for the secular changes dq , $d\dot{q}$, so that we have

$$(12) \quad \frac{dL}{dt} = \sum_k \dot{q}_k \frac{\partial L}{\partial q_k} + \sum_k \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k}.$$

On the other hand we emphasized at the same place that T is a homogeneous quadratic function^a of the \dot{q}_k . We can therefore apply the Euler rule

$$(13) \quad 2T = \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k}$$

for homogeneous functions. Differentiation with respect to the time gives

$$(14) \quad 2 \frac{dT}{dt} = \sum_k \dot{q}_k \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} + \sum_k \ddot{q}_k \frac{\partial T}{\partial \dot{q}_k}.$$

We now subtract (12) from (14). Because $L = T - V$, the left member becomes

$$\frac{dT}{dt} + \frac{dV}{dt}.$$

On the right the second terms cancel *provided* V is independent of \dot{q}_k . In that case, by means of Eq. (6), the first terms on the right cancel as well, so that we obtain

$$(14a) \quad \frac{dT}{dt} + \frac{dV}{dt} = 0$$

from which we conclude that

$$(15) \quad T + V = E.$$

The law of the conservation of energy is therefore a consequence of Lagrange's equations.

We must now examine the assumptions leading to this important conclusion.

a) From the meaning of T we can say that the kinetic energy is determined by the position and velocity of the system, hence by q and \dot{q} ; T could depend on t explicitly only as a result of the elimination of the equations

^a Even when this is not the case and L is instead assumed to be any desired function of the q_k and \dot{q}_k , a generalized conservation law of the form $H = \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L = \text{const.}$ can be given. In Chapter VIII we shall call the function H thus defined the "Hamiltonian"; the conservation law contained in Eq. (15c) is a special case of the above equation.

of constraint, in case the latter depend on t ⁴. Now we have already seen on p. 68 that such constraints do work on the system, and therefore upset the conservation of energy. It is then indeed necessary for the validity of the conservation law that T not contain the time explicitly.

b) The assumption that L does not depend explicitly on t therefore reduces to the assumption that V is independent of t . This condition, too, is necessary. Otherwise one would have to add the term

$$-\frac{\partial V}{\partial t}$$

on the right side of Eq. (12). This term would then reappear with opposite sign in the right-hand member of Eq. (14a). Instead of $T+V=\text{const.}$ we should then obtain

$$(15a) \quad \frac{d}{dt}(T+V) = \frac{\partial V}{\partial t},$$

that is, the law of the conservation of energy would be invalidated.

c) Suppose that V depends not only on the q_k but also on the \dot{q}_k . With the aid of (6) we obtain as the difference of the right members of (14) and (12)

$$(15b) \quad \sum \dot{q}_k \frac{d}{dt} \frac{\partial V}{\partial \dot{q}_k} + \sum \ddot{q}_k \frac{\partial V}{\partial \ddot{q}_k} = \frac{d}{dt} \sum \dot{q}_k \frac{\partial V}{\partial \dot{q}_k}.$$

This case does lead to a conservation law, which has, however, the unfamiliar form

$$(15c) \quad T+V - \sum \dot{q}_k \frac{\partial V}{\partial \dot{q}_k} = \text{const.}$$

One more conclusion can be drawn from the above which will be useful to us later. We shall calculate $L-2T = -(T+V)$, by using the expression (13) for $2T$ and reverting to the assumption that V is a function of only the q_k . We then arrive at

$$(T+V) = L - \sum \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = L - \sum \dot{q}_k \frac{\partial L}{\partial \dot{q}_k},$$

or

$$(16) \quad T+V = \sum \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L.$$

The total energy $T+V$ can be calculated from the expression for the Lagrangian.

⁴ Sometimes such time-dependent conditions are called *rheonomous* (fluid) as opposed to time-independent conditions which are characterized as *scleronomous* (fixed, rigid).

The rather abstract developments of this section will come to life with the examples of the following section. To prepare ourselves for these we shall specialize the two expressions

$$\frac{\partial L}{\partial \dot{q}_k} \quad \text{and} \quad \frac{\partial L}{\partial q_k}$$

occurring in (6) for the simplest case, the motion of an isolated mass point expressed in Cartesian coordinates x, y, z . We have

$$T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad \frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \text{ etc.}$$

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = X, \text{ etc.}$$

Since, according to this equation, $\frac{\partial L}{\partial \dot{x}}$ represents the x -coordinate of momentum, we shall, quite in general, call $\frac{\partial L}{\partial \dot{q}_k}$ the *component of generalized momentum belonging to q_k* . Since on the other hand $\frac{\partial L}{\partial x}$ furnishes the x -component of the force, we shall label the two terms resulting from $\frac{\partial L}{\partial q}$ as *q -components of generalized force*,

$$(17) \quad \frac{\partial T}{\partial q} - \frac{\partial V}{\partial q} = \frac{\partial T}{\partial q} - Q.$$

Q is an *external force* as in Eq. (7), whereas $\frac{\partial T}{\partial q}$ is a *fictitious Lagrange force* dependent on the way in which the q coordinate varies with position. In the case of Cartesian coordinates x, y, z where curves of constant q are parallel to each other, a given q_i is independent of the q_k ($k \neq i$) and the fictitious force vanishes.

§ 35. Examples Illustrating the Use of Lagrange's Equations

We have chosen examples which were treated earlier by elementary methods, in order to demonstrate the superiority of the Lagrange formalism.

(1) The Cycloidal Pendulum.

The obvious coordinate q in this case is the angle of rotation of the wheel generating the cycloids in Fig. 26. The Cartesian coordinates expressed in terms of this angle are, according to (17.2),

$$\begin{aligned} x &= a(\phi - \sin \phi), & \dot{x} &= a(1 - \cos \phi) \dot{\phi} \\ y &= a(1 + \cos \phi), & \dot{y} &= -a \sin \phi \dot{\phi}. \end{aligned}$$

From these we calculate

$$T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) = ma^2(1 - \cos \phi) \dot{\phi}^2$$

$$V = mgy = mga(1 + \cos \phi)$$

$$(1) \quad L = ma^2(1 - \cos \phi) \dot{\phi}^2 - mga(1 + \cos \phi).$$

This is all we need to know about the geometry and mechanics of our system. The Lagrange formalism automatically takes care of the rest:

$$\frac{\partial L}{\partial \dot{\phi}} = 2ma^2(1 - \cos \phi) \dot{\phi}, \quad \frac{\partial L}{\partial \phi} = ma^2 \sin \phi \dot{\phi}^2 + mga \sin \phi$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = 2ma^2(1 - \cos \phi) \ddot{\phi} + 2ma^2 \sin \phi \dot{\phi}^2$$

or, when substituted into the differential equation (6),

$$(1 - \cos \phi) \ddot{\phi} + \frac{1}{2} \sin \phi \dot{\phi}^2 = \frac{g}{2a} \sin \phi.$$

Introduction of the half angle and division by $2 \sin \frac{1}{2} \phi$ simplifies this to

$$(2) \quad \sin \frac{\phi}{2} \ddot{\phi} + \frac{1}{2} \cos \frac{\phi}{2} \dot{\phi}^2 = \frac{g}{2a} \cos \frac{\phi}{2}.$$

It can easily be verified that the left member equals $-2 \frac{d^2}{dt^2} \cos \frac{1}{2} \phi$. Our differential equation (2) is therefore identical to the previous Eq. (17.6), by means of which we were able to prove the rigorously isochronous behavior of the cycloidal pendulum.

(2) The Spherical Pendulum

Here the angles θ and ϕ , polar angle and geographic longitude respectively on the sphere of radius l , are the given coordinates of the mass point. The line element is

$$ds^2 = l^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

so that the kinetic energy becomes

$$T = \frac{m}{2} l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2).$$

As in (18.5a) the potential energy is $V = mgl \cos \theta$ and therefore

$$(3) \quad L = \frac{m}{2} l^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - mgl \cos \theta.$$

And now the automatic calculation along the Lagrange pattern sets in. After division by constant factors, the differential equations for θ and ϕ are

$$(4) \quad \ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 - \frac{g}{l} \sin \theta = 0$$

$$\frac{d}{dt} (l^2 \sin^2 \theta \dot{\phi}) = 0.$$

The second of these equations is the law of conservation of areal velocity, in agreement with (18.8). Note that we have here avoided the calculation which necessarily preceded this equation in the earlier treatment. With the help of the areal velocity constant C of Eq. (18.8), the first of Eqs. (4) can be written

$$\ddot{\theta} = \frac{C^2}{l^2} \frac{\cos \theta}{\sin^3 \theta} + \frac{g}{l} \sin \theta.$$

The second term on the right is equivalent to the gravitational torque $|L| = mgl \sin \theta$, this being the generalized component of force associated with the angle $q = \theta$ in the sense of (34.7). The first term is a fictitious Lagrange force in the sense of (34.17); the origin of this force is the fact that the lines along which the angle θ is measured on a sphere do not run parallel but diverge from the pole.

It is instructive to apply to this example the extension of Lagrange's equations for which provision was made in Eq. (34.11) by introducing the excess coordinate r together with the θ and ϕ . Now r is of course fixed through the relation $r = l$; nevertheless we are interested in this coordinate because it will give us, by means of the multiplier λ , the pressure of the mass point on the surface of the sphere, or, what amounts to the same thing, the tension in the suspension cord of the pendulum. In order to obtain the pertinent differential equation we need only replace (3) by

$$(5) \quad L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - mgr \cos \theta$$

and form a third Lagrange equation to be added to the two of (4),

$$(6) \quad \frac{d}{dt} m\dot{r} - mr\dot{\theta}^2 - mr \sin^2 \theta \dot{\phi}^2 + mg \cos \theta = \lambda r.$$

We have put the quantity $F_{k\mu}$ occurring in (34.11) equal to r , for in order to obtain agreement with Eq. (18.1) we have written the condition $r = l$ in the form

$$F = \frac{1}{2} (r^2 - l^2) = 0.$$

If we set $r = l$ and $\dot{r} = \ddot{r} = 0$, it follows from (6) that

$$(7) \quad \lambda l = mg \cos \theta - ml(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2).$$

This is in agreement with (18.6) if there we transform the rectangular coordinates to θ, ϕ . Such a calculation is once more avoided by the use of the Lagrange scheme.

(3) The Double Pendulum

Here the two angles ϕ and ψ of Fig. 38 are suitable coordinates q_k . In the notation of § 21 we write

$$(8) \quad \begin{aligned} X &= L \sin \phi, & x &= L \sin \phi + l \sin \psi \\ Y &= L \cos \phi, & y &= L \cos \phi + l \cos \psi. \end{aligned}$$

From these we get the following exact relations:

$$\begin{aligned} T &= \frac{M}{2} (\dot{X}^2 + \dot{Y}^2) + \frac{m}{2} (\dot{x}^2 + \dot{y}^2) \\ &= \frac{M+m}{2} L^2 \dot{\phi}^2 + \frac{m}{2} l^2 \dot{\psi}^2 + mLl \cos(\phi - \psi) \dot{\phi} \dot{\psi}, \end{aligned}$$

$$V = -MgY - mgy = -(M+m)gL \cos \phi - mgl \cos \psi.$$

The sign of the last expression is negative because (cf. Fig. 38) Y and y have been taken positive in the direction of the force of gravity. We shall here call Λ the Lagrangian formed from $T - V$ since we have used the letter L for the length of the pendulum suspension. We obtain

$$\begin{aligned} \frac{\partial \Lambda}{\partial \dot{\phi}} &= (M+m)L^2 \dot{\phi} + mLl \cos(\phi - \psi) \dot{\psi}, \\ \frac{\partial \Lambda}{\partial \dot{\psi}} &= ml^2 \dot{\psi} + mLl \cos(\phi - \psi) \dot{\phi}, \\ \frac{\partial \Lambda}{\partial \phi} &= -(M+m)gL \sin \phi - mLl \sin(\phi - \psi) \dot{\phi} \dot{\psi}, \\ \frac{\partial \Lambda}{\partial \psi} &= -mgl \sin \psi + mLl \sin(\phi - \psi) \dot{\phi} \dot{\psi}. \end{aligned}$$

In writing down the Lagrange equations from these relations we shall at once go over to small ϕ, ψ . $\dot{\phi}, \dot{\psi}$ are quantities of the same magnitude as ϕ, ψ ; their squares can therefore be neglected. The equations in question are then

$$(9) \quad \begin{aligned} \ddot{\phi} + \frac{g}{L} \phi &= -\frac{m}{M+m} \frac{l}{L} \ddot{\psi}, \\ \ddot{\psi} + \frac{g}{l} \psi &= -\frac{L}{l} \ddot{\phi}. \end{aligned}$$

These are identical with the Eqs. (21.3); we need merely switch back from coordinate angles ϕ , ψ to coordinate distances X , x by making use of the transformation equations (8) which, for small ϕ , ψ , simplify to

$$\phi = \frac{X}{L}, \quad \psi = \frac{x-X}{l}.$$

The identity is immediate for the second of Eqs. (9) and (21.3); the same is true of the first Eq. (9) and the first Eq. (21.3) provided we introduce for $\ddot{\psi}$ in the right member its value from the second Eq. (9). The discussion of the oscillation process following Eq. (21.3) is hence immediately applicable to our present Eqs. (9) and need not be repeated here.

In concluding we wish to emphasize that in the present purely formal treatment there was no mention whatever of the tension in the pendulum string l ; this tension is implicitly contained in the Lagrange equations of motion as an internal reaction of the system, as has already been stressed in the footnote on p. 112.

(4) The Heavy Symmetrical Top

The classical coordinates q_k of this problem are the Eulerian angles θ , ϕ and ψ [θ and ϕ have been introduced already in (25.4) and (26.5a)]. We shall define them and their corresponding angular velocities as follows (cf. Fig. 52):

1. θ is the angle between the vertical and the axis of the top; $\dot{\theta}$ is the angular velocity about the line of nodes which is perpendicular to both of these directions.
2. ψ is the angle which the line of nodes makes with a fixed direction in the horizontal plane, for instance the x -axis; $\dot{\psi}$ is the angular velocity about the vertical.
3. ϕ is the angle which the line of nodes makes with a fixed direction in the equatorial plane of the top, for example the X -axis; $\dot{\phi}$ is the angular velocity about the axis of symmetry of the top.

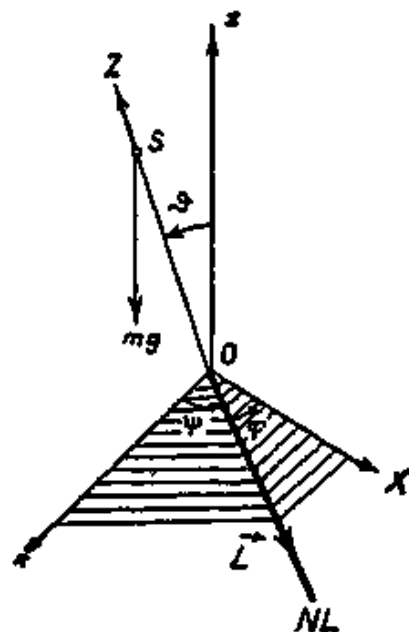


FIG. 52. Definition of the Eulerian angles θ , ϕ , ψ , and their sense. The labeling of the axes is in agreement with the systems of coordinates introduced on p. 139 (z =vertical, Z =axis of top, x =horizontal line fixed in space, X =line in the equatorial plane of the top, fixed in the top).

The $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$ are *holonomic* but *curvilinear* components of the angular velocity vector ω , as opposed to the ω_1 , ω_2 , ω_3 which were *rectilinear* but *non-holonomic* components of rotational velocity. Table (10) below shows the direction cosines between both sets of components. The table also gives the sense of rotation of $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$ (rule of right-handed screw):

(10)

	$\dot{\theta}$	$\dot{\phi}$	$\dot{\psi}$
ω_1	$\cos \phi$	0	$\sin \theta \sin \phi$
ω_2	$-\sin \phi$	0	$\sin \theta \cos \phi$
ω_3	0	1	$\cos \theta$

The first two columns follow in an obvious manner from what was said in 1 and 3. In order to understand the third column, note that the projection of the vertically oriented vector $\dot{\psi}$ in the equatorial plane is $\dot{\psi} \sin \theta$; this vector in turn is resolved in the equatorial plane into the two components indicated opposite ω_1 and ω_2 , viz., $\dot{\psi} \sin \theta \sin \phi$ and $\dot{\psi} \sin \theta \cos \phi$ respectively.

Notice that our table, unlike those in § 2, can be read only from left to right, not from top to bottom. From its rows we now obtain

$$\begin{aligned}
 \omega_1 &= \cos \phi \dot{\theta} + \sin \theta \sin \phi \dot{\psi}, \\
 \omega_2 &= -\sin \phi \dot{\theta} + \sin \theta \cos \phi \dot{\psi}, \\
 \omega_3 &= \dot{\phi} + \cos \theta \dot{\psi}.
 \end{aligned}
 \quad (11) \quad (11a) \quad \omega_1^2 + \omega_2^2 = \dot{\theta}^2 + \sin^2 \theta \dot{\psi}^2.$$

Putting $I_2 = I_1$, the expression (26.17) therefore becomes

$$(12) \quad T = \frac{I_1}{2} (\dot{\theta}^2 + \sin^2 \theta \dot{\psi}^2) + \frac{I_2}{2} (\dot{\phi} + \cos \theta \dot{\psi})^2.$$

By virtue of Eq. (25.6a) for the gravitational potential energy V we have

$$(13) \quad L = T - V = \frac{I_1}{2} (\dot{\theta}^2 + \sin^2 \theta \dot{\psi}^2) + \frac{I_2}{2} (\dot{\phi} + \cos \theta \dot{\psi})^2 - P \cos \theta,$$

$$P = mgs.$$

L is therefore *independent* of the position coordinates ϕ and ψ and depends only on their change with time. We say that ϕ and ψ are *cyclic coordinates*. The name has its origin in the dynamic behavior of a rotating wheel (Greek: $\kappaυκλoσ$) which is determined not by its instantaneous position but only by its speed of revolution. Hence

$$\frac{\partial L}{\partial \phi} = \frac{\partial L}{\partial \psi} = 0.$$

From Lagrange's equations the time derivatives of the quantities

$$\frac{\partial L}{\partial \dot{\phi}} \quad \text{and} \quad \frac{\partial L}{\partial \dot{\psi}}$$

must then vanish. At the end of the last section we called these quantities the generalized momenta associated with ϕ and ψ . From now on we shall always designate them by p . Thus we write in general

$$(14) \quad p_k = \frac{\partial L}{\partial \dot{q}_k}.$$

We can then assert that if the coordinates q_k are cyclic, the momenta p_k conjugate to cyclic coordinates are integrals of the motion (i.e., constants of integration). In our case we already know the significance of these constants from (25.6). We have

$$(15) \quad p_\phi = M'', \quad p_\psi = M'.$$

Previously, on p. 141, we lacked the expressions of these constants in terms of the position coordinates of the top. These can now be derived by application of the general rule (14):

$$(16) \quad \begin{aligned} p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = I_3 (\dot{\phi} + \cos \theta \dot{\psi}), \\ p_\psi &= \frac{\partial L}{\partial \dot{\psi}} = I_1 \sin^2 \theta \dot{\psi} + I_3 \cos \theta (\dot{\phi} + \cos \theta \dot{\psi}). \end{aligned}$$

Combination of (15) and (16) results in

$$(17) \quad \begin{aligned} \dot{\phi} + \cos \theta \dot{\psi} &= \frac{M''}{I_3}, \\ I_1 \sin^2 \theta \dot{\psi} &= M' - M'' \cos \theta. \end{aligned}$$

Eqs. (17) exhaust the content of two of the Lagrange equations. The third one expresses the rate of change of

$$(18) \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I_1 \dot{\theta}$$

and becomes, if (17) is used to eliminate $\dot{\phi}$ and $\dot{\psi}$,

$$(19) \quad I_1 \ddot{\theta} = \frac{(M' - M'' \cos \theta)(M' \cos \theta - M'')}{I_1 \sin^3 \theta} + P \sin \theta.$$

The right-hand member, which comes from $\frac{\partial L}{\partial \theta}$, contains not only the gravitational effect familiar to us from (25.4), but in addition a fictitious force which is a consequence of the nature of the coordinate system used, as we know from p. 192.

Eq. (19) has the character of a generalized pendulum equation. We need not be detained with its integration, for we can avail ourselves of the integral of energy

$$(20) \quad T + V = E$$

which must be identical with the result of a first integration of (19). Let us once more eliminate the quantities $\dot{\phi}$ and $\dot{\psi}$ of Eq. (12) with the help of (17). Then (20) yields

$$(21) \quad \frac{I_1}{2} \left\{ \dot{\theta}^2 + \left(\frac{M' - M'' \cos \theta}{I_1 \sin \theta} \right)^2 \right\} + \frac{M''^2}{2I_2} + P \cos \theta = E.$$

Since Eq. (21) contains three constants of integration, namely M' , M'' , and E , it must be the general integral of first order for the problem of the top. Finally, just as in § 18 for the spherical pendulum, we replace θ and $\dot{\theta}$ by

$$\cos \theta = u; \quad \dot{\theta} \sin \theta = -\dot{u}.$$

We then obtain

$$(22) \quad \left(\frac{du}{dt} \right)^2 = U(u)$$

where

$$(23) \quad U(u) = \left(\frac{2E}{I_1} - \frac{M''^2}{I_1 I_2} - \frac{2P}{I_1} u \right) (1 - u^2) - \left(\frac{M' - M'' u}{I_1} \right)^2.$$

Since $U(u)$ is a polynomial of third degree in u , the time t must be given by an elliptic integral of the first kind, as in the case of the spherical pendulum:

$$(24) \quad t = \int \frac{du}{U^{1/2}}.$$

The azimuth angle ψ is given from Eq. (17) by an elliptic integral of the third kind (cf. p. 100),

$$(25) \quad \psi = \int \frac{M' - M'' u}{I_1 (1 - u^2)} \frac{du}{U^{1/2}}.$$

We can now repeat the considerations following Fig. 29 on p. 99, and arrive at the picture of Fig. 53. The trace of the axis of the top on a unit sphere oscillates back and forth between the two circles of latitude $u = u_2$ and $u = u_1$, which it touches. At the points of tangency, as

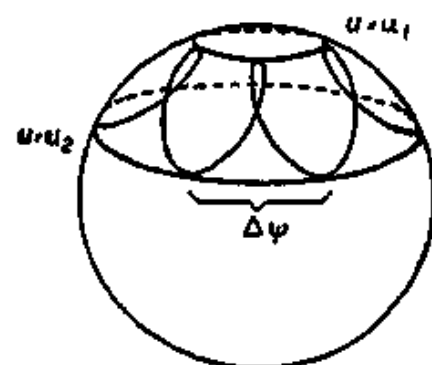


FIG. 53. Trace of the axis of the heavy symmetrical top on a sphere of unit radius.

At the points of tangency, as

shown in Fig. 53, the trace may either merely pass by, or make a loop; the loop may in turn degenerate into a cusp. During each oscillation the axis of the top advances by the same azimuth angle $\Delta\psi$, obtained from Eq. (25) by a complete elliptic integral of the third kind, similar to that in (18.15).

In particular, if the top is to describe a regular precession about the vertical, it is necessary that the parallel circles u_1 and u_2 become merged; the curve $U(u)$ of Fig. 29 (p. 99) must then touch the axis of abscissae from below. This shows that the regular precession of the heavy top is a particular form of motion (whereas, in the case of the top under no forces, it is the general form of motion).

If the two roots u_1 and u_2 do not coincide exactly, but only approximately, we still *seem* to have a uniform advance of the axis of the top about the vertical; on closer observation one notices, however, that *small nutations* are superposed on this uniform advance, giving rise to what we called a "pseudo-regular precession". This is the typical phenomenon that one observes in the usual experiments with tops: one first imparts the greatest possible angular momentum to the top about its axis by pulling a string off its rim, and then sets it point down in a socket pan, taking great care not to add a perceptible lateral impulse to the motion.

We explain this behavior as follows: in such an experiment the initial angular momentum M is close to the axis of symmetry; this also follows from the Poinsot method for the initial axis of rotation. Hence the axis of rotation describes at first a narrow circuit on the unit sphere of Fig. 43. The parallel circles $u=u_1$, $u=u_2$ touching this circuit are close neighbors and remain close during the entire course of the motion, as can be seen from our general illustration in Fig. 53. The angular momentum and hence also the angular velocity are at first very great; they, too, remain unchanged during the motion apart from frictional losses. The nutations are therefore very rapid and almost invisible. The top seems reluctant to yield to the influence of gravity, instead constantly "sidestepping" in a direction perpendicular to the force of gravitation. It is this paradoxical behavior which has for centuries attracted amateur and professional investigator alike to the theory of the spinning top.

§ 36. An Alternate Derivation of Lagrange's Equations

Even though the derivation of Lagrange's equations for generalized coordinates from Hamilton's principle is unsurpassed in clarity and brevity, we feel that it is somewhat artificial. The transformation properties of the various dynamic variables, which form the core of the Lagrange equations, are not brought to light. The following derivation will remedy this situation.

We focus our attention on a system of $\frac{n}{3}$ mass points (n being divisible by 3), subject to arbitrary constraints, chosen holonomic for the sake of simplicity. The number of constraints is equal to $n-f$, where f is the number of degrees of freedom of the system. Our notation will be that of Eq. (34.2). We shall number the coordinates, assumed orthogonal, $x_1, x_2 \dots x_n$; similarly we let $X_1, X_2 \dots X_n$ be the components of the external forces. Finally we shall call $\xi_1, \xi_2 \dots \xi_n$ the components of momentum of our mass points. We would have preferred to name them $p_1, p_2 \dots p_n$, as agreed to in (35.14); this notation must, however, be reserved for the generalized momenta. We have

$$(1) \quad \xi_i = m_i \dot{x}_i, \quad i=1, 2 \dots n,$$

where the m_i are of course equal in groups of three. The motion of our system is described by Lagrange's equations of the first kind (12.9) which are, in the present notation,

$$(2) \quad \frac{d\xi_i}{dt} = X_i + \sum_{\mu=f+1}^n \lambda_\mu \frac{\partial F_\mu}{\partial x_i}, \quad i=1, 2 \dots n.$$

We now introduce the generalized position coordinates $q_1, \dots q_f$, which can be and are to be chosen in such a way that, just as in (34.2), the $n-f$ conditions $F_\mu=0$ are identically satisfied. Then Eqs. (34.2b) must hold between the old and the new velocity coordinates; we solve these for the \dot{x} and write them as follows:

$$(3) \quad \dot{x}_i = \sum_{k=1}^f a_{ik} \dot{q}_k, \quad i=1, 2 \dots n.$$

The a_{ik} , called F_{ik} in (34.2b), are functions of the $x_1 \dots x_n$ and therefore also of the $q_1 \dots q_f$, as stressed in § 34. We see that whereas the old and new position coordinates are connected by an *arbitrary point transformation*, the velocity coordinates transform *linearly*, the coefficients depending on the position coordinates.

What is the transformation character of the *components of force*? We shall call the new force components Q_k and define them as in (34.7) by means of the invariance of virtual work, that is

$$(4) \quad \delta W = \sum_{i=1}^n X_i \delta x_i = \sum_{k=1}^f Q_k \delta q_k.$$

We now pass from virtual to real displacements and from these to the corresponding velocities. By virtue of (3), Eq. (4) becomes

$$(4a) \quad \sum_{k=1}^f Q_k \dot{q}_k = \sum_{i=1}^n X_i \sum_{k=1}^f a_{ik} \dot{q}_k.$$

The \dot{q}_k , unlike the \dot{x}_i , are independent of each other. Hence their coefficients on the right and left of (4a) must be equal, so that

$$(5) \quad Q_k = \sum_{i=1}^n a_{ik} X_i, \quad k=1, 2 \dots f.$$

This is the *transpose* of transformation (3); in (3) we sum over the k , in (5) over the i . Written explicitly,

$$\begin{aligned} \dot{x}_1 &= a_{11} \dot{q}_1 + a_{12} \dot{q}_2 \dots & Q_1 &= a_{11} X_1 + a_{21} X_2 + \dots \\ \dot{x}_2 &= a_{21} \dot{q}_1 + a_{22} \dot{q}_2 \dots & Q_2 &= a_{12} X_1 + a_{22} X_2 + \dots \end{aligned}$$

The transposition hence consists of an interchange of a_{ik} and a_{ki} . We say that *the components of force transform contravariantly*⁵ (or are "contragredient") to the velocity coordinates.

The *components of momentum transform like the components of force*, that is, covariantly to them. For we can think of the momenta as those impulsive forces which cause our mass points, initially at rest, to take on the required velocities. If we call the new momenta p_k , they can be expressed in terms of the old ξ_i by means of the relations

$$(6) \quad p_k = \sum_{i=1}^n a_{ik} \xi_i.$$

These are the defining equations for the p_k . The definition is rather clumsy, but can be converted quite readily to a more meaningful form. For this purpose let us consider, as on p. 186, the expressions for the kinetic energy as function of the \dot{q} on the one hand, and as function of the \dot{x} on the other. We shall distinguish the two expressions, wherever necessary, by writing

$$T_{\dot{q}} \text{ or } T_{\dot{x}}.$$

We then form

$$(7) \quad \frac{\partial T_{\dot{q}}}{\partial \dot{q}_k} = \sum_{i=1}^n \frac{\partial T_{\dot{x}}}{\partial \dot{x}_i} \left[\frac{\partial \dot{x}_i}{\partial \dot{q}_k} \right].$$

The bracket is to remind us that in differentiating with respect to \dot{q}_k we must keep the q_k as well as all \dot{q}_i ($i \neq k$) fixed. According to Eq. (3) the term in brackets is just a_{ik} . On the other hand the elementary expression

$$T_{\dot{x}} = \frac{1}{2} \sum m_i \dot{x}_i^2 \text{ evidently yields } \frac{\partial T_{\dot{x}}}{\partial \dot{x}_i} = \xi_i.$$

⁵ In the theory of general relativity it is customary to denote by a superscript (Q^k, p^k) those quantities which, like Q and the p (about to be defined), transform contravariantly (i.e., are "contragredient") to the \dot{q}_k . We believe, however, that this usage, so important in general relativity, can here be dispensed with,

Instead of (7) we then have

$$(8) \quad \frac{\partial T \dot{q}}{\partial \dot{q}_k} = \sum_{i=1}^n a_{ik} \dot{x}_i.$$

The right member is identical to that of (6). Hence the result:

$$(9) \quad p_k = \frac{\partial T \dot{q}}{\partial \dot{q}_k}$$

We can now assume that the external forces are derivable from a potential V independent of the \dot{q} , and introduce the Lagrangian $L = T - V$, so that (9) can be rewritten

$$(9a) \quad p_k = \frac{\partial L}{\partial \dot{q}_k}.$$

We have thus quite generally justified the definition of the p_k anticipated in (35.14).

We are now in a position to transform the equations of motion (2) to generalized coordinates. To this end we multiply them successively by the different a_{ik} ($k=1 \dots n$) and sum over i . By Eq. (5), the first term in the right member becomes

$$(10) \quad Q_k = - \frac{\partial V}{\partial q_k}.$$

In the second term on the right the factor of λ_μ is

$$(11) \quad \sum_{i=1}^n a_{ik} \frac{\partial F_\mu}{\partial x_i} \quad \text{for } \mu = f+1, \dots, n$$

Now Eq. (3) tells us that

$$(12) \quad a_{ik} = \frac{\partial x_i}{\partial \dot{q}_k}.$$

This becomes evident if one writes (3) in the equivalent form $dx_i = \sum a_{ik} d\dot{q}_k$, and holds all q except q_k fixed. Instead of (11) we can now also write

$$\sum_{i=1}^n \frac{\partial F_\mu}{\partial x_i} \frac{\partial x_i}{\partial \dot{q}_k} = \frac{\partial F_\mu}{\partial \dot{q}_k}.$$

But according to (34.2) it is precisely for $\mu = f+1, \dots, n$ that the F_μ have been made identically zero by our choice of the q_k , so that the partial derivatives of the F_μ with respect to the q_k vanish as well. Hence the right member of our equation reduces to (10).

The left member,

$$\sum_i a_{ik} \frac{d\xi_i}{dt},$$

is transformed into

$$(13) \quad \frac{d}{dt} \sum_i a_{ik} \xi_i - \sum_i \xi_i \frac{da_{ik}}{dt} = \frac{dp_k}{dt} - \sum_i \xi_i \frac{d}{dt} \frac{\partial x_i}{\partial q_k},$$

where we have made use of (6) and (12). The last sum can be written in the form

$$\sum_i m_i \dot{x}_i \frac{\partial x_i}{\partial q_k} = \frac{\partial}{\partial q_k} \frac{1}{2} \sum_i m_i \dot{x}_i^2 = \frac{\partial T}{\partial q_k},$$

where the index q of T is to remind us that T must be converted to a function of the q, \dot{q} before the differentiation with respect to q_k is carried out. The right side of (13) will then become

$$(13a) \quad \frac{dp_k}{dt} - \frac{\partial T}{\partial q_k}.$$

Since it is to be equal to (10), we finally obtain

$$(14) \quad \frac{dp_k}{dt} = \frac{\partial T}{\partial q_k} - \frac{\partial V}{\partial q_k} = \frac{\partial L}{\partial q_k}.$$

Referring back to (9a) we see that this is identical with the Lagrange equation in form (34.6), or, if we do not assume the existence of a potential energy, with that in the form (34.8).

We have thus convinced ourselves that we need not have recourse to Hamilton's principle to derive the Lagrange equations; we need merely make a thorough study of the transformation properties of the dynamic variables involved.

§ 37. The Principle of Least Action

In the conclusion to § 33 we spoke of the *teleological* character of our integral principles. "Teleological" means "shaped by a purpose," "directed toward an end." "Among all possible motions, Nature chooses that which reaches its goal with the minimum expenditure of action." This statement of the principle of least action may sound somewhat vague, but is completely in keeping with the form given it by its discoverer.

In the formulation of the principle not only teleological but also theological beliefs played a role. Maupertuis recommended his principle with the assertion that it best expressed the wisdom of the Creator. Leibniz, too, must have had such arguments in mind, as shown by the title of his *Theodicee* (justification of God).

Maupertuis published his principle in the year 1747. He was referred to a letter of Leibniz of the year 1707 (the original of this letter has been lost); he nevertheless defended his priority with passion, even throwing his weight as president of the Berlin Academy into the dispute. The principle acquired a mathematically definite form only later in the hands of Euler and especially Lagrange.

In the formulation of the principle of least action given above two things are not clear.

1. What is meant by the word "action"? Clearly not the same entity as in Hamilton's principle, for we are dealing with a formulation which, though related to that of Hamilton, is yet distinct from it.

2. What is meant by the phrase, "all possible motions"? It is quite essential to define precisely the class of all motions to be considered for comparison; only thus shall we be able to choose from this class the real motion as the most purposeful or favorable.

Regarding 1: Leibniz took the product $2T dt$ as his element of action. In what is to follow we, too, shall designate by *action integral* the quantity*

$$(1) \quad S = 2 \int_{t_1}^{t_2} T dt.$$

Maupertuis, who, like Descartes, regarded the momentum mv as basic in mechanics, took $mv ds$ to be the element of action. It is clear, however, that the definitions of Leibniz and Maupertuis are equivalent in the case of the single mass point, since

$$(2) \quad 2T dt = mv \cdot v dt = mv ds.$$

This equality carries over to arbitrary mechanical systems, provided that by action we understand the sum of the $m_k v_k ds_k$ for all the mass points of the system.

Regarding 2: in Hamilton's principle we had restricted the sum total of motions to be compared by means of conditions (1) and (2) of § 33. Here we shall keep (2), but alter (1). Instead of $\delta t = 0$ we shall now require that

$$(3) \quad \delta E = 0.$$

We shall therefore compare only trajectories of the same energy E as that of the real trajectory under investigation. This condition implies of course that our principle is now valid only for motions in which energy is conserved,

* The factor of 2 is of course unimportant as far as the minimum property of S is concerned. It is, however, convenient, especially for the formulations of § 44. Incidentally Leibniz was still in doubt as to whether he should take mv^2 or, as we nowadays do, $\frac{1}{2}mv^2$ as "live force."

i.e., motions caused by forces which have a potential. If we call the potential energy of the real path V , that of the varied paths $V + \delta V$, we must have because of (3),

$$(4) \quad \delta T + \delta V = 0, \quad \delta V = -\delta T, \quad \delta L = \delta T - \delta V = 2\delta T.$$

To visualize the change in the state of affairs caused by condition (3) we recall Fig. 51. There two points related by a variation δq belonged to the same time t . This is now no longer the case. The time of the varied point is $t + \delta t$ rather than t (cf. Fig. 54). Hence our varied path does not reach the end point at time $t = t_1$, but, according to the way our figure is drawn, at a later time. On the varied path a point Q is reached at a time $t = t_1$, whereas on the original path the corresponding point (also labeled Q) is reached at an earlier time $t_1 - \delta t_1$.

We now repeat the calculations of § 33. Eqs. (3) and (4) of that section remain valid, but Eq. (5) must be altered because, as stressed there, it is valid only for $\delta t = 0$. We find the condition replacing (5) by forming

$$(5) \quad \delta \dot{x} = \frac{d(x + \delta x)}{d(t + \delta t)} - \frac{dx}{dt}.$$

Let us transform the quotient of differentials on the right by writing

$$(6) \quad \frac{\frac{d(x + \delta x)}{dt}}{\frac{d(t + \delta t)}{dt}} = \frac{\frac{dx}{dt} + \frac{d}{dt}\delta x}{1 + \frac{d}{dt}\delta t} = \frac{dx}{dt} + \frac{d}{dt}(\delta x) - \dot{x} \frac{d}{dt}(\delta t) + \dots,$$

where we have neglected products in small quantities of order higher than the first. From (5) we therefore have

$$\delta \dot{x} = \frac{d}{dt}(\delta x) - \dot{x} \frac{d}{dt}(\delta t),$$

or

$$(7) \quad \frac{d}{dt}(\delta x) = \delta \dot{x} + \dot{x} \frac{d}{dt}(\delta t).$$

If we introduce this in (33.4), we have, with index k arbitrary,

$$(8) \quad \ddot{x}_k \delta x_k = \frac{d}{dt}(\dot{x}_k \delta x_k) - \dot{x}_k \delta \dot{x}_k - \dot{x}_k^2 \frac{d}{dt}(\delta t).$$

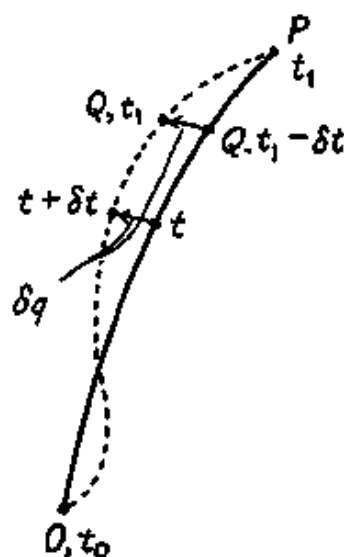


FIG. 54. Variation of the "trajectory" in the principle of least action. Since the energy is not varied, point q of the original path and $q + \delta q$ of the varied path belong to different times t and $t + \delta t$. To the endpoint P on the real path is assigned the point Q on the varied one.

Eq. (8) is valid for coordinates y and z as well as x . Therefore (33.3), rather than leading to (33.8) as before, yields in this instance

$$(9) \quad \frac{d}{dt} \sum m_k (\dot{x}_k \delta x_k + \dot{y}_k \delta y_k + \dot{z}_k \delta z_k) = \delta T + 2T \frac{d}{dt} (\delta t) + \delta W.$$

Here we make use of (4) to put

$$(9a) \quad \delta W = -\delta V = +\delta T,$$

thereby giving as the right member of (9)

$$(10) \quad 2\delta T + 2T \frac{d}{dt} (\delta t).$$

Let us now integrate (9) from t_0 to t_1 . In this process the left member vanishes because of condition (33.2); we then obtain, using (10),

$$(11) \quad 2 \int_{t_0}^{t_1} \delta T dt + 2 \int_{t_0}^{t_1} T d\delta t = 0.$$

This, however, is nothing but

$$(12) \quad 2\delta \int_{t_0}^{t_1} T dt = 0$$

or, recalling (1),

$$(12a) \quad \delta S = 0.$$

This concludes the explicit proof of the principle of least action, as envisaged by Maupertuis.

Let us subject the transition from (11) to (12) to some further scrutiny. In Hamilton's principle the two symbols

$$\delta \int T dt \quad \text{and} \quad \int \delta T dt$$

could be used interchangeably because of the condition $\delta t = 0$; use of this was made for instance in the transition from Eq. (33.10) to (33.11). From our present viewpoint the expressions are, however, different in character, as comparison of Eqs. (11) and (12) above will show.

In particular, let us consider a motion under no forces. In that case $T = E$, so that, with the help of (3), Eq. (12) gives

$$(13) \quad \delta \int_{t_0}^{t_1} dt = \delta(t_1 - t_0) = 0.$$

This is the *principle of least time* (principle of "earliest arrival") which Fermat formulated and applied to the refraction of light, after Heron, in ancient times, had treated the reflection of light in a similar fashion.

In the case of a single free mass point we can put $v = \text{const.}$ instead of $T = E$ and write in place of (12),

$$(14) \quad \delta \int v dt = \delta \int ds = 0.$$

This is the principle of the "shortest path." It determines the trajectory of a free mass point, for instance on a curved surface or — as in general relativity — in a manifold of arbitrary curvature. Such a trajectory is called a *geodesic*. We shall come back to this point in § 40.

In his celebrated Königsberg *Vorlesungen über Dynamik* of 1842 (published by Clebsch) Jacobi justified the necessity for completely eliminating the time t from the principle of least action. This is possible, because

$$T = E - V = \frac{1}{2} \sum m_k v_k^2 = \frac{1}{2} \frac{\sum m_k ds_k^2}{dt^2}$$

and therefore

$$dt = \left(\frac{\sum m_k ds_k^2}{2(E - V)} \right)^{\frac{1}{2}}.$$

Instead of (12) we can then require that

$$(15) \quad \delta \int [2(E - V)]^{\frac{1}{2}} [\sum m_k ds_k^2]^{\frac{1}{2}} = 0.$$

With E fixed, the variation here concerns only the spatial properties of the trajectory of the system; there is no longer any mention of the passage of time during the motion.

Let us come back once more to the teleological aspect of the principles of Hamilton and of least action. Notice that the "least action" may, under certain circumstances, also be a "greatest action"; for in demanding that $\delta \dots = 0$ we do not necessarily obtain a minimum, but rather in general only an *extremum*. We see this most simply in the example of the geodesics on the surface of a sphere, which are arcs of great circles. Suppose that initial point O and endpoint P lie on a specified hemisphere. Then the arc of a great circle connecting them directly is indeed shorter than all arcs lying in planes through O and P but not containing the center of the sphere. Yet the complementary arc, which proceeds from O to P in the opposite direction, traversing the hemisphere not containing the two endpoints, is also a geodesic; and this line is longer than all other arcs of circles which join O to P over this hemisphere. We therefore conclude that in general we do not need to think of the integral principles as demonstrating the "purposefulness" of Nature; they merely constitute an unusually impressive mathematical formulation of an extremal property common to the laws of dynamics.

Maupertuis claimed that his principle was generally valid for all laws of nature. Nowadays we are more inclined to accord this property to Hamilton's principle. We mentioned on p. 185 that Helmholtz made this principle the basis of his studies in electrodynamics. Since that time

integral variational principles of Hamiltonian form have been used in the most diverse fields.

In volume II we shall have direct recourse to this principle in order to gain deeper understanding of the concept of fluid pressure. A special advantage of this procedure will be that we shall obtain not only the differential equations—in this case partial differential equations—of the problem, but also the boundary conditions which the solutions of these equations must satisfy. The same turns out to be true for other problems with continuous mass distributions (capillarity, vibrating membrane, etc.). In many cases it is first necessary to look for the Lagrangian L of the problem at hand before L can be used in the variational principle. Such a case, for instance, is the motion of an electron in a magnetic field; there the force acting is not derivable from a potential V . Relativistic problems form another case; there one should not use the expression for the kinetic energy derived in (4.10) to build the Lagrangian. Instead the expression

$$(16) \quad m_0 c^2 \int (1 - \beta^2)^{\frac{1}{2}} dt$$

must be used as the kinetic contribution to the action principle. The Eulerian derivation of this term leads directly to the relativistic momentum \mathbf{p} of (3.19) and therefore also to the law of the velocity-dependent electron mass. In general, especially outside of mechanics, the search for the Lagrange function L which leads (via the variational principle) to given differential laws is an arduous problem for the solution of which there are no universally valid rules. The previously mentioned problem of the electron in a magnetic field was solved in a simple manner by Larmor and Schwarzschild. A separation of L into a kinetic and a potential contribution according to the pattern $L = T - V$ is then in general no longer feasible.

It is to be emphasized that the quantity under the integral of (16) is nothing but the element of *proper time* (2.17), which was recognized by Minkowski as the simplest invariant of the special theory of relativity; Einstein furthermore generalized it in the form of a world line element in the general theory of relativity. In the form (16) Hamilton's principle therefore automatically satisfies the invariance requirement of relativity theory. In this property Planck⁷ saw the "most brilliant success which Hamilton's principle has achieved."

⁷ Cf. the instructive article in *Die Kultur der Gegenwart*, Part III, § III, 1, p. 701 (B. G. Teubner, Leipzig 1915).

CHAPTER VII

DIFFERENTIAL VARIATIONAL PRINCIPLES OF MECHANICS

§ 38. Gauss' Principle of Least Constraint

Gauss was not only a very eminent mathematician, but also an astronomer and geodesist, and, as such, a passionate calculator of numerical results. It was he who founded the method of least squares, which he evolved with successively greater depth in three extensive treatises. If, as happened now and then, he was asked (against his will) to deliver a lecture at the University of Goettingen, his preferred topic was always the method of least squares.

His brief paper of 1829 entitled "On a New General Fundamental Principle of Mechanics"¹ is concluded with the characteristic sentence, "It is quite remarkable that Nature modifies free motions incompatible with the necessary constraints in the same way in which the calculating mathematician uses least squares to bring into agreement results which are based on quantities connected to each other by necessary relations."

Gauss called his new fundamental principle the *principle of least constraint*. He defined the measure of constraint as follows: consider a mass point of the system, and form the product of its mass by the "square of the deviation of this point from free motion." The sum of this product over all mass points of the system defines the constraint. Let us number the mass points and their rectangular coordinates as on p. 66. We then have

$$(1) \quad Z = \sum_{k=1}^{3n} m_k \left(\ddot{x}_k - \frac{X_k}{m_k} \right)^2$$

as the measure of the constraint of a system of n mass points; for the "free motion" which would occur were the internal constraints neglected is given by

$$\ddot{x}_k = \frac{X_k}{m_k}.$$

Thus the quantity contained in the parenthesis of (1) is indeed the "deviation from free motion" caused by the constraint on the k^{th} mass point.

¹ *Crelle's Journal f. Math.* 4, 232 (1829); *Werke* 5, 23.

It can (cf. p. 61) also be called the "lost force" divided by the mass, so that instead of (1) we can write

$$(2) \quad Z = \sum_{k=1}^{3n} \frac{1}{m_k} (\text{lost force})_k^2.$$

Notice that here the lost forces and reciprocal masses play the same role as the errors and weights in the calculation of errors.

We must now define what is meant by the expression "least constraint," that is, we must indicate what quantities are to be kept fixed and what quantities are to be varied in the calculation of $\delta Z = 0$.

We shall keep *fixed*

a) the instantaneous state of the system, i.e., the position and velocity of each of its mass points. We must therefore put

$$(3) \quad \delta x_k = 0, \quad \delta \dot{x}_k = 0.$$

b) the constraints to which the system is subject. If we take these in the holonomic form $F_i(x_1, x_2, \dots) = 0$, we must, in the variation δZ , take into account the secondary condition

$$(4) \quad \sum_{k=1}^{3n} \frac{\partial F_i}{\partial x_k} \delta x_k = 0, \quad i = 1, 2, \dots, r,$$

where r is the number of conditions, $3n - r = f$ therefore the number of degrees of freedom of the system. Let us differentiate Eq. (4) twice with respect to t . This yields terms in δx , $\delta \dot{x}$ and $\delta \ddot{x}$. Because of (3) we need keep only those in $\delta \ddot{x}$, that is,

$$(4a) \quad \sum_{k=1}^{3n} \frac{\partial F_i}{\partial x_k} \delta \ddot{x}_k = 0.$$

c) the forces acting on the system and, of course, the masses, so that we have

$$(5) \quad \delta X_k = 0, \quad \delta m_k = 0.$$

The remaining quantity \ddot{x}_k is then the only one to be *varied*.

Taking the secondary conditions (4a) into account by the method of Lagrange's undetermined multipliers, we obtain from (1)

$$(6) \quad \delta Z = 2 \sum_{k=1}^{3n} \left\{ m_k \ddot{x}_k - X_k - \sum_{i=1}^r \lambda_i \frac{\partial F_i}{\partial x_k} \right\} \delta \ddot{x}_k = 0.$$

Only $f = 3n - r$ of the $\delta \ddot{x}_k$ are independent. As on p. 66 we can, however, choose our λ_i in such a way as to make r of the $\{ \}$ vanish, so that only

f terms are left in (6). The $\delta\ddot{x}_k$ of these remaining f terms can now be treated as independent. It follows that their f associated $\{ \}$ must vanish. We therefore arrive at Lagrange's equations of the first kind in the form (12.9).

Clearly the proof extends without change to non-holonomic constraints. Thus we are indeed confronted by a "new general fundamental principle of mechanics," as claimed by Gauss in the title of his paper. This fundamental principle is fully equivalent to d'Alembert's principle. Like the latter it is a *differential* principle in that it deals only with the present behavior of the system, not its future or past behavior. Here we do not need the rules of the calculus of variations, but only those of the ordinary differential calculus in the determination of the maxima and minima.

§ 39. Hertz's Principle of Least Curvature

Strictly speaking this principle is but a special case of that of Gauss. Nevertheless Hertz was able to call his principle, if not new, at least completely general; the reason for this is that he succeeded in replacing all forces by means of connections between the system in question and other systems interacting with it (cf. p. 5). Hertz was hence able to restrict himself to *systems under no forces*. In order to give the principle its sought-for geometric interpretation, he found himself obliged, moreover, to assume all masses to be multiples of a *unit mass*, say of atomic origin. The factor m_k in Gauss' expression (38.1) then becomes 1, while X_k becomes 0. It follows that (38.1) goes over to

$$(1) \quad Z = \sum_{k=1}^N \ddot{x}_k^2.$$

Here we have indicated by means of the upper index N of the summation that the number of unit masses of the system to be summed has been augmented in an unspecified manner by a suitable number of unit masses corresponding to the interacting systems coupled to the given system.

Let us change (1) by writing

$$\frac{d^2 x_k}{ds^2} \text{ in place of } \ddot{x}_k, \text{ where } (2) \quad ds^2 = \sum_{k=1}^N dx_k^2.$$

This is permitted because of the special form of the principle of energy. This principle is a consequence of Lagrange's equations of the first kind, and hence also of the principle of least constraint. For our present specialization the principle of energy can be written

$$\frac{1}{2} \sum_{k=1}^N \left(\frac{dx_k}{dt} \right)^2 = E$$

or, more concisely,

$$\left(\frac{ds}{dt}\right)^2 = \text{const.}$$

A division of (1) by the square of this constant yields thus the quantity

$$(3) \quad K = \sum_{k=1}^N \left(\frac{d^2 x_k}{ds^2}\right)^2.$$

Hertz calls ds the *element of line*, $K^{\frac{1}{2}}$ the *curvature* of the trajectory described by the system, and postulates

$$(4) \quad \delta K = 0.$$

Every free system remains in a state of rest or of uniform motion along a path of least curvature.

The mode of expression (cf. Art. 309 of Hertz's book cited earlier) is chosen so as to recall Newton's formulation of the first law.

The mathematical treatment of postulate (4) follows that of Gauss and, on the basis of the conditions of variation stipulated under (a) and (b) on p. 211, evidently leads to Lagrange's equations of the first kind for a system under no forces (with $m_k = 1$).

What justifies Hertz in calling ds the "line element" and $K^{\frac{1}{2}}$ the "curvature"? Evidently these concepts are to be interpreted in a poly-dimensional sense. We are not in three dimensions, but in an N -dimensional Euclidean space of coordinates x_1, x_2, \dots, x_N . In this space the element of line is indeed given by (2). We shall now discuss the cases of two and three dimensions in order to show that the square of the curvature of a trajectory is quite generally given by (3).

According to Eq. (5.10) we have, in the space of coordinates x_1, x_2 ,

$$(5) \quad K = \frac{1}{\rho^2} = \left(\frac{\Delta\epsilon}{\Delta s}\right)^2.$$

From Fig. 4b, $\Delta\epsilon$ is the angle between two neighboring tangents to the path whose points of contact with the path are a distance Δs apart. These tangents have direction cosines

$$(6) \quad \frac{dx_1}{ds}, \quad \frac{dx_2}{ds} \quad \text{and} \quad \frac{dx_1}{ds} + \frac{d^2 x_1}{ds^2} \Delta s, \quad \frac{dx_2}{ds} + \frac{d^2 x_2}{ds^2} \Delta s, \quad \text{respectively.}$$

Now these direction cosines are at the same time the coordinates of the two points formed by the intersection of a unit circle about the origin of coordinates with two radii drawn from the origin parallel to the tangents; moreover the angle $\Delta\epsilon$ is measured by the arc of distance between these two points of intersection. According to (6) we therefore have

$$\Delta\epsilon^2 = \left[\left(\frac{d^2x_1}{ds^2} \right)^2 + \left(\frac{d^2x_2}{ds^2} \right)^2 \right] \Delta s^2$$

and from (5),

$$(7) \quad K = \left(\frac{d^2x_1}{ds^2} \right)^2 + \left(\frac{d^2x_2}{ds^2} \right)^2.$$

In the space of the three coordinates x_1, x_2, x_3 , $\Delta\epsilon$ is once again the angle between neighboring tangents to the three-dimensional trajectories. The unit circle is now replaced by a unit sphere through the center of which parallels to the two tangents are to be drawn. The distance between their points of intersection with the surface of the sphere measures $\Delta\epsilon$ in units of arc:

$$\Delta\epsilon^2 = \left[\left(\frac{d^2x_1}{ds^2} \right)^2 + \left(\frac{d^2x_2}{ds^2} \right)^2 + \left(\frac{d^2x_3}{ds^2} \right)^2 \right] \Delta s^2.$$

From (5) we thus obtain an expression for K which now has three terms.

The generalization to a space of N dimensions and to the equation (3) of N terms is now obvious.

With this we must conclude our report on the mechanics of Hertz. As mentioned on p. 5, his is an interesting and stimulating idea, carried out with great logic; because of the complicated replacement of forces by connections it has, however, borne little fruit.

§ 40. A Digression on Geodesics

We define as geodesics of an arbitrary curved surface the trajectories of mass points under no forces (hence no friction) constrained to move on the surface. Let the mass of a particle be equal to 1, and the equation of the surface $F(x, y, z) = 0$.

The principle of least action states that these geodesics are also the shortest possible lines or, more generally (cf. p. 208) lines whose lengths are extrema. Since conservation of energy holds, the velocity along the path is constant. By choosing the constant of energy properly we can put the velocity equal to 1 and therefore replace $\frac{d}{dt}$ by $\frac{d}{ds}$.

We obtain the basic definition of geodesics if we describe our trajectories by Lagrange's equations of the first kind. Written vectorially, these are, in our case,

$$(1) \quad \dot{v} = \lambda \text{ grad } F.$$

\dot{v} has the direction of the principal normal to the trajectory if, as in our case, $v = \text{const.}$ so that $\dot{v} = 0$ (cf. § 5, beginning of (3)); it follows (of. same place) that \dot{v} lies in the osculating plane. $\text{grad } F$, on the other hand, has

the direction of the normal to the surface, since for any translation (dx, dy, dz) on the surface we have

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0,$$

so that the direction

$$\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z}$$

is indeed normal to that of the displacement. Eq. (1) therefore contains the basic definition of geodesics which states that *the principal normal of a geodesic coincides with the normal to the surface*, or equivalently, *the osculating plane of a geodesic contains the normal to the surface*.

We now appeal to the principle of least curvature. According to it the geodesic has a smaller curvature than neighboring paths; the neighboring paths are, according to conditions (38.3), restricted to pass through the same point with the same tangent as the geodesic at the point considered. We obtain the total class of these neighboring paths by passing through the tangent in question all possible skew planes and determining their intersections with the surface; the plane containing the normal to the surface furnishes the geodesic. According to Hertz's principle these skew sections have a greater curvature than the normal section, or, equivalently, a smaller radius of curvature.

This fact is in agreement with Meusnier's theorem in the differential geometry of surfaces, which states that the radius of curvature of an oblique section equals the projection of the radius of curvature of the normal section on the plane of the oblique section. We thus recognize in Meusnier's theorem a quantitative expression of the general content of the principle of least curvature.

Let us finally apply Lagrange's equations of the second kind to our geodesics. We thereby enter the sphere of thought of Gauss' great treatise of 1827 ("Disquisitiones generales circa superficies curvas"), which, extended to four dimensions, is also the sphere of thought of the general theory of relativity.

While Lagrange introduces arbitrary curvilinear coordinates q , Gauss uses as coordinates on the surface two arbitrary families of curves which cover the surface with a "grid." As customary, we shall call them

$$(2) \quad u = \text{const.}, \quad v = \text{const.}$$

In these coordinates Gauss writes the line element ds in the form

$$(3) \quad ds^2 = E du^2 + 2F du dv + G dv^2.$$

The "first differential parameters" E , F and G are to be thought of as

functions of u and v . They are connected with the rectangular coordinates x, y, z of the points on the surface by the relations

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \quad G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2,$$

$$F = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v}.$$

The square of the line element divided by $2dt^2$ is the expression of the kinetic energy T of our (unit) mass point moving on the surface. We can thus transform the Lagrange equations for generalized coordinates to Gaussian notation by forming

$$p_u = \frac{\partial T}{\partial \dot{u}} = E\dot{u} + F\dot{v}$$

$$2\frac{\partial T}{\partial u} = \frac{\partial E}{\partial u}\dot{u}^2 + 2\frac{\partial F}{\partial u}\dot{u}\dot{v} + \frac{\partial G}{\partial u}\dot{v}^2.$$

If, finally, we put $\frac{ds}{dt}$ in place of $\frac{d}{dt}$, the differential equation of the geodesics is, according to the method of Lagrange,

$$(4) \quad \frac{d}{ds} \left(E \frac{du}{ds} + F \frac{dv}{ds} \right) = \frac{1}{2} \left\{ \frac{\partial E}{\partial u} \left(\frac{du}{ds} \right)^2 + 2 \frac{\partial F}{\partial u} \frac{du}{ds} \frac{dv}{ds} + \frac{\partial G}{\partial u} \left(\frac{dv}{ds} \right)^2 \right\}$$

for the u -coordinate. We need not write down the corresponding differential equation for the v -coordinate; by virtue of the principle of energy (in our case $\frac{ds}{dt} = 1$) it must be identical to (4).

Gauss derives Eq. (4) in Art. 18 of the cited treatise by means of the principle of the shortest path. Here we merely wanted to point out the fact that Gauss' method of general surface parameters (2) is equivalent to Lagrange's method of the mechanics of systems. Both methods are invariant with respect to an arbitrary transformation of coordinates and depend only on the intrinsic properties of the surface or of the mechanical system respectively.

CHAPTER VIII
THE THEORY OF HAMILTON
§ 41. *Hamilton's Equations*

In Lagrange's equations our independent variables were the q_k and \dot{q}_k . In Hamilton's equations, which we shall now derive in two different ways, the q_k and p_k are the independent variables; the latter is defined by Eq. (36.9a). Whereas the characteristic function of Lagrange's equations was the "free energy" $T - V$, regarded as function of the q_k and \dot{q}_k , in Hamilton's equations the characteristic function is the total energy $T + V$, regarded as function of the q_k and p_k . This function we call the Hamiltonian function or simply the *Hamiltonian*, and we designate it by $H(q, p)$ just as we called the free energy the Lagrangian and designated it by $L(q, \dot{q})$. Between H and L there exists relation (34.16), which we shall write

$$(1) \quad H = \sum p_k \dot{q}_k - L,$$

using the definition of the p_k .

Let us at once extend the basis of the theory by recalling the last part of § 37: we shall drop the decomposition of L into a kinetic and a potential contribution and, in addition, permit an explicit dependence on t . According to p. 190, such a dependence may arise if either the equations of constraint or the defining equations for the coordinates contain the time. We then write the Lagrangian in the generalized form

$$(1a) \quad L = L(t, q, \dot{q}).$$

Let us keep Eq. (1) as our definition of the Hamiltonian associated with L ,

$$(1b) \quad H = H(t, q, p)$$

although H then loses the meaning of total energy. As before, the p_k are given by the relation

$$(1c) \quad p_k = \frac{\partial L}{\partial \dot{q}_k}.$$

If we take Hamilton's principle

$$(1d) \quad \delta \int_{t_1}^{t_2} L dt = 0$$

as our fundamental principle of mechanics, we obtain Lagrange's equations

just as in § 34—in spite of the new extended meaning of L . For purposes of the following we shall write these equations in the form

$$(1e) \quad \dot{p}_k = \frac{\partial L}{\partial q_k}.$$

(1) Derivation of Hamilton's Equations from Lagrange's Equations

Let us write down the total differentials of H and L :

$$(2) \quad dH = \frac{\partial H}{\partial t} dt + \sum \frac{\partial H}{\partial q_k} dq_k + \sum \frac{\partial H}{\partial p_k} dp_k,$$

$$(2a) \quad dL = \frac{\partial L}{\partial t} dt + \sum \frac{\partial L}{\partial q_k} dq_k + \sum \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k,$$

and, by means of Lagrange's equations (1e) and the definition (1c) of the p_k , transform dL to

$$(2b) \quad dL = \frac{\partial L}{\partial t} dt + \sum \dot{p}_k dq_k + \sum p_k d\dot{q}_k.$$

Let us, on the other hand, form the total differential of (1) with the help of (2b):

$$(3) \quad dH = \sum \dot{q}_k dp_k + \sum p_k d\dot{q}_k - \frac{\partial L}{\partial t} dt - \sum \dot{p}_k dq_k - \sum p_k d\dot{q}_k.$$

Cancelling of the last term on the right against the second term yields

$$(3a) \quad dH = - \frac{\partial L}{\partial t} dt - \sum \dot{p}_k dq_k + \sum \dot{q}_k dp_k.$$

This expression for dH must, of course, be identical to that of Eq. (2). If we equate the coefficients of dt , we obtain

$$(3b) \quad \frac{\partial H}{\partial t} = - \frac{\partial L}{\partial t}.$$

Comparison of the coefficients of dq_k and dp_k yields

$$(4) \quad \dot{p}_k = - \frac{\partial H}{\partial q_k}, \quad \dot{q}_k = \frac{\partial H}{\partial p_k}.$$

These relations, exhibiting an amazing symmetry, are "Hamilton's ordinary differential equations" or, for short, *Hamilton's equations*.

Incidentally, they first occurred in the much earlier "*Mécanique analytique*" of Lagrange (Sec. 5, § 14), where they were, however, derived and put to use only for the special case of small vibrations.

(2) Derivation of Hamilton's Equations from Hamilton's Principle

In the light of (1) we write this principle in the form

$$\begin{aligned}
 (5) \quad & -\delta \int L dt = \delta \int [H(t, q, p) - \sum p_k \dot{q}_k] dt \\
 & = \sum_k \int \left(\frac{\partial H}{\partial q_k} \delta q_k + \frac{\partial H}{\partial p_k} \delta p_k - \dot{q}_k \delta p_k - p_k \delta \dot{q}_k \right) dt = 0,
 \end{aligned}$$

where we can transform the last term in the parenthesis by partial integration,

$$(6) \quad - \int_{t_0}^{t_1} p_k \delta \dot{q}_k dt = \int_{t_0}^{t_1} \dot{p}_k \delta q_k dt - p_k \delta q_k \Big|_{t_0}^{t_1}.$$

The integrated term vanishes because of the way in which the variation is carried out in Hamilton's principle. Substitution of (6) in (5), followed by collection of terms in δq_k and δp_k , yields

$$(7) \quad \sum_k \int \left(\left\{ \frac{\partial H}{\partial q_k} + \dot{p}_k \right\} \delta q_k + \left\{ \frac{\partial H}{\partial p_k} - \dot{q}_k \right\} \delta p_k \right) dt = 0.$$

If it were permitted to treat the δq_k and the δp_k as independent variations, one would be justified in putting the factors of δq_k and δp_k separately equal to 0 for every value of the index k , and so obtain Hamilton's equations (4). This, however, is not allowed; for while q_k and p_k enter in H as independent variables, they are related in time through Eq. (1c), a fact which might conceivably cause our Eq. (7) to be satisfied identically. We notice, however, that a partial differentiation of (1) with respect to p_k (q_k being held constant) causes the second $\{ \}$ of (7) to vanish identically. We conclude, therefore, that the first $\{ \}$ must vanish as well.

One of the reasons why we derived Hamilton's equations in the second way is that we wish now to make an important remark connected with it.

We know that Lagrange's equations are invariant under arbitrary "point transformations," i.e., that they keep their form if we replace the q_k by a new set of coordinates Q_k connected with the former by relations of the type

$$(8) \quad Q_k = f_k(q_1, q_2, \dots, q_f).$$

The associated P_k are then given by

$$(8a) \quad P_k = \frac{\partial L}{\partial \dot{Q}_k} = \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \dot{Q}_k} = \sum_i p_i a_{ik},$$

that is, by linear functions of the p_i whose coefficients a_{ik} are functions of the q_k , just as in (36.3).

We shall now show that Hamilton's equations are invariant under the much more general transformations

$$\begin{aligned}
 (9) \quad & Q_k = f_k(q, p) \\
 & P_k = g_k(q, p),
 \end{aligned}$$

where the f_k and g_k are arbitrary functions of the two sets of variables q_k and p_k —arbitrary, that is, to within a restriction to be mentioned below. In particular, the g_k need no longer be linear in the p_k .

Let us suppose that Eqs. (9) are solved for the q, p in terms of the Q, P [we must of course require that Eqs. (9) be so constituted that this is possible] and are substituted in the expression $H(q, p)$. Let us call \bar{H} this new transformed Hamiltonian. We then have

$$(10) \quad H(q, p) = \bar{H}(Q, P).$$

Let us, moreover, compare the quantity $\sum p_k \dot{q}_k$ occurring in (5) with $\sum P_k \dot{Q}_k$. It is easy to see that the two expressions would be equal in a transformation (8), (8a). We now require that this equality be maintained in a general transformation (9), apart from an additive term. The latter we require to be a complete time derivative of a function F' of the q and p , or, alternatively, of a function F of the q and Q ¹. We hence put

$$(11) \quad \sum p_k \dot{q}_k = \sum P_k \dot{Q}_k + \frac{d}{dt} F(q, Q)$$

with arbitrary F . This is the restriction on transformation (9) mentioned above.

In the substitution of Eqs. (10) and (11) in (5) the additional term $\frac{dF}{dt}$ vanishes in the integration and subsequent variation, since δq and δQ vanish at the endpoints; Eq. (5) then retains its earlier form, becoming

$$\delta \int (\bar{H}(Q, P) - \sum P_k \dot{Q}_k) dt = 0.$$

Furthermore, nothing is changed in our transformations (6) and (7); we conclude that Hamilton's equations remain valid in the new variables. In complete correspondence to our Eqs. (4) we now have

$$(12) \quad \dot{P}_k = - \frac{\partial \bar{H}}{\partial Q_k}, \quad \dot{Q}_k = \frac{\partial \bar{H}}{\partial P_k}.$$

Transformations (9), as subjected to restriction (11), are called *canonical transformations* or² *contact transformations*. The reason for the latter name

¹ If F' is originally given as a function of q and p , we can of course solve for p from the first Eq. (9) and substitute it in F' , thus obtaining a new function F of q and Q .

² The terms are not entirely synonymous, their difference being one of definition. We need not be detained with this difference, but remark that under suitable conditions either of the two transformations can be shown to be a special case of the other. Cf., for instance, Whittaker, *Analytical Dynamics* (Dover), Chapter XI, or Osgood, *Mechanics* (Macmillan), Chapter XIV.—TRANSLATOR.

is a geometric one. Let us consider a hypersurface in the f -dimensional space of the q_1, q_2, \dots, q_f , given by

$$(13) \quad z = z(q_1, \dots, q_f);$$

the quantities

$$p_k = \frac{\partial z}{\partial q_k}$$

determine the position of the tangent plane to the hypersurface and can, for this reason, be interpreted as the coordinates of this plane. We require that there exist a condition

$$(14) \quad dz = \sum_{k=1}^f p_k dq_k$$

between the coordinates of the point q_k and those of the plane p_k . This condition insures the "union of lineal elements," i.e., the continuity of the coordinates p_k as we pass from an arbitrary point of coordinates q_k to a neighboring point. Let us now introduce new coordinates Q_k, P_k by means of Eq. (9) and calculate (13) in terms of these new coordinates. Let the result be

$$z = Z(Q, P).$$

We now demand that this new expression again represent a hypersurface touched by the planes of coordinates P at the points determined by the Q . From (14) we must therefore have

$$(15) \quad dZ = \sum_{k=1}^f P_k dQ_k,$$

or, with ρ a factor of proportionality,

$$(16) \quad dZ - \sum P_k dQ_k = \rho (dz - \sum p_k dq_k).$$

Thus the contact between the surface and its tangent plane at a given point has been preserved in a transformation of the point. Let us compare condition (16) with Eq. (11), which, when multiplied by dt , can be written

$$(16a) \quad \sum p_k dq_k = \sum P_k dQ_k + dF.$$

If we put $dF = dz - dZ$ in (16a) and $\rho = 1$ in (16), the two conditions are in agreement. This may constitute adequate justification for the name "contact transformation."

In transformations of the generality of Eqs. (9) the meaning of the P_k as components of momentum is obscured. For this reason we prefer to

call the P_k, Q_k *canonical variables*; P_k and Q_k are then said to be *canonically conjugate*. Because Hamilton's equations are invariant under transformations (9) [with restriction (11)], they are often called *Hamilton's canonical equations*.

It is to this invariance under canonical transformations that Hamilton's equations owe their special significance in astronomical perturbation theory. They also play an important role in the statistical mechanics of Gibbs, a topic which we shall discuss in Vol. V.

We conclude our treatment of Hamilton's equations with a remark dealing with the principle of energy.

In agreement with Eq. (2) we have, quite generally,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_k \left(\frac{\partial H}{\partial q_k} \dot{q}_k + \frac{\partial H}{\partial p_k} \dot{p}_k \right).$$

According to (4), the parenthesis vanishes for all k . We then obtain, in general,

$$(17) \quad \frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

If, in particular, H does not depend *explicitly* on t , we arrive at the *conservation law*

$$(18) \quad \frac{dH}{dt} = 0, \quad H = \text{const.}$$

This law is more general than that of the conservation of energy, for, according to (1) and (1c), it states that

$$(18a) \quad \sum \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L = \text{const.}$$

where L must not depend explicitly on t , but otherwise can be quite arbitrary. It is this conservation law to which we alluded in footnote 3 of Chapter VI. Eq. (18a) leads to the *conservation of energy* if L can be split up into two contributions, a kinetic one homogeneous of second degree in the \dot{q}_k , and a potential one independent of the \dot{q}_k .

§ 42. Routh's Equations and Cyclic Systems

In Eqs. (10) and (11) of § 34 we considered a "mixed type" of equation resulting from a combination of Lagrange equations of the first and the second kind. We shall now become acquainted with a mixed type of equation arising from a combination of Lagrange's equations of the second

kind with those of Hamilton. The new equations bear the name of Routh³ who, for several decades, dominated the study of mechanics in Cambridge as "coach" and examiner in the "tripos." Somewhat later Helmholtz⁴ developed the same equations as the basis of his theory of monocyclic and polycyclic systems, a theory which he intended to use in the solution of the fundamental problems of thermodynamics.

We subdivide the degrees of freedom of the system into two groups. One group, containing $f-r$ degrees of freedom, can be described by Lagrange's position and velocity coordinates

$$q_1, q_2, \dots, q_{f-r}; \quad \dot{q}_1, \dot{q}_2, \dots, \dot{q}_{f-r},$$

and the other, containing r degrees of freedom, is to be represented in terms of Hamilton's canonical variables

$$q_{f-r+1}, q_{f-r+2}, \dots, q_f; \quad p_{f-r+1}, p_{f-r+2}, \dots, p_f.$$

Instead of the Lagrangian L or the Hamiltonian H we now construct a Routh function R , which is to be a function of the $2f$ variables enumerated above and, for the sake of generality, of the time as well:

$$(1) \quad R(t, q_1, q_2, \dots, q_f; \quad \dot{q}_1, \dot{q}_2, \dots, \dot{q}_{f-r}, p_{f-r+1}, \dots, p_f).$$

R is to be defined by the equation

$$(2) \quad R = \sum_{k=f-r+1}^f p_k \dot{q}_k - L(t, q_1, \dots, q_f; \dot{q}_1, \dots, \dot{q}_f).$$

We see that for $r=f$, R transforms to the Hamiltonian (41.1); for $r=0$, where the summation on the right vanishes, it goes over into the Lagrangian (apart from sign). Evidently we could have replaced definition (2) of R by the equivalent condition

$$(2a) \quad R = H(t, q_1, \dots, q_f; p_1, \dots, p_f) - \sum_{k=1}^{f-r} p_k \dot{q}_k.$$

We now proceed as in Eqs. (41.2) to (41.4). We form the total differential of R , on the one hand from (1),

³ In this connection we wish to mention the two volumes of Routh's *Treatise on the Dynamics of a System of Rigid Bodies*; I, Elementary Part, II, Advanced Part. It is a collection of problems of unique variety and richness. Routh first developed his form of the dynamical equations in the prize article *A Treatise of Stability of a Given State of Motion* (1877).

⁴ *Berliner Akad.* (1884) and *Crelle's Journal f. Math.* 97.

$$(3) \quad dR = \frac{\partial R}{\partial t} dt + \sum_{k=1}^f \frac{\partial R}{\partial q_k} dq_k + \sum_{k=1}^{f-r} \frac{\partial R}{\partial \dot{q}_k} d\dot{q}_k + \sum_{k=f-r+1}^f \frac{\partial R}{\partial p_k} dp_k,$$

and on the other, from (2),

$$(3a) \quad dR = \sum_{k=f-r+1}^f \dot{q}_k dp_k + \sum_{k=f-r+1}^f p_k d\dot{q}_k - dL,$$

For dL we can use the expression (41.2b), which we shall, for greater clarity, decompose into

$$(3b) \quad dL = \frac{\partial L}{\partial t} dt + \sum_{k=1}^f \dot{p}_k dq_k + \sum_{k=1}^{f-r} p_k d\dot{q}_k + \sum_{k=f-r+1}^f p_k d\dot{q}_k.$$

Substitution in (3a) causes the last term of (3b) to cancel against the middle term of (3a), so that we are left with

$$(4) \quad dR = -\frac{\partial L}{\partial t} dt - \sum_{k=1}^f \dot{p}_k dq_k - \sum_{k=1}^{f-r} p_k d\dot{q}_k + \sum_{k=f-r+1}^f \dot{q}_k dp_k.$$

A term-by-term comparison with (3) yields the relation

$$\frac{\partial R}{\partial t} = -\frac{\partial L}{\partial t}$$

and the scheme of equations given below:

for $k=1, 2, \dots, f-r$	for $k=f-r+1, f-r+2, \dots, f$
$\dot{p}_k = -\frac{\partial R}{\partial q_k}$ $p_k = -\frac{\partial R}{\partial \dot{q}_k}$	$\dot{p}_k = -\frac{\partial R}{\partial q_k}$ $\dot{q}_k = +\frac{\partial R}{\partial p_k}$

The $f-r$ equations on the left are of the Lagrange type with $L = -R$, whereas the r equations on the right are of the Hamiltonian type with $H = R$.

The application of these equations to cyclic systems, which Routh had in mind when he formulated them, proceeds as follows: we assume that the coordinates of the second group are cyclic, so that, from p. 197, they do not occur in the Lagrangian; in that case neither do they occur in the Routh function. The associated p_k are then constant [from the upper equation of the right group of Routh's equations (5) or, as remarked on p. 198, from Lagrange's equations]. We can now replace these constant values of

the p_k , and, with the help of Eq. (41.1c), those of the (generally not constant) associated \dot{q}_k in Eq. (2). We thus obtain a Routh function which depends only on the $f-r$ coordinates of the first group of q_k and \dot{q}_k . For these coordinates the left group of Eq. (5) above is valid. We have, therefore, reduced the problem to $f-r$ equations of the Lagrange type.

Routh used his method chiefly in the difficult problems of the stability of given states of motion. Let us instead illustrate the method with a reasonably simple example, that of the symmetrical top. The cyclic coordinates of this doubly cyclic problem are the Eulerian angles ϕ and ψ ; according to Eqs. (35.15) to (35.17), we have

$$\begin{aligned} p_\phi \ddot{\phi} + p_\psi \ddot{\psi} &= M'' \left(\frac{M''}{I_3} - \cos \theta \frac{M' - M'' \cos \theta}{I_1 \sin^2 \theta} \right) + M' \frac{M' - M'' \cos \theta}{I_1 \sin^2 \theta} \\ &= \frac{M''^2}{I_3} + \frac{(M' - M'' \cos \theta)^2}{I_1 \sin^2 \theta}; \end{aligned}$$

by virtue of (35.13) the Routh function then becomes

$$\begin{aligned} R &= \frac{M''^2}{I_3} + \frac{(M' - M'' \cos \theta)^2}{I_1 \sin^2 \theta} - \frac{I_1}{2} \dot{\theta}^2 - \frac{(M' - M'' \cos \theta)^2}{2I_1 \sin^2 \theta} - \frac{M''^2}{2I_3} + P \cos \theta \\ &= -\frac{I_1}{2} \dot{\theta}^2 + \Theta(\theta), \quad \Theta = \frac{M''^2}{2I_3} + \frac{(M' - M'' \cos \theta)^2}{2I_1 \sin^2 \theta} + P \cos \theta. \end{aligned}$$

With $q_k = \theta$, the lower equation in the left group of our present Eqs. (5) then yields

$$p_k = I_1 \dot{\theta}$$

and the upper equation of the same group gives

$$(6) \quad I_1 \ddot{\theta} = -\frac{\partial \Theta}{\partial \theta},$$

which is, of course, in agreement with the "generalized pendulum equation" (35.19). This example may serve to illustrate the usefulness of Routh's method, particularly for problems more difficult than the one presented.

In 1891 Boltzmann gave a series of lectures on Maxwell's electromagnetic theory at the University of Munich. He devoted his first lectures to the detailed consideration of a doubly cyclic mechanical system in order to illustrate the mutual inductive effect between two electrical circuits. The carefully worked mechanical model, consisting mainly of two pairs of beveled gears with centrifugal governors, is preserved in the museum of our Institute. To us it seems much more complicated than Maxwell's theory which it was intended to illustrate. Hence we shall not use it to clarify this theory, but instead take advantage of it in an exercise on the differential of an automobile, to which it is similar in its essential features.

Let us finally generalize the mathematical formalism which led us from Lagrange's to Hamilton's and to Routh's equations. We consider a function Z of two variables (or two sets of variables) x and y , and let

$$(7) \quad dZ(x, y) = X dx + Y dy.$$

If we wish to replace x, y by X, Y as independent variables, it is convenient to consider, instead of Z , the "modified function"

$$(8) \quad U(X, Y) = xX + yY - Z(x, y).$$

Indeed a differentiation of (8) at once gives, in view of (7),

$$(9) \quad dU(X, Y) = x dX + y dY.$$

Eqs. (7) and (9) are identical to the "reciprocity relations"

$$(10) \quad \begin{aligned} \frac{\partial Z}{\partial x} &= X, & \frac{\partial Z}{\partial y} &= Y, \\ \frac{\partial U}{\partial X} &= x, & \frac{\partial U}{\partial Y} &= y. \end{aligned}$$

If, on the other hand, we wish to replace only one of the original variables, say y , by its "canonically conjugate" Y , we shall have to "modify" (8) to

$$(11) \quad V(x, Y) = yY - Z,$$

which yields

$$(12) \quad dV(x, Y) = -X dx + y dY$$

with the "reciprocity relations"

$$(13) \quad \frac{\partial V}{\partial x} = -X, \quad \frac{\partial V}{\partial Y} = y.$$

The transition from Z to U can be compared with that from Lagrange to Hamilton, that from Z to V with the transition from Lagrange to Routh.

Such a change of independent variables and the attendant modification of the characteristic function is called a *Legendre transformation* and plays an extensive role in analysis. We have mentioned it chiefly in order to be able to refer to it in our study of thermodynamics (Vol. V).

§ 43. The Differential Equations for Non-Holonomic Velocity Parameters

Whereas the differential equations considered so far were all modeled after those of Lagrange for generalized coordinates, the theory of the spinning top brought us in contact with equations of an entirely different, much simpler structure, viz., Euler's Eqs. (26.4) for the angular velocities ω_1 ,

ω_1 and ω_3 . Let us determine what relation they bear to Lagrange's equations.

The difference between the two types stems from the fact that the $\omega_1, \omega_2, \omega_3$ are not holonomic coordinates like the θ, ψ, ϕ , but linear functions of these which are not integrable with respect to t . The connection between them is given by Eq. (35.11). Let us immediately consider the unsymmetrical top with kinetic energy

$$(1) \quad T = \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2),$$

and for brevity restrict ourselves to the case of a top under no forces.

We start out with Lagrange's equation for the ϕ -coordinate

$$(2) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = 0.$$

According to (35.11)

$$\begin{aligned} \frac{\partial \omega_1}{\partial \dot{\phi}} &= \frac{\partial \omega_2}{\partial \dot{\phi}} = 0, & \frac{\partial \omega_3}{\partial \dot{\phi}} &= 1, \\ \frac{\partial \omega_1}{\partial \phi} &= \omega_2, & \frac{\partial \omega_2}{\partial \phi} &= -\omega_1, & \frac{\partial \omega_3}{\partial \phi} &= 0, \end{aligned}$$

so that, in view of (1),

$$\begin{aligned} \frac{\partial T}{\partial \dot{\phi}} &= I_1 \omega_1 \frac{\partial \omega_1}{\partial \dot{\phi}} + I_2 \omega_2 \frac{\partial \omega_2}{\partial \dot{\phi}} + I_3 \omega_3 \frac{\partial \omega_3}{\partial \dot{\phi}} = I_3 \omega_3, \\ \frac{\partial T}{\partial \phi} &= I_1 \omega_1 \frac{\partial \omega_1}{\partial \phi} + I_2 \omega_2 \frac{\partial \omega_2}{\partial \phi} + I_3 \omega_3 \frac{\partial \omega_3}{\partial \phi} = (I_1 - I_2) \omega_1 \omega_2. \end{aligned}$$

From (2) we then have

$$(3) \quad I_3 \frac{d\omega_3}{dt} = (I_1 - I_2) \omega_1 \omega_2.$$

This is the third Euler Eq. (26.4).

A similar calculation for the θ -coordinate yields

$$\begin{aligned} \frac{\partial \omega_1}{\partial \dot{\theta}} &= \cos \phi, & \frac{\partial \omega_2}{\partial \dot{\theta}} &= -\sin \phi, & \frac{\partial \omega_3}{\partial \dot{\theta}} &= 0, \\ \frac{\partial \omega_1}{\partial \theta} &= \dot{\psi} \cos \theta \sin \phi, & \frac{\partial \omega_2}{\partial \theta} &= \dot{\psi} \cos \theta \cos \phi, & \frac{\partial \omega_3}{\partial \theta} &= -\dot{\psi} \sin \theta. \end{aligned}$$

From (1) we obtain

$$\begin{aligned} \frac{\partial T}{\partial \dot{\theta}} &= I_1 \omega_1 \cos \phi - I_2 \omega_2 \sin \phi, \\ \frac{\partial T}{\partial \theta} &= (I_1 \omega_1 \sin \phi + I_2 \omega_2 \cos \phi) \dot{\psi} \cos \theta - I_3 \omega_3 \dot{\psi} \sin \theta. \end{aligned}$$

Lagrange's equation

$$(4) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = 0$$

hence becomes

$$(5) \quad \begin{aligned} 0 = & I_1 \frac{d\omega_1}{dt} \cos \phi - I_1 \frac{d\omega_2}{dt} \sin \phi \\ & - I_1 \omega_1 \sin \phi (\dot{\phi} + \dot{\psi} \cos \theta) - I_2 \omega_2 \cos \phi (\dot{\phi} + \dot{\psi} \cos \theta) \\ & + I_3 \omega_3 \dot{\psi} \sin \theta. \end{aligned}$$

But according to (35.11),

$$\dot{\phi} + \dot{\psi} \cos \theta = \omega_3, \quad \dot{\psi} \sin \theta = \omega_1 \sin \phi + \omega_2 \cos \phi,$$

so that the second and third lines of (5) can be written

$$(I_2 - I_1) \omega_2 \omega_1 \sin \phi - (I_2 - I_3) \omega_2 \omega_3 \cos \phi$$

and, together with the first line,

$$(6) \quad 0 = \left\{ I_1 \frac{d\omega_1}{dt} - (I_2 - I_3) \omega_2 \omega_3 \right\} \cos \phi - \left\{ I_2 \frac{d\omega_2}{dt} - (I_3 - I_1) \omega_3 \omega_1 \right\} \sin \phi.$$

Finally the Lagrange Equation

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} = 0$$

becomes, after suitable transformation of variables and in view of (3),

$$(7) \quad 0 = \left\{ I_1 \frac{d\omega_1}{dt} - (I_2 - I_3) \omega_2 \omega_3 \right\} \sin \phi - \left\{ I_2 \frac{d\omega_2}{dt} - (I_3 - I_1) \omega_3 \omega_1 \right\} \cos \phi.$$

It follows from (6) and (7) that both $\{\}$ must necessarily vanish, so that we obtain the first and second Euler equations (26.4).

The transformation which we have carried out for one specific example can be performed quite generally⁵ in the case of an arbitrary number of non-holonomic velocity parameters defined as linear (or more general) functions of real velocity coordinates. If, as in the case of the rigid body, the kinetic energy takes an especially simple form when expressed in terms of these parameters, such transformations can be of signal value for the integration of the equations of motion; they can also be useful in that they may satisfy non-holonomic conditions. Boltzmann found it necessary

⁵ Cf., in particular, G. Hamel, *Math. Ann.* 59 (1904), and *Sitzungsber. der Berl. Math. Ges.* 37 (1938). Furthermore, *Encykl. d. Math. Wiss.* IV.2, Art. Prange No. 3 and ff.

to introduce the components of momentum corresponding to non-holonomic velocities in the kinetic theory of gases. He called these components "momentoids."

§ 44. The Hamilton-Jacobi Equation

At the beginning of the previous century the most burning question of theoretical physics was, "wave theory or corpuscular theory of light?" The wave theory was founded by Huygens and, at the time mentioned, found its confirmation in Thomas Young's discovery of the phenomenon of interference. The corpuscular theory, on the other hand, had Newton's seemingly authoritative backing. W. R. Hamilton, astronomer and profound mathematical thinker, was just then engaged in a study of the paths of light rays in optical instruments. The results of these studies* began to appear in print in 1827, at about the time at which the two greatest advocates of wave optics, Fraunhofer and Fresnel, died at almost the same early age. Hamilton's work on general dynamics, the results of which we shall briefly summarize in this section, came somewhat later, but it is intimately related to his investigations in ray optics†.

Let us add parenthetically that as a result of Planck's discovery of the elementary quantum of action the above-mentioned question must now be posed differently. We no longer ask, "waves or corpuscles?" but state, "waves *as well as* corpuscles!" It seems at first sight impossible to reconcile these apparently contradictory concepts; actually they are complementary rather than contradictory aspects both of optics and of dynamics. Their reconciliation, as Schrödinger has recognized, results from a logical extension of Hamilton's ideas and leads to *wave or quantum mechanics*.

Ray optics is the mechanics of light particles; in optically inhomogeneous media the paths of these particles are by no means straight lines, but are determined by Hamilton's ordinary differential equations or Hamilton's principle which is equivalent to them. From the viewpoint of wave optics, on the other hand, the rays of light are given by the orthogonal[‡] trajectories of a system of wave surfaces or wave fronts. According to Huygens' principle, these wave fronts are parallel surfaces. Hamilton under-

* Treatises on ray optics, *Trans. Roy. Irish Acad.* 1827, with supplements of 1830 and 1832. His work on dynamics appeared in the *Trans. Roy. Soc. London* 1834 and 1835.

† In the formulation by Jacobi this connection was lost. It was newly worked out in 1891 by F. Klein (*Naturforscher-Ges. in Halle; Ges. Abhandl.*, Vol. II, pp. 601, 603).

‡ This is true of optically isotropic media. In anisotropic media such as crystals the orthogonality between ray and wave front is no longer an ordinary Euclidean one, but a non-Euclidean, generalized tensor orthogonality.

took to represent the family of wave surfaces by a (perforce partial) differential equation and to extend this method to the polydimensional space of the q_k of an arbitrary mechanical system. As we shall see, the family of wave surfaces is then given by $S = \text{const.}$, where S is the least action function of Eq. (37.1). The trajectories orthogonal to the surfaces are determined by the equation

$$(1) \quad p_k = \frac{\partial S}{\partial q_k}.$$

(1) Conservative Systems

For the moment we deal with a mechanical system in which energy is conserved and can be resolved into a kinetic part T and a potential part V . T , V and H are hence not explicitly dependent on t .

We start out with Eq. (37.9), and replace δW in the right member by

$$-\delta V = \delta(T - E) = \delta T - \delta E.$$

The right member of (37.9) then becomes

$$(2) \quad 2\delta T + 2T \frac{d}{dt} \delta t - \delta E.$$

Next we transform the left member of the same equation to generalized coordinates p , q ,

$$(3) \quad \frac{d}{dt} \sum p_k \delta q_k.$$

Equating (3) and (2) yields

$$(4) \quad 2\delta T + 2T \frac{d}{dt} \delta t - \delta E = \frac{d}{dt} \sum p_k \delta q_k.$$

We integrate (4) with respect to t between the limits 0 and t to obtain

$$(5) \quad \delta S - t \delta E = \sum p \delta q - \sum p_0 \delta q_0,$$

where S is defined by Eq. (37.1) and p_0 and δq_0 refer to the lower limit $t=0$ of the integration, p and δq to its upper limit t .

Eq. (5) indicates that we must regard the action integral S as a function of the initial position q_0 , the final position q and the energy E , i.e., that we are to use the arbitrarily assigned total energy E as variable in place of the time t :

$$(6) \quad S = S(q, q_0, E).$$

According to (5), the motion as a function of time is then given by

$$(7) \quad t = \frac{\partial S}{\partial E}.$$

where q_0 and q are held fixed. If, instead, we keep E fixed and vary either q or q_0 , (5) yields

$$(8) \quad p = \frac{\partial S}{\partial q}, \quad p_0 = - \frac{\partial S}{\partial q_0}.$$

The first of these relations is in agreement with our assertion (1); as for the second, we shall soon transform it to a more convenient form.

It must be admitted that we have not gained much in the way of knowledge of the motion as long as S is not known in the form (6). Let us, however, recall the equation of energy

$$H(q, p) = E,$$

where we substitute the value of p from Eq. (8) to obtain

$$(9) \quad H\left(q, \frac{\partial S}{\partial q}\right) = E.$$

We regard (9) as the determining equation for the characteristic function S . It is called "Hamilton's partial differential equation" or the *Hamilton-Jacobi equation* for conservative systems. With T homogeneous of second degree in the p (V can be assumed independent of the p), it is of *second degree and first order*.

Let us suppose that we have found a *complete integral* of this equation, i.e., a solution that contains a number of assignable constants equal to the number of degrees of freedom of the problem. Call these constants

$$\alpha_1, \alpha_2, \dots, \alpha_f.$$

Since S itself does not occur in (9), it is determined by (9) only up to an additive constant. One of the above constants of integration, say α_1 , is, therefore, in excess and can be replaced by an additive constant which remains unassigned. We may replace α_1 by our energy parameter E , so that the complete integral can be written in the form

$$(10) \quad S = S(q, E, \alpha_2, \alpha_3, \dots, \alpha_f) + \text{const.}$$

The classic method used to arrive at such a complete solution is that of *separation of variables*—a method often, but not always, applicable. We shall deal with this method in § 46. In § 45 we shall show how Eq. (10) leads to a knowledge of the motion of the system.

(2) Dissipative Systems

We shall now adopt the general viewpoint that the Lagrangian L and hence the Hamiltonian H depend on t . In general it is then impossible to decompose L and H into T and V ; if, in particular, a potential energy V does exist, it will have to depend on the time. This case is important for

the perturbation problems of astronomy and quantum mechanics. There exists then no energy principle, hence no total energy constant E . It follows that we cannot use the principle of least action, but must revert to Hamilton's principle. Consequently we define a characteristic function S^* , given by the integral occurring in Hamilton's principle,

$$(11) \quad S^* = \int_{t_0}^t L dt,$$

and regard S^* as function of the initial and final positions and of the *time of travel* t ,

$$(12) \quad S^* = S^*(q, q_0, t).$$

This is to be compared to Eq. (6) where the constant total energy E (non-existent in the present case) took the place of t .

Let us now form $\frac{dS^*}{dt}$, first by means of (11),

$$(13) \quad \frac{dS^*}{dt} = L,$$

next by means of (12),

$$(14) \quad \frac{dS^*}{dt} = \sum \frac{\partial S^*}{\partial q_k} \dot{q}_k + \frac{\partial S^*}{\partial t} = \sum p_k \dot{q}_k + \frac{\partial S^*}{\partial t}.$$

The relation analogous to (8) used here,

$$(15) \quad p_k = \frac{\partial S^*}{\partial q_k},$$

is easily verified. Merely differentiate (11) with respect to q_k and recall Eq. (41.1e).

In view of the general definition (41.1) of H , the comparison of (13) and (14) now yields

$$(16) \quad \frac{\partial S^*}{\partial t} + H = 0;$$

from Eq. (15) we have, therefore,

$$(17) \quad \frac{\partial S^*}{\partial t} + H\left(q, \frac{\partial S^*}{\partial q}, t\right) = 0.$$

This is the *Hamilton-Jacobi equation in general form*. It includes our earlier Eq. (9) as a special case. To show this, let us assume, as in (a), that H is independent of t . From (17) it follows that S^* is linear in t . Hence we put

$$S^* = at + b$$

and conclude from (16) that $-a=H$, i.e., equal to the energy constant E which now exists; b proves to be identical to our former characteristic function S . Thus Eq. (17) reduces to the special form (9) in this case.

The remarks made in (a) concerning the integration of (9) apply equally well to the more general Eq. (17). The complete integral of the latter now contains $f+1$ constants, one of which is again additive. Instead of (10) we can then write

$$(18) \quad S^* = S^*(q, t, \alpha_1, \alpha_2, \dots, \alpha_f) + \text{const.}$$

§ 45. Jacobi's Rule on the Integration of the Hamilton-Jacobi Equation

We stated in connection with Eqs. (44.8) that the second of these did not lend itself to ready integration. The reason for this is that we integrated our partial differential equation, not in the form (44.6), but in the forms (44.10) and (44.18), respectively. In Eq. (44.7),

$$(1) \quad t = \frac{\partial S}{\partial H},$$

on the other hand, we had obtained an equation describing, very directly, *the motion in time*. We shall now prove that if we differentiate S with respect to the constants of integration $\alpha_2, \alpha_3, \dots, \alpha_f$ instead of E , we obtain equations

$$(2) \quad \beta_k = \frac{\partial S}{\partial \alpha_k}, \quad k=2, 3, \dots, f$$

which describe *the geometric configuration of the path of the system, provided that we regard the β_k as a second set of constants of integration*. This is Jacobi's rule for the case (a). In the case (b) it takes the even simpler form

$$(3) \quad \beta_k = \frac{\partial S^*}{\partial \alpha_k}, \quad k=1, 2, \dots, f.$$

Here we have f equations of uniform structure which give both the *temporal* and the *spatial course of the motion of the system*.

We can introduce the same simplicity into case (a) by formally writing

$$(3a) \quad \beta_1 = \frac{\partial S}{\partial \alpha_1},$$

where we have put $t = \beta_1$ and $E = \alpha_1$.

We shall restrict our proof to case (a). Let us recall the definition (41.11) of a contact transformation, which we shall write, for purposes of the following,

$$(4) \quad dF(q, Q) = \sum p_k dq_k - \sum P_k dQ_k.$$

Let us compare this with the total differential of the characteristic function (44.10),

$$dS(q, E, \alpha) = \sum_{k=1}^f \frac{\partial S}{\partial q_k} dq_k + \frac{\partial S}{\partial E} dE + \sum_{k=2}^f \frac{\partial S}{\partial \alpha_k} d\alpha_k,$$

which becomes, with substitution from (44.8) and (2), (3a) of this section,

$$(5) \quad dS(q, \alpha) = \sum_{k=1}^f p_k dq_k + \sum_{k=1}^f \beta_k d\alpha_k.$$

This equation agrees with (4) if we identify

$$(6) \quad F \text{ with } S, \quad Q_k \text{ with } \alpha_k, \quad P_k \text{ with } -\beta_k.$$

Now we know that, by means of a transformation $q_k, p_k \rightarrow Q_k, P_k$ satisfying condition (4), we pass from Hamilton's equations (41.4)

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad \dot{q}_k = \frac{\partial H}{\partial p_k}$$

to Eqs. (41.12),

$$\dot{P}_k = -\frac{\partial \bar{H}}{\partial Q_k}, \quad \dot{Q}_k = \frac{\partial \bar{H}}{\partial P_k}.$$

In view of (6) these become, in our case,

$$(7) \quad -\dot{\beta}_k = -\frac{\partial \bar{H}}{\partial \alpha_k}, \quad \dot{\alpha}_k = -\frac{\partial \bar{H}}{\partial \beta_k}.$$

But from (41.10),

$$\bar{H}(Q, P) = H(q, p),$$

or, by virtue of (6),

$$\bar{H}(\alpha, -\beta) = E = \alpha_1.$$

It follows that

$$(9) \quad \frac{\partial \bar{H}}{\partial \alpha_k} = \begin{cases} 1 & \text{for } k=1, \\ 0 & \text{for } k>1; \end{cases} \quad \frac{\partial \bar{H}}{\partial \beta_k} = \begin{cases} 0 & \text{for } k=1, \\ 0 & \text{for } k>1. \end{cases}$$

Thus Eqs. (7) become

$$(10) \quad \dot{\beta}_k = \begin{cases} 1 & \text{for } k=1, \\ 0 & \text{for } k>1; \end{cases} \quad \dot{\alpha}_k = \begin{cases} 0 & \text{for } k=1, \\ 0 & \text{for } k>1. \end{cases}$$

These equations tell us nothing new regarding the α_k ; they merely confirm that the α_k are constants of integration. The same can be said of the

equation for β_1 ; from $\dot{\beta}_1=1$ we simply have $\beta_1=t$ (to within an additive constant of no importance), nothing new in view of Eq. (3a). Eqs. (10) for β_k with $k>1$, on the other hand, furnish the proof of Jacobi's rule; they state that, like the α_k , the β_k are integration constants.

The proof can be extended without important changes to the case (b) provided we make the definition of a contact transformation somewhat more general. Since we do not need this result in the following, we shall not be detained by it.

§ 46. Classical and Quantum-Theoretical Treatment of the Kepler Problem

In this section we wish to show how the Hamilton-Jacobi method of integration leads unambiguously and directly to the solution of the planetary problem of astronomy. We shall, furthermore, discover with surprise that the same method is made to order for the requirements of atomic physics and leads in a natural way to the (older) quantum theory.

We begin with the Lagrangian of the two-body problem with fixed sun M , expressed in polar coordinates,

$$(1) \quad L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2) + G \frac{mM}{r}$$

from which we calculate the momenta

$$(1a) \quad p_r = m\dot{r}, \quad p_\phi = mr^2\dot{\phi}.$$

Substitution of these in (1) and a change of sign in the potential energy yield the Hamiltonian

$$(1b) \quad H = \frac{1}{2m} \left(p_r^2 + \frac{1}{r^2} p_\phi^2 \right) - G \frac{mM}{r}$$

and, from (44.9), the Hamilton-Jacobi equation

$$(2) \quad \left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \phi} \right)^2 = 2m \left(E + G \frac{mM}{r} \right).$$

Let us apply in this example the method of "separation of variables" mentioned on p. 231.

We try a solution of the differential equation (2) of the form

$$(3) \quad S = R + \Phi$$

in which R depends only on r and Φ only on ϕ . If we replace the right member of (2) by the general function $f(r, \phi)$, we obtain

$$(3a) \quad \left(\frac{dR}{dr}\right)^2 + \frac{1}{r^2} \left(\frac{d\Phi}{d\phi}\right)^2 = f(r, \phi).$$

In general, such a relation does not hold. If, however, f is independent of ϕ , as in our case, we need merely put $\frac{d\Phi}{d\phi}$ equal to a constant, say C (called the "separation constant"). R is then determined by the equation

$$(4) \quad \left(\frac{dR}{dr}\right)^2 = f(r) - \frac{C^2}{r^2},$$

which is solved by quadrature, yielding a complete integral. The assumption that f is independent of ϕ is evidently equivalent to the fact that in our case ϕ is cyclic, that is, it does not occur explicitly in the differential equation. We see that the method of separation of variables is based on special symmetry properties of the given differential equation, symmetry properties which are often, though by no means always, realized.

We now follow the general pattern of § 45, put $C = \alpha_2$ and separate (2) into

$$(5) \quad \frac{\partial S}{\partial \phi} = \alpha_2$$

$$(6) \quad \frac{\partial S}{\partial r} = \left[2m \left(E + G \frac{mM}{r} \right) - \frac{\alpha_2^2}{r^2} \right]^{\frac{1}{2}}.$$

Eq. (5) is the law of conservation of angular momentum, that is, Kepler's second law; the separation constant α_2 is the constant angular momentum, essentially identical to the areal velocity constant used in Eq. (6.2). Eq. (6) gives the variable radial momentum.

To calculate the characteristic function S , we integrate (5) and (6) and form (3). Replacing E by α_1 , we obtain

$$(7) \quad S = \int_{r_0}^r \left[2m \left(\alpha_1 + G \frac{mM}{r} \right) - \frac{\alpha_2^2}{r^2} \right]^{\frac{1}{2}} dr + \alpha_2 \phi + \text{const.}$$

The lower limit of integration can be chosen arbitrarily since it merely affects the magnitude of the additive constant.

Let us, for the present, focus our attention on the geometric trajectory, i.e., on Kepler's first law. To do this we follow (45.2) and form

$$(8) \quad \beta_2 = \frac{\partial S}{\partial \alpha_2} = -\alpha_2 \int_{r_0}^r \left[2m \left(\alpha_1 + G \frac{mM}{r} \right) - \frac{\alpha_2^2}{r^2} \right]^{-\frac{1}{2}} \frac{dr}{r^2} + \phi.$$

It is evidently convenient to introduce $s = \frac{1}{r}$ instead of r as variable of integration and to rewrite (8)

$$(9) \quad \begin{aligned} \beta_2 - \phi &= \alpha_2 \int_{s_0}^s \left[2m \left(\alpha_1 + GmMs \right) - \alpha_2^2 s^2 \right]^{-\frac{1}{2}} ds \\ &= \int_{s_0}^s \frac{ds}{[(s - s_{\min})(s_{\max} - s)]^{\frac{1}{2}}}. \end{aligned}$$

Here s_{\min} and s_{\max} are the reciprocals of the distances from sun to aphelion and perihelion. Comparison of the two integrals yields

$$(10) \quad \begin{aligned} s_{\min} s_{\max} &= -\frac{2m\alpha_1}{\alpha_2^2} \\ s_{\min} + s_{\max} &= \frac{2Gm^2 M}{\alpha_2^2}. \end{aligned}$$

Now we wish to obtain (9) in convenient trigonometric form; the transformation

$$(11) \quad s = \frac{s_{\min} + s_{\max}}{2} + \frac{s_{\max} - s_{\min}}{2} u,$$

suggests itself. It takes $s = s_{\max}$ into $u = +1$ and $s = s_{\min}$ into $u = -1$. From (9) we then have

$$(12) \quad \beta_2 - \phi = \int_{u_0}^u \frac{du}{(1-u^2)^{\frac{1}{2}}}$$

and, making the assignable lower limit of integration equal to 1,

$$(13) \quad \phi - \beta_2 = \cos^{-1} u, \quad u = \cos(\phi - \beta_2).$$

Finally we return from u to s via (11) and take note that, according to p. 42, Fig. 7,

$$s_{\min} = \frac{1}{a(1+\epsilon)}, \quad s_{\max} = \frac{1}{a(1-\epsilon)},$$

so

$$s = \frac{1}{a(1-\epsilon^2)} + \frac{\epsilon}{a(1-\epsilon^2)} u.$$

From (13) we then obtain the equation of an ellipse in the familiar form

$$(14) \quad s = \frac{1}{r} = \frac{1 + \epsilon \cos(\phi - \beta_2)}{a(1-\epsilon^2)},$$

where the constant β_2 can be absorbed into the definition of ϕ .

For experimental reasons the astronomer is interested not so much in the geometrical form of the trajectory as in the motion as a function of time. Here again the Hamilton-Jacobi method gives the answer in the most systematic fashion, namely, by means of Eq. (45.1),

$$t = \frac{\partial S}{\partial H} = \frac{\partial S}{\partial \alpha_1},$$

from which we obtain, after substitution of the variable s ,

$$(15) \quad t = -\frac{m}{\alpha_2} \int_{s_0}^s \frac{ds}{s^2 [(s-s_{\min})(s_{\max}-s)]^{\frac{1}{2}}}.$$

With this equation we complete our former treatment in § 6, where the position of the planet as a function of time was left undetermined. With the help of the "eccentric anomaly" of Problem I.16 as the new variable of integration [its symbol u should not be confused with the auxiliary u in Eq. (11)] equation (15) can be solved by elementary integration and leads directly to the celebrated Kepler equation

$$nt = u - \epsilon \sin u$$

mentioned in the cited problem.

It is well-known that two- and several-body problems play a central role in modern atomic physics as well. In the hydrogen atom the electron moves about the nucleus, the proton, like a planet about the sun. Here, too, the Hamilton-Jacobi method has proved of surprising value. It literally shows us the point at which *quantum numbers* must be introduced.

In the older quantum theory, whenever the k th degree of freedom was separable from the remaining ones, one defined a *phase integral* (also called "action variable") of the k th degree of freedom given by

$$(16) \quad J_k = \int p_k dq_k.$$

The integral was to be taken over the whole range of values of the variable q_k . One then asked that J_k be an integral multiple of Planck's elementary quantum of action (cf. p. 181),

$$(16a) \quad J_k = n_k h.$$

With p_k in (16) expressed in terms of the characteristic function S , one obtains

$$(17) \quad \int \frac{\partial S}{\partial q_k} dq_k = \Delta S_k = n_k h.$$

ΔS_k is the k th "modulus of periodicity" of the function S , i.e., the change suffered by S when q_k runs through a complete cycle of its values.

The electron of a hydrogen atom has coordinates $q_1 = \phi$ and $q_2 = r$. The differential equation (2) for S and its solution (7) can be transferred directly from astronomy to atomic physics, provided we replace the gravitational potential energy by the Coulomb energy $-\frac{e^2}{r}$.

Since the range of the coordinate ϕ extends from 0 to 2π , we obtain from (7) and (17)

$$(18) \quad \Delta S_\phi = 2\pi \alpha_\phi = n_\phi \hbar.$$

n_ϕ is the *azimuthal quantum number*; α_ϕ , as we know, is identical to the azimuthal moment of momentum p_ϕ .

The range of values of the r -coordinates extend from r_{\min} to r_{\max} and back. Eqs. (7) and (17) therefore yield

$$(19) \quad \Delta S_r = 2 \int_{r_{\min}}^{r_{\max}} \left[2m \left(E - \frac{e^2}{r} \right) - \frac{n_\phi^2 \hbar^2}{4\pi^2 r^2} \right]^{\frac{1}{2}} dr = n_r \hbar.$$

n_r is the *radial quantum number*. The quadrature is best performed by complex integration in the r -plane; once this is done, (19) becomes

$$(20) \quad -n_\phi \hbar + 2\pi i \frac{me^2}{(2mE)^{\frac{1}{2}}} = n_r \hbar.$$

The energy of the hydrogen electron in the quantum state $n = n_r + n_\phi$ is, therefore,

$$(21) \quad E = - \frac{2\pi^2 m e^4}{\hbar^2 n^2}.$$

It is negative because the energy was set equal to zero for infinite electron-proton distance (see the above expression for the potential energy).

Eq. (21), together with Bohr's postulate of the radiation of energy in quantum jumps, led to the first understanding of the hydrogen spectrum (the so-called Balmer series) and from there to the modern theory of spectral lines in general.

Present-day developments of atomic theory have gone beyond the description of electronic orbits here presented. As mentioned at the beginning of § 44, investigations following Hamilton's line of thought have resulted in a more profound wave-mechanical conception of atomic processes.

PROBLEMS

Chapter I

I.1. *Elastic collision*¹. n equal masses M are placed in contact with each other along a straight line. Two masses M , both having velocity v , collide with the row of n masses from the left. Evidently the laws of momentum and energy are satisfied if the two masses on the left transfer their velocities to the last two masses on the right. Show that these laws cannot be satisfied if only one mass is expelled on the right, or if the two last masses on the right are set in motion with different velocities v_1, v_2 .

I.2. *Elastic collision with unequal masses*. Let the last mass m on the right be smaller than the remaining masses. Let a mass M collide from the left with velocity v_0 . Show from the principles of energy and momentum that it is impossible for m to be the only mass set in motion. If it is assumed that only two masses are set in motion, what must be their velocities?

I.3. *Elastic collision with unequal masses*. Let the last mass M' on the right be greater than the remaining ones. Make the same assumptions as in Problem 2, taking notice, however, that the next-to-last mass on the right transfers its momentum toward the left. What is the velocity of M' and of the first mass M at the left end of the row? What happens if M' is very large?

I.4. *Inelastic collision between an electron and an atom*. An electron m , of velocity v , collides centrally with an atom M initially at rest. The atom is excited and is raised from its ground state to an energy level E units above it. What is the minimum initial velocity v_0 that the electron must have?

You will find one quadratic equation each for the final velocities v of the electron and V of the atom. The minimum value v_0 results from the requirement that the radical occurring in the solutions for v and V be real. The value of v_0 is somewhat higher than would be expected if only the conservation of energy were called into play, although the difference is not observable because the ratio $M/m \geq 2000$ is very high.

¹It is essential that the student carry out the experiments described in Problems I.1 to I.3 himself. This can be done with coins on a smooth support, with elastic spheres so suspended on strings that they touch in the position of rest, or finally with marbles in a trough.

If the colliding particle is of the same, or approximately the same, mass as the struck one, the required minimum energy is about twice that expected from conservation of energy alone.

1.5. Rocket to the moon. A rocket with continuous exhaust shoots vertically upward. Let the exhaust velocity be a relative to the rocket and $\mu = -\dot{m}$ be the mass expelled per second, and assume both constant in time. Assume that the motion occurs under constant gravitational acceleration g , friction being neglected. Set up the equation of motion and integrate it under the assumption that the initial velocity of the rocket on the surface of the earth is zero. If $\mu = \frac{1}{100}$ of the initial mass m_0 and $a = 2000$ meter \cdot sec $^{-1}$, what height has the rocket reached at $t = 10, 30, 50$ sec.?

1.6. Water drop falling through saturated atmosphere. A spherical water droplet falls, without friction and under the influence of gravity, through an atmosphere saturated with water vapor. Let its initial radius ($t = 0$) be c , its initial velocity, v_0 . As a result of condensation the water drop experiences a continuous increase in mass proportional to its surface; as will be shown, its radius then increases linearly with time. Integrate the differential equation of the motion by introducing r instead of t as independent variable. Show that for $c = 0$ the velocity increases linearly with time.

1.7. Falling chain. A chain lies pushed together at the edge of a table, except for a piece which hangs over it, initially at rest. The links of the chain start moving one at a time; neglect friction. The energy written in the usual form is here no longer an integral of the motion. Instead, the impulsive (Carnot) energy loss must be taken into account in writing the balance of energy.

1.8. Falling rope. A rope of length l slides over the edge of a table. Initially a piece x_0 of it hangs without motion over the side of the table. Let x be the length of rope hanging vertically at time t . The rope is assumed to be perfectly flexible. Show that the principle of energy in the form $T + V = \text{const.}$ gives an integral of the motion.

1.9. Acceleration of moon due to earth's attraction. The moon's distance to the earth is about 60 earth radii. Assume that the lunar orbit is a circle, once traversed in 27 days, 7 hours and 43 minutes. From this the acceleration of the moon toward the earth (centripetal acceleration) can be calculated. Comparison of this value with that from Newton's law of gravitation gave the first confirmation of this law.

1.10. The torque as a vector quantity. Consider a rectangular coordinate system (x, y, z) , with \mathbf{r} the radius vector of the point of application of a force \mathbf{F} . We now pass to a second coordinate system (x', y', z') , obtained from the former by rotation. Show that the moment of force \mathbf{F} about the origin of the first coordinate system transforms like a vector, i.e., like

$\mathbf{r} = (x, y, z)$. To prove this one must assume that both coordinate systems are of similar sense (both right-handed or both left-handed).

I.11. *The hodograph of planetary motion.* From Eq. (6.5) with $A=0$, the hodograph of planetary motion is given by

$$\xi = \dot{x} = -\frac{GM}{O} \sin \phi, \quad \eta = \dot{y} = +\frac{GM}{O} \cos \phi + B,$$

where M is the mass of the sun, O the angular momentum constant, ϕ the true anomaly (cf. Fig. 6). Show that the trajectory is a hyperbola or ellipse, depending on whether the "pole" $\xi=\eta=0$ of the hodograph is excluded from or included by the hodograph. Also describe the limiting cases of parabola and circle in terms of the position of this pole.

I.12. *Parallel beam of electrons passing through the field of an ion. Envelope of the trajectories.* A source located at infinity shoots off electrons (charge e , mass m) along parallel paths with constant initial velocity v_0 . An ionized atom A (charge E , mass M) is fixed at the origin. If e and E have the same sign, what area surrounding A is never touched by the electrons?

Take the y -axis as the direction of the incident particles; treat the problem as plane. It will be easiest to write the trajectory of an electron in polar coordinates with A as pole of the coordinate system and focus of the hyperbolic trajectory. The boundary of the above-mentioned area is obtained as the envelope of the electronic trajectories. Because of $M \gg m$ one can consider A to be at rest.

Show that if e and E have opposite signs, the envelope of trajectories seems to yield the same boundary, but that it is now devoid of physical meaning.

I.13. *Elliptical trajectory under the influence of a central force directly proportional to the distance.* Consider a mass m under the influence of a force directed toward a fixed point O (center of force)

$$\mathbf{F} = -k\mathbf{r}$$

($\mathbf{r} = \vec{Om}$, $k = \text{const.}$). Show that the following three laws hold for the motion of m :

1. m describes an ellipse with O as center.
2. The radius vector \mathbf{r} sweeps out equal areas in equal times.
3. The period T is independent of the shape of the ellipse, depending only on the force law, i.e., the values of k and m .

I.14. *Nuclear disintegration of lithium* (Kirchner, Bayer. Akad. 1933). If a hydrogen nucleus (proton, mass m) collides with velocity v_p with a nucleus of Li^7 (lithium of atomic weight 7), the latter splits into two α -particles (mass $m_\alpha = 4m_p$). These two α -particles fly off in almost (but

not exactly) diametrically opposite directions. Assume that the α -particles fly off symmetrically with respect to the line of collision, and with equal velocities, and calculate the angle 2ϕ between them. Notice that in addition to the kinetic energy E_p of the proton, there occurs another energy E liberated as a result of the mass defect, which is greatly in excess of E_p , and is likewise transmitted to the two α -particles. Thus the final answer for $\cos\phi$ contains not only m_p and m_α , but also the kinetic energy E_p of the proton, and E .

In the units customary in atomic physics, $E=14\cdot10^6$ ev (electron-volt). In an experiment $E_p=0.2\cdot10^6$ ev; what are the values of v_α and 2ϕ ?

I.15. *Central collisions between neutrons and atomic nuclei; effect of a block of paraffin.* Neutrons are slowed down but little by a lead plate 50 cm thick; a layer of paraffin about 20 cm deep, on the other hand, absorbs them completely. This can easily be understood if one remembers that in a central collision the kinetic energy of the neutron (mass $m=1$) is completely transferred to one of the hydrogen nuclei of the paraffin (proton mass $M_1=1$); whereas the amount of energy transferred in a central collision with a lead nucleus (mass $M_2=206$) is hardly worth mentioning. Draw a curve showing the kinetic energy which the initially motionless atomic nucleus (of mass M) receives from the neutron (mass m) in a central collision, as a function of the ratio M/m .

I.16. *Kepler's equation.* The secular variation of the motion of the planet in its orbit is determined, in differential form, by the principle of angular momentum. In order to obtain the secular variation in integral form, one can, following Kepler, proceed as follows (Fig. 55).

Draw a circle about the center of the ellipse with the major axis as diameter. We now associate a point K on the circle with the planet located at point E of the ellipse at the time t . If we take the principal axes of the ellipse as coordinate axes, point K has the same abscissa as E . Whereas E is given by its polar coordinates r, ϕ (pole S), K is determined by polar coordinates a, u (pole M). Thus with the true anomaly ϕ we associate

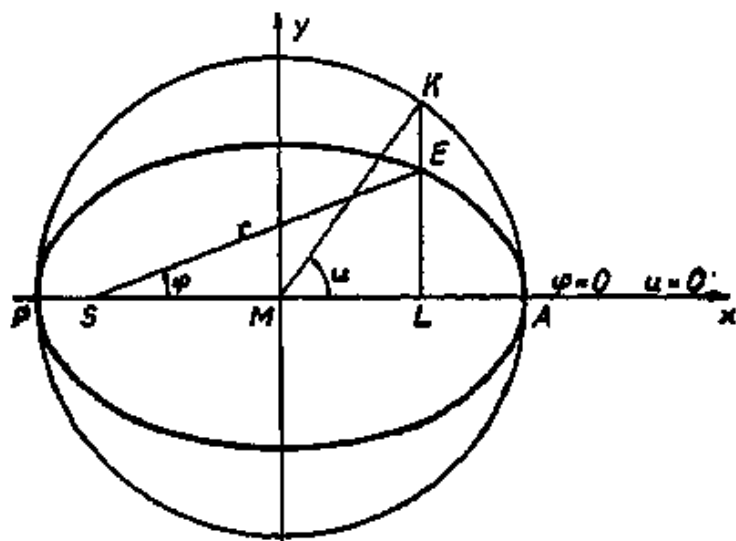


FIG. 55. Kepler's construction of the eccentric anomaly u and its connection with the true anomaly ϕ .

the *eccentric anomaly* u (as in the text, we count both from aphelion in the direction of motion, at variance with astronomical usage, where the anomalies are counted from perihelion, though, of course, also in the direction of motion of the planet).

The coordinates x and y of the planet E can be expressed, on the one hand, in terms of r and ϕ , and on the other, in terms of one of the semi-axes of the ellipse and the eccentric anomaly u , so that, with K given, E is also given. The course of the motion of point K on the circle is then determined by the celebrated Kepler equation

$$nt = (u - \epsilon \sin u).$$

Here ϵ is the eccentricity of the elliptical trajectory, and $n = \left(\frac{GM}{a^3}\right)^{\frac{1}{2}} = \frac{C}{ab}$, where a , b are the semi-axes, G the gravitational constant, M the mass of the sun, C the areal velocity constant.

In order to derive Kepler's equation, start out with the polar equation of the ellipse, referred to S as pole and the ray SA (aphelion) as polar axis,

$$r = \frac{p}{1 - \epsilon \cos \phi}$$

where p is the "parameter" $a(1 - \epsilon^2)$. Now use the transformation relations referred to above to introduce u in place of ϕ , and obtain the equation

$$r = a(1 + \epsilon \cos u).$$

Differentiation of these two equations, elimination of r and ϕ , introduction of the principle of angular momentum and of Eq. (6.8) finally yield Kepler's equation by an integration, provided we stipulate that at $t=0$ the planet is at aphelion.

Chapter II

II.1. Non-holonomic conditions of a rolling wheel. A sharp-edged wheel, of radius a , rolls without sliding on a rough plane support (think, for example, of a hoop rolling on an even road). Its instantaneous position is determined by assigning values to

1. coordinates x , y of the point of contact of the wheel with the support, referred to a rectangular coordinate system x , y , z whose xy -plane coincides with the support;
2. angle θ between axle of wheel and z -axis;
3. angle ψ formed between the tangent to the wheel (intersection of the plane of the wheel with that of the support) and the x -axis;

4. angle ϕ that the radius toward the instantaneous point of contact of the wheel makes with an arbitrary fixed radius, this angle to be counted positive, say, in the direction of rotation.

In finite motion the wheel therefore has five degrees of freedom. The mobility of the wheel is, however, restricted by the condition of pure rolling (without slipping) brought about by the static friction between wheel and support; indeed, with the wheel moving along its instantaneous direction, the distance δs moved along the direction of the tangent must equal $a\delta\phi$. By projecting this equation on the coordinate axes we then obtain the conditions of constraint

$$(1) \quad \delta x = a \cos \psi \delta\phi, \quad \delta y = a \sin \psi \delta\phi$$

which the displacements δx , δy and $\delta\phi$ must satisfy.

Hence the rolling wheel has only three degrees of freedom in infinitesimal motion.

Show that conditions (1) cannot be reduced to equations between the coordinates themselves. To do this, show that the existence of an equation $f(x, y, \phi, \psi) = 0$ [θ does not occur in (1)] is incompatible with conditions (1).

II.2. *Approximate design of a flywheel for a double-acting one-cylinder steam engine* (cf. also § 9 (4)). A double-acting piston engine is one in which steam is introduced alternately on both sides of the piston, so that work is done during both strokes of a cycle.

Let us assume, for simplicity, that the steam pressure remains constant during each stroke (full pressure or Diesel cycle), and let us, moreover, suppose that the connecting rod is of infinite length. The variable torque as a function of crank angle ϕ transmitted from the piston to the crank shaft is then given by

$$L = L_0 \sin \phi$$

for the half-cycle in which the crank moves from the backward to the dead forward position [cf. Eq. (9.5)]. Here L_0 is a constant; ϕ is counted in the sense of rotation from the dead backward position. During the second half-cycle, from forward to dead backward position, under the same assumptions as made above (viz., 1. double-acting engine, 2. operation under full pressure, 3. infinite connecting rod), the torque changes according to the same law, provided ϕ is now measured in the sense of rotation from the dead forward position.

Let the load on the engine be given by the constant torque W , corresponding to a power of N HP with n r.p.m. Thus the driving torque L is variable, while the load torque W is constant. As a result the angular velocity of the engine fluctuates between a maximum value ω_{\max} and a minimum value ω_{\min} , its mean value ω_m being given approximately by

$$\omega_m = \frac{\omega_{\max} + \omega_{\min}}{2}.$$

It is the purpose of the flywheel to prevent the relative fluctuation, that is the degree δ of unbalance of the engine, given by

$$\delta = \frac{\omega_{\max} - \omega_{\min}}{\omega_m},$$

from exceeding a given limit. How great must the moment of inertia of the flywheel be if the inertial effect of the moving masses (piston, piston rod, cross head, connecting rod and crank) is neglected?

II.3. Centrifugal force under increased rotation of the earth. How fast must the earth rotate and how long would the day be in order that centrifugal force and gravity just cancel at the equator?

II.4. Poggendorff's experiment. From one end of the beam of a balance we suspend a weightless pulley which can rotate without friction. A string U passes over the pulley; on one side the string carries the weight P , on the other, the weight $P+p$, where p is a small additional weight, just as in Atwood's machine. Initially p is fastened to the axle of the pulley by means of a thread u . On the other side of the balance these weights are suitably equilibrated in a pan. The thread u is then burned.

(a) With what acceleration do the weights P and $P+p$ rise and fall, respectively?

(b) Is the beam of the balance displaced in this process?

(c) What is the tension in the string U ?

II.5. Accelerated inclined plane. An inclined plane is moved in a vertical direction according to a given dependence on the time. Investigate the motion of a body of mass m sliding down the plane without friction; in particular, consider the case that the inclined plane is moved with the constant accelerations $+g$ and $-g$.

II.6. Products of inertia for the uniform rotation of an unsymmetrical body about an axis. An unsymmetrical body rotates uniformly about an axis whose ends rest in bearings A and B . What reactions A and B must be exerted by the bearings? Calculate them from d'Alembert's principle; show that they result from the total centrifugal force applied at the center of gravity and from the resultant moment of the centrifugal forces acting on the individual mass elements.

From p. 55 we know the reactions caused by the weight of the body alone; we can therefore omit their effect here.

II.7. Theory of the Yo-yo. A disk-shaped body of mass M and moment of inertia I is provided with a deep symmetrical groove in the median plane

perpendicular to its axis. A string is wound on the shaft of radius r in the groove. The loose end of the string is held in the hand. One then lets the body fall, with the string taut at all times. As the body descends, it acquires a rotational acceleration until the string is unwound. Then follows a transition stage, not to be considered in detail here, the result of which is that the body shifts from one side of the string to the other. The string now winds around the shaft in the opposite sense, and the body climbs with rotational deceleration, and so on. What is the string tension

(a) in descent?

(b) in ascent?

Assume that r is so small compared to the distance of the axis from the loose end of the string that the string can at all times be considered to be vertical.

II.8. *Particle moving on the surface of a sphere.* A mass point moves on the outside of the upper half of a sphere. Let its initial position z_0 and initial velocity v_0 be arbitrary, except that the latter is to be tangential to the surface of the sphere. The motion is to be frictionless, occurring solely under the influence of gravity. At what height does the mass point leave the sphere?

Chapter III

III.1. *Spherical pendulum with infinitesimal oscillations.* In general, the nodal points of the trajectory of a spherical pendulum advance during the course of the motion. For sufficiently small oscillations, however, the nodal points must be fixed, for we are then dealing with an harmonic elliptical motion. Estimate in what order the advance $\Delta\phi$ of the nodal points vanishes with vanishing area of the ellipse.

III.2. *Position of the resonance peak of forced, damped oscillations.* In a forced oscillation with damping the maximum amplitude of oscillation lies, not at $\omega = \omega_0$ as in the case of no damping, but at a value below ω_0 (cf. Fig. 33) depending on the amount of damping.

Find for what value of ω $|C|$ is a maximum.

[Show that the maximum of the velocity amplitude $|C|\omega$ (or of the time average of the kinetic energy) occurs exactly at $\omega = \omega_0$.]

III.3. *The galvanometer.* A galvanometer is connected through a switch with a direct-current source of constant EMF E . At time $t=0$, the switch is closed. After a sufficiently long time the galvanometer deflection reaches its final value α_∞ . What is its motion between the initial position of rest, $\alpha=0$, $\dot{\alpha}=0$, and the final position, $\alpha=\alpha_\infty$?

Three effects have to be taken into account. First, an external torque proportional to the electric current and hence to the *EMF* acts on the galvanometer of moment of inertia I . Second, there acts a damping torque proportional to the angular velocity, which tends to slow down the motion. Third, the torsion in the suspension acts as a restoring torque and is proportional to the deflection α . Let ρ be the factor of proportionality of the damping torque, ω_0^2 that of the restoring torque.

Distinguish and explain graphically the three cases

- (a) weak damping ($\rho < \omega_0$),
- (b) aperiodic ("critical") damping ($\rho = \omega_0$),
- (c) strong damping ($\rho > \omega_0$).

III.4. *Pendulum under forced motion of its point of suspension.*

(a) A particle is suspended from an inextensible string and oscillates without damping under the influence of gravity. The point of suspension is moved along a straight horizontal line according to some given law of displacement $\xi = f(t)$.

What are the equations of motion of the system, the mass of the string being neglected? Derive them either from d'Alembert's principle or from Lagrange's equations of the first kind.

The equations of motion become considerably simplified if we pass to small oscillations, i.e., retain terms of only the first order.

If we make the additional assumption that the displacements of the point of suspension are harmonic in time, the equations of motion can easily be integrated. As the pendulum is set moving, say by a motion of the point of suspension, its proper frequency is excited; the amplitude of this proper frequency is gradually damped out (though we shall neglect damping in the analysis), thus leading to a steady state of oscillation with the same frequency as that forced on the point of suspension. Show that when the motion has thus become stationary, suspension point and mass m move in the same sense below the resonance frequency, in opposite senses above it.

(b) Make a similar analysis of the case in which the point of suspension is subjected to a vertical displacement η , with special emphasis on the case that the acceleration of the point is constant. What is the period of oscillation if the point of suspension is displaced with accelerations $+g$ and $-g$?

III.5. *Practical arrangement of coupled pendulums, sketched in Fig. 56.*

Between two fixed supports A and B is stretched a weightless, flexible and elastic wire. Its tension S is regulated by an adjustable weight G attached to the loose end of the wire hanging over the angle iron B . Two pendulums are suspended *bifilarly* at points C and D which divide the

segment AB into three, let us say, equal parts. The bifilar suspensions, indicated as simple suspensions in the sketch, enable the pendulums to swing out with good accuracy in a transverse direction, i.e., normal to the plane of the drawing. By increasing G the coupling between the two pendulums is made weaker (not stronger, as might at first be thought!). In what is to follow, we shall assume the coupling to be weak, which means that S is large compared with the weight of the pendulum bobs. We further suppose the angles of deflection ϕ_1 and ϕ_2 of the pendulums with

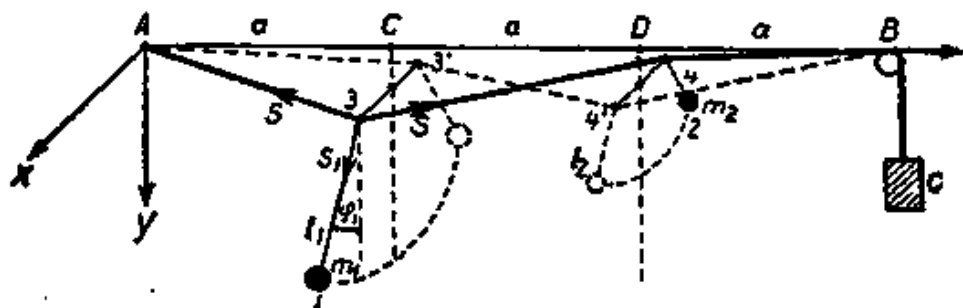


FIG. 56. Wire $AODB$ is held taut by the weight G . It is deformed into $A34B$ or, for the opposite deflection, into $A3'4'B$, the deflection being caused not only by the gravitational action on masses m_1 and m_2 , but also by the inertial effects of the pendulums. The latter are labeled 1 and 2, are of lengths l_1 and l_2 , and suspended bifilarly, so that they swing out normally to the plane of the drawing (the bifilar suspensions are not shown in the figure). ϕ_1 and ϕ_2 are the instantaneous deflections from the vertical.

respect to the vertical to be small. (Refer to Fig. 56 for notation; $3'$ and $4'$ are the deflections of the points of suspension C and D symmetrically opposite to 3 and 4.) These angles are then approximated by

$$\sin \phi_1 = \phi_1 = \frac{x_2 - x_3}{l_1}, \quad \cos \phi_1 = 1$$

$$\sin \phi_2 = \phi_2 = \frac{x_3 - x_4}{l_2}, \quad \cos \phi_2 = 1.$$

With neglect of the y -component of the small oscillations we have for m_1 and similarly for m_2 ,

$$(1) \quad m_1 g = S_1 \cos \phi_1 = S_1 \quad m_2 g = S_2 \cos \phi_2 = S_2$$

$$(2) \quad m_1 \ddot{x}_1 = -S_1 \sin \phi_1 = -\frac{m_1 g}{l_1} (x_3 - x_1), \quad m_2 \ddot{x}_2 = -S_2 \sin \phi_2 = -\frac{m_2 g}{l_2} (x_4 - x_2).$$

At the points of suspension C and D , S_1 and S_2 respectively must, at any instant, be in equilibrium with the tension S ; the latter is altered negligibly little by S_1 and S_2 . This yields two more conditions between x_1 , x_2 , x_3 and x_4 . We can solve these for x_3 and x_4 and substitute them in (2). We then

obtain the simultaneous differential equations of coupled pendulums. Verify that these are indeed in agreement with Eqs. (20.10).

III.6. The oscillation quencher. A system capable of oscillating in the x -direction (mass M , proportionality constant of the restoring force, K) is coupled by means of a spring (constant k) to a mass m , in such a way that m , too, can oscillate in the x -direction. We require that when an external force $P_x = c \cos \omega t$ acts on the mass M , this mass M stay at rest. What conditions must be satisfied by the system (m, k)?

Chapter IV

IV.1. Moments of inertia of a plane mass distribution. Prove that for any plane mass distribution the moment of inertia about the "polar" axis (perpendicular to the plane) equals the sum of the moments of inertia about two mutually perpendicular "equatorial" axes (in the plane of the mass distribution, intersecting in the polar axis). Specialize the foregoing to a circular disk.

IV.2. Rotation of a top about its principal axes. According to Fig. 46a, b, the rotations of an unsymmetrical top about the axes of the largest and smallest moments of inertia are stable, that about the axis of the intermediate moment of inertia is unstable. Prove this analytically. Start out with Euler's equations of motion and put the angular velocity about the axis of rotation $\omega_1 = \text{const.} = \omega_0$. Angular velocities ω_2 and ω_3 about the other two principal axes are initially zero, but, due to a perturbation, acquire values different from zero. If we suppose the perturbation small, the first Euler equation states that to a first approximation ω_1 remains unchanged, $= \omega_0$. From the other two equations one obtains a system of two linear differential equations of first order in ω_2 and ω_3 . Put $\omega_2 = ae^{\lambda t}$ and $\omega_3 = be^{\lambda t}$ with arbitrary constants a and b , and substitute in the two equations. The discussion of the resulting quadratic equation for λ yields the proof of the above assertion.

IV.3. High and low shots in a billiard game. Follow shot and draw shot. A billiard ball is hit with horizontal cue in its median plane, i.e., without "English." At what height h above the center must the cue hit the ball so that pure rolling (no slipping) will ensue? Work out a theory of balls struck high and low, taking into account the kinetic friction between ball and cloth. By how much does the velocity of the center of mass grow in a high shot during the total time friction is acting, and by how much does it decrease in a low shot? What time elapses before only pure rolling remains?

The same method can be used to explain the phenomena taking place on collision with another ball, i.e., follow and draw shots.

IV.4. *Parabolic motion of a billiard ball.* How must a ball be struck so that the initial motion of its center of gravity and the axis of rotation are not normal to each other? Show that the direction of the force of friction is constant as long as the ball slides. What is the trajectory of the center of the ball? After what time does pure rolling ensue?

Chapter V

V.1. *Relative motion in a plane.* A plane rotates with variable angular velocity ω about its normal at one of its points O .

What forces in addition to the centrifugal force must be applied to a mass point so that its equations of motion in the rotating plane take on the same form as in the inertial frame of a spatially fixed plane? It will be convenient to introduce complex variables $x+iy$ in the spatially fixed plane, $\xi+i\eta$ in the rotating plane.

V.2. *Motion of a particle on a rotating straight line.* A mass point moves without friction on a straight line which in its turn rotates with constant angular velocity ω about a fixed horizontal axis intersecting, and perpendicular to, the straight line. Calculate the motion on the rotating straight line as a function of time, and show that the force of constraint (guiding force) and the component of the gravitational attraction along this force just balance the Coriolis force.

V.3. *The sleigh as the simplest example of a non-holonomic system* [after C. Carathéodory, *Z. angew. Math. Mech.* 13, 71 (1933)]. The sleigh is regarded as a rigid plane system with three degrees of freedom in finite motion, one degree in infinitesimal motion. (Cf. the rolling wheel in Problem II.1, which had five degrees of freedom in finite motion, three in infinitesimal motion.)

Neglect the sliding friction on the snow, or, alternatively, think of it as permanently compensated by the pull of a horse. One must, however, take into account the friction F exerted laterally by the snow tracks against the runners of the sleigh, for it prevents any lateral motion of these. Let this friction be concentrated at one point of application O .

An $\xi\eta$ -system is fixed in the sleigh. The ξ -axis runs horizontally along the center line of the runners and passes through the center of mass G with coordinates $\xi=a$, $\eta=0$, and the η -axis passes horizontally through the point of application of F . In the horizontal plane of the snow we fix an xy -system. Let ϕ be the angle between the axes of ξ and x , $\omega=\dot{\phi}$ the

instantaneous angular velocity of the sleigh about the vertical; M is the mass of the sleigh, I its moment of inertia about the vertical through the mass center; u, v are the components of the velocity of the point O ($\xi=\eta=0$) along ξ and η .

(a) Derive the three simultaneous differential equations for the quantities u, v, ω with F as external force, using the method of complex variables of Problem V.1.

(b) Simplify them by introducing the non-holonomic condition $v=0$ and determine F from them.

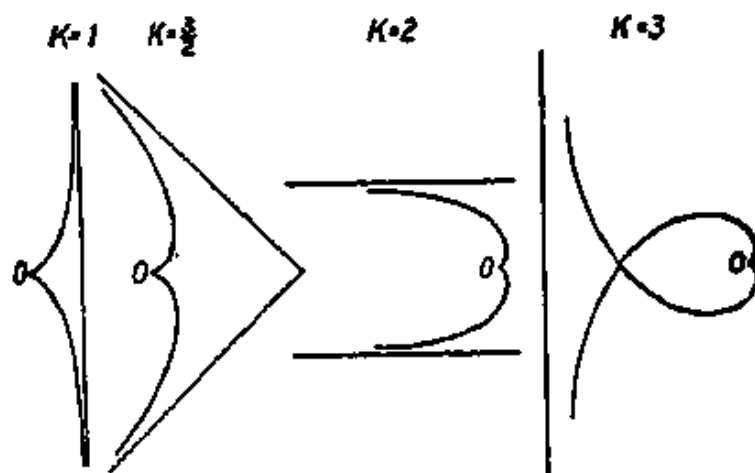


FIG. 57. Trajectory of a sleigh for various values of k , after Carathéodory.

(c) Integrate them by introducing, instead of ϕ , an auxiliary angle proportional to ϕ .

(d) Verify that the kinetic energy of the sleigh is constant (because F does no work).

(e) Show that, with suitable choice of the time scale, the trajectory of point O in the xy -plane possesses a cusp at $t=0$ and asymptotically approaches straight lines for $t=\pm\infty$, as shown by the curves of Fig. 57 borrowed from Carathéodory.

Chapter VI

VI.1. Example illustrating Hamilton's principle. Calculate the value of Hamilton's integral between the limits $t=0$ and $t=t_1$

(a) for the real motion of a falling particle, $z=\frac{1}{2}gt^2$;

(b) for two fictitious motions $z=ct$ and $z=at^3$, where the constants c and a must be so determined that initial and end positions coincide with those of the real path, in agreement with the rules of variation in Hamilton's

principle. Show that the integral has a smaller value for the real motion (a) than for the fictitious ones (b).

VI.2. *Once more the relative motion in a plane and the motion on a rotating straight line.* Treat Problems V.1 and V.2 by means of the Lagrange method.

VI.3. *Once more the free fall on the rotating earth and Foucault's pendulum.* Verify that these problems, too, can be treated by Lagrange's method without knowledge of the laws of relative motion. This procedure is interesting and simpler in thought than that of Chapter V; it does, however, require a careful inspection of the numerous small terms occurring; only *after* the differentiations $\frac{d}{dt} \frac{\partial}{\partial \dot{q}}$ and $\frac{\partial}{\partial q}$ have been carried out should the usual approximations of large terrestrial radius and small angular velocity be made; until then, *all* terms must be carried.

Start out with ordinary spherical polar coordinates r, θ, ψ , where r is measured from the earth's center. Next compare these with coordinates ξ, η, ζ introduced in Fig. 49. Let R be the earth's radius, θ_0, ψ_0 the coordinates of the projection on the earth of the initial position of the freely falling body or of the point of suspension of the pendulum. We then have the following relations between coordinates r, θ, ψ and ξ, η, ζ of the falling or oscillating particle m ,

$$(1) \quad \xi = R(\theta - \theta_0), \quad \eta = R \sin \theta (\psi - \psi_0), \quad \zeta = r - R,$$

with

$$(2) \quad \psi_0 = \omega t, \quad \theta_0 = \frac{\pi}{2} - \phi = \text{colatitude}.$$

From this

$$\dot{\xi} = R\dot{\theta}, \quad \dot{\eta} = R \sin \theta (\dot{\psi} - \omega) + \frac{\cos \theta}{\sin \theta} \eta \dot{\theta}, \quad \dot{\zeta} = \dot{r}$$

and, conversely,

$$(3) \quad r\dot{\theta} = \left(1 + \frac{\zeta}{R}\right) \dot{\xi},$$

$$r \sin \theta \dot{\psi} = \left(1 + \frac{\zeta}{R}\right) \dot{\eta} + \omega R \left(1 + \frac{\zeta}{R}\right) \sin \theta - \frac{\cos \theta}{\sin \theta} \left(1 + \frac{\zeta}{R}\right) \frac{\eta}{R} \dot{\xi}, \quad \dot{r} = \dot{\zeta},$$

where the angle θ occurring on the right must, according to (1), be regarded as a function of ξ .

These values are to be replaced in the expression

$$T = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\psi}^2)$$

for the kinetic energy, which becomes, as a result, a function of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, ξ , η and ζ . If we denote the terms later to be dropped by \dots , we can, for instance, calculate from T

$$(4) \quad \frac{\partial T}{\partial \dot{\xi}} = m \left(1 + \frac{\zeta}{R}\right)^2 \dot{\xi} - m \frac{\cos \theta}{\sin \theta} \left(1 + \frac{\zeta}{R}\right) \frac{\eta}{R} \left\{ \dots + \omega R \left(1 + \frac{\zeta}{R}\right) \sin \theta + \dots \right\}$$

$$(5) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{\xi}} = m \ddot{\xi} - m \omega \cos \theta \dot{\eta} + \dots$$

$$(6) \quad \frac{\partial T}{\partial \xi} = \frac{1}{R} \frac{\partial T}{\partial \theta} = + m \omega \cos \theta \dot{\eta} + \dots$$

As our potential energy we can take

$$(7) \quad V = mg(r - R) = mg\zeta.$$

Verify that in this manner we obtain Eqs. (30.5) for the free fall and Eqs. (31.2) for Foucault's pendulum, from which follow the results developed earlier.

VI.4. "Wobbling" of a cylinder rolling on a plane support. A circular cylinder of radius a has an inhomogeneous mass distribution, so that the center of mass G has the distance s from the axis of the cylinder. The cylinder rolls on a horizontal plane under the influence of gravity. Let m be the mass of the cylinder, I its moment of inertia about an axis through the mass center parallel to its axis of symmetry. Investigate the motion with the help of Lagrange's method, introducing the angle ϕ rotated through as generalized coordinate q . In calculating the kinetic energy, put the point of reference

(a) in the center of mass,

(b) in the geometrical center,

of the cylinder and verify that in both cases the same differential equation in ϕ results.

Show by means of the "method of small oscillations" that the equilibrium of the cylinder is stable with G in the lowest, and unstable with G in the highest position.

VI.5. Differential of an automobile. If the driven wheels of an automobile are not to slide, they must be able to turn at different speeds on a curve. This is achieved by the differential (Fig. 58). The engine drives the driving wheel (Ω) in which axle A is fixed. Two bevel gears (ω) sit on A in such a way that they can rotate independently about A . They, in turn, are in mesh with the pair of bevel gears (ω_1 , ω_2) on which they may roll (cf. Fig. 58, left) as A turns.

The axle of the rear wheels of an automobile is cut at the center (Fig. 58, right). Fixed to the left end of its right half is the bevel gear (ω_1), to the right end of its left half, the bevel gear (ω_2). The two halves of the rear

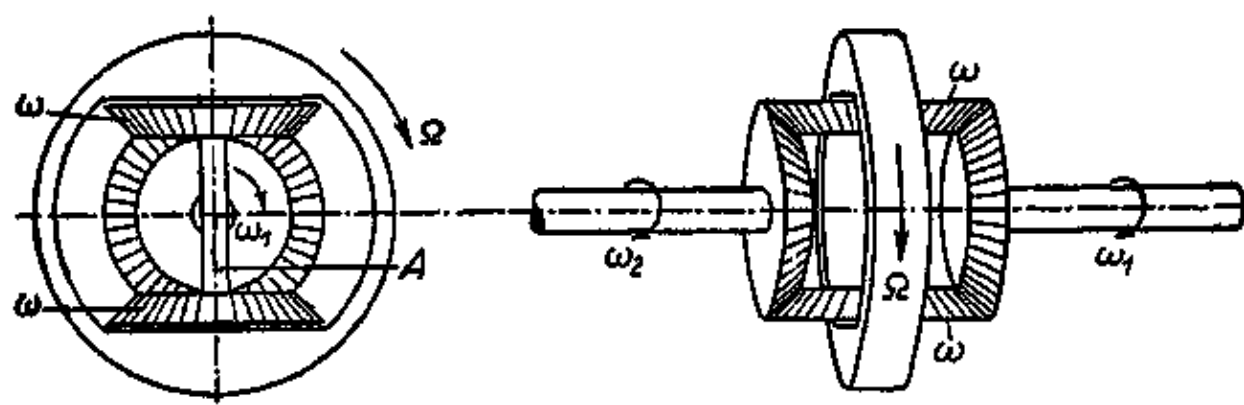


FIG. 58. The differential of an automobile, at the same time a model (after Boltzmann) for the induction effect of two coupled circuits. Left: view along rear axle of vehicle. Right: side view of this axle.

axle are therefore coupled by the differential in such a way that they can turn with different angular velocities.

Set up the kinematic relations between angular velocities Ω , ω , ω_1 and ω_2 . Next make use of the principle of virtual work to derive the condition of equilibrium between the driving torque L acting on (Ω) and the torques L_1 and L_2 acting on (ω_1) and (ω_2).

What is the equation of motion of the system? Let I_1 and I_2 be the moments of inertia of (ω_1) and (ω_2), I that of the pair of gears (ω) about the axis of A , I' that of (ω) about the axis of the driving wheel. Neglect the contribution of (Ω) to I' .

If one rear wheel is accelerated, for instance by decreasing friction, the other wheel is retarded, even if driving torque and frictional torque remain equal there.

HINTS FOR SOLVING THE PROBLEMS

Almost all numerical calculations occurring in these problems can be carried out to sufficient accuracy by means of a slide rule. Let us call express attention to this useful tool for quick approximate calculations.

I.1. The proof that $v_1=v_2=v$ can be derived either algebraically or geometrically. In the latter case use v_1 and v_2 as rectangular coordinates in a plane diagram.

I.2. The velocities of the expelled masses are, respectively,

$$\frac{2M}{M+m}v_0 \quad \text{and} \quad \frac{M-m}{M+m}v_0.$$

I.3. Here we obtain the formulas of I.2 with a change of sign.

I.4. Verify that the quadratic equation for V leads to the same minimum value v_0 as that for v .

I.5. The differential equation to be integrated is

$$m\dot{v} - \mu a = -mg.$$

With the independent variable $m=m_0-\mu t$ instead of t one obtains

$$v = -a \ln\left(1 - \frac{\mu}{m_0}t\right) - gt$$

and, by an additional integration (z =height above earth's surface),

$$(1) \quad z = \frac{am_0}{\mu} \left\{ \left(1 - \frac{\mu}{m_0}t\right) \ln\left(1 - \frac{\mu}{m_0}t\right) + \frac{\mu}{m_0}t \right\} - \frac{1}{2}gt^2.$$

For small t we obtain, by neglecting higher terms in t ,

$$(2) \quad z = \left(\frac{\mu a}{m_0} - g\right) \frac{t^2}{2}.$$

The numerical calculation with equation (1) yields

$t = 10$	30	50 sec
$z = 0.54$	5.65	18.4 km

I.6. Since water has specific gravity 1, the mass of the drop is $m = \frac{4}{3}\pi r^3$, i.e., $dm = 4\pi r^2 dr$. In condensation, on the other hand, with α a factor of proportionality, $dm = 4\pi r^2 \alpha dt$; it follows that $dr = \alpha dt$. In terms of r the differential equation is then

$$\alpha \frac{d}{dr}(r^2 v) = r^2 g.$$

By virtue of the initial condition $v=v_0$ for $r=c$ its solution is

$$v = \frac{g}{\alpha^2} + \frac{c^2}{r^2} \left(v_0 - \frac{g}{\alpha^2} \right);$$

for $c=0$ and $v_0=0$ we have, respectively,

$$v = \frac{g}{\alpha^2}, \quad v = \frac{g}{\alpha^2} \left(1 - \frac{c^2}{r^2} \right).$$

I.7. Let x be the instantaneous length of chain hanging down. With the mass of the chain per unit length put equal to 1, the equation of motion is

$$\frac{d}{dt}(x\dot{x}) = x\ddot{x} + \dot{x}^2 = gx.$$

Since its integration is somewhat difficult—after the substitution $x=u^2$ it leads to an elliptic integral—we shall be satisfied with expressing the quantities \dot{T} , \dot{V} and \dot{Q} (Carnot energy loss per unit time) in terms of x , \dot{x} and \ddot{x} and showing that by the equation of motion we have

$$\dot{T} + \dot{V} + \dot{Q} = 0, \text{ and hence, } \dot{T} + \dot{V} \neq 0.$$

I.8. Our equation of motion is $\ddot{x} = gx$. This linear differential equation with constant coefficients has a solution of the form (3.24b). The validity of the principle of energy can be read off either in differential form from the equation of motion, or in integrated form from its solution

$$x = a(e^{\alpha t} + e^{-\alpha t}), \quad \alpha^2 = \frac{g}{l}, \quad a = \frac{x_0}{2}.$$

I.9. The numerical data given in the problem permit the calculation of the centripetal acceleration of the moon in msec^{-2} . For the radius r of the earth we can take the original definition of the meter, $r = \frac{2}{\pi} 10^7 \text{m}$.

The law of gravitation, on the other hand, yields $\frac{g}{80^2}$ as the centripetal acceleration after the gravitational constant G has been eliminated by means of g as on p. 20. The two numerical values thus obtained are in satisfactory agreement.

I.10. Set up the transformation equations for the coordinates as in (2.5), but with $\alpha_0 = \beta_0 = \gamma_0 = 0$. The components of the transformed moment L' are found to be linear expressions of the components of L with coefficients equal to the cofactors of the transformation scheme. For the latter we have the relations

$$\rho\gamma_1 = \begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix}, \quad \rho\gamma_2 = \begin{vmatrix} \alpha_3 & \alpha_1 \\ \beta_3 & \beta_1 \end{vmatrix}, \dots$$

to be proved from the orthogonality conditions. Here $\rho = \pm 1$, according as the transformed system has the same sense ("unimodular transformation") as the original system, or the opposite sense.

I.11. From Eqs. (6.8) we have [according to Fig. 7 and Eq. (6.5), B is negative]

$$\epsilon = \frac{-B}{\frac{GM}{O}} = \frac{|B|}{\frac{GM}{O}}.$$

It follows that for the ellipse ($\epsilon < 1$) $\frac{GM}{O} > |B|$, for the hyperbola ($\epsilon > 1$) $\frac{GM}{O} < |B|$. Now $R = \frac{GM}{O}$ is the radius of the hodograph circle, $|B|$ the distance of the center from the pole. From this the assertion made in the statement of the problem follows at once.

The table below, in which

$$v_0 = \frac{GM}{O} + |B|$$

signifies the magnitude of the planet's velocity at perihelion, shows how the limiting cases of the circle and the parabola fit into the scheme.

Planetary Trajectory	ϵ	$ B $	Hodograph	v_0
circle	$= 0$	$= 0$	center at pole	$= \frac{GM}{O}$
ellipse	< 1	$< R$	hodograph includes pole	$< \frac{2GM}{O}$
parabola	$= 1$	$= R$	hodograph passes through pole	$= \frac{2GM}{O}$
hyperbola	> 1	$> R$	hodograph excludes pole	$> \frac{2GM}{O}$

I.12. In the differential equations (6.4) we must replace GM by $\pm \frac{eH}{m}$, where the upper sign (attraction) corresponds to the case of the positive ion, the lower sign (repulsion) to that of the negative ion. Note that here $\dot{x}=0$, $\dot{y}=-v_0$, and the meaning of ϕ is the same as in Fig. 6, so that Eqs. (6.5) give for $\phi = \frac{\pi}{2}$,

$$A = \pm \frac{eH}{m} C, \quad B = -v_0.$$

Eq. (6.6) then becomes

$$(1) \quad \frac{1}{r} = \pm \frac{eE}{m_0 C^2} (1 - \sin \phi) - \frac{v_0}{C} \cos \phi.$$

C changes from trajectory to trajectory with the distance of the firing direction from the y -axis. It follows that (1) represents a family of curves. To obtain the envelope of this family, differentiate Eq. (1) with respect to C , then eliminate C from this and the original equation to obtain

$$(2) \quad x^2 = p^2 - 2py, \quad p = \pm \frac{4eE}{m_0 v_0^2}.$$

Note that any electron path consists of only one branch of a hyperbola, whereas (1) represents both branches; verify—most simply by a sketch of the corresponding families of curves—that Eq. (2) is the envelope of the actual electron paths only in the case of repulsion.

I.13. It will be easiest to use the method of harmonic oscillations of § 3, (4). It is, however, instructive to check that the methods of § 6 also lead to the desired end.

I.14. The nuclear reaction treated here is not an elastic collision; nor is it an inelastic one. It is, so to speak, a “superelastic” one, in that the nuclear binding energy E is to be added to the primary energy E_p . The kinetic energy of the α -particles can be calculated in the classical form $E_\alpha = \frac{1}{2} m_\alpha v_\alpha^2$.

Elimination of v_α from the equations of energy and momentum then yields Kirchner's result for the symmetrical case,

$$\cos \phi = \left(\frac{m_p}{2m_\alpha} \frac{E_p}{E + E_p} \right)^{\frac{1}{2}}$$

1 ev is that energy which a potential drop of 1 volt ($= 10^9$ electromagnetic units of potential) imparts to the electronic charge e ($= 1.6 \cdot 10^{-20}$ electromagnetic units of charge), so that $1 \text{ ev} = 1.6 \cdot 10^{-12}$ erg.

The mass of the proton is $m_p = 1.65 \cdot 10^{-24}$ g, that of the α -particle, is hence $m_\alpha = 6.6 \cdot 10^{-24}$ g. The latter is needed in order to pass from E_α , at first expressed in ev and then converted to erg, to the velocity v_α . The value of v_α thus found shows that the classical form for E_α was justified and that the relativity correction of Eq. (4.11) is negligible.

I.15. In the second Eq. (3.27) we put $V_0 = 0$ and, say, $v_0 = 1$, so that one can immediately calculate the kinetic energy $\frac{1}{2} M V^2$ of the struck particle after the collision as function of $x = \frac{M}{m}$; in particular one finds it to be a maximum for $x = 1$ and to be small—only 1.9 % of the maximum value—for $x = 206$.

Proceeding from such considerations, Fermi in 1935 worked out his method for the production of "thermal" neutrons, i.e., slow neutrons of uniform velocity, which by frequent collisions have reached equilibrium with the protons of thermal energy contained in the paraffin.

I.16. The coordinates of E are

$$(1a) \quad \begin{aligned} x &= ML = a \cos u \\ &= SL - SM = r \cos \phi - \epsilon a \end{aligned}$$

$$(1b) \quad y = EL = r \sin \phi = b \sin u.$$

Write the polar equation of the ellipse in r, ϕ in the form

$$(1) \quad r = \epsilon r \cos \phi + p, \quad p = a(1 - \epsilon^2).$$

Substitute the value of $r \cos \phi$ from (1a) to obtain

$$(2) \quad r = \epsilon(a \cos u + \epsilon a) + a(1 - \epsilon^2) = a(1 + \epsilon \cos u).$$

A differentiation of (2) gives

$$(3) \quad dr = -\epsilon a \sin u du.$$

A differentiation of (1) gives

$$\epsilon \sin \phi d\phi = -p \frac{dr}{r^2}.$$

From this

$$(4) \quad \frac{-p}{\epsilon \sin \phi} \dot{r} = r^2 \dot{\phi} = C \quad (C = \text{areal velocity constant}).$$

Eq. (4) is transformed by (1b) and (3) into

$$\frac{pa}{b} r \dot{u} = C.$$

Finally replace r from (2), to arrive at the differential equation

$$(5) \quad (1 + \epsilon \cos u) du = n dt \qquad (6) \quad n = \frac{Cb}{pa^2}.$$

Integration of (5) yields

$$u - \epsilon \sin u = nt.$$

The integration constant vanishes because we agreed to measure time in such a way that for $u=0$ we have $t=0$. nt is called the *mean anomaly* and, like the other anomalies, is measured from perihelion in astronomy. The name comes from the fact that by means of Eq. (6.9) the right member of (6) can be transformed to $\frac{2\pi}{T}$.

II.1. Reduce

$$\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial \phi} \delta \phi + \frac{\partial f}{\partial \psi} \delta \psi$$

by means of condition (1) of the problem, to obtain for the right member

$$\left(\frac{\partial f}{\partial x} a \cos \psi + \frac{\partial f}{\partial y} a \sin \psi + \frac{\partial f}{\partial \phi} \right) \delta \phi + \frac{\partial f}{\partial \psi} \delta \psi.$$

Now $\delta \phi$ and $\delta \psi$ can individually be put $=0$, so that

$$(2) \quad \frac{\partial f}{\partial \psi} = 0 \quad \text{and} \quad (3) \quad a \frac{\partial f}{\partial x} \cos \psi + a \frac{\partial f}{\partial y} \sin \psi + \frac{\partial f}{\partial \phi} = 0.$$

The latter equation is valid for all ψ and can therefore be differentiated with respect to ψ . With the help of (2) this gives

$$(4) \quad -a \frac{\partial f}{\partial x} \sin \psi + a \frac{\partial f}{\partial y} \cos \psi = 0$$

and, after a second differentiation with respect to ψ ,

$$(5) \quad a \frac{\partial f}{\partial x} \cos \psi + a \frac{\partial f}{\partial y} \sin \psi = 0.$$

From (4) and (5) it follows that

$$(6) \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

According to (3) we must then also have

$$(7) \quad \frac{\partial f}{\partial \phi} = 0.$$

(2), (6) and (7) show that there does not exist a condition $f=0$ dependent on x, y, ϕ, ψ , i.e., that our system is non-holonomic. Proof of G. Hamel, "Elementare Mechanik," 2nd Ed., Leipzig 1922.

II.2. Draw the work diagram of the engine, that is, the L -curve and the W -line with the crank angle from 0 to π as abscissa. Note that the areas enclosed between the L -curve and the abscissa and the W -line and the abscissa must be equal. This yields a relation between L and W . The angles ϕ_2 and ϕ_1 belonging to ω_{\max} and ω_{\min} are the points of intersection of the L - and W -curves in the diagram, $\sin \phi_1 = \sin \phi_2 = \frac{2}{\pi}$, $\phi_2 = \pi - \phi_1$, $\phi_1 = 39^\circ 33' = 0.69$ radians. Determine the difference in the kinetic energy of the flywheel between angles ϕ_2 and ϕ_1 and express it in terms of

I , ω_m and δ . The equation of energy written for the same interval yields the magnitude of the required I in the form

$$I = \frac{W}{\delta \omega_m^2} (\pi \cos \phi_1 - \pi + 2\phi_1) = \frac{0.86}{\delta \omega_m^2} W.$$

With

$$N = \frac{W\omega}{75} HP \quad \text{and} \quad n = \frac{60}{2\pi} \omega \text{ r.p.m.}$$

one obtains

$$I \cong 43,400 \frac{N}{\delta m^2} kg \cdot m \cdot sec^2$$

in the practical system of units.

II.3. For the magnitude of the earth's radius see Problem I.9. For the numerical calculation of the length of the day put $(8\pi)^{\frac{1}{2}} = 5$.

II.4. (a). If one thinks of the beam as held fixed in position, one need only consider the equilibrium of gravity and inertial forces at the pulley in a virtual rotation $\delta\phi$ of the pulley (torque equation). From this one obtains the acceleration \ddot{z} of the weights as a small fraction of g .

(b). Add a virtual rotation of the beam to the foregoing. Here the moments of the inertial forces about the fulcrum of the balance beam enter. One finds that equilibrium does not prevail. The beam is deflected downward on the side of the pan as long as the weight p is falling. In estimating the excess weight the diameter of the pulley may be neglected in comparison with the length of the balance beam. Another procedure using the same approximation is to compare the load on the pan with the load due to weights and inertial forces on the other side of the beam.

II.5. Let the equation of the inclined plane be

$$(1) \quad F(z, x, t) = z - ax - \phi(t) = 0.$$

$a = \tan \alpha$ determines the constant inclination α of the plane to the horizontal; $\phi(t)$ is its intersection with the z -axis which varies with time. Lagrange's equations of the first kind (12.9a) give

$$(2) \quad \ddot{x} = -\lambda a, \quad \ddot{z} = \lambda - g.$$

In order to determine λ , differentiate (1) twice with respect to t ,

$$(3) \quad \ddot{z} - a\ddot{x} = \ddot{\phi}(t).$$

Substitution of (2) in (3) yields λ ; the integration of (2) can now easily be carried out. With initial conditions $\dot{x}=\dot{z}=0$, $x=x_0$, $z=z_0$ at $t=0$ one obtains

$$x = x_0 - \frac{\alpha}{1+\alpha^2} \left(\phi(t) - \phi(0) - \dot{\phi}(0)t + g \frac{t^2}{2} \right),$$

$$z = z_0 + \frac{1}{1+\alpha^2} \left(\phi(t) - \phi(0) - \dot{\phi}(0)t - g\alpha^2 \frac{t^2}{2} \right).$$

From this we have for $\ddot{\phi} = +g$

$$x = x_0 - g \frac{t^2}{2} \sin 2\alpha, \quad z = z_0 + g \frac{t^2}{2} \cos 2\alpha$$

and for $\ddot{\phi} = -g$,

$$x = x_0, \quad z = z_0 - g \frac{t^2}{2},$$

as for a free fall. $\lambda=0$ only under the last assumption; otherwise λ acts as a pressure against the sliding body and hence does work.

The problem can be solved by means of d'Alembert's principle without introducing λ . Since the time is not to be varied (cf. p. 68), we have, from (1), $\delta z = \alpha \delta x$ for the virtual displacements. From d'Alembert's principle it follows that

$$\ddot{x} + (g + \ddot{z})\alpha = 0,$$

which, together with (3), allows one to calculate \ddot{x} and \ddot{z} directly. This example illustrates that d'Alembert's method leads to a solution more directly and simply than do Lagrange's equations; the latter, on the other hand, have the advantage that the forces of constraint are quantitatively determined.

II.6. In § 11, (1), d'Alembert's principle was used to derive the equation of acceleration of a system rotating under the influence of an external torque. We introduced a virtual rotation $\delta\phi$ about the axis of rotation, which we shall here take as our x -axis. Only the tangential inertial forces were relevant, since the normal ones, the centrifugal forces, did no work in the rotation $\delta\phi$.

Here we ask for the forces exerted on bearings A and B in a uniform rotation, or, instead, for their reactions A and B . It is precisely the centrifugal forces which are relevant, whereas the tangential inertial forces drop out in a uniform rotation. If we introduce the virtual translations δy , δz , the virtual work becomes equal to the product of δy and δz by the sum of

the y - and z -components of the centrifugal forces acting on the single mass elements, these forces being

$$dm y \omega^2, \quad dm z \omega^2.$$

An integration yields the inertial components Y and Z of the ordinary swinging motion of the total mass m , which are to be thought of as applied at the center of mass.

Next we introduce virtual rotations $\delta\phi_y$ and $\delta\phi_z$ about the y - and z -axis respectively. The virtual work done in these is given by

$$-\delta\phi_y \int dm x z \omega^2 \quad \text{and} \quad \delta\phi_z \int dm x y \omega^2.$$

They correspond to the torques

$$L_y = -I_{xz} \omega^2 \quad \text{and} \quad L_z = I_{xy} \omega^2.$$

To determine the bearing reactions A and B , fix the origin of coordinate system xyz , say, at the bearing A , designate by l the distance between the two bearings and by η and ζ the coordinates of the mass center along the y - and z -directions. We then obtain the two component equations

$$(1) \quad \begin{aligned} A_y + B_y &= -m\eta\omega^2, \\ A_z + B_z &= -m\zeta\omega^2 \end{aligned}$$

and the two moment equations

$$(2) \quad \begin{aligned} lB_z &= -I_{xz}\omega^2, \\ lB_y &= -I_{xy}\omega^2 \end{aligned}$$

for the determination of the four unknowns A_y , A_z and B_y , B_z .

Clearly these periodically varying reactions in the bearings are undesirable from the engineering standpoint. To avoid them it is not only necessary that the center of mass lie on the axis of rotation $\eta = \zeta = 0$, Eq. (1), but also that the axis of rotation be a principal axis of the mass distribution, $I_{xz} = I_{xy} = 0$, Eq. (2); see, in this connection, Ch. IV, § 22, near Eq. (15a). The fulfillment of this second condition is just as important as that of the first. The fulfillment of both conditions is called the "balancing" of the rotating body.

II.7. Let S be the tension of the string, z the portion of its length that is unwound at any given instant. We have during (a):

$$I\dot{\omega} = Sr, \quad S = m(g - \ddot{z}).$$

\dot{z} and \ddot{z} are positive; because of $\dot{z} = r\omega$,

$$(1) \quad \ddot{z} = r\dot{\omega} = \frac{Sr^2}{I},$$

$$(2) \quad S = \frac{mg}{1 + \frac{mr^2}{I}};$$

during (b):

The rotation ω retains its sense. The torque of the string tension acts against ω . \dot{z} becomes negative and we have

$$(3) \quad \dot{z} = -r\omega, \quad \ddot{z} = -r\dot{\omega} = +\frac{Sr^2}{I},$$

$$(4) \quad S = \frac{mg}{1 + \frac{mr^2}{I}}.$$

In both cases (a) and (b) the string tension is the same, and constant in time; it is smaller than the weight of the rotating body.

In the transition stage between (a) and (b) one perceives a very noticeable pull in the hand which corresponds to the transition from positive momentum $m\dot{z}$ to negative. During this interval S becomes greater than that given by Eq. (2).

II.8. The condition that the particle jump off is, according to (18.7),

$$\lambda = 0 \quad \text{or} \quad R_n = 0$$

so that, from (18.6),

$$(1) \quad mg\frac{z}{l} = -\frac{m}{l}(x\ddot{x} + y\ddot{y} + z\ddot{z}).$$

Now for every path on the sphere

$$x\dot{x} + y\dot{y} + z\dot{z} = 0, \quad \text{i.e.,} \quad x\ddot{x} + y\ddot{y} + z\ddot{z} = -(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = -v^2,$$

so that, instead of (1), we can write

$$(2) \quad \frac{mgz}{l} = \frac{mv^2}{l}.$$

The right side does not equal the centrifugal force along the path, since the path is not a geodesic in our case. In agreement with Meusnier's theorem of § 40 it is equal to the projection of this centrifugal force on the normal to the spherical surface.

From the equation of energy

$$(3) \quad v^2 = v_0^2 - 2g(z - z_0).$$

Eq. (2) can therefore be rewritten in terms of the initial values v_0, z_0 :

$$(4) \quad 3z = 2z_0 + \frac{v_0^2}{g} = 2(z_0 + h_0),$$

where $h_0 = \frac{1}{2} \frac{v_0^2}{g}$ is the height of free fall corresponding to velocity v_0 .

III.1. For a pendulum hanging almost vertically coordinates x and y are small quantities of the first order; z equals $-l$ to quantities of the second order. For this reason the third Eq. (18.2) gives, to quantities of the second order,

$$(1) \quad \lambda = -\frac{mg}{l}$$

and the first two Eqs. (18.2) define, as in Problem I.13, a harmonic elliptical motion of circular frequency

$$(2) \quad \frac{2\pi}{T} = \left(\frac{g}{l}\right)^{\frac{1}{2}}.$$

For the areal velocity constant of the elliptical motion we have

$$(3) \quad C = \frac{2\pi ab}{T} = \left(\frac{g}{l}\right)^{\frac{1}{2}} ab \rightarrow 0,$$

and for the energy constant (initial state $\theta_0 = \epsilon, \dot{\theta}_0 = 0$)

$$(4) \quad E = T + V = mgl \left(-1 + \frac{\epsilon^2}{2}\right).$$

With $u = \eta - 1$ one has, from (18.11),

$$U = -\frac{4g}{l} \left(\eta - \frac{\epsilon^2}{2}\right) \eta - \frac{C^2}{l^2} = \frac{4g}{l} (\eta_1 - \eta) (\eta - \eta_2),$$

$$\eta_{1,2} = \frac{\epsilon^2}{4} \pm \left(\frac{\epsilon^4}{16} - \frac{C^2}{4gl^2}\right)^{\frac{1}{2}}.$$

From (18.15) we then have

$$(5) \quad 2\pi + \Delta\phi = \frac{C}{4(lg)^{\frac{1}{2}}} \int_{\eta_2}^{\eta_1} \frac{d\eta}{\eta[(\eta_1 - \eta)(\eta - \eta_2)]^{\frac{1}{2}}}.$$

A substitution modelled after Eq. (46.11) transforms the integral of (5) into the known integral

$$\int_0^\pi \frac{dv}{A + B \cos v} = \frac{\pi}{(A^2 - B^2)^{\frac{1}{2}}}, \quad A = \frac{\epsilon^2}{4}, \quad B = \left(\frac{\epsilon^4}{16} - \frac{C^2}{4gl^2}\right)^{\frac{1}{2}}.$$

Thereupon (5) yields $\Delta\phi = 0$, which was to be proved.

III.2. The first assertion of the problem is proved immediately by differentiation of Eq. (19.10) for $|C|$ with respect to ω ; the second assertion is similarly proved by differentiation of $|C|\omega$ with respect to ω .

III.3. Let us designate the proportionality factors of the damping torque and the restoring torque by $2\rho I$ and $\omega_0^2 I$ respectively. One then obtains Eq. (19.9) as the equation of motion of the galvanometer, with the difference that the right member is now a constant C and α replaces x in the notation. Fit the constants a and b of the general solution

$$\alpha = C + e^{-\rho t} (a \cos [(\omega_0^2 - \rho^2)^{1/2} t] + b \sin [(\omega_0^2 - \rho^2)^{1/2} t])$$

to the conditions $\alpha = \dot{\alpha} = 0$ at $t = 0$, and the constant C to the condition $\alpha \rightarrow \alpha_\infty$ as $t \rightarrow \infty$.

In the case (a) one obtains a transient motion with decreasing oscillations, in the case (c), a monotonic transient motion towards the final position. Case (b) should be treated as limiting case of either (a) or (c); in it we arrive at a secular term containing t as a factor.

III.4. In part (a) of the problem d'Alembert's principle (x, y = coordinates of the oscillating mass point, y positive upward) demands

$$(1) \quad \ddot{x} \delta x + (\ddot{y} + g) \delta y = 0.$$

The equation of constraint is

$$(2) \quad (x - \xi)^2 + y^2 = l^2.$$

Its variation (t , and hence also ξ , being held fixed) gives

$$(3) \quad (x - \xi) \delta x + y \delta y = 0.$$

Combination of (1) and (3) results in

$$(4) \quad y \ddot{x} - (x - \xi) (\ddot{y} + g) = 0.$$

Differentiating (2) twice with respect to t yields a second equation for \ddot{x} and \ddot{y} which, together with (4), furnishes the exact differential equation of the problem.

When passing to small vibrations, one must remember that $x - \xi$ is a small quantity of first order so that, according to (2), $y \approx \pm l$ to small quantities of second order. \dot{y} and \ddot{y} are then also small quantities of second order, so that (4) becomes

$$(5) \quad l \ddot{x} + (x - \xi) g = 0.$$

With $x - \xi = u$ one obtains the inhomogeneous pendulum equation

$$(6) \quad \ddot{u} + \frac{g}{l} u = -\ddot{\xi},$$

showing that $-m\ddot{\xi}$ acts as driving force. The integration is performed as on p. 101. The phase relation between the motion of the point of suspension and that of the mass point emphasized in the text of the problem corresponds to Fig. 31. It will be instructive to make an experiment with a string whose lower end carries a weight, and whose upper end is moved horizontally to and fro by the hand. For fast motion of the hand (case above resonance) the out-of-phase motion of the two points can be very clearly observed.

Using the method of Lagrange's equations of the first kind, from Lagrange's equation for y one finds $\lambda = -\frac{g}{l}$ up to small quantities of second order, and from the x -equation one obtains Eq. (5).

In part (b) of the problem Eq. (1) remains valid. Condition (2) becomes

$$(7) \quad x^2 + (y - \eta)^2 = l^2.$$

Its variation yields, instead of (4),

$$(8) \quad (y - \eta)\ddot{x} - x(\ddot{y} + g) = 0.$$

If x is treated as a small quantity of first order, (7) gives to quantities of second order

$$(9) \quad y - \eta = -l, \quad \ddot{y} = \ddot{\eta}.$$

By this, (8) becomes

$$(10) \quad \ddot{x} + \frac{\ddot{\eta} + g}{l} x = 0.$$

The same follows from Lagrange's equations of the first kind, since the y -equation yields the value

$$(11) \quad \lambda = -\frac{\ddot{\eta} + g}{l}$$

if approximation (9) is used, so that the x -equation becomes identical to (10).

If the point of suspension is moved upward with constant acceleration $+g$ it follows that the force of gravity seems doubled; if the point is moved downward with $-g$, it seems to be annulled. This points to an equivalence between gravity and acceleration, which, together with the equality of the gravitational and inertial mass (p. 19), formed the foundation of Einstein's theory of gravitation.

III.5. Equilibrium of the tensions at points C and D (necessary because the wire is weightless!) demands that

$$(3) \quad S_1 \frac{x_1 - x_3}{l_1} = S \frac{x_3}{a} + S \frac{x_3 - x_4}{a}, \quad S_2 \frac{x_3 - x_4}{l_2} = S \frac{x_4}{a} + S \frac{x_4 - x_3}{a},$$

so that, from Eq. (1) of the problem, with $\sigma_1 = \frac{m_1 g}{S} \frac{a}{l_1}$, $\sigma_2 = \frac{m_2 g}{S} \frac{a}{l_2}$,

$$(4) \quad \begin{aligned} \sigma_1 x_1 &= (2 + \sigma_1)x_3 - x_4 \\ \sigma_2 x_2 &= (2 + \sigma_2)x_4 - x_3. \end{aligned}$$

We have presupposed weak coupling, so that σ_1 and σ_2 are small numbers; they can be cancelled in the right members of (4). Solving for x_3 , x_4 then yields

$$(5) \quad \begin{aligned} x_3 &= \frac{2}{3} \sigma_1 x_1 + \frac{1}{3} \sigma_2 x_2 \\ x_4 &= \frac{2}{3} \sigma_2 x_2 + \frac{1}{3} \sigma_1 x_1 \end{aligned}$$

and substitution in (2) gives

$$(6) \quad \begin{aligned} \ddot{x}_1 + \frac{g}{l_1} (1 - \sigma_1) x_1 &= \frac{1}{3} \frac{g}{l_1} (\sigma_2 x_2 - \sigma_1 x_1) \\ \ddot{x}_2 + \frac{g}{l_2} (1 - \sigma_2) x_2 &= \frac{1}{3} \frac{g}{l_2} (\sigma_1 x_1 - \sigma_2 x_2). \end{aligned}$$

These simultaneous differential equations are to be treated just like (20.10). The meaning, for our problem, of the quantities ω_1 , ω_2 , k_1 , k_2 introduced there can be found by comparison with Eq. (6) above.

III.6. The effect of m on M is represented by $k(X - x)$, that of M on m by $k(x - X)$. In the two resulting simultaneous differential equations for X and x put $X = 0$. It will be found that the condition required—that only m take part in the oscillation—is given by the resonance requirement that the circular frequency of the proper oscillation of system (m, k) be equal to the circular frequency ω of the external force.

Such an arrangement is used in engineering practice as an “oscillation quencher.” It may thus be used on a crank shaft with a flywheel rotating with constant angular velocity ω ; there the quencher is a device capable of variable rotation; its purpose is to absorb the oscillations of the crank with which it is coupled. In such a case the angle rotated through takes the place of coordinate x of our problem.

IV.1. Moments of inertia of plane mass distributions are important for the torsion and bending of beams in elasticity theory (Vol. II). Because of $r^2 = x^2 + y^2$ we have

$$I_p = \int r^2 dm = \int x^2 dm + \int y^2 dm = I_x + I_y$$

In problems of elasticity the mass is to be thought of as uniformly distributed with density 1 over the cross section of the beam, so that $dm = dS =$ element of area. For a circular disk of radius a and area $S = \pi a^2$ one then obtains

$$I_p = \int r^2 dS = 2\pi \int_0^a r^3 dr = \frac{1}{2} S a^2 \text{ and hence, } I_x = I_y = \frac{1}{4} S a^2.$$

IV.2. We leave the ratio of the magnitudes of the three principal moments of inertia arbitrary to the very last; we thus embrace, in one and the same calculation, the three cases in which A is the greatest, smallest, and intermediate principal moment of inertia.

IV.3. The impulse Z imparts to the ball (radius a) both a translational and a rotational momentum,

$$(1) \quad Mv = Z \quad \text{and} \quad (2) \quad I\omega = Zh,$$

where h is the height above the center at which the horizontally held cue strikes the ball. The axis of ω is perpendicular to the median plane. The peripheral velocity u at the lowest point lies in the median plane and equals $a\omega$. This is true not only at $t=0$ (time of the impact), but also for $t>0$.

According to (11.12a), $I = \frac{2}{5} Ma^2$, so that, for $t=0$, by Eqs. (2) and (1),

$$(3) \quad \frac{2}{5} Mau = Zh = Mvh.$$

$v=u$ means pure rolling, and from (3) requires $h = \frac{2}{5} a$. Notice that we have counted u positive in the direction opposite to v . For high shots $h > \frac{2}{5} a$ the sliding velocity $u-v$ of the point of contact between ball and cloth is >0 and opposed to v ; friction is therefore directed along v and of magnitude μMg . Its moment about the center, μMga , acts against the rotation ω .

For low shots the friction is directed in the opposite way. In general, we can associate the upper sign with a high shot, the lower sign with a low shot, and write for $t>0$,

$$(4) \quad \dot{v} = \pm \mu g,$$

$$(5) \quad \dot{u} = \mp \frac{5}{2} \mu g.$$

Discussion by means of graph: draw v and u as ordinates against t as abscissa; both are represented by straight lines which intersect in the case of high shots as well as that of low shots. At the point of intersection

$u=v$ pure rolling takes place. From then on the graphs of u and v run on in coincidence along a horizontal straight line. The abscissa of the intersection is

$$(6) \quad \tau = \pm \frac{5h-2a}{7a} \frac{Z}{\mu g M}.$$

Note that for a low shot the first numerator is negative since h lies between $-a$ and $\frac{2}{5}a$; the negative sign of the right member of (6) is therefore only a formal one. The increase or decrease of velocity for high and low shots respectively is given by $\Delta v = \pm \mu g \tau$. The final velocity of pure rolling becomes

$$v + \Delta v = \frac{5}{7} \frac{h+a}{a} \frac{Z}{M}$$

i.e., proportional to the height $h+a$ of the point of impact above the cloth.

Theory of the follow shot. The ball struck high meets a second ball in a central collision during the time interval $t < \tau$ in which $u > v$. Let u_0 and v_0 be the values of u and v at the moment of impact. v_0 is transferred to the second ball. According to (4), the first ball is then accelerated from $v=0$. From (5), its u decreases from u_0 on down. A new graph shows that there is an intersection at which pure rolling begins to take place. Abscissa of the point of intersection and velocity of pure rolling are, respectively,

$$(7) \quad \tau_1 = \frac{2}{7} \frac{u_0}{\mu g}, \quad v_1 = \mu g \tau_1 = \frac{2}{7} u_0.$$

Theory of the draw shot. Again the driven ball meets a second one in the interval $t < \tau$, where now, however, $u < v$. For an extremely low shot, which we shall presuppose, u is, as a matter of fact, negative, that is, has the same direction as v . Let u_0 and v_0 be the values of u and v just before impact. v_0 is again transmitted to the second ball. From (4), the first ball is accelerated from $v=0$ in the negative sense: it runs backward. Eq. (5) tells us that u increases from its negative initial value u_0 toward positive values, i.e., its absolute value decreases. The two straight lines of v and u intersect (new diagram); the abscissa of the point of intersection and the final velocity of pure rolling now become

$$(8) \quad \tau_2 = \frac{2}{7} \frac{|u_0|}{\mu g}, \quad |v_2| = \frac{2}{7} |u_0|.$$

IV.4. The cue is no longer held horizontally as in IV.3, but forms an angle with the horizontal plane; evidently the cue must hit the ball at a point of the upper hemisphere, as in our earlier "high shots." Put

the x -axis along the horizontal component of the impulse and the z -axis along the vertical. The components of the impulse Z become $(Z_x, 0, Z_z)$ and the components of the impulsive torque N referred to the center of the ball (which is also the origin of the xyz -system),

$$N_x = yZ_z, \quad N_y = zZ_x - xZ_z, \quad N_z = -yZ_x.$$

Here x, y and z are the coordinates of the point of impact of cue and ball. From the N_x, N_y we obtain the angular velocities

$$\omega_x = \frac{5}{2} \frac{N_x}{Ma^2}, \quad \omega_y = \frac{5}{2} \frac{N_y}{Ma^2}.$$

The associated peripheral velocities at the lowest point P of the ball are

$$(1) \quad u_x = -a\omega_y, \quad u_y = +a\omega_x.$$

N_z and ω_z do not interest us; they do not generate any sliding at P , but merely a "boring" friction to be neglected. Let the sliding motion at the cloth have components

$$(2) \quad v_x - u_x = -\rho \cos \alpha, \quad v_y - u_y = -\rho \sin \alpha.$$

It creates a friction R making an angle $\pi + \alpha$ with the x -axis and having magnitude $\mu g M$. Its influence on the translation and rotation for $t > 0$ is determined by

$$\begin{aligned} M\dot{v}_x &= R_x, & M\dot{v}_y &= R_y, \\ I\dot{\omega}_x &= aR_y, & I\dot{\omega}_y &= -aR_x. \end{aligned}$$

It follows that

$$\dot{v}_x = -\mu g \cos \alpha, \quad \dot{v}_y = -\mu g \sin \alpha$$

and, by virtue of (1) and (2),

$$(4) \quad \dot{u}_y = -\frac{5}{2}\mu g \sin \alpha, \quad \dot{u}_x = -\frac{5}{2}\mu g \cos \alpha;$$

$$(5) \quad \dot{v}_x - \dot{u}_x = -\frac{d}{dt}(\rho \cos \alpha) = -\frac{7}{2}\mu g \cos \alpha,$$

$$\dot{v}_y - \dot{u}_y = -\frac{d}{dt}(\rho \sin \alpha) = -\frac{7}{2}\mu g \sin \alpha.$$

Solution for $\dot{\alpha}$ and $\dot{\rho}$ from the last two members of Eqs. (5) gives

1. $\dot{\alpha} = 0$. The friction has constant direction; since it also has constant magnitude, the path of point P in the horizontal plane becomes a parabola.

The axis of the parabola is parallel to the initial direction α of the sliding motion, which can be gathered from the components of Z and N .

2. $\dot{\rho} = -\frac{7}{2}\mu g$, $\rho = \rho_0 - \frac{7}{2}\mu g t$, $\rho = 0$ for $t = \tau = \frac{2}{7} \frac{\rho_0}{\mu g}$. ρ_0 is the initial magnitude of sliding velocity which can likewise be determined from Z and N . For $t > \tau$ the sliding and friction are permanently 0. The ball pursues a straight course tangent to the parabola.

V.1. Let ϕ be the instantaneous angle by which the rotating plane has turned with respect to the fixed plane. We put

$$(1) \quad x + iy = (\xi + i\eta)e^{i\phi}.$$

Two differentiations with respect to t give, with $\dot{\phi} = \omega$,

$$(2) \quad \ddot{x} + i\ddot{y} = \{\ddot{\xi} + i\ddot{\eta} + 2i\omega(\dot{\xi} + i\dot{\eta}) + i\dot{\omega}(\xi + i\eta) - \omega^2(\xi + i\eta)\}e^{i\phi}.$$

$\xi + i\eta$ is the (complex) vector \mathbf{r} as observed from the rotating plane, $\dot{\xi} + i\dot{\eta} = \dot{\mathbf{r}}$ its velocity observed from the same plane, etc. Since $i(\dot{\xi} + i\dot{\eta}) = (\dot{\xi} + i\dot{\eta})e^{i\frac{\pi}{2}}$ is a vector perpendicular to the latter, we can write,

$$(3) \quad 2i\omega(\dot{\xi} + i\dot{\eta}) = 2\omega \mathbf{X} \dot{\mathbf{r}}, \quad i\dot{\omega}(\xi + i\eta) = \dot{\omega} \mathbf{X} \mathbf{r},$$

where ω is of course directed along the normal to the complex plane. As on p. 165, let us call \mathbf{w} the velocity $\dot{x} + i\dot{y}$, as observed from the fixed plane; we shall, however, retain the designation of superscript dots for the time derivatives referred to the rotating plane, as written in Eq. (3) above. Eq. (2) then transforms to the following equation analogous to (29.4):

$$(4) \quad \dot{\mathbf{w}} = \{\ddot{\mathbf{r}} + 2\omega \mathbf{X} \dot{\mathbf{r}} + \dot{\omega} \mathbf{X} \mathbf{r} - \omega^2 \mathbf{r}\} e^{i\phi}.$$

If $\mathbf{F} = F_x + iF_y$ is the force referred to the fixed plane, $\Phi = F_\xi + iF_\eta$ that referred to the rotating one, we have, from (1), $\mathbf{F} = \Phi e^{i\phi}$, so that

$$(5) \quad \Phi = \mathbf{F} e^{-i\phi}.$$

In the light of (4) and (5) we then have from $m\dot{\mathbf{w}} = \mathbf{F}$ that

$$(6) \quad m \{\ddot{\mathbf{r}} + 2\omega \mathbf{X} \dot{\mathbf{r}} + \dot{\omega} \mathbf{X} \mathbf{r} - \omega^2 \mathbf{r}\} = \Phi.$$

With this we have determined the additional forces required in the problem. In particular, one identifies the second term on the left with the Coriolis force.

We have intentionally treated this problem in complex notation in order to emphasize that two-dimensional vectors are best represented by complex variables.

V.2. Let us choose the plane in which the straight line rotates as the xy -plane; x -axis horizontal, y -axis vertically upward. Let $\phi = \omega t$ be the angle of the straight line with the x -axis. One can reduce the problem to the foregoing one by associating with the rotating straight line a vertical $\xi\eta$ -plane in which the line is fixed. This $\xi\eta$ -plane must then rotate in the xy -plane with constant angular velocity ω . It is convenient to put the ξ -axis along the rotating straight line. In order to keep the mass point on the ξ -axis one must exert on it a force of constraint in the η -direction. Our external force Φ is therefore the sum of the force of constraint, which we shall call mb , and the force of gravity mg . From Eq. (5) of the foregoing problem the contribution of the latter to Φ is $-img e^{-i\phi}$. Summing, we then have

$$\Phi = \Phi_{\xi} + i\Phi_{\eta} = -mg \sin \omega t - img \cos \omega t + imb.$$

In Eq. (6) of the previous problem one can put $r = \xi$ and, by virtue of (3) *ibid.*, $2\omega \times \dot{r} = 2i\omega \dot{\xi}$; further one must put $\dot{\omega} = 0$. One obtains

$$(1) \quad \ddot{\xi} + 2i\omega \dot{\xi} - \omega^2 \xi = -mg \sin \omega t + i(b - g \cos \omega t).$$

Its real part gives

$$(2) \quad \ddot{\xi} - \omega^2 \xi = -g \sin \omega t,$$

a differential equation with solution

$$(3) \quad r = A \cosh \omega t + B \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t.$$

If one puts the imaginary part of (1) equal to zero, one obtains the relation between force of constraint, gravity and Coriolis force given in the problem, *viz.*,

$$(4) \quad b = g \cos \omega t + 2\omega \dot{\xi}.$$

V.3. (a) Let $x_0 + iy_0$ determine the position of O in the xy -plane. We then have

$$(1) \quad \begin{aligned} \dot{x}_0 + i\dot{y}_0 &= (u + iv)e^{i\phi} \\ \ddot{x}_0 + i\ddot{y}_0 &= \{u + iv + i\omega(u + iv)\}e^{i\phi}. \end{aligned}$$

Let $x + iy$ determine the position of G in the xy -plane. We have

$$(1) \quad \begin{aligned} x + iy &= x_0 + iy_0 + ae^{i\phi} \\ \dot{x} + i\dot{y} &= (u + iv + i\omega a)e^{i\phi} \end{aligned}$$

$$(2) \quad \ddot{x} + i\ddot{y} = [u + iv + i\omega a + i\omega(u + iv) - \omega^2 a] e^{i\phi}.$$

In the xy -plane there corresponds to the external force R the complex quantity

$$(2') \quad F = R i e^{i\phi}.$$

From (2) and (2') the Second Law, $\ddot{x} + i\ddot{y} = \frac{F}{M}$ leads to

$$\ddot{u} + i\ddot{v} + i\dot{\omega}a + i\omega(u + iv) - \omega^2a = i\frac{R}{M}$$

or, resolved into components,

$$(3) \quad \ddot{u} - \omega v - \omega^2a = 0,$$

$$(4) \quad \ddot{v} + \dot{\omega}a + \omega u = \frac{R}{M}.$$

In addition we have, from the law of angular momentum,

$$(5) \quad I\dot{\omega} = -Ra.$$

(b) Conditions $v=0$, $\dot{v}=0$ simplify (3) and (4) to

$$(3') \quad \ddot{u} - \omega^2a = 0, \quad (4') \quad \dot{\omega}a + \omega u = \frac{R}{M}.$$

Elimination of R from (4') and (5) gives

$$(6) \quad \dot{\omega}a \left(1 + \frac{I}{Ma^2}\right) + \omega u = 0.$$

Now put $I = Mb^2$ (b =radius of gyration) and

$$(7) \quad k^2 = 1 + \frac{b^2}{a^2} > 1,$$

which transforms (6) into

$$(6') \quad k^2 \dot{\omega}a + \omega u = 0.$$

After integration of the simultaneous Eqs. (3') and (6') R is determined by (4') or (5).

(c) Elimination of u from (3') and (6') yields

$$(8) \quad k^2 \frac{d}{dt} \frac{\dot{\omega}}{\omega} = -\omega^2.$$

After multiplication by $\frac{\dot{\omega}}{\omega}$ this equation becomes integrable and furnishes

$$(9) \quad k^2 \left(\frac{\dot{\omega}}{\omega}\right)^2 = k^2 c^2 - \omega^2, \quad (9') \quad k\dot{\omega} = \omega(k^2 c^2 - \omega^2)^{\frac{1}{2}},$$

where c is a constant of integration. One gets rid of the square root by putting

$$(10) \quad \omega = kc \cos \psi.$$

With suitable choice of the sign of the square root, (9') becomes

$$(10') \quad \dot{\psi} = c \cos \psi$$

or

$$c dt = \frac{d\psi}{\cos \psi}, \quad (11) \quad ct = \frac{1}{2} \ln \frac{1 + \sin \psi}{1 - \sin \psi}.$$

We have thus determined ψ as a function of t . We can now express all quantities in terms of ψ ; ω from (10), u and R from (6') and (4'):

$$(12) \quad u = ak^2 c \sin \psi, \quad (12') \quad R = \frac{M}{2} ak(k^2 - 1)c^2 \sin 2\psi.$$

This completes the integration.

Because of $\omega = \dot{\phi}$, comparison of (10) and (10') finally yields the relation $\dot{\psi} = \frac{\dot{\phi}}{k}$. Our auxiliary angle ψ is hence proportional to the angle of rotation ϕ ,

$$(13) \quad \psi = \frac{\phi}{k},$$

since the constant of integration can be made zero by suitable choice of the arbitrary direction of the x -axis.

(d) From (1'), for $v=0$,

$$(14) \quad \begin{aligned} |\dot{x} + i\dot{y}|^2 &= \dot{x}^2 + \dot{y}^2 = u^2 + \omega^2 a^2, \\ T &= \frac{M}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\omega^2 = \frac{M}{2}(u^2 + \omega^2 a^2) + \frac{M}{2}(k^2 - 1)a^2 \omega^2 \\ &= \frac{M}{2}(u^2 + k^2 a^2 \omega^2). \end{aligned}$$

From (10) and (12) this equals

$$(15) \quad T = \frac{M}{2} a^2 k^4 c^2 (\sin^2 \psi + \cos^2 \psi) = \text{Const.}$$

(e) From (1) and (12)

$$\dot{x}_0 = ak^2 c \sin \psi \cos \phi, \quad \dot{y}_0 = ak^2 c \sin \psi \sin \phi,$$

so that, by virtue of (10') and (13),

$$(16) \quad \frac{dx_0}{d\phi} = ak \tan \psi \cos \phi, \quad \frac{dy_0}{d\phi} = ak \tan \psi \sin \phi.$$

Eq. (11) tells us that

$$\text{for } \psi = 0, \quad t = 0$$

$$\text{for } \psi = \pm \frac{\pi}{2}, \quad t = \pm \infty.$$

The whole trajectory takes place between $-\frac{\pi}{2} < \psi < +\frac{\pi}{2}$, $-k\frac{\pi}{2} < \phi < +k\frac{\pi}{2}$.

At $t=0$ a cusp occurs; for, according to (16) with $\psi=0$, $\phi=0$,

$$\frac{dx_0}{d\phi} = \frac{dy_0}{d\phi} = \frac{d^2y_0}{d\phi^2} = 0; \text{ on the other hand, } \frac{d^2x_0}{d\phi^2} \text{ and } \frac{d^3y_0}{d\phi^3} \neq 0;$$

the cusp has tangents parallel to the x -axis on both of its branches.

For $t=\pm\infty$ the path becomes asymptotic, for ϕ becomes stationary: from (16), quite generally,

$$\frac{dx_0}{d\phi} = \frac{dy_0}{d\phi} = \pm\infty.$$

In addition (16) yields

$$\frac{dy_0}{dx_0} = \tan \phi = \pm \tan k\frac{\pi}{2},$$

so that the asymptotes are situated symmetrically with respect to the x -axis, with angles $\pm k\frac{\pi}{2}$ as shown by Fig. 57 of p. 252 for $k=1, \frac{3}{2}, 2, 3$.

VI.1. With z taken positive in the sense of fall, i.e. downward, $V = -mgz$. Initial position $z=0$ for $t=0$ lies above the final position $z=z_1$ at $t=t_1$.

(a) For $z=\frac{1}{2}gt^2$ we obtain

$$\int L dt = \int_0^{t_1} \left[\frac{m}{2} (gt)^2 + mg \cdot \frac{g}{2} t^2 \right] dt = \frac{1}{8} mg^2 t_1^3.$$

(b) For $z=ct$ we must choose c in such a way that for $t=t_1$

$$z=z_1 = g\frac{t_1^2}{2}; \text{ we therefore have } c = \frac{gt_1}{2}.$$

With this value we find

$$\int L dt = \int_0^{t_1} \left[\frac{m}{2} \left(\frac{gt_1}{2} \right)^2 + mg \frac{gt_1}{2} t \right] dt = \frac{3}{8} mg^2 t_1^3.$$

For $z=at^3$, $a=\frac{1}{2}\frac{g}{t_1}$, on the other hand,

$$\int L dt = \int_0^{t_1} \left[\frac{m}{2} \left(\frac{3g}{2t_1} \right)^2 t^4 + mg \frac{g}{2t_1} t^3 \right] dt = \frac{7}{20} mg^2 t_1^3.$$

Whereas in Hamilton's principle we compare paths differing only by infinitesimal amounts, here the trajectories of (b) in the phase space of the q, \dot{q} (here z, \dot{z}) differ by finite amounts from the real motion (a). Nevertheless even now the value of Hamilton's integral is smaller for (a) than for (b), as

$$\frac{1}{8} < \frac{3}{8} \quad \text{and} \quad \frac{1}{8} < \frac{7}{20}.$$

This is true here even for arbitrary lengths of path, which need not be the case as a general rule (cf. p. 208).

VI.2. As in Problem V.1, let ξ and η be the coordinates fixed in the rotating plane; let $\mathbf{u}=(\dot{\xi}, \dot{\eta})$ be the velocity measured with respect to this plane. The velocity relative to the fixed plane is then

$$\mathbf{w}=\mathbf{u}+\mathbf{v}, \quad \mathbf{v}=\boldsymbol{\omega} \times \mathbf{r}$$

[cf., for instance, the first line of the table of p. 139]. Resolution into components gives

$$w_{\xi}=\dot{\xi}-\omega\eta, \quad w_{\eta}=\dot{\eta}+\omega\xi,$$

$$|\mathbf{w}|^2=\dot{\xi}^2+\dot{\eta}^2+2\omega(\xi\dot{\eta}-\eta\dot{\xi})+\omega^2(\xi^2+\eta^2).$$

From this it follows, with $T=\frac{1}{2}m|\mathbf{w}|^2$, that

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}}=m\frac{d}{dt}(\dot{\xi}-\omega\eta)=m(\ddot{\xi}-\omega\dot{\eta}-\dot{\omega}\eta)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\eta}}=m\frac{d}{dt}(\dot{\eta}+\omega\xi)=m(\ddot{\eta}+\omega\dot{\xi}+\dot{\omega}\xi)$$

$$\frac{\partial T}{\partial \xi}=m(\omega\dot{\eta}+\omega^2\xi), \quad \frac{\partial T}{\partial \eta}=m(-\omega\dot{\xi}+\omega^2\eta).$$

Let Φ_{ξ} , Φ_{η} be the components of the external force \mathbf{F} with respect to the moving axes ξ , η ; we then obtain Lagrange's equations

$$m(\ddot{\xi}-2\omega\dot{\eta}-\dot{\omega}\eta-\omega^2\xi)=\Phi_{\xi},$$

$$m(\ddot{\eta}+2\omega\dot{\xi}+\dot{\omega}\xi-\omega^2\eta)=\Phi_{\eta}.$$

This is in exact agreement with Eq. (6) of Problem V.1, provided we resolve the latter into its components.

In the guiding on a rotating straight rod treated in Problem V.2 we have

$$v^2=\frac{dr^2+r^2d\phi^2}{dt^2}=\dot{r}^2+r^2\omega^2, \quad L=\frac{m}{2}(\dot{r}^2+r^2\omega^2)-mgr\sin\omega t;$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}}=m\dot{r}, \quad \frac{\partial L}{\partial r}=mr\omega^2-mg\sin\omega t.$$

The Lagrange equation resulting from this is identical to Eq. (2) of V.2. It immediately leads to the solution (3) of that problem. With the present method we need not speak of Coriolis and similar forces, though, on the other hand, we do not learn anything about the force of constraint.

VI.3. The terms left out in Eq. (4) of the problem and indicated by . . . are

$$\left(1 + \frac{\xi}{R}\right) \dot{\eta} \quad \text{and} \quad -\frac{\eta}{R} \left(1 + \frac{\xi}{R}\right) \frac{\cos \theta}{\sin \theta} \dot{\xi}.$$

After multiplication by the factor of $\{\}$ they would give, on differentiation with respect to t , terms of second or higher order in the ξ , η , ζ or their derivatives. In connection with the differentiated equations (5) and (6) we should remark that terms of second order such as $\zeta \ddot{\xi}$, $\dot{\zeta} \dot{\xi}$, etc., have, of course, been omitted. It is worth noting that through this omission the radius R of the earth disappears from the results. In the complete Eq. (6) we would actually obtain, in addition to the term written down, a term in ω^2 , viz.,

$$R \sin \theta \cos \theta \omega^2,$$

which evidently represents the ξ -component of the ordinary centrifugal force; the corresponding ζ -component would occur in $\frac{\partial T}{\partial \zeta}$. These terms must, however, be omitted because they have already been included in the effective gravitational acceleration g , Eq. (30.1).

In the case of Foucault's pendulum one should evidently use not the ordinary form (34.6) of Lagrange's equations, but the mixed type (34.11), coupled with the equation of constraint (31.1).

Incidentally, note that due to the definition of η and ψ_0 in (1) and (2) our problem belongs to the class of problems dependent on the time discussed on p. 217.

VI.4. The center of mass describes a "curtate" cycloid in a plane normal to the axis of the cylinder. Its parametric equations in terms of the angle of rotation ϕ are obtained from Eq. (17.1) for a "common" cycloid by replacing a of Eq. (17.1) in part by s ,

$$\begin{aligned} \xi &= a\phi - s \sin \phi, & \dot{\xi} &= (a - s \cos \phi) \dot{\phi}, \\ \eta &= a - s \cos \phi, & \dot{\eta} &= s \sin \phi \dot{\phi}. \end{aligned}$$

(a) If we take the mass center as reference point O , we have

$$\begin{aligned} T_{\text{transl}} &= \frac{m}{2} (\dot{\xi}^2 + \dot{\eta}^2) = \frac{m}{2} (a^2 + s^2 - 2as \cos \phi) \dot{\phi}^2, \\ T_{\text{rot}} &= \frac{I}{2} \dot{\phi}^2, \quad T_m = 0, \quad V = mg\eta = mg(a - s \cos \phi). \end{aligned}$$

Notice that $\omega = \dot{\phi}$ is originally the angular velocity of the cylinder about its axis of symmetry, but that, according to (23.8), it is also the angular

velocity about a parallel axis through the mass center. Putting $I = mb^2$ (b = radius of gyration) and $c^2 = a^2 + s^2 + b^2$, we have

$$(1) \quad L = T_{\text{transl}} + T_{\text{rot}} - V = \frac{m}{2}(c^2 - 2as\cos\phi)\dot{\phi}^2 - mg(a - s\cos\phi)$$

$$\frac{1}{m} \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = (c^2 - 2as\cos\phi)\ddot{\phi} + 2as\sin\phi\dot{\phi}$$

$$\frac{1}{m} \frac{\partial L}{\partial \phi} = as\sin\phi\dot{\phi}^2 - gs\sin\phi.$$

Hence the equation of motion is

$$(2) \quad (c^2 - 2as\cos\phi)\ddot{\phi} + as\sin\phi\dot{\phi}^2 + gs\sin\phi = 0.$$

(b) If we choose the center of the cross section through the mass center as reference point O , the latter moves horizontally with velocity $a\dot{\phi}$; with $I' = I + ms^2$ [cf. (16.8)] we have

$$T_{\text{transl}} = \frac{m}{2}a^2\dot{\phi}^2, \quad T_{\text{rot}} = \frac{I'}{2}\dot{\phi}^2, \quad V \text{ as above,}$$

but now T_m is not 0; from Eq. (22.11) it is given by

$$T_m = -ma\dot{\phi}^2s\cos\phi.$$

As a result

$$(3) \quad L = T_{\text{transl}} + T_{\text{rot}} + T_m - V = \frac{m}{2}(c^2 - 2as\cos\phi)\dot{\phi}^2 - mg(a - s\cos\phi),$$

which is in agreement with (1), so that we obtain, once more, the equation of motion (2). For small oscillations about $\phi = 0$ it yields

$$\ddot{\phi} + \frac{g}{l_1}\phi = 0, \quad l_1 = \frac{c^2 - 2as}{g} = \frac{(a-s)^2 + b^2}{g} \dots : \text{stability;}$$

for small oscillations about $\phi = \pi$, on the other hand, with $\psi = \pi + \phi$,

$$\ddot{\psi} - \frac{g}{l_2}\psi = 0, \quad l_2 = \frac{c^2 + 2as}{g} = \frac{(a+s)^2 + b^2}{g} \dots : \text{instability.}$$

VI.5. 1. *Relations between the angular velocities.* The derivation of these relations is simplest if one remembers that at the points at which the bevel gears (ω) are in mesh with gear (ω_1) on the one hand and gear (ω_2) on the other, the peripheral velocities must, at any instant, be equal. Gears (ω) rotate about axle A with angular velocity ω ; in addition, this axle rotates together with (ω) about the common geometric axis of (Ω), (ω_1) and (ω_2) with angular velocity Ω . If r , r_1 and r_2 are the mean radii of bevel gears (ω), (ω_1), (ω_2), we must have at point of contact (ω , ω_1)

$$r\omega + r_1\Omega = r_1\omega_1$$

and at point of contact (ω , ω_2)

$$-r\omega + r_2\Omega = r_2\omega_2.$$

With $r_1=r_2$ we obtain from this the relations

$$(1) \quad \begin{aligned} 2\Omega &= \omega_1 + \omega_2 \\ 2\omega &= \frac{r_1}{r}(\omega_1 - \omega_2). \end{aligned}$$

Of course these relations can also be derived by introducing virtual rotations.

2. *Relations between the torques.* The virtual work of L must always equal the sum of the virtual work of L_1 and L_2 , i.e.,

$$L\Omega\delta t = L_1\omega_1\delta t + L_2\omega_2\delta t.$$

We now replace Ω in terms of ω_1 and ω_2 by means of (1) to arrive at

$$\left(\frac{L}{2} - L_1\right)\omega_1 + \left(\frac{L}{2} - L_2\right)\omega_2 = 0.$$

This is possible for arbitrary ω_1 , ω_2 only if

$$(2) \quad \frac{1}{2}L = L_1 = L_2.$$

It is seen that the driving torque of the engine is transferred in equal amounts to each rear wheel at all times, no matter what the values of angular velocities ω_1 and ω_2 .

3. *Equation of motion of the system.* Here it will be found simplest to use Lagrange's equations of the second kind. We have

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I\omega^2 + I'\Omega^2).$$

We replace ω and Ω by their expressions in terms of ω_1 and ω_2 and introduce abbreviations

$$\begin{aligned} L_{11} &= I_1 + \frac{I'}{4} + \frac{I r_1^2}{4r^2}, \\ L_{22} &= I_2 + \frac{I'}{4} + \frac{I r_2^2}{4r^2}, \\ L_{12} &= L_{21} = \frac{I'}{4} - \frac{I r_1^2}{4r^2}. \end{aligned}$$

Lagrangian's equations then become

$$(3) \quad \begin{aligned} \frac{d}{dt}(L_{11}\omega_1 + L_{12}\omega_2) &= \frac{L}{2} - W_1, \\ \frac{d}{dt}(L_{21}\omega_1 + L_{22}\omega_2) &= \frac{L}{2} - W_2. \end{aligned}$$

W_1 and W_2 are two resisting torques acting at the two rear wheels; they have their origin in the static friction at the ground and may, if one wishes, include the other resistances (air, etc.).

If L , W_1 and W_2 are given as functions of the time, one can calculate the parentheses in the left members of (3) as time integrals of the right members, so that ω_1 and ω_2 become known functions of the time.

Averaged over the time, the right members of (3) are equal to zero, so that ω_1 and ω_2 are constant. If, however, the resistance acting on one wheel is decreased, which happens, for instance, if the wheel jumps off a bump in the road and momentarily turns in the air ($W=0$), this wheel is accelerated, whereas the other is decelerated.

4. *Analogy to electrodynamics.* Eqs. (3) are so written that they remind one of the interaction of two inductively coupled currents (see the remarks on p. 225 concerning Boltzmann). If we identify the L_{ij} with the coefficients of induction of the two circuits, ω_1 and ω_2 with the currents flowing in them, the left members of (3) are the electrodynamic induction effects. $\frac{1}{2}L$ corresponds to the "impressed EMF" acting in the circuits, and

$$T = \frac{1}{2} L_{11} \omega_1^2 + L_{12} \omega_1 \omega_2 + \frac{1}{2} L_{22} \omega_2^2$$

is the total magnetic field energy. According to p. 197, one calls cyclic systems those whose Lagrangian contains only the derivatives of the coordinates with respect to time (here $\omega_1 = \dot{\phi}_1$, $\omega_2 = \dot{\phi}_2$). They therefore constitute the mechanical analogue of stationary electric currents. Both the differential mechanism and the symmetrical top are doubly cyclic systems.

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MECHANICS OF DEFORMABLE BODIES

Mechanics of Deformable Bodies

Lectures on Theoretical Physics, Vol. II

BY ARNOLD SOMMERFELD

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Translated from the second German edition

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PREFACE*

It is due in part to the encouragement of former students of mine that I have decided to publish my general course of lectures on theoretical physics which I gave regularly during thirty-two years at the University of Munich.

They were of an introductory nature, attended not only by the physics majors of the University and the Polytechnic Institute, but also by the many candidates for degrees in the teaching of mathematics and physics—usually in their fourth to eighth semesters—as well as by astronomers and some physical chemists. Classes were held four times a week and supplemented by a two-hour problem period. Special courses in modern physics which were given concurrently have not been included in this series of books; their subject matter has found its way into my papers and treatises. While it is true that quantum mechanics always hovers in the background, reference being made to it now and then, the actual substance of these lectures is classical physics.

The order of subjects, which is retained in their publication, was

1. Mechanics
2. Mechanics of Deformable Media
3. Electrodynamics
4. Optics
5. Thermodynamics and Statistical Mechanics
6. Partial Differential Equations in Physics

The lectures on mechanics were given in alternate semesters by myself and by my colleagues in mathematics. Concurrent courses in hydrodynamics, electrodynamics and thermodynamics were taught by younger members of the faculty. Vector analysis was given in a separate course so that its systematic development could be omitted from my lectures.

Here, as in my classes, I shall not detain myself with the mathematical

* From the forewords to Vol. I and Vol. II, first and second edition.

foundations, but proceed as rapidly as possible to the physical problems themselves. My aim is to give the reader a vivid picture of the vast and varied material that comes within the scope of theory when a reasonably elevated vantage-point is chosen. With this purpose in mind I shall not be too concerned if I have left some gaps in the systematic justification and axiomatic structure of the work. At any rate, I have avoided frightening the student with drawn-out investigations of a mathematical or logical nature and distracting his attention from that which is physically interesting. I think that this attitude has proved its worth in my lectures; it has therefore been retained in the printed text. And when Planck's lectures are impeccable in their systematic organization, then I may perhaps claim for my own lectures a greater variety of subject matter and a more flexible handling of the mathematical apparatus. However, I gladly refer the reader to the more complete and often more thorough treatment of Planck, especially for thermodynamics and statistical mechanics.

The problems collected at the end of each volume should be considered as supplements to the text. They were presented by the students during the problem periods after they had worked them out and handed them in in writing. Elementary numerical calculations, such as are found in great number in textbooks and collections of exercises, have, in general, not been included. The problems are numbered by chapter. Sections are numbered through in each volume, and equations in each section. Within each volume references to earlier equations can thus be made by merely giving the numbers of section and equation.

In this second volume a fairly complete development of certain mathematical methods had to be given. These methods are often taken up in a separate introductory course on theoretical physics; their incorporation in Vol. II accounts for its larger size. But the actual subject of this volume is the mechanics of systems with an infinite number of degrees of freedom. The place of ordinary differential equations (governing the mechanics of systems with a finite number of degrees of freedom) is here taken by partial differential equations, the place of vector algebra by vector analysis, which is briefly summarized in Chap. 1. Besides, it was necessary to develop the fundamentals of tensor analysis, being an indispensable tool in the theory of elastic solids and viscous fluids. This has been done for Cartesian coordinates and to some extent also for orthogonal curvilinear coordinates.

Some points may be mentioned here in which this presentation seems to be more complete than the one generally found in textbooks on the same level: In Chap. I.2 it is proved that the curl is an axial vector (or antisymmetric tensor). Following O. Reynolds we consider in II.10 *two* laws of similitude and *two* corresponding invariants, viz. a dimensionless

number S characterizing the pressure dependence in addition to the usual Reynolds number. In III.15 we discuss the quasi-elastic body (gyroscopic ether); its logical place is among the continuous media that are compatible with the fundamental theorem of kinematics in I.1. Rather than explain Maxwell's equations by a mechanical model, we want to show in this discussion the fundamental difference between electrodynamics and mechanics. In Chap. V, Sections 27 and 28 deal with the somewhat involved problems of circular waves and ship waves. Complete calculations are supplied using the method of stationary phase which is a simplified version of the method of steepest descent. In the plate and jet problems of Chap. VI the dimensions of plate and of orifice etc. are carried as parameters throughout the entire calculation. This form of analysis has perhaps a stronger physical appeal than the usual one which employs dimensionless quantities. Kármán's vortex street (32) is extended according to Maue to include the unsymmetrical case in which the flow is not parallel to the street. The hydrodynamic theory of journal bearings is briefly dealt with in VII.36. Riemann's theory of shock waves is discussed in 37 with a particular view to the results that Bechert obtained in certain elementary integrable cases. Sec. 38 is a report on the history and present situation of the difficult turbulence problem and includes also Burger's mathematical model of turbulence. In VIII.43 one finds, in the problem of the helical spring, an example of combined bending and torsion. In 44 the boundary conditions for an oscillating parallelepiped are discussed, and the foundations laid for the quantum-theoretical thermodynamics of the solid body.

It is obvious that not all the topics found in this volume could have been touched upon in the brief period of one term; several of the subjects mentioned before have in fact been added for the print.

The *second edition* of Vol. II has been supplemented by a representation of general tensor calculus limited to three dimensions and orthogonal line elements. Tensor calculus does not hold an advantage over the simpler vector analytic formulas for the cases considered here (cf. App. IV), but, because of its importance in the general theory of relativity, it cannot be entirely omitted if a fairly complete exposition of the mathematical methods of theoretical physics is the goal.

The discussion of the turbulence problem, which already presented great difficulties in the first edition, had to be revised on the basis of so-far unpublished work by C. F. von Weizsäcker and W. Heisenberg. My opinion of long standing that turbulence would finally be accounted for by integration of the Navier-Stokes equations in their complete, non-linear, form, proved wrong in the special case of "isotropic turbulence" investigated by the two authors; here, as in the kinetic theory of gases, statistical methods have shown their superiority. It was, of course, impossible to

review the new results in detail, but the previous representation had to be corrected in many places in accordance with the new point of view.---

Munich, July 1946

Arnold Sommerfeld

In this translation of the second volume of Professor Sommerfeld's "Vorlesungen über theoretische Physik", the text of the second edition (1947, D. V. B., Wiesbaden) has been followed. Disregarding a few non-technical remarks that hold interest only for German readers, nothing has been omitted, and no changes have been made apart from necessary adaptations of the notation. Several brief footnotes have been inserted, however, to bridge over certain differences, mostly of a terminological nature. Topical references to the first volume have been supplemented by corresponding references to well-known American and British texts with the aim of making this volume independent from the preceding one, and a new index has been prepared.

G.K.

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CHAPTER I

KINEMATICS OF DEFORMABLE BODIES

1. A Fundamental Theorem of Kinematics

Helmholtz establishes the following theorem at the beginning of his paper on vortex motions:¹ The most general motion of a sufficiently small element of a deformable (i.e., not rigid) body can be represented as the sum of

1. a translation,
2. a rotation,
3. an extension (contraction) in three mutually orthogonal directions.

The proof is based on the Taylor expansion of the relative displacement of two neighboring points P and O in terms of their original coordinate differences.

Let P be a point of the volume element under consideration and x, y, z its coordinates in a rectangular system the origin of which, O , lies within the element. In a general motion of the body the points P and O will both experience changes of position, which we denote by ξ, η, ζ and ξ_0, η_0, ζ_0 respectively, referring to the chosen space-fixed coordinate system. Taylor's formula then gives for the displacement of P

$$\begin{aligned} \xi &= \xi_0 + \frac{\partial \xi}{\partial x} x + \frac{\partial \xi}{\partial y} y + \frac{\partial \xi}{\partial z} z + \dots \\ \eta &= \eta_0 + \frac{\partial \eta}{\partial x} x + \frac{\partial \eta}{\partial y} y + \frac{\partial \eta}{\partial z} z + \dots \\ \zeta &= \zeta_0 + \frac{\partial \zeta}{\partial x} x + \frac{\partial \zeta}{\partial y} y + \frac{\partial \zeta}{\partial z} z + \dots \end{aligned}$$

Introducing for brevity

$$(2) \quad \frac{\partial \xi}{\partial x} = a_{11}, \quad \frac{\partial \xi}{\partial y} = a_{12}, \quad \frac{\partial \eta}{\partial x} = a_{21}, \quad \frac{\partial \eta}{\partial y} = a_{22}, \quad \dots,$$

¹Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. Crelles J. 55, 25 (1858).

we write each of the quantities a_{ik} as a sum of an "odd" and an "even" term

$$a_{ik} = \frac{a_{ik} - a_{ki}}{2} + \frac{a_{ik} + a_{ki}}{2};$$

"odd" and "even" refer, of course, to the commutation of the subscripts.

Now, (1) may be rewritten in the following form

$$\begin{aligned} \xi &= \xi_0 + \left| 0 + \frac{a_{12} - a_{21}}{2} y + \frac{a_{13} - a_{31}}{2} z \right| \\ &\quad + \left\| a_{11}x + \frac{a_{12} + a_{21}}{2} y + \frac{a_{13} + a_{31}}{2} z \right\| \\ \eta &= \eta_0 + \left| \frac{a_{21} - a_{12}}{2} x + 0 + \frac{a_{23} - a_{32}}{2} z \right| \\ (3) \quad &\quad + \left\| \frac{a_{21} + a_{12}}{2} x + a_{22}y + \frac{a_{23} + a_{32}}{2} z \right\| \\ \zeta &= \zeta_0 + \left| \frac{a_{31} - a_{13}}{2} x + \frac{a_{32} - a_{23}}{2} y + 0 \right| \\ &\quad + \left\| \frac{a_{31} + a_{13}}{2} x + \frac{a_{32} + a_{23}}{2} y + a_{33}z \right\|, \end{aligned}$$

where terms higher than first order in x, y, z have been omitted. Let us introduce the symbol \mathbf{s} for the total change of position of P ; the vertical separation lines in (3) indicate that the displacement \mathbf{s} is compounded of three partial motions $\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2$, or

$$(4) \quad \mathbf{s} = \mathbf{s}_0 + \mathbf{s}_1 + \mathbf{s}_2.$$

The displacement \mathbf{s}_0 , with the components ξ_0, η_0, ζ_0 , is the same for all points P of a volume element and is therefore a *translation*.

The central portion of the set (3), \mathbf{s}_1 , is a *rotation*. Upon introducing the vector $\mathbf{\hat{\phi}}$, with the components

$$(5) \quad \varphi_x = \frac{a_{32} - a_{23}}{2}, \quad \varphi_y = \frac{a_{13} - a_{31}}{2}, \quad \varphi_z = \frac{a_{21} - a_{12}}{2},$$

and the position vector $OP = \mathbf{r}$, we obtain simply

$$(6) \quad \mathbf{s}_1 = \mathbf{\hat{\phi}} \times \mathbf{r}.$$

This displacement is well known from rigid body kinematics (Vol. I², Eq. 22.3) and corresponds to the infinitesimal rotation ϕ (the appropriate notation would be $\delta\phi$) whose axis and magnitude are given by the components ϕ_x, ϕ_y, ϕ_z .³

The infinitesimal rotation is not a vector in the proper sense, such as a polar vector that characterizes a translatory displacement. One may, nevertheless, denote it by the usual vectorial symbolism, associating with it a polar vector ϕ that points in the direction of the axis of rotation and has a length equal to the (infinitesimal) angle of rotation. More about axial vectors is found in 2.

There is, of course, *no change in the length of the vector* OP due to the displacement s_1 . This is self-evident if we consider s_1 from the point of view of rigid body kinematics, but will be shown here independently. The square of the distance \overline{OP} is

$$\overline{OP}^2 = |r + s_1|^2 = |r|^2 + 2r \cdot s_1 + |s_1|^2,$$

but according to (6)

$$r \cdot s_1 = r \cdot (\phi \times r) = \phi \cdot (r \times r) = 0.$$

Disregarding second order terms as in (3), we find

$$|r + s_1|^2 = |r|^2 = r^2$$

(It will be noticed that here and in all subsequent arguments of a similar nature, "no change" means "no change in first approximation.")

In order to pass from infinitesimal to finite quantities the displacements must be considered in their time dependence, which amounts to introducing velocities instead of displacements, viz.

$$(7) \quad s = v\Delta t, \quad \phi = \omega\Delta t,$$

or in components

$$(8) \quad \begin{aligned} \xi &= u\Delta t, & \eta &= v\Delta t, & \zeta &= w\Delta t, \\ \varphi_x &= \omega_x\Delta t, & \varphi_y &= \omega_y\Delta t, & \varphi_z &= \omega_z\Delta t. \end{aligned}$$

Here v is the velocity of the particle or volume element considered, and ω its vortex vector; the physical importance of the latter quantity was first recognized by Helmholtz.

²A. Sommerfeld, *Vorlesungen über Theoretische Physik*, Akademische Verlagsgesellschaft, Leipzig, 1944, Bd. I, quoted henceforth as Vol. I. English translation in press (Academic Press Inc., New York).

³See, e.g., J. L. Synge and B. A. Griffith, *Principles of Mechanics*, McGraw-Hill, New York, 1942 Sec. 10.5.

The components of the vortex vector in the system x, y, z are according to (8), (5), and (2)

$$(9) \quad \omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right), \quad \omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right), \quad \omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right).$$

A more complete analysis of this fundamental definition belongs also to the review of vector analysis in the following article.

We turn now to the third partial motion s_2 indicated in the right hand part of our scheme (3). Here the deformability becomes significant, the two other partial motions having been recognized as the elements of the general motion of a rigid body.

The displacement s_2 is a *linear vector function* of the position vector \mathbf{r} , in the sense of Vol. I, 22.⁴ We denote the components of s_2 by ξ_2, η_2, ζ_2 , and introduce the notation

$$(10) \quad \begin{aligned} \xi_2 &= \epsilon_{xx}x + \epsilon_{xy}y + \epsilon_{xz}z, \\ \eta_2 &= \epsilon_{yx}x + \epsilon_{yy}y + \epsilon_{yz}z, \\ \zeta_2 &= \epsilon_{zx}x + \epsilon_{zy}y + \epsilon_{zz}z. \end{aligned}$$

The meaning of the coefficients ϵ_{ik} follows from (2) and (3):

$$(11) \quad \begin{aligned} \epsilon_{xx} &= \frac{\partial \xi}{\partial x}, & \epsilon_{xy} &= \epsilon_{yx} = \frac{1}{2} \left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x} \right), \\ \epsilon_{yy} &= \frac{\partial \eta}{\partial y}, & \epsilon_{yz} &= \epsilon_{zy} = \frac{1}{2} \left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y} \right), \\ \epsilon_{zz} &= \frac{\partial \zeta}{\partial z}, & \epsilon_{zx} &= \epsilon_{xz} = \frac{1}{2} \left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z} \right). \end{aligned}$$

The quantities ϵ are the components of the *strain tensor*. The tensor itself may be symbolized in a similar way as the moment of inertia in Vol. I, 22.13b⁵ by the quadratic array

$$(12) \quad \epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}$$

⁴See, e.g., L. Brand, *Vector and Tensor Analysis*, Wiley, New York, 1947, §§60 and 62.

⁵See, e.g., Synge and Griffith, *op. cit.*, Sec. 11.3.

In the present case the tensor is *symmetric* (i.e., the components are symmetric to the main diagonal that connects the upper left and lower right corners), a fact which is an immediate consequence of the definition of the third partial motion s_3 in (3). On the other hand, the scheme of coefficients of the partial motion s_1 represents an *antisymmetric tensor*. Its definition, given by Eq. (6) or Eq. (3) (the part between the vertical lines), yields the array

$$(12a) \quad \mathfrak{L} = \begin{pmatrix} 0 & \varphi_{xy} & \varphi_{xz} \\ \varphi_{yx} & 0 & \varphi_{yz} \\ \varphi_{zx} & \varphi_{zy} & 0 \end{pmatrix}, \quad \varphi_{ih} = -\varphi_{hi},$$

and our change of notation $\varphi_x = \varphi_{xx} = -\varphi_{xx}$ etc. establishes accordance with the double subscript notation introduced in (10). Note here that the antisymmetric tensor can always be represented by a vector, which is, of course, not true for a symmetric tensor, as will be proved in 2.

In the analysis of the strain tensor we make use of the concept of the *tensor quadric* as in Vol. I, 22.15 in the analysis of the moment of inertia. Consider for this purpose the scalar product

$$(13) \quad s_2 \cdot r = \epsilon_{xx}x^2 + 2\epsilon_{xy}xy + \epsilon_{yy}y^2 + \dots = f(x, y, z).$$

By putting $f(x, y, z) = \text{const}$, we obtain a surface of second order, the *strain* (or *deformation*) *quadric*, also called the *ellipsoid of strain* or of *dilatation*; this terminology, of course, does not exclude contraction, nor does it imply that the quadric is necessarily an ellipsoid as in the case of the inertia tensor. It may be any one of the surfaces of second order, e.g., a hyperboloid of one or two sheets or a degenerate surface such as a pair of planes.

In order to decide this, we refer the strain quadric to its principal axes. Upon introducing the corresponding rectangular coordinates X_1, X_2, X_3 , Eq. (13) takes the form

$$(14) \quad F(X_1, X_2, X_3) = \epsilon_1 X_1^2 + \epsilon_2 X_2^2 + \epsilon_3 X_3^2 = \text{const.}$$

The coefficients $\epsilon_1, \epsilon_2, \epsilon_3$ are called the *principal extensions* (or *contractions*, if negative).

If (13) is used, the linear vector function (10) may be rewritten in the form

$$(15) \quad \xi_1 = \frac{1}{2} \frac{\partial f}{\partial x}, \quad \eta_2 = \frac{1}{2} \frac{\partial f}{\partial y}, \quad \zeta_3 = \frac{1}{2} \frac{\partial f}{\partial z},$$

and in terms of the principal coordinates we have the correspondingly simplified linear vector function

$$(16) \quad \begin{aligned} \Xi_1 &= \frac{1}{2} \frac{\partial F}{\partial X_1} = \epsilon_1 X_1, & \Xi_2 &= \frac{1}{2} \frac{\partial F}{\partial X_2} = \epsilon_2 X_2, \\ \Xi_3 &= \frac{1}{2} \frac{\partial F}{\partial X_3} = \epsilon_3 X_3. \end{aligned}$$

The principal components of displacement Ξ_i introduced here are the projections of the vector \mathbf{s}_2 upon the principal axes X_i .

The proof of Helmholtz's theorem has been completed by the establishment of relations (16), which indicate that the third partial motion is composed of three extensions (contractions) in the mutually orthogonal directions of the principal axes of the deformation quadric. In fact, any point P of our volume element with coordinates X_i in the principal system is carried over, according to (16), into a point P' with coordinates

$$(17) \quad X_i + \Xi_i = X_i(1 + \epsilon_i).$$

The tensor quadric has been introduced here by formal definition, but a direct physical interpretation is possible when coordinate differentials or the equivalent direction cosines are introduced in place of the coordinates x, y, z [see Eq. (4.21)].

We now proceed from the linear extensions ϵ_i to the *cubical* extension or *dilatation* Θ which is defined as the specific change of volume (i.e. the volume change per unit of original volume)

$$(18) \quad \Theta = \frac{\Delta V' - \Delta V}{\Delta V}.$$

Here ΔV is the original and $\Delta V'$ the expanded (or compressed) volume of the element. The calculation is easy for a volume element that has the shape of a rectangular cell with sides parallel to the principal axes. Let one corner coincide with the point 0 and denote the sides before and after dilatation by a_i and a'_i . Then $\Delta V = a_1 a_2 a_3$, $\Delta V' = a'_1 a'_2 a'_3$. According to (17), $a'_i = a_i(1 + \epsilon_i)$, therefore $\Delta V' = \Delta V(1 + \epsilon_1)(1 + \epsilon_2)(1 + \epsilon_3)$ and, according to (18),

$$(19) \quad \Theta = (1 + \epsilon_1)(1 + \epsilon_2)(1 + \epsilon_3) - 1 = \epsilon_1 + \epsilon_2 + \epsilon_3.$$

(The product terms have again been omitted.)

The representation of Θ by (19) is valid in any Cartesian system: take for example the original system x, y, z . We assert that there is always

$$(20) \quad \Theta = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}.$$

For the proof we have to use certain properties of the transformation by which the surface $f = \text{const}$ of Eq. (13) is transformed into $F = \text{const}$ of Eq. (14). Writing the transformation in the schematic form

$$(21) \quad \begin{array}{c|ccc} & x & y & z \\ \hline X_1 & \alpha_1 & \beta_1 & \gamma_1 \\ X_2 & \alpha_2 & \beta_2 & \gamma_2 \\ X_3 & \alpha_3 & \beta_3 & \gamma_3 \end{array} \quad \begin{array}{l} \sum \alpha_i^2 = 1, \quad \sum \alpha_i \beta_i = 0, \dots \\ (21a) \quad \alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1, \dots \\ \alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2 = 0, \dots \end{array}$$

and reading the scheme (21) from left to right (i.e., solving for the X_i), we obtain by substituting in (14)

$$\begin{aligned} F &= \sum \epsilon_i X_i^2 = \sum \epsilon_i (\alpha_i x + \beta_i y + \gamma_i z)^2 \\ &= x^2 \sum \epsilon_i \alpha_i^2 + 2xy \sum \epsilon_i \alpha_i \beta_i + y^2 \sum \epsilon_i \beta_i^2 + \dots \end{aligned}$$

This expression must be identical with (13), therefore

$$(22) \quad \begin{aligned} \epsilon_{xx} &= \alpha_1^2 \epsilon_1 + \alpha_2^2 \epsilon_2 + \alpha_3^2 \epsilon_3, \\ \epsilon_{yy} &= \beta_1^2 \epsilon_1 + \beta_2^2 \epsilon_2 + \beta_3^2 \epsilon_3, \\ \epsilon_{zz} &= \gamma_1^2 \epsilon_1 + \gamma_2^2 \epsilon_2 + \gamma_3^2 \epsilon_3. \\ \epsilon_{xy} &= \alpha_1 \beta_1 \epsilon_1 + \alpha_2 \beta_2 \epsilon_2 + \alpha_3 \beta_3 \epsilon_3, \\ (22a) \quad \epsilon_{yz} &= \beta_1 \gamma_1 \epsilon_1 + \beta_2 \gamma_2 \epsilon_2 + \beta_3 \gamma_3 \epsilon_3, \\ \epsilon_{xz} &= \gamma_1 \alpha_1 \epsilon_1 + \gamma_2 \alpha_2 \epsilon_2 + \gamma_3 \alpha_3 \epsilon_3. \end{aligned}$$

On adding Eqs. (22) and taking into account the orthogonality relations (21a), we finally obtain

$$(23) \quad \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \epsilon_1 + \epsilon_2 + \epsilon_3.$$

The result is: The sum of the elements in the principal diagonal of the strain tensor is independent of the choice of the system of coordinates; it is an *invariant of the tensor*. In 4 this result will be supplemented by a search for all invariants of a tensor. Also the relations (22a) not fully exploited so far will reappear there.

By (19), (20), and (11) the dilatation Θ can be put in the simple form

$$(24) \quad \Theta = \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}.$$

Having established the geometrical significance of the *principal* extensions ϵ_i , our next task is the geometrical interpretation of the *general* strain components ϵ_{xx} and ϵ_{xy} . For the diagonal components ϵ_{xx} , \dots the case is not different from that of the quantities ϵ_1 , \dots : ϵ_{xx} is the increment in length of an x -fiber per unit of original length. If such a fiber is cut from the body between the points $x = 0$ and $x = \Delta l$, its length after straining $\Delta l' = \Delta l + \xi - \xi_0$, whence the specific change of length is

$$\frac{\Delta l' - \Delta l}{\Delta l} = \frac{\xi - \xi_0}{\Delta l}.$$

But the last expression is seen to be equal to

$$\frac{\partial \xi}{\partial x} = \epsilon_{xx},$$

when we substitute the special values $x = \Delta l$, $y = z = 0$ in Eq. (1).

For the purpose of interpreting the components ϵ_{xy} , we consider an " x,y -lamina" instead of an " x -fiber". Let 0123 denote an infinitesimal rectangular part of the lamina with sides a and b (Fig. 1). In the process

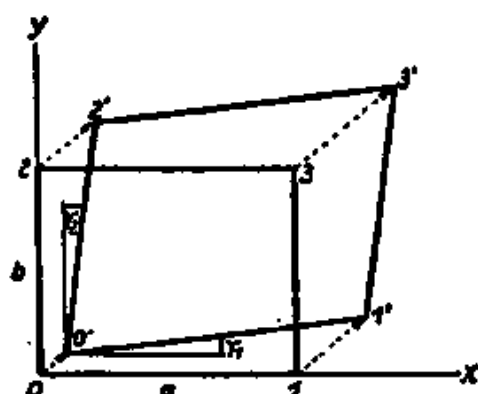


FIG. 1. Geometric interpretation of the strain component ϵ_{xy} as angular change in an originally orthogonal volume element. The diagram gives the projection in the x,y -plane.

of straining, the points 0, 1, 2 are displaced to $0'$, $1'$, $2'$, and the rectangle is deformed into a parallelogram. (Note that displacements in z -direction are disregarded; think of the strained figure as projected on the x,y -plane: the projection $3'$ of the strained point 3 will then coincide with the fourth corner of the parallelogram subtended by $0'1'$ and $0'2'$, if higher order terms are omitted. Note the analogous omission also in the case of the ϵ_{xx} .) The angles γ_1 and γ_2 indicated in Fig. 1 can now be calculated, again omitting higher order terms:

$$\begin{aligned} \gamma_1 &= \operatorname{tg} \gamma_1 = \frac{\eta_1 - \eta_0}{a + \xi_1 - \xi_0} = \frac{\eta_1 - \eta_0}{a} = \frac{\partial \eta}{\partial x} \\ (25) \quad \gamma_2 &= \operatorname{tg} \gamma_2 = \frac{\xi_2 - \xi_0}{b + \eta_2 - \eta_0} = \frac{\xi_2 - \xi_0}{b} = \frac{\partial \xi}{\partial y} \\ \gamma_1 + \gamma_2 &= \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} = 2\epsilon_{xy}. \end{aligned}$$

In this equation, subscripts on ξ , η refer to the displaced points. The angle $\gamma = \gamma_1 + \gamma_2$ is equal to the change of the originally right angle at O due to the distortion of the rectangle $O123$; ϵ_{xy} , which is half of that angle, is the *shearing strain* or, simply the *shear*. The quantities ϵ_y , and ϵ_{xz} have the same significance relative to the y,z - and z,x -laminae.

A short compilation of the more common notations of the strain tensor follows.

This book	ϵ_{xx}	ϵ_{yy}	ϵ_{zz}	$\epsilon_{xy} = \epsilon_{yx}$	$\epsilon_{yz} = \epsilon_{zy}$	$\epsilon_{zx} = \epsilon_{xz}$
Love and some American authors . .	e_{xx}	e_{yy}	e_{zz}	$e_{xy} = e_{yx}$	$e_{yz} = e_{zy}$	$e_{zx} = e_{xz}$
Kirchhoff and Planck	x_x	y_y	z_z	$\frac{1}{2} x_y = \frac{1}{2} y_x$	$\frac{1}{2} y_z = \frac{1}{2} z_y$	$\frac{1}{2} z_x = \frac{1}{2} x_z$
Some English authors	e	f	g	$\frac{1}{2} a$	$\frac{1}{2} b$	$\frac{1}{2} c$
Engineering usage . .	ϵ_x	ϵ_y	ϵ_z	$\frac{1}{2} \gamma_{xy} = \frac{1}{2} \gamma_{yx}$	$\frac{1}{2} \gamma_{yz} = \frac{1}{2} \gamma_{zy}$	$\frac{1}{2} \gamma_{zx} = \frac{1}{2} \gamma_{xz}$

The inclusion of the factor $\frac{1}{2}$ in our definition of the ϵ_{xy} , . . . is suggested by the system of notation of general tensor analysis, but we shall see in 40 that Kirchhoff's choice of the x_x , . . . which sets this factor in evidence has a certain advantage when simple expressions for energetic quantities are desired.

2. Review of Vector Analysis

Throughout this volume we shall make continual use of *vector analysis*, that is, calculus applied to vector quantities. Thus, while familiarity with *vector algebra* and with the basic concepts of *vector analysis* is assumed on the part of the reader, a clarification of some of the fundamental points seems appropriate.

First, we repeat here the definition of vector and scalar given in Vol. I, following 22.6. A quantity is a vector if, in an orthogonal transformation of the coordinate system, it follows the same transformation rules as the position vector $\mathbf{r} = x, y, z$; it is a scalar if it is invariant in such transformations.⁶

Let us put this definition to use in the proof that the quantity Θ , derived from the vector $\mathbf{s} = x, y, z$ in (1.24), is a scalar. We replace \mathbf{s} by

⁶For this definition compare H. and B. S. Jeffreys, *Methods of Mathematical Physics*, Cambridge University Press, Cambridge, 1946, Chap. 2.023.

an arbitrary vector $\mathbf{A} = A_x, A_y, A_z$,⁷ and consider instead of Θ more generally the quantity

$$(1) \quad \operatorname{div} \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z},$$

called the divergence of the vector field \mathbf{A} .

The invariance of $\operatorname{div} \mathbf{A}$, which we now prove, amounts to the following equality:

$$\operatorname{div}' \mathbf{A} = \frac{\partial A_{x'}}{\partial x'} + \frac{\partial A_{y'}}{\partial y'} + \frac{\partial A_{z'}}{\partial z'} = \operatorname{div} \mathbf{A}$$

For the sake of brevity, the coordinates, primed or unprimed, will be numbered rather than lettered. Transformation (1.21) reads in the simplified notation

$$(2) \quad \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x'_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ x'_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ x'_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{array} \quad (2a) \quad \begin{aligned} \sum_k \alpha_{ik} \alpha_{jk} &= \delta_{ij}, \\ \sum_i \alpha_{ik} \alpha_{il} &= \delta_{kl}. \end{aligned}$$

The meaning of the symbol δ_{ij} (and correspondingly δ_{kl}) is the usual one:

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}.$$

For an arbitrary function $U(x_1, x_2, x_3)$, which may at the same time be considered as a function of x'_1, x'_2, x'_3 because of (2), one has

$$(3) \quad \frac{\partial U}{\partial x'_i} = \sum_k \frac{\partial U}{\partial x_k} \frac{\partial x_k}{\partial x'_i}.$$

This can be rewritten as

$$(3a) \quad \frac{\partial U}{\partial x'_i} = \sum_k \alpha_{ik} \frac{\partial U}{\partial x_k}$$

if the scheme of coefficients (2) is read vertically downward. According to our vector definition we may apply the transformation rule for \mathbf{r} , that is

⁷In the original German text, vectors and their components are printed in the same Gothic types. The more usual way of making a typographical distinction between the two has been adopted for this translation.

the scheme (2), to the vector **A**. Thus, on reading (2) from left to right, it follows

$$(4) \quad A'_i = \sum_j \alpha_{ji} A_j .$$

We now identify the arbitrary function U in (3a) with the vector component A_i . Doing this for $i = 1, 2, 3$ and adding leads us to

$$\sum_i \frac{\partial A'_i}{\partial x'_i} = \sum_i \sum_k \sum_l \alpha_{ik} \alpha_{li} \frac{\partial A_l}{\partial x_k} = \sum_k \sum_l \delta_{kl} \frac{\partial A_l}{\partial x_k} .$$

Here the first member is $\text{div}' \mathbf{A}$ and the last member $\text{div} \mathbf{A}$ if proper account is taken of the δ -symbol, hence the invariance of the divergence has been proved. The cubical dilatation (1.24) can now be written as $\Theta = \text{div} \mathbf{s}$.

The divergence operator derives a scalar from a vector. The gradient operator, on the other hand, is the simplest differential operator that derives a vector from a scalar. Denoting the scalar by U (a scalar point function as before), we construct the vector

$$(5) \quad \text{grad } U = \frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \frac{\partial U}{\partial x_3} .$$

Its vector character follows at once from (3a) which can be written by means of the symbol "grad"

$$(5a) \quad \text{grad}'_i U = \sum_k \alpha_{ik} \text{grad}_k U .$$

Thus the gradient transforms in the same way as the position vector.

We now turn to the differential operation introduced in (1.9) for which the symbol "curl" is used. Writing (1.9) with this symbol, we have

$$(6) \quad \mathbf{B} = \text{curl } \mathbf{A}, \quad \text{curl}_x \mathbf{A} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad \text{etc.},$$

where \mathbf{B} stands for $2\omega^s$ and \mathbf{A} for $\mathbf{v} = u, v, w$. Eq. (1.6) already contains a definition of the vector ω . On setting as in (1.7)

$$\mathbf{s}_1 = \mathbf{v}_1 \Delta t, \quad \phi = \omega \Delta t,$$

(1.6) is carried over into

$$(7) \quad \mathbf{v}_1 = \omega \times \mathbf{r} .$$

^sWe shall later come back to this factor $\frac{1}{2}$ that is now included in the definition of \mathbf{B} .

The developments in 1 make it evident that the definition of ω by (6) [or (1.9)] agrees with that by (7) [or (1.6)]; this will be checked by direct calculation in problem I.1.

Let us now write (7) with the same lettering as (6); we can again use \mathbf{B} instead of 2ω , but we may use \mathbf{A} also for \mathbf{v}_1 , since in what follows we are only concerned with the vector character of this quantity and $\text{curl } \mathbf{v}_1$ is identical with $\text{curl } \mathbf{v}$. In this way (7) becomes

$$(7a) \quad 2\mathbf{A} = \mathbf{B} \times \mathbf{r}.$$

Our aim is to show that the quantity \mathbf{B} has, at least in a certain sense, vector character. Such a proof, which furnishes a better understanding of the concept "curl", is by no means superfluous although it is omitted in most textbooks.

The formal proof can be based directly on the definition (6), in which case it amounts to a rather cumbersome calculation (problem I, 2), but a clarification of the issue can also be obtained if one starts from Eq. (7a).⁹ There the components of \mathbf{B} appear as coefficients of a linear *vector function*, like the quantities ϵ_{ik} in (1.10). Introducing double subscripts and numbering the vector components (instead of lettering), we write

$$(8) \quad \begin{aligned} B_x &= B_1 = B_{32} = -B_{23}, & A_x &= A_1, \\ B_y &= B_2 = B_{13} = -B_{31}, & A_y &= A_2, \\ B_z &= B_3 = B_{21} = -B_{12}, & A_z &= A_3. \end{aligned}$$

This notation for the components of \mathbf{B} corresponds to the representation of Φ by (1.12a); in both cases the diagonal terms (B_{ii} , φ_{ii}) vanish and the "vector" \mathbf{B} is really an *antisymmetric tensor* like φ .

Eq. (7a) now takes the simple form

$$(9) \quad 2A_i = \sum_k B_{ik}x_k$$

which we can check, say, for $i = 1$; we find

$$2A_1 = 2A_x = B_{11}x_1 + B_{12}x_2 + B_{13}x_3 = -B_y y + B_z z = (\mathbf{B} \times \mathbf{r})_x$$

in agreement with (7a).

In order to define the quantities B for the rotated coordinate system x' , we write Eqs. (9) in this system

$$(10) \quad 2A'_i = \sum_k B'_{ik}x'_k$$

⁹Note, however, that Eq. (6) by no means implies Eq. (7a) in the case of a general vector field \mathbf{A} . Observe the careful formulation of problem I.1.

and reintroduce x and A by

$$x'_k = \sum_i \alpha_{ki} x_i, \quad A'_i = \sum_k \alpha_{ik} A_k$$

according to (2). This transforms (10) into

$$(11) \quad 2 \sum_k \alpha_{ik} A_k = \sum_k \sum_l B'_{ik} \alpha_{kl} x_l.$$

On substituting for the components A in the first member of this equation according to (9) which, obviously, can be written as $2A_k = \sum_l B_{kl} x_l$ by a mere relettering of the subscripts, Eq. (11) becomes

$$\sum_k \sum_l B_{kl} \alpha_{ik} x_l = \sum_k \sum_l B'_{ik} \alpha_{kl} x_l,$$

which is an identity in the three independent variables x_1, x_2, x_3 . By comparing the coefficients, the following three equations between the B and B' are obtained

$$(12) \quad \sum_k B_{kl} \alpha_{ik} = \sum_k B'_{ik} \alpha_{kl}, \quad l = 1, 2, 3,$$

from which the B' must be determined. This is done by multiplying Eqs. (12) with α_{jl} and summing over l . The second members of (12) add up to

$$(12a) \quad \sum_l \sum_k B'_{ik} \alpha_{kl} \alpha_{jl} = \sum_k B'_{ik} \sum_l \alpha_{kl} \alpha_{jl} = \sum_k B'_{ik} \delta_{kj} = B'_{ij}.$$

Thus the new components B' have been found, viz.

$$(13) \quad \sum_l \sum_k B_{kl} \alpha_{ik} \alpha_{jl} = B'_{ij}.$$

As it stands, each B' is a linear function of all nine components B , but, with $B_{ki} = -B_{ik}$ and $B_{kk} = 0$, the number of terms reduces to three. Written more completely, the expressions for B' are

$$(14) \quad \begin{aligned} B'_{ij} = & B_{32}(\alpha_{i3}\alpha_{j2} - \alpha_{i2}\alpha_{j3}) + B_{13}(\alpha_{i1}\alpha_{j3} - \alpha_{i3}\alpha_{j1}) \\ & + B_{21}(\alpha_{i2}\alpha_{j1} - \alpha_{i1}\alpha_{j2}). \end{aligned}$$

Let us write out one of them, say, for $i = 3, j = 2$:

$$(15) \quad \begin{aligned} B'_{32} = & B_{32}(\alpha_{33}\alpha_{22} - \alpha_{32}\alpha_{23}) + B_{13}(\alpha_{31}\alpha_{23} - \alpha_{33}\alpha_{21}) \\ & + B_{21}(\alpha_{32}\alpha_{21} - \alpha_{31}\alpha_{22}). \end{aligned}$$

The coefficients of the quantities B in (15) are nothing else but the cofactors of the elements in the first row of the determinant of the transformation (2). Depending on whether this determinant is $+1$ or -1 , the cofactors are *equal* or *opposite* to the elements to which they belong¹⁰ (see, e.g., Vol. I, problem I.10.). Eq. (15) may therefore be rewritten as

$$\pm B'_1 = \alpha_{11}B_1 + \alpha_{12}B_2 + \alpha_{13}B_3$$

where the single subscript notation has been reintroduced for the B . In the same way B'_2 and B'_3 are obtained. The result can be written in the form

$$(16) \quad \pm B'_i = \sum_j \alpha_{ij}B_j,$$

which is exactly the vector transformation rule of Eq. (4) except for the ambiguity of the sign.

Two things can be learned from this result:

1. *The quantity $\mathbf{B} = \text{curl } \mathbf{A}$ behaves like an ordinary (polar) vector in a rotation, that is, in an orthogonal coordinate transformation with determinant $+1$, hence the general use of the simple terminology "curl vector" which we shall also adopt.*
2. *The curl is, strictly speaking, not a polar, but an axial vector for the following reason: In a transition from a dextral to a sinistral system or, more generally, in a rotation with a subsequent change of orientation (determinant -1) the new components of the curl are the negatives of the new components of the (polar) vector that represents the curl in the original system.*

We can now easily see why the use of double subscripts is an advantage in the foregoing proof: An axial vector is not appropriately represented by an arrow in the direction¹¹ of the axis; the correct symbol should be a circular arrow about the axis. In this way the plane of the circular arrow, say the y, z -plane, takes over the function of the x -axis, and a better way to write the left member of Eq. (6) would be

$$(17) \quad \text{curl}_{yz} \mathbf{A} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

The notation $\text{curl}_x \mathbf{A}$ represents the axial vector by a polar vector. This is possible in pure rotation according to 1. and is indeed quite often an aid to the imagination.

In this connection a well-known and often applied rule may be men-

¹⁰Cf., e.g., Brand, *op. cit.*, §149.

¹¹In fact, two orientations are possible and only one is chosen in accordance with the orientation of the coordinate system.

tioned that helps to keep in mind the structure of the expressions (6) or (17) and in particular the sequence of signs in those formulas. It reads:

$$(18) \quad \text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}.$$

The letters \mathbf{i} , \mathbf{j} , \mathbf{k} denote here unit vectors¹² in the directions of the x -, y -, z -axes. Multiplication with $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ means differentiation with respect to the corresponding variable. The expansion of the determinant (18) yields

$$(18a) \quad \text{curl } \mathbf{A} = \mathbf{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right),$$

which represents the correct way to form the curl components.

It should be noted, however, that the mapping of an axial upon a polar vector is possible only in *three-dimensional space*. This can be inferred from the number of components ν which is, of course, equal to the number of dimensions n in the case of the polar vector, while for an axial vector in general

$$\nu = \frac{n(n-1)}{2}, \quad \text{or} \begin{cases} \text{for } n = 1, 2, 3, 4, \dots \\ \nu = 0, 1, 3, 6, \dots \end{cases}$$

The number ν is correlated with the double subscript notation of the axial vector: it is the number of combinations of the n directions in space in groups of two. That ν coincides with n for $n = 3$ is, one might say, a coincidence. The number of components $\nu = 6$ in four-dimensional space plays a decisive part in electrodynamics, where it coincides with the *number of components of the electromagnetic field* (3 electric + 3 magnetic components).

We return to our starting point, Eq. (19), which we write in vector form

$$(19) \quad \boldsymbol{\omega} = \frac{1}{2} \text{curl } \mathbf{v}.$$

¹²The \mathbf{i} , \mathbf{j} , \mathbf{k} in the original German text have not been printed as vectors. The symbolic notation $\mathbf{V} = iV_x + jV_y + kV_z$ has then the character of a "higher complex number" and indicates more strongly the conceptual relationship between vectors, ordinary complex numbers, and Hamilton's quaternions.

It cannot be denied that the factor $\frac{1}{2}$ in this formula is a flaw. The meaning of ω is, as we know now, the whirl or rotation associated with the velocity distribution \mathbf{v} . Our formal introduction of the symbol curl in (6) implies that the "physical rotation" is only half of the "mathematical rotation". It is not feasible to remove this paradox by absorbing the factor $\frac{1}{2}$ in the definition of the curl symbol; that would have the most uncomfortable consequences for general vector analysis, in particular for electrodynamics. Nothing can be done but to point out the flaw and apologize.

The three operators grad, div, and curl have so far been defined by *differential* operations. There exists, however, a very convenient way of introducing div and curl by means of *integral* operations. Let us start with the *divergence*.

Visualize the vector field \mathbf{A} as a field of flow, that is, let magnitude and direction of the flow be given everywhere by the magnitude and direction of \mathbf{A} . Let the field point P be surrounded by a closed surface and $\Delta\tau$ be the enclosed volume. If \mathbf{n} is the outward normal associated with the surface element $d\sigma$, A_n represents the outflow per unit of surface, normal to the surface. On forming $(1/\Delta\tau) \int A_n d\sigma$, the limit for vanishing $\Delta\tau$ (or σ) becomes

$$(20) \quad \operatorname{div} \mathbf{A} = \lim_{\Delta\tau \rightarrow 0} \frac{1}{\Delta\tau} \int A_n d\sigma.$$

The agreement between this definition and (1) is evident in the special case that $\Delta\tau$ is chosen as a rectangular cell with center P and sides Δx , Δy , Δz (Fig. 2). If A_x at the point $x + \Delta x$ is expanded after Taylor and

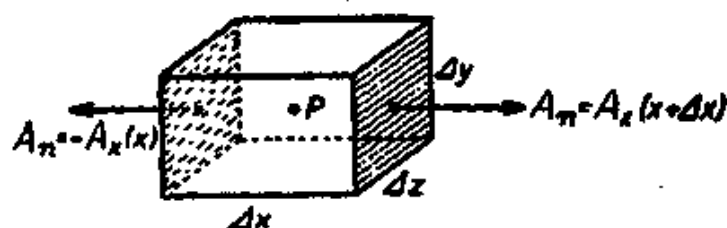


FIG. 2. Calculation of $\operatorname{div} \mathbf{A}$ by an integral over the surface of an infinitesimal parallelepiped.

higher powers of Δx are neglected as before, one finds for the two x -surfaces (that is, the surfaces normal to the x -axis)

$$\int A_n d\sigma = \int (A_x(x + \Delta x) - A_x(x)) dy dz = \int \frac{\partial A_x}{\partial x} \Delta x dy dz = \frac{\partial A_x}{\partial x} \Delta\tau$$

and two corresponding terms for the two pairs of y - and z -surfaces; altogether one obtains the expression for $\operatorname{div} \mathbf{A}$. In writing (20) mathematicians would add the proviso that the limit indicated therein actually

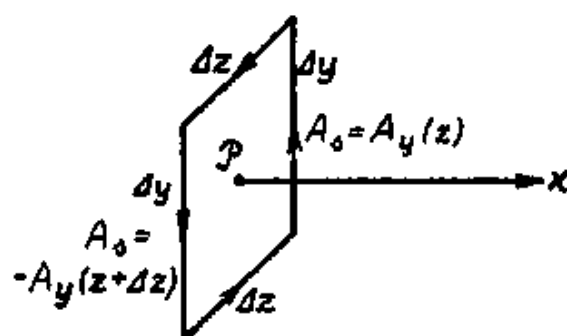
exists. This we have omitted, since the restrictions as to continuity imposed on the vector field by such a proviso certainly would not be more severe than those which are inherent in definition (1). Thus, as a definition, (20) is certainly equivalent to (1); it should be preferred, however, as it is of a less formalistic nature.

The curl can be defined by the following construction in our field of flow. Let an arbitrarily chosen, oriented line a pass through the point P ; in the plane that contains P and is normal to a , draw a closed curve s surrounding P , and denote the enclosed area by $\Delta\sigma$. Let A_s be the component of \mathbf{A} in direction of the arc element ds , taken in that sense of direction which forms a right hand screw with the axis a . Consider now the line integral $\oint A_s ds$, for which Lord Kelvin introduced the appropriate name *circulation*. The limit of the ratio of the circulation to the area $\Delta\sigma$ is

$$(21) \quad \text{curl}_a \mathbf{A} = \lim_{\Delta\sigma \rightarrow 0} \frac{1}{\Delta\sigma} \oint A_s ds.$$

The agreement of this definition with the previous one, (17), can again be seen quite easily if the curve s is appropriately chosen. Suppose one wants to verify (21) for the y, z -component of the curl. The line a then coincides with the $+x$ -direction, and s may be chosen as a rectangle with center P and sides $\Delta y, \Delta z$ (see Fig. 3). On expanding the contributions

FIG. 3. Calculation of curl \mathbf{A} by a line integral over an orthogonal circuit.



of each pair of sides as before, the first pair (which is parallel to the y -axis) yields

$$\int A_s ds = \int (-A_y(z + \Delta z) + A_y(z)) dy = -\frac{\partial A_y}{\partial z} \Delta y \Delta z,$$

and the second pair

$$\int A_s ds = \frac{\partial A_z}{\partial y} \Delta y \Delta z.$$

Altogether one obtains according to (21)

$$\text{curl}_x \mathbf{A} = \text{curl}_{xx} \mathbf{A} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

The other components come out if the line α is made to coincide with the other coordinate axes, and the results are found by cyclic permutation ("rotation") of the letters.

The "integral" definitions put geometrical meaning into the operators div and curl previously introduced by formal analytic means, but a visualization of the operation $\text{grad } U$ is still lacking. Imagine for this purpose the level surfaces (that is, the surfaces $U = \text{const}$) and their orthogonal trajectories marked in space. The direction of the gradient is everywhere tangential to the trajectory; its magnitude, $\partial U / \partial n$, expresses itself in the concentration of the level surfaces as they follow each other: for constant δU , $\partial U / \partial n$ is the larger, the smaller δn .

Compared with the original definitions (1), (5), and (17), the advantage of the geometrical definitions is not only their less formalistic nature, but also their validity in *general* coordinates, which permits in particular an easy transition to any required system of curvilinear *orthogonal* coordinates.

We show this for the general case of a system of three mutually orthogonal families of surfaces, while the more elementary special cases such as polar coordinates will be treated in prob. I.3. Let p_1, p_2, p_3 be the parameters of the three families of surfaces. As a consequence of the mutual orthogonality, the analytic expression for the square of the distance of two neighbored points contains no mixed terms of the type $dp_i dp_j$. Following the notation g_{ik} of the general theory of relativity, we denote the coefficients of the quadratic terms by g_i^2 , so that the square of the element of length is

$$(22) \quad ds^2 = g_1^2 dp_1^2 + g_2^2 dp_2^2 + g_3^2 dp_3^2,$$

which implies in particular the following expressions for the differentials in the three coordinate directions:

$$(23) \quad ds_1 = g_1 dp_1, \quad ds_2 = g_2 dp_2, \quad ds_3 = g_3 dp_3.$$

These expressions replace dx, dy, dz , if orthogonal curvilinear coordinates are used instead of the usual Cartesian coordinates. The components of the gradient, heretofore

$$\text{grad } U = \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z},$$

now become

$$(24) \quad \text{grad } U = \frac{1}{g_1} \frac{\partial U}{\partial p_1}, \frac{1}{g_2} \frac{\partial U}{\partial p_2}, \frac{1}{g_3} \frac{\partial U}{\partial p_3}.$$

The expression for the divergence in the new coordinates is obtained from (20) if the volume $\Delta\tau$ is chosen as an infinitesimal cell with (curvilinear) sides $\Delta s_1, \Delta s_2, \Delta s_3$:

$$(25) \quad \operatorname{div} \mathbf{A} = \frac{1}{g_1 g_2 g_3} \left\{ \frac{\partial}{\partial p_1} (g_2 g_3 A_1) + \frac{\partial}{\partial p_2} (g_3 g_1 A_2) + \frac{\partial}{\partial p_3} (g_1 g_2 A_3) \right\}.$$

The expression for the 1-component of the curl is obtained by choosing the surface that contains the "rectangular" circuit orthogonal to the coordinate direction p_1 :

$$(26) \quad \operatorname{curl}_1 \mathbf{A} = \frac{1}{g_2 g_3} \left(\frac{\partial (g_3 A_2)}{\partial p_1} - \frac{\partial (g_2 A_3)}{\partial p_1} \right),$$

The 2- and 3-components follow by "rotating" the numbers 1, 2, 3.

3. The Theorems of Gauss, Stokes, and Green

It is not difficult to proceed from the integral definitions of divergence and curl in (2.20) and (2.21) to the integral theorems of Gauss and Stokes.

Let the vector field \mathbf{A} be differentiable inside the closed surface σ so as to establish the value of $\operatorname{div} \mathbf{A}$ everywhere in the interior. The following integral relation then constitutes *Gauss's theorem*:

$$(1) \quad \int \operatorname{div} \mathbf{A} \, d\tau = \int A_n \, d\sigma.$$

Here n is the outward¹³ normal associated with the surface element $d\sigma$.

To prove this relation we divide the volume τ in a number of small cells $\Delta\tau_m$; in accordance with the mathematical definition of the definite integral, a set of such divisions with decreasing cell size may be used to write the first member of (1) as the limit

$$(1a) \quad \lim_{M \rightarrow \infty} \sum_{m=0}^M \operatorname{div} \mathbf{A} \, \Delta\tau_m,$$

where the value of $\operatorname{div} \mathbf{A}$ may be taken at an arbitrary point of the cell $\Delta\tau_m$. If the integral definition (2.20) is substituted for $\operatorname{div} \mathbf{A}$ the sum in (1a) becomes

$$(1b) \quad \sum_m \Delta\tau_m \lim_{\Delta\tau \rightarrow 0} \frac{1}{\Delta\tau} \int A_n \, d\sigma.$$

There is no objection against writing the factor $\Delta\tau_m$, which is still finite, to the right of the limit sign; also the summation, which so far refers only

¹³This convention may cause some inconvenience, as in the case $\mathbf{A} = \operatorname{grad} U$, where $\partial U / \partial n$ is then in general not defined outside of σ ; we may, however, replace $\partial U / \partial n$ by $-\partial U / \partial n'$ where n' is now the inward normal.

to a finite number of terms, may be interchanged with the limit sign. The expression (1b) then takes the form

$$(1c) \quad \lim_{\Delta\tau \rightarrow 0} \sum_m \frac{\Delta\tau_m}{\Delta\tau} \int A_n d\sigma.$$

Let it now be agreed upon that the cell $\Delta\tau$ which serves to define the divergence be *identical* with the cell $\Delta\tau_m$ that belongs to the definite integral.¹⁴ At the same time let us write $d\sigma_m$ instead of $d\sigma$, to indicate that $d\sigma_m$ is now an element of the surface of $\Delta\tau_m$. After these changes (1c) becomes

$$(1d) \quad \lim_{\Delta\tau_m \rightarrow 0} \sum_m \int A_n d\sigma_m.$$

In carrying out the summation over m the parts of the σ_m -surfaces that lie *inside* σ cancel each other in pairs because of the opposite signs of the normals n . Only the contributions of the surface elements that are adjacent to the σ -surface are left over; hence the sum in (1d) may simply be written

$$(1e) \quad \int A_n d\sigma,$$

where the integral refers to the *surface of the originally given region in space* as in the integral (1). Since this integral is independent of the division into cells $\Delta\tau_m$, the limiting process required in (1d) does not change anything, neither does the summation required in (1a). Expression (1e), as it stands, is therefore the value of the expression (1a) which, in turn, was only another way of writing the volume integral on the left side of (1). Since (1e) is identical with the right member of (1), everything is proved.

The formal proof of Gauss's theorem usually found in textbooks is based on a partial integration in rectangular coordinates; it can be replaced, as we have seen, by an argument that uses only the general principles of integral calculus and vector analysis. The same is true for Stokes's theorem, for which we shall outline only the general idea of the proof, referring to the foregoing for the formal details.

In terms of fluid flow, the second member of (1) means the outflow through the boundary surface (inflow, if negative). The first member of (1) represents the algebraic sum of the strengths of all sources or sinks, continuously distributed throughout the interior. The equality of the two members is thus intuitively evident.

¹⁴To be sure, this means that the limiting processes $\Delta\tau \rightarrow 0$ and $\Delta\tau_m \rightarrow 0$ (or $m \rightarrow \infty$) are carried out simultaneously, whereas the two processes should be carried out independently according to (1b).

In *Stokes's theorem* we consider an arbitrary, in general curved, surface σ bounded by a closed, oriented curve s (circuit). At each surface element $d\sigma$ construct the normal n and let it point in the direction that forms a right hand screw with the orientation of s . The content of Stokes's theorem is then: For any vector field A , continuous in the neighborhood of σ ,

$$(2) \quad \int \text{curl}_n A \, d\sigma = \oint A_s \, ds.$$

For a proof divide σ into small surface elements that can be considered as parts of the associated tangential planes, and let their boundaries s_m be oriented in accordance with the orientation of s . Write down equation (2.21) for each of the elements $\Delta\sigma_m$, multiply with $\Delta\sigma_m$ and sum over m . This leads to

$$(3) \quad \lim_{m \rightarrow \infty} \sum_m \text{curl}_n A \, \Delta\sigma_m = \lim_{m \rightarrow \infty} \sum_m \oint \lim_{\Delta\sigma \rightarrow 0} \frac{\Delta\sigma_m}{\Delta\sigma} A_s \, ds_m.$$

On making $\Delta\sigma = \Delta\sigma_m$ and proceeding as in Eqs. (1a)-(1d), the contributions of all internal boundaries cancel each other, since each segment belongs to two adjacent surface elements and is therefore traversed twice in opposite directions. Only the contributions of the external boundaries remain and sum up to the line integral on the right hand side of (2). In terms of fluid flow, Eq. (2) states: the circulation around the boundary curve s equals the flux of the vortex vector through an *arbitrary* surface σ subtended by s .

The integral taken over the boundary curve vanishes if the surface integral in Stokes's theorem refers to a *closed* surface. We indicate this as follows:

$$(4) \quad \oint \text{curl}_n A \, d\sigma = 0$$

where we assume, of course, the continuity of $\text{curl } A$, a point which has been emphasized before.

If the vector A is, in particular, chosen as a gradient, $A = \text{grad } U$, say, then the differential expression

$$(5) \quad A_s \, ds = A_x \, dx + A_y \, dy + A_z \, dz$$

becomes a total differential dU . In this case

$$(6) \quad \oint A_s \, ds = 0$$

for any boundary curve. The first member of (2) must then vanish for any surface σ and any normal direction n . This implies that quite generally

$$(7) \quad \text{curl grad } U = 0,$$

a formula which can also be read from the differential definition of the curl, Eq. (2.6).

The equation $\text{curl } \mathbf{A} = 0$, written in rectangular coordinates, is equivalent to the three conditions for the differential expression $A_x dx + A_y dy + A_z dz$ to be a total differential [see Eq. (5)]. These same conditions already appeared in Vol. I, Eq. (18.17) for the existence of a potential in a force field; they can now be written in the comprehensive form $\text{curl } \mathbf{A} = 0$.

Note, by the way, that the operation curl grad has a well defined meaning, but grad curl has not, since the gradient, by its definition, cannot be applied to a vector.

When Gauss's theorem is applied to a curl vector, that is, when $\text{curl } \mathbf{B}$ is substituted for \mathbf{A} in (1), the right member of (1) becomes an integral of the form (4), hence it vanishes. Consequently the expression on the left side of (1) is zero for any closed surface σ in τ . This means that generally

$$(8) \quad \text{div curl } \mathbf{B} = 0,$$

an identity which can also be read directly from the differential definition (2.1) and (2.6). Again the converse symbol $\text{curl div } \mathbf{B}$ carries no meaning since the operation curl cannot be applied to a scalar.

Let us also have a look at the symbols div grad and grad div . The first applies to a scalar, say U . We write

$$(9) \quad \text{div grad } U = \nabla^2 U.$$

In Cartesian coordinates the operator ∇^2 is

$$(9a) \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In orthogonal curvilinear coordinates p_1, p_2, p_3 the relations (2.24) and (2.25) yield

$$(9b) \quad \nabla^2 U = \frac{1}{g_1 g_2 g_3} \left\{ \frac{\partial}{\partial p_1} \left(\frac{g_2 g_3}{g_1} \frac{\partial U}{\partial p_1} \right) + \frac{\partial}{\partial p_2} \left(\frac{g_3 g_1}{g_2} \frac{\partial U}{\partial p_2} \right) + \frac{\partial}{\partial p_3} \left(\frac{g_1 g_2}{g_3} \frac{\partial U}{\partial p_3} \right) \right\}.$$

The symbol ∇^2 is called the *Laplace Operator* or *Laplacian*, also the *second differential parameter*. The *first differential parameter* is defined by

$$(9c) \quad \begin{aligned} DU = (\nabla U)^2 &= \left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial U}{\partial y} \right)^2 + \left(\frac{\partial U}{\partial z} \right)^2 \\ &= \frac{1}{g_1^2} \left(\frac{\partial U}{\partial p_1} \right)^2 + \frac{1}{g_2^2} \left(\frac{\partial U}{\partial p_2} \right)^2 + \frac{1}{g_3^2} \left(\frac{\partial U}{\partial p_3} \right)^2. \end{aligned}$$

Both symbols operate on scalar point functions. Other notations for the

Laplacian are Δ^2 and Δ , the latter mainly in French and German literature.¹⁵

The converse symbol, grad div, applies to vectors. Let it here be sufficient to mention that it appears in the useful formula

$$(10) \quad \text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2 \mathbf{A}.$$

Caution is necessary in the application of (10) since in this formula the symbol $\nabla^2 = \text{div grad}$ operates on a vector, contrary to its definition. Actually, the formula refers to the *Cartesian components* of the vector \mathbf{A} , and ∇^2 operates on the individual component of \mathbf{A} , considered as a scalar. Accordingly, the proof of (10) must be carried out in Cartesian coordinates and consists in checking the equality of the two members by an easy calculation; the operator ∇ can also be used.

Formula (10), conversely, serves to transform into general coordinates the quantity $\nabla^2 \mathbf{A}$, which so far is only known for Cartesian components. We define

$$(10a) \quad \nabla^2 \mathbf{A} = \text{grad div } \mathbf{A} - \text{curl curl } \mathbf{A}.$$

The right member is known in arbitrary orthogonal coordinates according to (2.24)-(2.26) so as to give a definite meaning to the left member. We shall see this procedure applied in 9, after Eq. (18). In problem I.4 we discuss and interpret the difference between the correctly and incorrectly calculated components of $\nabla^2 \mathbf{A}$ in cylindrical coordinates.

We turn now to *Green's theorem*¹⁶ which occupies a *unique place* among

¹⁵The sign ∇ (read "del" or "nabla", the latter being the name of a Hebrew musical instrument of inverted triangular shape) is symbolically defined by

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}.$$

∇^2 is meant as a scalar product:

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The product of ∇ with a scalar φ is the gradient of φ : The scalar product $\nabla \cdot \mathbf{A}$, if evaluated according to the formal rules of vector algebra, gives the divergence of \mathbf{A} :

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \text{div } \mathbf{A}.$$

The vector product $\nabla \times \mathbf{A}$ means, accordingly, the curl of \mathbf{A} ; one obtains, with $\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}$, $\mathbf{i} \cdot \mathbf{i} = 0$, etc., as in (2.18b),

$$\nabla \times \mathbf{A} = \mathbf{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) = \text{curl } \mathbf{A}.$$

¹⁶George Green, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, published in Nottingham, 1828. The theorem is announced and established in full generality in art. 3 of this paper, the calculations are, of course, carried out in (Cartesian) coordinates, not in vectors. The notations U , V , used in (11), are Green's.

the vector-analytic integral theorems. Its applications in mathematical physics are very numerous. It is of fundamental importance in potential theory for which Green first devised it; indispensable in hydrodynamics, electrodynamics, and optics, it is also highly useful in pure mathematics. Riemann based his theory of complex functions upon the two-dimensional form of Green's theorem; it is likely to turn up at any time in the calculus of variations and in the theory of eigenfunctions and integral equations. Green proved his theorem by partial integrations as it is often done at present. We shall derive it more simply from Gauss's theorem which, apparently, was not known to Green. Let U and V be two scalar point functions satisfying the necessary continuity requirements (existence of the second and continuity of the first partial derivatives) and substitute for A in (1)

$$(11) \quad A = U \operatorname{grad} V - V \operatorname{grad} U,$$

and, consequently, for A_n

$$(12) \quad A_n = U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n}.$$

We calculate¹⁷

$$(13a) \quad \operatorname{div} (U \operatorname{grad} V) = \operatorname{grad} U \cdot \operatorname{grad} V + U \operatorname{div} \operatorname{grad} V,$$

$$(13b) \quad \operatorname{div} (V \operatorname{grad} U) = \operatorname{grad} V \cdot \operatorname{grad} U + V \operatorname{div} \operatorname{grad} U$$

and find for $\operatorname{div} A$ from (11)

$$(14) \quad \operatorname{div} A = U \operatorname{div} \operatorname{grad} V - V \operatorname{div} \operatorname{grad} U = U \nabla^2 V - V \nabla^2 U.$$

On using (14) and (12), Gauss's theorem (1) becomes

$$(15) \quad \int (U \nabla^2 V - V \nabla^2 U) d\tau = \int \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right) d\sigma,$$

which constitutes *Green's theorem in its first form*. (Unfortunately, the nomenclature is not uniform. Many authors refer to (15) as *Green's second formula* which seems to be the preferred usage in current American literature.)

Green's theorem is correct for any two functions U and V and any boundary σ of the volume τ . Note that the positive direction of n points outward.

It was assumed so far that the volume τ has only *one* exterior boundary surface. If an inner boundary exists also as in the case of a body with

¹⁷For an arbitrary scalar ψ and an arbitrary vector A one easily verifies the formula $\operatorname{div} \psi A = \psi \operatorname{div} A + A \cdot \operatorname{grad} \psi$.

one or more cavities, the integration on the right side of (15) must include the inner boundary. This happens in particular if the interior contains points where one of the functions U or V becomes infinite or otherwise singular, so that these points must be excluded from the integration by inner boundaries σ_i . The corresponding integrals then appear in addition on the right side of (15).

If one substitutes $A = U \text{ grad } V$, that is, $A_n = U \partial V / \partial n$ in Gauss's theorem, (1) becomes on account of (13a)

$$(16) \quad \int \text{grad } U \cdot \text{grad } V \, d\tau + \int U \nabla^2 V \, d\tau = \int U \frac{\partial V}{\partial n} \, d\sigma$$

which constitutes the *second form of Green's theorem* (or *Green's first formula* if the other terminology is used). On identifying U and V one obtains the form

$$(16a) \quad \int (\nabla U)^2 \, d\tau + \int U \nabla^2 U \, d\tau = \int U \frac{\partial U}{\partial n} \, d\sigma,$$

which contains both differential parameters (9b, c).

The last identity leads directly to conclusions of fundamental importance in potential theory, which is, mathematically speaking, nothing else but the theory of the differential equation

$$(17) \quad \nabla^2 U = 0$$

the solutions of which are known as *harmonic* or *potential functions*. From formula (16a) one derives at once the following lemma: *A potential function that vanishes on a closed surface σ and is regular inside, vanishes everywhere in the interior of σ . In fact, Eq. (16a) implies*

$$(17a) \quad \int (\nabla U)^2 \, d\tau = 0 \quad \text{inside } \sigma$$

because of (17) and the boundary condition $U = 0$; but (17a) requires $\nabla U = 0$, the integrand being non-negative. This means $U = \text{const}$, and since U vanishes on σ , $U = 0$.

The *uniqueness of solution of the boundary value problem* of potential theory can also be proved by means of (16a). We formulate this problem as follows: required an integral U of (17) that assumes given values on a given boundary surface and is regular everywhere inside. If there were two solutions U_1 and U_2 satisfying the conditions of the problem, the difference $U = U_1 - U_2$ would satisfy the conditions of the foregoing lemma; thus $U = 0$ inside σ as before, and $U_1 = U_2$.

The result holds, with a slight modification, also in the case that instead of U the normal derivative $\partial U / \partial n$ is prescribed on the boundary (see problem I.5).

These few remarks should give a general idea of the scope of Green's theorem; 20 will bring a characteristic example of how it is put to work.

4. Some Remarks on Tensor Analysis

One distinguishes between *symmetric*, *antisymmetric*, and general or *asymmetric* tensors. A general tensor can be considered as the sum of a symmetric and an antisymmetric tensor in exactly the same way as the coefficients a_{ik} in 1. The antisymmetric tensor, in turn, is representable by an (axial) vector as in (1.12a), hence we have only to deal with *symmetric* tensors in this article. As we shall limit our discussion to second order tensors (i.e. tensors whose components carry not more than two subscripts), the representation (1.12) of the strain tensor may be taken as a starting point. Our remarks will be formulated so as to refer to the strain tensor, but they are of course valid for any symmetric tensor of second order, in particular for the stress and friction tensors of Chapter II.

The behavior of a tensor in a transformation of coordinates, which is so closely connected with the general significance of the tensor concept, can be studied either by means of its representation through the scheme of coefficients of a linear vector function as in (1.10), or by means of its relation to the tensor quadric (1.13). In rewriting these two equations we replace x, y, z by x_1, x_2, x_3 ; $\epsilon_{xx}, \epsilon_{xy}, \dots$ by ϵ_{ik} ; the special vector ξ_1, η_2, ζ_3 by the general vector symbol A with components A_1, A_2, A_3 , and obtain:

$$(1) \quad A_i = \sum_k \epsilon_{ik} x_k, \quad (2) \quad \sum_i \sum_k \epsilon_{ik} x_i x_k = \text{const.}$$

We compare these with the transformed equations

$$(1') \quad A'_i = \sum_k \epsilon'_{ik} x'_k, \quad (2') \quad \sum_i \sum_m \epsilon'_{im} x'_i x'_m = \text{const.},$$

in order to find the relations that connect the tensor components ϵ and ϵ' . We shall again use the relations between the vector components x, A and x', A' known from (2.2), viz.

$$(3) \quad A'_i = \sum_k \alpha_{ik} A_k, \quad A_i = \sum_k \alpha_{ki} A'_k,$$

$$(4) \quad x_i = \sum_l \alpha_{il} x'_l, \quad x_k = \sum_m \alpha_{mk} x'_m.$$

The comparison of (1) with (1') need not be carried out here, since the procedure is formally identical with the transformation of the antisymmetric tensor, carried out in (2.9)-(2.13); one has only to write ϵ_{ik} instead of B_{ik} , A instead of $2A$, and to observe that the antisymmetry of

the B is not made use of until (2.14). Then the result (2.13) reads in the present notation

$$(5) \quad \epsilon'_{ij} = \sum_l \sum_k \alpha_{il} \alpha_{jl} \epsilon_{kl}.$$

This same result can also be obtained by comparison of (2) with (2'): Substitute, according to (4), x_i and x_k in (2) and reverse the order of summation. The expression $\sum_l \sum_m \sum_i \sum_k \epsilon_{ik} \alpha_{il} \alpha_{mk} x'_l x'_m = \text{const}$ must be identical with (2'), therefore

$$(6) \quad \epsilon'_{lm} = \sum_i \sum_k \epsilon_{ik} \alpha_{il} \alpha_{mk},$$

which is indeed the same as (5), except for the lettering of the subscripts.

The second method to establish the transformation rules is, of course, much simpler, but could not have been applied in the case of the anti-symmetric tensor of rotation. The attempt to associate a quadric with such a tensor fails as the quadric vanishes identically: the coefficients of the squares vanish ($\varphi_{ii} = 0$), the product terms cancel in pairs because of $\varphi_{ik} = -\varphi_{ki}$.

Our result (6) may be expressed in form of the following proposition: *The components of the symmetric tensor transform as the products and squares of the coordinates*, a statement which, conversely, can be considered as the definition of a tensor. The corresponding statement for a polar vector would be: *The components transform as the coordinates themselves*.

Eqs. (6) are generalizations of our previous relations (1.22) and (1.22a); they are sums of six terms, while (1.22) and (1.22a) are sums of three terms. The reason is evidently that the strain tensor was assumed in "diagonal form", that is, referred to its principal axes. But a rectangular cell cut parallel to the principal axes remains rectangular after straining; its angular changes and, therefore, its shears vanish. Thus three of the six tensor components in (6) are zero whenever the ϵ refer to principal axes.

We shall now discuss the *invariants of a tensor*, supplementing what was learned in Eq. (1.23) about the invariance of the diagonal sum, and prove first that the determinant of the coefficients of the quadric (2) is invariant in orthogonal coordinate transformations.

Denoting the determinant by D we have to show

$$(7) \quad D = \begin{vmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{vmatrix} = D',$$

where D' stands for the determinant of the ϵ' in Eq. (6). The proof is based on the multiplication theorem for determinants, which we apply here as follows: when the determinant of the ϵ is multiplied with the determinant of the transformation coefficients α , the result can again be written as a determinant whose elements β_{ik} are $\sum_i \epsilon_{ik} \alpha_{il}$. On multiplying the determinant of the β once more with the determinant of the α , the elements of the resulting determinant are

$$\sum_k \beta_{ki} \alpha_{mk} = \sum_k \sum_l \epsilon_{kl} \alpha_{li} \alpha_{mk}.$$

According to (6) this is identical with the transformed tensor component ϵ'_{ik} , the determinant of which was denoted by D' . Thus we have found

$$(7a) \quad D' = D \cdot |\alpha_{il}| |\alpha_{mk}|,$$

but this does not differ from assertion (7) as the determinant of the α equals unity.

When the surface $f = \sum \sum \epsilon_{ik} x_i x_k = \text{const}$ is orthogonally transformed into $f' = \sum \sum \epsilon'_{ik} x'_i x'_k = \text{const}$, the quadratic form $\varphi = \sum x_i^2$ is transformed into $\varphi' = \sum x_i'^2$. (The "circle at infinity" $\varphi = 0$ which is the gauge quadric of Euclidean metric, transforms into itself in orthogonal transformations.) Consequently a specified quadric of the family $f + \lambda\varphi = \text{const}$ transforms into $f' + \lambda\varphi' = \text{const}$ and the parameter λ has the same value in these two equations of the same surface of second order. But since the determinant of the coefficients of *this* surface is also invariant, it follows that

$$(8) \quad D_\lambda = \begin{vmatrix} \epsilon_{11} + \lambda & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} + \lambda & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} + \lambda \end{vmatrix} = D'_\lambda,$$

and this must hold for any choice of the arbitrary parameter λ . Thus a generalization of (7) has been proved.

The determinant D_λ may now be expanded in powers of λ ,

$$(8a) \quad D_\lambda = D + \lambda\Delta + \lambda^2\Theta + \lambda^3.$$

Here Θ is our previous diagonal sum of (1.23) and Δ the quadratic expression

$$(9) \quad \Delta = \epsilon_{11}\epsilon_{22} + \epsilon_{22}\epsilon_{33} + \epsilon_{33}\epsilon_{11} - \epsilon_{12}\epsilon_{21} - \epsilon_{23}\epsilon_{32} - \epsilon_{31}\epsilon_{13}.$$

On doing the same with D'_λ and denoting the coefficients by D' , Δ' , Θ' in analogy to (8a), we obtain because of (8)

$$D + \lambda\Delta + \lambda^2\Theta + \lambda^3 = D' + \lambda\Delta' + \lambda^2\Theta' + \lambda^3.$$

This must hold for any value of λ , therefore

$$(10) \quad D = D', \quad \Delta = \Delta', \quad \Theta = \Theta'.$$

The symmetrical tensor possesses three invariants, that is, three characteristic scalars that are independent of the incidental way in which the tensor is represented by its components. The geometrical relations between the invariants and the tensor quadric are the object of problem I.6. The invariant Θ is, in mechanical interpretation, equivalent to the cubical dilatation, as previously shown. In general, Θ is known as the *first scalar* or *spur* of the tensor.

The preceding argument relates to quadratic forms in a general way and must be valid for the antisymmetric as well as for the symmetric tensor. Hence, also the antisymmetric tensor ought to have three invariants, but the invariant Θ is identically zero (the diagonal terms vanish) and the determinant D vanishes (so does any antisymmetric determinant of odd order). There remains only the invariant Δ for which (9) supplies the special form

$$(11) \quad \Delta = \epsilon_{23}^2 + \epsilon_{31}^2 + \epsilon_{12}^2,$$

which is equal to the sum of squares of the components of the associated axial vector. Hence in the case of the axial vector only *one quantity exists that is independent of the choice of the coordinate system*, exactly as in the case of the polar vector. For the polar vector this is the length and for the axial vector the magnitude of the "twist", which equals the length of the polar vector by which the axial vector is represented.

As in the case of the moment of inertia¹⁸ (Vol. I, Fig. 40 a, b, c), we shall now discuss some particular cases of the tensor quadric. The significance of a *spherical* tensor quadric is evident. In terms of strain it means *uniform extension or contraction in all directions*: no shears occur since every axis may be considered as principal axis; all right angles are preserved.

Unidirectional extension, e.g., in x -direction with vanishing extensions in all directions normal to x is associated with the quadric $x^2 = \text{const}$, which is a *pair of planes*. Note, however, that this is not the state of strain caused by unidirectional stress in x -direction, since an additional contraction normal to x is produced in that case. The strain quadric is then a *hyperboloid of one sheet*, axisymmetric with respect to the x -axis.

A state of strain that presents some interest arises when the points of the body are displaced along concentric circles about an invariable axis in such a way that the displacements are inversely proportional to the

¹⁸ Cf. Synge and Griffith, *op. cit.*; Sec. 11.3 contains a table.

distance from the axis, r . It prompts itself to introduce cylindrical coordinates and to define the displacement vector by

$$(12) \quad s_r = 0, \quad s_\varphi = \frac{A}{r}, \quad s_z = 0.$$

The strain components in the same coordinates are determined on the basis of formulas (26) and (28) to be derived presently; they are correct for arbitrary orthogonal curvilinear coordinates and yield in the present case

$$(13) \quad e_{r\varphi} = -\frac{A}{r^2}, \quad e_{rr} = e_{\varphi\varphi} = e_{zz} = e_{\varphi z} = e_{zr} = 0.$$

We now write the tensor quadric for a point r, φ, z in (local) coordinates ξ, η, ζ oriented parallel to the increments $dr, d\varphi, dz$:

$$(14) \quad 2e_{r\varphi}\xi\eta = \text{const.}$$

The coefficient $e_{r\varphi}$ varies with the distance r , but has a fixed value in Eq. (14), which refers to a certain point of the body. We may then include the factor $2e_{r\varphi}$ in the constant, and have instead of (14)

$$(15) \quad \xi\eta = \text{const.},$$

an equilateral hyperbola. Hence the tensor quadric is quite generally an *equilateral hyperbolic cylinder*. The axes of the hyperbola (15) form an angle of 45° with the asymptotes $\xi = 0, \eta = 0$.

A volume element cut parallel to $dr, d\varphi, dz$ is after straining in a state

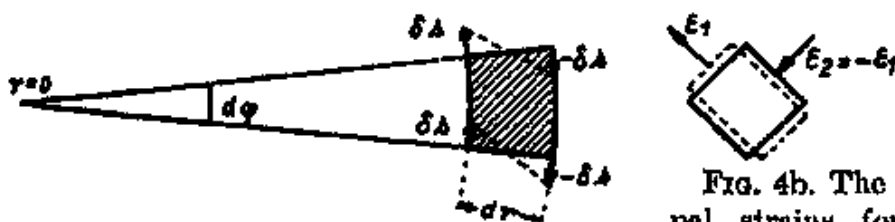


FIG. 4a. In the state of pure shear an originally rectangular volume element is transformed into a rhomboid.

FIG. 4b. The principal strains form an angle of 45° with the directions of the shear and are equal and opposite to each other.

of *pure shear*. The face that originally had four right angles is distorted into a rhomboid-like shape (see Fig. 4a). By setting

$$s(r) = s_0 + \delta s, \quad s(r + dr) = s_0 - \delta s, \quad (\varphi = \text{const})$$

it is seen that only δs contributes to the distortion, while the effect of s_0 is a mere translation (not indicated in the figure). On the other hand, the volume element drawn in Fig. 4b which is oriented parallel to the *principal*

axes of the tensor quadric is not only *free of shear strain* (as required by the definition of the principal axes) but also *free of dilatation*.¹⁹

This is a consequence of the "equilateral shape" of the tensor quadric: from (13) it is seen that the spur Θ vanishes; hence it must also vanish for the volume element cut parallel to the principal axes (invariance of the spur). Since $\Theta = \epsilon_1 + \epsilon_2 + \epsilon_3$ and $\epsilon_3 = \epsilon_{xx} = 0$, we have $\epsilon_2 = -\epsilon_1$. The state of *pure shear* is characterized by *opposite principal extensions*.

In conclusion we derive the formulas that express *the strain tensor in curvilinear coordinates* (we have already used them to obtain (13)). These formulas cannot be written down in such a direct way as those for grad, div, and curl in Eqs. (2.24)-(2.26), because there exists no definition of the tensor that is intuitively meaningful or carries self-evident geometrical invariance such as the definitions of grad, div, curl in 2. The following analysis²⁰ should make up for that and contribute to a deeper understanding of the tensor quadric. It is somewhat laborious, but will prove a great help later, e.g. in the torsion problem, 42.

Let ds be the distance of two neighboring points P and P' of the medium under consideration, let dx, dy, dz be the differences of their Cartesian and dp_1, dp_2, dp_3 the differences of their curvilinear coordinates. According to (2.22) one has

$$(16) \quad ds^2 = dx^2 + dy^2 + dz^2 = g_1^2 dp_1^2 + g_2^2 dp_2^2 + g_3^2 dp_3^2.$$

In the deformation of the medium all points suffer displacements given by (1.10), which we shall now denote by δq . By dq we denote, on the other

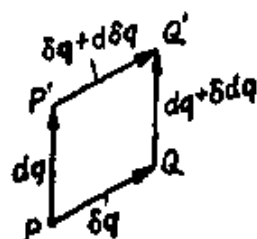


FIG. 5. To illustrate the transformation of the strain tensor in general orthogonal coordinates: a line element dq is shifted along the displacement vector δq .

hand, the relative position vector of two neighboring points before the deformation, so that $|dq| = ds$. The symbol $d\delta q$ denotes the change of the displacement δq caused by the transition d from one field point to a neighboring field point. With the notation of Fig. 5, this is the difference of the displacement vectors $P'Q'$ and PQ . The symbol δdq means, again in terms of Fig. 5, the change of the position vector PP' caused by the

¹⁹Note also that this state of strain is formally the same as in a bar of circular cross section subjected to torsion, but the generators of the tensor quadric are perpendicular to the bar axis in the latter case [compare (4.13) with (42.3)].

²⁰Cf. E. Beltrami, *Annali di Matematica* 10, 188 (1881). *Opere Matematiche*, III, 383.

displacement operation δ , which carries P into Q and P' into Q' ; $\delta d\mathbf{q}$ is thus the difference of the position vectors QQ' and PP' , after and before deformation. But the quadrilateral $PQQ'P'$ supplies the vector relation $PP' + P'Q' = PQ + QQ'$ or $P'Q' - PQ = QQ' - PP'$, which is equivalent to the commutative relation

$$(17) \quad d\delta\mathbf{q} = \delta d\mathbf{q}.$$

We return now to (1.10) and have in Cartesian coordinates

$$(18) \quad d\delta\mathbf{q} = \begin{cases} d\delta x = \epsilon_{xx} dx + \epsilon_{xy} dy + \epsilon_{xz} dz, \\ d\delta y = \epsilon_{yx} dx + \epsilon_{yy} dy + \epsilon_{yz} dz, \\ d\delta z = \epsilon_{zx} dx + \epsilon_{zy} dy + \epsilon_{zz} dz. \end{cases}$$

These relations give the change of the displacement $d\delta\mathbf{q}$ as a linear vector function of the Cartesian components of the vector $d\mathbf{q}$. If we decompose $d\mathbf{q}$ into its curvilinear components given by (2.23), viz.

$$(19) \quad ds_1 = g_1 dp_1, \quad ds_2 = g_2 dp_2, \quad ds_3 = g_3 dp_3$$

we obtain again a linear vector function for $d\delta\mathbf{q}$ the coefficients of which are the tensor components we look for:

$$(20) \quad d\delta\mathbf{q} = \begin{cases} d\delta s_1 = \epsilon_{11} ds_1 + \epsilon_{12} ds_2 + \epsilon_{13} ds_3, \\ d\delta s_2 = \epsilon_{21} ds_1 + \epsilon_{22} ds_2 + \epsilon_{23} ds_3, \\ d\delta s_3 = \epsilon_{31} ds_1 + \epsilon_{32} ds_2 + \epsilon_{33} ds_3. \end{cases}$$

Our task is to calculate the "curvilinear tensor components" ϵ_{ik} from the displacement components $\delta q_1, \delta q_2, \delta q_3$ in directions p_1, p_2, p_3 as in (1.11).

We first determine in Cartesian coordinates the change of the square of the element of length ds^2 produced by the displacement $\delta\mathbf{q}$. From (16), (17), and (18) we infer

$$\begin{aligned} \frac{1}{2} \delta ds^2 &= dx \delta dx + dy \delta dy + dz \delta dz \\ (21) \quad &= \epsilon_{xx} dx^2 + \epsilon_{yy} dy^2 + \epsilon_{zz} dz^2 \\ &\quad + 2\epsilon_{xy} dx dy + 2\epsilon_{yz} dy dz + 2\epsilon_{zx} dz dx. \end{aligned}$$

The last member is identical with the polynomial of the tensor quadric except for the notation of the variables dx, dy, dz that replace the former x, y, z . The first member of (21) may be written in the form $ds \delta ds$. (On dividing by ds^2 we obtain $\delta ds/ds$ which is the *extension of the ds -fiber per*

unit of length; the variables are now the quantities dx/ds , dy/ds , dz/ds . Hence the *tensor quadric* gives directly the *specific extension* of the ds -fiber in terms of the *direction cosines* dx/ds , dy/ds , dz/ds and thus lends intuitive physical meaning to the concept of the symmetrical tensor.

The procedure leading to Eq. (21) can just as well be carried out in curvilinear coordinates p_1, p_2, p_3 . Using (19) we write the length element in the form $ds^2 = ds_1^2 + ds_2^2 + ds_3^2$. Eqs. (20) and (17) then yield

$$\begin{aligned} \frac{1}{2} \delta ds^2 &= ds_1 \delta ds_1 + ds_2 \delta ds_2 + ds_3 \delta ds_3 \\ (22) \quad &= \epsilon_{11} ds_1^2 + \epsilon_{22} ds_2^2 + \epsilon_{33} ds_3^2 \\ &\quad + 2\epsilon_{12} ds_1 ds_2 + 2\epsilon_{23} ds_2 ds_3 + 2\epsilon_{31} ds_3 ds_1. \end{aligned}$$

On dividing this equation by ds^2 the extension of the ds -fiber is obtained as before.

The required relations between the ϵ_{ik} and the displacement components δq_i are now obtained by the direct calculation of $\frac{1}{2} \delta ds^2$ in terms of the ds_i and subsequent comparison with the right member of (22). Writing (16) in condensed form, we obtain first

$$(22a) \quad \frac{1}{2} \delta ds^2 = \sum_i g_i \delta q_i dp_i^2 + \sum_i g_i^2 dp_i \delta dp_i.$$

Here we have to express the variations, imposed on the coefficients g_i and on the coordinate differentials dp_i by the displacement δq_i in terms of the variations and differentials of the coordinates themselves; in other words, we have to substitute in (22a)

$$\delta g_i = \sum_k \frac{\partial g_i}{\partial p_k} \delta p_k, \quad \delta dp_i = d\delta p_i = \sum_k \frac{\partial \delta p_i}{\partial p_k} dp_k$$

and obtain

$$(23) \quad \frac{1}{2} \delta ds^2 = \sum_i \sum_k g_i \frac{\partial g_i}{\partial p_k} \delta p_k dp_i^2 + \sum_i \sum_k g_i^2 \frac{\partial \delta p_i}{\partial p_k} dp_i dp_k.$$

Now we are ready to introduce the ds_i . According to (19), we write ds_i/g_i for the dp_i . The "curvilinear" displacement components δp_i are obviously connected with the components δq_i by

$$\delta q_i = g_i \delta p_i, \quad \text{or} \quad \delta p_i = g_i^{-1} \delta q_i.$$

In this way the first of the two sums on the right side of (23) becomes

$$(23a) \quad \sum_i \sum_k \frac{\delta q_k}{g_k} \frac{\partial g_i}{\partial p_k} ds_i^2;$$

the second sum becomes

$$\sum_i \sum_k \frac{g_i}{g_k} \frac{\partial}{\partial p_k} \left(\frac{\delta q_i}{g_i} \right) ds_i ds_k$$

or, on carrying out the differentiations,

$$(23b) \quad \sum_i \sum_k \frac{1}{g_k} \frac{\partial \delta q_i}{\partial p_k} ds_i ds_k - \sum_i \sum_k \frac{\delta q_i}{g_i g_k} \frac{\partial g_i}{\partial p_k} ds_i ds_k.$$

Now we can compare the coefficients of $ds_i ds_k$ in (22) with those in the sum of (23a) and (23b). Let us do this first for the diagonal terms: the coefficient of a *specified* ds_i^2 in (23a) is

$$(24a) \quad \sum_k \frac{\delta q_k}{g_i g_k} \frac{\partial g_i}{\partial p_k};$$

in (23b), k must be identified with the given i . The coefficient is then simply

$$(24b) \quad \frac{1}{g_i} \frac{\partial \delta q_i}{\partial p_i} - \frac{\delta q_i}{g_i^2} \frac{\partial g_i}{\partial p_i}.$$

Upon adding (24a) and (24b), two terms cancel and one obtains

$$(25) \quad \frac{1}{g_i} \left(\frac{\partial \delta q_i}{\partial p_i} + \sum_{k \neq i} \frac{\delta q_k}{g_k} \frac{\partial g_i}{\partial p_k} \right).$$

This, then, is the value of ϵ_{ii} . Written in extenso, (25) means

$$(26) \quad \begin{aligned} \epsilon_{11} &= \frac{1}{g_1} \left(\frac{\partial \delta q_1}{\partial p_1} + \frac{\delta q_2}{g_2} \frac{\partial g_1}{\partial p_2} + \frac{\delta q_3}{g_3} \frac{\partial g_1}{\partial p_3} \right) \\ \epsilon_{22} &= \frac{1}{g_2} \left(\frac{\partial \delta q_2}{\partial p_2} + \frac{\delta q_3}{g_3} \frac{\partial g_2}{\partial p_3} + \frac{\delta q_1}{g_1} \frac{\partial g_2}{\partial p_1} \right) \\ \epsilon_{33} &= \frac{1}{g_3} \left(\frac{\partial \delta q_3}{\partial p_3} + \frac{\delta q_1}{g_1} \frac{\partial g_3}{\partial p_1} + \frac{\delta q_2}{g_2} \frac{\partial g_3}{\partial p_2} \right). \end{aligned}$$

In the determination of the shears ϵ_{ik} , we note first that (23a) gives no contribution. In (23b), either sum yields two terms if i and k are specified, viz. one term $\dots ds_i ds_k$ and another $\dots ds_k ds_i$. This amounts to the four term expression

$$(27) \quad \frac{1}{g_k} \frac{\partial \delta q_i}{\partial p_k} + \frac{1}{g_i} \frac{\partial \delta q_k}{\partial p_i} - \frac{1}{g_i g_k} \left(\delta q_i \frac{\partial g_i}{\partial p_k} + \delta q_k \frac{\partial g_k}{\partial p_i} \right)$$

which, on the other hand, equals $2\epsilon_{ik}$. If we write (27) in extenso, we obtain

$$\begin{aligned}
 \epsilon_{12} &= \frac{1}{2} \left(\frac{1}{g_2} \frac{\partial \delta q_1}{\partial p_2} + \frac{1}{g_1} \frac{\partial \delta q_2}{\partial p_1} \right) - \frac{1}{2g_1 g_2} \left(\delta q_1 \frac{\partial g_1}{\partial p_2} + \delta q_2 \frac{\partial g_2}{\partial p_1} \right) \\
 (28) \quad \epsilon_{23} &= \frac{1}{2} \left(\frac{1}{g_3} \frac{\partial \delta q_2}{\partial p_3} + \frac{1}{g_2} \frac{\partial \delta q_3}{\partial p_2} \right) - \frac{1}{2g_2 g_3} \left(\delta q_2 \frac{\partial g_2}{\partial p_3} + \delta q_3 \frac{\partial g_3}{\partial p_2} \right) \\
 \epsilon_{31} &= \frac{1}{2} \left(\frac{1}{g_1} \frac{\partial \delta q_3}{\partial p_1} + \frac{1}{g_3} \frac{\partial \delta q_1}{\partial p_3} \right) - \frac{1}{2g_3 g_1} \left(\delta q_3 \frac{\partial g_3}{\partial p_1} + \delta q_1 \frac{\partial g_1}{\partial p_3} \right).
 \end{aligned}$$

Formulas (26) and (28) can be cast in a much more comprehensive form if the notation of general tensor calculus is employed as in Appendix I.

For the present we apply these formulas only to verify our Eqs. (13). If we specify the p_i as cylindrical coordinates by setting $p_1 = r$, $p_2 = \varphi$, $p_3 = z$, it follows that $g_1 = g_3 = 1$, $g_2 = r$, and the displacement (12) reads

$$\delta q_1 = \delta q_3 = 0, \quad \delta q_2 = \frac{A}{r}.$$

As a consequence, only those terms in (26) and (28) contribute something that contain δq_2 or $\partial \delta q_2 / \partial p_1$ and do not have a vanishing coefficient $\partial g_i / \partial p_k$ (note that all these derivatives vanish except $\partial g_2 / \partial p_1$ which equals unity). Only ϵ_{12} contains such terms, viz.

$$\frac{1}{2g_1} \frac{\partial \delta q_2}{\partial p_1} = -\frac{1}{2} \frac{A}{r^2} \quad \text{and} \quad -\frac{1}{2g_1 g_2} \delta q_2 \frac{\partial g_2}{\partial p_1} = -\frac{1}{2} \frac{A}{r^2}.$$

Thus the system (26) and (28) reduces in our case to: $\epsilon_{12} = -A/r^2$, and all other ϵ vanish, in agreement with (13).

Let us finally interpret the displacement field (12) as a velocity field by setting

$$(29) \quad v_r = 0, \quad v_\varphi = \frac{A}{r}, \quad v_z = 0.$$

This motion is of fundamental importance in the theory of vortices. We shall see that it is *irrotational* wherever $r > 0$, that is, at all points outside the axis. This is true in spite of our inclination to "see" (29) as a whirl about the axis $r = 0$. The steady motion characterized by the velocity distribution (29) is known as a *circulating motion*, also as a (straight) *line vortex* or *simple vortex*. The latter terminology makes it evident that the points of the axis have a particular significance: they form a vortex line, that is, a concentrated vortex filament. This will be discussed more fully in 19, p. 140, where the irrotationality of the vortex motion will also be shown; the latter point, however, can be verified right now by means of

the tabulated answer to problem I.3. One has only to identify our vector (29) with the vector **A** in the formulas giving the curl components in cylindrical coordinates.

The tensor concept has been developed in this article by studying the behavior of a tensor in the transition from one Cartesian coordinate system to another. If more general transformations are to be considered, the methods of general tensor analysis must be employed of which a short account is found in Appendix I-IV.

CHAPTER II

STATICS OF DEFORMABLE BODIES

5. Concept of Stress; General Classification of Deformable Bodies

The term statics will be used throughout this book in the same sense as in the mechanics of a finite number of degrees of freedom, where it was applied according to the following definition: Statics treats of the forces to which matter is subjected regardless of the motion caused by them (cf. Vol. I, 5).

One distinguishes between external and internal forces. Once a mechanical system has been delimited, the *internal* forces of the system are those which obey Newton's third law (equality of action and reaction). Among the *external* forces, gravity and centrifugal force¹ are commonly encountered in mechanics of deformable bodies. Capillary forces occur also; they act between internal and surface molecules or between a solid boundary and the molecules of the surrounding fluid. Examples of internal forces are the reactions (tension and pressure forces, surface forces) that have their origin in the distribution of matter over a limited space. They play the most important part² in the mechanics of deformable bodies.

In solids the reactions measured per unit of area are known as *stresses*; in the case of fluids we usually speak of *pressures* (negative stresses); both have the dimension of a force per unit area, dyne/cm² in the CGS system.

To analyze the stress concept in the case of an elastic solid, we imagine a plane surface cutting through the body and consider the interaction between the two parts of the body across this surface. In Fig. 6 such a surface of separation has been drawn normal to the x -axis, and the two parts of the body have been disjoined for the sake of clarity in the drawing. On the part to the left a "positive x -surface" has been marked by cross hatching. (A surface with an outward normal pointing in the positive x -direction is called positive x -surface for brevity.) We assume the x -surface

¹The proper place for the centrifugal force, which is an inertia effect, is in dynamics (Chap. III), but it may be considered in statics since it is equivalent to an external force for an observer moving with the body.

²The distinction between external and internal forces in a specified system does not in general coincide with the distinction between forces of physical origin (impressed forces) and forces of geometrical origin (due to spatial constraints to which the system is subjected).

in this figure bounded in the shape of a rectangle with the sides Δy , Δz . The x -surface bounding the right part of the body, which in reality coincides with the positive x -surface, has also been cross hatched. It is a negative x -surface since its outward normal points in the negative x -direction.

Consider now the forces which the negative x -surface exerts upon the positive x -surface. When these forces are parallel to the positive x -direction we speak of a positive *normal stress* σ_{xx} ; its magnitude equals the sum of the forces divided by the area $\Delta y \Delta z$. In general, however, the total force will by no means be perpendicular to the x -surface. We then resolve the force and the associated stress into components in the positive x -, y -, and z -directions. The y and z components are known as *tangential* or *shear stresses* and denoted by σ_{xy} , σ_{xz} . This notation indicates the position of the surface by the first subscript and the direction of the stress by the second.

The principle of action and reaction requires us to draw for the right part in Fig. 6 the stress arrows that act across the negative x -surface in

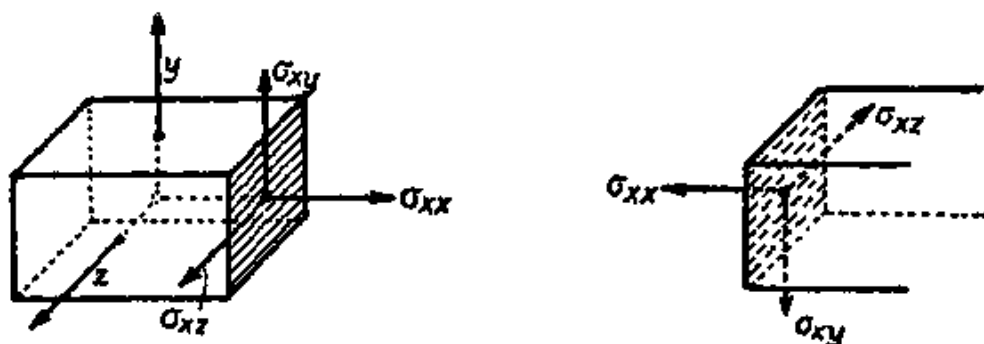


FIG. 6. Normal and shear stresses acting on a volume element of an elastic body. Positive and negative x -surfaces.

opposite direction. A positive σ_{xx} means here a stress in the negative x -direction. The same is true for the negative x -surface of the body element in the left part of the figure which is obtained by a mere displacement (through $-\Delta x$) of the negative x -surface considered before. In speaking of the entirety of quantities σ_{ik} characteristic for a volume element, we use the term *stress tensor* although the proof of the tensor character will be given in 8 only, by showing the symmetry of the σ_{ik} and their transformation rules. The term stress tensor is, by the way, redundant since the word tensor in itself implies tension.

The forces exerted by the surrounding elements of the body may be directed either toward the exterior or the interior of the body element under consideration. In the latter case (compression) it is more appropriate to speak of pressures than of stresses. We shall, however, retain the uniform terminology "stress" in the case of an elastic *solid*, and consider pressures as negative stresses.

Not so in the case of *fluids*. The cohesive forces in a fluid are weak, its "ultimate strength" under ordinary circumstances is exceedingly small if compared with that of a solid. In other words, the resistance of fluids against tension is so small that positive stresses practically never occur; we then prefer to deal in our equations with *pressures* p rather than stresses σ . In a sketch such as Fig. 6, the positive p_{xx} would have to be drawn as an arrow directed inward.³

The totality of pressures p_{ik} acting upon a fluid element will again be called a *tensor*. There is, however, an essential difference between the normal pressures p_{ii} and the tangential pressures p_{ik} , at least in the state of rest or of slow motion. A fluid layer may be moved along the adjacent layer without expenditure of a noticeable amount of energy provided the motion is slow; this means that the tangential pressures p_{ik} are quite small if the two layers are slowly displaced parallel to each other. By extension, we must assume that the tangential pressures vanish altogether in the state of rest. In this way a fundamental simplification of the pressure tensor p is accomplished which will be applied in 6 (hydrostatics).

For larger relative velocities of the fluid particles the complete pressure tensor must be studied and its dependence on the velocity gradient investigated. This will be done in 10 (viscous flow).

The concept of the fluid state is meant to cover also *gases* and *vapors*, the latter two being characterized by larger compressibility. Fluids in the restricted sense, or *liquids*, are essentially incompressible. Of course, they yield to a certain extent if subjected to strong external forces, but their compressibility is very small in comparison with gases or vapors and can be neglected in a great number of problems; so is water often considered as *incompressible*.

The *condition for incompressibility* reads $\Theta = 0$ according to (1.24); on passing from the displacement vector s to the velocity vector v as in (1.7), we obtain the relation

$$(1) \quad \operatorname{div} v = 0,$$

expressing the fact that the balance of outflow and inflow for a given volume element $\Delta\tau$ is zero at any time. If Δm denotes the mass contained in $\Delta\tau$, we have

$$(2) \quad \Delta m = \rho \Delta\tau$$

where ρ stands for the *density*. For homogeneous incompressible fluids ρ is constant in time as well as in space.

³It is only consequent to invert the direction of the arrows also for p_{xy} , p_{yz} , etc. so that p_{xy} is counted *positive* if it points in the *negative* y -direction. This rule will be followed from now on.

What corresponds to Eq. (1) in the case of a *compressible fluid* where not the volume but the mass is preserved? The mass Δm cannot decrease unless part of it passes through the boundary of the volume element $\Delta \tau$. The mass flow at any point of its boundary is represented by the normal component of the vector $\rho \mathbf{v}$. The total outflow of mass during the time element Δt is therefore [cf. e.g. (2.20)]

$$(3) \quad \int \rho v_n d\sigma \Delta t = \operatorname{div} (\rho \mathbf{v}) \Delta \tau \Delta t.$$

On the other hand, the loss of mass during the same time is found from (2) as

$$- \frac{\partial \rho}{\partial t} \Delta t \Delta \tau.$$

Comparison with (3) yields the following *condition for the conservation of mass*

$$(4) \quad \frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \mathbf{v}) = 0,$$

also known under the name of *equation of continuity*. It may be written in the form:

$$(4a) \quad \frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0$$

The first term in this equation is the *material differential quotient* which will be discussed more fully at the beginning of 11. Its definition is

$$(4b) \quad \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \operatorname{grad} \rho.$$

If ρ is constant in time *and* space, Eq. (4) takes again the form of the incompressibility condition (1). In an incompressible inhomogeneous fluid (ρ variable in space) we have by (4b)

$$(4c) \quad \frac{d\rho}{dt} = 0 \quad \text{that is} \quad \frac{\partial \rho}{\partial t} = -\mathbf{v} \cdot \operatorname{grad} \rho,$$

which describes the temporal fluctuations of ρ at a fixed point of observation, while the material rate of change of ρ vanishes.

We add here Eqs. (1), (4), and (4a) spelled out in Cartesian coordinates:

$$(1') \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$(4') \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0,$$

$$(4a') \quad \frac{d\rho}{dt} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0.$$

In problems involving capillary forces we are concerned with the fluid layers that are closest to the surface (surface film). The *surface of separation* introduced previously (see Fig. 6) is now to be replaced by a *linear cut* through the film. The object is again to study the reaction forces transferred from one bank of the cut to the other. Referring the forces to the unit length of the cut we speak of *surface tension*; its dimension is dyne/cm.

6. Equilibrium of Incompressible Fluids (Hydrostatics).

The subject of this section is, of course, not only water ("hydor" in Greek) but any liquid in equilibrium since differences in the rheological characteristics have no bearing on the *state of rest*: shear stresses, as already observed, are zero in equilibrium, or

$$(1) \quad p_{ik} = 0, \quad i \neq k.$$

This equation actually amounts to a *definition of the fluid* in the state of rest—in contrast to the state of solidification: glass which has no crystalline structure must be characterized physically as a *fluid in the solid state* for which Eq. (1) is, of course, incorrect.

A fundamental conclusion from (1) is that the hydrostatic pressure is associated with a *spherical* tensor quadric. Any diameter of the quadric may be taken as a principal axis, because (1) must be true in any coordinate system. The complete symmetry of the tensor of hydrostatic pressure can be expressed as follows. *At a given point the pressure acts on any surface element in the direction of its normal i , its magnitude being the same for all directions i ; $p_{11} = p_{22} = p_{33}$ for any three orthogonal directions.* Pascal⁴, it appears, was the first to perceive this law.

We can summarize this result in the statement that the hydrostatic pressure is a *scalar quantity*. It will be denoted by p , as is usually done with omission of the subscripts that are now abundant. Note that there is no preferential loading of the *horizontal* cross sections if a *vertical* force such as gravity acts on the fluid; the same pressure p is transferred

⁴Blaise Pascal (1624-1662), the great geometer (Pascal's theorem) and mathematician (Pascal's triangle, foundation of the calculus of probabilities), a great writer and theologian too; he studied the laws of air pressure and at his instigation the first barometrical determination of an altitude was carried out.

through any cross-section (also a vertical one). *The unidirectional character of the force does not interfere with the symmetry of the pressure.*

In order to calculate the pressure if magnitude and direction of the external force are given, let \mathbf{F} denote the external force per unit of fluid volume, so that the dimension of \mathbf{F} is dyne/cm³. Then the volume element $\Delta\tau$ is acted upon by the force $\mathbf{F}\Delta\tau$. On delimiting the volume element as rectangular cell ($\Delta\tau = \Delta x\Delta y\Delta z$) and marking the negative x -surface by the value x of the x -coordinate and, consequently, the positive x -surface by $x + \Delta x$, the forces in positive x -direction are:

	on the neg. x -surface	on the pos. x -surface	on the vol. element
(2)	$p(x)\Delta y\Delta z$	$-p(x + \Delta x)\Delta y\Delta z$	$F_x\Delta\tau$

The first and second contributions add up to

$$(3a) \quad -\frac{\partial p}{\partial x}\Delta x\Delta y\Delta z = -\frac{\partial p}{\partial x}\Delta\tau$$

and yield after division by $\Delta\tau$

$$(3b) \quad -\frac{\partial p}{\partial x} + F_x = 0;$$

this can be written in the form

$$(4) \quad \text{grad } p = \mathbf{F},$$

which is correct for any direction at any point of the liquid.

The relations (3b) or (4) for the force components obviously constitute a necessary condition for the equilibrium. From statics of rigid bodies a second condition is known requiring that the resultant moment be zero. This condition must also apply to non-rigid bodies, for, if we remove or relax the internal constraints of a rigid mechanical system, we obtain a non-rigid system with a vastly increased number of degrees of freedom. Hence, *quite generally, any equilibrium condition for a rigid body must be valid a fortiori for a deformable body.*

It is, however, easily seen that the second or moment equilibrium condition is automatically fulfilled in the case of a liquid. The normal pressure cannot produce a moment about any axis. Moments about the x -axis, for instance, could be produced only by the shearing stresses p_{xy} and p_{yx} , but they vanish. The external force \mathbf{F} cannot produce a moment either; this is certainly true when we assume a continuous distribution of lines of force; such a distribution can be always considered as parallel in any infinitesimal region (the field of gravity is a simple example for what is meant). If, on the other hand, the lines of the field \mathbf{F} have a singular

point at our volume element (a point where the field lines intersect), then \mathbf{F} must needs vanish there, and so does the moment of \mathbf{F} . Eq. (4) is in fact not only a *necessary* but also a *sufficient* condition of equilibrium.

Eq. (4) includes a very remarkable theorem: *equilibrium is only possible if the external force has a potential*, that is, if \mathbf{F} can be represented as the gradient of a scalar function. In this case we may write

$$(5) \quad \mathbf{F} = - \text{grad } U$$

where the minus sign is prompted by the relation to the potential energy, cf. Vol. I, appendix to 18.⁵ The *existence* of the potential function U is not sufficient, U must also be *single valued within the space occupied by the liquid*. Only under this assumption is it possible to calculate p from (4) as a single-valued scalar point function in the form

$$(6) \quad p + U = \text{const.}$$

The integration constant in (6) must be determined by special conventions concerning the normalization of the potential and the pressure.

Postponing the further discussion of this theorem to the end of this article, we deal for the present with the regular case in which the force \mathbf{F} has a single-valued potential and take gravity as an example. Let $z = 0$ be the surface of the liquid (call it "water" for simplicity) and orient the positive z -axis vertically downward. Then

$$(7a) \quad \mathbf{F} = F_z \mathbf{k}; F_z = \rho g = \gamma,$$

$$(7b) \quad U = -\rho g z = -\gamma z.$$

Here $\gamma (= \rho g)$ denotes the *specific weight* of the water as distinguished from the *specific mass* or density ρ . In writing (7b) the arbitrary constant in U has been adjusted so as to give zero potential at the surface. We obtain now from (6)

$$(8) \quad p = \gamma z + \text{const.},$$

where *const* is the pressure value at the surface $z = 0$, equal to the atmospheric pressure; however, p can also be given the meaning of the overpressure relative to the atmosphere (gauge pressure), which is more convenient for what follows. Let p have this meaning, then (8) becomes

$$(8a) \quad p = \gamma z.$$

This equation contains the well known elementary rules about communicating vessels, viz. equal pressure at equal depth in each of the

⁵See e.g., T. C. Slater and N. H. Frank, *Mechanics*, McGraw-Hill, New York, 1947, Sec. III,2.

interconnected vessels regardless of shape. The same equation may also be obtained directly from the differential formula (4). One follows the shape of the vessel along a broken line consisting of a series of vertical and horizontal segments inside the water and integrates. Along the horizontal segments $dp = 0$, along the vertical ones $dp = \gamma dz$. Summation of the vertical contributions gives immediately $p = \gamma z$ at the depth z .

Let us insert here a few basic numerical constants: normal atmospheric pressure = 76 cm Hg, density of Hg 13.596 gr/cm³; hence from (8a), $p_{atm} = 76 \times 13.596 \times g \text{ gr/cm}^2 = 1033 g \times \text{gr/cm}^2 = 1.033 \text{ kg-wt./cm}^2$. One cm² of a water surface carries a pressure load of about 1 kg-weight which makes 10 metric tons per m². The height of a water column corresponding to 76 cm Hg would be 10.33m, since for this height $p_{atm} = 1033 \times 1 \times g \text{ gr/cm}^2 = 1.033 \text{ kg-wt./cm}^2$ where $\rho = 1$ has been used.

We now consider an example that combines centrifugal force and gravity: a liquid in a drum (centrifuge) rotates with constant angular velocity ω about a vertical axis. The centrifugal force per unit of volume is

$$(9) \quad \mathbf{F} = F_r \mathbf{r}; F_r = \rho r \omega^2; \mathbf{r} \text{ is a unit vector } \perp \text{ to the axis.}$$

Its potential is given by

$$(9a) \quad U = -\frac{1}{2} \rho r^2 \omega^2.$$

The total potential of gravity and centrifugal force is (except for a constant) given by

$$U = -\rho g z - \frac{1}{2} \rho r^2 \omega^2 = -\gamma \left(z + \frac{r^2 \omega^2}{2g} \right).$$

From (6), the pressure in the rotating liquid is found as

$$(10) \quad p = \gamma \left(z + \frac{r^2 \omega^2}{2g} \right) + \text{const.}$$

To determine the constant in (10) let us first mark the water level in the axis of the drum, i.e., at $r = 0$, by z_0 . If p is again the overpressure, we have for $z = z_0$ and $r = 0$ from (10)

$$0 = \gamma z_0 + \text{const}, \quad \text{const} = -\gamma z_0$$

and in general

$$(11) \quad p = \gamma \left(z - z_0 + \frac{r^2 \omega^2}{2g} \right).$$

The free surface, characterized by $p = 0$, has the equation

$$(12) \quad z_0 - z = \frac{r^2 \omega^2}{2g}.$$

It is the well known paraboloid of rotation indicated in Fig. 7. Since z is positive downward, $z_0 - z$ is the ordinate of the parabolic meridian curve counted positive upward. On denoting by h the difference of the water

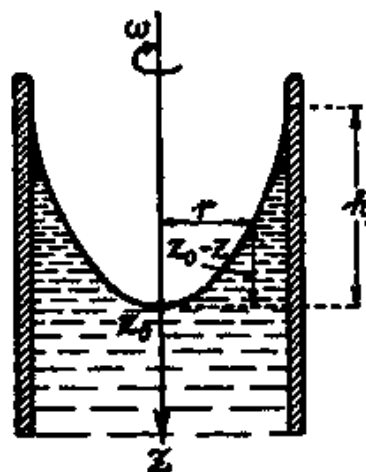


FIG. 7. The paraboloid surface of a fluid mass in rotation. The rise along the wall can be considered as velocity head.

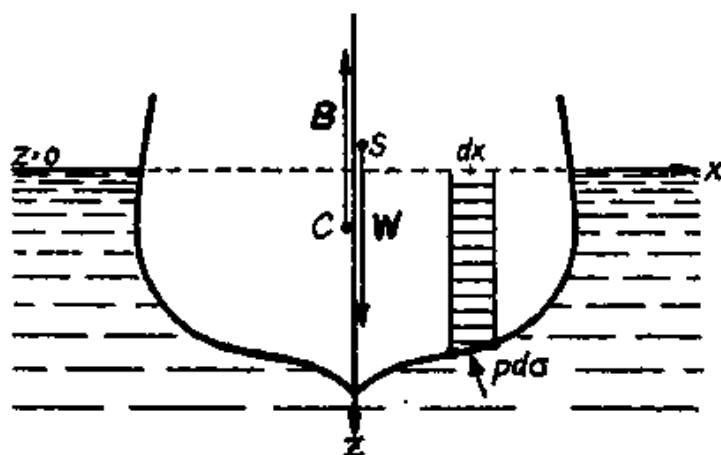


FIG. 8. The equilibrium of a prismatic hull. The buoyancy is the resultant of all pressure forces acting on the surface.

levels at the circumference and the center, and by v the circumferential velocity, h appears in the well known form of the velocity head,

$$(13) \quad h = \frac{v^2}{2g}.$$

The surfaces of constant pressure are congruent paraboloids, obtained by displacing the free surface vertically downward.

Next we turn to the problem of the *equilibrium of a boat* where again gravity is the only external force present. Fig. 8 represents a cross section normal to the longitudinal axis of the boat (positive y -axis forward). Let us think of the hull as a cylindrical surface with generatrices parallel to the y -axis. S is the center of mass of the boat, C the center of mass of the displaced water or of the "displacement", when the boat is upright.

Defining the buoyancy of the boat as the resultant of all pressure forces acting upon the wetted surface, we can show that this resultant passes through C . The point C is thus called the center of buoyancy.

Let $d\sigma$ be a surface element of the hull. According to the simplified form of the hull, $d\sigma = ds dy$ where ds is the line element of the cross section. The pressure γz acts in line with the surface normal of $d\sigma$, thus the pressure force is $\gamma z d\sigma$ and its vertical component is $\gamma z d\sigma \cos(n, z)$, where n is the inward normal of $d\sigma$. With $d\sigma \cos(n, z)$ being numerically

equal to the projection of $d\sigma$ upon the horizontal, the vertical component of the force element can be written $\gamma z \, dx \, dy$. The buoyancy is then found as

$$(14) \quad B = \gamma \int z \, dx \, dy = \gamma V,$$

where V is the volume of the displacement; $B = \gamma V$ expresses, of course, the principle of Archimedes. The vector \mathbf{B} points vertically upward and its *line of action* is obtained in exactly the same way as if the elements $\gamma z \, dx \, dy$ were vertical vectors; \mathbf{B} can therefore be found as the resultant of the gravity forces that act upon the columns $z \, dx \, dy$ and thus passes through the center of gravity C of the displacement. In equilibrium, C and S lie on the same vertical and \mathbf{B} balances the weight of the boat \mathbf{W} as indicated in Fig. 8.

The condition $B = \gamma V = W$ determines the draught of the boat; naturally, the draught increases with the ship load. Note that also the horizontal components of the pressure forces acting on two opposite elements $d\sigma_1$ and $d\sigma_2$ at the same depth z balance each other. This remains true, even if the boat is unsymmetrically loaded, and follows immediately from the equality of the two projections upon the vertical plane $\cos(n_1, x) \, d\sigma_1 = \cos(n_2, x) \, d\sigma_2$.

We now want to learn something about the *oscillations of the boat* if the equilibrium position is disturbed. This is still a question well within the bounds of hydrostatics; for, in good approximation to reality, we may assume that the pressure distribution at any phase of the motion does not differ greatly from the hydrostatic pressure distribution that would prevail if this phase were a possible rest position.

Let us apply this idea to the rolling motion of the boat (rotation about the longitudinal or y -axis). The symmetry plane of the boat now subtends a variable angle with the vertical which will be denoted by ϑ . During the roll the shape of the displacement and the position of C will change. Assume a roll of small amplitude and let C' be the center of mass of the displacement at the end of the roll, C'' the corresponding point when the roll has gone in the opposite direction. Let \mathfrak{M} be the center of curvature of the curve $C'C''$ at the point C . (In Fig. 9, \mathfrak{M} has been constructed in an approximate way by intersecting the normals at the points C' and C'' .) This point \mathfrak{M} is known as the *metacenter*. Our figure illustrates the instant of the motion when the buoyancy \mathbf{B} passes through C' . As it is, \mathbf{B} is to the right of the weight vector \mathbf{W} which always passes through S . The moment \mathbf{M} of the couple formed by \mathbf{B} and \mathbf{W} is indicated in the figure by a circular arrow that acts so as to decrease the angle of roll ϑ . This is the necessary and sufficient condition for the *stability of*

the boat. It is fulfilled whenever the metacenter \mathfrak{M} is *above* the center of mass S , (so drawn in the figure). If \mathfrak{M} were below S , the moment M would tend to increase the disturbance and finally make the boat capsize.

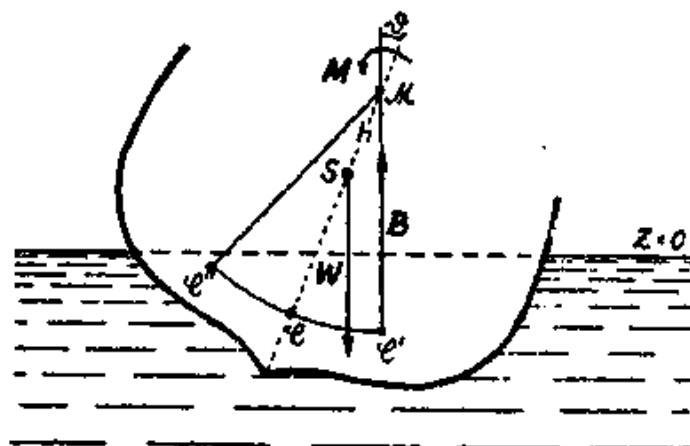


FIG. 9. Metacenter and restoring moment for the tilted hull.

The distance between S and \mathfrak{M} is called the metacentric height h . According to the figure one has for small ϑ

$$|M| = M_r = hW \sin \vartheta \sim hW\vartheta.$$

The differential equation of the rolling motion is therefore the same as that of a compound pendulum (see e.g. Vol. I, 16).^o With Θ denoting the moment of inertia of the boat about its longitudinal axis,

$$(15) \quad \Theta \ddot{\vartheta} = M_r = -hW\vartheta,$$

and the circular frequency of the oscillations

$$(16) \quad \omega = \sqrt{\frac{hW}{\Theta}}.$$

Practical values of h are about 3 to 5 times the distance CS .

The validity of the foregoing is limited, strictly speaking, to infinitesimal amplitudes, where only one metacenter \mathfrak{M} has to be considered. \mathfrak{M} is actually the limiting position of the metacenter for $\vartheta \rightarrow 0$. In naval architecture one has to investigate the stability also for finite ϑ values, and takes into account that the point \mathfrak{M} moves in reference to the boat when ϑ changes.

We now turn to a discussion of what happens in a field of force that has *no single-valued potential*. Instead of an exposition in general terms

^oCf. Synge and Griffith, *op. cit.*, Sec. 7.2.

let us consider a simple example which can be easily shown in an actual experiment. A weakly conducting fluid such as a solution of CuSO_4 is placed in a shallow cylindrical container with insulating bottom and conducting side wall. A copper wire runs along the axis of the container. A potential difference of a few volts is put between center wire and side wall so as to maintain a current flowing from the axis through the fluid and spreading out toward the wall. Let I be the current and \mathbf{J} the vector of the current density, $|\mathbf{J}|$ being the current passing radially through the unit of area at the distance r from the axis. Obviously

$$(17) \quad |\mathbf{J}| = \frac{I}{2\pi r h}$$

where h is the height of the fluid layer. Now we impose a fairly homogeneous magnetic field \mathbf{H} whose lines of force pass perpendicularly through the fluid layer. The electromagnetic force acting upon a fluid element of unit volume is then given by

$$(18) \quad \mathbf{F} = \mathbf{J} \times \mathbf{H},$$

a vector that is normal to \mathbf{J} and \mathbf{H} and points therefore in the tangential direction. On introducing the polar angle φ and considering (17) and (18), we have

$$(19) \quad |\mathbf{F}| = F_\varphi = \frac{A}{r}, \quad A = \frac{I \cdot H}{2\pi h}.$$

This force has the potential

$$(20) \quad U = -A\varphi.$$

The negative gradient of U taken with respect to the cylindrical coordinates r, z is indeed zero and for the φ -direction

$$-\text{grad}_\varphi U = -\frac{1}{r} \frac{\partial U}{\partial \varphi} = \frac{A}{r} = F_\varphi.$$

Yet this potential is obviously not single-valued in the (doubly connected!) domain of the fluid, since it changes its value by the amount $2\pi A$ for each revolution of the variable φ about the axis $r = 0$.

Since the hydrostatic pressure must be a single-valued point function, a pressure distribution that balances this potential in the sense of Eq. (6) cannot exist. Hence the fluid yields to the force F_φ and begins to move. The velocity pattern of the motion is the single-vortex motion described in (4.29); the circumferential velocity is in our case

$$|\mathbf{v}| = v_\varphi = \frac{A}{r} \frac{t}{\rho}$$

where t is the time through which the magnetic field has been acting and ρ the density of the fluid. The velocity would grow continually if there were no friction.

The mathematician who studies a multi-valued function tries to arrange its set of values in a number of different branches. He leads an appropriate cut through the domain of definition of the function to restrict the possible types of curves along which the independent variable may be changed, and thereby establishes a single-valued branch of the function. We can do the analogous thing in our experiment by putting a separating wall (wood or cardboard) into the fluid between the axis and the side wall. The fluid builds up on one side and recedes from the other, and the potential is now a single-valued function in the domain of the fluid; a hydrostatic pressure $p = A\phi$ is built up so as to balance the potential U , and the motion stops. The pressure distribution can be inferred from the shape of the surface; on one side of the separating wall, the level is by the amount $2\pi A$ higher than on the other side. The level difference is thus equal to the modulus of periodicity of the potential.

7. Statics of Compressible Fluids

The density of compressible gases and vapors depends on the pressure. Let the relation between the two have the form $p = C\rho^n$ which can be set up in reference to a normal state p_0, ρ_0 as follows:

$$(1) \quad \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^n.$$

The curve corresponding to this equation is known in technical and astrophysical applications as a *polytrope*; n is called the *polytropic exponent*.

In the isothermal state (constant temperature throughout the fluid), n equals 1. This is an immediate consequence of the equation of state

$$(2) \quad p = \rho \frac{R}{\mu} T,$$

where R = universal gas constant, μ = molar weight⁷ of the gas (e.g.,

⁷As is well known the number of grams in a *mole* equals the sum of the atomic weights of the gas. Instead of *molar* weight the term *molecular* weight is frequently used, but we wish to save this term in order to apply it to the actual weight (or more properly, the mass) of the single molecule. As R. W. Pohl has pointed out (Z.f.Phys., 121, 5-13, 1943) the definition of the mole depends on the definition of the unit of mass. If gr is replaced by kg, the mole changes into the "kilomole". The molar volume and Loschmidt's number per mole (Avogadro's number) change correspondingly (see later).

$\mu = 32$ gr for O_2), T = absolute temperature. In the *adiabatic* state (convective equilibrium such as results from a vigorous mixing process without heat exchange) $n = 1.4$ for diatomic gases.* In aeronautics a mean value of $n = 1.2$ is sometimes assumed (polytropic exponent of the atmosphere, see below).

It is known that the exponent $n = 1.4$ of the adiabatic state is equal to the ratio c_p/c_v of the specific heats at constant pressure and constant volume. This ratio follows from the first law of thermodynamics by means of an application of classical statistics (equipartition theorem of energy). The theoretical value is obtained as $c_p/c_v = 1 + 2/f$, where f is the number of degrees of freedom of the gas molecule. For a diatomic molecule such as N_2 or O_2 this number equals 5 (three degrees of freedom of translation and two of rotation; the rotation about the line connecting the two atoms cannot be excited for quantum-theoretical reasons), hence the value $n = 1 + 2/5 = 1.4$.

In the case of a compressible fluid, it is convenient to follow a different convention in measuring the external force. Previously referred to the *unit of volume* and denoted by F , we refer the external force now to the *unit of mass* and denote it by P . Thus

$$(3a) \quad F\Delta\tau = P\Delta m, \quad \text{hence } F = \rho P.$$

In the case of gravity where $|F| = \rho g$ we have simply

$$(3b) \quad \pm |P| = P_z = \pm g,$$

the sign depending on the choice of the orientation of the z -axis; in the case of the centrifugal force where $|F| = \rho r\omega^2$ as in (6.9) we have

$$(4) \quad |P| = r\omega^2.$$

The advantage of the new notation becomes apparent if we consider the

In CGS units the gas constant $R = 8.31 \times 10^7$ erg/centigrade; on the other hand, if one uses Meter-Kilogram-Second (MKS) units as Pohl does, the powers of ten drop out. In the same way as the unit of force in the MKS system is defined by 1 Dyne = 10^5 dyne, we can introduce as unit of energy in the MKS system

$$1 \text{ Erg} = 1 \text{ M}^2\text{KS}^{-2} = 1 \text{ M} \cdot 1 \text{ Dyn} = 10^7 \text{ erg} = 1 \text{ Joule}.$$

The gas constant is then $R = 8.31$ Erg/centigrade. Cf. the remarks about MKS-units in Vol. I, end of 1, also Slater and Frank, *op. cit.*, Sec. I.5.

*The pressure distribution within the gaseous mass is such that an arbitrary adiabatic displacement of a particle does not disturb its temperature equilibrium with the neighboring particles, cf. problem II, 1a.

equilibrium condition (6.4) which is correct for both compressible and incompressible fluids. It reads now

$$(5) \quad \frac{1}{\rho} \text{grad } p = \mathbf{P},$$

where the second member is a known quantity and the first member can be expressed in terms of either p or ρ , if a p, ρ -relation is known.

For the p, ρ -relation (1), Eq. 5 takes the form

$$(6) \quad \text{grad } \Phi = \mathbf{P}, \quad \text{where} \quad (6a) \quad \Phi = \frac{p_0^{1/n}}{\rho_0} \frac{p^{1-(1/n)}}{1 - (1/n)}.$$

The same Eq. (6) is correct for any non-polytropic p, ρ -relation provided we explain the quantity Φ by

$$(6b) \quad \Phi = \int_A^B \frac{dp}{\rho}.$$

Here A is a fixed point, B is the variable field point with coordinates x, y, z .

In the same way as (6.4), Eq. (6) implies that *in a compressible fluid, too, equilibrium is only possible when the external force, that is in the present case, the external force per unit of mass \mathbf{P} , has a potential*. On denoting the potential by V , we draw from (6)

$$(7) \quad \text{grad } (\Phi + V) = 0, \quad \Phi + V = \text{const.}$$

For the field of gravity $V = gz$, where z is counted positive upward. When the polytropic relation (1) is used, \mathbf{P} is given by (6), and the second relation (7) takes the form

$$(8) \quad \frac{p_0^{1/n}}{\rho_0} p^{1-(1/n)} = \frac{n-1}{n} (\text{const} - gz).$$

At $z = 0$, $p = p_0$ (reference pressure). The constant in (8) is found from $p_0/\rho_0 = \text{const } (n-1)/n$. Eq. (8) can then be written as

$$(9) \quad p = p_0 \left(1 - \frac{n-1}{n} \frac{\rho_0}{p_0} gz \right)^{n/(n-1)}.$$

On expressing p by ρ we obtain

$$(9a) \quad \rho = \rho_0 \left(1 - \frac{n-1}{n} \frac{\rho_0}{p_0} gz \right)^{1/(n-1)}.$$

Formulas (9) and (9a) represent the distribution of pressure and density in a polytropic atmosphere; its altitude h can be determined by setting $p = 0$ or $\rho = 0$:

$$(9b) \quad h = \frac{n}{n-1} \frac{p_0}{\rho_0 g}.$$

By reintroducing h in (9) and (9a) these equations can now be simplified and read

$$(10) \quad p = p_0 \left(1 - \frac{z}{h}\right)^{n/(n-1)}, \quad \rho = \rho_0 \left(1 - \frac{z}{h}\right)^{1/(n-1)}.$$

The distribution of the temperature can be found when the ratio p/ρ is calculated from (10) and the equation of state (2) is applied:

$$(11) \quad RT = \mu \frac{p_0}{\rho_0} \left(1 - \frac{z}{h}\right).$$

According to this, the temperature distribution in the polytropic atmosphere is a linear function of the altitude, dropping from $T_0 = \mu p_0 / \rho_0 R$ to $T = 0$ between $z = 0$ and $z = h$.

Numerical results for the altitude of the polytropic atmosphere are found from (9b): in the adiabatic case, $n = 1.4$, the altitude is only 28 km; in the polytropic case, when one chooses $n = 1.2$, one obtains 48 km; the altitude of the isothermal atmosphere is, of course, *infinite*.

The assumption $T = \text{const}$ is not permissible for the entire atmosphere. In the troposphere, the lowest part of the atmosphere, there is a very noticeable temperature lapse as one goes up; in the stratosphere, the temperature first remains constant over a considerable interval and increases later on. The boundary between troposphere and stratosphere is located at an average height of 12 km, at the poles it is lower than at the equator. For minor altitude differences the assumption of constant temperature may be maintained. The pressure and density distribution resulting in this case is well known under the name of the *barometric formula*. Since this formula is of general interest, it will be derived directly from the fundamental Eq. (5).

On setting $P_z = -g$ and writing (2) in the form

$$(12) \quad \rho = Cp, \quad C = \frac{\mu}{RT},$$

we obtain from (5)

$$\frac{1}{p} \frac{dp}{dz} = -Cg.$$

This gives on integration

$$(13) \quad \log p = -Cgz + \log p_0,$$

where p_0 is the pressure at the ground ($z = 0$); on solving for p , one has

$$(14) \quad p = p_0 e^{-Cgz} = p_0 e^{-\mu g z / RT}.$$

The same formula holds for the distribution of the density [comp. (12) or $\rho/\rho_0 = p/p_0$]:

$$(15) \quad \rho = \rho_0 e^{-\mu g z / RT}.$$

According to (13), the altitude may be read directly from a barometer if the instrument is furnished with an appropriate scale.*

One can interpret Eq. (15) in the sense of a sedimentation equilibrium of the air masses. It was Jean Perrin who successfully applied this idea to make a laboratory model of the atmosphere using a homogeneous emulsion of mastic droplets in water. His aim was to determine in this way a fundamental constant of atomic theory. That this can be done is evident as soon as μ in Eq. (15) is replaced by Lm , where L is Loschmidt's⁹ (or Avogadro's) number, i.e. the number of molecules per mole, and m is the mass of the single molecule; mg would therefore be the molecular weight in the proper sense of the word. Eq. (15) then becomes

$$(15a) \quad \rho = \rho_0 e^{-Lmgz/RT}.$$

While it is impossible to measure directly m of an actual molecule, Perrin's model gas permits of a direct determination of *its* m (e.g. by weighing a sufficiently large number of mastic droplets). Eq. (15a) can then be

*Eq. (14) or (15) can be derived from the general Eq. (9) or (9a) by making n approach the limit 1 and using

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n.$$

⁹Sometimes one uses the term "Loschmidt's number" in the sense of the number of molecules per cm³ and distinguishes it from "Avogadro's number" = number of molecules per mole, a definition of Loschmidt's number that was proposed by Boltzmann in a memorial address for Loschmidt (L. Boltzmann, *Populäre Schriften*, Leipzig 1905, p. 243). It is true that Loschmidt had focused his attention only on the first of the two numbers which he was able to estimate by several gas-kinetic methods, but he could have easily derived the second from the first number on the basis of Avogadro's law. Yet one must admit that not even the crudest estimate of the order of magnitude of either one of the numbers was within Avogadro's reach, if one considers the general state of knowledge of his time. Since the number of molecules *per mole* is a fundamental constant of all atomic physics, it appears reasonable to give a characteristic name to this number alone. Should it not be the name of the man who was the first to devise methods for its determination?

used for a determination of L if the densities ρ and ρ_0 have been observed. The value obtained by Perrin in this way is 68×10^{22} , which is remarkably close to the value $L = 60.2 \times 10^{22}$ accepted today.

When we introduce in (15a) instead of R the gas constant per molecule¹⁰

$$(15b) \quad k = \frac{R}{L},$$

the exponential in (15a) becomes

$$(15c) \quad e^{-mgs/kT} = e^{-V/kT}.$$

Note that V stands here for the potential energy of a mass point m that has been elevated to the altitude z in the field of gravity. The right member of (15c) is called the *Boltzmann factor*; it involves a specified energy V , but its physical meaning can be extended to the energy states of any physical system and represents, generally speaking, the relative probability of such a state in reference to a standard state of equal temperature.

The Boltzmann factor is fundamental in all statistical investigations; the purpose of this short digression into atomic theory has been to point to its connection with the elementary barometric formula.

Since the molar weight μ occurs in formula (15), a possibility arises to separate heavy from light molecules in a mixture of gases by sedimentation. Practically, this cannot be done in the field of gravity, but a centrifugal field may be effectively used for that purpose.

The potential of the centrifugal force (4) per unit of mass is

$$(16) \quad V = -\frac{1}{2}r^2\omega^2.$$

According to (7) we have for the quantity Φ

$$(16a) \quad \Phi = \text{const} + \frac{1}{2}r^2\omega^2.$$

Here we assume ω to be constant [as in (6.9a)] all over the inside of the centrifuge. On the other hand, the integral Φ for an isothermal gas or mixture of gases is found from (6b) and (12)

$$(17) \quad \Phi = \frac{RT}{\mu} \int \frac{dp}{p} = \frac{RT}{\mu} \log \frac{p}{p_0}.$$

Here p_0 is the pressure at the axis of the centrifuge ($r = 0$), and the con-

¹⁰Usually called Boltzmann's constant although Planck, following Boltzmann's ideas, was the first to introduce it definitely.

stant in (16a) vanishes. From (16a) and (17) the pressure distribution inside the centrifuge is obtained:

$$(18) \quad \log \frac{p}{p_0} = \frac{\mu}{RT} \cdot \frac{1}{2} r^2 \omega^2, \quad p = p_0 \exp \left(\frac{\mu}{2} \frac{r^2 \omega^2}{RT} \right).$$

The pressure is seen to increase exponentially with increasing r .

Consider now a mixture of two gases with molar weights μ_1 and μ_2 . Eq. (18) gives two different partial pressures p_1 and p_2 with two different constants p_{01} , p_{02} that depend on the mixing ratio. The total pressure is $p = p_1 + p_2$. According to Eq. (18), the partial pressure that belongs to the larger molar weight increases more rapidly than that belonging to the smaller μ , when r increases. Since for a given temperature the pressures and densities change proportionally, the heavier component of the mixture is concentrated in the proximity of the circumference of the centrifuge. The numerical magnitude—or rather minuteness—of this effect for air (21% O₂, 79% N₂) will be the object of problem II.2.

As in the incompressible case, we may consider gravity in addition to the centrifugal force; the *gravity potential* $-gz$ is then to be added in Eq. (16). Instead of (18) we have

$$(19) \quad p = p_0 \exp \left\{ \frac{\mu}{RT} \left(-gz + \frac{1}{2} r^2 \omega^2 \right) \right\}.$$

The surfaces of constant pressure (hence constant density) are again paraboloids, but p_0 is now the pressure at the particular paraboloid $gz = r^2 \omega^2 / 2$, from which the other isobaric surfaces are obtained by an upward or downward translation. Note that there does not exist a free surface toward vacuum ($p = 0$) since the exponential function does not become zero for finite values of the argument.

Design and construction of centrifuges of enormous efficiency have become possible through the researches of Th. Svedberg. The speed of these devices which now are commercially available under the name of ultracentrifuges goes up to 100,000 r.p.m., the strength of the centrifugal field reaches 750,000g. One of the purposes of these machines is to sort protein molecules (molar weight about 30,000). Their principal field of application is in colloidal chemistry and biology where the objects of investigation are approximately incompressible mixtures of liquids rather than compressible gases.

8. The State of Stress of an Elastic Solid

The mathematical characterization of the stresses in a solid body is complete when the stress is recognized as a symmetrical tensor of second

order. In other words, it must be shown that the array of stresses

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

is symmetric about the principal diagonal,

$$(1) \quad \sigma_{xy} = \sigma_{yx}, \text{ etc.,}$$

and that the components transform in the same way as the squares and products of the coordinates in passing to another Cartesian system [cf. Eq. (4.6)]

The symmetry can be easily inferred from the condition of moment equilibrium. Let us first set up the moment about the z -axis which acts on the volume element with the sides Δx , Δy , Δz , cf. Fig. 10. The normal

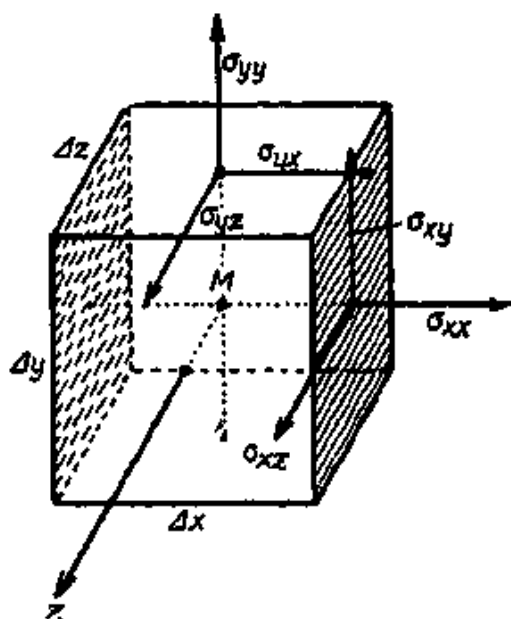


FIG. 10. The symmetry of the strain tensor. The resulting moment of the shear forces must vanish.

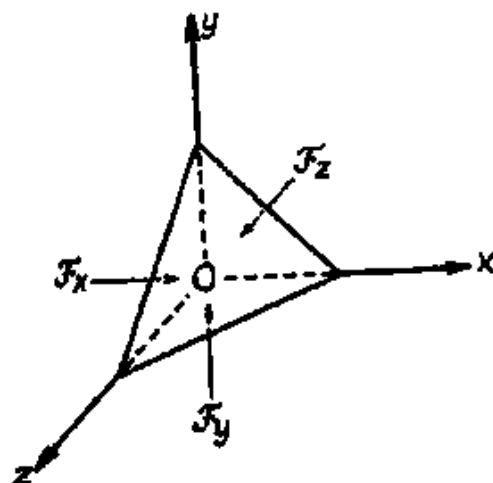


FIG. 11. Illustrating the equilibrium of forces acting on a tetrahedral volume element. F denotes the area.

stresses do not contribute anything to this moment since their vectors intersect with the z -axis. Consider now the tangential stresses σ_{xy} and σ_{yx} indicated in the figure. The corresponding forces transmitted to the volume element and the arms of these forces are

$$\sigma_{xy}\Delta y\Delta z, \quad \frac{1}{2}\Delta x \quad \text{and} \quad \sigma_{yx}\Delta z\Delta x, \quad \frac{1}{2}\Delta y.$$

For the signs of the resulting moments refer to Fig. 10, observing the convention that the direction of a stress component be indicated by its second index, when it acts on a "positive" surface element (cf. p. 38). The sum of the moments due to σ_{xy} and σ_{yx} then becomes

$$(2) \quad \frac{\Delta\tau}{2} (\sigma_{xy} - \sigma_{yx}).$$

This does not take care of the contributions of the shear stresses acting across the "negative" x - and y -surfaces of our volume element. Let σ'_{xy} and σ'_{yx} be these stresses; on expanding the first in terms of Δx , the second in terms of Δy , one sees immediately that $\sigma'_{xy} = -\sigma_{xy}$ and $\sigma'_{yx} = -\sigma_{yx}$ if terms of the order Δx and Δy are neglected. The contributions of these stresses to the moment about the z -axis is therefore once more the quantity (2) if only terms of the order of magnitude $\Delta\tau$ are retained. An additional external force \mathbf{F} (per unit of volume) will in general also be present; its moment is $|\mathbf{F}| \Delta\tau l$ where the arm l is of a smaller order of magnitude than the side length [see 6, after Eq. (4)]: it cannot possibly balance the reactive moments. Thus the equilibrium condition requires that the expression (2) vanishes by itself, but this is identical with the first symmetry condition in (1). The other two follow in the same way by considering moments about the x - and y -axes.

The proof of our second point, the *tensor character of the stress*, is more involved. It must be based on the *vector character* of the *reactive forces* that are transmitted across a surface element, just as the corresponding proof for the strain tensor was based on the vector character of the displacement.

We first set up the conditions for the equilibrium of forces acting on a tetrahedron, three faces of which coincide with the coordinate planes while the fourth is formed by a plane of arbitrary normal direction n (Fig. 11). Let the area of the oblique face be F_n , and F_x , F_y , F_z the areas of the faces lying in the planes $x = 0$, $y = 0$, $z = 0$ and consider these areas as projections of F_n :

$$(3) \quad F_x = F_n \cos(n, x), \quad F_y = F_n \cos(n, y), \quad F_z = F_n \cos(n, z).$$

The equilibrium condition for the x -direction is then¹¹

$$(4) \quad F_n \sigma_{nx} = F_x \sigma_{xx} + F_y \sigma_{yx} + F_z \sigma_{zx},$$

which because of (3) may be written

$$(5) \quad \sigma_{nx} = \alpha_1 \sigma_{xx} + \beta_1 \sigma_{yx} + \gamma_1 \sigma_{zx}.$$

¹¹One checks with the aid of Fig. 11 that F_x , F_y , F_z are *negative* x -, y -, z -surfaces; hence, in the total x -force, the terms with factors σ_{xx} , σ_{yx} , σ_{zx} are originally negative. In Eq. (4) they have been transferred to the right side with the *positive* sign.

Here α_1 , β_1 , γ_1 have been written for the direction cosines in (3) as indicated in the following scheme:

	n	t	t'
x	α_1	α_2	α_3
y	β_1	β_2	β_3
z	γ_1	γ_2	γ_3

Eq. (5) expresses the x -component of the stress acting across the n -surface. The y - and z -components can be determined analogously by setting up the equilibrium conditions for the y - and z -directions. One obtains

$$(6) \quad \sigma_{ny} = \alpha_1 \sigma_{xy} + \beta_1 \sigma_{yy} + \gamma_1 \sigma_{zy},$$

$$(7) \quad \sigma_{nz} = \alpha_1 \sigma_{xz} + \beta_1 \sigma_{yz} + \gamma_1 \sigma_{zz}.$$

The *normal stress* on the n -surface results now from (5), (6) and (7) by projecting the stress vector

$$(8a) \quad \sigma_{nx} \mathbf{i} + \sigma_{ny} \mathbf{j} + \sigma_{nz} \mathbf{k}$$

on the n -direction. The projection equals

$$(8) \quad \begin{aligned} \sigma_{nn} &= \alpha_1 \sigma_{nx} + \beta_1 \sigma_{ny} + \gamma_1 \sigma_{nz} \\ &= \alpha_1^2 \sigma_{xx} + \beta_1^2 \sigma_{yy} + \gamma_1^2 \sigma_{zz} + 2\alpha_1 \beta_1 \sigma_{xy} + 2\beta_1 \gamma_1 \sigma_{yz} + 2\gamma_1 \alpha_1 \sigma_{xz}. \end{aligned}$$

The directions t and t' already introduced may be realized as the legs of a right angle lying in the n -surface in an otherwise arbitrary position. We now project the stress vector (8a) also upon the directions t and t' and obtain the *tangential* or *shear stress* across the n -surface, resolved in a t - and t' -component:

$$(9) \quad \begin{aligned} \sigma_{nt} &= \alpha_2 \sigma_{nx} + \beta_2 \sigma_{ny} + \gamma_2 \sigma_{nz} \\ &= \alpha_1 \alpha_2 \sigma_{xx} + \beta_1 \beta_2 \sigma_{yy} + \gamma_1 \gamma_2 \sigma_{zz} \\ &\quad + (\beta_1 \alpha_2 + \alpha_1 \beta_2) \sigma_{xy} + (\gamma_1 \beta_2 + \beta_1 \gamma_2) \sigma_{yz} + (\alpha_1 \gamma_2 + \alpha_2 \gamma_1) \sigma_{xz} \end{aligned}$$

(for $\sigma_{nt'}$, change subscript 2 into 3). When numerals are introduced instead of the subscripts x, y, z and n, t, t' and α_{ik} is written for the nine

direction cosines, (8) and (9) and the corresponding relation for $\sigma_{n'}$ can be written in the condensed form

$$(10) \quad \sigma'_{i'h} = \sum_i \sum_m \alpha_{i1} \alpha_{hm} \sigma_{1m}.$$

This equation contains no reference to the size of the tetrahedron and gives, therefore, normal and tangential stresses acting across an arbitrary n -plane through O in Fig. 11. Its full significance is obtained if we disregard the particular roles of n , t , and t' . Eq. (10) is then seen to express the stresses acting on a cell parallel to the arbitrary orthogonal triad n, t, t' by the stresses on a cell parallel to x, y, z .

Since (10) has the form of the tensor transformation formulas as derived for the strain tensor in (4.5), we may from now on refer to the *stress tensor* and visualize the state of stress in the neighborhood of a point by means of a stress quadric (ellipsoid, hyperboloid, etc.) Magnitude and direction of the principal stresses are determined by the principal axes of the quadric. Any state of stress can be represented for a sufficiently small neighborhood of the field point as the sum of three mutually "orthogonal" tensors of simplest structure, viz. unidirectional tensions or compressions in correspondence with Helmholtz's theorem for the strains in 1 Eq. (16).

In setting up the equilibrium condition for the infinitesimal *tetrahedron* in (4), only the reactive forces transmitted through the surfaces, but no body forces have been considered. This is justified since the contributions of the latter are proportional to the *volume* while the stresses have the order of magnitude of the areas of the *faces*. The situation is different if we now turn to the equilibrium of external and internal forces for a small *rectangular cell*.* We want to establish a relation that is analogous to the fundamental equation (6.4) of hydrostatics. To secure equilibrium in x -direction as in (6.3) we have to consider the contributions of the normal stresses acting on the positive and negative x -surfaces and, *in addition*, those due to the tangential x -stresses across the positive and negative y - and z -surfaces; the latter do not occur in hydrostatics. The sum must be balanced against the contribution of the body forces represented, per unit of volume, by the vector \mathbf{F} . This yields, after cancellation of the factor $\Delta\tau = \Delta x \Delta y \Delta z$, the equilibrium condition

$$(11) \quad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + F_x = 0.$$

The corresponding equations for the y - and z -directions are obtained by "rotating" the letters. Note that in (11) the \mathbf{F} term has the same sign as

*Since the area terms due to opposite faces now cancel [cf. the remark following Eq. (2)].

the stress derivatives while in (6.4) the signs were opposite. The reason is obviously that the positive sign now denotes a tension, while it denoted a pressure previously.

The triplet of equations that corresponds to (11) reads in condensed form:

$$(12) \quad \text{Div } \sigma + F = 0.$$

The symbol Div stands for "vector divergence"; it derives a vector from a tensor while the operation div, formally explained in the same way, derives a scalar from a vector. In the general tensor analysis, where tensors of arbitrary order are considered, certain operations that decrease the order of a tensor are studied. The present case occurs there under the heading "differentiation with subsequent contraction".¹²

The definition of the symbol Div contained in (12) is obviously limited to Cartesian coordinates. It can be generalized for arbitrary orthogonal systems by the introduction of the limit of a surface integral as in (2.20). Enclose the field point, at which the p -component of the divergence vector associated with the tensor field σ is to be calculated, in a closed surface f with outward normal n . The generalized definition of Div is then

$$(13) \quad \text{Div}_p \sigma = \lim_{\Delta\tau \rightarrow 0} \frac{1}{\Delta\tau} \int \sigma_{np} df$$

where $\Delta\tau$ is the volume enclosed by f , and σ_{np} the component of the stress that acts across the n -surface element, taken in the direction p .

In the particular case of Cartesian coordinates we take for $\Delta\tau$ the rectangular cell $\Delta x, \Delta y, \Delta z$, and verify the new definition (13) in the same way as in the argument leading from Eq. (2.20) to (2.20a). In the general case of orthogonal coordinates [cf. Eq. (2.22)], the surface f is again chosen as the boundary of the curvilinear rectangular cell $g_1\Delta p_1, g_2\Delta p_2, g_3\Delta p_3$. The component of the vector divergence in the direction dp_3 follows from (13) as

$$(14) \quad \text{Div}_3 \sigma = \frac{1}{g_1 g_2 g_3} \left\{ \frac{\partial}{\partial p_1} (g_2 g_3 \sigma_{13}) + \frac{\partial}{\partial p_2} (g_3 g_1 \sigma_{23}) + \frac{\partial}{\partial p_3} (g_1 g_2 \sigma_{33}) \right\}.$$

The meaning of σ_{ik} is, of course, the component of the stress in the direction of dp_k acting across a surface element that is perpendicular to the direction of dp_i .

To obtain the k^{th} component of the equilibrium condition the expression (14) must be augmented by the component of F acting in the direction

¹²Cf. Appendix I-IV. General tensor analysis has been developed to serve as the adequate mathematical language of the general theory of relativity.

of the coordinate p_k . This sum replaces the left member of Eq. (12) in the case of general coordinates.

But the conditions of equilibrium are not sufficient to determine the tensor field σ from the vector field \mathbf{F} , since the tensor has six independent components while Eq. (12) is only equivalent to three scalar equations. The three missing equations are not supplied by the conditions of the moment equilibrium as one might be tempted to believe, since that condition has already been exploited in establishing the *symmetry* of the stress tensor. For the *determination* of the stress tensor the strain-stress relations must be used as will be done in the following §.

In this connection one may ask for supplementary conditions which together with (12) would ensure that the six quantities σ represent a physically possible state of stress. The answer is found in the *conditions of compatibility*. The same question could have been asked in the case of the strain field, since the six functions ϵ depend on the three displacement components ξ, η, ζ and, of course, may not be chosen arbitrarily.¹³

In conclusion we have again compiled the more frequent notations for the stress tensor:

This book	σ_{xx}	σ_{yy}	σ_{zz}	$\sigma_{xx} = \sigma_{yx}$	$\sigma_{yy} = \sigma_{xy}$	$\sigma_{zz} = \sigma_{zz}$
Many American authors	τ_{xx}	τ_{yy}	τ_{zz}	$\tau_{xx} = \tau_{yx}$	$\tau_{yy} = \tau_{xy}$	$\tau_{zz} = \tau_{zz}$
Love and many English authors	X_x	Y_y	Z_z	$Y_x = X_y$	$Z_y = Y_z$	$X_z = Z_x$
Kirchhoff and Planck	$-X_x$	$-Y_y$	$-Z_z$	$-Y_x = -X_y$	$-Z_y = -Y_z$	$-X_z = -Z_x$
Some English authors	P	Q	R	S	T	U
Engineering usage	σ_x	σ_y	σ_z	$\tau_{xy} = \tau_{yx}$	$\tau_{yz} = \tau_{zy}$	$\tau_{zx} = \tau_{xz}$

9. Strain-Stress Relations, Elastic Constants, Elastic Potential

In the present section the loads are supposed to be sufficiently small so that second and higher powers of the stress and strain components may be neglected. Then the general form of the relations between stress

¹³ Cf. Love, *Mathematical Theory of Elasticity*, 4th ed., Cambridge, 1927, pp. 40 and 135.

and strain components becomes *linear*. Hooke, a contemporary of Newton, was the first to state this "law", although in simpler formulation. About the limits of the linear law (proportional and elastic limits) some remarks will be found in Chap. VIII, 39.

We further assume that the elastic body is isotropic: its physical properties should be the same in all directions. Now it is well known that the structure of most solids such as minerals, metals and engineering materials in general is crystalline or microcrystalline. The following analysis refers therefore to "volume elements" that are large compared to the microstructure of the material, that is, what we call a volume element contains a great number of irregularly oriented microcrystals. The elastic properties of macroscopic crystals will be discussed in Chap. VIII.

A volume element cut parallel to the principal axes of the stress ellipsoid makes a convenient starting point for our analysis. The three pairs of faces are subjected to the principal stresses $\sigma_1, \sigma_2, \sigma_3$, while shear stresses σ_{ik} do not occur. It is then a first consequence of the assumed isotropy, that the shape of the volume element after loading is still rectangular. Thus there are no angular changes, the shear strains ϵ_{ik} are zero, and the state of strain consists in pure extensions along the principal axes of the stress ellipsoid. *Hence the principal axes of the stress quadric coincide with the principal axes of the strain quadric.* The principal extensions will be denoted by $\epsilon_1, \epsilon_2, \epsilon_3$, as in (1.14).

Now by the proposed linearity of the stress-strain relations the following relation between a stress component and the strain components is implied:

$$(1) \quad \sigma_1 = a\epsilon_1 + b\epsilon_2 + c\epsilon_3.$$

Since no axis is preferred, two further equations must be obtainable simply by rotation of the subscripts:

$$(1a) \quad \sigma_2 = a\epsilon_2 + b\epsilon_3 + c\epsilon_1, \quad \sigma_3 = a\epsilon_3 + b\epsilon_1 + c\epsilon_2.$$

Again for reasons of isotropy, c must equal b , in Eq. (1), otherwise the directions (2) and (3) would not be equivalent with regard to the stress σ_1 . It is convenient to rewrite (1) by adding $\pm b\epsilon_1$, viz.

$$\sigma_1 = (a - b)\epsilon_1 + b(\epsilon_1 + \epsilon_2 + \epsilon_3),$$

or, with a change in notation¹⁴

$$(2) \quad \sigma_1 = 2\mu\epsilon_1 + \lambda(\epsilon_1 + \epsilon_2 + \epsilon_3).$$

¹⁴Replacing the factor of ϵ_1 in Eq. (2) by 2μ is an advantage, as will be seen below [Eqs. (16) and (18)].

$$\epsilon_{xx} = \frac{1}{2\mu} \left(\sigma_{xx} - \frac{\lambda}{2\mu + 3\lambda} \Sigma \right) = 2\mu' \sigma_{xx} + \lambda' \Sigma$$

(6a)

$$\epsilon_{xy} = \frac{1}{2\mu} \sigma_{xy} = 2\mu' \sigma_{xy}.$$

or, condensed,

$$(7a) \quad \epsilon_{ik} = 2\mu' \sigma_{ik} + \lambda' \delta_{ik} \Sigma.$$

These are the general strain-stress relations and the constants appearing in (7a) are, in terms of μ and λ ,

$$(7b) \quad 2\mu' = \frac{1}{2\mu}, \quad \lambda' = -\frac{1}{2\mu} \frac{\lambda}{2\mu + 3\lambda}.$$

Eqs. (6) and (6a) show that the isotropic elastic body has *two elastic constants* which are physical characteristics of the material of the body and depend on the temperature. An anisotropic crystal has more than two elastic parameters; depending on the character of its asymmetry there may be up to 21 elastic constants, as will be seen in Chap. VIII. The quantities λ and μ , introduced by Lamé, are known as *Lamé's constants*, or better, *Lamé's moduli*.¹⁶ For general investigations they are the most convenient characteristics of the elastic behavior, but their physical meaning is not immediately obvious. The quantities λ and μ will now be expressed by two other constants that refer in a straightforward way to the simplest type of tension or compression experiment.

A vertical prismatic bar with cross-section F , the upper end of which is rigidly fixed, is subjected to a load P that acts at the lower end. Let the load P be uniformly distributed over the end cross-section. The tension across the end section (and across any other parallel section) is $\sigma = P/F$; σ is a *principal stress* since no side forces, and therefore no shear stresses, are active. The associated *principal extension* ϵ equals $\Delta l/l$, where l is the length of the bar and Δl the displacement of the end cross-section. It turns out that the ratio of stress to extension is a *constant of the material*, that is, it is independent of P , F , and l , as long as the loading is not excessive. This ratio is denoted by E and known as *Young's modulus*.¹⁷

¹⁶It stands to reason to have two different words for the coefficients of the stress-strain and those of the strain-stress relations. Following Voigt, the former, like (λ, μ) , could be called *moduli*, the latter, like (λ', μ') , *constants of elasticity* [cf. 40].

¹⁷Introduced 1807 by Thomas Young, the same who discovered the interference of light.

or *modulus of elasticity*. Thus, for a bar under tension, we have

$$(8) \quad E = \frac{\sigma}{\epsilon},$$

where E has the same dimension [force/area] as σ (ϵ is dimensionless). For steel and wrought iron an average value of E is 2×10^8 kg.-wt./cm².

Elongation, however, is not the only response to the loading that can be observed in the tension experiment; actually, it is always accompanied by a contraction of the cross-section. It is this same contraction which we notice on a much larger scale when a rubber band is stretched, and we take it as the natural adaptation of the material to the loading. Denoting the contraction, which is the same for all fibers of the cross-section, by $-\epsilon'$, in this case a positive number, we define quite generally

$$(9) \quad -\frac{\epsilon'}{\epsilon} = \nu \quad \text{or, using (8),} \quad \epsilon' = -\frac{\nu}{E} \sigma.$$

The quantity ν is also a *constant of the material*. It is called *Poisson's ratio* of transverse contraction to longitudinal extension.¹⁸

Poisson not only introduced this number, but actually determined its value as $\frac{1}{4}$, on the basis of a rather limited molecular theory. Modern lattice theory of the solid state has again taken up this problem and improved on Poisson's calculation; one of the results is that there is no universal theoretical value of ν . In engineering applications the accepted value for iron is $\nu = 1/m = 0.3$.

When the bar is under compression instead of tension, longitudinal contraction is accompanied by transverse dilatation, but E and ν remain the same in spite of the changes of sign that have occurred, provided, of course, that the loading is within the elastic limit (see also 39).

We now apply what we have learned in the simple tension experiment to analyze the general state of stress, which we consider as a superposition of three unidirectional stresses in the principal directions; the effect of each stress is the same as in the bar problem. Because of the linearity of all occurring relations we may superimpose the stresses as well as the associated strains. However, the extension along the principal axis 1 is not solely determined by the stress σ_1 , but in addition (if considered as a transverse contraction) by the stresses acting in the principal directions 2 and 3. In this way one obtains from (8) and (9)

$$\epsilon_1 = \frac{\sigma_1}{E} - \frac{\nu}{E} (\sigma_2 + \sigma_3) = \frac{1+\nu}{E} \sigma_1 - \frac{\nu}{E} (\sigma_1 + \sigma_2 + \sigma_3)$$

¹⁸The preferred letter for Poisson's ratio in American and English literature seems to be ν .

or, in condensed writing,

$$(10) \quad \epsilon_i = \frac{1+\nu}{E} \sigma_i - \frac{\nu}{E} \Sigma$$

On summing the relations (10) for $i = 1, 2, 3$, one has

$$(10a) \quad \Theta = \frac{1-2\nu}{E} \Sigma, \quad \Sigma = \frac{E}{1-2\nu} \Theta,$$

and by substituting Σ in (10) the result

$$(10b) \quad \sigma_i = \frac{E}{1+\nu} \left(\epsilon_i + \frac{\nu}{1-2\nu} \Theta \right)$$

is obtained. The relations which we wish to find result from a comparison between (10b) and (3). This leads to

$$(11) \quad 2\mu = \frac{E}{1+\nu}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},$$

which express Lamé's parameters by E and ν whose physical significance is more immediate.

Let us now take up another particularly simple case of a stress field, viz. *equal pressure in all directions* as it is realized, e.g., inside the cylinder of a hydraulic press. Both the stress and the strain ellipsoid are now spheres [see also 4 after (11)]:

$$\sigma_1 = \sigma_2 = \sigma_3 = -p, \quad \Sigma = -3p, \quad \epsilon_1 = \epsilon_2 = \epsilon_3 = -\frac{1}{3} |\Theta|.$$

Eq. (10b) gives in this case

$$(12) \quad p = \frac{E |\Theta|}{3(1-2\nu)}.$$

We define now in analogy to (8) a *modulus of compression* K by the equation

$$(13) \quad K = \frac{p}{|\Theta|}$$

and infer from (12) and (13) at once

$$(14) \quad K = \frac{1}{3} \frac{E}{1-2\nu}.$$

The *incompressible case* $\Theta = 0$ corresponds to $K = \infty$ or $\nu = \frac{1}{2}$. This value represents the upper limit for the possible values of Poisson's ratio. For $\nu > \frac{1}{2}$ the behavior of the body becomes *unstable*: its response to an

external pressure would be an increase of volume ($\Theta > 0$) instead of a decrease.

We conclude this discussion with another particular stress field, that of pure shearing stress. Take a rectangular parallelepiped, the z -surfaces of which are free of stress while the shearing stress $\sigma_{xz} = \sigma_{zx} = \tau$ acts across the x - and y -surfaces. Our example is one of plain stress, that is, the stress field is independent of z and can be represented by simply drawing a cross-section parallel to the x - y -plane as in Fig. 12a. The

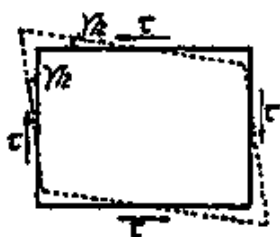


FIG. 12a. Change of shape in pure shear loading.

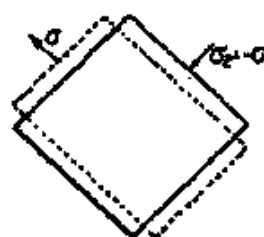


FIG. 12b. The principal stresses in pure shear loading are equal and opposite.

deformation due to τ changes the original rectangle into a rhomboid, one pair of right angles being diminished, the other pair increased by the angle γ . The stress quadric is, like the strain quadric in pure shear (cf. 4 Fig. 4a, b), a cylinder parallel to the z -axis the basis of which is an equilateral hyperbola. The principal axes form an angle of 45° with the x - and y -directions and the two principal stresses are numerically equal and of opposite sign: $\sigma_2 = -\sigma_1$; σ_3 is, of course, zero. This is a consequence of the invariance of the first scalar of the stress tensor, Σ : for the element considered originally (Fig. 12a), $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$ by hypothesis, hence an element cut parallel to the principal axes must take tension and compression in equal amounts to make again $\Sigma = \sigma_1 + \sigma_2 = 0$.

The state of pure shear stress serves to define the *shear modulus* which we provisionally denote by G , a notation generally accepted in engineering practice. In partial analogy to (8) and (13) we set

$$(15) \quad G = \frac{\tau}{\gamma}.$$

The analogy is not complete: the denominators in (8) and (13) represent the *extension* and (cubical) *compression* that correspond to the stresses in the numerators, but in the present case, (15), the denominator is the *angular change* produced by the stress that stands in the numerator, and this change is *twice* the strain component obtained according to the definition (1.25).

To find G , that is, to express it by the elastic constants already introduced, we go back to the general relation (6), $\sigma_{xy}/2\epsilon_{xy} = \mu$, which reads in our present notation $\tau/\gamma = \mu$. Thus (15) and (11) yield at once

$$(16) \quad G = \mu = \frac{1}{2} \frac{E}{1 + \nu}.$$

The shear modulus G is identical with Lamé's modulus μ so that the notation G is no longer needed. Also the name *torsion modulus* is in use for this constant because of its significance for the problem of torsional waves (see p. 107) and the torsion problem of a bar, 42.

We are now able to determine completely the stress from the equilibrium condition (8.12). This problem was posed at the end of 8, and it was pointed out that the *three* equilibrium conditions were insufficient for a complete determination of the *six* stress components σ_{ik} . But if we express the σ_{ik} by the ϵ_{ik} and write for the ϵ_{ik} their original definitions (1.11) in terms of the displacements, we obtain three equations just sufficient to determine the displacement vector.

That is an easy job if Cartesian coordinates are used. The x -component of the vector divergence in (8.12) transforms, according to the stress-strain relation (6), into

$$\text{Div}_x \sigma = 2\mu \left(\frac{\partial \epsilon_{xx}}{\partial x} + \frac{\partial \epsilon_{yx}}{\partial y} + \frac{\partial \epsilon_{zx}}{\partial z} \right) + \lambda \frac{\partial \Theta}{\partial x},$$

which becomes, on going back to displacements by means of (1.11),

$$\text{Div}_x \sigma = \mu \left(2 \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \eta}{\partial y \partial x} + \frac{\partial^2 \xi}{\partial z^2} + \frac{\partial^2 \zeta}{\partial z \partial x} \right) + \lambda \frac{\partial \Theta}{\partial x}.$$

This is easily rearranged into

$$(17) \quad \text{Div}_x \sigma = \mu \nabla^2 \xi + (\mu + \lambda) \frac{\partial \Theta}{\partial x}.$$

On substituting (17) in the equilibrium conditions (8.12) and rotating the letters we obtain finally

$$(18) \quad \begin{aligned} \mu \nabla^2 \xi + (\mu + \lambda) \frac{\partial \Theta}{\partial x} + F_x &= 0, \\ \mu \nabla^2 \eta + (\mu + \lambda) \frac{\partial \Theta}{\partial y} + F_y &= 0, \\ \mu \nabla^2 \zeta + (\mu + \lambda) \frac{\partial \Theta}{\partial z} + F_z &= 0. \end{aligned}$$

These three fundamental equations determine the displacement vector ξ, η, ζ everywhere in the interior of the body if its behavior along the body surface is determined by appropriate boundary conditions. When the ξ, η, ζ have been found, the ϵ follow by differentiation and the σ by way of the stress-strain relations. Thus Eqs. (18) are the definitive differential equations of elastic equilibrium. The quantity μ occurs in (18) without the factor 2, which again indicates that the choice of notation in Eq. (9.2) was to the purpose.

Eqs. (18) are limited to Cartesian coordinates. If we wish to write them in general coordinates, we have to interpret the symbol ∇^2 as $\text{grad div} - \text{curl curl}$ according to (3.10a). On denoting the displacement vector ξ, η, ζ by δq as in (4.17)-(4.28), the legitimate vectorial form of Eq. (18) is obtained:

$$\mu \text{ grad div } \delta q - \mu \text{ curl curl } \delta q + (\mu + \lambda) \text{ grad } \Theta + F = 0,$$

Instead of $\text{div } \delta q$ we may write Θ and obtain

$$(19) \quad (2\mu + \lambda) \text{ grad } \Theta - \mu \text{ curl curl } \delta q + F = 0.$$

The transition to general coordinates p_1, p_2, p_3 is now governed by the general expressions (2.24)-(2.26) for grad , etc. Setting $P = \text{curl } \delta q$ for brevity, one has immediately

$$\begin{aligned} (2\mu + \lambda) \frac{1}{g_1} \frac{\partial \Theta}{\partial p_1} - \frac{\mu}{g_2 g_3} \left(\frac{\partial g_3 P_3}{\partial p_2} - \frac{\partial g_2 P_2}{\partial p_3} \right) + F_1 &= 0, \\ (20) \quad (2\mu + \lambda) \frac{1}{g_2} \frac{\partial \Theta}{\partial p_2} - \frac{\mu}{g_3 g_1} \left(\frac{\partial g_1 P_1}{\partial p_3} - \frac{\partial g_3 P_3}{\partial p_1} \right) + F_2 &= 0, \\ (2\mu + \lambda) \frac{1}{g_3} \frac{\partial \Theta}{\partial p_3} - \frac{\mu}{g_1 g_2} \left(\frac{\partial g_2 P_2}{\partial p_1} - \frac{\partial g_1 P_1}{\partial p_2} \right) + F_3 &= 0. \end{aligned}$$

The quantities Θ and P that occur here have according to (2.25) and (2.26) the meaning

$$\begin{aligned} (20a) \quad \Theta &= \frac{1}{g_1 g_2 g_3} \left\{ \frac{\partial}{\partial p_1} (g_2 g_3 \delta q_1) + \dots \right\}, \\ P_1 &= \frac{1}{g_3 g_3} \left(\frac{\partial}{\partial p_2} (g_3 \delta q_3) - \frac{\partial}{\partial p_3} (g_2 \delta q_2) \right), \dots, \dots \end{aligned}$$

The boundary conditions to be added to Eqs. (18) or (20) may refer either to displacements or to stresses, or, as in the problem of 44, partly to

the one and partly to the other (mixed boundary value problem). When the *displacements* are prescribed along the entire surface of the elastic body by

$$(21) \quad \xi = \xi_0, \quad \eta = \eta_0, \quad \zeta = \zeta_0,$$

where ξ_0, η_0, ζ_0 are point functions given along the surface, the boundary value problem is similar to that of potential theory (p. 25) but considerably more complicated. In order to prescribe the *stresses* one assumes a system of external forces distributed over the surface of the body in the form $\mathbf{F} = \mathbf{f}d\sigma$ so that the "surface density" \mathbf{f} of the external forces is a finite point vector given along the surface. The surface stresses developed by the body must be balanced by the given vector \mathbf{f} everywhere on the surface. Hence we have for each surface element with normal direction n

$$(21a) \quad \sigma_{nx} + f_x = 0, \quad \sigma_{ny} + f_y = 0, \quad \sigma_{nz} + f_z = 0.$$

In the particular case that *no external forces* act on the surface, the boundary conditions become

$$(21b) \quad \sigma_{nx} = \sigma_{ny} = \sigma_{nz} = 0.$$

Such is the case for the sides of the bent beam in 41 or for the bar in torsion in 42. When the components of σ are expressed by the ξ, η, ζ in (21a, b), three boundary conditions result for the first derivatives of the ξ, η, ζ . Thus the integration problem is similar to that of the second boundary value problem of potential theory (cf. problem I, 5), but again considerably more difficult.

The uniqueness of solution of the problems mentioned, in particular for a simply connected body, can be proved as in potential theory essentially by an application of Green's theorem (cf. p. 25). In such a proof, homogeneity of the material, hence uniform temperature, and strictly elastic behavior are assumed. In reality, however, internal stresses are present even without external loads: they are caused by imperfections of the casting, inhomogeneous cooling etc. and are a constant source of trouble for the engineer; in applied elasticity, they constitute an unknown element which makes the strict uniqueness proofs of the mathematical theory somewhat unrealistic.

We turn now to the energetic aspect of the stress-strain relation and determine first the *strain-energy* function. The calculation proceeds again in Cartesian coordinates x, y, z and displacement components ξ, η, ζ .

Consider the two x -surfaces of the volume element $\Delta\tau = \Delta x \cdot \Delta y \cdot \Delta z$ while an infinitesimal increment of the displacement

$$\xi, \eta, \zeta \rightarrow \xi + d\xi, \quad \eta + d\eta, \quad \zeta + d\zeta$$

is imposed on the strain field. The work done by the stresses across the negative and positive x -surface is, in the same order,

$$dW_x = -(\sigma_{xx}d\xi + \sigma_{xy}d\eta + \sigma_{xz}d\zeta)\Delta y\Delta z$$

and

$$\begin{aligned} dW_{x+\Delta x} = & +(\sigma_{xx}d\xi + \sigma_{xy}d\eta + \sigma_{xz}d\zeta)\Delta y\Delta z \\ & + \frac{\partial}{\partial x}(\sigma_{xx}d\xi + \sigma_{xy}d\eta + \sigma_{xz}d\zeta)\Delta x\Delta y\Delta z. \end{aligned}$$

The algebraic sum of the two expressions is clearly the second line of the second equation. On adding the work done by the external force \mathbf{F} in x -direction, one obtains

$$(22) \quad \left\{ \frac{\partial}{\partial x}(\sigma_{xx}d\xi + \sigma_{xy}d\eta + \sigma_{xz}d\zeta) + F_x d\xi \right\} \Delta x\Delta y\Delta z.$$

Two analogous expressions are obtained for the work done in the displacement of the y - and z -surfaces. On adding all three contributions and carrying out the indicated differentiations, the terms that contain the (not differentiated) factors $d\xi$, $d\eta$, $d\zeta$ cancel the F -terms, if we make use of the equilibrium condition (8.12). The work dW per unit of volume is thus found as

$$\begin{aligned} (23) \quad dW = & \sigma_{xx}d\frac{\partial\xi}{\partial x} + \sigma_{yy}d\frac{\partial\eta}{\partial y} + \sigma_{zz}d\frac{\partial\zeta}{\partial z} + \sigma_{xy}d\left(\frac{\partial\xi}{\partial y} + \frac{\partial\eta}{\partial x}\right) \\ & + \sigma_{yz}d\left(\frac{\partial\eta}{\partial z} + \frac{\partial\zeta}{\partial y}\right) + \sigma_{zx}d\left(\frac{\partial\zeta}{\partial x} + \frac{\partial\xi}{\partial z}\right). \end{aligned}$$

In this calculation we have employed the symmetry of the stress tensor and the fact that $\partial d\xi$ (the gradient of the incremental displacement) is commutative with $d\partial\xi$ (the increment of the gradient of the displacement) (cf. 4, Fig. 5; d here corresponds to δ there). Eq. (23) can be written in the form

$$\begin{aligned} (24) \quad dW = & \sigma_{xx}d\epsilon_{xx} + \sigma_{yy}d\epsilon_{yy} + \sigma_{zz}d\epsilon_{zz} + 2\sigma_{xy}d\epsilon_{xy} \\ & + 2\sigma_{yz}d\epsilon_{yz} + 2\sigma_{zx}d\epsilon_{zx} = \sum_i \sum_k \sigma_{ik}d\epsilon_{ik}, \end{aligned}$$

when the strain tensor is introduced.

It is possible to express relation (24) in terms of the strain tensor alone, by means of the stress-strain relations. Substitution according to (7) yields the form

$$(25) \quad dW = 2\mu \sum_i \sum_k \epsilon_{ik}d\epsilon_{ik} + \lambda\theta d\theta.$$

This expression is a total differential as long as the "constants" λ and μ remain truly constant in the process of loading. We may then speak of a strain-energy function, without referring to the particular way of loading that starts from the (stress-free) initial state and leads to the final (stressed) state. In other words, the strain-energy is a *function of state*. It reads in terms of the strain components:

$$(26) \quad W = \mu \sum \sum \epsilon_{ik}^2 + \frac{\lambda}{2} \Theta^2.$$

As a function of state, W must be independent of the particular choice of coordinates x, y, z . This can be seen immediately through the following rearrangement:

$$\begin{aligned} \sum \sum \epsilon_{ik}^2 &= (\epsilon_{11} + \epsilon_{22} + \epsilon_{33})^2 \\ &\quad + 2(\epsilon_{12}^2 + \epsilon_{23}^2 + \epsilon_{31}^2 - \epsilon_{11}\epsilon_{22} - \epsilon_{22}\epsilon_{33} - \epsilon_{33}\epsilon_{11}) \\ &= \Theta^2 - 2\Delta, \end{aligned}$$

where Δ is the second scalar of the strain tensor, as explained in Eq. (4.9). Thus, instead of (26), we have the invariant representation

$$(26a) \quad W = \frac{2\mu + \lambda}{2} \Theta^2 - 2\mu\Delta.$$

A more symmetrical expression for W which also makes the invariance evident is

$$(26b) \quad W = \frac{1}{2} \sum \sum \sigma_{ik} \epsilon_{ik}.$$

The right member in this equation is the simplest *simultaneous invariant* of the two tensors σ and ϵ . On expressing σ by ϵ or vice versa [Eqs. (7) or (7a)] one obtains the quadratic forms

$$(27) \quad W(\epsilon) \quad \text{or} \quad W(\sigma).$$

The first is, of course, identical with (26), the second can be obtained from (26) by formally replacing ϵ, Θ by σ, Σ and μ, λ by μ', λ' from (7b).

The factor $\frac{1}{2}$ that occurs, e.g., in (26b) is in obvious connection with Hooke's law, that is, with the assumed *linear* relation between stress and strain.

From the quadratic forms (27) the stress and strain components can be reobtained; they are the partial derivatives of the energy W with respect to the corresponding strain and stress components:

$$\sigma_{ik} = \frac{\partial W(\epsilon)}{\partial \epsilon_{ik}}, \quad \epsilon_{ik} = \frac{\partial W(\sigma)}{\partial \sigma_{ik}}.$$

For this reason, the negative strain-energy W is also called the *elastic potential* per unit of volume.

We have already observed that μ and λ are temperature-dependent parameters. If the temperature is kept constant while the load is applied (isothermal transition), then μ and λ are true constants, and the preceding determination of W is correct. If on the other hand the transition is carried out without heat transfer to or from the individual volume elements of the body (adiabatic transition), then the strain energy can again be shown to be a function of state, but the "adiabatic" values of μ and λ are different from the "isothermal" ones. This difference is, as one would expect, especially noticeable in the gaseous state and will be discussed in 13 when the velocity of sound is calculated. In the more general case, as in a body with non-uniform temperature distribution, the strain energy is *not a function of state*, but depends on the path of the transition; the elastic potential does not exist.

10. Viscous Pressures and Dissipation, Particularly in Incompressible Fluids

The following analysis is based on an assumption about the nature of the viscous forces which, in the special form to be discussed first, is due to Newton. Let the velocity of the fluid $u(y)$ be everywhere parallel to the x -axis, increasing in some way with increasing y . The reactions due to viscous friction that are transferred through a surface normal to the y -axis are, according to Newton, proportional to the velocity gradient, the factor of proportionality μ depending on the nature of the fluid. We refer the reactive forces to the unit of area and designate them as frictional or viscous pressures. Newton's assumption then is

$$(1) \quad p_{yy} = -\mu \frac{\partial u}{\partial y}.$$

The minus sign as well as the notation p_{yy} requires some explanation. In Fig. 13 a surface SS has been indicated which, relative to the lower part of the fluid, is a positive y -surface, since the outward normal points in the direction of the positive y -axis. The fluid above SS , flowing faster, tries to take with it the fluid below SS ; the upper small arrow in Fig. 13 represents therefore the retarding action exerted on the upper fluid body by the lower one. Either arrow should be designated by σ_{yy} if we were to use stresses in our analysis, the upper one representing the $+x$ -traction across a $+y$ -surface, the lower one the $-x$ -traction across a $-y$ -surface. We use pressures, however, and this explains the minus sign in Eq. (1);

p_{yz} is negative for the positive y -surface (since $\partial u/\partial y$ is positive) and positive for the negative y -surface.¹⁹

Our aim is to generalize Newton's formula for the case of an arbitrary fluid flow. To do this we must go back to the fundamental theorem of kinematics of 1, which deals with the resolution of an infinitesimal displacement into translation, rotation, and deformation. The first two, as we know, correspond to rigid body motions and cannot involve internal friction. The friction is entirely due to the third part that can be de-

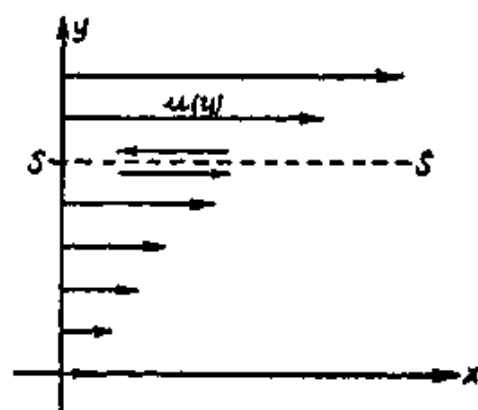


FIG. 13. Transfer of friction pressures in laminar flow. Newton's hypothesis.

scribed in terms of the strain tensor ϵ . Now, in a fluid the time rate of displacement rather than the displacement itself is the physically important quantity (see 5, p. 39), accordingly, we shall not consider ϵ , but $\dot{\epsilon}$ as responsible for the friction. In anticipation of the result of the subsequent discussion we assume now:

$$(2) \quad p_{ik} = -2\mu \dot{\epsilon}_{ik}, \quad \left. \begin{matrix} i \\ k \end{matrix} \right\} = x, y, z.$$

The factor of proportionality μ which has already occurred in (1) characterizes the internal friction and is called the *coefficient of viscosity*. The value of μ depends on the nature of the fluid and changes very strongly with the temperature (in liquids, increasing temperature makes μ decrease).

We show first that the set (2) is equivalent to (1) for the special flow of Fig. 13 where

$$u = u(y), \quad v = 0, \quad w = 0.$$

Applying (1.11) and the array (1.12) to the deformation velocities u, v, w rather than to the deformations ξ, η, ζ , we obtain

¹⁹Cf. the footnote on p. 30.

$$\begin{pmatrix} 0 & \frac{1}{2} \frac{\partial u}{\partial y} & 0 \\ \frac{1}{2} \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the content of Eq. (2) reduces to the following equations which are identical with (1):

$$p_{xy} = p_{yx} = -\mu \frac{\partial u}{\partial y};$$

the other p_{ik} vanish.

Assumption (2) is in agreement with the tensor-analytic principles considered in the preceding article. This is at least true for *incompressible fluids*, and we wish to limit the validity of (2) to that case. Then $\Theta = 0$ by (5.1), and the stress-strain relations (9.7) become identical with (2) as soon as the stresses are replaced by the pressures (change of sign) and the strains by the rates of strain. Hence one would conclude from the uniqueness of the stress-strain relations that a tensor-invariant relation between p and $\dot{\epsilon}$ other than (2) cannot exist. (The physical meaning of the viscosity coefficient μ has, of course, nothing to do with Lamé's modulus μ in the last article.)

First, however, the following point must be clarified: in the theory of the elastic solid we were able to show by an equilibrium consideration that σ is a *symmetric* tensor. The corresponding argument is questionable in the case of the viscous tensor p , since the necessary transition to a sufficiently small volume element is no longer legitimate in the present case, as we shall see below. Suppose then one would admit the possibility that

$$p_{ik} \neq p_{ki}.$$

Now, an unsymmetric pressure tensor could always be resolved into a symmetric part \bar{p} and an antisymmetric part π such that

$$\bar{p}_{ik} = \bar{p}_{ki} \quad \text{and} \quad \pi_{ik} = -\pi_{ki}.$$

The latter may be represented by an (axial) vector [cf. Eq. (1.12a)] and can be invariantly connected only with a quantity of the same character. The vortex velocity ω is such a quantity, and it is the only one that can be used. One would then have to postulate a static-kinematic relation of the form

$$(3) \quad \pi_{ik} = -\kappa \omega_{ik}.$$

Hence to admit the possibility of the asymmetry of the viscous tensor p amounts to the acceptance of a "vortex friction" as in (3), but this concept is inadmissible since it leads to a nonsensical result even in the simplest case.

Suppose a drum filled with some viscous liquid is made to revolve about its axis. The liquid starts to rotate at the circumference of the drum and the motion is gradually communicated to the bulk of the fluid. While this goes on a moment is required to overcome the internal friction of the liquid in addition to the moment required for the acceleration of the moving mass. In the *steady state* which is finally reached, the angular velocity ω is everywhere the same; at least, one may safely consider this as the final state since no experience points to the contrary. The liquid rotates as a rigid body and surely *no moment* is needed to sustain the motion if bearing friction etc. is disregarded. In contrast to that, Eq. (3) contends the presence of friction also in the steady state. Originating in the fluid, the friction would be conveyed to the drum and would have to be balanced by an external moment.

This whole discussion is of a *phenomenological nature* in accordance with the general point of view taken in this volume, and the same is true for the preceding tensor-geometric considerations. The deformable medium is being considered as a *continuum* and no attention given to the *molecular* origin of the stresses or pressures. The molecular theory of *compressible* fluids will be dealt with in the kinetic theory of gases. It furnishes a *direct explanation of the frictional pressures* in terms of the transport of momentum by thermal motion or, as it is sometimes put, in terms of the diffusion of momentum. Such a theory is physically altogether more profound; for compressible fluids it leads of *necessity* to the relation (21) between pressure and rate of deformation,⁴⁰ which will be found at the end of this article. That more general relation comprises (2) as a special case if the fluid is incompressible ($\Theta = \dot{\Theta} = 0$). On the other hand, the gas-kinetic concepts do not directly apply to *incompressible* fluids, that is, not unless arbitrary additional assumptions are made. There is, however, no doubt that also in the incompressible case the single volume element is exchanging momentum with more distant volume elements, and that this constitutes the physical cause of fluid friction. But it is just this idea of interaction by exchange of molecules that is in contradiction with the transition to an *isolated* small fluid element as in 8.

In addition to the frictional pressures and superimposed on them, there is a uniform normal pressure of the same nature as the pressure in the

⁴⁰Cf. in particular the critical remarks of E. Fues, Z. f. Physik 118, 409 (1941) and 121, 58 (1943) which start from ideas of Maxwell and Boltzmann, like most investigations in that field.

fluid at rest when the external force \mathbf{F} is present. We denote it by p as before (no subscripts, since it is a scalar), although it is not identical with the hydrostatic p (see below). The total pressure tensor is then

$$(4) \quad P = \begin{pmatrix} p + p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p + p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p + p_{zz} \end{pmatrix}$$

We are now ready to state the relation that is the analogue of the equilibrium condition (8.12). Although it is true that we are not concerned with an elastic body at *rest*, but with a *flow* of fluid particles, we may first assume that (in the frame of reference under consideration) the motion proceeds without change of magnitude and direction of the velocity. Each particle is then in a state of mechanical equilibrium, and we can formulate an "equilibrium condition" also in the present case. It is modelled after (8.12) and reads (with the proper change of sign)

$$(5) \quad \text{Div } P = \mathbf{F}.$$

Here Div stands for the "vector divergence", explained in (8.13) in an invariant way. Written in Cartesian coordinates, (5) takes the form

$$(6) \quad \begin{aligned} \frac{\partial p}{\partial x} + \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} &= F_x, \\ \frac{\partial p}{\partial y} + \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} &= F_y, \\ \frac{\partial p}{\partial z} + \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} &= F_z. \end{aligned}$$

These equations determine a steady fluid motion as soon as we express the six unknowns p_{ik} through the three velocity components u, v, w by means of (2). We obtain first

$$(7) \quad \frac{\partial p}{\partial x} - 2\mu \frac{\partial^2 u}{\partial x^2} - \mu \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \right) - \mu \left(\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 u}{\partial z^2} \right) = F_x,$$

which reduces on account of the condition of incompressibility

$$(8) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

to

$$(8a) \quad \frac{\partial p}{\partial x} - \mu \nabla^2 u = F_x.$$

The corresponding equations for the y - and z -directions are

$$(8b) \quad \frac{\partial p}{\partial y} - \mu \nabla^2 v = F_v,$$

$$\frac{\partial p}{\partial z} - \mu \nabla^2 w = F_z.$$

Eqs. (8), (8a), (8b) form a system of *four* equations for the four unknowns u, v, w, p . The pressure p is, of course, not equal to the hydrostatic pressure value, but is to be found from the quadruplet of equations simultaneously with the unknowns u, v, w .

The same Eqs. (8)-(8b) can also be considered as approximate equations for a flow where the particle accelerations do not exactly vanish, but are so small that their omission appears to be justifiable. This type of fluid motion—"creeping" one might call it—will be studied in 35, 36.

In both cases (either exactly or approximately free of acceleration), boundary conditions must be added to the differential equations to specify a problem, but there is a fundamental difference between the boundary conditions to be imposed on a perfect and on a viscous fluid. While in the first case only the boundary condition $v_n = 0$ is to be fulfilled and arbitrary tangential velocities v_t may be admitted along the rigid walls, we must stipulate in the second case that the fluid adheres to the surface of the rigid boundaries. This implies, of course, that v_t approaches zero *continuously* in the neighborhood of the boundary, since a jump of v_t would mean an infinitely large gradient of v_t and, consequently, *infinitely large values of the viscous pressures p_{xx}* . Experiments on the flow in narrow tubes (see below) have justified this form of boundary conditions which we state once more:

$$(9) \quad v_n = 0, \quad v_t = 0.$$

The fundamental difference between perfect and viscous fluids makes it understandable that the transition from small friction to the limit of vanishing friction is not without analytical difficulties. We shall come back to this point in 33 (Prandtl's boundary layer).

We now discuss the simplest application of the equilibrium conditions (8) and boundary conditions (9), viz., the classical Hagen-Poiseuille flow³¹ in a capillary tube. Let the horizontal tube have circular cross section with a radius a , so small as to make the flow proceed in straight stream

³¹G. Hagen, Poggendorffs Ann. 46 (1839). J. Poiseuille, Comptes Rendus 11 (1840) 12 (1841). Hagen was superintendent of works in Berlin, Poiseuille a physician in Paris. Both found independently the law (15), essentially by experimental investigations.

lines parallel to the axis of the tube (for more details about laminar flow and the conditions of its stability, cf. 16). When the axis of the tube is taken as x -axis and the distance $r = \sqrt{y^2 + z^2}$ from the x -axis is introduced, the velocity distribution is

$$(10) \quad v = w = 0, \quad u = u(r).$$

Eq. (8) shows then that u is independent of x ; because of the cylindrical symmetry, u is a function of r alone. Gravity may be neglected in our case, hence the force \mathbf{F} is zero. Eq. (8b) gives now

$$(11) \quad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0, \quad \text{therefore} \quad p = p(x).$$

Since the first term of (8a) depends only on x , the second only on r , and their difference is zero, either term must be a constant, say $-A$. Thus we have

$$(11a) \quad \frac{dp}{dx} = -A, \quad \nabla^2 u = -\frac{A}{\mu}.$$

According to this, the constant A is the pressure gradient along the tube, or, if multiplied with the length of the tube l , the pressure difference Π between beginning and end of the tube. We substitute for $\nabla^2 u$ its value in cylinder coordinates (problem I, 3) and note that there is no dependence on φ and on the axial coordinate:

$$\nabla^2 u = \frac{1}{r} \frac{d}{dr} r \frac{du}{dr}.$$

The conclusions following from (11a) are then

$$\frac{d}{dr} r \frac{du}{dr} = -\frac{A}{\mu} r, \quad r \frac{du}{dr} = -\frac{A}{\mu} \frac{r^2}{2} + C_1.$$

Here C_1 must vanish since otherwise du/dr would become infinite for $r = 0$. A second integration yields

$$(12) \quad u = -\frac{A}{4\mu} r^2 + C_2.$$

The constant C_2 is to be determined from the boundary condition (9) which now reads

$$(13) \quad u = 0 \quad \text{for} \quad r = a.$$

The liquid sticks to the wall of the tube. There is no doubt that (13) is correct: it follows unequivocally from the observations and holds for water as

well as for non-wetting liquids such as mercury on glass. According to (12),

$$C_2 = \frac{A}{4\mu} a^2,$$

so that (12) takes the form

$$(14) \quad u = \frac{A}{4\mu} (a^2 - r^2).$$

The velocity profile is parabolic (cf. Fig. 14). The particles that are at the time $t = 0$ on a plane cross-section, lie at $t > 0$ on a paraboloid of rotation



FIG. 14. Laminar flow in a capillary tube; the liquid adheres to the wall, the velocity profile is parabolic.

the vertex of which travels in the direction of the flow while the points of contact with the wall remain fixed.

The volume discharge Q per unit of time is

$$(15) \quad Q = 2\pi \int_0^a ur \, dr = \frac{\pi}{8} \frac{A}{\mu} a^4.$$

This formula has served as the basis for most determinations of viscosity constants in the last hundred years. The velocity of flow averaged over the cross section

$$(16) \quad u_m = \frac{Q}{\pi a^2} = \frac{A}{8\mu} a^2$$

is half the maximum velocity u_{\max} , [Eq. (14) with $r = 0$]; the pressure loss follows from (15) and (16) as

$$(17) \quad \Pi = 8\mu l \frac{u_m}{a^2}.$$

The pressure loss due to friction is proportional to the mean flow velocity and inversely proportional to the square of the radius. This law which, in modified form, is also correct for other laminar flows (cf. problems II, 3 and 4 and Figs. 19 and 19a on p. 121) is in characteristic contrast to the law for turbulent flow (see 16) where the pressure loss is approximately proportional to the square of the average velocity and inversely proportional to the radius of the tube.

We turn to the discussion of the energetic aspect of viscous flow. To calculate the work done by the friction forces we can use Eq. (9.22) as a starting point. Replacing the stresses σ by the pressures p (change of sign)

and the *virtual* displacements $d\xi$, $d\eta$, $d\zeta$, by the quantities $u dt$, $v dt$, $w dt$, which now are *actual motions occurring in the time element dt* , we obtain in place of (9.22)

$$\left\{ -\frac{\partial}{\partial x} [(p + p_{xx})u + p_{xy}v + p_{xz}w] + F_x u \right\} dt \Delta x \Delta y \Delta z.$$

This is the work done by the uniform pressure p and the friction pressures p_{ik} on the x -faces augmented by the work of the x -component of the external force. On adding the corresponding expressions in y and z and carrying out the differentiations, the resulting expression can be reduced by applying the conditions of equilibrium and incompressibility (6) and (8), and becomes after division by $dt \Delta x \Delta y \Delta z$

$$\begin{aligned} & -p_{xx} \frac{\partial u}{\partial x} - p_{xy} \frac{\partial v}{\partial y} - p_{xz} \frac{\partial w}{\partial z} \\ (18) \quad & -p_{xy} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - p_{yz} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) - p_{zx} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right). \end{aligned}$$

This energy is, as it were, abundant, it has no mechanical equivalent since no energy is needed for the maintenance of a steady fluid motion; it must therefore appear in the form of heat. *Expression (18) thus represents the heat produced by friction per unit of time and of volume (dissipation).*

Introducing a notation similar to the thermodynamic usage, we designate by dq the heat transferred to the unit of volume during the time dt . Using the definition of the strain components ϵ_{ik} in (1.11), we can transform (18) into

$$(19) \quad \frac{dq}{dt} = - \sum_i \sum_k p_{ik} \dot{\epsilon}_{ik}.$$

On applying the relations (2) between the p_{ik} and the $\dot{\epsilon}_{ik}$, we obtain for the dissipation

$$(20) \quad \frac{dq}{dt} = 2\mu \sum_i \sum_k \dot{\epsilon}_{ik}^2.$$

The quantity dq defined by (20) is, of course, *not* a total differential like the elastic dW of the corresponding equation (9.25), whence there is no thermal variable of state $\int dq$ that would correspond to the elastic strain energy (9.26).

In conclusion we wish to point out very briefly the changes which the foregoing considerations require in the case of *compressible* fluids. In the first place, comparison with the stress-strain relations in the form of (9.7) shows that the assumption (2) should be replaced by

$$(21) \quad p_{ik} = -2\mu \dot{\epsilon}_{ik} - \lambda \delta_{ik} \dot{\Theta}$$

because of $\dot{\Theta} = 0$. Thus we have a second coefficient of viscosity,²² λ , which is associated with the rate of dilatation $\dot{\Theta}$.

The equilibrium conditions (5) and (6) remain valid, if only \mathbf{F} is replaced by $\rho\mathbf{P}$ as in (7.3), but the form (8a) of these conditions must now be changed to

$$(22) \quad \frac{\partial p}{\partial x} - \mu \nabla^2 u - (\mu + \lambda) \frac{\partial}{\partial x} \dot{\Theta} = \rho P_x.$$

This equation together with the two others that are obtained by rotating the letters, and the equation of continuity (5.4) determine the four unknowns u , v , w , and p provided that a p, ρ -relation is prescribed. That the present set of equations is more involved is seen even in the case of the Hagen-Poiseuille problem for gases; one does not obtain an equation for u as above, but two simultaneous equations for u and ρ . As for the dissipation (20), it changes, as one would expect, to

$$(23) \quad \frac{dq}{dt} = 2\mu \sum \dot{\epsilon}_{ik}^2 + \lambda \dot{\Theta}^2.$$

²²No further determination of this quantity is intended here. In the literature one finds usually $\lambda = -2\mu/3$ on the basis of a gas-kinetic argument valid for monatomic gases (Enskog, Uppsala thesis, 1917). It is not difficult to draw the general conclusion from (21) that in the case of uniform compression an isotropic frictional pressure of the magnitude $(2\mu/3 + \lambda) \cdot \dot{\Theta}$ arises. The number $2\mu/3 + \lambda$ can therefore be designated as volume viscosity, in distinction from μ which could be called laminar viscosity. According to Enskog the volume viscosity of monatomic gases is zero, but the generalization of this result to gases whose molecules contain more than one atom is not justified.

CHAPTER III

DYNAMICS OF DEFORMABLE BODIES

11. Euler's Equations for a Perfect Incompressible Fluid

The transition from statics to dynamics is accomplished by adding the inertial resistances to the external forces in accordance with d'Alembert's principle. The inertial resistance of a particle with mass Δm is

$$-\Delta m \frac{d\mathbf{v}}{dt} = -\rho \Delta \tau \frac{d\mathbf{v}}{dt}.$$

Hence the inertial resistance per unit of volume becomes

$$(1) \quad -\rho \frac{d\mathbf{v}}{dt}.$$

This quantity must be added to the external force per unit of volume \mathbf{F} in the equilibrium conditions (6.4) if one wishes to obtain the equations of motion. Transposing the inertia term (1) to the other side of the equation, we have

$$(2) \quad \rho \frac{d\mathbf{v}}{dt} + \text{grad } p = \mathbf{F}.$$

Here one must carefully distinguish between the *total* or *material* acceleration $d\mathbf{v}/dt$ and the *local* acceleration $\partial\mathbf{v}/\partial t$. Take for instance the x -component of the velocity $u(x, y, z, t)$. While the particle moves through $d\mathbf{s} = i dx + j dy + k dz$ in the time dt , the change of u is

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz,$$

where the particle coordinates x, y, z are to be considered as functions of t . Accordingly, the material acceleration in x -direction along the path $dx = u dt, dy = v dt, dz = w dt$ becomes

$$(3) \quad \frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z},$$

and the difference between material and local acceleration in x -direction is

$$(3a) \quad \frac{du}{dt} - \frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z},$$

the meaning of the local acceleration being the rate of change of the velocity at a *specified point* of the flow. To illustrate the difference by an example, take the steady flow through a pipe of varying cross-section. Let the axis of the pipe coincide with the x -axis; then we are mainly concerned with the velocity component u . By hypothesis the flow is steady: hence $\partial u/\partial t = 0$ everywhere, but by no means du/dt . On the contrary, the velocity of flow increases where the pipe becomes narrower and decreases where it widens. The difference between the two accelerations is given by the right member of (3a), in particular, by the first term because of the assumed preponderance of u . The terms on the right side of (3a) are often called the *convective terms of the acceleration*.

Writing (2) in Cartesian components by means of (3) and the corresponding expressions obtained by rotation of letters, one finds

$$\begin{aligned}
 & \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = F_x, \\
 (4) \quad & \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) + \frac{\partial p}{\partial y} = F_y, \\
 & \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = F_z.
 \end{aligned}$$

The fourth differential equation is the condition of incompressibility

$$(4a) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The quadruplet of equations (4), (4a) constitute *Euler's equations¹ of perfect fluid motion*. The pressure p is not to be confused with the hydrostatic pressure; it appears as the fourth unknown beside u , v , w .

It suggests itself to abbreviate Euler's equations by writing them in the symbolic form

$$(5) \quad \rho \left(\frac{\partial}{\partial t} + (\mathbf{v} \text{ grad}) \right) \mathbf{v} + \text{grad } p = \mathbf{F},$$

where the symbol $(\mathbf{v} \text{ grad})$ could be replaced by $(\mathbf{v} \nabla)$ as on p. 23. But this form is misleading as soon as one tries to use it, as it stands, for non-Cartesian coordinates (see problem III, 1). The operation grad applies only to scalars and must not operate on vectorial quantities. We may,

¹Leonhard Euler (1707-1783). His first two papers on the equilibrium and motion of fluids appeared 1755 in Vol. 11 of the Berlin Academy, a later treatment 1770 in Vol. 14 of the Petrograd Academy.

however, define the *pseudo-vectorial symbol* $(\mathbf{v} \text{ grad}) \mathbf{v}$ by explaining it through a *legitimate vector formula*. The definition we have in mind reads

$$(6) \quad (\mathbf{v} \text{ grad}) \mathbf{v} = \text{grad} \frac{\mathbf{v}^2}{2} - \mathbf{v} \times \text{curl} \mathbf{v}.$$

In fact, the x -component of the first member of (6) can be rearranged in the form

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\ = \frac{1}{2} \frac{\partial}{\partial x} (u^2 + v^2 + w^2) + v \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + w \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \end{aligned}$$

which establishes right away the identity with the x -component of the second member of (6). The corresponding relations for the y - and z -components follow by rotating the letters.

Now we may replace Eq. (5) by

$$(7) \quad \rho \left(\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \text{curl} \mathbf{v} \right) + \text{grad} \left(\rho \frac{\mathbf{v}^2}{2} + p \right) = \mathbf{F}.$$

Adding the condition of incompressibility

$$(7a) \quad \text{div} \mathbf{v} = 0,$$

we have obtained Euler's equations in invariant form that can be specialized for any sort of curvilinear coordinates (e.g. polar coordinates) by the use of the expressions for grad, div, and curl in (2.24)-(2.26), or of the ready formulas in the tabulated solutions of problem I, 3.

Considering the mathematical character of Euler's equations we notice immediately their *non-linearity* which distinguishes them from the many linear equations of mathematical physics, as e.g. in potential theory, heat conduction, electrodynamics, etc. The non-linearity which is caused by the presence of the convective terms

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \quad \text{etc., or} \quad \mathbf{v} \times \text{curl} \mathbf{v} + \text{grad} \frac{\mathbf{v}^2}{2}$$

makes the integration incomparably more difficult, for we can no longer use the *principle of superposition* of solutions by which more general integrals are found in the form of a combination of particular integrals. The integration of Euler's hydrodynamic equations is thus a considerably more difficult mathematical problem than, for example, that of the seemingly more complicated equations of Maxwell in electromagnetic theory.

Only in the case of *irrotational* flow is it possible to give immediately a first integral of Euler's equations. The condition is

$$(8) \quad \omega = \frac{1}{2} \operatorname{curl} \mathbf{v} = 0;$$

it is sufficient that Eq. (8) be fulfilled at a certain time instant only, since in an inviscid fluid it then remains permanently fulfilled as will be shown in Chap. IV.

We shall at first consider a still more restricted case, viz. steady flow, so that

$$(8a) \quad \frac{\partial \mathbf{v}}{\partial t} = 0.$$

It is further assumed as in (6.5) that the external force \mathbf{F} has a potential

$$(8b) \quad \mathbf{F} = - \operatorname{grad} U.$$

Then (7) becomes

$$\operatorname{grad} \left(\rho \frac{v^2}{2} + p + U \right) = 0$$

and can be integrated to

$$(9) \quad \rho \frac{v^2}{2} + p + U = \text{const}$$

which is Bernoulli's famous equation.² It was found *before* the discovery of Euler's equations by an argument which could be considered as an anticipation of the energy principle. For Bernoulli, no other potential energy than that of gravity would count; with $U = \rho g z$ (z pos. upwards), Eq. (9) would read

$$(9a) \quad \frac{v^2}{2} + \frac{p}{\rho} + g z = \text{const.}$$

Bernoulli's equation is the most important theorem in elementary fluid dynamics and finds application in the solution of numerous technical problems of turbine design, aerodynamics, etc. The first term of Eq. (9) is the kinetic energy per unit of volume, the third term the potential energy of the external force, again per unit of volume. The pressure p in (9) appears, as it were, as the potential energy of internal forces that are active in the unit volume cell and account for the dynamic interaction

²Daniel Bernoulli, *Hydrodynamica*, Strasbourg 1738. This work contains also the first attempt at a molecular theory of gases and the first treatment of the bending of a beam (cf. 41).

between neighboring incompressible fluid elements. A deeper mathematical interpretation of p will be the object of the next article.

Let us now illustrate the physical content of (9a) by some simple examples. Imagine a tank filled with liquid having an orifice at the level of the bottom. Let the free surface be $z = 0$ and the bottom $z = -h$. As long as the orifice is closed, $v = 0$ everywhere in the tank and $p = 0$ at the surface (where p denotes the overpressure relative to the atmosphere). The constant in (9a) is then equal to zero, the pressure $p = \rho gh = \gamma h$ at the bottom and is identical with the hydrostatic pressure in 6. When the orifice is opened we may assume that, at the surface, v remains approximately zero so that $\text{const} = 0$ as before. But since p is now zero also at the orifice, Eq. (9a) gives for $z = -h$

$$(10) \quad v = \sqrt{2gh}$$

as in the case of a freely falling body. This is the well known content of *Torricelli's theorem*.³

Another example is the steady flow through a horizontal tube of variable cross-section. Since the volume flux through each cross section is the same for an incompressible fluid, v increases with decreasing cross section and vice versa, as already observed on p. 84. (The present v is essentially the same as the axial component u there). According to (9a) the pressure must exhibit the converse change; this is easily demonstrated if the tube is furnished with a series of vertical open manometers in which the liquid is allowed to rise. The levels in the manometer tubes indicate the variations of the pressure along the tube, showing smaller pressure when the cross-section is smaller. The comparison with the behavior of a crowd of people trying to force their way through a narrow passage is inevitable: the way the fluid does it is more rational.

The following little experiment which can be improvised at any time may also serve as an illustration of Bernoulli's equation. Put a piece of paper (say 2×2 inches) on your left hand and hold the second and third finger of your right hand closely above it. Blow now vigorously through the narrow slit formed by the two fingers against the center of the paper. Contrary to expectation, the paper is not pressed against the left hand but lifted toward the fingers of the right hand where it remains floating for some time. Explanation: the air that emerges between the fingers and the paper flows in a channel of increasing cross section (cf. Fig. 15). Its velocity decreases, hence the pressure increases. At the end of the channel there is atmospheric pressure, p_0 , hence the pressure in

³Torricelli was a pupil of Galilei, he lived before Bernoulli's time and could not have used Bernoulli's equation in obtaining this result.

the channel $p < p_0$. Since underneath the paper there is atmospheric pressure p_0 , an overpressure $p_0 - p$ acts from below, or as one might also put it, there is suction from above. When this experiment is carried out on a larger scale with a jet of compressed air directed against a plate that is constrained to move normal to the jet, then the plate can be heavily loaded, e.g., with one's own weight.

In this experiment and in the preceding one we have applied Bernoulli's equation to air, although the formulation given in (9) is only correct for incompressible fluids. Yet the error caused in this way is not very great as long as the velocity of flow is not too large. In general, the

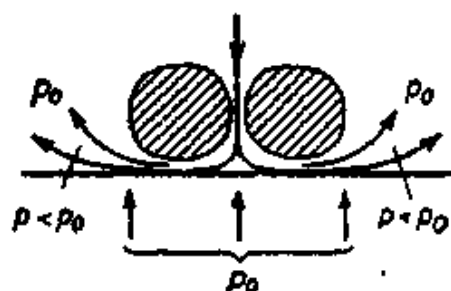


FIG. 15. Suction produced by blowing against a piece of paper through the slit formed by two fingers. The paper is not blown away but lifted toward the fingers.

corrections due to compressibility are small when the velocity of flow is small compared to the velocity of sound (cf. 13, and Appendix).

We have still to set up the generalized form of Bernoulli's equation in the case of a non-steady flow. Accordingly, we drop assumption (8a), but maintain (8) (irrotationality) and (8b) (\mathbf{F} has a potential). Now (8) is the necessary and sufficient condition for the expression $u dx + v dy + w dz$ to be a total differential, hence we may put

$$(11) \quad u dx + v dy + w dz = -d\Phi$$

and call Φ the *velocity potential*. Then

$$u = -\frac{\partial \Phi}{\partial x}, \quad v = -\frac{\partial \Phi}{\partial y}, \quad w = -\frac{\partial \Phi}{\partial z},$$

or, more briefly,

$$(12) \quad \mathbf{v} = -\text{grad } \Phi \quad \text{and therefore also} \quad \frac{\partial \mathbf{v}}{\partial t} = -\text{grad } \frac{\partial \Phi}{\partial t}.$$

It will be noticed that the negative sign in (11) and (12) is unessential;⁴ it is put in only to maintain the analogy with the potential of a force as in (8b).

Because of $\text{div } \mathbf{v} = 0$, Φ satisfies the potential equation (3.17)

$$(13) \quad \nabla^2 \Phi = 0.$$

⁴Its use has, in fact, been discontinued by many authors.

On account of (8), (8b), and (12), Euler's equations (7) take now the form

$$(14) \quad \text{grad} \left(-\rho \frac{\partial \Phi}{\partial t} + \rho \frac{v^2}{2} + p + U \right) = 0,$$

which is integrated to

$$(15) \quad -\frac{\partial \Phi}{\partial t} + \frac{1}{2} D\Phi + \frac{1}{\rho} (p + U) = \text{const.}$$

Here we have used the notation (3.9c) for the first differential parameter.

Eq. (15) is the *generalization of Bernoulli's equation for non-steady motions*. Like (9), it is a first integral of Euler's equations. Since (15) is found by spatial integration of (14), the constant in (15) is certainly independent of x, y, z , but it may in general still depend on the time. In other words, *const* is a function of t that has a uniform value for all points of the fluid. It must be found from the boundary conditions which may very well change with time. It will be noticed that this time dependence can be included in the definition of the velocity potential Φ since the potential equation (13) determines only the spatial, but not the temporal behavior of Φ .

So far we have only considered the case of *irrotational motion*; the question remains what can be done in the more general case where $\text{curl } \mathbf{v}$ is not zero.

A glance at equation (7) shows that also in this case integration becomes possible if one integrates in the *direction of a stream line*, or, in the non-steady case along the field line of the vector field \mathbf{v} . When this is done, the integral over the second term of the left member of (7) vanishes since $\mathbf{v} \times \text{curl } \mathbf{v}$ is perpendicular to \mathbf{v} and does not contribute anything to the integration. If we, moreover, maintain conditions (8a) and (8b) (steady flow in a field of force that has a potential) Eq. (9) is still correct, but in a different sense: the constant on the right side has no longer a uniform value for the entire space filled by the fluid, but changes from *one streamline to the next*. We shall call the equation (9), if understood in this sense, the *modified Bernoulli equation*. Its generalization for non-steady motion will not be discussed here.

12. Derivation of Euler's Equations from Hamilton's Principle The Pressure, a Lagrange Multiplier

There is no doubt that the concept of pressure in an *incompressible* fluid presents certain difficulties to the physical understanding. The pressure is considered a variable of state, but is denied any influence upon the way in which the fluid occupies the space, a difficulty, which is not

present in a compressible fluid where the pressure determines the density. We shall therefore try another approach toward understanding the concept of pressure in the case of an incompressible fluid, using the ideas of Hamiltonian mechanics.

We subject the mass particles of our fluid in motion to a *virtual* displacement

$$\delta \mathbf{s} = \delta \xi, \delta \eta, \delta \zeta,$$

which must not violate, however, the condition of incompressibility, $\Theta = 0$ (1.24). The expression

$$(1) \quad \delta \Theta = \frac{\partial \delta \xi}{\partial x} + \frac{\partial \delta \eta}{\partial y} + \frac{\partial \delta \zeta}{\partial z} = \text{div } \delta \mathbf{s}$$

must therefore vanish. This can be taken care of by providing (1) with a Lagrange multiplier λ and adding it to the integrand of Hamilton's principle. This we write in the form of Eq. (33.10) of Vol. I and obtain⁵

$$(2) \quad \int_{t_0}^{t_1} dt \int d\tau (\delta T + \delta W + \lambda \delta \Theta) = 0.$$

The integration with respect to $d\tau$ refers to the entire volume of the fluid. The variation of the kinetic energy δT and the virtual work of the external forces δW refer to the unit of volume in the same way as $\delta \Theta$ in (1). Hence we have

$$(3) \quad T = \frac{\rho}{2} \mathbf{v}^2, \quad \delta T = \rho \mathbf{v} \cdot \delta \mathbf{v}$$

and obtain for the virtual work

$$(4) \quad \delta W = \mathbf{F} \cdot \delta \mathbf{s}.$$

In (3) we substitute for the actual velocity

$$(4a) \quad \mathbf{v} = \frac{d\mathbf{s}}{dt}$$

and, accordingly, for its virtual variation

$$(4b) \quad \delta \mathbf{v} = \delta \frac{d\mathbf{s}}{dt} = \frac{d}{dt} \delta \mathbf{s}.$$

We now transform the term δT by partial integration with respect to t , observing that $\delta \mathbf{s}$ vanishes at the limits t_0 and t_1 according to the operational rules of Hamilton's principle. Thus we obtain

⁵See Jeffreys and Jeffreys, *op. cit.*, 10.06.

$$(5) \quad \int_{t_0}^{t_1} \rho \mathbf{v} \cdot \delta \mathbf{v} \, dt = \int_{t_0}^{t_1} \rho \mathbf{v} \cdot \frac{d}{dt} \delta \mathbf{s} \, dt = - \int_{t_0}^{t_1} \rho \frac{d\mathbf{v}}{dt} \cdot \delta \mathbf{s} \, dt.$$

The last term in the integrand of (2) is transformed according to the relation

$$\lambda \operatorname{div} (\delta \mathbf{s}) = \operatorname{div} (\lambda \delta \mathbf{s}) - \operatorname{grad} \lambda \cdot \delta \mathbf{s}.$$

On integrating and using Gauss's theorem we obtain

$$(6) \quad \int \lambda \operatorname{div} (\delta \mathbf{s}) \, d\tau = \int \lambda \delta s_n \, d\sigma - \int \operatorname{grad} \lambda \cdot \delta \mathbf{s} \, d\tau.$$

The surface integral on the right side will be taken up eventually; right now we are only concerned with

$$(7) \quad \int \lambda \operatorname{div} (\delta \mathbf{s}) \, d\tau = \dots - \int \operatorname{grad} \lambda \cdot \delta \mathbf{s} \, d\tau.$$

Introducing (1), (4), (5) and (7) in Hamilton's integral (2), we have

$$(8) \quad \int_{t_0}^{t_1} dt \int d\tau \left[\left(-\rho \frac{d\mathbf{v}}{dt} + \mathbf{F} - \operatorname{grad} \lambda \right) \cdot \delta \mathbf{s} \right] + \dots = 0.$$

Thanks to the introduction of the multiplier λ the variation $\delta \mathbf{s}$ may be chosen arbitrarily within the volume element τ and the time interval t_0 to t_1 ; but, with arbitrary displacements, Eq. (8) can only be fulfilled, if the volume and the surface integral vanish separately. Thus the factor of $\delta \mathbf{s}$ in the scalar product must be zero everywhere inside the domain of integration:

$$(9) \quad \rho \frac{d\mathbf{v}}{dt} + \operatorname{grad} \lambda = \mathbf{F}.$$

This is Euler's equation written in the form (11.2) provided λ is identified with the pressure p ; hence from the point of view of general mechanics, the hydrodynamic pressure represents the reaction against the condition of incompressibility, since λ was introduced to eliminate that condition. This corresponds to the way in which λ was interpreted in the case of the spherical pendulum in Vol. I. Eq. (18.7) where the constraint was a rigid spherical surface on which the mass point had to remain. The multiplier λ turned out to be a measure of the force that keeps the mass point on the constraint. In the present investigation, the condition of incompressibility has also been shown to have the character of a rigid constraint without energetic consequence; this may help to remove the difficulty associated with the concept of pressure in an incompressible fluid.*

*See e.g., Joos, G., *Theoretical Physics*, Hafner, New York, 1934, p. 110.

We should, however, be able to base our argument on the usual form of the incompressibility condition and its variation

$$(10) \quad \operatorname{div} \mathbf{v} = 0. \quad \delta \operatorname{div} \mathbf{v} = \operatorname{div} \delta \mathbf{v} = 0$$

rather than on condition (1). Retaining the expressions (3) and (4) for δT and δW , we rewrite Hamilton's principle (2) in the form

$$(11) \quad \int_{t_0}^{t_1} dt \int d\tau \{ \rho \mathbf{v} \cdot \delta \mathbf{v} + \mathbf{F} \cdot \delta \mathbf{s} + \lambda' \operatorname{div} \delta \mathbf{v} \} = 0.$$

We have denoted the multiplier by λ' to indicate that it is not identical with our previous λ (if for no other, then for dimensional reasons). We must now transform the middle term in (11) so as to have the same independent variation $\delta \mathbf{v}$ everywhere. We put

$$(12) \quad \bar{\mathbf{F}} = \int \mathbf{F} dt$$

and obtain

$$\int_{t_0}^{t_1} dt \int d\tau \mathbf{F} \cdot \delta \mathbf{s} = - \int_{t_0}^{t_1} dt \int d\tau \bar{\mathbf{F}} \cdot \delta \mathbf{v};$$

in the last transformation Eq. (4b) was used and the fact applied that the terms, which occur in the partial integration and refer to the limits t_0 and t_1 , vanish as before in (5). When we also transform the term with the factor λ' as in (7), the volume integral in (11) becomes

$$\int_{t_0}^{t_1} dt \int d\tau (\rho \mathbf{v} - \bar{\mathbf{F}} - \operatorname{grad} \lambda') \cdot \delta \mathbf{v}.$$

The same conclusion that leads from (8) to (9) leads now to

$$(13) \quad \rho \mathbf{v} - \bar{\mathbf{F}} - \operatorname{grad} \lambda' = 0.$$

It is only necessary to differentiate this equation with respect to t and to observe the meaning of $\bar{\mathbf{F}}$ in (12), to reobtain Euler's equations (ρ is, of course, independent of t because of the incompressibility). The result is

$$(14) \quad \rho \frac{d\mathbf{v}}{dt} + \operatorname{grad} p = \mathbf{F} \quad \text{with} \quad p = - \frac{d\lambda'}{dt}.$$

In this argument the pressure is again given by the *Lagrange multiplier* λ' although in a form different from the one we had before. This should not be surprising in view of the remark following (11).

We still have to consider the surface integral occurring in Eq. (6) which should give us the *surface conditions* required for the complete

determination of the pressure. On replacing λ by p and δs_n by δv , the integral reads

$$(15) \quad \int p \delta v d\sigma.$$

Before continuing, we wish to broaden the physical basis of this argument by considering another expression of the same structure as (15) which takes care of the cohesion of the fluid. In doing so we must for once resort to ideas taken from molecular physics. The fluid particles act upon each other not only with the pressure p (a consequence of their permanent volume as we may put it now), they also interact with their close neighbors through short range attractive forces which have their origin in the electrical structure of the molecules. For an inner volume element these forces cancel each other because they are in average directionally uniform. Not so for an element adjacent to the surface, where these forces add up to a resultant N in the direction of the inward surface normal. If N is referred to the unit of surface layer, the virtual work done by N in the displacement δv equals

$$\delta W = -N d\sigma \delta v.$$

Thus there appears in Hamilton's principle in addition to the *volume* integral over δW , in which the cohesive forces cancel, the *surface* integral

$$(16) \quad \int \delta W d\sigma = - \int N \delta v d\sigma.$$

This and the surface integral (15) (which resulted from the condition of incompressibility) can be written as one integral, viz.

$$(17) \quad \int (p - N) \delta v d\sigma.$$

According to Hamilton's principle the time integral of (17) must vanish together with the first term in (8), whence

$$(18) \quad \int_{t_0}^{t_1} dt \int (p - N) \delta v d\sigma = 0.$$

The conclusions to be drawn from (18) can be arranged according to the following three types of possible boundaries:

- a) The fluid is bounded by a rigid wall.
- b) The fluid has a free surface; it is bounded by vacuum or air.
- c) The fluid is bounded by another fluid, e.g., oil above water, but the boundary surface may have arbitrary shape.

a) A rigid wall prevents the fluid from moving perpendicularly to it; if this be true for the real motion, it must also be valid for the virtual displacement. Hence $\delta v = 0$ and condition (18) is automatically fulfilled. *The rigid wall does not provide a surface condition for the pressure.*

b) At a free surface δv is arbitrary. Eq. (18) requires then

$$(19) \quad p = N$$

for each element $d\sigma$ of the free surface at any time t .

c) At the surface of separation between fluid 1 and 2 with hydrodynamic pressures p_1 , p_2 and cohesive pressures N_1 , N_2 , we first require according to (18) that for any $d\sigma$ and any t

$$(20) \quad (p_1 - N_1)\delta v_1 + (p_2 - N_2)\delta v_2 = 0.$$

However, the contact of the fluids must not be disturbed by the virtual displacements. In other words, there must be

$$\delta v_1 = -\delta v_2,$$

the negative sign originating in our convention about the direction of n . Thereupon (20) gives

$$(21) \quad p_1 - p_2 = N_{12}, \quad N_{12} = N_1 - N_2.$$

Along the surface of separation of two fluids there is a pressure difference caused by the cohesive forces. It depends on the nature of the fluids and on the shape of the surface of separation, but is independent of the motion of the fluids.

Thus not only the differential equations of the problem (in the present case Euler's equations) follow from Hamilton's principle, but also such surface conditions for the pressure as may be imposed, can be found in the same way; this will be taken up again in 17 (surface tension).

13. Euler's Equations for the Perfect Compressible Fluid and Their Application to Acoustics

The transition from statics to dynamics is made as before in 11 by adding the inertial resistance to the external force. We have found it an advantage in the compressible case, to refer the external force to the unit of mass rather than to the unit of volume, hence we should do the same with the inertial resistance which is then simply $-dv/dt$. Replacing P in the equilibrium condition (7.5) by $P - dv/dt$, we obtain

$$(1) \quad \frac{dv}{dt} + \frac{1}{\rho} \text{grad } p = P.$$

These equations supplemented by the equation of continuity (5.4)

$$(2) \quad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$

constitute *Euler's equations for the compressible fluid*. The p, ρ -relation may be represented by the sufficiently general polytropic relation (7.1). The pressure p as well as the density ρ do now depend on x, y, z , and t .

In acoustics we are usually concerned only with *small oscillations* of the air. The difficulties arising from the non-linear terms (11.3a) in Euler's equations can then be avoided: we replace

$$\frac{d\mathbf{v}}{dt} \quad \text{by} \quad \frac{\partial \mathbf{v}}{\partial t}.$$

Quadratic terms must be kept, however, when we investigate, as in 37, shock-like phenomena such as occur in the firing of a gun. In the present case it is useful to take the normal atmospheric pressure p_0 as reference pressure for p ; p_0 will then not occur in (1), since it is constant in space and drops out if the gradient is formed, while p has the character of a small disturbance of p_0 . As we neglect terms of second order, we may write ρ_0 instead of ρ in (1) and in the second term of (2), ρ_0 being the normal density. In other words, we consider p , the derivatives of ρ , and the components of \mathbf{v} as small quantities. Finally, it is permitted in acoustics to neglect body forces altogether, i.e., to set $\mathbf{P} = 0$.

Thus we obtain the simple system of equations

$$(3) \quad \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad} p = 0,$$

$$(4) \quad \frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \mathbf{v} = 0.$$

Instead of $\operatorname{grad} p$ we may write

$$(5) \quad \operatorname{grad} p = \left(\frac{dp}{d\rho} \right)_0 \operatorname{grad} \rho = c^2 \operatorname{grad} \rho, \quad c^2 = \left(\frac{dp}{d\rho} \right)_0,$$

where the constant c is related to the normal state of the atmosphere. On substituting in (3) we obtain

$$(6) \quad \rho_0 \frac{\partial \mathbf{v}}{\partial t} + c^2 \operatorname{grad} \rho = 0.$$

We are now ready to eliminate \mathbf{v} from (4) and (6). Taking the partial derivative of (4) with respect to t and substituting for $\rho_0 \partial \mathbf{v} / \partial t$ according

to (6), we obtain

$$(7) \quad \frac{\partial^2 \rho}{\partial t^2} = c^2 \operatorname{div} \operatorname{grad} \rho = c^2 \nabla^2 \rho.$$

The same equation is valid for p , since $\operatorname{grad} p$, and therefore $\nabla^2 p$ and $\partial^2 p / \partial t^2$ are equal to the corresponding expressions in ρ if the latter quantities are multiplied with c^2 [cf. (5)]. Thus

$$(8) \quad \frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p.$$

The same differential equation also governs the velocity field. It is, of course, not the velocity \mathbf{v} that satisfies equation (8), but the velocity potential Φ , provided it exists. For, on putting $\mathbf{v} = -\operatorname{grad} \Phi$ according to (11.12), (4) and (6) become⁷

$$\frac{\partial \rho}{\partial t} = \rho_0 \nabla^2 \Phi \quad \text{and} \quad \rho_0 \frac{\partial \Phi}{\partial t} = c^2 \rho.$$

Here ρ can be eliminated by differentiation as before, and the result is

$$(8) \quad \frac{\partial^2 \Phi}{\partial t^2} = c^2 \nabla^2 \Phi.$$

Equations of the type (7), (8), (9) are called *wave equations*. They play a very important part in the fundamentals of mathematical physics in the form of the *equation of the vibrating string* (one-dimensional, $\nabla^2 = \partial^2 / \partial x^2$) or *vibrating membrane* (two-dimensional, $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$).

Our present problem reduces to the equation of the vibrating string if we consider a process that depends on one coordinate only. Eq. (8), for instance, takes the form

$$(10) \quad \frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}.$$

The integral of this equation can be given in the form of *d'Alembert's solution*

$$(11) \quad p = F_1(x + ct) + F_2(x - ct),$$

where F_1 and F_2 are real functions, entirely arbitrary except for certain continuity requirements (existence of first and second derivatives). In fact, any function of $x \pm ct$ satisfies the differential equation (10); the

⁷In the second of the two following equations a "constant," independent of x, y, z but dependent on t should be added; it can be absorbed in the definition of Φ , however. (cf. p. 89).

same is true for a linear combination of such functions thanks to the fact that we neglected the quadratic terms in Euler's equations.

Expression (11) is the *general* solution of the one-dimensional wave equation, since it can be adapted to an arbitrarily given initial state. Let it be required that for $t = 0$

$$p = f_1(x), \quad \frac{\partial p}{\partial t} = f_2(x).$$

Then we have only to make

$$F_1(x) + F_2(x) = f_1(x), \quad F_1'(x) - F_2'(x) = \frac{1}{c} f_2(x).$$

This gives directly

$$(12) \quad F_{1,2}(x) = \frac{1}{2} \left(f_1(x) \pm \frac{1}{c} \int_{x_0}^x f_2(\xi) d\xi \right).$$

The result is illustrated by Fig. 16 in which $f_2 = 0$ is assumed. Half of the initial pressure disturbance $f_1(x)$ moves to the right and half of it to the left, both travel without change of shape (undistorted, as one says) with the velocity c . The physical meaning of the constant c becomes now apparent; it is the *velocity of sound*. In Fig. 16 the initial disturbance is

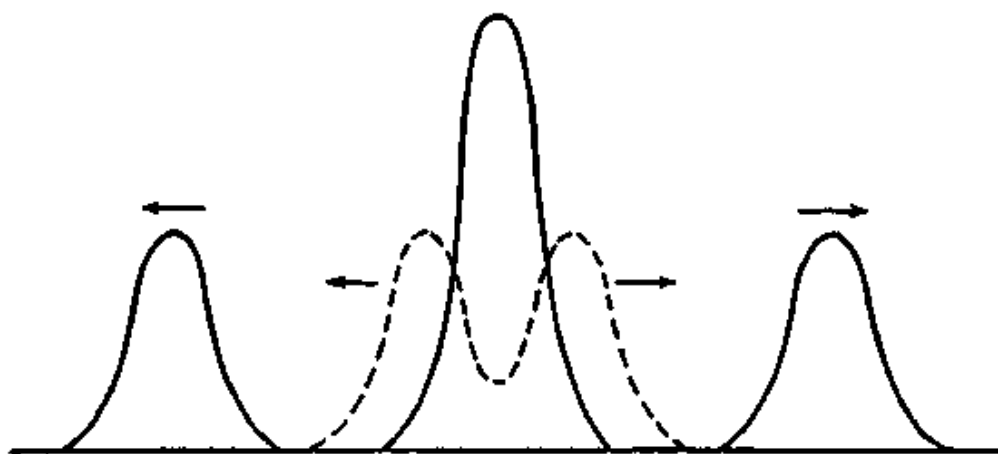


FIG. 16. Illustrating d'Alembert's solution. The initial pressure distribution (solid curve at the center) travels with constant velocity to both sides so that the ordinates of the two pressure hills are half the original ordinates, the width being the same. At first, the two hills partly overlap, (dotted curve) later they leave an undisturbed space between each other (solid curves left and right).

represented by the "hill" in the middle. A later state is illustrated by the half size pressure hills left and right. In terms of acoustics, the figure illustrates the propagation of a *noise*; the initial disturbance may have any shape as a function of x , the pressure being the same at all points of a plane normal to x . Or, in terms of the vibrating string, the string is

plucked (rather, distorted into an arbitrary plane curve) at the time $t = 0$ and then left alone. In the latter case we should have to complete the picture by adding the reflections that the disturbances suffer at the ends of the (finite) string.

In acoustics of speech and music one is not so much concerned with a single but with a periodically sustained excitation of the air. Denoting the pitch (= frequency) by $\nu = 1/\tau$ and introducing $\omega = 2\pi\nu$, we assume F_1 and F_2 in the form of trigonometric functions

$$\begin{aligned} F_2(x - ct) &= a \cos(kx - \omega t + \alpha), \\ F_1(x + ct) &= b \cos(kx + \omega t + \beta). \end{aligned} \quad (13)$$

(a and b are amplitudes, α and β phase constants, $k = \omega/c = 2\pi/\lambda$ is the *wave number*, that is, the number of waves on a segment of length 2π). The plane wave F_2 travels in the positive and the wave F_1 in the negative x -direction. A superposition of the two waves results in a standing wave if the amplitudes a , b are equal. Note the formula

$$c = \frac{\omega}{k} = \frac{\lambda}{\tau} = \lambda\nu. \quad (14)$$

First we wish to learn something about the actual motion of the particles whose oscillations constitute these waves. According to (5), the gradient of ρ has the same direction as the gradient of p , that is, the $\pm x$ -direction; the particle acceleration $d\mathbf{v}/dt$ points in this direction at any time according to (6), hence the vector \mathbf{v} has the $\pm x$ -direction. The direction of the oscillations coincides therefore with the direction of propagation (or with the opposite direction, depending on the phase): *our waves are longitudinal; transversal waves do not exist in perfect fluids*. The process illustrated in Fig. 16 is, of course, also a longitudinal wave.

Complex notation of periodic phenomena is the most convenient symbolism and very widely used today. When complex exponentials are substituted for the trigonometric functions, solution (13) takes the form

$$\begin{aligned} Ae^{i(kx - \omega t)}, \quad A &= ae^{i\alpha}, \\ Be^{-i(kx + \omega t)}, \quad B &= be^{-i\beta}. \end{aligned} \quad (15)$$

Physically meaningful is, of course, only the real part of either expression; it agrees with the corresponding right member in (13). This leaves the sign of i still undetermined; it has been chosen in (15) so as to make the time dependent term in both expressions equal to $e^{-i\omega t}$. One may even suppress the time factor altogether, and obtains then the following symbols

(15a) Ae^{ikx} , a plane wave traveling in positive x -direction

Be^{-ikx} , a plane wave traveling in negative x -direction

This representation is valid for pressure, density and velocity potential with the correspondingly changed meaning of A and B . It will contribute greatly to shorten our formulas in Chap. V.

We still have to say something about the *numerical value of the sound velocity*. Under the assumption of an isothermal change of state the polytropic exponent n equals 1. Then from (7.1)

$$\frac{dp}{d\rho} = \frac{p_0}{\rho_0},$$

and by (5)

$$(16) \quad c = \sqrt{\frac{p_0}{\rho_0}}.$$

On the basis of standard pressure and temperature we have (p. 44)

$$p_0 = 1033 \times 981 \frac{\text{gr-wt}}{\text{cm sec}^2}, \quad \rho_0 = 1.293 \times 10^{-3} \frac{\text{gr}}{\text{cm}^3},$$

and consequently

$$c = \sqrt{\frac{1012}{1.293}} \times 10^3 \frac{\text{cm}}{\text{sec}} = 280 \frac{\text{m}}{\text{sec}}.$$

Newton had already obtained this value which is much too small. The theoretical formula upon which he based his calculation is

$$(16a) \quad c = \sqrt{\frac{\text{Elasticity}}{\text{Density}}},$$

which requires some clarification. Actually, the term elasticity means in the present case the same as modulus of compression. For an elastic solid, this is given by (9.13); the corresponding definition for a gas would be

$$K = \left| \frac{dp}{d\Theta} \right| \quad \text{with} \quad d\Theta = - \frac{dp}{\rho},$$

or,

$$(16b) \quad K = \rho \frac{dp}{d\rho}.$$

Thus (16a) becomes

$$c = \sqrt{\frac{dp}{d\rho}} = \sqrt{\left(\frac{dp}{d\rho}\right)_0},$$

in accordance with definition (5).

The question arises, however, whether the assumption of an *isothermal* change of state can be justified. It was Laplace who first noticed that heat exchange between the fluid particles should be practically impossible because of the rapidity of the oscillations; hence the change of state should be considered *adiabatic*. The polytropic exponent is then no longer 1 but 1.4 (p. 50) and we have from (7.1)

$$(17) \quad \left(\frac{dp}{d\rho}\right) = n \frac{p_0}{\rho_0},$$

The value of the velocity of sound takes up the factor $\sqrt{1.4}$ and turns out to be

$$(17a) \quad c = \sqrt{1.4} \times 280 = 332 \frac{\text{m}}{\text{sec}},$$

in satisfactory agreement with the experiment.^a

In conclusion we wish to generalize d'Alembert's method of integration for a *spherical wave*. Let the origin of the wave coincide with the origin O of a system of polar coordinates r, ϑ, φ , and assume a spherically symmetric distribution of pressure (no dependence on ϑ and φ). The expression for $\nabla^2 p$ (problem I, 3) is then

$$\nabla^2 p = \frac{1}{r} \frac{\partial^2(rp)}{\partial r^2}$$

and equation (8) takes the form

$$\frac{\partial^2(rp)}{\partial t^2} = c^2 \frac{\partial^2(rp)}{\partial r^2}.$$

Exactly as in (11) the integral is

$$rp = F_1(r + ct) + F_2(r - ct).$$

Any distribution of p and $\partial p/\partial t$ given for $t = 0$ and $r > 0$ can be represented according to this formula and continued for $t > 0$; the values of F_2 for negative argument must be adjusted so as to keep p finite for $r = 0$.

^aThis difference in the two values of c , which is due to a difference in K , illustrates well the fact that the parameters of elasticity depend on the character of the change of state (see end of §).

Here again, we are mainly concerned with the periodic case which we now write according to (15), but without the time factor $e^{-i\omega t}$

$$(18) \quad p = \frac{A}{r} e^{ikr} \quad \text{and} \quad (18a) \quad p = \frac{B}{r} e^{-ikr} \quad \text{respectively.}$$

The same representation is valid for ρ and Φ . Since the gradient of Φ coincides with the r -direction the oscillation is *longitudinal* also in this case.

Eq. (18) represents an outgoing spherical wave: the spherical surfaces of equal phase $kr - \omega t = \text{const}$ travel away from O with a phase velocity

$$\frac{dr}{dt} = \frac{\omega}{k} = c.$$

Such a radiation process can be considered as produced by a pulsating sphere at O .

On the other hand, (18a) means an incoming spherical wave. The surfaces of equal phase $-kr - \omega t = \text{const}$ travel with the velocity

$$\frac{dr}{dt} = -\frac{\omega}{k} = -c$$

toward the origin, but this radiation pattern can hardly be realized physically.

Appendix

*Comparison of Compressible and Incompressible Flows.**

Since the analysis of flows of compressible fluids presents much greater difficulties because of the variable density, the question arises: *when is it permissible to consider a compressible medium as incompressible within given limits of accuracy?* This means that a certain prescribed error is admitted for the quantity of interest which may be either ρ or p or v .

We restrict the following discussion to steady flow and start from the equation of continuity. Cross-sections of stream filaments in the compressible case do not change in the same way as in the incompressible case, which is a consequence of the different form of the continuity equation, viz., $\text{div}(\rho \mathbf{v}) = 0$ according to (5.4). On integrating the divergence of the mass flow over a stream tube between the small cross-sections F and F_0 , assumed to be orthogonal to the streamlines, one obtains by Gauss's theorem

* Cf. also Prandtl-Tietjens, *Fundamentals of Hydro- and Aeromechanics*, McGraw-Hill, New York, p. 224.

$$(19) \quad \rho F v = \rho_0 F_0 v_0 = \text{const} = M$$

where v , v_0 and ρ , ρ_0 are the absolute velocity and density values at F and F_0 .

In the incompressible case, (19) reduces to

$$(19a) \quad F v = F_0 v_0 = \text{const} = V.$$

The quantities M and V in (19) and (19a) are the mass flux and the volume flux through the stream tube.

We now compute the pressure in the steady flow of a compressible fluid and compare it with the pressure in an incompressible flow. Bernoulli's equation (with no other body force than gravity) is according to (11.9a) and (7.6b)

$$(20) \quad \frac{1}{2} (v^2 - v_0^2) + g(h - h_0) + \int_{v_0}^v \frac{dp}{\rho} = 0,$$

but in an incompressible medium one would have instead of (20)

$$(20a) \quad \frac{1}{2} (v^2 - v_0^2) + g(h - h_0) + \frac{p - p_0}{\rho} = 0.$$

The quantities without subscripts refer to an arbitrary cross-section of a stream tube while the subscript 0 denotes the reference cross-section of the same stream tube. The evaluation of the integral in (20) according to the polytropic assumption $p = C\rho^n$ yields

$$(21) \quad \begin{aligned} \int_{v_0}^v \frac{dp}{\rho} &= \frac{n}{n-1} \left(\frac{p}{\rho} - \frac{p_0}{\rho_0} \right) = \frac{n}{n-1} \frac{p_0}{\rho_0} \left[\left(\frac{p}{p_0} \right)^{n-1} - 1 \right] \\ &= \frac{n}{n-1} \frac{p_0}{\rho_0} \left[\left(\frac{p}{p_0} \right)^{(n-1)/n} - 1 \right]. \end{aligned}$$

Substituting this in (20) and solving for p , we obtain first

$$(21a) \quad p = p_0 \left[1 - \frac{n-1}{n} \frac{\rho_0}{p_0} \left\{ \frac{1}{2} (v^2 - v_0^2) + g(h - h_0) \right\} \right]^{n/(n-1)}.$$

If we assume

$$(21b) \quad \frac{1}{2} (v^2 - v_0^2) + g(h - h_0) < \frac{n}{n-1} \frac{p_0}{\rho_0},$$

we may expand the right member of (21a) according to the binomial theorem and have

$$(21c) \quad p = p_0 \left[1 - \frac{\rho_0}{p_0} \left\{ \frac{1}{2} (v^2 - v_0^2) + g(h - h_0) \right\} + \frac{1}{2n} \frac{\rho_0^2}{p_0^2} \left\{ \quad \right\}^2 \pm \dots \right].$$

If all but the linear term of the series are neglected, one reobtains the result (20a) as one would expect.

Let us now estimate the error in the important case of the flow past a rigid body. The reference point p_0, v_0 is assumed to be at a great distance from the body, and differences in altitude are from now on neglected so that $h = h_0$. With a view to the later Figs. 46 and 48 we compute the pressure p at the stagnation point or simply the "stagnation pressure" $p - p_0$. Since $v = 0$ [cf. also the definition of the stagnation point in (29.7)], we obtain from (21c)

$$p - p_0 = \frac{1}{2} \rho_0 v_0^2 + \frac{1}{2n} \frac{\rho_0}{p_0} \frac{\rho_0 v_0^4}{4} \mp \dots$$

Here we introduce the *sound velocity associated with the polytrope under consideration*. At the reference point which we identify with what was called the normal state in (13.17), the sound velocity is

$$(21d) \quad c_0 = \sqrt{\left(\frac{dp}{d\rho}\right)_0} = \sqrt{n \frac{p_0}{\rho_0}}.$$

Thus the previous equation takes the form

$$(22) \quad p - p_0 = \frac{1}{2} \rho_0 v_0^2 \left(1 + \frac{v_0^2}{4c_0^2} \mp \dots\right),$$

and can be compared with (20a), which we now write

$$(22a) \quad p - p_0 = \frac{1}{2} \rho_0 v_0^2.$$

The error incurred in computing the stagnation pressure from (22a) instead of from (22) amounts to 1% at the most if $v_0 \leq 0.2c_0$. In the case of standard air ($n = 1.4$, $c_0 = 332$ m/sec) this means $v_0 \leq 66.4$ m/sec; if, on the other hand, an error of 10% is admitted, v_0 should be kept smaller than $0.63c_0 = 209$ m/sec. These estimates have been carried out on the basis of the quadratic term in (22).

The change of density associated with the pressure change is obtained from (21a) by the polytropic relation. The analogous expansion yields

$$(23) \quad \rho = \rho_0 \left(1 - \frac{v^2 - v_0^2}{2c_0^2} + \dots\right).$$

We now determine the change of v along the stream tube. In the incompressible case (20a) gives with $h = h_0$

$$(24) \quad v = \sqrt{v_0^2 + \frac{2\Delta p}{\rho_0}} \quad \text{with} \quad \Delta p = p_0 - p \quad \text{and} \quad \rho = \rho_0.$$

On the other hand, (20) gives, with the last value of the integral (21),

$$(24a) \quad v = \sqrt{v_0^2 + \frac{2n}{n-1} \frac{p_0}{\rho_0} \left[1 - \left(\frac{p}{p_0} \right)^{(n-1)/n} \right]}$$

in the compressible case. On putting again $p = p_0 - \Delta p$ and expanding as before, one has

$$(24b) \quad v = \sqrt{v_0^2 + \frac{2\Delta p}{\rho_0} \left(1 + \frac{1}{2n} \frac{\Delta p}{p_0} + \dots \right)},$$

which agrees with (24) except for the quadratic term in Δp .

Returning once more to the stream tube cross-section in Eqs. (19) and (19a), we obtain by logarithmic differentiation of the equation of continuity (19)

$$(25) \quad \frac{d\rho}{\rho} + \frac{dF}{F} + \frac{dv}{v} = 0,$$

Bernoulli's equation (20) with $h = h_0$ can also be written in differential form, viz.

$$(26) \quad v dv + \frac{dp}{\rho} = 0,$$

and so can the polytropic equation:

$$(27) \quad \frac{dp}{p} = n \frac{d\rho}{\rho}.$$

On substituting for the differentials dp and $d\rho$ in (25) from (27) and (26), one has

$$(28) \quad \frac{dF}{F} = \frac{dp}{\rho v^2} \left(1 - \frac{\rho v^2}{np} \right) = \frac{dp}{\rho v^2} \left(1 - \frac{v^2}{c^2} \right).$$

Note however that c is now the sound velocity at the cross-section of interest, F . The corresponding relation for the incompressible case is obtained by putting $d\rho = 0$ in (27), or $c \rightarrow \infty$ in (28):

$$(28a) \quad \frac{dF}{F} = \frac{dp}{\rho v^2}.$$

Comparison of the two results shows first that the laws governing the change of cross-section are practically identical as long as $v \ll c$, similarly to the laws for the pressure changes given in (22) and (22a). But Eqs. (28) and (28a) allow another important conclusion to be drawn. *For an incompressible fluid, dF and dp have always the same sign, i.e., the cross-*

section enlarges with increasing pressure and becomes smaller with decreasing pressure. For a compressible fluid this is only true if $v < c$, i.e., in subsonic flow. In supersonic flow ($v > c$) the cross-section enlarges with decreasing pressure and becomes smaller with increasing pressure. It is this fact that is responsible for the fundamental difference between supersonic and subsonic flows of compressible fluids, a difference which does not exist for incompressible fluids for which the sound velocity is infinitely large according to the definition $c = \sqrt{dp/d\rho}$. We also infer from (28) that the cross-section of a stream tube has an extremum when the sound velocity is reached ($v = c$), and we can easily see that it must be a *minimum*.

Formulas (19) and (19a) play an important role in hydraulics. They serve to determine such flow cross-sections as occur in piping systems, valves, turbine bladings, etc. Eqs. (24) and (24a) serve to calculate the velocity of discharge from the nozzle of a pressure tank where p_0 is the (constant) inside pressure and p the (constant) external pressure; v_0 may be assumed ~ 0 in good approximation.—Formula (24a) is named after de Saint-Venant and Wantzel (1839).

The Laval¹⁰ nozzle is of particular interest in this connection. It serves to change energy of high pressure and low velocity into energy of low pressure and high velocity as any nozzle does, but the peculiar feature of the Laval nozzle is that the velocity of discharge can exceed the sound velocity. In accordance with our results, its design must be such that the cross-sections decrease until sonic speed is reached and then increase again.

14. Dynamics of the Elastic Body

The dynamic equations of the elastic body follow again from the static equilibrium condition by adding the inertia force to the external force \mathbf{F} . Displacements in a solid can always be considered as small (only at the yield point, that is, beyond the limit of purely elastic behavior considerable displacements occur, cf. 39). The material derivative may, therefore, be identified with the local derivative and the inertia force per unit of volume becomes $-\rho \partial^2 \mathbf{s} / \partial t^2$, where $\mathbf{s} = i\xi + j\eta + k\zeta$.

Writing the static equilibrium condition in the form (8.12), we obtain as the differential equation of elasticity

$$(1) \quad \rho \frac{\partial^2 \mathbf{s}}{\partial t^2} = \text{Div } \boldsymbol{\sigma} + \mathbf{F},$$

¹⁰Devised by the Swedish engineer de Laval, who also invented a good separator and the first usable steam turbine.

or, spelled out in Cartesian coordinates as in (8.11),

$$\begin{aligned}
 (1a) \quad \rho \frac{\partial^2 \xi}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + F_x, \\
 \rho \frac{\partial^2 \eta}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_y, \\
 \rho \frac{\partial^2 \zeta}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z.
 \end{aligned}$$

Using, however, form (9.18) of our fundamental equation which is preferable for what follows, we obtain the elastic differential equation in the form

$$(1b) \quad \rho \frac{\partial^2 \mathbf{s}}{\partial t^2} = (\mu + \lambda) \text{grad } \Theta + \mu \nabla^2 \mathbf{s} + \mathbf{F}, \quad \Theta = \text{div } \mathbf{s}.$$

This equation refers to Cartesian coordinates; a more general equation is obtained when one starts from Eq. (9.19) instead of (9.18).

We are here mainly interested in free oscillations, therefore put $\mathbf{F} = 0$. From (1b) it follows directly that there will be waves of dilatation as well as of distortion, the first having *longitudinal*, the latter *transversal character*.

Waves of dilatation. On taking the divergence of every term in (1b), we obtain

$$\rho \frac{\partial^2 \Theta}{\partial t^2} = (\mu + \lambda) \nabla^2 \Theta + \mu \text{div } \nabla^2 \mathbf{s}$$

and may replace

$$\text{div } \nabla^2 \mathbf{s} \quad \text{by} \quad \nabla^2 \text{div } \mathbf{s} = \nabla^2 \Theta.$$

In this way we obtain the simple scalar equation

$$(2) \quad \rho \frac{\partial^2 \Theta}{\partial t^2} = (2\mu + \lambda) \nabla^2 \Theta.$$

By comparison, e.g. with (13.8), the *velocity of propagation of the waves of dilatation*, which we denote by a , is seen to be

$$(3) \quad a^2 = \frac{2\mu + \lambda}{\rho}.$$

For a one dimensional or spherically symmetric state the periodic solutions obtained by d'Alembert's method (13.11) for the infinite elastic body are of the form

$$(4) \quad e^{ikx} \quad \text{or} \quad \frac{1}{r} e^{ikr}, \quad k = \frac{\omega}{a}.$$

They are plane or spherical waves with circular frequency ω that travel with the constant velocity a .

Waves of distortion. On taking the curl of Eq. (1b) term by term and putting as in (2.19)

$$\text{curl } \mathbf{s} = 2\dot{\phi},$$

Eq. (1b) yields because of $\text{curl grad} = 0$

$$(5) \quad \rho \frac{\partial^2 \phi}{\partial t^2} = \mu \nabla^2 \phi.$$

Here ϕ represents in magnitude and axis the angle of torsion of the volume element under consideration. Comparison with (13.8) shows that the velocity of propagation of the waves of distortion, which we denote by b , is given by

$$(6) \quad b^2 = \frac{\mu}{\rho}.$$

This is the reason why Lamé's modulus μ is sometimes called the torsion modulus, as already mentioned on p. 68. Again d'Alembert's method gives plane and spherical waves that are represented by (4) in the same way as the compression waves, provided that the wave number k is now ω/b . The comparison of (3) and (6) shows immediately that

$$a > b, \quad \text{since} \quad \frac{a^2}{b^2} = 2 + \frac{\lambda}{\mu} = \frac{2 - 2\nu}{1 - 2\nu};$$

as for the last term, cf. (9.11). For iron with ν about 0.3 (see p. 65), we obtain $a/b = \sqrt{7/2} = 1.87$.

The meaning of this is obvious: the elastic resistance of solids against changes of volume is considerably larger than that against changes of the relative orientation. The absolute value of b in a solid is, of course, still much larger than the sound velocity c in air. With the data of pp. 65, 68, and with $\rho = 7.8 \text{ gr/cm}^3$ for iron, the value of b is found to be 3100 m/sec. Both formulas (3) and (6) for a and b can be subsumed under Newton's rule (13.16a), provided the term elasticity is suitably specified in each case.

The compressional waves and the waves of distortion considered here are *three-dimensional waves* progressing in an elastic body without boundaries. In addition, there exist *surface waves* that are bound to the free surface of the body and have a special importance for seismic phenomena; they are dealt with in 45. The surface waves travel more slowly than the three-dimensional waves, that is to say, the earth puts up more elastic resistance against the latter than the former. They constitute the principal parts of the observed earthquakes, while the three-dimensional waves that arrive earlier are observed as preliminaries.

15. The Quasi-Elastic Body as Model of the Ether

In 19th century physics, a material carrier was assumed for the optical phenomena, equipped as far as possible with the properties of ordinary elastic bodies. This construction, however, led to difficulties even in the most elementary problem of reflexion and refraction, about which more in 45. As early as 1839 MacCullagh tried to drop the connection with the ordinary theory of elasticity with the aim to develop a representation of optics that would be free of the difficulties mentioned. It turned out later that his theory agreed formally with Maxwell's electro-magnetic optics (1864), in particular as far as the optics of transparent bodies is concerned. The following remarks should be considered as an interpretation of MacCullagh's equations.

Let us go back to the beginning of 1. There the general locomotion of a continuous medium was decomposed into the three parts of translation, rotation, and deformation. The elastic body responds to a deformation with a stress tensor which is determined by the deformation tensor; it is not sensitive to rotation (and, of course, not to translation). We now try to imagine a "quasi-elastic" body, supposedly insensitive to deformations but *responsive to rotations relative to absolute space*! Since the rotation has the character of an antisymmetric tensor (1.12a) we shall assume that the stress acting on the volume element as a result of the rotation is also an antisymmetric tensor, as indicated in the following array:

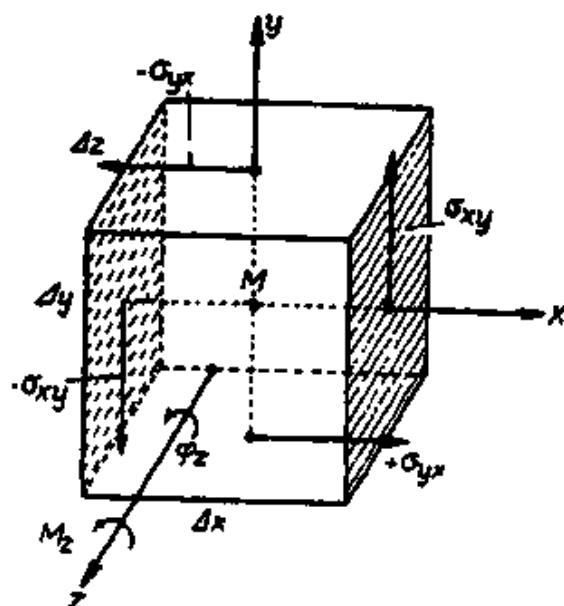
$$(1) \quad \begin{pmatrix} 0 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & 0 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & 0 \end{pmatrix}, \quad \sigma_{ih} = -\sigma_{hi}.$$

The *stress-rotation relation* here assumed is illustrated in Fig. 17. Let the volume element $\Delta\tau$ be twisted through the angle φ_z (indicated by the right hand screw arrow about the positive z -axis). To produce or to maintain this rotation we need—this is our hypothesis—a moment about the z -axis

$$(2) \quad M_z = k\varphi_z\Delta\tau.$$

The quantity k could be called the "twist modulus" of the quasi-elastic body. The moment is associated with two shear stresses σ_{xy} and $-\sigma_{yx}$ across the positive x - and y -surfaces and with corresponding anti-parallel shear stresses across the negative x - and y -surfaces, all indicated in the figure. In order to be in accordance with (2) and (1) we have to put

FIG. 17. Strain and rotation in the "quasielastic" body.



$$(3) \quad \sigma_{xy} = -\sigma_{yx} = \frac{k}{2} \varphi_z.$$

This assumed, the moment originating at the two x -surfaces (see the analogous argument that accompanies Fig. 10) is

$$2\sigma_{xy}\Delta y\Delta z \cdot \frac{\Delta x}{2} = \frac{k}{2} \varphi_z \Delta \tau$$

and the moment originating at the two y -surfaces

$$-2\sigma_{yx}\Delta x\Delta z \cdot \frac{\Delta y}{2} = \frac{k}{2} \varphi_z \Delta \tau,$$

so that the total moment is in fact that of equation (2). By rotation of the letters we obtain from (3)

$$(3a) \quad \sigma_{yz} = -\sigma_{zy} = \frac{k}{2} \varphi_x, \quad \sigma_{zx} = -\sigma_{xz} = \frac{k}{2} \varphi_y.$$

The equation of motion of the quasi-elastic body follows now from (14.1a). In applying this equation we have to assign inertia to the volume element of the ether and to consider its displacement as small, since we previously neglected the quadratic convective terms in putting $du/dt = \partial u/\partial t$. We also disregard any external forces ($F = 0$). In this way we obtain from (14.1a) with the use of (3) and (3a)

$$\rho \frac{\partial u}{\partial t} = \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = -\frac{k}{2} \left(\frac{\partial \varphi_x}{\partial y} - \frac{\partial \varphi_y}{\partial z} \right),$$

or, written in vector form,

$$(4) \quad \rho \frac{\partial \mathbf{v}}{\partial t} = -\frac{k}{2} \text{curl } \boldsymbol{\phi}.$$

This is the equation of motion of the ether; it must be supplemented by the relation between \mathbf{v} and the angular velocity $\boldsymbol{\omega}$, which reads

$$(5) \quad \frac{\partial \boldsymbol{\xi}}{\partial t} = \frac{1}{2} \text{curl } \mathbf{v}$$

where $\boldsymbol{\omega} = d\boldsymbol{\xi}/dt$ was replaced by $\partial \boldsymbol{\xi}/\partial t$. As a further assumption we add the incompressibility of the ether; we also note that $\boldsymbol{\xi}$, being the curl of the displacement vector, has no divergence. Thus \mathbf{v} and $\boldsymbol{\xi}$ are subject to the following conditions

$$(6) \quad \text{div } \mathbf{v} = 0, \quad \text{div } \boldsymbol{\xi} = 0.$$

The system of equations (4), (5) and (6) is of impressive simplicity and symmetry. It is formally identical with *Maxwell's equations for the empty space!*

Before elaborating this statement, let us introduce the following notations: \mathbf{E} = electric field strength, \mathbf{H} = magnetic field strength; α, β = factors of proportionality that depend on the choice of units for E and H . Now we put

$$\text{either} \quad a) \quad \mathbf{v} = \pm \alpha \mathbf{E}, \quad \boldsymbol{\xi} = \mp \beta \mathbf{H},$$

$$\text{or} \quad b) \quad \mathbf{v} = \pm \alpha \mathbf{H}, \quad \boldsymbol{\xi} = \pm \beta \mathbf{E}.$$

The signs in these equations depend on the signs chosen for the units of the electric charge and magnetic pole strength. Eqs. (4), (5), and (6) assume then the form

$$(7) \quad \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \text{curl } \mathbf{H}, \quad \text{div } \mathbf{E} = 0,$$

$$\mu_0 \frac{\partial \mathbf{H}}{\partial t} = - \text{curl } \mathbf{E}, \quad \text{div } \mathbf{H} = 0$$

for the choice a) as well as for the choice b). The abbreviations ϵ_0, μ_0 are known as the dielectric constant and permeability of the vacuum; in our notation they are given by

$$(a) \quad \epsilon_0 = \frac{\rho}{k} \frac{2\alpha}{\beta}, \quad \mu_0 = \frac{2\beta}{\alpha},$$

$$(b) \quad \mu_0 = \frac{\rho}{k} \frac{2\alpha}{\beta}, \quad \epsilon_0 = \frac{2\beta}{\alpha}.$$

Their product is independent of the choice of the units α, β , since we have in either case

$$(8) \quad \epsilon_0 \mu_0 = \frac{4\rho}{k} = \frac{1}{c^2}.$$

The quantity c defined in (8) has, physically, the meaning of the *velocity of light in vacuum*. It can be included in Newton's rule (13.16a) in analogy to the sound velocities in 13 and 14, when the word elasticity is given the meaning of one fourth of the "twist modulus" k .

It is by no means our intention to assign any physical reality to this "ether model". Physicists had convinced themselves by the turn of the century that all attempts at a mechanical explanation of Maxwell's equations were doomed to failure. What we mean here is not a mechanical *explanation* but, at best, a mechanical *analogy*. Maxwell's equations are among the fundamentals of the electrical theory of matter, so one should not expect that they can be derived from the macroscopic properties of ponderable bodies. On the contrary, they seem to stem from the same root as general gravitation, that is, from the space-time metric, according to a recent paper by Schrödinger.¹¹ Our remarks, however, may have some justification inasmuch as they show: if we were to construct an "ether" as a substratum for Maxwell's equation, then we would have to furnish it with qualities that are diametrically opposed to those of ordinary matter, viz., an *absolute directional orientation relative to space* in contrast to the *relative orientation of the volume elements toward each other* possessed by elastic bodies.

The following historical remark may be of interest. In an extension of MacCullagh's ideas Lord Kelvin,¹² in the eighties, developed the concept of the quasi-elastic or, as he sometimes put it, "quasi-rigid" ether. He was not satisfied, only to postulate an ether with reactive responses (2), but attempted to construct a gyroscopic model that actually would react in the required way. As is well known, a fast spinning top can be so arranged as to acquire directional stability and will then respond to fairly strong moments with small angular changes only. But an ether model, based on gyroscopic effects, becomes desperately complicated. Each volume element has to be equipped with several tops that must be oriented relative to each other in such a way that the desired rotational stiffness is achieved not only for one but for all three axes. A construction as complicated as that would be the only way to realize a "gyrostatic" ether.

Lord Kelvin took the point of view marked by a) on p. 110 and correlated the "twist" of his gyrostatic ether to the magnetic vector \mathbf{H} . He

¹¹E. Schrödinger, Proc. R. Irish Academy 49, 43 (1943).

¹²Sir William Thomson (Lord Kelvin), Mathematical and Physical Papers, Vol. 3, Art. 49, 50, 52.

did not commit himself to a definite interpretation of the electric vector \mathbf{E} , thus abstaining from a complete mechanical representation of Maxwell's equations. There is no doubt that standpoint a) is physically more evident, since \mathbf{H} has the character of an axial vector like \mathfrak{g} , while \mathbf{E} possesses that of a polar vector like \mathbf{v} . On the other hand, standpoint b) seemed to recommend itself because it took care of the pure ether and the insulators, but also left space for the conductors in the quasi-elastic picture.¹³ It was Boltzmann¹⁴ who pointed out the difficulties involved in standpoint b) which are caused by the existence of the *true electric charge*. Disregarding the electrodynamics of conductors, we shall only pursue assumption a) somewhat further in the following. In 20 we shall come back to the correlation b) in connection with Helmholtz's vortex theory.

For the ponderable insulator the two principal equations (7) are still valid as in vacuum, only the values ϵ_0 , μ_0 must now be replaced by ϵ , μ . The divergence conditions, however, undergo an essential change. Instead of $\text{div } \mathbf{H} = 0$ we have now

$$(9) \quad \text{div } \mathbf{B} = 0, \quad \mathbf{B} = \mu \mathbf{H} = \text{magnetic induction}$$

and we would therefore correlate \mathbf{B} rather than \mathbf{H} with the rotation \mathfrak{g} , which would introduce no difficulty. On the other hand the condition $\text{div } \mathbf{E} = 0$ must be altered into

$$(10) \quad \text{div } \mathbf{D} = \rho_e, \quad \mathbf{D} = \epsilon \mathbf{E} = \text{dielectric displacement.}$$

Here ρ_e is the spatial density of the true electric charge. When we now correlate \mathbf{D} instead of \mathbf{E} to the flow velocity \mathbf{v} and set up suitable relations between the constants ϵ , μ and k , ρ , α , β we are still able to obtain complete formal analogy with Maxwell's equations for the ponderable insulator. It is clear, however, that the existence of the true charge causes difficulty also for the standpoint a), if it is desired to maintain the incompressibility of the ether. There is, of course, a mathematically permissible but physically rather drastic remedy for this difficulty. One simply prescribes that the quasi-elastic fluid leaves or enters the field at points that carry true charges according to the signs. Where the fluid goes or whence it comes is left in the dark. As an apology for this assumption one could quote the example of the great Bernhard Riemann¹⁵ who makes the same hypothesis in an attempt at a theory of gravitation and electrostatics, in his paper: "Neue mathematische Prinzipien der Naturphilosophie", but

¹³A. Sommerfeld, Ann. d. Phys. 46, 139 (1892).

¹⁴L. Boltzmann, Ann. d. Phys. 47, 743 (1892), and Vorlesungen über Maxwells Theorie, Leipzig 1893, Vol. II, 1, p. 6.

¹⁵Ges. Werke, 2nd ed., Teubner, Leipzig, 1892, p. 523. The paper was written shortly after Riemann had obtained the doctorate, and published posthumously.

it seems to be more correct to admit that no mechanical or quasi-mechanical picture is a suitable representation of the fundamental fact of the electric charge.

We shall have no reason to come back in Vol. III to the model of the ether discussed here. The electric charge and the structure of the electromagnetic field must be accepted as entities that transcend mechanics.

16. Dynamics of Viscous Fluids. Hydrodynamics and Hydraulics. Reynolds' Criterion of Turbulence

In order to set up the general equations of motion of viscous fluids, we have to introduce the inertia force in the equations of uniform motion (10.8a,b). \mathbf{F} must be replaced by

$$\mathbf{F} - \rho \frac{d\mathbf{v}}{dt} = \mathbf{F} - \rho \left\{ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \text{ grad}) \mathbf{v} \right\}.$$

In this way we obtain the Navier-Stokes equations¹⁰

$$(1) \quad \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \text{ grad}) \mathbf{v} - \mu \Delta \mathbf{v} + \text{grad } p = \mathbf{F}$$

which are written here in pseudo-vectorial form. In order to put them in invariant shape one has to make use of Eqs. (11.6) and (3.10a), viz.

$$(\mathbf{v} \text{ grad}) \mathbf{v} = \text{grad } \frac{v^2}{2} - \mathbf{v} \times \text{curl } \mathbf{v}$$

and

$$\nabla^2 \mathbf{v} = \text{grad div } \mathbf{v} - \text{curl curl } \mathbf{v}.$$

Eq. (1) has to be supplemented in the incompressible case by

$$(1a) \quad \text{div } \mathbf{v} = 0.$$

In the case of compressible fluids, (1a) must be replaced by (5.4); at the same time the additional term $-(\mu + \lambda) \text{grad } \dot{\theta}$, that is $-(\mu + \lambda) \text{grad div } \mathbf{v}$, has to be added to the left member of (1) according to (10.22).

Today we consider the Navier-Stokes equations as fundamental for the entire theory of fluid flow. Engineers throughout the 19th century were, on the contrary, convinced that there was a big discrepancy between theoretical hydrodynamics on the one side and hydraulics on the other side. It was the British engineer and physicist Osborne Reynolds

¹⁰Navier, 1822, Stokes, 1845.

who bridged the gap with his profound experimental¹⁷ and theoretical¹⁸ investigations.

He experimented with glass tubes of various diameters at varying pressure gradients which amounts to changing the average velocities of the flow. (Average velocity means here the mean value of the velocity with respect to the cross-section.) Introducing a colored fluid filament into the entrance cross-section he observed the behavior of this filament throughout the tube. At small diameters and not too large average velocities the colored filament remains straight and parallel to the axis of the tube. At a larger diameter or larger velocity there appear irregular deviations that have a tendency to fill the whole tube. The initially straight filament becomes fuzzy and finally indistinguishable. In the first case the motion is called *laminar* (proceeding in regular laminations as in the Hagen-Poiseuille flow), in the last case it is called *turbulent*, as suggested by Reynolds.

Reynolds would not have been able to bring order in his experimental results, if he had not considered them from the point of view of a "*law of similitude*." Considerations of this character already appear in the first beginnings of western physics, as in the work of Galilei.¹⁹ Generally speaking, they consist in the comparison of two experimental arrangements differing only in scale.

In our case, consider two tubes of radii a_1 and a_2 . Putting

$$(2a) \quad a_2 = \alpha a_1,$$

the number α is the change of the scale of length in the transition from experiment 1 to experiment 2. It pertains not only to the radii, but to all dimensions of length, such as the coordinates of two "corresponding points" in the two tubes. Any two such points are thus connected by

$$x_2 = \alpha x_1, \quad y_2 = \alpha y_1, \quad z_2 = \alpha z_1.$$

The average velocities in corresponding cross-sections being v_1 and v_2 , we put

$$(2b) \quad v_2 = \beta v_1.$$

¹⁷An Experimental Investigation of the Circumstances which determine whether the Motion of Water shall be direct or sinuous, and the Law of Resistance in Parallel Channels, Phil. Trans. R. Soc. 174 (1883).

¹⁸On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion, Phil. Trans. R. Soc. 186 (1895). In connection with this H. A. Lorentz, Ges. Werke, Vol. I, Leipzig, p. 43.

¹⁹In his last work "Discorsi e Dimostrazioni Matematiche," 1638.

The dimension of velocity being length/time, the number β determines also the change of the time scale in our transition; this change is evidently α/β , according to (2a) and (2b). In addition we allow the two tubes to carry two different liquids of different density and viscosity and put

$$(2c) \quad \rho_2 = \gamma \rho_1.$$

The factor γ defines the change of the mass scale which is $\gamma\alpha^3$ because of (2a) and (2c).

It is more convenient to use instead of the coefficient of viscosity μ the *kinematic* viscosity ν defined by

$$\nu = \frac{\mu}{\rho}.$$

Again we put

$$(2d) \quad \nu_2 = \delta \nu_1.$$

We finally compare the pressures in corresponding cross-sections by setting

$$(2e) \quad p_2 = \epsilon p_1.$$

Obviously, ϵ could be expressed by α , β , γ , because of the dimension of p .

We now write Eq. (1) after division by ρ once more in a schematic way and omit the external force which is here immaterial:

$$(3) \quad \text{acceleration} \quad -\nu \nabla^2 \mathbf{v} + \frac{1}{\rho} \text{grad } p = 0.$$

The dimension of the acceleration term may be written either as velocity/time or velocity²/length. In the transition from (1) to (2), this term takes up, according to (2a) and (2b), the factor

$$(4a) \quad \frac{\beta^2}{\alpha};$$

according to (2b) and (2d) the second term of (3) takes up the factor

$$(4b) \quad \delta \frac{\beta}{\alpha^2},$$

since the repeated differentiation with respect to the coordinates occurring in $\nabla^2 \mathbf{v}$ means obviously a repeated division by a length. The change of scale in the last term of (3) is finally [by (3), (2a), (2c), (2e)]

$$(4c) \quad \frac{1}{\gamma} \frac{\epsilon}{\alpha}.$$

Suppose now, Eq. (3) is obeyed by the flow in the experimental arrangement 1. If it is to be obeyed also by the flow 2, the ratio of the three factors (4a, b, c) must be equal to one:

$$(5) \quad \frac{\beta^2}{\alpha} : \delta \frac{\beta}{\alpha^2} : \frac{1}{\gamma} \frac{\epsilon}{\alpha} = 1 : 1 : 1.$$

This amounts to the following two relations:

$$(5a) \quad \frac{\beta\alpha}{\delta} = 1. \quad (5b) \quad \frac{\epsilon}{\gamma\beta^2} = 1.$$

The meaning of (5a) in terms of the experimental parameters involved is because of (2a, b, d)

$$(6) \quad \frac{v_1 a_1}{\nu_1} = \frac{v_2 a_2}{\nu_2}.$$

Similarly, (5b) leads by (2b, c, e) to

$$(7) \quad \frac{p_1}{\rho_1 v_1^2} = \frac{p_2}{\rho_2 v_2^2}.$$

Eqs. (6) and (7) constitute the results of Reynolds theory of similitude. In the literature the term *Reynolds criterion* is usually applied to equation (6) only, although (7) is a part of it as well, as we have just seen. It is the two relations together that furnish the necessary and sufficient conditions of hydrodynamic similitude. In what follows we shall make extensive use also of equation (7).

If these two criteria are satisfied, the flows 1 and 2 are either both laminar or both turbulent, since the experimental arrangement 2 is nothing but a "mechanical image" of 1 on a different scale.

The dimensionless expression defined by (6) is known as Reynolds number and denoted²⁰ by R :

$$(8) \quad R = \frac{va}{\nu},$$

where the length a may designate any linear dimension characteristic of the experiment (such as a distance of two plates, a depth of a channel, radii of falling spheres, etc.). The number defined by (7) will be denoted by S so that

$$(9) \quad S = \frac{p}{\rho v^2}.$$

²⁰Sometimes denoted by Re .

The transition from laminar to turbulent flow in particular is a mechanically *similar* event for the tubes 1 and 2 which is therefore characterized by the same numerical value of R . This is the *critical Reynolds number*:

$$(10) \quad R_{crit} = \left(\frac{va}{\nu} \right)_{crit}.$$

Its value is rather sensitive to the particular form of the entrance section of the tube. When a well rounded entrance section in the form of a trumpet bell is used, a laminar flow is established from the start and stays laminar up to rather large values of R . If the fluid enters the tube from the tank through a straight, sharp edged connection, the initial flow is disturbed by lateral components which are not damped out by friction at once; the transition to full turbulence occurs then at a comparatively low R . The critical Reynolds number is between 1200 (very irregular inlet) and 20,000 (mouth piece with a well-rounded fairing). Thus our statement regarding the constancy of the critical Reynolds number should actually be restricted to flows with similar initial conditions.

The question arises: what is the cause of the transition from laminar to turbulent flow and vice versa? Our treatment of the laminar flow in capillary tubes on p. 78ff does not point to anything like a limitation of the flow pattern to small diameters or small velocities; on the contrary, it appears that the Hagen-Poiseuille flow is a possible form of the motion under all circumstances, but no longer *stable* if $R > R_{crit}$. Thus Reynolds's criterion appears as a *stability criterion* dictated by experience; the following remarks should help to bring it closer to our physical understanding:

Viscosity tends to smooth out lateral components either initially present or produced by the roughness of the tube, it thus favors the laminar pattern. The inertia of the fluid requires the conservation of such side components once they have come into existence, thus it acts in favor of turbulence. In the static treatment of the capillary flow in 10, there was no inertia, and side components were entirely out of consideration; thus it appeared as if the results obtained there could be extended to arbitrarily large tubes and arbitrarily fast flows. Note also the antagonism between viscosity and inertia that becomes apparent even in the structure of the expression $\nu = \mu/\rho$ which occurs in the denominator of (8): An increase of μ permits one to increase the value of va without leaving the region of stability; increasing ρ diminishes that value and favors turbulence.

This view is also in accordance with the apparent lack of definition of the transition point, a fact that could be compared with the well known thermo-dynamic phenomenon of super-cooled water. Water can be cooled

below 0°C without freezing provided that it is left entirely undisturbed. Similarly, the regime of the laminar flow can be extended to larger R by carefully avoiding the production of lateral components at the inlet or by the roughness of the walls.

In order to decide whether there exists at all a sharply defined ideal value of the critical Reynolds number one would have to proceed as follows: one studies the flow in a sufficiently long tube with sufficiently smooth walls at a sufficiently large distance from the inlet. Initial flow irregularities in the region of the inlet will have enough time to attenuate in the stable case without the interference of side components newly created at the wall. If now at certain R -values laminar and at other R -values turbulent flow is observed and if the upper limit of the *former* R -values agrees with the lower limit of the *latter* R -values, the common limit would appear as the ideal value of R_{crit} .²¹ Such an experiment cannot be carried out practically, therefore the large variation of the values of R_{crit} given in the literature.²²

The transition to turbulent flow becomes apparent not only in the change of the flow pattern, but also in the change of the *pressure-law*, to which we now turn. Here the second law of similitude (9) comes into the foreground. We replace p in (9) by the pressure difference Π between the beginning and the end of the tube observing that the dimensions of Π and p are, of course, the same. In doing so the length l of the tube must now also be considered (the tube in Eq. (9) could have had any arbitrary length). Now l cannot be related to the dimensionless number in (9), but we may use the ratio l/a to rewrite the definition of S as

$$(11) \quad \Pi = \rho v^2 \frac{l}{a} S,$$

which is valid for the similar motions 1 and 2 at the same S -value but for different subscripts of Π , ρ , v , l , a .

We now compare (11) with the formula used in hydraulics to compute the loss of pressure head in pipe or channel systems:

$$(11a) \quad h = \lambda \frac{v^2}{2g} l \frac{U}{F}.$$

²¹It seems not impossible that with improved experimental conditions (better regulation at the inlet and smoother walls) R_{crit} may be raised to arbitrarily high values. If this should be true, an ideal value would not exist and Reynolds's stability limit would only be of practical value, depending on experimental conditions. On the other hand, there seems to exist a definite Reynolds number below which turbulent flow cannot be maintained.

²²They have been systematically investigated according to inlet conditions by L. Schiller.

Here λ is a pure number, the "hydraulic friction coefficient", F is the cross-sectional area of the flow, U the wetted circumference of the conduit. For a circular cross-section filled with fluid, U/F becomes $2/a$ so that

$$(11b) \quad h = \lambda \frac{v^2 l}{g a}.$$

On identifying S with λ and expressing Π by the hydrostatic level difference h ($\Pi = \gamma h = \rho g h$), (11b) becomes identical with (11).

If, on the other hand, S is identified with the pure number $8/R$, so that

$$S = \frac{8\nu}{va} = \frac{8\mu}{\rho va},$$

the Hagen-Poiseuille formula

$$(12) \quad \Pi = 8\mu l \frac{v}{a^3},$$

is obtained with the only difference that, previously, we had the more precisely defined average velocity u_m where we now have v .

In (11b) and (12) the different forms of the pressure dependence in laminar and turbulent flow that was already mentioned on p. 80, become again apparent: in (11b) the dependence on v and a is in the form v^2/a , in (12) in the form v/a^3 . We have been able to overcome this contrasting behavior by choosing the number S in two different ways. (Note that the definition of S in (9) does not, in itself, teach anything about the interdependence of the flow parameters; only the identification of S with another non-dimensional combination of flow parameters has the character of a physical law.)

It will be noticed that the hydraulic formula (11a) has the same structure as a law of air resistance that had already been used by Newton. According to this law, a body having the velocity v relative to air experiences a resistance proportional to v^2 (the same is true for the wind pressure against a surface at rest). Also in ballistics the resistance law is, at subsonic velocities, nearly a v^2 -law.

A more accurate representation of the pressure losses observed in turbulent flow through smooth pipes is possible within the framework of Eq. (11) if a *weak* dependence of S on R is assumed, (that is, S is supposed neither independent of R as in (11b) nor $\sim R^{-1}$ as in (12)). The following assumptions have been tested for larger R -values

$$(13) \quad S = \lambda R^{-\kappa} \quad \text{and} \quad S = \lambda_0 + \lambda_1 R^{-\kappa}$$

with $\kappa = 1/4$ to $1/5$ in the first formula (13) and with somewhat larger values of κ in the second one.

The hydraulic v^2 -law (11a) is modified by the first formula (13) (Blasius formula, $\kappa = 1/4$) to $v^{2-\kappa}$ which agrees almost exactly with old observations of Hagen and Reynolds (cf. also problem III, 2). For more details about the pressure in the transition region and the influence of roughness see L. Prandtl's "Strömungslehre", Braunschweig 1949, which can be considered as a source book in this field.²³

The change of the pressure law at the critical limit is schematically represented in Fig. 18: At small velocities $v < v_{crit}$ the linear law of

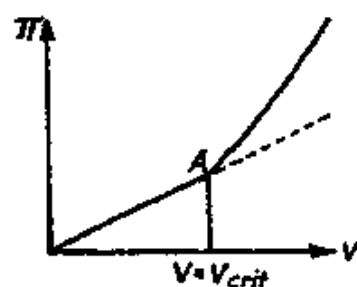


FIG. 18. The pressure gradient as a function of the mean velocity v . At the critical point A the laminar flow becomes turbulent.

Hagen-Poiseuille, at velocities $v > v_{crit}$ the increase according to $v^{2-\kappa}$. The broken line that continues the straight branch beyond A represents the possible increase of the stability limit A .

The change of the pressure law at A is accompanied ("caused" would perhaps be the better word) by a noticeable change of the velocity distribution over the cross-section. In contrast to the parabolic distribution of the laminar case, the time average of the velocity in turbulent flow is practically constant over the interior of the tube; only in the close vicinity of the wall it drops quickly to zero, satisfying the boundary condition $v = 0$.

The contrast laminar-turbulent occurs not only in circular tubes or pipes with other cross-sections, but can also be observed in the flow between two plates at rest. This flow pattern, a two-dimensional analogy to the Hagen-Poiseuille flow, was treated in problem II, 3a. The parabolic velocity profile which is also characteristic for this case is shown in Fig. 19.

If the parabolic cylinder of Fig. 19 is cut along the symmetry plane $y = h$, that is, if only the lower half of the flow pattern is considered, one obtains the velocity distribution in a stream in laminar motion (strictly speaking, the stream is infinitely wide). Let the bottom of the stream form the angle α with the horizontal (Fig. 19a), so that the component of gravity $\rho g \sin \alpha$ replaces the pressure gradient of the previous figure. L. Hopf²⁴ (Munich thesis, 1909) made a careful investigation of

²³For pipe flow compare J. C. Hunsaker and B. G. Rightmire, Engineering application of Fluid Dynamics, McGraw-Hill, New York, 1947, Chap. VIII.

²⁴Partly published in Ann. d. Phys. 32, 777 (1910).

the experimental realization of such a laminar stream stabilized by viscous forces, and of the limits of its stability. The critical Reynolds number in this investigation was found to be $R_{crit} = \nu h/\nu \cong 330$.

An even simpler case of laminar motion develops from Fig. 19 when one of the plates is at rest while the other moves with constant velocity

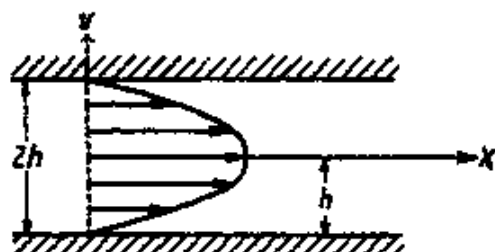


FIG. 19. Laminar flow between two plates at rest.



FIG. 19a. The "river in laminar flow" under the influence of gravity; free surface at $y = 0$.

U . The velocity profile is here linear for laminar flow (cf. Fig. 19b). On denoting the flow velocity by u , we have

$$(14) \quad u = \frac{y}{h} U \quad \text{and, of course,} \quad v = w = 0.$$

The flow proceeds parallel to the x -direction; it should be noticed, however, that this flow is not irrotational although there seems to be no rotation involved in it:

$$2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{U}{h}.$$

Because of its particular analytic simplicity, (14) is the preferred flow for theoretical investigations of stability (cf. 38).

Following Couette,²⁵ an elegant realization of this flow can be made by means of two coaxial cylinders. When the inner cylinder is at rest and the outer cylinder moves with a circumferential velocity U , transition to turbulent flow occurs according to Couette at a Reynolds number $R_{crit} = Uh/\nu = 1900$ where h is the thickness of the fluid layer. The converse case where the inner cylinder moves has been thoroughly investigated by Taylor.²⁶

In this discussion we have only scratched the surface of the turbulence problem. Central questions such as: "What is the reason for the instability?" "Is it possible to give a mathematical description of the turbulent fluctuations of the velocity?" "Is the turbulent state contained among

²⁵M. Couette, Ann. Chim. Phys. 21 (1890).

²⁶Sir G. I. Taylor, Phil. Trans. 223, 289 (1923).

the solutions of the Navier-Stokes equations?" have not been touched. No complete answer to these questions to which we shall return in 38 seems possible today.

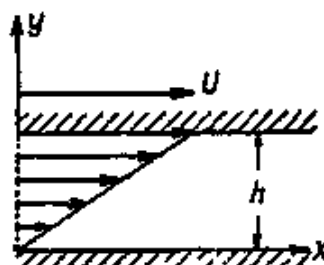


FIG. 19b. The laminar rectilinear flow between a plate at rest and in motion may be considered as the limiting case of the Couette flow between coaxial cylinders.

17. Some Remarks on Capillarity²⁷

The molecular forces of cohesion that are responsible for the phenomena of capillarity have already been encountered at the end of 12 in the form of the normal pressure N acting on the fluid surface (Laplace's theory). For the purpose of a short survey, however, it is more convenient not to adopt Laplace's²⁸ starting point but to follow Gauss²⁹ who proceeds from a minimum principle for the surface energy; the normal pressure N of Eq. (12.16) will be derived from this minimum principle.

We stipulate that there acts within the surface F of the fluid an everywhere equal *surface tension* T in tangential direction: F is, as it were, covered by a membrane which does not by itself possess elasticity of form. Its resistance against deformation (comparable with that of a thin rubber skin) is furnished solely by a tangential tension T that acts along the border of the membrane in the direction that is normal to the border and lies in the tangential plane.

We start with a simplified case, restricting ourselves to a *cylindrical* fluid surface or, which amounts to the same, to a membrane of cylindrical form. The tension equilibrium in such a membrane is the same as that of a *vibrating string* the instantaneous shape of which coincides with the profile of the cylinder.

Let u be the displacement of the string between two neighboring points P and P' and Δs their distance. We separate this segment from the rest of the string by cuts at P and P' (cf. Fig. 20a). To make up for the

²⁷Since we wish to treat the phenomena of capillarity only to such extent as is indispensable in a textbook on general hydrodynamics, we refer for a more complete representation to H. Minkowski's carefully written article in *Enzyklopädie d. Math. Wiss.* Vol. V, 1 p. 558; there the reader will also find a molecular-theoretical explanation of capillarity which is more complete and cuts deeper than our purely phenomenological remarks.

²⁸*Mécanique céleste*, Supplément de l'action capillaire, 1806.

²⁹*Principia generalia theoriae fluidorum in statu aequilibrii*, (1830).

effect of the two remaining pieces, we introduce the tensions T and T' acting in the tangential directions of the string.³⁰ They are both equal in magnitude to the tension to which the string is subjected (say, by the peg at the end) but are not parallel to each other. Let $\Delta\epsilon$ be the change

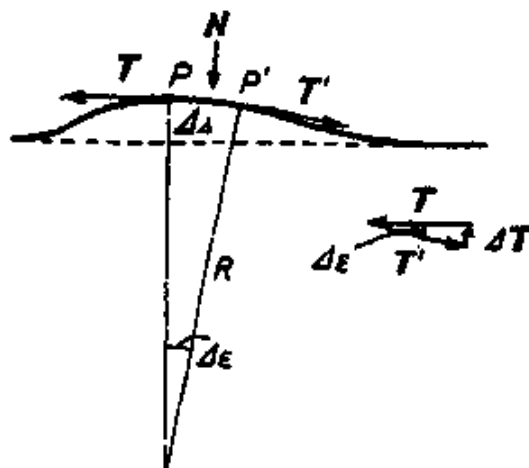


FIG. 20a. The tension T of a vibrating string and the corresponding restoring force N . At the same time this is a cross-sectional diagram of a cylindrical liquid surface.

of the slope corresponding to Δs , and ΔT the closing side of the force triangle with sides T and T' (insert of Fig. 20a). The magnitude of the vector ΔT is then

$$\Delta T = T |\Delta\epsilon|$$

and its direction is perpendicular to the arc element Δs . The resultant force acting on Δs equals $-\Delta T$. Instead of this, we introduce the resultant force per unit of length N so that $N\Delta s = -\Delta T$ and consequently

$$(1) \quad N\Delta s = T |\Delta\epsilon|.$$

On the other hand, the curvature $1/R$ is given by

$$(2) \quad \pm \frac{1}{R} = \lim_{\Delta s \rightarrow 0} \frac{\Delta\epsilon}{\Delta s} = \frac{\frac{\partial^2 u}{\partial x^2}}{\left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)^{3/2}}$$

where u is the deflection of the string and x the abscissa taken along the undeflected string; in the denominator of the last member we must take the positive value of the square root if s increases with x , which we assume.

Eqs. (1) and (2) determine the magnitude of N :

$$N = \frac{1}{|R|} T.$$

The direction of N points always to the concave side of the string.

³⁰Note that the dimension of T here is that of a force while in the following two-dimensional case T has the dimension force/length.

Let us now slightly extend the meaning of N by providing it with a $+$ or $-$ sign according to whether $\Delta\epsilon$ is negative or positive. (In Fig. 20a, N is thus positive.) If, in addition, we adopt the lower sign in (2) which makes $R \geq 0$ if $\Delta\epsilon \leq 0$, we may rewrite the last equation for the modified N :

$$(3) \quad N = \frac{1}{R} T,$$

or, for sufficiently small deflection,

$$(4) \quad N = - \frac{\partial^2 u}{\partial x^2} T.$$

From (4) we derive immediately the differential equation of the vibrating string mentioned in the context of Eq. (13.9). We have only to state the equilibrium between the external force in $+u$ -direction (that is $-N$) and the inertia force of the unit length of the string $-\rho \partial^2 u / \partial t^2$, and obtain

$$(5) \quad \rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2}$$

where ρ is the linear density of the string.

If we now pass from a cylindrical to a surface of double curvature, (4) has to be replaced by

$$(6) \quad N = \left(\frac{1}{R_1} + \frac{1}{R_2} \right) T,$$

where R_1 and R_2 are the two principal radii of curvature at the surface point under consideration. While in general the surface normals at two infinitesimally distant surface points do not intersect, but are skew to

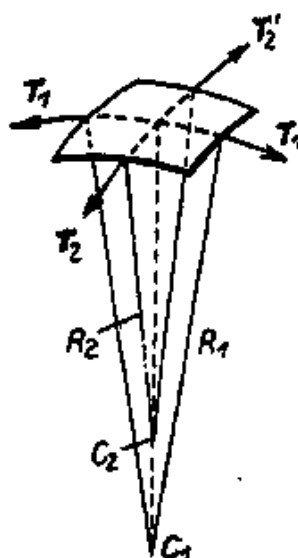


FIG. 20b. A vibrating membrane of double curvature; R_1 and R_2 are the radii of curvature of the two principal sections. At the same time this represents a liquid surface of the same shape.

each other, surface normals along the principal directions do intersect, and it is exactly this circumstance that defines the principal directions. The radii R_1 , R_2 are then explained as the reciprocals of the curvatures of the two principal (normal) sections. Denoting their centers of curvature by C_1 and C_2 respectively (see Fig. 20b), we now investigate the equilibrium of the surface tensions T acting on the infinitesimal rectangle $\Delta s_1 \Delta s_2$ of Fig. 20b oriented parallel to the principal directions 1 and 2. If one constructs the force triangles for either principal section as in Fig. 20a one obtains two concluding sides ΔT_1 and ΔT_2 and from them two normal pressures N_1 and N_2 as in equation (4). The normal pressure N in Eq. (6) is evidently obtained by adding N_1 and N_2 .

From (6), the differential equation of the vibrating membrane follows directly in the form mentioned in the context of Eq. (13.9), if one puts according to (3) and (4)

$$(7) \quad \frac{1}{R_1} \sim -\frac{\partial^2 u}{\partial x^2}, \quad \frac{1}{R_2} \sim -\frac{\partial^2 u}{\partial y^2}$$

and substitutes for N the inertia force

$$N = -\rho \frac{\partial^2 u}{\partial t^2}, \quad (\rho = \text{surface density}).$$

The equation thus obtained from (6)

$$(8) \quad \frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad c^2 = \frac{T}{\rho}$$

is not only correct for coordinates taken in the principal directions, but because of the invariance of the ∇^2 -operator for arbitrary Cartesian coordinates x, y , too.

The quantity

$$(9) \quad M = \frac{1}{R_1} + \frac{1}{R_2}$$

that occurs in (6) is the *mean curvature* of the surface, while the measure of the curvature³¹ introduced by Gauss

$$(9a) \quad K = \frac{1}{R_1 R_2}$$

is called *Gauss's or total curvature*.

³¹Minkowski *loc. cit.* defines the mean curvature $M = 1/2 (1/R_1 + 1/R_2)$ which corresponds better to the idea of a "mean" curvature. We retain the usual definition (9), which is, of course, irrelevant for the following results.

Our remarks about the vibrating string and membrane seem to be a digression from the subject of capillarity, but actually Eqs. (4) or (6) are applicable, as they stand, for the capillary surface skin in a cylindrical or more general form. They can be introduced in Eqs. (12.19) or (12.21), which were derived from Hamilton's principle, and determine therefore the surface condition for the hydrodynamic pressure p .

Let us first take a *free surface* such as water bounded by air; then $p = N$ according to (12.19). For a cylindrical surface (realized by a capillary wave traveling, e.g., in x -direction), N is given by (3) or (4), and we have

$$(10) \quad p = \frac{T}{R} \quad \text{or} \quad p = -T \frac{\partial^2 u}{\partial x^2},$$

where in the second equation the displacement u relative to the rest position is supposed to be sufficiently small. We shall come back to this equation in 25. Likewise we have for a surface of general shape according to (6)

$$(11) \quad p = T \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

Here as well as in (10) p is the (hydrodynamic) pressure in the fluid over which the capillary lamina is spread out.³²

A soap film subtended by a rigid boundary curve (a wire loop bent in some way) can be considered as an isolated capillary lamina devoid of the fluid body by which it is otherwise supported. Since the air pressure on both sides cancels, $p = 0$, and according to (11) the mean curvature M is then also zero. The same condition would, of course, be true for an actual membrane in equilibrium if subjected to the same boundary conditions; the equilibrium condition $\nabla^2 u = 0$ as it follows from (8) agrees with (11) when $p = 0$, provided the membrane is approximately plane.

The vanishing of M means that the two principal radii of curvature are numerically equal but opposite. A surface with this property is called a *minimal surface*. Its area F is smaller than that of any neighboring surface with the same boundary curve. This is easily understood from the capillary standpoint: The tensional energy U stored in the capillary film is

$$(12) \quad U = T \cdot F.$$

³²As a supplement to Eqs. (10) and (11) we should like to emphasize that there is also a pressure load on the plane, undisturbed surface termed internal pressure by van der Waals. As indicated on p. 93, it is a consequence of the molecular attraction. The surface tension T is present on a plane surface too, but it becomes active only on a wavy surface where it changes direction from point to point.

A virtual displacement at constant T with the boundary kept fixed produces a change of energy

$$(13) \quad \delta U = T\delta F.$$

This, on the other hand, is the virtual work which is supposed to vanish in equilibrium. Therefore $\delta F = 0$ is the equilibrium condition, indicating that the surface F occurring in the energy expression (12) must be a surface of smallest area.

This is shown in the following well known experiment:³³ on a plane soap film with an otherwise arbitrary boundary a loop of thread is placed. The film inside the loop is destroyed, e.g., by piercing it. Upon that, the loop is immediately pulled out into circular form. Since the circle is the curve of maximum area for a given circumference the remaining part of the membrane has the smallest area for the given (rigid) boundary and length of thread.

In the interior of a *closed soap bubble* there is overpressure relative to the external pressure. The shape of the bubble is spherical if gravity is neglected; since for reasons of symmetry $R_1 = R_2$, the radius of the bubble is by (11)

$$R = \frac{2T}{p}.$$

In a structure consisting of many adjacent bubbles (*foam*), the pressures in the single bubbles or cells will be slightly different. The pressure difference along the walls of the structure is therefore constant for any region that bounds not more than two cells. According to (11) the entire structure is then composed of surface pieces of *constant mean curvature*.

Condition (13) applies also to phenomena involving the *angle of contact*. Consider e.g. an arrangement as in Fig. 21 and suppose that, of the three

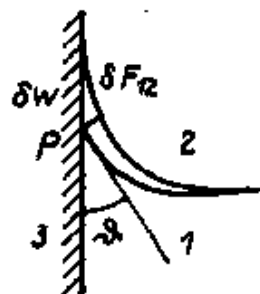


FIG. 21. Equilibrium between a liquid (1), air (2), and a rigid wall (3); ϑ is the angle of contact.

media abutting on each other at P , 1 is a liquid, 2 is air and 3 a solid. P is a point of the contact line which we imagine perpendicular to the plane of the drawing; if the point P is given a virtual displacement up-

³³A great many other experiments with accurate description of the soap bubble technique in C. V. Boys, *Soap Bubbles*, London (1931).

ward, δw , the surface of separation F_{12} is increased per unit of length of the contact line by

$$\delta F_{12} = \cos \vartheta \cdot \delta w \cdot 1,$$

where ϑ is the angle of contact, and the surface of contact F_{13} is increased by

$$\delta F_{13} = \delta w \cdot 1.$$

As in (13) we stipulate

$$T_{12}\delta F_{12} + T_{13}\delta F_{13} = (T_{12} \cos \vartheta + T_{13})\delta w = 0,$$

or, since δw is an arbitrary variation,

$$(14) \quad \cos \vartheta = - \frac{T_{13}}{T_{12}}.$$

In order to obtain an acute angle for ϑ (water and glass), T_{13} must be negative and $|T_{13}| < T_{12}$. For non wetting liquids (mercury and glass) ϑ is obtuse.

Capillary elevation and depression are caused by the combined action of gravity and surface tension, cf. problem III, 5. As familiar a phenomenon, however, as a liquid droplet under the action of gravity leads to a complicated differential equation that admits only approximate solutions. A less complicated problem is the theory of the beautiful phenomena known as "water bells".³⁴

A complete theory of capillarity is not possible without the use of thermodynamic considerations (Gibbs).

We should like to add that in the original theory of Laplace there is, beside the surface contribution to the capillary energy [our equation (12)], a second term which is proportional to the *volume* of the liquid. This term is essential in van der Waals's theory of the liquid state; it corresponds to the constant a that occurs in van der Waals's equation of state of a real gas.

³⁴A vertical jet is directed centrally against a circular plate. The liquid leaves the plate as a thin film in the form of a surface of revolution, sometimes in the shape of a bell. See E. Buchwald and H. König, Ann. d. Phys. 23, 557 (1935) and 26, 650 (1936).

CHAPTER IV

VORTEX THEORY

18. Helmholtz's Vortex Theorems

In the paper of 1858 in which Helmholtz gives the kinematic analysis of vortex motion (cf. the footnote on p. 1) he also completes the dynamic theory of vortices in its essentials. Simplifications in method were found in the following decades, but new results were not discovered.

The main content of Helmholtz's theory are the conservation laws:¹ *It is impossible to produce or destroy vortices*, or, expressed in more general terms, *the vortex strength is constant in time*. This theorem is correct under the following conditions: the fluid is inviscid and incompressible; the external forces possess a single-valued potential within the space filled by the fluid. Apart from the conservation of the vortex strength *in time* we shall see that there is also a *spatial* conservation: the vortex strength is constant *along each vortex line or vortex tube*, which must be either closed or end at the boundary of the fluid.

1. The Differential Form of the Conservation Theorem

Following Helmholtz we start from Euler's equations in the form (11.2). Since the external force \mathbf{F} has a potential U by hypothesis, Euler's equations can be written

$$(1) \quad \frac{d\mathbf{v}}{dt} = - \text{grad} \left(\frac{p + U}{\rho} \right).$$

The differential quotient on the left side is again the material acceleration, that is, the total rate of change of the velocity of the material particle under consideration. This leads to the form (11.7) of Euler's equations which can now be written in the following way:

$$(2) \quad \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \text{curl } \mathbf{v} = - \text{grad} \left(\frac{\mathbf{v}^2}{2} + \frac{p + U}{\rho} \right).$$

¹Corresponding theorems have been found in posthumous papers of Lejeune Dirichlet, +1859, and were published by Dedekind who edited Dirichlet's collected works.

We take the curl of this equation: the right member vanishes because of $\text{curl grad} = 0$. On setting $\omega = \frac{1}{2} \text{curl } \mathbf{v}$ we obtain

$$(3) \quad \frac{\partial \omega}{\partial t} - \text{curl} (\mathbf{v} \times \omega) = 0.$$

We make use of the following well known formula to transform the vector product of any two vectors ω and \mathbf{v} :

$$(4) \quad \text{curl} (\omega \times \mathbf{v}) = (\mathbf{v} \text{ grad}) \omega - (\omega \text{ grad}) \mathbf{v} + \omega \text{ div } \mathbf{v} - \mathbf{v} \text{ div } \omega.$$

(This formula is proved, just as the related formula (11.6), by verification in Cartesian coordinates.)

In our case, (4) simplifies on account of

$$\text{div } \mathbf{v} = 0 \quad \text{and} \quad \text{div } \omega = \text{div curl } \mathbf{v} = 0,$$

and one obtains by substitution in (3)

$$(5) \quad \frac{\partial \omega}{\partial t} + (\mathbf{v} \text{ grad}) \omega = (\omega \text{ grad}) \mathbf{v}.$$

The left member is the material rate of change of ω so that (5) can also be written as

$$(6) \quad \frac{d\omega}{dt} = (\omega \text{ grad}) \mathbf{v}.$$

The material rate of change of ω is zero wherever ω is zero. From this point Helmholtz immediately proceeds to the conclusion, that *fluid particles which were vortex-free at some time remain so for ever*. The reasoning he may have had in mind could be this: Let $\omega = 0$ for $t = t_0$, then by (6) $d\omega/dt = 0$. These two statements seem to imply that, for the mass particle under consideration, ω is still zero for $t = t_0 + \Delta t$. Then (6) implies $d\omega/dt = 0$ also for $t_0 + \Delta t$, which is equivalent to $\omega = 0$ for $t = t_0 + 2\Delta t$, and so on.

For a rigorous mathematical proof we need more preparation: Let \mathbf{A} be a vector associated with the moving fluid (not a "point function",



FIG. 22. The flux of a vector through a surface element σ that moves with the fluid.

but a "particle function") and $d\sigma$ a surface element that likewise moves with the fluid, changing its shape and position in the course of time. Let A_n denote the component of \mathbf{A} normal to $d\sigma$ and consider the flux of \mathbf{A} through $d\sigma$ at successive instances t_1 and t_2 . Corresponding line elements

ds_1 and ds_2 of the boundaries of $d\sigma_1$ and $d\sigma_2$ are correlated to each other by the displacement $w = vdt$, which is the length of the infinitesimal region in Fig. 22. Its top and bottom surfaces are $d\sigma_2$ and $d\sigma_1$ and its volume $d\tau = w_n d\sigma$. (In view of the limiting process we are going to perform, it does not make any difference whether we identify $d\sigma = d\sigma_1$ and $w_n \perp d\sigma_1$, or $d\sigma = d\sigma_2$ and $w_n \perp d\sigma_2$).

Gauss's theorem or the definition of the divergence (both amount to the same) yields

$$(7) \quad \int A_n d\sigma = \text{div } \mathbf{A} d\tau = (\text{div } \mathbf{A}) w_n d\sigma.$$

The integral to the left refers to the boundary of the volume swept by $d\sigma$ in the time dt . Note, however, that the vector field \mathbf{A} is for the present kept constant in time. (Think of it as "frozen" at the instant t_1 .)

In Fig. 22 one element of the *side surface* of $d\tau$ has been marked. Its contribution to the integral is

$$\mathbf{A} \cdot (d\mathbf{s} \times \mathbf{w}) = d\mathbf{s} \cdot (\mathbf{w} \times \mathbf{A}) = ds(\mathbf{w} \times \mathbf{A})_n,$$

where the subscript s to the last term denotes the component in the direction of $d\mathbf{s}$, which makes a right hand screw with the outward normal of the top surface. The integral over the side wall thus becomes equal to the line integral over the boundary of the surface

$$(7a) \quad \int ds (\mathbf{w} \times \mathbf{A})_n = d\sigma \text{curl}_n (\mathbf{w} \times \mathbf{A}),$$

a transformation that corresponds to Stokes's theorem.

The contributions of the top and bottom surfaces to the integral in (7) are simply

$$(A_n d\sigma)_2 - (A_n d\sigma)_1 = \delta(A_n d\sigma)$$

(note the opposite orientation of the normals). The right member $\delta(A_n d\sigma)$ denotes the change of the \mathbf{A} -flux due to the motion and distortion of $d\sigma$. From (7) and (7a) this change is now found as

$$(7b) \quad \delta(A_n d\sigma) = (\mathbf{w} \text{div } \mathbf{A} - \text{curl } (\mathbf{w} \times \mathbf{A}))_n d\sigma.$$

We have still to take account of the change of the field \mathbf{A} itself. The change of flux due to this cause is during the time dt

$$\frac{\partial A_n}{\partial t} dt d\sigma$$

On denoting the total change of flux by

$$\frac{d}{dt} (A_n d\sigma) dt$$

and substituting for the displacement \mathbf{w} its value $\mathbf{v} dt$, we obtain finally

$$(7c) \quad \frac{d}{dt} (A_n d\sigma) = \left(\frac{\partial A}{\partial t} + \mathbf{v} \operatorname{div} \mathbf{A} - \operatorname{curl} (\mathbf{v} \times \mathbf{A}) \right)_n d\sigma.$$

This formula, too, was found by Helmholtz,² but in another connection. It appears in his posthumous paper about the principle of least action in electrodynamics.

Upon identifying the vector \mathbf{A} with the vortex vector $\boldsymbol{\omega}$, Eq. (7c) which is correct for any direction of the normal \mathbf{n} of $d\sigma$, becomes

$$(8) \quad \frac{d}{dt} (\omega_n d\sigma) = \left(\frac{\partial \omega}{\partial t} - \operatorname{curl} (\mathbf{v} \times \boldsymbol{\omega}) \right)_n d\sigma$$

since $\operatorname{div} \boldsymbol{\omega} = 0$. The right member vanishes because of (3), thus

$$(9) \quad \frac{d}{dt} (\omega_n d\sigma) = 0.$$

This equation expresses the theorem of *conservation of vortices*. The flux of the vortex vector has a certain unalterable value for every surface element that moves with the fluid. *Vortices cannot be created or destroyed, or, the vorticity is a convective quantity of the flow; it adheres to the individual fluid particle and moves along with it.*

2. The Integral Form of the Conservation Theorem

We start from the concept of *circulation* (p. 17) established by Lord Kelvin³ with the aim of simplifying Helmholtz's vortex theory. Identifying the general vector \mathbf{A} of (2.21) with our \mathbf{v} , we consider the circulation

$$(10) \quad \Gamma = \oint \mathbf{v} \cdot d\mathbf{s} = \oint (\mathbf{v} \cdot d\mathbf{s}) = \oint (u dx + v dy + w dz)$$

around a circuit, i.e., a closed oriented curve in the fluid; we think of it as of a "material curve" that consists, like a string of beads, of *material fluid particles*, and floats along with them. Let C be the circuit at the

²Ges. Werke, Vol. III, p. 476, cf. particularly the equations (8a), (8b) that refer to the vector of the virtual displacement $\mathbf{A} = \{\delta\xi, \delta\eta, \delta\zeta\}$. Helmholtz writes everything in components, which does not add to lucidity; our notation and derivation corresponds to H. A. Lorentz's presentation in *Enzykl. d. Math. Wiss.* Vol. V, 2, p. 75, Eq. (5); there the equation is shown to be of fundamental importance in the electrodynamics of moving bodies.

³W. Thomson (Lord Kelvin), On Vortex Motion, *Transactions R. Soc. Edinburgh*, Vol. 25 (1869). Reprinted in *Mathem. and Phys. Papers*, Vol. IV, p. 13. In the same volume on p. 1, there is also the paper on vortex atoms often referred to in the popular literature: the vortex atoms left no impression on the development of atomic physics.

instant t and C' the distorted circuit at $t + \Delta t$ (see Fig. 23). Let ds be an arc element of C and ds' the corresponding element of C' . The straight arrows in the figure represent the paths of the fluid particles during Δt , drawn for two neighboring particles i and $i + 1$; the lengths are

$$v_i \Delta t \quad \text{and} \quad v_{i+1} \Delta t.$$

The arrows correspond, of course, to the vector \mathbf{w} in Fig. 22. Fig. 23 differs from Fig. 22 only in that the infinitesimal boundary curves of the elements $d\sigma$ are now pulled out into finite circuits C and C' . The following calculation is likewise closely related to the previous one.

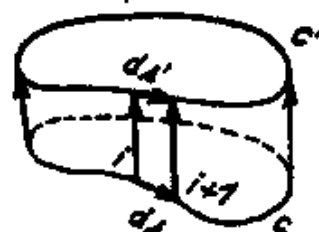


FIG. 23. Illustrating the concept of circulation.

First, we see from the quadrilateral formed by the arrows in Fig. 23 that

$$v_i \Delta t + ds' = ds + v_{i+1} \Delta t$$

or

$$(11) \quad \frac{ds' - ds}{\Delta t} = v_{i+1} - v_i.$$

We now calculate

$$(12) \quad \frac{d\Gamma}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Gamma' - \Gamma}{\Delta t} = \frac{d}{dt} \oint \mathbf{v} \cdot d\mathbf{s} = \oint \frac{d\mathbf{v}}{dt} \cdot d\mathbf{s} + \oint \mathbf{v} \cdot \frac{d}{dt} d\mathbf{s},$$

where Γ' stands for the circulation about C' . The expression that appears in the last integrand

$$\frac{d}{dt} d\mathbf{s}$$

is nothing else but the limit of the first member of (11) for $\Delta t \rightarrow 0$. An approximate value of the integral itself is thus found in the sum

$$\sum_i \mathbf{v}_i \cdot (\mathbf{v}_{i+1} - \mathbf{v}_i).$$

If we now go to the limit $i \rightarrow \infty$, the polygon $\cdots i, i + 1 \cdots$ approaches the circuit C and our sum approaches the line integral over C

$$\oint \mathbf{v} \cdot d\mathbf{v} = \frac{1}{2} v^2 \Big|_A^B,$$

where A is the arbitrary starting point (corresponding to $i = 0$) and B the end point of the integration, which coincides with A . Thus the second integral in the last member of (12) vanishes.

But the first integral vanishes likewise. According to Euler's Eqs. (1) it equals the line integral of $-\text{grad} [(p + U)/\rho]$, that is

$$\frac{p + U}{\rho} \Big|_B^A = 0.$$

Thus we have according to (12)

$$(13) \quad \frac{d\Gamma}{dt} = 0, \quad \Gamma = \text{const.}$$

The circulation taken about an arbitrary circuit that consists of material particles retains its initial value throughout the flow.

The circulation is directly connected with the vorticity present in the fluid as we shall show now. According to Stokes's theorem (3.2) we have

$$(14) \quad \Gamma = \oint \mathbf{v} \cdot d\mathbf{s} = \oint v_s ds = \int \text{curl}_s \mathbf{v} d\sigma = 2 \int \omega_s d\sigma.$$

If we now consider an arbitrary surface bounded by the circuit C and take at every element $d\sigma$ the normal component of the vortex vector ω , Γ equals twice the vorticity that passes the surface σ . Hence the flux of the vortex vector through the surface σ remains constant throughout the flow. Helmholtz's vortex theorem appears here in the form of Kelvin's circulation theorem; the contents of both are identical.*

It is, of course, not necessary to follow a particular surface σ chosen in (14) throughout its motion, since any surface through C gives the same vortex flux $\omega_s d\sigma$. This is again a consequence of Stokes's theorem which

*It is now easy to see that the assumption $\rho = \text{const}$ is not necessary for the validity of the conservation theorem. There is only one point in the foregoing proof that has not a purely kinematic character, and this is the vanishing of $\oint (dv/dt) \cdot ds$ in (12); here the compressibility might have an effect. But in the compressible case the integrand may be written in the form $-\text{grad} (\Phi + V)$ when the notation of Eqs. (7.6b) and (7.7) is used, provided that a p, ρ -relation is given.

The condition for the validity of Helmholtz's (or Kelvin's) theorem is therefore the existence of a single-valued "acceleration potential"

$$\int_{P_1}^{P_2} \frac{1}{\rho} (\text{grad } p + \mathbf{F}) \cdot d\mathbf{s},$$

and it does exist when

1. the fluid is inviscid
2. the external force per unit of mass, \mathbf{F}/ρ , has a single-valued potential V
3. a p, ρ -relation exists.

permits us to use an *arbitrary* surface σ . But it is required that the circuit C consist permanently of the same material particles.

Eqs. (13) and (14) become obviously identical with the previous Eq. (9) when σ becomes infinitesimal. Conversely, we could have derived the present result from (9) by an integration over $d\sigma$, but the reasoning used here is simpler and less abstract than the previous one and might be considered as a supplement not entirely superfluous.

While Eqs. (1) and (2) represent the *differential form* of Euler's equations, the circulation theorem (13) could be called their *integral form*, using a terminological distinction taken from electrodynamics. Likewise Eqs. (7) appearing in 15 represent the *differential form of Maxwell's equations*. Besides, there is an integral form of the same equations expressing their physical content in the form of relations between line and surface integrals, more suitable for a number of purposes.

3. The Spatial Distribution of the Vorticity

Following an idea of Helmholtz we consider now the field lines of the vector field ω : The vector ω determines at a given time at every point of the field of flow a certain positive direction of the vorticity. Proceeding from one point to the next along these directions we describe a *vortex line* in exactly the same way as we describe a line of force in a field of force by proceeding along the force directions. Now we take a surface element Δq perpendicular to the vortex line which we let pass through a border point of Δq , and construct the vortex lines that go through the other border points of Δq . In this way a *vortex tube* is obtained. The circulation about the boundary of Δq is according to (14)

$$(15) \quad \Gamma = 2\mu \quad \text{with} \quad \mu = \omega \Delta q.$$

The flux of the vortex vector through the tube or the *vortex strength*,* as we shall also call it, has been denoted by μ . According to (13), μ is constant in time regardless of the changes experienced by the vortex tube and the vortex lines of which its boundary consists.

However, the vortex strength remains constant also in space, that is,

*The vortex carried by an element Δq can be characterized by $2\omega\Delta q$ as well as by $\omega\Delta q$. When the first possibility is chosen, the term *vorticity* is frequently used as a name for $2\omega\Delta q$, so that the differential circulation $\Delta\Gamma$ is directly equal to the vorticity. In this book $\omega\Delta q$ has been chosen as the characteristic quantity and the term *vortex strength* introduced to denote it, while the term vorticity is regarded as synonymous with the term vortex vector $\omega = \frac{1}{2} \text{curl } v$.

This definition of μ also presupposes that the boundary of Δq is oriented so as to form a right hand screw with ω ; μ is then essentially positive. In the two-dimensional case it is more convenient to give a sign to μ [cf. (19.8) and 21].

as long as one proceeds along the same vortex tube (cf. Fig. 24). Since $\text{div } \omega = 0$, we obtain, by applying Gauss's theorem to a piece of the vortex tube between the normal cross-sections Δq and $\Delta q'$,

$$(16) \quad 0 = \int \text{div } \omega \, d\tau = \int \omega_n \, d\sigma$$

where the integration to the right includes Δq and $\Delta q'$ and the side walls of the tube segment. But ω_n vanishes on the side wall according to the

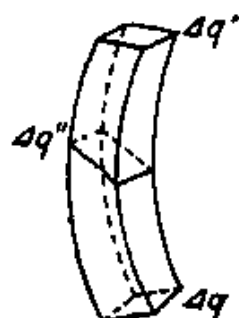


FIG. 24. The flux of the vortex vector through an orthogonal and an oblique cross-section of a vortex tube.

definition of the vortex tube. Furthermore, the component ω_n has the value $-\omega$ at Δq and the value $+\omega'$ at $\Delta q'$, where ω and ω' are the amounts of ω at the respective places. Thus Eq. (16) means

$$(17) \quad 0 = \omega' \Delta q' - \omega \Delta q, \quad \text{that is} \quad \mu' = \mu.$$

This finishes the proof. Note also that the same value μ is obtained when the tube is cut in oblique direction. With $\Delta q''$ denoting the area of such a cut and ω''_n the normal component of ω at $\Delta q''$, Gauss's theorem gives as before

$$\omega''_n \Delta q'' = \omega \Delta q = \mu.$$

Thus the product of the area of the cut and the normal component of the vortex vector is the same for an arbitrary cut through the vortex tube. This points to an *analogy* between vortex tubes and electric currents which will be discussed in 20.

It is an immediate consequence of the constancy of the vortex strength that a vortex tube can never end within the fluid. *It either reaches the boundary or it must be closed.*

So far we have silently assumed that ω is continuously distributed in space; the vortex strength μ is then of the same order of magnitude as the tube cross-section Δq . But we can consider the limiting case, that ω is concentrated at single, infinitely narrow, linear regions, in analogy to the so-called linear conductors in electricity. The vortex vector ω becomes infinitely large within those regions while it may have any finite value outside, in particular it may be zero. In the latter case we speak of a

vortex filament. Its strength has a finite value that does not go to zero with Δq and can be written as

$$(18) \quad \mu = \lim_{\Delta q \rightarrow 0} \omega \Delta q.$$

19. Two- and Three-dimensional Potential Flow

A flow that is in general irrotational is called a *potential flow*. The term "*in general*" means here "with the possible exception of singular points or lines." As discussed already in 11, p. 88, the condition of irrotationality, $2\omega = \text{curl } \mathbf{v} = 0$, implies the existence of a *velocity potential* Φ ; in the preceding article we found in addition that a flow remains irrotational if it starts out as an irrotational flow. In the present article, however, we shall consider only *steady* flow problems so that the question of time dependence of Φ does not arise. Since we shall further assume an incompressible fluid all through this chapter, the potential equation (11.13)

$$(1) \quad \nabla^2 \Phi = 0;$$

is in general fulfilled in the space filled by the fluid. (For the meaning of "in general" see above.)

We wish to investigate the properties of the flow that result from (1). This is particularly simple in the case of two-dimensional flow where

$$(2) \quad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0.$$

Let us first compare (2) with the wave equation (13.9)

$$(2a) \quad \frac{\partial^2 \Phi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0.$$

By putting $t = y$ and $c^2 = -1$, (2a) is changed into (2). Since the general integral of (2a) is known, viz. d'Alembert's solution (13.11)

$$(2b) \quad \Phi = F_1(x + ct) + F_2(x - ct),$$

we also know the *general solution* of (2); it is:

$$(3) \quad \Phi = \frac{1}{2} \{f(x + iy) + f^*(x - iy)\}.$$

The two arbitrary functions of (2b) must be assumed as conjugate complex in (3) (indicated by *) in order to make Φ real; besides, we have added the factor $\frac{1}{2}$ so that Φ is directly equal to the real part of $f(x + iy)$.

The general solution of (2) is therefore given by the *real part of an arbitrary analytic function* f of the complex variable $z = x + iy$. It is then convenient to write this function in the form

$$(4) \quad f(z) = \Phi(x, y) + i\Psi(x, y).$$

The imaginary part Ψ , that is, the function conjugate to Φ , is sometimes called the conjugate potential. In hydrodynamics ψ is known as the *stream function*.

To find interrelations between Φ and Ψ we differentiate (4) once with respect to x and once with respect to y , indicating the differentiation with respect to the complex argument z by $f'(z)$ as usual. This gives

$$(4a) \quad \frac{\partial}{\partial x} f(z) = f'(z) \frac{\partial z}{\partial x} = f'(z) = \frac{\partial \Phi}{\partial x} + i \frac{\partial \Psi}{\partial x},$$

$$\frac{\partial}{\partial y} f(z) = f'(z) \frac{\partial z}{\partial y} = i f'(z) = \frac{\partial \Phi}{\partial y} + i \frac{\partial \Psi}{\partial y}.$$

By elimination of $f'(z)$ results

$$\frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial y} + i \left(\frac{\partial \Psi}{\partial x} + \frac{\partial \Phi}{\partial y} \right) = 0.$$

Since Φ and Ψ are real, this relation is equivalent to the two relations

$$(5) \quad \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \quad \frac{\partial \Phi}{\partial y} = - \frac{\partial \Psi}{\partial x},$$

the *Cauchy-Riemann equations of theory of complex functions*. They can be expressed in the comprehensive form of one symbolic equation

$$(5a) \quad \frac{\partial \Phi}{\partial s} = \frac{\partial \Psi}{\partial n},$$

valid for an arbitrary pair of orthogonal line elements of the x, y -plane that follow each other in the same sense as the positive real and the positive imaginary axes. When ds coincides with the x -direction (and dn consequently with the y -direction), (5a) becomes identical with the first

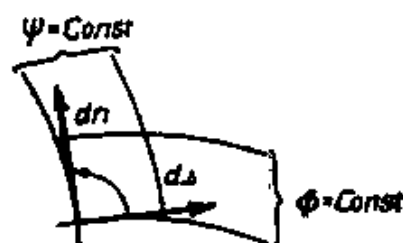


FIG. 25. Equipotential lines $\Phi = \text{const}$ and streamlines $\Psi = \text{const}$ in two-dimensional irrotational flow.

equation (5); when ds coincides with the y -direction and dn with the negative x -direction, (5a) becomes the second Eq. (5). That the relation between the directional derivatives (5a) is correct for any other pair ds, dn can be easily derived from (5). Thus (5a) is the appropriate vectorial

expression of the Cauchy-Riemann equations in invariant writing. When the direction ds is made to coincide with a contour line of the velocity potential as indicated in Fig. 25, the derivative $\partial\Phi/\partial s = 0$; hence, by (5a), the contour line of the stream function coincides with the n -direction since in this case

$$(6) \quad \frac{\partial\Psi}{\partial n} = 0.$$

Thus it is seen that the *two families of curves* $\Phi = \text{const}$ and $\Psi = \text{const}$ are *orthogonal to each other*. This is usually shown by cross multiplication of Eqs. (5), resulting in

$$(6a) \quad \text{grad } \Phi \cdot \text{grad } \Psi = 0,$$

but (5a) is a direct expression of that fact as we have just seen.

To justify the term *stream function*, we turn again to Fig. 25. If ds coincides with the contour line of the velocity potential as above, the vector $\mathbf{v} = -\text{grad } \Phi$ points in the direction of dn , but this is the direction of the curve $\Psi = \text{const}$ by (6); hence the lines $\Psi = \text{const}$ are the *stream lines* of the flow.

The families $\Phi = \text{const}$ and $\Psi = \text{const}$ are not only *orthogonal* but also *isometric*, that is, the rate at which the functions Φ and Ψ increase in the directions ds and dn respectively are equal; in other words, if the increments of both functions are equal, say $\delta = d\Phi = d\Psi$, then $ds = dn$ according to (5a). This means geometrically that the curves $\Phi = c_1$, $\Psi = c_2$, $\Phi = c_1 + \delta$, $\Psi = c_2 + \delta$ form an infinitesimal *square*. As a consequence, the image of an infinitesimal figure drawn in the $x + iy$ -plane appears geometrically similar in the $\Phi + i\Psi$ -plane: The mapping of the $x + iy$ -plane on the $\Phi + i\Psi$ -plane is conformal, that is, geometrically similar in the small. The scale of the mapping is given by $|f'(x + iy)|$; it changes from point to point, but, at a given point, is independent of the direction.⁵

⁵It is no exaggerated claim that the theory of analytic functions of a complex variable is identical with two-dimensional potential theory or, in terms of hydrodynamics, with two-dimensional theory of potential flow. In particular the methods introduced by Riemann in his dissertation (1851), where analytic functions are characterized by their intrinsic properties such as singularities, connectivities at branch points, etc. rather than by their analytic expressions, have been prompted by his deep interest in mathematical physics. The following story reveals in a characteristic way the attitudes of two great minds of the epoch toward Riemann's discovery. On a vacation trip in Switzerland, sometime in the seventies, Helmholtz meets Weierstrass, the great mathematician. The latter, recently engaged in the study of Riemann's dissertation, complains that R.'s methods were so hard to understand. Upon this Helmholtz borrows the paper and when he meets Weierstrass next time, he declares that Riemann's ideas appear to him perfectly natural and self-evident.

The simplest example in which we apply the methods of the theory of functions to hydrodynamics is the *line vortex* discussed on p. 35. We put

$$(7) \quad f(z) = iA \log z \quad (A \text{ real}).$$

Setting $z = re^{i\varphi}$ we obtain

$$f(z) = \Phi + i\Psi = -A\varphi + iA \log r,$$

and consequently

$$(7a) \quad \Phi = -A\varphi, \quad \Psi = A \log r.$$

The stream lines are given by $\Psi = \text{const}$ or $r = \text{const}$. The flow velocity follows from $\mathbf{v} = -\text{grad } \Phi$. Since in our case Φ depends only on φ , the velocity

$$(7b) \quad \pm v = v_\varphi = -\text{grad}_\varphi \Phi = -\frac{1}{r} \frac{\partial \Phi}{\partial \varphi} = \frac{A}{r}, \quad v_r = 0$$

in agreement with (4.29).

The origin is a singular point of the flow since v becomes infinite there. This singularity appears already in Eq. (7) since $f(z)$ goes logarithmically to infinity for $z = 0$.

The flow is everywhere irrotational except for $r = 0$. This must be so by virtue of its derivation from a complex function f , and can be verified by computing the vortex vector ω , which is here normal to the x, y -plane, in application of the results of problem I, 3:

$$2\omega = \frac{1}{r} \frac{\partial(rv_\varphi)}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \varphi} = 0.$$

The corresponding calculation in Cartesian coordinates is unnecessarily involved; it is carried out here for once:

$$v_x = \mp v \sin \varphi = \mp v \frac{y}{r} = -A \frac{y}{r^2},$$

$$v_y = \pm v \cos \varphi = \pm v \frac{x}{r} = +A \frac{x}{r^2}.$$

Thus we have at every point with the exception of the origin

$$2\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{A}{r^2} - \frac{2Ax^2}{r^4} + \frac{A}{r^2} - \frac{2Ay^2}{r^4} = 0.$$

According to the invariant definition in (2.21), ω can always be computed from the circulation around an infinitely small circuit enclosing the point under consideration. This implies that *the circulation around any*

point with the exception of $r = 0$ vanishes provided the circuit is sufficiently close to the point. If on the other hand the circulation is taken around any finite (arbitrarily small) circle enclosing the origin one obtains from (18.10)

$$(8) \quad \Gamma = \oint \mathbf{v} \cdot d\mathbf{s} = \int_0^{2\pi} v r d\varphi = 2\pi A.$$

Γ is independent of r and therefore the same for any two circles r_1 and r_2 ; this is necessarily so since the circular ring between the two is vortex-free. The entire vorticity is thus concentrated at the singular origin. The vortex strength* at this point is according to our definition of μ in (18.15)

$$(8a) \quad \mu = \pi A.$$

Viewed in three dimensions, the singular point corresponds to a *vortex filament* normal to the x, y -plane. A second vortex filament of opposite strength $\mu = -\pi A$ is at infinity, when infinity is considered as a point of the complex plane as one usually does in theory of functions. Note the formula following from (7b) and (8a):

$$(9) \quad v_\varphi = \frac{\mu}{\pi r}.$$

We can have both vortex filaments located in a finite domain, e.g., at the points $z = \pm c$, if we subject z to a bilinear transformation. Consider instead of (7) the function

$$(10) \quad f(z) = iA \log \frac{z - c}{z + c}.$$

and put

$$(10a) \quad z - c = r_1 e^{i\varphi_1}, \quad z + c = r_2 e^{i\varphi_2},$$

Defining the *bipolar* coordinates ρ and φ by

$$(10b) \quad \rho = \log \frac{r_1}{r_2}, \quad \varphi = \varphi_1 - \varphi_2,$$

we obtain according to (4)

$$(11) \quad \Phi = -A\varphi, \quad \Psi = A\rho = A \log \frac{r_1}{r_2}.$$

Since all angles φ subtended by the segment $(-c, +c)$ have their vertices on a circular arc, the equipotential lines $\Phi = \text{const}$ are circles through the singular points $z = \pm c$; but the streamlines $\Psi = \text{const}$ are

*In the two-dimensional case, we take all circulation integrals in the same sense ("enclosed" area to the left hand); the vortex strength μ can then be positive or negative.

also circles since the locus of constant ratio of distances from two given points is a circle. Note that the intersection points of any circle $\Psi = \text{const}$ with the real axis separate the points $-c, +c$ harmonically. Fig. 26 shows the orthogonality of the two families of circles, it also gives an idea of the network of infinitesimal quadratic meshes that is obtained if one allows the parameters of either family of circles to change in equal increments δ .

Problem IV, 1 shows how to handle bipolar coordinates. They are infrequently used in mathematical physics, one of the reasons being that

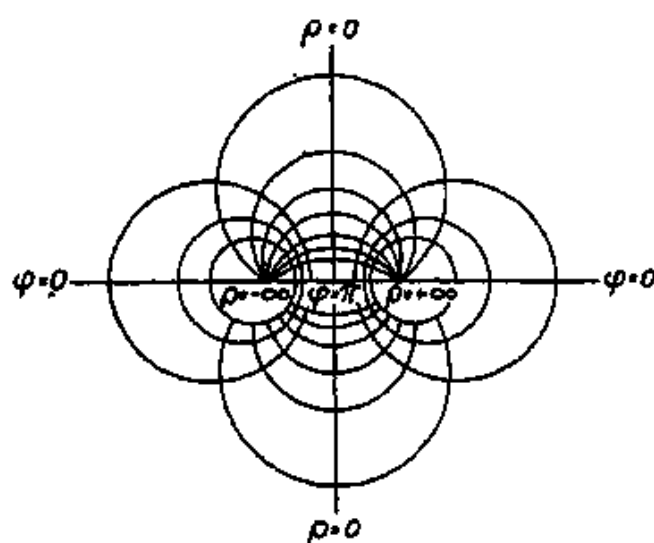


FIG. 26. Superposition of two vortices of equal strength and opposite orientation. The diagram also shows bipolar coordinate lines.

the two-dimensional wave equation is not "separable" in these coordinates; this is discussed in the same problem.

Our present function $f(z)$ can be obtained by the superposition of two simple vortices of the type (7), one having the center $z = +c$ and the constant $+A$, the other the center $z = -c$ and the constant $-A$:

$$(12) \quad f(z) = iA \log(z - c) - iA \log(z + c).$$

A circuit of arbitrary shape enclosing $+c$, but not $-c$, gives according to (8)

$$(12a) \quad \Gamma_1 = 2\pi A$$

A circuit of arbitrary shape enclosing $-c$, but not $+c$, gives according to (8)

$$(12b) \quad \Gamma_2 = -2\pi A$$

A circuit that encloses both or none of the two singularities gives $\Gamma = 0$. As in (8a), the strength of the vortex filament in $+c$ or $-c$ is

$$(13) \quad \mu_1 = \pi A \quad \text{or} \quad \mu_2 = -\pi A.$$

The field belonging to our present function $f(z)$ can also be obtained by the superposition of two fields of the type (9) if r is replaced by $|z - c|$ and $|z + c|$, respectively. One obtains by superposition

$$(14) \quad \mathbf{v} = \frac{\mu \hat{\phi}_1}{\pi |z - c|} - \frac{\mu \hat{\phi}_2}{\pi |z + c|} = \frac{\mu \hat{\phi}_1}{\pi r_1} - \frac{\mu \hat{\phi}_2}{\pi r_2}$$

Here $\hat{\phi}_1$ and $\hat{\phi}_2$ are unit vectors in the direction of increasing angles φ_1 and φ_2 . This result will be compared in problem IV, 2 with the representation of the same flow in bipolar coordinates according to (11)

$$(14a) \quad \mathbf{v} = A \operatorname{grad} \varphi = \frac{\mu}{\pi} \operatorname{grad} \varphi.$$

We turn now to confocal elliptic coordinates, which are of importance for the flow patterns to be treated in Chapter VI. A simple approach to these orthogonal coordinates takes on the form of a geometrical analysis of the mapping of the plane $z = x + iy$ on the plane $\zeta = \xi + i\eta$ given by the following relation

$$(15) \quad z = c \cosh \zeta$$

which may also be written as

$$(15a) \quad x + iy = \frac{c}{2} (e^{\xi+i\eta} + e^{-\xi-i\eta}).$$

The substitution

$$e^{-i\eta} = \cos \eta \pm i \sin \eta,$$

leads immediately to

$$x + iy = c (\cosh \xi \cos \eta + i \sinh \xi \sin \eta).$$

Separation of real and imaginary parts on both sides gives now

$$(16) \quad x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta.$$

From (16), η can be eliminated by the use of $\cos^2 \eta + \sin^2 \eta = 1$, resulting in

$$(17a) \quad \frac{x^2}{c^2 \cosh^2 \xi} + \frac{y^2}{c^2 \sinh^2 \xi} = 1.$$

The elimination of ξ from (16) by the use of $\cosh^2 \xi - \sinh^2 \xi = 1$ gives

$$(17b) \quad \frac{x^2}{c^2 \cos^2 \eta} - \frac{y^2}{c^2 \sin^2 \eta} = 1.$$

For constant ξ , (17a) represents an ellipse with the semi-axes

$$a = c \cosh \xi, \quad b = c \sinh \xi.$$

Its eccentricity is given by

$$(18a) \quad \sqrt{a^2 - b^2} = c,$$

and thus independent of ξ . Thus, if ξ varies, one obtains a system of confocal ellipses with foci $x = \pm c, y = 0$. On the other hand, for constant η (17b) represents a hyperbola with the semi-axes

$$a = c \cos \eta, \quad b = c \sin \eta;$$

its eccentricity is

$$(18b) \quad \sqrt{a^2 + b^2} = c$$

therefore independent of η and equal to that of the ellipses given in (18a). With variable parameter η , (17b) represents a system of hyperbolas, confocal among themselves and with the system of ellipses, cf. Fig. 27.

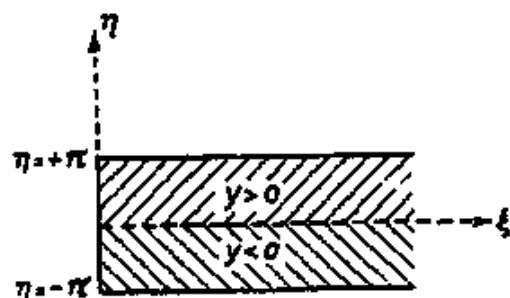
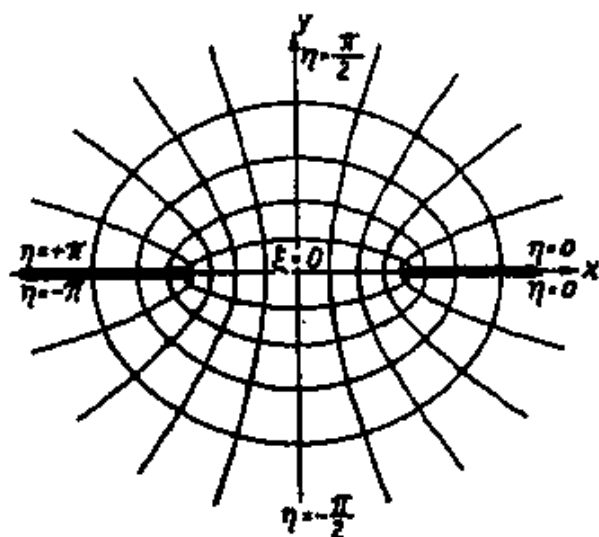


FIG. 27 (left). The system of confocal ellipses and hyperbolas $\xi = \text{const}$ and $\eta = \text{const}$.

FIG. 27a (above). The mapping of the $(x + iy)$ -plane on the $(\xi + i\eta)$ -plane.

It is of interest to consider the limiting parameter values $\xi = 0$ and $\xi = \infty$ in the family of ellipses: the value $\xi = 0$ gives by (17a) the focal line

$$y = 0, \quad -c < x < +c$$

and the value $\xi = \infty$ gives an infinitely extended ellipse.

As regards the parameter η , note the particular values

$$\eta = 0, \quad \eta = \pm \frac{\pi}{2}, \quad \eta = \pm \pi.$$

The value $\eta = 0$ characterizes, by (17b), the part of the real axis to the right of c : $y = 0, x > c$. $\eta = \pm \pi/2$ corresponds to $x = 0$, the hyperbola coincides with the y -axis. $\eta = \pm \pi$ gives the part of the x -axis to the left of c : $y = 0, x < -c$.

In order to obtain a unique relation between the coordinates ξ , η and the points of the x, y -plane, ξ and η must be restricted to a certain domain. One way of doing this is to assign the values $0 < \eta < \pi$ to the upper halves of the hyperbolas, and the values $-\pi < \eta < 0$ to the lower halves. By this rule the entire x, y -plane is mapped on the strip of the ξ, η -plane

$$(19) \quad 0 < \xi < +\infty, \quad -\pi < \eta < +\pi$$

indicated in Fig. 27a. Note that the one-to-one correspondence between the x, y -plane and the ξ, η -strip does not include the boundary of the strip; either the upper or the lower half of the borderline must be omitted. Or one might also consider boundary points that are mapped on the same point in the x, y -plane as identical (viz. $0, \pm\eta$; $\xi \neq 0, \pm\pi$).

The advantage of elliptic over bipolar coordinates with regard to the separability of the wave equation will be discussed in connection with problem IV,3. In Chapter VI problems arising from the flow past a plate will be treated by means of elliptic coordinates. We shall use a complex potential of the form

$$(20) \quad \Phi + i\Psi = f(\zeta), \quad \text{where} \quad f(\zeta) = \text{const} \cdot \sinh \zeta.$$

Here, ζ is thought to be related to z according to Eq. (15) so that $f(\zeta)$ becomes a complex function of z .

Let us return once more to the simple potential (7), but without the factor i on the right hand side. Then

$$(21) \quad f(z) = A \log z, \quad \Phi = A \log r, \quad \Psi = A\varphi.$$

The rays $\varphi = \text{const}$ originating at O are now the streamlines. The contour lines of the potential are the circles $r = \text{const}$ about O . Depending on the sign of A , O is either a source or a sink. Note in this connection that the general central symmetric solution of the potential equation is the logarithmic potential $\Phi = \log r$, save for a multiplicative and an additive constant.

A few remarks about the potential flow in space follow here as a supplement.

The central symmetric solution of the potential equation in space is the so-called Newtonian potential

$$(22) \quad \Phi = \frac{A}{r}.$$

No historical implication is intended, and Newton never spoke of this potential; the term should imply that (22) is associated with the field of the Newtonian attractive force

$$(22a) \quad \mathbf{F} = -\text{grad } \Phi = -A \text{ grad } \frac{1}{r}.$$

The hydrodynamical meaning of Φ is again the velocity potential of a source or sink at O .

In general, no stream function Ψ can be associated with the velocity potential in the three-dimensional case. Only in the case of an axial symmetric flow is it possible to construct a stream function.⁶ For this purpose cylindrical coordinates ρ, φ, z are introduced, z being the axis of symmetry. If the problem is axial-symmetric, Φ will not depend on φ but only on ρ and z . The potential equation (1), the mathematical equivalent of incompressibility, reads then (see problem I, 3)

$$(23) \quad \frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Phi}{\partial \rho} + \frac{\partial^2 \Phi}{\partial z^2} = 0.$$

It may also be written in the form

$$(23a) \quad \frac{\partial}{\partial \rho} \rho \frac{\partial \Phi}{\partial \rho} + \frac{\partial}{\partial z} \rho \frac{\partial \Phi}{\partial z} = 0.$$

From this form we can conclude that there exists a function $\Psi = \Psi(\rho, z)$ such that

$$\frac{\partial^2 \Psi}{\partial \rho \partial z} = \frac{\partial}{\partial \rho} \rho \frac{\partial \Phi}{\partial \rho} = - \frac{\partial}{\partial z} \rho \frac{\partial \Phi}{\partial z}.$$

If we now put in particular

$$(24) \quad \frac{\partial \Psi}{\partial z} = \rho \frac{\partial \Phi}{\partial \rho}, \quad \frac{\partial \Psi}{\partial \rho} = - \rho \frac{\partial \Phi}{\partial z},$$

we obtain by cross multiplication

$$\rho \left(\frac{\partial \Phi}{\partial \rho} \frac{\partial \Psi}{\partial \rho} + \frac{\partial \Phi}{\partial z} \frac{\partial \Psi}{\partial z} \right) = 0.$$

This implies that in any meridian plane $\varphi = \text{const}$ the curves $\Phi = \text{const}$ and $\Psi = \text{const}$ are mutually orthogonal. In other words, in such a plane the lines $\Psi = \text{const}$ coincide with the direction $\text{grad } \Phi$ and are *stream lines*. The function Ψ may therefore be considered as a stream function. From (24) and $\mathbf{v} = -\text{grad } \Phi$, one obtains the velocity components in the form

$$(25) \quad v_\rho = - \frac{1}{\rho} \frac{\partial \Psi}{\partial z}, \quad v_z = \frac{1}{\rho} \frac{\partial \Psi}{\partial \rho}.$$

It is impossible, however, to combine Φ and Ψ to form a complex function $\Phi + i\Psi$ of the variable $\rho + iz$ as in the two-dimensional case. If this could be done, then the relations (24) should coincide with the Cauchy-Rie-

⁶Stokes, Trans. Cambridge Phil. Soc. Vol. 7, 1842, p. 439.

mann Eqs. (5), which is not the case because of the factor ρ in (24). Also, Φ and Ψ would have to satisfy the equation $\partial^2 u / \partial \rho^2 + \partial^2 u / \partial z^2 = 0$, but actually Φ satisfies Eq. (23) and Ψ the equation obtained from $\text{curl } \mathbf{v} = 0$ by (25):

$$(26) \quad \frac{\partial^2 \Psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \Psi}{\partial \rho} + \frac{\partial^2 \Psi}{\partial z^2} = 0.$$

The powerful tool of the theory of complex functions cannot be used in three-dimensional potential theory.⁷

20. A Fundamental Theorem of Vector Analysis

The theorem which we wish to prove here is this: *A continuous vector field \mathbf{V} , defined everywhere in space and vanishing at infinity together with its first derivatives, can be represented as the sum of an irrotational field \mathbf{V}_1 and a solenoidal field \mathbf{V}_2 :*

$$(1) \quad \mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2,$$

where

$$(1a) \quad \text{curl } \mathbf{V}_1 = 0, \text{ div } \mathbf{V}_2 = 0.$$

Our decomposition charges all sources and sinks of the given field \mathbf{V} to the component field \mathbf{V}_1 and all vortices to the component field \mathbf{V}_2 so that:

$$(1b) \quad \text{div } \mathbf{V}_1 = \text{div } \mathbf{V}, \quad \text{curl } \mathbf{V}_2 = \text{curl } \mathbf{V}.$$

The representation (1) is *unique except for a vectorial constant*.

This fundamental theorem was proved in its essentials by Stokes⁸ in

⁷The great mathematician David Hilbert (1862-1943) sharply characterized the futility of all attempts in this direction by the following remark: time is one-dimensional, space is three-dimensional, however the number, that is, the perfect complex number, has two dimensions.

⁸In his big paper: On the dynamical theory of diffraction, Trans. Cambridge Phil. Soc. Vol. 9, p. 1, reprinted in Math. and Phys. Papers, Vol. II, Cambridge, 1883, p. 243 and particularly p. 254 ff. Stokes does not yet give an explicit definition of the vector potential \mathbf{A} . Instead, \mathbf{V}_2 is calculated by the following formula obtained through a combination of our equations (5) and (9)

$$4\pi \mathbf{V}_2 = \int \frac{\text{curl curl } \mathbf{V}}{r} d\tau.$$

For a rigorous proof see: O. Blumenthal, Ueber die Zerlegung unendlicher Vektorfelder, (Mathem. Annalen 61, 235, 1905). His only restriction is that \mathbf{V} and its first derivatives vanish at infinity while no additional assumption is made about how quickly they vanish. It turns out that the component fields \mathbf{V}_1 and \mathbf{V}_2 need not vanish themselves, they may even become in a restricted way infinite. In the following we shall make the somewhat vague assumption that \mathbf{V} vanishes "sufficiently strongly" at infinity.

1849. In a more complete form, it is the basis of Helmholtz's paper on vortex motion of 1858, where it is also proved (implicitly however, and, of course, without the use of vector notation). This theorem penetrates deeper into the integration methods of mathematical physics than anything we have done so far; it is of great importance in hydrodynamics, but even more so in electro-magnetic theory. We shall prove it in three steps.

1. *Calculation of V_1 .* The irrotational vector V_1 can be derived from a scalar potential Φ in the form

$$(2) \quad V_1 = -\text{grad } \Phi + \text{const.}$$

The potential Φ satisfies the differential equation

$$\text{div grad } \Phi = \nabla^2 \Phi = -\text{div } V_1,$$

which because of our condition (1b) may also be written as

$$(3) \quad \nabla^2 \Phi = -\text{div } V.$$

The constant added in (2) for the sake of generality means nothing for the determination of Φ . From the *inhomogeneous potential equation* (3) the right member of which is a known function, Φ is determined by Green's theorem in the form

$$(4) \quad 4\pi\Phi = \int \frac{\text{div } V}{r} d\tau.$$

(see 1a below). Here $r = r_{PQ}$ is the distance of the field point P , at which Φ is to be determined, from the "source" point Q which designates the place of the volume element $d\tau$, that is to say, the coordinates of Q are the integration variables. If Φ is known, V_1 is also known from (2) except for a constant.

2. *Calculation of V_2 .* One introduces a *vector potential* A by putting

$$(5) \quad V_2 = \text{curl } A + \text{const}$$

and adds as a further condition

$$(6) \quad \text{div } A = 0.$$

Note: if A fulfills (5), $A + \text{grad } \lambda$, with λ being an arbitrary point function, does the same; hence an additional condition like (6) may be stipulated.

Because of (5)

$$\text{curl curl } A = \text{curl } V_2,$$

which, according to (1b), may be written

$$(7) \quad \text{curl curl } A = \text{curl } V.$$

Also here the additive constant in (5) is immaterial. To (7) we now apply the vector rule (3.10a) valid for Cartesian components:

$$\nabla^2 \mathbf{A} = \text{grad div } \mathbf{A} - \text{curl curl } \mathbf{A}.$$

Because of (6) this reduces to $\nabla^2 \mathbf{A} = -\text{curl curl } \mathbf{A}$, hence (7) is equivalent to

$$(8) \quad \nabla^2 \mathbf{A} = -\text{curl } \mathbf{V}.$$

The components of \mathbf{A} then satisfy, like Φ , inhomogeneous potential equations that can be integrated by the following formula, analogous to (4):

$$(9) \quad 4\pi \mathbf{A} = \int \frac{\text{curl } \mathbf{V}}{r} d\tau.$$

We shall prove in 2a that this representation of \mathbf{A} satisfies by itself the condition (6). When \mathbf{A} is known, \mathbf{V}_2 is obtained by (5) except for a constant.

3. *Uniqueness of the resolution* $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$. Suppose there were another resolution $\mathbf{V}'_1, \mathbf{V}'_2$ beside $\mathbf{V}_1, \mathbf{V}_2$. The irrotational vector $\mathbf{V}'_1 - \mathbf{V}_1$ would then at the same time be solenoidal, since we should have according to (1b)

$$\text{div} (\mathbf{V}'_1 - \mathbf{V}_1) = 0.$$

In the same way we conclude that the solenoidal vector $\mathbf{V}'_2 - \mathbf{V}_2$ should at the same time be irrotational since, again by (1b),

$$\text{curl} (\mathbf{V}'_2 - \mathbf{V}_2) = 0.$$

The calculation of the potentials Φ and \mathbf{A} for the vectors $\mathbf{V}'_1 - \mathbf{V}_1$ and $\mathbf{V}'_2 - \mathbf{V}_2$ carried out as in (4) and (9) would yield $\Phi = 0, \mathbf{A} = 0$. Hence, by (2) and (5), we have

$$\mathbf{V}'_1 - \mathbf{V}_1 = \text{const}, \quad \mathbf{V}'_2 - \mathbf{V}_2 = \text{const}.$$

Thus the resolution is indeed *unique* except for a vectorial constant. For more details see Blumenthal, loc. cit.

1a. *The calculation of Φ and \mathbf{A} by Green's theorem.* Let us put in Green's theorem (3.15)

$$U = \Phi, \quad V = \frac{1}{r}$$

and apply it to a large sphere with radius R and center O . We know from (19.22) that $\nabla^2(1/r) = 0$ except for the singular point $r = 0$ which has to be excluded from the integration by a small spherical surface with

radius ρ . First we calculate the surface integral over the ρ -sphere which appears on the right side of (3.15). Let Ω denote the solid angle, that is, $d\Omega$ be the surface element of the unit sphere. The surface element of the ρ -sphere is $d\sigma = \rho^2 d\Omega$. Since dn is the outward normal seen from the region of integration, we have

$$dn = -d\rho, \quad \frac{\partial U}{\partial n} = -\frac{\partial \Phi}{\partial \rho}, \quad \frac{\partial V}{\partial n} = -\frac{d}{d\rho} \frac{1}{\rho} = +\frac{1}{\rho^2}.$$

That part of the right member of (3.15) which refers to the ρ -sphere becomes therefore

$$(10) \quad \Phi \int d\Omega + \int \frac{\partial \Phi}{\partial \rho} \rho d\Omega.$$

The second integral in (10) vanishes for $\rho \rightarrow 0$, and the first equals 4π . (Note that the factor Φ tends to the value of Φ at $\rho = 0$).

We now consider the surface integral over the R -sphere. The same consideration as before leads to the following expression for this integral

$$(10a) \quad - \int \Phi d\Omega - \int \frac{\partial \Phi}{\partial R} R d\Omega.$$

But this time both integrals vanish provided Φ goes to zero with $R \rightarrow \infty$ and $\partial \Phi / \partial R$ vanishes more strongly than $1/R$. This is what we want to enforce by requiring that V should vanish "sufficiently strongly" at infinity. The sum of the surface integrals over both spheres becomes then $4\pi\Phi$.

On the other hand, the first member of (3.15) is, because of (3) and $\nabla^2(1/r) = 0$,

$$\int \frac{\text{div } \mathbf{V}}{r} d\tau.$$

This finishes the proof of Eq. (4); the proof of (9) follows exactly the same lines.

2a. Proof that $\text{div } \mathbf{A} = 0$. We take the divergence of either side of equation (8)

$$\text{div } \nabla^2 \mathbf{A} = -\text{div } \text{curl } \mathbf{V} = 0.$$

Since the operators div and ∇^2 are commutative, one concludes that $u = \text{div } \mathbf{A}$ satisfies the homogeneous equation $\nabla^2 u = 0$. We carry out the integration as before under 1a) and notice that the surface integral over the R -sphere has again the limit zero, while the volume integral that appears in Green's representation of u is now identically zero. This shows that the additional condition (6) for $\text{div } \mathbf{A}$ is actually fulfilled.

3a. *More about the uniqueness.* When the vector field \mathbf{V} is not defined everywhere in space, but only inside a boundary surface Σ , additional surface integrals over Σ appear in place of the vanishing surface integrals over the surface of the infinitely large R -sphere. In the calculation of Φ , for instance, we have instead of (10a)

$$(12) \quad \int_{\Sigma} \Phi \frac{\partial}{\partial n} \frac{1}{r} d\Sigma - \int_{\Sigma} \frac{1}{r} \frac{\partial \Phi}{\partial n} d\Sigma.$$

This invalidates the representation (4) as well as our uniqueness proof, and the same is true for the representation (9) and the transformation (11). In the case of a finite domain it is indeed necessary to prescribe suitable boundary conditions on the surface Σ , in order to give a definite meaning to the resolution "irrotational-solenoidal".

Note that in the case of an infinitely extended vector field that is everywhere defined the constants in (2) and (5) must cancel each other, since the sum $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$ is supposed to vanish at infinity, according to our assumption at the beginning of this article.

Our uniqueness theorem has a well known counterpart in two dimensions: a function $f(x + iy)$ that is everywhere regular in the complex plane including the point ∞ is necessarily a constant (Liouville's theorem). Both real and imaginary parts of such a function represent a two dimensional potential that is irrotational and solenoidal at the same time; both parts must therefore be constants, making f a complex constant.

4. *Representation of a velocity field with given vortices.* For the sake of simplicity we assume first the entire space filled with a moving fluid and identify the vector \mathbf{V} with the fluid velocity \mathbf{v} . According to the condition of incompressibility, $\text{div } \mathbf{v} = 0$; hence by (4) $\mathbf{V}_1 = 0$, and $\mathbf{V} = \mathbf{v} = \mathbf{V}_2$ (the additive constant has been omitted). Upon introducing the vortex vector $\boldsymbol{\omega}$ which, for the present, we assume continuously distributed in space, we have by (9) and (5)

$$(13) \quad 2\pi\mathbf{A} = \int \frac{\boldsymbol{\omega}}{r} d\tau, \quad \mathbf{v} = \text{curl } \mathbf{A}.$$

The velocity \mathbf{v} can be represented by a superposition of the contributions of the individual vortex elements $\boldsymbol{\omega} d\tau$. Consider now the vortex tube (cf. p. 136) associated with a certain vortex element $\boldsymbol{\omega} d\tau$, a segment of which having the length $|\Delta s|$ and cross-section Δq is shown in Fig. 28. If the vortex strength $\mu = \boldsymbol{\omega} \Delta q$ [see (18.15)] is introduced, the contribution \mathbf{A}' of our vortex element to the vector potential \mathbf{A} is given by

$$(14) \quad 2\pi\mathbf{A}' = \frac{\mu}{r} \Delta s.$$

The corresponding contribution \mathbf{v}' to the velocity of the flow is then

$$(15) \quad \mathbf{v}' = \text{curl } \mathbf{A}' = \frac{1}{2\pi} \text{curl} \left(\frac{\mu}{r} \Delta \mathbf{s} \right).$$

Here the vector $\mu \Delta \mathbf{s}/r$ has the direction of $\Delta \mathbf{s}$ which, according to the definition of the vortex tube is identical with the direction ω . The operation curl to which this vector is subjected refers to the coordinates of the field point P for which \mathbf{v}' is to be computed. It is now convenient to introduce

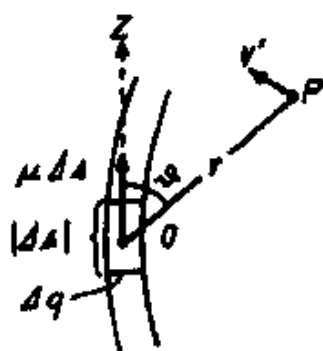


FIG. 28. The contribution of the element $\Delta \mathbf{s}$ of a vortex tube to the vector potential \mathbf{A} and the flow vector \mathbf{v} .

an auxiliary coordinate system, taking $\Delta \mathbf{s}$ as the z -axis of a Cartesian system with origin O at the volume element of the vortex tube under consideration.⁹ With $A' = A'_z$, $A'_x = A'_y = 0$, we obtain from (15)

$$(16) \quad 2\pi v'_x = 2\pi \frac{\partial A'_z}{\partial y} = -\frac{\mu}{r^3} \Delta s \frac{y}{r},$$

$$2\pi v'_y = -2\pi \frac{\partial A'_z}{\partial x} = +\frac{\mu}{r^3} \Delta s \frac{x}{r}, \quad v'_z = 0.$$

These three equations can be written in condensed form:

$$(17) \quad 2\pi \mathbf{v}' = \frac{\mu}{r^3} \Delta \mathbf{s} \times \mathbf{r}.$$

The direction of \mathbf{v}' is orthogonal to $\Delta \mathbf{s}$ as well as to the position vector $\mathbf{r} = \mathbf{OP}$; its magnitude is given by

$$(18) \quad 2\pi |\mathbf{v}'| = \frac{\mu}{r^3} |\Delta \mathbf{s}| \sin \vartheta,$$

where ϑ is the angle between $\Delta \mathbf{s}$ and \mathbf{r} . The orientation of \mathbf{v}' follows from the right hand rule for vector products.

The motion at the field point P due to the vortex element $\mu \Delta \mathbf{s}$ is de-

⁹Instead of that we could also use the general formula $\text{curl } f\mathbf{a} = f \text{curl } \mathbf{a} + (\text{grad } f) \times \mathbf{a}$, which gives (17) if one puts $\mathbf{a} = \Delta \mathbf{s} = \text{const}$ and $f = \mu/r$.

terminated by the law (17). The total velocity at P is the vectorial sum of the elemental velocities \mathbf{v}' :

$$(19) \quad \mathbf{v} = \sum \mathbf{v}'.$$

5. *The electro-magnetic interpretation of solenoidal fields.* We now compare (18) with the well known classical law of Biot-Savart

$$(20) \quad |d\mathbf{F}| = \frac{i ds}{r^2} \sin \vartheta,$$

where i is the current intensity in the element ds of the conductor and $d\mathbf{F}$ the force acting on the magnetic unit pole at the field point P ; ϑ and r have the same meaning as in Fig. 28; also the oriented direction of $d\mathbf{F}$ coincides with the oriented direction of \mathbf{v}' , given by the vector product rule.

This amounts to a perfect analogy between hydrodynamics and electrodynamics, which, according to a remark of Helmholtz, has helped greatly in the development of either science.

The current intensity i corresponds to the vortex strength μ , the current density \mathbf{J} to the vortex vector $\boldsymbol{\omega}$. The Biot-Savart force $d\mathbf{F}$ acting between an individual current element and the unit pole is the analogue of our elemental velocity $2\pi\mathbf{v}'$, the magnetic field strength (that is the force on the unit pole due to the total system of currents) corresponds (except for the factor 2π) to the total velocity vector $\mathbf{v} = \sum \mathbf{v}'$. As a system of electric currents is surrounded by magnetic field lines, so a system of vortex tubes is surrounded by stream lines. In electrodynamics we speak of the field \mathbf{F} as *induced* by the current density \mathbf{J} ; it is convenient to use the same expression in hydrodynamics and to speak of the velocity \mathbf{v} as *induced* by the vorticity $\boldsymbol{\omega}$.

This analogy corresponds to the correlation b) on p. 110 that coordinates the flow \mathbf{v} to the magnetic field strength and the rotation (previously $\boldsymbol{\xi}$ and now $\boldsymbol{\omega}$) to the electric field strength.

In the electromagnetic case all these things become particularly simple if one considers *linear conductors*, that is, wires of sufficiently small cross-section. Their counterparts in hydrodynamics are the *vortex filaments* introduced on p. 141 which will be treated in the following article.

6. *Boundary value problems in the finite space.* The determination of the velocity distribution in the case of the continuous flow of an infinitely extended incompressible fluid has been represented under 4) as a *summation problem*, provided the flow vector vanishes at infinity. The vector potential \mathbf{A} is found by integration over the distributed vorticity which is considered as given. The scalar potential is here immaterial since it

was possible to assume the flow vector to be free of sources and sinks throughout the space. This is different when the flow is restricted to a finite domain, where we have to consider boundary conditions for the pressure (e.g. $p = 0$ at a free surface) and for the velocity (e.g. $v_n = 0$ at a rigid wall). If sources are present in the flow, they must be excluded from the domain of validity of Euler's equations and the excluding surfaces must be considered as boundary surfaces of the fluid. The simple integral representation of A is no longer sufficient and a scalar potential Φ appears in addition, as earlier in 3a. As a consequence of the boundary conditions we face now a *boundary value problem* which can no longer be solved by a simple superposition of sources and vortices, but calls for the application of special mathematical methods suitable for the given boundary conditions (e.g. conformal mapping in the two-dimensional case).

21. Straight and Parallel Vortex Filaments

In this article we consider the following two-dimensional problem: Given a number of straight vortex filaments normal to the x, y -plane. What can be said about their mutual interaction in the course of time? With such a question we enter the domain of vortex dynamics while we have so far only dealt with the kinematic aspect of vortex theory (for instance in our analysis of the induced velocity field). The basis of vortex dynamics is found in Helmholtz's conservation theorem of 18 where we learned about the convective character of the vorticity. Hence, if we know the velocity imparted to the particles of an individual vortex filament by the other vortex filaments, we are also informed about the velocity with which that vortex filament moves along and about its contingent change of shape. Dynamics and kinematics of vortex filaments are thus directly connected. In the two-dimensional case to which we restrict ourselves in this article, it is understood that the vortex filament remains straight.

1. The Single Vortex Filament

As already learned in (19.7), the velocity field of the single vortex is given by

$$(1) \quad \Phi + i\Psi = iA \log(x + iy),$$

$$(1a) \quad \pm v = v_\varphi = -\text{grad}_\perp \Phi = \frac{A}{r} = \frac{\mu}{\pi r}.$$

The stream lines are circles about the origin; the vortex proper, that is to say the point O , does not take part in the motion.

This is the place to examine the connection between the field due to a simple vortex and the Biot-Savart law of the last article. To do this we must leave the complex plane and introduce the z -coordinate along the vortex filament, which reaches from $-\infty$ to $+\infty$. Let $x, y, 0$ and $0, 0, z$ be the coordinates of the field point and of the "source" point that supplies the integration variable as before. The line element, previously $|\Delta s|$, is now dz , its distance from the field point, previously called r , is now $R = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$ where the new r stands for $\sqrt{x^2 + y^2}$ as in (1a). From (20.17) and (20.19) we obtain after evaluation of the vector product

$$(2) \quad v_x + iv_y = \sum (v'_x + iv'_y) = \frac{\mu}{2\pi} (-y + ix) \int_{-\infty}^{+\infty} \frac{dz}{(r^2 + z^2)^{3/2}},$$

while v_z is obviously zero. To evaluate the improper integral in (2) we start with the well known formula

$$\int_0^a \frac{dz}{\sqrt{r^2 + z^2}} = \log(a + \sqrt{r^2 + a^2}) - \log r$$

and obtain by differentiation under the integral sign with respect to r

$$-r \int_0^a \frac{dz}{(r^2 + z^2)^{3/2}} = \frac{r}{\sqrt{r^2 + a^2}} \frac{1}{a + \sqrt{r^2 + a^2}} - \frac{1}{r}.$$

If $a \rightarrow \infty$, the first term to the right vanishes, and we have

$$\int_{-\infty}^{+\infty} \frac{dz}{(r^2 + z^2)^{3/2}} = 2 \int_0^{\infty} \frac{dz}{(r^2 + z^2)^{3/2}} = \frac{2}{r^2}.$$

Thus Eq. (2) yields

$$(2a) \quad v_x + iv_y = \frac{\mu}{\pi} \frac{-y + ix}{r^2}, \quad \pm |\mathbf{v}| = v_\varphi = \frac{\mu}{\pi r}$$

in agreement with (1a) in sense and direction.

On denoting by \mathbf{u} a vector in $\pm z$ -direction* of length $|\mu|$ and by \mathbf{r} the position vector in the x, y -plane, Eq. (2a) can be written as

$$(2b) \quad \mathbf{v} = \frac{1}{\pi r^2} \mathbf{u} \times \mathbf{r}$$

The components v_x and v_y calculated from (2b) are, in fact, identical with the values resulting from (2a). Eq. (2b) differs from the three-dimensional formula for \mathbf{v}' in (20.17) in the missing factor 2 and in the denominator of the right member which is now r^2 instead of r^3 . The difference is caused by the preceding integration with respect to z .

*Depending on the sign of μ .

2. Two Vortex Filaments of Equal Strength and Opposite or Equal Sense

The case of two opposite vortex filaments was dealt with in Eq. (19.12) et seq. From (19.14) or directly from Fig. 29a the following result is obtained: The velocity v_1 imparted to the filament F_1 by F_2 is equal and parallel to the velocity v_2 imparted to F_2 by F_1 :

$$(3) \quad v_1 = v_2 = \frac{\mu}{2\pi c} = v \quad (2c = \text{distance } F_1F_2).$$

On the other hand, the velocity v_0 at the center M of the segment F_1F_2 , subject to the common action of F_1 and F_2 equals

$$(3a) \quad v_0 = \frac{\mu}{\pi c} + \frac{\mu}{\pi c} = 4v.$$

Thus we see: both vortex filaments proceed with common velocity v ; the fluid between them moves in the same direction in which F_1 and F_2 move, but with

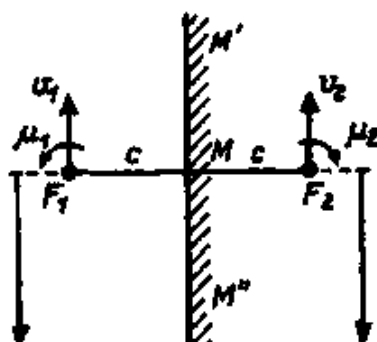


FIG. 29a. Two equally strong vortices of opposite orientation: the plane of symmetry can be replaced by a rigid surface of separation (wall).

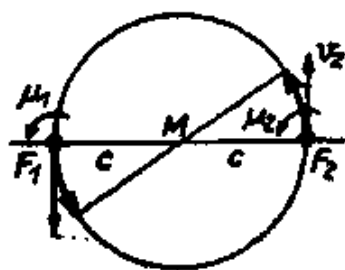


FIG. 29b. Two equally strong vortices of the same orientation move on a circle in diametrical position.

larger velocity. Fluid particles in line with F_1 and F_2 but not between them move in the opposite direction; their velocity becomes very large in the immediate neighborhood of F_1 and F_2 .

Not only at M , but at all points M' , M'' of the bisectrix of F_1F_2 the resultant velocity due to both filaments is parallel to the bisectrix. One may therefore replace it by a rigid wall since the flow along the bisectrix is just the one required by the boundary condition along a rigid wall. Hence an isolated vortex filament, originally at rest, starts to move if a wall is brought toward it; it travels parallel to the wall, the faster the closer the wall. The vortex is, as it were, pushed forward by its virtual image that is obtained by reflexion in the wall. More about that will be learned in the more complicated case of Fig. 31.

The case of *two vortex filaments of equal orientation* is also represented by (19.12), when the sign of the second term to the right is reversed. According to (3) F_2 induces at F_1 a velocity v that is opposite to the velocity which F_1 induces at F_2 . While F_1 travels downward on a circle with radius c , F_2 travels upward by the same amount. The diameter F_1F_2 turns about the center M which is constant in space. Both vortices travel therefore on the same circle, the one being diametrically opposite to the other.

3. A Theorem Concerning the "Center of Mass" of Two or More Vortices

When the strengths μ_1 and μ_2 of the two vortex filaments are different, the vortex points F_1 , F_2 still travel on concentric circles, but the radii are different. *The common center is the center of mass of μ_1 and μ_2* , when we think of μ_1 and μ_2 in terms of masses carried by the points F_1 and F_2 . The center of mass is, of course, on the line through F_1 and F_2 , *inside* the segment F_1F_2 if μ_1 and μ_2 have the *same* orientation (Fig. 30a), and

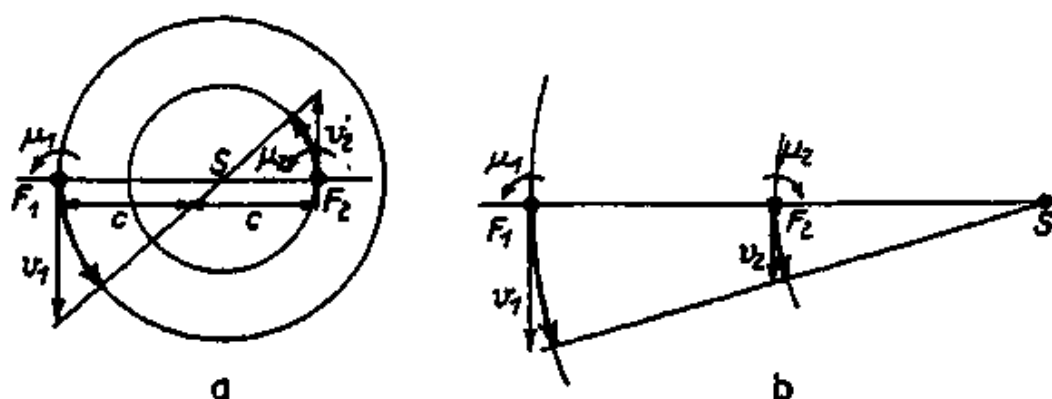


FIG. 30a (left), b (right). Two vortices move in circles about their "centroid." The centroid does (a), or does not (b), separate the vortices depending on whether the orientations of the vortices are equal or opposite.

outside if the orientation is *opposite* (Fig. 30b). In the latter case one of the masses must be counted negative. Figures 30a,b have been drawn under the assumption $\mu_2 > \mu_1$ and $|\mu_2| > \mu_1$, respectively.

The common center S of the circular paths described by F_1 and F_2 is found by connecting the end point of the velocity vector v_1 with the end point of v_2 (see Fig. 30a). This line intersects the line F_1F_2 in S . Since v_1 is due to F_2 and v_2 to F_1 , one has

$$(4) \quad v_1 = \frac{\mu_2}{2\pi c}, \quad v_2 = \frac{\mu_1}{2\pi c}.$$

For the position of the intersection point S we have $\overline{SF_1}/v_1 = \overline{SF_2}/v_2$, which yields because of (4)

$$(5) \quad \mu_1 \overline{SF_1} = \mu_2 \overline{SF_2}.$$

Hence S is indeed the centroid of the masses μ_1 and μ_2 .

The same construction is carried out in Fig. 30b for the case of opposite vortices. If here $|\mu_2| = \mu_1$, the point S is shifted to infinity: both circles degenerate into parallel lines and the first case of 2. is reobtained. The second case of 2. obviously corresponds to $\mu_1 = \mu_2$, so that the two circles of Fig. 30a coincide.

It should be realized that the point S so constructed is by no means at rest if considered as a point of the fluid. Actually the velocity at S in the case of Fig. 30a is given by

$$v_s = \frac{\mu_1}{\pi \overline{SF_1}} - \frac{\mu_2}{\pi \overline{SF_2}} = \frac{\mu_1}{\pi \overline{SF_1}} \left(1 - \frac{\mu_2}{\mu_1} \frac{\overline{SF_1}}{\overline{SF_2}} \right).$$

The expression in parentheses is because of (5)

$$1 - \frac{\mu_2}{\mu_1} \neq 0.$$

The statement that S does not move means only that the geometrical center of the circular paths of the vortices F_1 and F_2 is at rest and does not say anything about the motion of a fluid particle that happens to be at S . The term *center of mass at rest* will be used in this sense also in the remainder of this section.

Let us now consider a system of a number of vortex filaments $F_1, F_2, \dots, F_i, \dots, F_k$. The velocity induced by F_i at an arbitrary field point P can be written by means of (2b) as

$$(6) \quad \mathbf{v} = \frac{1}{\pi} \frac{\boldsymbol{\mu}_i \times (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^2}$$

where $\mathbf{r} - \mathbf{r}_i$ is the relative position vector of P with respect to F_i . Let P now coincide with F_k and sum over all i with the exception of $i = k$ since F_k is not affected by its own induction. Thus

$$(7) \quad \mathbf{v}_k = \frac{1}{\pi} \sum_i' \frac{\boldsymbol{\mu}_i \times (\mathbf{r}_k - \mathbf{r}_i)}{|\mathbf{r}_k - \mathbf{r}_i|^2}$$

is obtained where the apostrophe at the summation sign means the exclusion of the value k from the possible i -values. We introduce now

$$\mathbf{r}_{ik} = \mathbf{r}_k - \mathbf{r}_i, \quad r_{ik} = |\mathbf{r}_k - \mathbf{r}_i|$$

(7a)

$$\mathbf{e}_{ik} = \frac{\mathbf{r}_k - \mathbf{r}_i}{r_{ik}} = -\mathbf{e}_{ki}$$

where \mathbf{e}_{ik} is a unit vector in the x,y -plane. We shall further need a unit vector \mathbf{e} normal to the x,y -plane to indicate the direction of the vector \mathbf{y} : $\mathbf{y}_i = \mu_i \mathbf{e}$. With these notations, Eq. (7) transforms into

$$(7b) \quad \mathbf{v}_k = \frac{1}{\pi} \sum_i' \mu_i \frac{\mathbf{e} \times \mathbf{e}_{ik}}{r_{ik}}.$$

Now we form

$$(8) \quad \sum \mu_k \mathbf{v}_k = \frac{1}{\pi} \sum_k \sum_i' \mu_i \mu_k \frac{\mathbf{e} \times \mathbf{e}_{ik}}{r_{ik}}.$$

In the double sum to the right, \mathbf{e}_{ik} and therefore $\mathbf{e} \times \mathbf{e}_{ik}$ is antisymmetric in i and k while the products $\mu_i \mu_k$ and the distances r_{ik} are, of course, symmetric. Hence the terms cancel in pairs, and we have the result

$$(9) \quad \sum \mu_k \mathbf{v}_k = 0,$$

which implies that the velocity \mathbf{v}_s of the centroid of our vortex system (but only in the sense pointed out above) vanishes

$$(10) \quad \mathbf{v}_s = \frac{\sum \mu_k \mathbf{v}_k}{\sum \mu_k} = 0.$$

4. The Law of Areas for a System of Vortex Filaments

In a certain sense, there exists also an analogue to the law of areas in general mechanics; as we shall see, the *total angular momentum of the system of vortices is constant*, in the absence of external forces.

We define the momentum of a single vortex filament by $\mu_k \mathbf{v}_k$ and the moment of momentum or angular momentum of the system of vortices relative to an arbitrary origin O by

$$\mathbf{M} = \sum_k \mu_k \mathbf{r}_k \times \mathbf{v}_k.$$

Substituting from (7b) we obtain

$$(11) \quad \mathbf{M} = \frac{1}{\pi} \sum_k \sum_i' \frac{\mu_i \mu_k}{r_{ik}} \mathbf{r}_k \times (\mathbf{e} \times \mathbf{e}_{ik}).$$

On interchanging i and k and considering the antisymmetric character of the \mathbf{e}_{ik} we obtain

$$(11a) \quad \mathbf{M} = -\frac{1}{\pi} \sum_k \sum_i' \frac{\mu_i \mu_k}{r_{ik}} \mathbf{r}_i \times (\mathbf{e} \times \mathbf{e}_{ik}).$$

We add now Eqs. (11) and (11a) and find by applying the definition of \mathbf{e}_{ik} in (7a) the following form of \mathbf{M} :

$$(11b) \quad \mathbf{M} = \frac{1}{2\pi} \sum_k \sum_i' \mu_i \mu_k \mathbf{e}_{ik} \times (\mathbf{e} \times \mathbf{e}_{ik}).$$

According to a well known vector identity

$$(12) \quad \mathbf{e}_{ik} \times (\mathbf{e} \times \mathbf{e}_{ik}) = \mathbf{e}(\mathbf{e}_{ik} \cdot \mathbf{e}_{ik}) - \mathbf{e}_{ik}(\mathbf{e} \cdot \mathbf{e}_{ik}),$$

but the first term to the right equals \mathbf{e} and the second is zero (\mathbf{e}_{ik} is a unit vector, so is \mathbf{e} , and they are normal to each other). Thus it is seen that the right member of (11b) depends only on the vortex strengths μ_i and has the constant direction \mathbf{e} ; it is therefore a constant vector

$$(13) \quad \mathbf{M} = \frac{\mathbf{e}}{2\pi} \sum_i \sum_k' \mu_i \mu_k = \text{const.}$$

5. General Remarks on the Dynamics of Vortices

The dynamics of vortices which we have studied here is indeed a very peculiar one and deviates decisively from the dynamics of mass points.

To begin with, Newton's first law is altered. The isolated vortex (which is therefore not subjected to "forces") *remains in a state of rest*. A uniform rectilinear motion can only be acquired by association with a second vortex of equal strength but opposite sense of rotation or under the action of a wall at rest. Thus the relativity principle of classical mechanics according to which the state of rest and of uniform motion are equivalent is no longer valid. The reason is, of course, that the fluid to which the vortex belongs plays the role of a preferred frame of reference.

The modification of the second law is even more remarkable. The external action originating in a second vortex does not determine the *acceleration* but the *velocity*. The content of the law of motion of the mass center is shifted accordingly: not the *acceleration*, but the *velocity* of the mass center vanishes. As far as the *law of areas* is concerned the angular momentum of the vortex system is constant as in the case of a mechanical system in the absence of external forces, but the constant is entirely determined by the vortex strengths according to Eq. (13), in contradistinction to the mechanics of masses where this constant is a constant of integration that depends on initial conditions which may be chosen freely.

6. Atmospheric Vortices

At first thought it seems impossible to realize infinitely long vortex filaments experimentally. But, as Helmholtz noticed, in a fluid layer bounded by two parallel planes normal to the vortex filaments, the same vortex motions can be produced as in a fluid of infinite "thickness".

Such a fluid layer is realized to a certain extent by our atmosphere if one disregards the curvature of the earth. In fact, water spouts and tornadoes can be considered as vortex filaments on a tremendous scale

(filament diameter 5-500m), whereas atmospheric cyclones and anti-cyclones, the horizontal dimensions of which may well extend over 1000 km, being many times larger than their vertical extensions (10-15 km), combine the properties of vortices with those of waves.

For applications in meteorology the classical theory of vortices has to be modified in two points. The normal state of the atmosphere is not a state of rest but of uniform rotation; for an observer connected with the surface of the earth the fluid in which the vortex motion takes place is not free of external forces as we have assumed so far, but under the action of the Coriolis force. In addition to this dynamical correction there is also a thermodynamical one which is due to the compressibility of the air: since the density of the air is connected with the air pressure by an equation of state $\rho = F(p, T, \dots)$ which contains not only the pressure, but also the temperature T and possibly other variables such as humidity, the expression curl $(1/\rho \text{ grad } p)$ does in general not vanish. That means the presence of processes which can generate or destroy vorticity.

Those are the two extensions of the classical vortex theory that are considered in the meteorological literature when more general vortex theorems are formulated. (V. Bjerknes,¹⁰ H. Ertel¹¹).

22. Circular Vortex Rings

Closed vortices, in particular those of circular shape, differ from the straight vortices of the preceding section in that they occur or can be easily made to occur on a small scale. As smoke rings, they are well known to smokers, but they may also be seen above chimney tops. For the purpose of demonstration the following device will serve well: A circular hole is made in the front wall of a cardboard box, the rear wall is removed and a piece of somewhat elastic cloth is stretched over that end. Into the box is put a dish of hydrochloric acid and another of ammonium hydroxide; thus a dense smoke of ammonium chloride is produced. Smoke is ejected through the hole by a sharp tap against the rear wall and curls up in the form of a circular vortex as it passes the edge of the aperture. Fig. 31a illustrates this process. The vortex ring thus created moves straight forward for a distance of several yards with considerable speed.

According to the fundamental theorem of vortex theory (cf. p. 132) the vortex adheres to the fluid, that is, the air particles are carried along with the vortex. This becomes apparent when a candle is put in the way of the vortex: The flame is blown out by the "whirlwind" with a soft whistle.—Decreasing the aperture in the box wall makes for a *smaller*

¹⁰V. Bjerknes, *Meteorolog. Z.* 1900, pp. 97 and 145, also 1902, p. 96.

¹¹H. Ertel, *Physikal. Z.* 1942, p. 526, also *Meteorol. Z.* 1942, pp. 277 and 385.

radius of the vortex ring and at the same time for *larger* speed. When the circular aperture is replaced by a rectangular one the vortex, rectangular when it emerges, develops regular pulsations that indicate a tendency toward the circular shape; the latter is the *stable* one, while the rectangular shape is *unstable*.—The circular vortex increases its radius when approaching a wall parallel to the plane of the ring (Fig. 32).

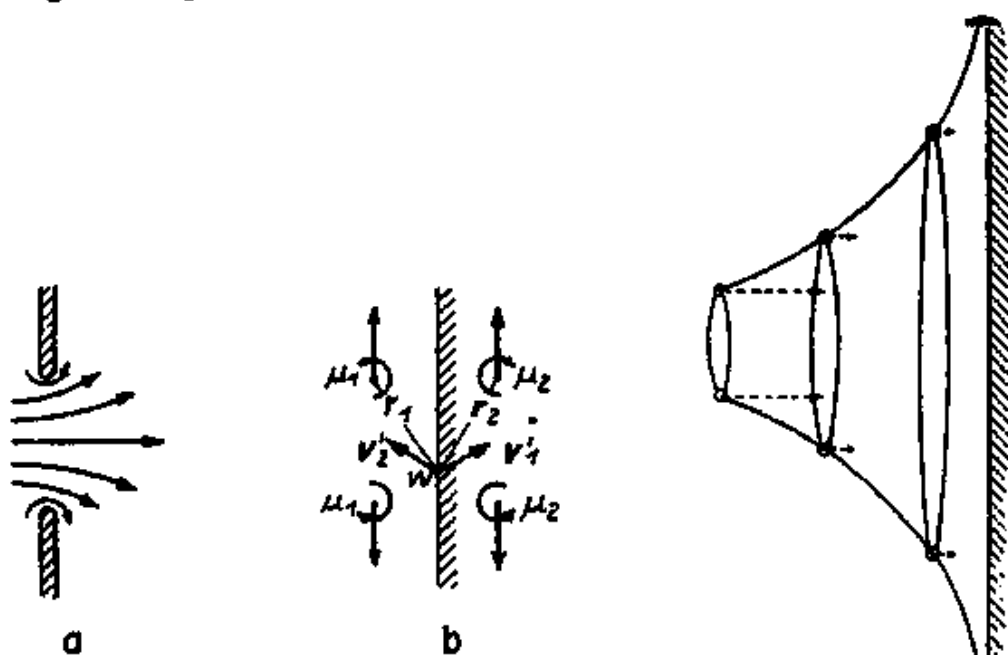


FIG. 31. a. Formation of a vortex ring when an air current is ejected through a hole. b. The vortex ring approaches a plane wall.

FIG. 32. The vortex ring widens when it approaches a wall parallel to its plane; its translatory velocity decreases.

The following experiment, not easily performed with our device, presents little difficulty to a smoker practiced in the art of blowing rings: A small ring is made to follow a bigger one. Moving with greater speed, it catches up with the bigger vortex ring and slips through it. Thereupon the smaller ring grows larger while the bigger one contracts. It has by now become the quicker of the two and will pass through the other, and, theoretically, the game should repeat over and over again. For this experiment, vortices in a liquid are more suitable than smoke rings. One uses colored liquid which is dropped upon the surface of a water reservoir from a pipette. When a drop strikes the surface, it pulls apart in the shape of a ring which penetrates into the water and expands at the same time. The next drop, or rather the vortex ring which it becomes, is initially smaller and overtakes the larger ring, etc. This arrangement permits one to observe the passing-through phenomenon several times in succession.

Turning now to the interpretation of these experiments, we start with the discussion of some aspects of the problem that can be dealt with by elementary means. This is the case if no account is taken of the inter-

action of the vortex ring with itself. The vortices, therefore, will for the present be considered as *infinitely thin* filaments.

Eq. (20.17), the equivalent of the Biot-Savart law, serves as a starting point and gives the velocity contribution \mathbf{v}' induced by a single vortex element $\mu\Delta s$ at given field point. For the midpoint M of the vortex ring in particular, all contributions become equal in amount and direction since in this case (20.17) becomes

$$v' = \frac{\mu\Delta s}{2\pi a^2}$$

where a denotes the radius of the ring. By summation over all Δs the velocity v_M at the midpoint is obtained:

$$(1) \quad v_M = \frac{\mu}{2\pi a^2} \sum \Delta s = \frac{\mu}{a}$$

Thus the velocity at M is seen to increase with decreasing a . As we observed before and shall prove later, the translatory velocity of the vortex ring increases also with decreasing a .

We take now as field point an arbitrary point A on the axis of symmetry of the ring, denote its distance from the center M by z (the distance from any point of the ring being $r = \sqrt{a^2 + z^2}$), and consider the components of \mathbf{v}' parallel and normal to the axis. The normal components due to any two diametrically opposite vortex elements will cancel each other. The axial component of \mathbf{v}' is $v'a/r$. By summation over all Δs , Eq. (20.17) leads to

$$(2) \quad v_A = \frac{\mu}{2\pi r^3} \frac{a}{r} \sum r \Delta s = \frac{\mu a^2}{r^3} = \mu \frac{a^2}{(a^2 + z^2)^{3/2}}.$$

Hence the velocity decreases along the axis on either side of M and goes to zero as z grows indefinitely.

For a field point in general position the summations that lead from \mathbf{v}' to \mathbf{v} are no longer quite so elementary, but require integrations of elliptic type. The same is true, as is well known, for the mathematically analogous calculation of the magnetic field induced by a circular current loop.

On the other hand, it is now possible to interpret some of the phenomena described in a *qualitative* way.

Fig. 31a indicates schematically the formation of a vortex as the air flows through the aperture, Fig. 31b illustrates the conditions that prevail when the vortex ring meets a wall parallel to its own plane. At the wall, the boundary condition is $v_n = 0$. It can be satisfied by employing the *method of images*, a procedure of general usefulness in problems of this character. To the physically real vortex on the left side of the wall one

adds a *virtual* vortex on the right side, in this way extending to infinity the hydrodynamic field that, physically, is bounded by the wall. An air particle W adjacent to the wall receives the velocity contribution \mathbf{v}'_1 from the real vortex μ_1 and \mathbf{v}'_2 from the virtual vortex μ_2 ; according to Eq. (20.17), \mathbf{v}'_1 is perpendicular to the plane subtended by the vector \mathbf{r}_1 and the axis of the vortex element μ_1 . Evidently, \mathbf{v}'_1 and \mathbf{v}'_2 add up to a motion *parallel* to the wall, thus *satisfying the condition* $v_n = 0$. This, obviously, is true not only for the two pairs of vortex elements in the plane of the diagram, but for any pair of elements in image position.

We now wish to find out about the influence exerted by the image vortex upon the real vortex,¹² e.g. upon the upper vortex element μ_1 in Fig. 31b. The strongest action is to be expected from the closest neighbor, that is, the upper element μ_2 . Again according to (20.17), this action consists in an induced velocity directed *vertically upward*, indicated in the diagram by a vertical arrow at the upper left element. All the other elements μ_2 will partly aid in this action or partly, but in a smaller degree, counteract it (e.g. the lower element μ_2). The same argument applied to the lower element μ_1 leads to the vertical downward arrow at the place of that element. In the diagram the influence of the real vortex upon its image is also indicated.

We see then that the real vortex is *enlarged* when it approaches the wall because of the hydrodynamic action of the image vortex. The same is true for the image vortex. As the radius of the ring grows, so the forward motion directed to the wall attenuates, since the translatory motion of the vortex ring is essentially in inverse proportion to the radius. (This has been mentioned before but has not been proved.) Hence the vortex growing ever larger and weaker has evanesced to infinity before it reaches the wall.

Using the same qualitative methods we can also understand the mutual slipping through of a pair of vortices (Fig. 33a, b, c, d). In Fig. 33a the radii and speeds of the two vortices are equal. Having been generated one after the other by the same device, their sense of rotation is the same, in contrast to Fig. 31a in which the senses of rotation were opposite. This time, the interaction of the two vortices consists in the enlargement of the more advanced vortex and, simultaneously, in the diminution of the rear vortex as indicated by the vertical arrows in Fig. 33a. In Fig. 33b and c the vortices are shown before and after passing which becomes possible on account of the greater speed of the rear vortex (cf. the horizontal arrows in Fig. 33b). In Fig. 33c, the interaction of the vortex pair at

¹²Speaking physically, the influence is, of course, all due to the wall, but we have just convinced ourselves that this influence is correctly described by the assumption of a virtual image vortex.

this stage is again indicated by vertical arrows: The more advanced vortex grows, the rear vortex contracts, hence the first is decelerated, the second accelerated. The result is Fig. 33d, a repetition of Fig. 33a: The game is ready to start anew.

But these qualitative methods are no longer sufficient for problems that involve the *interaction of the vortex with itself*. The translatory motion

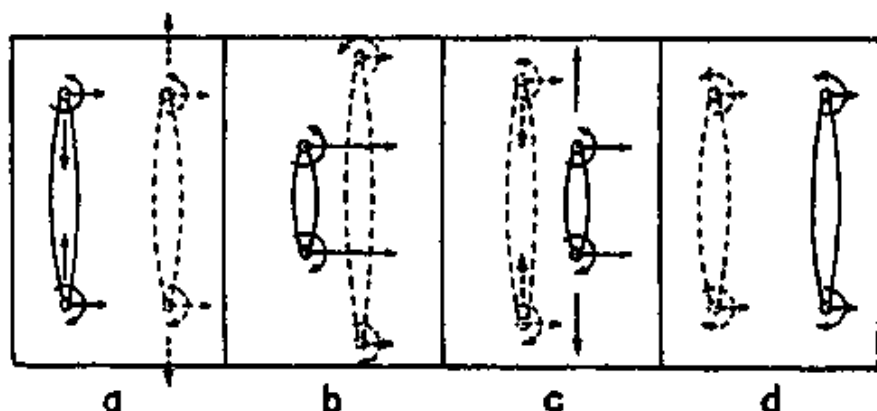


FIG. 33. The mutual threading of two vortex rings.

of a circular vortex is of that kind. Now the cross section of the vortex ring must be assumed finite, e.g. as a circle of radius c very small in comparison with the ring radius a . Within the circular cross section one might perhaps try a finite (e.g. constant) rotation ω ; the vortex strength μ that so far has been used only for *filamentous* vortices is now connected with ω by

$$\mu = \pi c^2 \omega.$$

The most convenient approach is through the stream function Ψ (19.24); within the vortex cross section the differential equation for Ψ is not Eq. (19.26) but the inhomogeneous equation

$$\frac{\partial^2 \Psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \Psi}{\partial \rho} + \frac{\partial^2 \Psi}{\partial z^2} = 2\omega.$$

The representation of Ψ (valid outside the cross section) by complete elliptical integrals of the first and second kind has already been given by Helmholtz. It serves as basis for the computation of the translatory velocity v_T , a computation that is mathematically involved and at the same time physically unsatisfactory. It will be omitted here. The result is¹³

$$(3) \quad v_T = \frac{\mu}{2\pi a} \left(\log \frac{8a}{c} - \frac{1}{4} \right).$$

¹³For a critical review of literature see A. E. H. Love, *Encykl. d. Mathem. Wiss.* IV. 3. p. 118

The term written here as $\frac{1}{2}$ is uncertain; it depends on the assumed ω -distribution over the cross-section. Our previous statement, repeatedly used in the discussion of the experiments, that v_T increases with decreasing a and decreases with increasing a is confirmed by Eq. 3, but the physically undetermined quantity a/c makes the result appear rather unrealistic. A comparison between (3) and (1) shows that v_T may be larger or smaller than v_M , depending on the choice of a/c .

Similar difficulties and a result formally quite analogous to (3) occur in the computation of the inductance of a wire (cf. vol. III), but in the electrical problem the radius c of the wire is of course physically determined.

How could we remove this uncertainty in the case of the circular vortex? There is hardly any other way but to investigate more completely the process of vortex formation at the edge of the circular aperture. This should result, if not in a definite radius c , then at least in a definite ω -distribution over the cross section (of course, ω should decrease radially). However, such an investigation could hardly be carried out and would certainly not be worth the labor; hence we have to leave the attractive and successful theory of vortex rings incomplete in an essential point.

A more direct grasp of the translatory motion of the vortex may be achieved by comparison with the motion of the straight vortex pair in 21, Fig. 29a. A vortex pair may be obtained from a vortex ring by restricting one's attention to two diametrically opposite ring elements. Lengthening of these two elements makes up, in a certain way, for the disregard of the other elements and finally leads to a pair of parallel straight filaments. To be sure, the similarity of the two vortex arrangements is only superficial, but their translatory motions are remarkably analogous. If we replace in Eq. (21.3) the letter c , which denotes half the distance of the vortex pair, by the ring radius a , the translatory velocity of the vortex pair becomes

$$v = \frac{\mu}{2\pi a}.$$

This is indeed the first factor on the right side of our Eq. 3 for v_T . The direction of v was perpendicular to the plane of the vortex pair; our v_T is perpendicular to the plane of the ring, as it should be.

A peculiar vortex motion may be observed in rowing. At the places where the oar breaks the surface of the water just previous to being lifted, small depressions of the surface appear—we could call them “dimples”—that run along on the surface. They are the endpoints of a vortex arc that has been escorting the submerged contour of the oar while it was pulled through the water. The dimple-like shape has its cause in the combined

action of centrifugal force and gravity. The rotating water surface must be normal to the resultant force in the same way as it is normal to gravity if no rotation is present. Now, while the surface form of a mass of water rotating with constant ω is the well known paraboloid, in our case ω diminishes quickly in radial direction and vanishes at a finite distance from the vortex center; the depression is thus restricted to the immediate neighborhood of the end of the vortex filament. The running along of a pair of dimples is illustrative of the translatory motion v_r of the vortex arc connecting the two dimples underneath the water surface.

Since it has become certain through optical investigations (Zeeman effect) that the solar matter in the sun spots is engaged in vigorous vortex motion, one would conclude that, there too, a vortex filament has emerged to the surface, coming out of the interior of the sun; and one would assume that to each sun spot there belongs a companion spot with converse sense of rotation, the two being connected by a vortex filament of roughly semi-circular shape that passes through the interior of the sun. On the other hand, one should be aware of the fact that the solar matter is not a perfect inviscid fluid. Thus it appears rather a bold step to apply our ideas about vortex motion literally to the interpretation of solar phenomena.

THEORY OF WAVES

Ever since waves were studied, water waves have served the natural scientist as a model for wave theory in general, although they are much more complicated than acoustical or optical waves. As *surface waves* they are bound to the common surface of two media, while the ordinary acoustic and optical waves are *three dimensional waves*.

There is this fundamental difference between vortices and waves: vortices pull the matter along in their own motion while in a wave the average locomotion of an individual fluid particle vanishes; it is not *matter* that travels but *energy* and *phase*.

We shall discuss in this chapter waves with different symmetry characteristics, such as *plane waves*, *circular waves*, *ship waves*, and *Mach waves*, starting out with the simplest type, the *plane progressive waves*. According to the nature of the restoring force we distinguish *gravity waves* and *capillary waves*. Gravity waves are the large, conspicuous waves which one usually has in mind in talking of water waves.

23. Plane Gravity Waves in Deep Water

We assume the wave as a completely periodic phenomenon and express the time dependence as on p. 98 in the form $e^{-i\omega t}$; waves of a more general time dependence can be obtained by superposition of partial waves having different circular frequencies (cf. 26).

We further assume that the wave motion is generated *out of the state of rest*, say, by a gust, a mechanical disturbance or the like (in problem VI, 3 we shall investigate under what circumstances an air current that grazes along a horizontal water surface can produce a wave motion). Since the fluid can be considered as *inviscid*, and since we shall consider in this and in the following article only the *potential field* of gravity, it follows from the conservation law of 18 that the motion possesses a *velocity potential*. This can depend in the case of a *plane wave* only on *two* spatial coordinates x and y , where x is the direction in which the wave progresses and y the depth coordinate. The problem is independent of the third spatial coordinate z which is horizontal and orthogonal to the direction of propagation. Φ is thus a two-dimensional potential as far as the space coordinates are concerned, and, according to 19, has the form $f(x \pm iy)$ in an *incompressible fluid*.

Let y be counted positive downward; since we wish to obtain a train of waves advancing in the positive x -direction, the x -dependence of f must be of a *trigonometrical* form, or, of the form of an imaginary exponential when written in this more convenient way. This leads to the following possibilities

$$(1) \quad \Phi = f(x + iy) e^{-i\omega t} = A e^{i(kx - \omega t)} e^{-ky},$$

or

$$(1a) \quad \Phi = f(x - iy) e^{-i\omega t} = B e^{i(kx - \omega t)} e^{+ky}.$$

Here, k is again the wave number, and there is

$$(2) \quad k = \frac{2\pi}{\lambda}, \quad \omega = \frac{2\pi}{\tau},$$

where λ is the wave length and τ the period.

We first assume the water *infinitely deep*, that is, the y -coordinate of the ground $y = h$ should be very large compared to λ .

At the ground,

$$(2a) \quad ky = 2\pi \frac{h}{\lambda} \rightarrow \infty.$$

This shows that the potential (1a) is not usable since it would yield infinite velocity amplitudes at the ground, but the potential (1) satisfies all conditions that have been imposed so far.

The representation (1) contains three parameters, A , ω , and k . A determines the amplitude of the wave for $y = 0$; both A and the (circular) frequency ω depend on the particular form of the excitation. While these two quantities can be chosen freely, the wave number k must be determined in its ratio to ω , for, according to (2), the ratio

$$(3) \quad \frac{\omega}{k} = \frac{\lambda}{\tau} = V,$$

is the *velocity of propagation* of the waves.

For the determination of k we must utilize the *condition for the free surface*:

$$(4) \quad p = 0,$$

(the atmospheric pressure is taken as zero). Eq. (4) follows from (12.19) if capillarity is neglected; it introduces a dynamic element into our theory while our argument so far has been wholly of a kinematic nature.

The pressure p and the potential Φ are connected by Euler's equations, which we shall use in the integrated form of Bernoulli's equation. Since

our problem involves a velocity potential variable in time, we have to take Bernoulli's equation in the form (11.15). We shall, however, neglect the quadratic term $(\nabla\Phi)^2$, since we consider the amplitude factor A as a *small quantity* (the usual procedure in mechanics of small oscillations). The abridged form of Bernoulli's equation is then

$$(5) \quad -\frac{\partial\Phi}{\partial t} + \frac{1}{\rho}(p + U) = \text{const.}$$

As explained on p. 89, the constant in (5) is independent of the space coordinates, but in general dependent on time. The only function of time which in our case does not upset the periodicity and the uniform advancement of the wave is

$$\text{const} = F(t) = 0.$$

Under these circumstances (5) assumes the simple form

$$(5a) \quad \frac{\partial\Phi}{\partial t} = \frac{U}{\rho}.$$

Here U is the gravity potential per unit of volume taken at the surface. Since y is counted positive downward, we have in general

$$(5b) \quad U = -\rho gy.$$

Let the equation of the surface profile be $y = \eta$, where η is a function of x and t . A positive η means a depression, a negative η an elevation, of the surface. From (5a) we have

$$(6) \quad \frac{\partial\Phi}{\partial t} = -g\eta.$$

The function η must have again the form of a progressive wave, like the one we have set up for the velocity potential Φ . Thus

$$(7) \quad \eta = a e^{i(kx - \omega t)}.$$

The constant a introduced here is in general complex since it includes amplitude *and* phase of the surface function; also, a has the character of a small quantity like A . If we substitute (7) and (1) in (6), we obtain after cancellation of the common exponential factor

$$(8) \quad i\omega A e^{-k\eta} = ga.$$

We now expand $e^{-k\eta}$ in powers of $k\eta$ and neglect the products $A\eta$, $A\eta^2$, etc. as small quantities of higher order; Eq. (8) then simplifies to

$$(9) \quad i\omega A = ga.$$

This is a relation between A and a , but it is not the relation between k and ω we require. The latter is obtained by introducing a further *kinematic* condition: We stipulate that the motion of the surface must coincide at any time with the motion of those fluid particles that happen to be at the surface at this time. That such a condition must be satisfied is rather obvious; we specify, however, that the components of the two motions taken in the normal direction n of the surface should be equal, since a motion of the fluid particles in the tangential plane does not change the shape of the surface, hence is immaterial for our problem. On denoting the velocity of the surface with V and the particle velocity with v as usual, our condition reads

$$(10) \quad V_n = v_n.$$

The component v_n , expressed by the velocity potential Φ , is

$$v_n = -\frac{\partial \Phi}{\partial n}.$$

However, if A is sufficiently small (the wave sufficiently flat), we can replace within a "cosine error", that is, with disregard of terms of second order

$$(10a) \quad \frac{\partial \Phi}{\partial n} \quad \text{by} \quad \frac{\partial \Phi}{\partial y}.$$

V is treated correspondingly: we replace

$$(10b) \quad V_n \quad \text{by} \quad \frac{\partial \eta}{\partial t},$$

that is by the "sinking speed" of the surface. With these simplifications, condition (10) reads

$$(11) \quad \frac{\partial \eta}{\partial t} = -\frac{\partial \Phi}{\partial y}.$$

Substituting for η and Φ from (7) and (1), we obtain, again after cancellation of the exponential on both sides,

$$(12) \quad -i\omega a = kA.$$

Now, the comparison of (9) and (12) yields at once

$$(13) \quad \frac{A}{a} = \frac{g}{i\omega} = -\frac{i\omega}{k}.$$

Our conclusions from (13) are:

1. There is a phase difference of $\pi/2$ between the a -wave and the A -

wave. If A is chosen a real quantity, which is permissible, a becomes purely imaginary; or, employing real representation, we can write for Φ in accordance with (1)

$$(14) \quad \Phi = A \cos(kx - \omega t)e^{-ky},$$

and Eq. (7) becomes now [cf. (13)]

$$(14a) \quad \eta = -\frac{\omega}{g} A \sin(kx - \omega t)$$

or

$$(14b) \quad \eta = -\frac{k}{\omega} A \sin(kx - \omega t).$$

2. The relation between k and ω is given by Eq. (13):

$$(15) \quad \omega^2 = gk.$$

Introducing here the velocity of propagation, we obtain [cf. (3)]

$$(16) \quad V^2 = \frac{\omega^2}{k^2} = \frac{g}{k} = \frac{g\lambda}{2\pi}, \quad V = \sqrt{\frac{g\lambda}{2\pi}}.$$

*The velocity of propagation depends on the wave length; long waves travel faster than smaller ones.*¹

When the propagation velocity of a wave depends on the wave length as in Eq. (16), we speak of dispersion, using an expression borrowed from

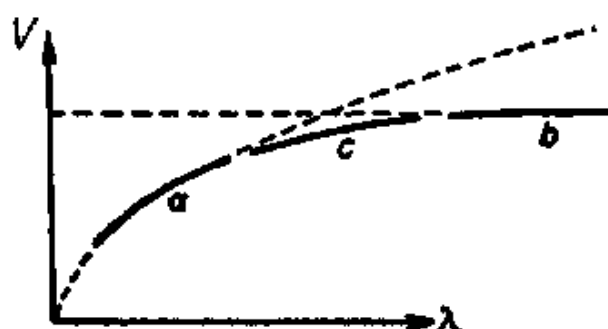


FIG. 34. The phase velocity V as a function of the wave length λ . The diagram gives the dispersion of gravity waves for $\lambda \ll h$ (segment a), $\lambda \gg h$ (segment b), $\lambda \approx h$ (segment c); h = depth of the water.

optics. The dispersion in a medium is *normal* when longer (red) waves have larger velocities (smaller index of refraction) than shorter (violet) waves. The behavior of *gravity waves in deep water thus corresponds to the case of normal dispersion in optics.*

¹We should like to recommend to readers who are irritated by the complex representation of periodic phenomena, to carry out the preceding calculation once more with real quantities and to convince themselves that they obtain again Eq. (16) although in a somewhat more cumbersome way. One would have to replace (1) by (14) and (7) by $\eta = |a| \cos(kx - \omega t + \alpha)$, where α is the phase difference between φ and η that is to be determined.

Fig. 34 should make this clearer. V is represented by the upper half of an ordinary parabola which has the V -axis as a tangent at $\lambda = 0$. Only the middle part a of the parabola has been drawn as a solid line, this being the region for which our assumptions are actually valid. For, if the wave length λ keeps increasing, it finally becomes of the same order of magnitude as the depth of the water h , and our assumption (2a) is no longer valid. This case will be taken up in the next article. On the other hand, if one goes to very small values of λ , gravity is no longer the decisive dynamic parameter, but *surface tension* takes the lead; this brings about a fundamental change of the dispersion law (see 25).

24. Plane Gravity Waves in Shallow and Moderately Deep Water

If we now assume the depth h as finite, the second form of the potential (23.1a) need no longer be rejected, but is as good as the form (23.1); hence a linear combination of (1) and (1a) may be taken for the velocity potential Φ :

$$(1) \quad \Phi = e^{i(kx - \omega t)} \{Ae^{-ky} + Be^{+ky}\}.$$

The new boundary condition at the bottom is: $v_y = 0$, or, $\partial\Phi/\partial y = 0$ for $y = 0$. This is, by (1), equivalent to

$$-Ae^{-kh} + Be^{+kh} = 0,$$

a condition for A and B which is more conveniently handled, if we introduce a constant C defined by

$$\frac{1}{2} C = Ae^{-kh} = Be^{+kh}$$

so that

$$(1a) \quad A = \frac{1}{2} Ce^{+kh}, \quad B = \frac{1}{2} Ce^{-kh}.$$

Then the potential Φ assumes the form

$$\begin{aligned} \Phi &= e^{i(kx - \omega t)} \frac{C}{2} \{e^{k(h-y)} + e^{-k(h-y)}\} \\ (2) \quad &= e^{i(kx - \omega t)} C \cdot \cosh k(h - y). \end{aligned}$$

The expression (23.7) for the surface depression remains unchanged; we rewrite it here:

$$(3) \quad \eta = ce^{i(kx - \omega t)}.$$

Likewise, the dynamic and kinematic surface conditions (23.6) and (23.11) remain valid. They give for $y = 0$, because of (2) and (3),

$$(4) \quad i\omega C \cosh kh = gc,$$

$$(5) \quad -i\omega c = Ck \sinh kh.$$

We first discuss the case of *shallow* water, that is,

$$kh \ll 1, \quad \cosh kh = 1, \quad \sinh kh = kh.$$

Eqs. (4) and (5) yield in this case

$$i\omega C = gc, \quad -i\omega c = Ck^2 h,$$

or

$$(6) \quad \frac{C}{c} = \frac{g}{i\omega} = -\frac{i\omega}{k^2 h}.$$

The formula for the velocity of propagation is now

$$(7) \quad V^2 = \frac{\omega^2}{k^2} = gh, \quad V = \sqrt{gh}.$$

(One is reminded of the definition of the velocity head $h = v^2/2g$ on p. 45, but it is merely a formal analogy, due to dimensional reasons.)

In shallow water the velocity of propagation is independent of the wave length; there is no dispersion.

On the basis of Eq. (7) we can now complete Fig. 34 for large λ values. The curve should approach a horizontal asymptote for $\lambda > h$ in the distance \sqrt{gh} from the λ -axis, as indicated by the branch *b* of this line.

When λ and h are of the same order of magnitude (*moderately deep* water), Eqs. (4) and (5) still govern the behavior of the wave. In this case, they supply directly

$$(8) \quad \frac{C}{c} = \frac{g}{i\omega \cosh kh} = -\frac{i\omega}{k \sinh kh},$$

from which we infer

$$(9) \quad \tanh kh = \frac{\omega^2}{gk}.$$

On introducing $x = kh$ and $d = \omega^2 h/g$, we obtain

$$(9a) \quad \tanh x = \frac{d}{x}.$$

The solution of this transcendental equation can be obtained graphically: We plot the equilateral hyperbola d/x and the curve $\tanh x$ above x as

abscissa (the latter is a monotonic increasing function with a horizontal asymptote at the distance 1 from the x -axis). There is then only one intersection point of the two curves, and its abscissa $x = x_0$ is the root of Eq. (9a); the required value of k is x_0/h . The *velocity of propagation in moderately deep water* is therefore

$$(10) \quad V = \frac{\omega}{k} = \frac{\omega h}{x_0}.$$

We have obtained an implicit representation of the dispersion law which should be supplemented by a chart or table of the solutions of Eq. (9a) in function of the parameter d . We also see in what way Fig. 34 is to be completed: the gap between segment a ($h \gg \lambda$) and segment b ($h \ll \lambda$) is bridged by segment c ($h \cong \lambda$) corresponding to Eq. (10). Since this

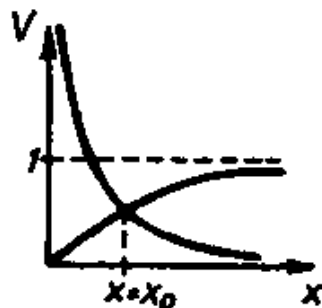


FIG. 35. Graphical solution of the transcendental Eq. (9a).

part of the curve increases, too, one sees that the *dispersion is normal* throughout.

The transition from a to c can also be read from the following formula which is a direct consequence of (9)

$$(10a) \quad V^2 = \frac{\omega^2}{k^2} = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}.$$

One obtains from (10a)

$$\text{for } h \gg \lambda: \quad V^2 \rightarrow \frac{g\lambda}{2\pi}, \quad \text{Eq. (23.16),}$$

$$\text{for } h \ll \lambda: \quad V^2 \rightarrow gh, \quad \text{Eq. (24.7).}$$

A remark on *surf formation* may be inserted here which is of a purely qualitative nature and includes perhaps an over-interpretation of our equations. Surf is bound to a region of very small h and, consequently, small $kh = x$ so that $\tanh x \cong x$. The transcendental Eq. (9a) becomes then algebraic: $x = d/x$. The root is

$$x_0 = \sqrt{d} = \omega \sqrt{\frac{h}{g}},$$

the velocity of propagation according to (10) is therefore

$$(11) \quad V = \sqrt{gh},$$

which is our formula (7) for shallow water. Now, our analysis is only valid for water of *uniform* depth, but we may try to apply it to the case of decreasing depth in the vicinity of the shore (cf. Fig. 36). Besides, our results apply only to *small amplitudes* since the amplitude squares have been omitted; nevertheless, we use them now for finite amplitudes, assuming *two* values for h , viz., h_c characterizing the crest and h_r characterizing the trough of a wave (cf. Fig. 36), and obtain from (11)

$$(11a) \quad \frac{V_c}{V_r} = \sqrt{\frac{h_c}{h_r}} > 1.$$

Hence the velocity of propagation of the crest is larger than that of the trough, which would modify the shape of the wave as indicated in Fig. 36

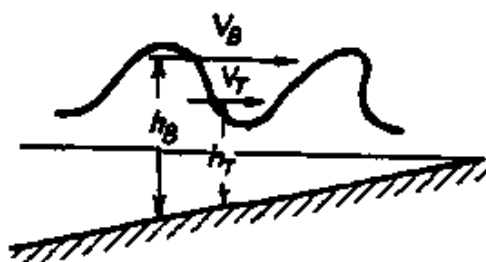


FIG. 36. Surf formation.

by the crest to the right. This offers an explanation of the surf phenomenon on the basis of our equations, although it is doubtful whether the conclusions are legitimate.

We turn to the very attractive problem of the path of an individual particle in wave motion which shall be discussed under the assumption of moderately deep water. Let x, y be the particle coordinates in water at rest and x', y' the same in water disturbed by the wave motion. The displacement from the rest position is then

$$(12) \quad \xi = x' - x, \quad \eta = y' - y;$$

where η no longer denotes the surface disturbance as before, but the disturbance at an arbitrarily specified depth y , (similarly ξ). The particle velocity according to (2) is

$$(13) \quad \begin{aligned} v_x = \dot{\xi} &= -\frac{\partial \Phi}{\partial x} = -ike^{(kx - \omega t)} C \cosh k(h - y), \\ v_y = \dot{\eta} &= -\frac{\partial \Phi}{\partial y} = ke^{(kx - \omega t)} C \sinh k(h - y). \end{aligned}$$

It will be noticed that Φ is here differentiated with respect to x, y , while it should be differentiated with respect to the variable coordinates x', y' ; also, we should write in (13) x', y' or $x + \xi, y + \eta$ instead of x, y . The difference, however, would be of second order in the quantities C, ξ and η and may therefore be neglected.

Integrating (13) with respect to t , which amounts to a division by $-i\omega$, we obtain

$$(14) \quad \xi = \frac{k}{\omega} e^{i(kx - \omega t)} C \cosh k(h - y),$$

$$\eta = i \frac{k}{\omega} e^{i(kx - \omega t)} C \sinh k(h - y).$$

where the constants of integration must be taken as zero because of the periodicity of the displacement. We introduce the notations (they will justify themselves presently)

$$(15) \quad a = \frac{kC}{\omega} \cosh k(h - y), \quad b = \frac{kC}{\omega} \sinh k(h - y)$$

and write the *real part* of (14), using these abbreviations:

$$(16) \quad \xi = a \cos(kx - \omega t), \quad \eta = -b \sin(kx - \omega t).$$

To find the equation of the particle path we have to eliminate t from (16). By squaring ξ and η we obtain

$$(17) \quad \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1,$$

which is the equation of an ellipse. The reason why we had to change back to the real representation becomes now apparent: the complex writing preserves its physical significance only in linear operations!

The ratio of the minor to the major axis follows from (15) as

$$(17a) \quad \frac{b}{a} = \tanh k(h - y) = \begin{cases} 0 & \text{for } y = h, \text{ bottom} \\ \tanh kh & \text{for } y = 0, \text{ surface.} \end{cases}$$

The excentricity of the ellipses is the same at any depth:

$$(17b) \quad e = \sqrt{a^2 - b^2} = \frac{kC}{\omega}.$$

Fig. 37a represents the position and shape of the ellipses in the general case, Fig. 37b in the case of deep water, Fig. 37c in the case of shallow

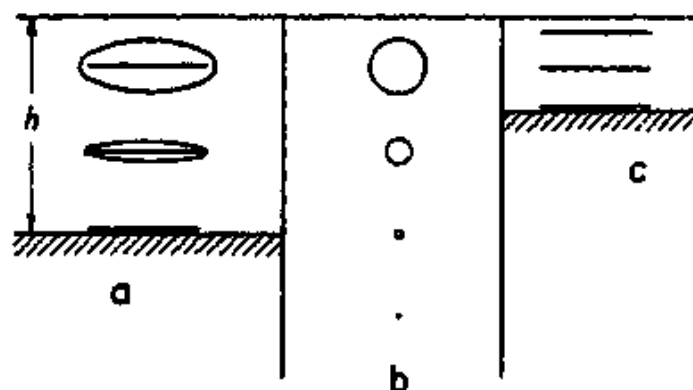


FIG. 37. The path curves of water particles (a) in moderately deep water, (b) in deep water, (c) in shallow water.

water. The ellipses of Fig. 37a are *confocal* if they are coaxially superimposed upon each other. In the limiting case $kh \rightarrow \infty$ we have by (17a)

$$\frac{b}{a} = \tanh(\infty) = 1.$$

The ellipses thus become *circles* as in Fig. 37b. The radii are determined by computing a or b from (15) and the first equation (1a):

$$\begin{aligned} a &= \frac{kC}{\omega} \cosh k(h - y) = \frac{k}{\omega} A e^{-kh} (e^{k(h-y)} + e^{-k(h-y)}) \\ &= \frac{k}{\omega} A (e^{-ky} + e^{-k(2h-y)}) \rightarrow \frac{k}{\omega} A e^{-ky}, \\ b &= \frac{kC}{\omega} \sinh k(h - y) = \frac{k}{\omega} A e^{-kh} (e^{k(h-y)} - e^{-k(h-y)}) \\ &= \frac{k}{\omega} A (e^{-ky} - e^{-k(2h-y)}) \rightarrow \frac{k}{\omega} A e^{-ky}. \end{aligned}$$

The radii of the circles are seen to decrease very quickly with increasing depth. In the other limiting case where kh and, *a fortiori*, $k(h - y)$ go to zero we obtain from (17a)

$$\frac{b}{a} = \tanh 0 = 0.$$

The ellipses degenerate into horizontal *straight segments* (cf. Fig. 37c); in shallow water there is, so to speak, no space left for a vertical displacement. The amplitude of the horizontal oscillation, e , is independent of the depth y [cf. (17b)].

The stream lines of the velocity field are no less interesting than the

path curves of the particles. Since in the present case of plane waves we have a two-dimensional velocity field, the stream function Ψ is obtained by (19.5) from the following equations

$$\frac{\partial \Psi}{\partial x} = -\frac{\partial \Phi}{\partial y}, \quad \frac{\partial \Psi}{\partial y} = \frac{\partial \Phi}{\partial x}.$$

On substituting the general² expression for Φ from (2), we obtain

$$\frac{\partial \Psi}{\partial x} = ke^{i(kx - \omega t)} C \sinh k(h - y),$$

$$\frac{\partial \Psi}{\partial y} = ike^{i(kx - \omega t)} C \cosh k(h - y).$$

The first or second of these expressions is now integrated over x or y respectively and the integration constant set equal to zero as before; this leads to

$$\Psi = -ie^{i(kx - \omega t)} C \sinh k(h - y).$$

To find the stream lines we have to consider this equation at a specified instant t . Let us then take the real part for $t = 0$:

$$(18) \quad \Psi = C \sin kx \cdot \sinh k(h - y).$$

For comparison, we write down the surface depression η according to (3), express c by C through (6), and take again the real part at $t = 0$:

$$(19) \quad \eta = -\frac{\omega}{g} C \sin kx.$$

A glance at (18) shows: the streamline $\Psi = 0$ requires $y = h$: the bottom is therefore a part of the streamline, which is fairly obvious; but for $kx = 0, \pm\pi, \pm 2\pi$, we have again $\Psi = 0$, whatever y is. At the same abscissas, η vanishes likewise, as is seen from (19). Hence the flow pattern is divided into rectangular domains which contain the stream lines. Furthermore we conclude from (18) that $\partial \Psi / \partial x = 0$ if $\cos kx = 0$, whatever y is. This happens at the abscissas $kx = \pi/2, 3\pi/2, \dots$, for which η has a maximum or a minimum; at these points all streamlines are horizontal.—Altogether this is sufficient information to form an idea of the general pattern of the streamlines (cf. Fig. 38).

The patterns of Fig. 38 and 37a are convincingly verified by actual photographs of path curves and streamlines. For the purpose of photography the fluid is kept in a narrow glass container with parallel walls, and small light absorbing particles such as metal filings mixed in; one

²In the case of shallow water the stream lines are obviously horizontal.

photographs across the container and obtains the *streamlines* by a *short exposure*. In viewing the photographs the eye involuntarily connects the line elements traced by the dark particles and recognizes the continuous arcs of the stream lines, and also their position relative to the fluid surface which appears moderately sharp in the picture. With a *longer exposure*, e.g., as long as a half period, and a less dense interspersation of metal particles, the *path curves* can well be distinguished as elliptical arcs, but the surface does not show up distinctly. This second picture gives also evidence

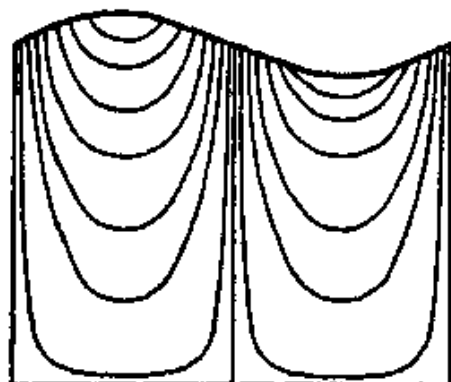


FIG. 38. The streamlines of gravity waves in water of finite depth.

of the propagation of the phase, if one compares the starting points (or end points) of the arcs left and right in the photograph.

The particles remain essentially at their places; this, at least, is true if the amplitude is sufficiently small, as assumed in our calculation, but is no longer quite correct for a finite amplitude. Therefore it is only the phase that travels with the velocity V which we have computed; phase velocity would thus be a more accurate designation for V .

25. Plane Capillary Waves and Combined Capillary-Gravity Waves

In the following discussion we shall first disregard gravity entirely, taking only the *surface tension* T into account. The surface is no longer free of forces, but under the action of the normal pressure N caused by T . Accordingly, the condition for the hydrodynamic pressure at the surface is not $p = 0$ as in (23.4), but, in the case of a plane wave, according to (17.10)

$$(1) \quad p = T \frac{\partial^2 \eta}{\partial x^2}.$$

(Note that u of (17.10) must be replaced by the *surface elevation* $-\eta$.) The value of p given by (1) is to be introduced in Bernoulli's equation (23.5) together with $U = 0$ (gravity is disregarded for the time being). Hence we obtain

$$(2) \quad \frac{\partial \Phi}{\partial t} = \frac{T}{\rho} \frac{\partial^2 \eta}{\partial x^2}.$$

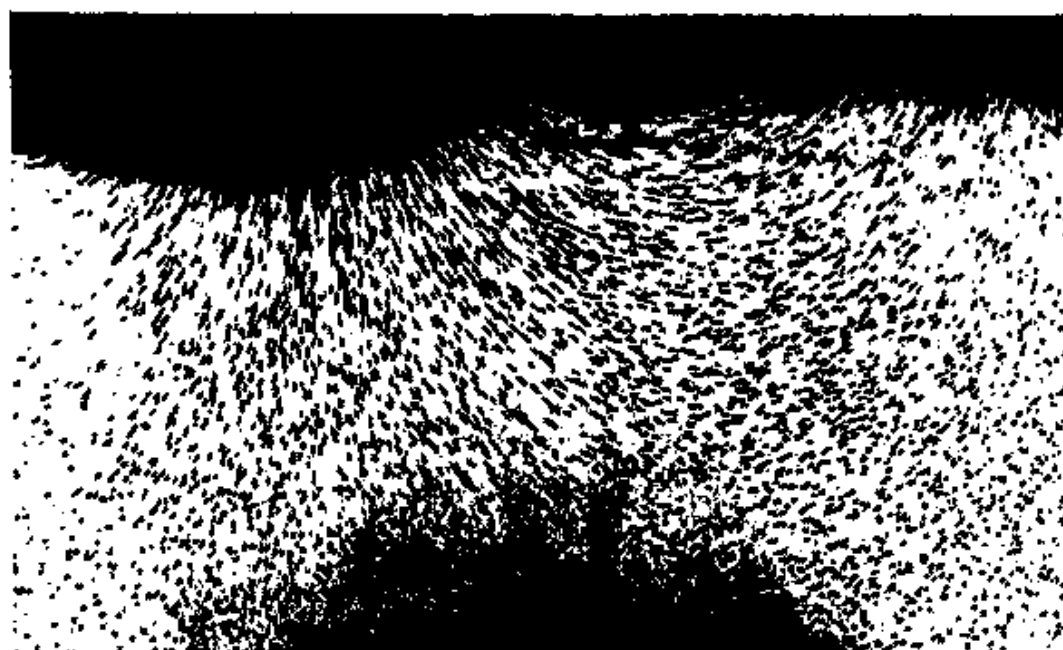


FIG. 39a. A photograph of the streamlines (short exposure).

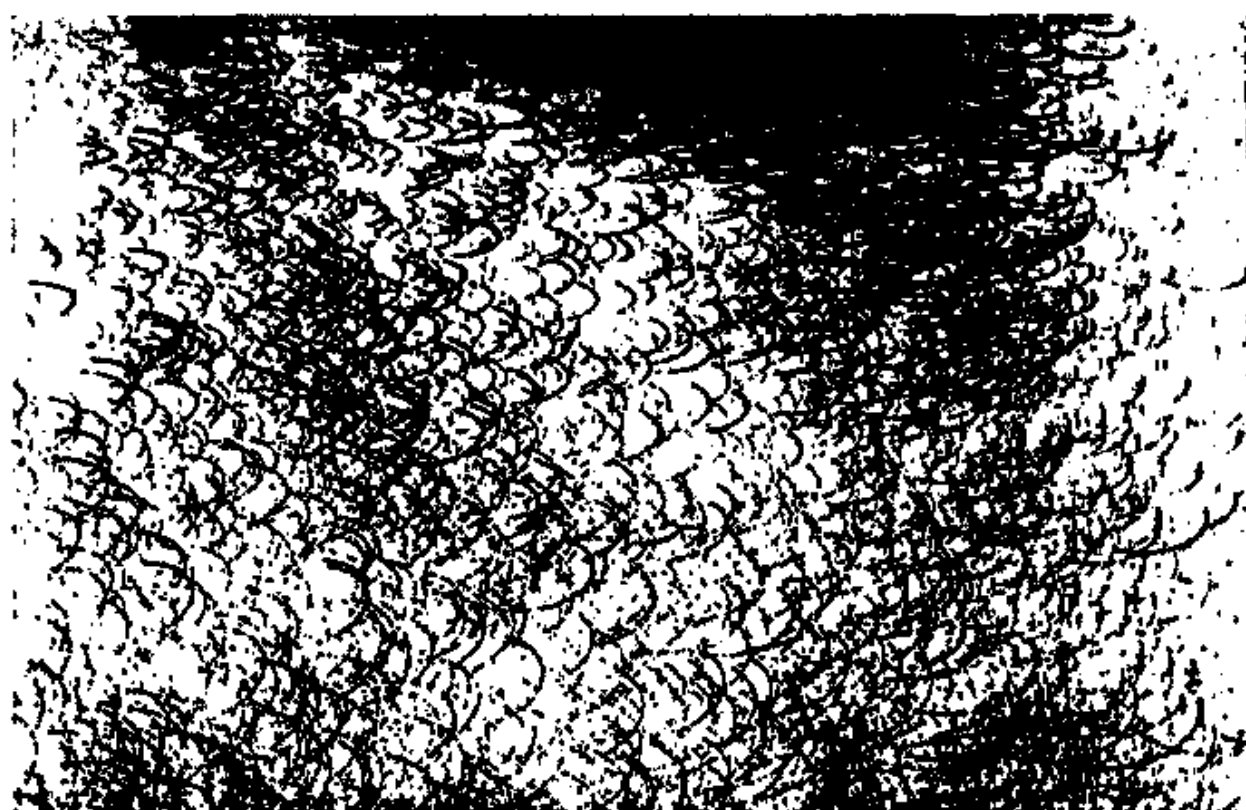


FIG. 39b. A photograph of the path curves (longer exposure).

The kinematic considerations at the beginning of 23 have dealt with the character of a plane wave in general and are consequently still valid. We therefore can take over the expressions (23.1) for Φ and (23.7) for η . Eq. (2) becomes after cancellation of the exponential and omission of second order terms

$$(3) \quad i\omega A = \frac{T}{\rho} k^3 a.$$

Also the kinematic condition (23.11) is in force; it gives, as in (23.12), a second equation

$$(4) \quad -i\omega a = kA.$$

From (3) and (4) we have

$$(5) \quad \frac{A}{a} = \frac{T}{\rho} \frac{k^3}{i\omega} = -\frac{i\omega}{k}$$

and, therefore, the following relation between ω and k

$$(6) \quad \omega^2 = \frac{T}{\rho} k^3.$$

The velocity of the capillary waves is

$$(7) \quad V^2 = \frac{\omega^2}{k^2} = \frac{T}{\rho} k, \quad V = \sqrt{\frac{T}{\rho} \frac{2\pi}{\lambda}}.$$

This then is a case of *anomalous dispersion* since V increases with *decreasing* λ , contrary to the gravity waves which show *normal* dispersion at all finite wave lengths. In Fig. 40, we have plotted V as a function of

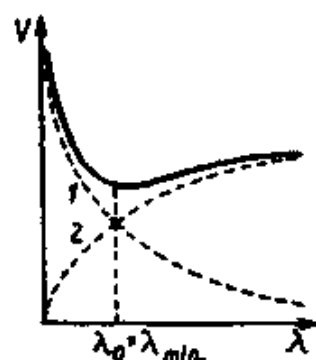


FIG. 40. Superposition of the phase velocities of capillary and gravity waves as functions of λ .

λ , obtaining the descending segment 1 (broken line). For comparison we have added the characteristic of the gravity waves from Fig. 34 as branch 2 (broken line). Let the intersection point which exists under all circumstances have the abscissa $\lambda = \lambda_0$. For $\lambda < \lambda_0$ the curve 1 is above curve 2, since the propulsive force of the capillary waves depends on the curvature of the surface profile which, for a given amplitude, is the larger the smaller the wave length. (This is also the reason why in Fig. 34 the segment α was not continued beyond a certain lower limit.) For $\lambda > \lambda_0$ the curve 2 is above 1. The unimportance of capillarity for large wave lengths which expresses itself in the small V -values justifies our use of the velocity potential for *deep water* throughout the present article. Even at a moderate

depth h the quotient h/λ is large for capillary waves, and so is *a fortiori* hk .

It is now easy to treat the general case of *combined action of capillarity and gravity*. We have only to introduce in Bernoulli's equation (23.5) the value p from (1) and the value for the gravity potential from (23.5b) $U = -\rho g \eta$. In place of (3) we obtain

$$(8) \quad i\omega A = \left(\frac{T}{\rho} k^3 + g \right) a.$$

From (4) and (8) follows now, instead of (5),

$$(9) \quad \frac{A}{a} = \frac{1}{i\omega} \left(\frac{T}{\rho} k^3 + g \right) = -\frac{i\omega}{k},$$

and therefore, instead of (6) and (7),

$$(10) \quad \omega^2 = \frac{T}{\rho} k^3 + gk,$$

$$(11) \quad V^2 = \frac{\omega^2}{k^2} = \frac{T}{\rho} k + \frac{g}{k} = V_1^2 + V_2^2.$$

Here V_1 is the velocity of the capillary wave according to (7), and V_2 is the velocity of the gravity wave according to (23.16); the velocity of the combination wave is found by a *quadratic superposition* formula.

Let us first compute from (11) the minimum of V as a function of λ . We obtain

$$2V \frac{dV}{d\lambda} = \frac{d}{dk} \left(\frac{T}{\rho} k + \frac{g}{k} \right) \frac{dk}{d\lambda} = 0,$$

and therefore

$$(12) \quad \frac{T}{\rho} = \frac{g}{k^2}, \quad \lambda_{\min} = \frac{2\pi}{k} = 2\pi \sqrt{\frac{T}{\rho g}}.$$

But λ_{\min} is at the same time the abscissa of the intersection point of the branches 1 and 2, which has been denoted by λ_0 in Fig. 40. In fact, one obtains from (7) and (23.16) the following equation for λ_0

$$\sqrt{\frac{T}{\rho} \frac{2\pi}{\lambda_0}} = \sqrt{\frac{g\lambda_0}{2\pi}},$$

hence $\lambda_0 = \lambda_{\min}$. The corresponding value of V_{\min} is found as

$$(13) \quad V_{\min}^2 = 2V_1^2 = 2V_2^2 = 2\sqrt{\frac{Tg}{\rho}}.$$

The general value (11) of V^2 can be put in the elegant form

$$(14) \quad \frac{V^2}{V_{\min}^2} = \frac{1}{2} \left(\frac{\lambda_{\min}}{\lambda} + \frac{\lambda}{\lambda_{\min}} \right),$$

which is easily verified.

We finally turn to the *numerical* values which in the present problem are not without interest. The constant of surface tension between water and air³ is

$$T = 72 \frac{\text{dynes}}{\text{cm}} = 72 \frac{\text{gr}}{\text{sec}^2}.$$

From (12) and (13) one obtains with $\rho = 1$, $g = 981$:

$$\lambda_{\min} = 2\pi \sqrt{\frac{72}{981}} = 1.73 \text{ cm}, \quad V_{\min} = \sqrt{2 \sqrt{72 \times 981}} = 23.2 \frac{\text{cm}}{\text{sec}}.$$

Water waves with a velocity of less than 23 cm/sec do not exist; waves of larger and of smaller wave length than $\lambda_{\min} = 1.73$ cm travel with greater velocity than 23 cm/sec. Lord Kelvin proposed for waves $\lambda < \lambda_{\min}$ the name "ripples"; one sometimes observes big gravity waves the faces of which are covered with fine capillary ripples.

26. The Concept of Group Velocity

At the end of 24 the term phase velocity was proposed in place of velocity of propagation V and it was pointed out that the *phase* of the wave advances with the velocity V . The phase is given in our representation [(23.1), (23.7), (24.1), etc.] by the exponent $i(kx - \omega t)$; if it is kept constant, that is, if one looks for the locus of equal phase as time proceeds, one obtains the condition

$$(1) \quad k dx - \omega dt = 0, \quad \text{therefore} \quad \frac{dx}{dt} = \frac{\omega}{k} = V.$$

For waves of a given invariable frequency (or, borrowing an expression from optics, for *monochromatic* waves), this is the only velocity that can come into consideration.

That is no longer true if waves of different frequency are superposed on each other. We speak then of a *group of waves*, particularly in the case where the frequencies are closely together. (Today one prefers, particularly in wave mechanics, the term "wave packet", an expression,

³Note that the most accurate method to determine T consists in the measurement of λ of standing capillary waves excited by a tuning fork. Elevation measurements in capillary tubes tend to be inaccurate because of impurities along the walls of the tube.

which is perhaps not quite so elegant as appropriate.) The wave group progresses with the *group velocity* U which, in general, is different from the phase velocity V . It is convenient to write the expression for U in the following way that will be justified later:

$$(2) \quad U = \frac{d\omega}{dk}.$$

If the waves proceed *without dispersion*, that is, if V is independent of the wave length λ (or of the wave number k) one obtains from (1) $d\omega = Vdk$ and therefore

$$(2a) \quad U = V.$$

Group and phase velocity coincide only for wave processes free of dispersion.

In general, however, we have

$$d\omega = V dk + k \frac{dV}{dk} dk,$$

or, by (2),

$$(2b) \quad U = V + k \frac{dV}{dk} = V - \lambda \frac{dV}{d\lambda}.$$

For normal dispersion ($dV/d\lambda > 0$), the group velocity is smaller than the phase velocity, for anomalous dispersion ($dV/d\lambda < 0$) the converse statement holds.

For gravity waves in deep water we have by (23.17)

$$V = \sqrt{\frac{g\lambda}{2\pi}}, \quad \frac{dV}{d\lambda} = \frac{1}{2\lambda} V,$$

and by (2b)

$$(2c) \quad U = V - \frac{1}{2} V < V.$$

On the other hand, for pure capillary waves (25.7) yields

$$V = \sqrt{\frac{T}{\rho} \frac{2\pi}{\lambda}}, \quad \frac{dV}{d\lambda} = -\frac{1}{2\lambda} V,$$

hence

$$(2d) \quad U = V + \frac{1}{2} V > V.$$

For the combined capillary-gravity waves we have according to Fig. 40 for $\lambda < \lambda_{\min}$ anomalous dispersion, therefore $U > V$; for $\lambda > \lambda_{\min}$ the

dispersion is normal and $U < V$. For $\lambda = \lambda_{\text{min}}$, $U = V$, corresponding to the horizontal tangent in Fig. 40 and to the vanishing factor $dV/d\lambda$ in (26).

The possibility of U and V having opposite signs has also been discussed.⁴

The concept of group velocity is originally a hydrodynamical one (Stokes 1876), but has proved of fundamental importance in optics (Lord Rayleigh). In wave mechanics phase velocity has formal significance while the group velocity is the important physical quantity, viz. the velocity of the particle represented by the wave packet (L. de Broglie 1924).

Let us first reproduce here the usual elementary derivation of formula (2), following Stokes. We superpose two waves that advance in positive x -direction, having the same amplitude, but slightly different frequency, hence slightly different wave number:

$$(3) \quad \eta = a\{\sin(k_1x - \omega_1t) + \sin(k_2x - \omega_2t)\}.$$

The wave group so produced has the character of a *beat*: at points where the difference of two phases is equal to $2n\pi$, there is reinforcement; when the phase difference is $(2n+1)\pi$ there is neutralization. This can be seen if (3) is written in the following form

$$(3a) \quad \eta = 2a \cos\left(\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right) \sin\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right).$$

The cosine term vanishes or equals ± 1 , depending on whether the phase difference of the two partial waves is an uneven or even multiple of π . Introducing

$$\frac{k_1 + k_2}{2} = k_0, \quad \frac{\omega_1 + \omega_2}{2} = \omega_0, \quad k_1 - k_2 = \Delta k, \quad \omega_1 - \omega_2 = \Delta \omega$$

we write for (3a)

$$(3b) \quad \eta = C \sin(k_0x - \omega_0t), \quad C = 2a \cos \frac{1}{2}(\Delta kx - \Delta \omega t).$$

The introduction of the "amplitude-factor" C for the cosine factor should suggest that it is a *slowly variable* quantity. We call it the group amplitude. The phase velocity of the compound wave as it follows from the sine factor in (3b) is

$$V = \frac{\omega_0}{k_0}$$

⁴Cf. Sir Arthur Schuster, *Theory of Optics*, London, 1924, p. 330.

and thus not sensibly different from the phase velocities of the component waves

$$V_1 = \frac{\omega_1}{k_1} \quad \text{and} \quad V_2 = \frac{\omega_2}{k_2}.$$

On the other hand, the velocity of propagation of the "amplitude" C is found by setting

$$\Delta kx - \Delta \omega t = \text{const},$$

which yields when differentiated

$$(3c) \quad \Delta k \, dx - \Delta \omega \, dt = 0, \quad \frac{dx}{dt} = \frac{\Delta \omega}{\Delta k}.$$

In the limit $\Delta k \rightarrow 0$ we therefore obtain our formula (2).

We can go beyond the very special assumption (3) when we consider an arbitrary group of waves whose frequencies are spread over a small frequency band. Since it is now convenient to return to complex representation, we put η in the form

$$(4) \quad \eta = \int_{k_0-\epsilon}^{k_0+\epsilon} a(k) e^{i(kx - \omega t)} dk.$$

The amplitude of the partial wave is now $a(k)dk$ and the small band width of the whole group 2ϵ , if measured in wave number units, so as to be concentrated about the central wave number k_0 . We rewrite the exponent in (4) accordingly:

$$kx - \omega t = k_0 x - \omega_0 t + (k - k_0)x - (\omega - \omega_0)t.$$

If this is done, (4) assumes the form

$$(4a) \quad \eta = C e^{i(k_0 x - \omega_0 t)}, \quad C = \int_{k_0-\epsilon}^{k_0+\epsilon} a(k) e^{i[(k-k_0)x - (\omega-\omega_0)t]} dk.$$

To find the velocity of propagation of the amplitude C in this more general case, we have to set the exponential in C , which alone contains x and t , constant. This yields for the whole wave group the sensibly constant value

$$\frac{dx}{dt} = \frac{\omega - \omega_0}{k - k_0} = \frac{\Delta \omega}{\Delta k}$$

in agreement with Eq. (2), if Δk is sufficiently small.

It should be noticed that the more general formula (4) as well as the expression (3) constitute a superposition of infinitely long wave trains, but do not represent an isolated hill or a few crests and troughs. To express an isolated wave group (a wave packet) by analytical means would require

the use of Fourier integrals; the integration in (4) would have to be carried out over all wave numbers from $-\infty$ to $+\infty$ instead of over the narrow range 2ϵ .

A wave packet can proceed *without change of shape* only if there is no dispersion. In the general case the group dissolves on account of the dispersion since its partial waves do not keep together; this tendency is of particular importance for the particle concept of wave mechanics. The conservation of the packet in the absence of dispersion follows directly from d'Alembert's solution (13.11), at least in the case of a sufficiently small amplitude, when this solution applies to dispersion-free hydrodynamics as well as to acoustics. An arbitrary initial disturbance $\eta = F(x)$ is then given by $\eta = F(x - Vt)$ at any later time if the initial velocity $\partial\eta/\partial t$ has been suitably chosen. This means that the wave can advance *without change of shape*.

In Fig. 41a and b a simple construction of the group velocity is shown that uses Eq. (2b) and the relation $\tan \alpha = \pm dV/d\lambda$.

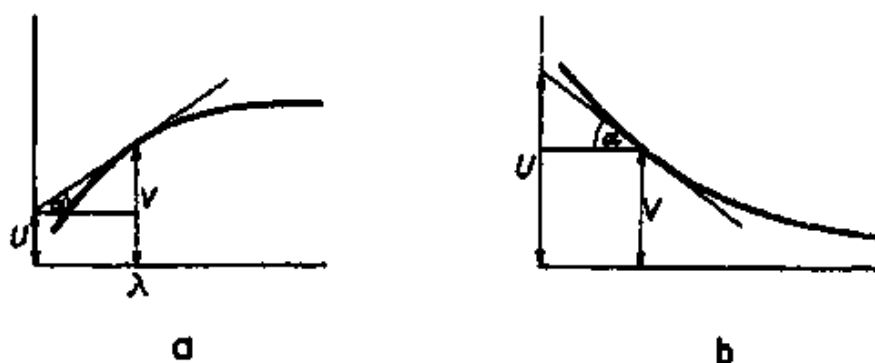


FIG. 41. Construction of the group velocity for a given phase velocity when the dispersion curve is given; (a) normal, (b) anomalous dispersion.

Reynolds⁵ and, simultaneously, Lord Rayleigh⁶ have pointed out the relation between the group velocity and the *transport of energy* and thus enhanced the physical understanding of the concept of group velocity. In their theory the ratio of group velocity U to phase velocity V appears as the ratio of the energy flow S through a specified cross-section of the wave during the time τ to the energy E contained in the fluid region ahead of this cross-section per length $V\tau$:

$$(5) \quad \frac{U}{V} = \frac{S}{E}.$$

⁵Reynolds, O., Papers on math. and phys. Subj., Cambridge 1901, Vol. I, p. 198, from Nature 46 (1877) 343.

⁶Lord Rayleigh, Scientific papers, Cambridge 1901, Vol. I, p. 322 and Theory of sound, Vol. I, appendix from Proc. London Math. Soc. 9, 21.

It is convenient for what follows, to identify the time τ with the period of oscillation, so that $V\tau$ is equal to the wave length λ . The energy E is more accurately defined as the energy *surplus* of the oscillating fluid over the fluid at rest.

Let us examine these relations for deep water waves of sufficiently small amplitude. Since we are concerned with energy, that is with a quadratic function of the rates of displacement, we cannot use the complex representation. Taking the real equations (23.14) and (23.14b)

$$\begin{aligned}\Phi &= A \cos(kx - \omega t)e^{-ky}, \\ \eta &= -\frac{k}{\omega} A \sin(kx - \omega t),\end{aligned}\tag{6}$$

we can obtain the potential energy of gravity contained between the undisturbed surface $y = 0$ and the wave surface $y = \eta$ at a specified time t , per length λ in x -direction and per width 1 in z -direction:

$$\begin{aligned}E_{\text{pot}} &= g\rho \int_0^\lambda dx \int_0^\eta y dy \\ &= g\rho \frac{k^2}{\omega^2} \frac{A^2}{2} \int_0^\lambda \sin^2(kx - \omega t) dx = g\rho \frac{k^2}{\omega^2} \frac{A^2}{4} \lambda.\end{aligned}\tag{7}$$

Using the dispersion formula (23.15), we rewrite this result:

$$E_{\text{pot}} = \frac{\rho}{4} k A^2 \lambda = \frac{\rho}{2} \pi A^2.\tag{7a}$$

This is at the same time the difference between the potential energy of the oscillating fluid and the fluid at rest for the *entire* depth which reaches from $y = \eta$ to $y = \infty$ for the moving fluid and from $y = 0$ to $y = \infty$ for the fluid at rest.

The kinetic energy contained in this region has the same magnitude, as is always the case for small oscillations; this will be discussed in detail in problem V, 1. The total energy, or more accurately, the energy excess over the fluid at rest becomes therefore

$$E = E_{\text{kin}} + E_{\text{pot}} = \rho\pi A^2.\tag{8}$$

In order to determine the energy flow S through the cross-section at $x = \lambda$, which may be replaced by the cross-section $x = 0$, we have to know the pressure p at this cross-section. This is found from Bernoulli's equations (23.5) and (23.5b) as

$$p = \rho \frac{\partial \Phi}{\partial t} + \rho g y + \text{const.}\tag{9}$$

Since p acts normal to the plane $x = 0$, its product with the x -component of the velocity, multiplied with $dy dt$ expresses the work dW done at the surface element $dy \cdot 1$ during dt :

$$(9a) \quad dW = pv_x dy dt = -p \frac{\partial \Phi}{\partial x} dy dt.$$

The work performed at the cross-section $x = 0$ during the time τ represents the energy flow through this cross-section in this time. It is by (9a) and (9)

$$(10) \quad S = - \int_0^\tau dt \int_{-\infty}^{\infty} dy p \frac{\partial \Phi}{\partial x} = - \int_0^\tau dt \int_{-\infty}^{\infty} dy \left(\rho \frac{\partial \Phi}{\partial t} + \dots \right) \frac{\partial \Phi}{\partial x}.$$

The terms $\rho gy + \text{const}$ not written out in (10) have the periodic factor $\partial \Phi / \partial x$. Since they are independent of t , their contributions cancel when integrated over the period τ . Substitution from (6) gives

$$S = \rho \omega k A^2 \int_0^\tau dt \sin^2 \omega t \int_{-\infty}^{\infty} e^{-2ky} dy$$

or, after integration,

$$(11) \quad S = \frac{\rho}{2} \pi A^2 e^{-2ky} = \frac{\rho}{2} \pi A^2.$$

Here, e^{-2ky} has been replaced by 1 which means omission of terms of higher order than A^2 .

Comparing (11) and (8), one obtains

$$(12) \quad \frac{S}{E} = \frac{1}{2}.$$

This is indeed the value of the ratio U/V for deep water waves obtained in (2c). We have thus confirmed in this simplest case the general energetic definition (5) of the group velocity. It also holds in the considerably more complicated case of moderately deep water where Eqs. (2b) and (24.10a) lead to

$$(13) \quad \frac{U}{V} = \frac{1}{2} \left(1 + \frac{2kh}{\sinh 2kh} \right).$$

Let us now have a look at this result from a more qualitative point of view; we again consider the wave train of length λ ending at $x = 0$. If no dispersion were present, the entire energy content of the wave would flow through the cross-section during the period τ ; in other words, we should have $S = E$ and $U = V$. Actually, only a part of this energy is transported in the case of normal dispersion (half of it with deep water

waves), whereas the other part remains in the oscillating fluid ahead of the cross-section. This makes the case of anomalous dispersion appear rather peculiar since more energy passes through the cross-section than is present in the λ -train ahead of it; the energy remaining in the train would be negative. To maintain the wave motion in such a case requires expending more energy than travels with the wave; in the first case less excitation energy is required for the same amount of energy transported. *Energy transport is thus easier in a medium of normal dispersion and more difficult in a medium of anomalous dispersion.*

27. Circular Waves

The waves that are produced when a stone is thrown into water form a series of concentric circular crests and troughs; their amplitudes are not constant, nor are the distances between the crests. What one observes is a sharp decrease of the amplitude and an increase of the distance between two subsequent crests, which seems to follow a peculiar law. This problem that appears so simple requires for its solution a considerable mathematical apparatus: we not only need Bessel functions and Fourier integrals, but we should have to use the method of steepest descent if we were to treat it in full accuracy. A systematic treatment of these devices will be given in Vol. VI. At this point we shall limit ourselves to such explanations as are directly required here.

For our analysis we shall replace the stone that hits the water surface by a "standard disturbance": at $r = 0$ a cylindrical piston of radius r_0 is immersed in the water to a distance a from the surface and suddenly withdrawn at the time $t = 0$. If we again denote the surface depression by η , the initial state is given by

$$(1) \quad \begin{aligned} \eta &= a, & r < r_0. \\ \eta &= 0, & r > r_0. \end{aligned}$$

In preparation for the problem of a single disturbance we first consider the much simpler case of a periodic excitation.

1. The Periodic Case. Introduction of Bessel Functions

The excitation we have in mind works in a similar way as the device that produces waves in a swimming pool: a straight board subjected to a periodic motion excites plane progressive gravity waves advancing normal to the board. We have investigated waves of this type in 23 for deep water and wish now to transfer our previous results to the present circular symmetric problem.

Introducing cylindrical polar coordinates r, φ, y (y positive downward), we write the condition of incompressibility for the velocity potential Φ (cf. problem I, 3) in the form

$$(2) \quad \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0.$$

If we at first assume circular symmetry, Φ does not depend on φ and can be written as follows

$$(3) \quad \Phi = A f(r) e^{-k y} e^{-i \omega t}.$$

Now the following differential equation results for $f(r)$:

$$(4) \quad \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} + k^2 f = 0.$$

We substitute $kr = \rho$ and obtain

$$(4a) \quad \frac{d^2 f}{d\rho^2} + \frac{1}{\rho} \frac{df}{d\rho} + f = 0.$$

Let $J_0(\rho)$ be the solution of this equation which is regular at $\rho = 0$ and assumes there the value 1. The power series expansion of this function can be easily found from (4a) by the method of undetermined coefficients:

$$(5) \quad J_0(\rho) = 1 - \left(\frac{\rho}{2}\right)^2 + \frac{1}{2!^2} \left(\frac{\rho}{2}\right)^4 - \frac{1}{3!^2} \left(\frac{\rho}{2}\right)^6 + \dots$$

If, on the other hand, Φ is not independent of φ , then we replace $f(r)$ by $f_n(r) e^{i n \varphi}$ and obtain instead of (4) and (4a)

$$(6) \quad \frac{d^2 f_n}{dr^2} + \frac{1}{r} \frac{df_n}{dr} + \left(k^2 - \frac{n^2}{r^2}\right) f_n = 0$$

and

$$(6a) \quad \frac{d^2 f_n}{d\rho^2} + \frac{1}{\rho} \frac{df_n}{d\rho} + \left(1 - \frac{n^2}{\rho^2}\right) f_n = 0.$$

Let $J_n(\rho)$ denote that solution of (6a) which is regular at $\rho = 0$ and admits the following expansion,

$$(7) \quad J_n(\rho) = \frac{1}{n!} \left(\frac{\rho}{2}\right)^n - \frac{1}{1!(n+1)!} \left(\frac{\rho}{2}\right)^{n+2} + \frac{1}{2!(n+2)!} \left(\frac{\rho}{2}\right)^{n+4} - \dots$$

which may be considered as a generalization of (5). The functions J_n are known as *Bessel Functions of n^{th} order*; they are entire transcendental functions.

The following relation between J_1 and J_0

$$(8) \quad J_1(\rho) = -\frac{d}{d\rho} J_0(\rho),$$

can be immediately verified from (5) and (7).

On writing Eq. (4a) in the form

$$\frac{d}{d\rho} \left(\rho \frac{dJ_0}{d\rho} \right) + \rho J_0(\rho) = 0,$$

where $J_0(\rho)$ has been written for f , one has because of (8),

$$\rho J_0(\rho) = \frac{d}{d\rho} (\rho J_1(\rho)).$$

This relation, integrated to an arbitrary upper limit, yields the following formula

$$(8a) \quad \int_0^{\rho_0} \rho J_0(\rho) d\rho = \rho_0 J_1(\rho_0).$$

We shall also need the integral representation

$$(9) \quad J_0(\rho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i\rho \cos \alpha} d\alpha.$$

It is not difficult to verify that this expression satisfies Eq. (4a) and has the value 1 for $\rho = 0$, wherefore it must be identical with (5). The same result can also be obtained, though less directly by series expansion of the exponential function in (9). Obviously, (9) can also be written in the form

$$(9a) \quad J_0(\rho) = \frac{1}{\pi} \int_0^{\pi} e^{i\rho \cos \alpha} d\alpha.$$

We now substitute $f = J_0(kr)$ in the formula for the potential (3) and obtain

$$(10) \quad \Phi = A J_0(kr) e^{-ky} e^{-i\omega t}.$$

This formula satisfies the condition for deep water: $\Phi \rightarrow 0$ for $y \rightarrow \infty$, but it should also satisfy the conditions (23.6) and (23.11) for the surface $y = 0$. The analytical form of the surface depression is assumed in correspondence to the form of Eq. (10)

$$(10a) \quad \eta = a J_0(kr) e^{-i\omega t}.$$

Our expression for Φ and η leads, by exactly the same steps as in Eqs. (23.9), (23.12) and (23.13), after cancellation of $J_0(kr) e^{-i\omega t}$, to

$$(11) \quad \frac{A}{a} = \frac{g}{i\omega} = -\frac{i\omega}{k}.$$

Thus we reobtain the dispersion law (23.15)

$$(11a) \quad \omega = \sqrt{gk}$$

and the final expression for the potential becomes

$$(12) \quad \Phi = \frac{ag}{i\omega} J_0(kr) e^{-ky - i\omega t}.$$

2. Single Disturbance. The Fourier-Bessel Integral

The content of the Fourier integral theorem is this: an arbitrary function $F(x)$ (provided it is not "too irregular") can be represented by superposition of trigonometric functions $\sin kx$ or by the equivalent exponential functions e^{ikx} in the form

$$(13) \quad F(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikx} \int_{-\infty}^{+\infty} d\xi F(\xi) e^{-i k \xi}, \quad -\infty < x < +\infty.$$

This theorem has an analogue in the following representation of F by Bessel functions

$$(13a) \quad F(r) = \int_0^{\infty} k dk J_0(kr) \int_0^{\infty} \xi d\xi F(\xi) J_0(k\xi), \quad 0 < r < +\infty.$$

Both theorems will be proved in Vol. VI, where (13a) will be obtained as a consequence of (13).

We now apply (13a) to the initial state (1). Our function $F(r)$ is then given by

$$F(r) = \eta = \begin{cases} a & \text{for } 0 < r < r_0, \\ 0 & \text{for } r_0 < r < \infty. \end{cases}$$

Its representation according to (13a) is

$$(14) \quad \eta_{t=0} = a \int_0^{\infty} k dk J_0(kr) \int_0^{r_0} \xi d\xi J_0(k\xi).$$

For the evaluation of the inner integral Eq. (8a) is used, which gives, with $k\xi = \rho$ and $k r_0 = \rho_0$,

$$\int_0^{r_0} \xi d\xi J_0(k\xi) = \frac{1}{k^2} \int_0^{\rho_0} \rho d\rho J_0(\rho) = \frac{r_0}{k} J_1(kr_0).$$

Instead of the double integral (14) we have got

$$(14a) \quad \eta_{t=0} = ar_0 \int_0^\infty J_0(kr) J_1(kr_0) dk.$$

Now we can show that η and Φ at any later instant $t > 0$ have the following forms:

$$(15) \quad \eta = ar_0 \int_0^\infty J_0(kr) J_1(kr_0) \exp(-i \sqrt{gk} t) dk,$$

$$(16) \quad \Phi = -iagr_0 \int_0^\infty J_0(kr) J_1(kr_0) \exp(-ky - i \sqrt{gk} t) dk / \sqrt{gk}.$$

For a proof, we observe first that (15) takes the value of (14a) if $t = 0$. Now, the expressions (15) and (16) are obtained by applying the same "operator"

$$r_0 \int_0^\infty J_1(kr_0) dk \dots$$

to the periodic solutions (10a) and (12), with due regard to the dispersion law (11a). Since (10a) and (12) satisfy the differential equation of our problem and the boundary conditions for $y = 0$ and $y = \infty$, the same must be true for our expressions (15) and (16). Furthermore (15) and (16) satisfy the initial condition for $t = 0$ and are thus the required solutions.

3. Integration with Respect to k . The Method of the Stationary Phase

The quantities of physical interest in circular waves are all connected with the surface function η to which we therefore limit the following discussion. To evaluate η we first replace J_0 in (15) by its integral representation (9a) and reverse the order of integration, obtaining the double integral

$$(17) \quad \eta = \frac{ar_0}{\pi} \int_0^\pi d\alpha \int_0^\infty dk J_1(kr_0) \exp(ikr \cos \alpha - i \sqrt{gk} t).$$

We denote the inner integral by K , put $2\tau = t \sqrt{gr_0}/r$ and introduce a new integration variable by $p = \sqrt{kr_0}$; with these notations we have

$$(18) \quad K = \frac{2}{r_0} \int_0^\infty p dp J_1(p^2) \exp[i(p^2 \cos \alpha - 2p\tau)r/r_0].$$

Here r/r_0 is a very large number, the exponential is, therefore, a rapidly varying function of p . In general, the positive and negative oscillations of

the integrand will cancel except at those places where the factor of ir/r_0 is *slowly* variable. If we abbreviate this factor by

$$(19) \quad f(p) = p^2 \cos \alpha - 2pr,$$

this will happen when

$$(19a) \quad f'(p) = 0, \quad p = p_0 = \frac{\tau}{\cos \alpha}.$$

This observation leads to a method of estimating the value of the integral known as the *method of the stationary phase*. Since the "phase" of the quickly oscillating exponential function becomes stationary at p_0 , one limits the range of integration to the neighborhood of this value and thus achieves the integration by elementary means. It was particularly Lord Kelvin who applied this method expertly to many problems of hydrodynamics and optics. The mathematically exact form of this device⁷ is called the *method of steepest descent* which we mentioned before.

Eq. (19) can be written in the form

$$f(p) = \cos \alpha \{ (p - p_0)^2 - p_0^2 \}$$

The integral (18) then becomes

$$(20) \quad K = \frac{2}{r_0} \exp(-ip_0^2 \cos \alpha \cdot r/r_0) \int_{p_0-\epsilon}^{p_0+\epsilon} J_1(p^2) p \exp[i \cos \alpha (p - p_0)^2 r/r_0] dp.$$

where the range of integration is now an ϵ -neighborhood of p_0 . The factors $J_1(p^2)$ and p in the integrand are *slowly variable* compared with the exponential function and may be replaced by $J_1(p_0^2)$ and p_0 . We finally substitute in (20)

$$(20a) \quad s = q(p - p_0), \quad q^2 = \frac{\tau}{r_0} \cos \alpha$$

and obtain

$$(21) \quad K = (2/r_0) \exp(-ip_0^2 \cos \alpha \cdot r/r_0) J_1(p_0^2) \frac{p_0}{q} \int_{-\epsilon q}^{+\epsilon q} e^{is^2} ds.$$

As to the position of the critical value $p = p_0$, (19a) gives, since τ is positive,

$$\begin{aligned} p_0 > 0 & \quad \text{for} \quad 0 < \alpha < \frac{\pi}{2}, \\ p_0 < 0 & \quad \text{for} \quad \frac{\pi}{2} < \alpha < \pi. \end{aligned}$$

⁷The problem of the circular waves has been treated according to that method in the Munich thesis of H. Widenbauer, Z. angew. Mathem. u. Mech. 14, 321 (1939). Our Fig. 42 has been taken from this paper.

In the first case p_0 and, consequently, the value $s = 0$ fall in the range of integration $0 < k < \infty$; not so in the second case. Thus the representation (21) is valid only for $\alpha < \pi/2$. Otherwise we have

$$(21a) \quad K = 0; \quad \frac{\pi}{2} < \alpha < \pi,$$

which means complete extinction by "interference".

The final evaluation of (21) is achieved by means of the well known formula

$$(22) \quad \int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}.$$

On replacing t by

$$t = e^{-i(\pi/4)} s, \quad t^2 = -is^2, \quad dt = e^{-i(\pi/4)} ds,$$

we are led^{*} to

$$(22a) \quad \int_{-\infty}^{+\infty} e^{t^2} ds = e^{i(\pi/4)} \sqrt{\pi}.$$

The left member is our integral in (21) whose limits become $\pm \infty$ if r/r_0 is made to approach infinity at constant ϵ and $\cos \alpha \neq 0$. If we introduce the result (22a) in (21) we obtain

$$(23) \quad K = (2 \sqrt{\pi/r_0}) \exp(-ip_0^2 \cos \alpha \cdot r/r_0 + i\pi/4) J_1(p_0^2) \frac{p_0}{q}.$$

Now the value of p_0 following from (17a) and (19a),

$$p_0 = \sqrt{\frac{gr_0 t^2}{4r^2}} \frac{1}{\cos \alpha},$$

is very small, provided

$$(24) \quad \frac{gr_0 t^2}{4r^2} \ll 1$$

(but we must exclude a small finite neighborhood of $\alpha = \pi/2$). The Bessel function J_1 is, for small p_0 , sufficiently well approximated by the first term of (7):

$$J_1(p_0^2) = \frac{p_0^2}{2}, \quad p_0 J_1(p_0^2) = \frac{p_0^3}{2} = \frac{1}{2} \left(\frac{gr_0 t^2}{4r^2} \right)^{3/2} \frac{1}{\cos^3 \alpha}.$$

^{*}It will be noticed that the original path of the integration in (22a) is the bisectrix of the 1st and 3d quadrant in the complex s -plane which forms also the limit of convergence of the integral; this path may be replaced by the real axis and two circular arcs; the contributions of the latter vanish if $R \rightarrow \infty$.

Taking the value of q from (20a) we obtain finally for K

$$(25) \quad K = \sqrt{\frac{\pi}{\cos^7 \alpha}} \frac{r_0}{r^{\frac{7}{2}}} \left(\frac{gt^2}{4r} \right)^{3/2} \exp(-igt^2/4r \cos \alpha + i\pi/4).$$

4. Integration with Respect to α . Discussion of a Limiting Case

We return now to (17), where we may restrict the integration interval to $0 < \alpha < \pi/2$ because of (21a). On introducing the volume displaced by the initial impulse, $V_0 = \pi r_0^2 a$, (17) is transformed into

$$(26) \quad \eta = \frac{V_0}{r^2} \left(\frac{gt^2}{4\pi r} \right)^{3/2} \int_0^{\pi/2} \frac{d\alpha}{\cos^{7/2} \alpha} \exp(-igt^2/4r \cos \alpha + i\pi/4).$$

We shall finally have to make $r_0 \rightarrow 0$. In order to obtain a finite effect in the limit, V_0 must be kept constant, that is to say, the depth of immersion a must approach ∞ in a definite way. We shall come back to this eventually.

The representation of η by (26) depends essentially on the variable

$$(27) \quad u = \frac{gt^2}{4r}.$$

Our aim is to determine the asymptotic behavior of η if $u \rightarrow \infty$. Now for large u the exponent in (26) becomes once more a rapidly varying function of α , so that the method of the stationary phase can again be applied. Similarly as before, we denote the factor of u in the exponent of (26) by $f(\alpha)$ and have

$$f(\alpha) = \frac{1}{\cos \alpha}, \quad f'(\alpha) = \frac{\sin \alpha}{\cos^2 \alpha}.$$

The critical α -value is thus

$$\alpha = \alpha_0 = 0, \quad f(\alpha_0) = 1, \quad f(\alpha) = 1 + \frac{\alpha^2}{2}, \quad \frac{1}{\cos^{7/2} \alpha_0} = 1,$$

and consequently

$$(28) \quad \eta = \frac{V_0}{r^2} \left(\frac{u}{\pi} \right)^{3/2} \exp(-iu + i\pi/4) \int_0^\epsilon \exp(-iu\alpha^2/2) d\alpha.$$

We need the value of the integral in (28) for large u . Keeping ϵ constant, we apply the same argument that led us from (21) to (23). This time the integral in (28) transforms as follows:

$$\int_0^\epsilon \exp(-iu\alpha^2/2) d\alpha = \sqrt{\frac{2}{u}} \int_0^\infty \exp(-is^2) ds = \sqrt{\frac{\pi}{2u}} \exp(-i\pi/4).$$

By substitution in (28) we finally obtain

$$(29) \quad \eta = \frac{V_0}{\sqrt{2\pi r^3}} u e^{-iu}.$$

The dimensions in this equation are easily checked: u is a number according to (27), V_0/r^3 is a length and so is η .

Our method seems to break down at the upper limit $\pi/2$ of the integral in (26) because of the denominator $\cos \alpha$, but according to (21a) this is at the same time the lower limit at which K vanishes. Thus one is probably right to assume that a special investigation of the behavior of (26) at the upper limit is unnecessary.

The real part of (29) is the surface equation in which we are interested; we have

$$\eta = \frac{V_0}{\sqrt{2\pi r^3}} u \cos u.$$

η becomes infinite for $r = 0$, which is quite understandable since the depth of immersion a has become infinite in the limiting process $r_0 \rightarrow 0$ for fixed V_0 . The amplitudes of the crests decrease according to u/r^3 or r^{-3} , the crests follow each other at the distance

$$\Delta r = \frac{8\pi r^2}{g t^3},$$

as is easily seen, if the phases of neighboring crests $u = 2\pi n$, $u = 2\pi(n+1)$ are compared for constant t . Hence the wave length is no longer a constant as in our previous examples of wave motion, but increases at

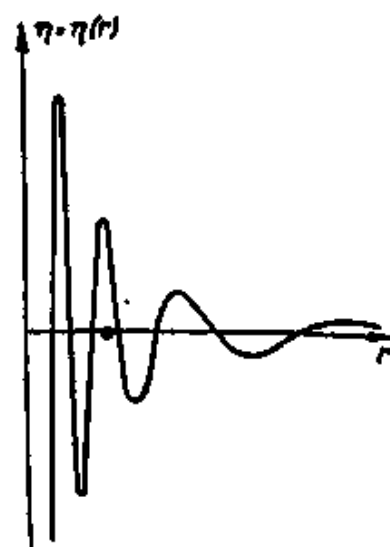


FIG. 42. Shape of the water surface t seconds after the ring waves were excited; t must satisfy the condition $u \gg 1$.

constant t with r^2 and decreases at constant r with t^3 . Fig. 42 is a diagram of the surface profile η . Its appearance agrees well with that of a water surface disturbed by the fall of a small object like a stone or a rain drop.

Our problem was set as a prize problem by the academy of Paris in 1816 and solved by Cauchy. About this and a later paper of Poisson cf. the report of H. Burkhardt.*

28. Ship Waves (*Kelvin's Limit Angle and Mach's Angle*)

The wave pattern that is left behind by a ship at sea consists of a system of waves that envelopes the hull lengthwise and is interwoven with a system of cross waves. The two systems advance with the boat so as to be stationary relative to it. The beauty of the pattern is most impressive when viewed from an airplane or from the top of a high cliff, but the same phenomenon on a more modest scale develops behind a duck swimming in a pond.

In our analysis the object that produces the waves will be considered as a point. The problem can then be formulated in the following way: The instantaneous location of the boat is the origin of a system of circular waves; this origin is in uniform rectilinear motion, its velocity being the speed of the boat v . Our task is to find *the result of the superposition of the successive circular waves*. That it will be stationary relative to the boat is evident, but the detail structure of the wave pattern is surprising enough and can only be unraveled by a careful analysis.

With a view to Fig. 43, let O be the location of the boat at the time $t = 0$, and Q its location t seconds earlier so that $QO = vt$. We wish to find the ordinate of the water surface η at the field point P . It is to be compounded of all ordinates η_i that were produced at earlier instants t by means of the formula

$$(1) \quad \eta = \beta \int_{-\infty}^0 \eta_i dt.$$

The factor β in (1) must have the dimension of a reciprocal time; we put it equal to v/l . For the length l there is hardly any choice other than the cube root of the initial displacement V_0 in Eq. (27.29). Thus we obtain from (1) and (27.29)

$$(2) \quad \eta = C \int_{-\infty}^0 \frac{1}{r_i^2} u_i \exp(-iu_i) dt, \quad C = \frac{V_0^{2/3} v}{\sqrt{2\pi}}$$

where r_i is the distance QP in Fig. 43, that is, the distance between the field point P and the location of the source of disturbance, t seconds ago.

Now let P have the polar coordinates r and ϑ relative to the pole O ; r and ϑ are therefore independent of t . According to Fig. 43 we obtain

*Jahresbericht der deutschen Math. Ver. Vol. X, p. 429 (1908).

$$(3) \quad r_i^2 = r^2 + v^2 t^2 + 2rvt \cos \vartheta,$$

where t is negative. As in (27.27) we put

$$(4) \quad u_i = \frac{g}{4r_i} \frac{t^3}{r_i} = f(t).$$

The representation (27.29) which we have applied in (2) was computed under the assumption $u \gg 1$. With this condition, $f(t)$ becomes again a rapidly varying function so that the method of stationary phase can be applied. We then have to find the roots of the equation $f'(t) = 0$. From (4) and (3) we obtain

$$(5) \quad \frac{4}{g} f'(t) = \frac{2t}{r_i} - \frac{t^2}{r_i^3} (v^2 t + rv \cos \vartheta) = \frac{t}{r_i^3} (v^2 t^2 + 3rvt \cos \vartheta + 2r^2).$$

Hence the roots of $f'(t) = 0$ are

$$t = -\frac{3}{2} \frac{r}{v} \cos \vartheta \pm \sqrt{\frac{9}{4} \frac{r^2}{v^3} \cos^2 \vartheta - 2 \frac{r^2}{v^3}}.$$

If ϑ is an acute angle, the roots t_1 and t_2 are both negative

$$(6) \quad \begin{aligned} t_1 &= -\frac{3}{2} \frac{r}{v} \left(\cos \vartheta - \sqrt{\cos^2 \vartheta - \frac{8}{9}} \right), \\ t_2 &= -\frac{3}{2} \frac{r}{v} \left(\cos \vartheta + \sqrt{\cos^2 \vartheta - \frac{8}{9}} \right). \end{aligned}$$

Now, in order to fall in our integration interval $-\infty < t < 0$, the roots not only have to be negative, they must also be real. This implies

$$\cos^2 \vartheta > \frac{8}{9}, \quad |\vartheta| < \vartheta_0,$$

where ϑ_0 denotes the limiting angle

$$(7) \quad \cos^2 \vartheta_0 = \frac{8}{9} \quad \text{or} \quad \operatorname{tg} \vartheta_0 = \frac{1}{\sqrt{8}}, \quad \vartheta_0 = 19^\circ 28'.$$

This angle was first determined by Lord Kelvin. For $|\vartheta| > \vartheta_0$ there is no such t -value as would make the phase stationary, that is to say, *the whole wave pattern is bounded on either side by a straight line forming the angle ϑ_0 with the direction of motion of the boat*. This is shown in Fig. 44 which is taken from Lamb's¹⁰ *Hydrodynamics*.

The interference pattern itself can be understood on the basis of the integral (2) which essentially reduces to the two contributions of the neigh-

¹⁰H. Lamb, *Hydrodynamics*, Cambridge, 5th edition, p. 409 ff.

borhoods of t_1 and t_2 . These contributions contain the phase factors $\exp[-if(t_1)]$ and $\exp[-if(t_2)]$. By putting $f(t_1)$ and $f(t_2)$ constant, one obtains the two systems of curves, the lengthwise and transverse waves mentioned before. In Fig. 44 the successive crests of the two systems have been drawn. We shall discuss these curves by giving the field point P a

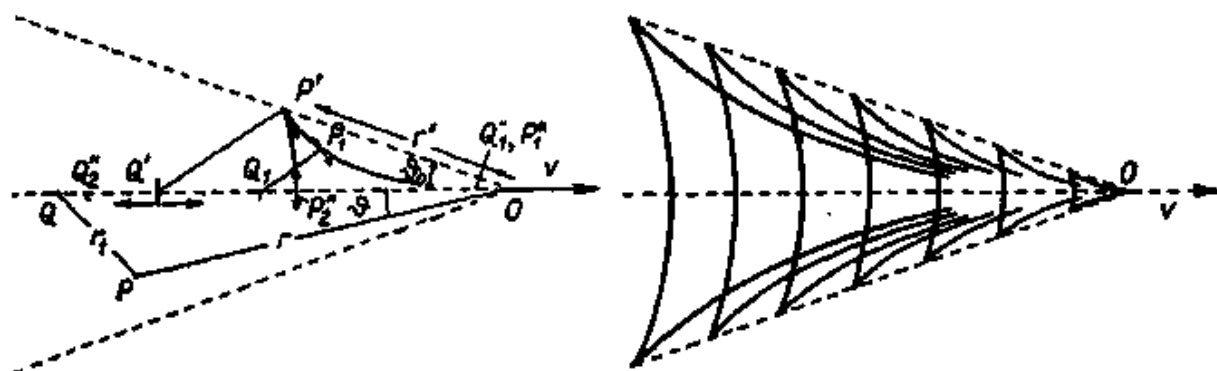


FIG. 43 (left). Illustrating the generation of ship waves. O is the present position of the ship, $Q, Q' \dots$ are previous positions of the ship, $P, P' \dots$ are specified positions of the field point.

FIG. 44 (right). Lengthwise and transverse ship waves.

variety of positions, having in mind that the disturbance at P at the time $t = 0$ is essentially caused by two annular waves that issued from the source in the positions Q' and Q'' at the times t_1 and t_2 respectively.

Let us start with a field point position $P = P'$ on the limiting line $\vartheta = \vartheta_0$. The two values t_1 and t_2 in (6) coincide. The common source position associated with this case is the point Q' in Fig. 43. Its distance from O is by (6)

$$OQ' = -vt'_1 = -vt'_2 = \frac{3}{2} r' \cos \vartheta_0,$$

where r' is the distance of P' from O . The direction of the curves of uniform phase $f(t_1) = f(t_2) = \text{const}$ in P' is given by a circular arc element with center Q' and radius $Q'P'$. Both curves of constant phase pass through P' in the same forward direction. Considered as one curve they form a *cusp* at P' .

If now ϑ is decreased and r so chosen that $f(t_2)$ remains constant the source position associated with t_2 moves to the *left* from Q' (see Fig. 43) since $-vt_2$ increases according to (6). When ϑ reaches the value zero, Q' reaches the position Q'' , so that

$$OQ'' = -vt_2 = \frac{3}{2} r \left(1 + \frac{1}{3}\right) = 2r.$$

Here r is the distance between O and the field point P_2'' at which the transverse wave intersects the course of the boat. The direction of the phase curve is obviously perpendicular to the course; it coincides with the circular element of radius $Q_2''P_2'' = r$ about the center Q_2'' . This gives a general idea of the shape of the phase curves $f(t_2) = \text{const.}$

If, on the other hand, we choose the r -values for decreasing ϑ so that $f(t_1)$ remains constant, $-vt_1$ decreases [see (6)] and the point Q' in Fig. 43 moves *right* to the new position Q_1 . The slope of the lengthwise wave through P' becomes flatter, as is shown by the circular arc about Q_1 with radius Q_1P_1 . The distance OQ_1'' becomes for very small ϑ

$$-vt_1 = \frac{3}{2}r \left(1 - \frac{\vartheta^2}{2} - \sqrt{\frac{1}{9} - \frac{\vartheta^2}{2}} \right) = r \left(1 + \frac{3}{8}\vartheta^2 \right)$$

so that Q_1'' nearly coincides with the field point P_1'' when both are close to O . The lengthwise waves should become tangential to the direction of travel at O if our method were still valid in the neighborhood of O . This, however, is not the case: Our method of stationary phase breaks down for a short running time t . Nevertheless the general shape of the lengthwise waves has thus been clarified.

The phase curves $f(t_1) = \text{const.}$, $f(t_2) = \text{const.}$ are essentially identical with the actual wave pattern observed. Along either curve the contribution of the integral whose phase is *not* constant shows up as a secondary ripple, at least at some distance from O . For a comprehensive analytical representation of both families of curves see Lamb loc. cit. and L. Hopf in his Munich thesis quoted on p. 120; there the evaluation is carried out by complex integration.

Looking back at our result it appears strange that there should exist a limiting angle ϑ_0 independent of the traveling speed v . This seems to disagree with the well known photographs of projectiles moving with supersonic speed, first obtained by the Austrian philosopher and physicist Ernst Mach (1838-1916) with the so-called schlieren method. The projectile in Mach's beautiful theory is shrunk to a moving point, just as the steam boat before, from which compression waves originate continually. At the time of observation the wave that has originated t seconds ago has now spread over a spherical surface of radius $r = ct$ where c is the sound velocity. In the meantime the projectile has traveled a distance $x = vt$. The spherical shells so produced have an enveloping circular cone, the *Mach cone*. Half its apex angle is called the *Mach angle* and given by

$$\sin \vartheta_0 = \frac{r}{x} = \frac{c}{v}.$$

The Mach angle is approaching zero with increasing v , in contrast to the limiting angle $\vartheta_0 = 19^\circ 18'$ in the case of the ship waves.

The reason for this different behavior is found in the *dispersion*. The sound waves travel at fixed velocity c without dispersion. The deep water

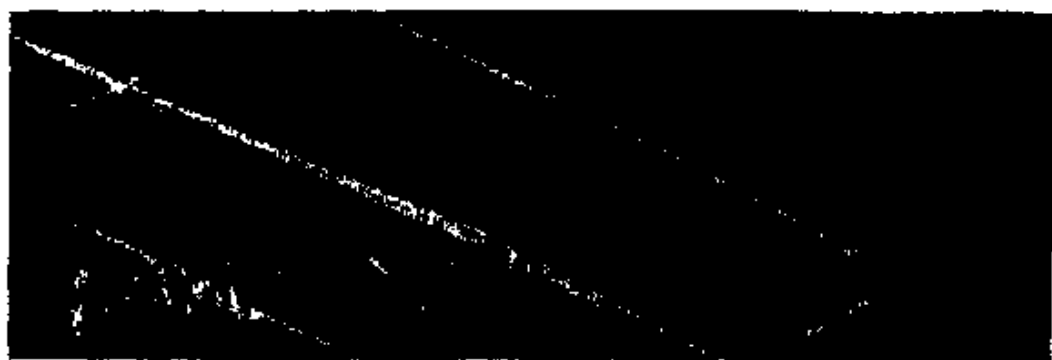


FIG. 45. A projectile moves with supersonic velocity between two parallel walls, the Mach cone is reflected at the walls.



FIG. 45a. The projectile has pierced a thin wooden wall. The wood splinters also produce Mach cones of varying apex angles, according to their velocities.

waves follow the dispersion law $V = \sqrt{g/k}$. With the velocity thus depending on the wave length, there exist waves which at any given speed of the boat run along with the boat, while in the Mach phenomenon all waves are overtaken by the projectile. Thus the fact that ϑ_0 is independent of v becomes understandable.

We could substantiate this interpretation if we treated the *ship waves in shallow water*, in the sense of 24. According to (24.7) there is no dis-

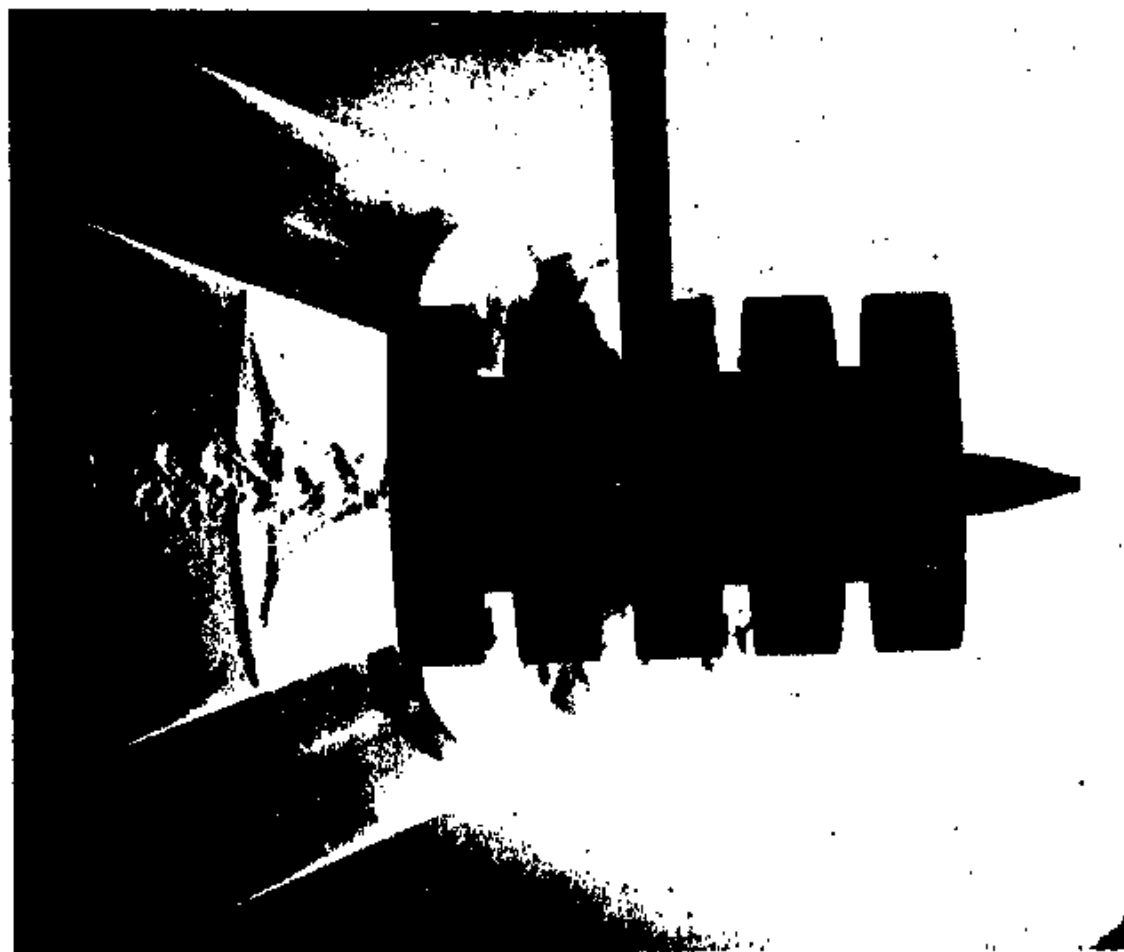


FIG. 45b. The projectile passes a cylinder with circular slots. The spherical waves originating at the openings are tangential to the Mach cone of the projectile.

persion in this case, and Mach's purely geometrical argument may be applied. If $v > V$ the limiting angle should go to zero with increasing v . This problem, however, in which we would have to assume a flat disturbance (a flat-bottomed boat) seems to us rather artificial. However, the classical investigations of annular waves on which the theory of ship waves is based (Euler 1759, Laplace 1776, Lagrange 1787) assumed shallow water.

We close this chapter with some of the less well known pictures of the Mach phenomenon which were taken in the ballistic laboratory of

C. Cranz. Fig. 45 shows a projectile flying between two walls: the Mach cones suffer regular reflection at the walls, also, on this and the other photographs a tail of eddies is shed from the rear end of the projectile. Fig. 45a shows a shot through a thin wooden wall which is located at the left border of the picture: the projectile itself which has somewhat deviated from its path after piercing the wall and the numerous splinters flying along show their Mach cones. Note that the splinters possess Mach cones with larger apex angles than the projectile in accordance with their smaller velocities which, however, are still larger than the sound velocity c . Fig. 45b is very instructive: the projectile was shot through a cylinder furnished with circular slots and is just emerging from the other end. Spherical waves come from the slots whose envelopes fit the Mach cone of the projectile accurately. Here we see a direct experimental realization of the mental picture we have of the formation of the Mach cone as the envelop of individual spherical waves. At the same time the photograph is a beautiful visualization of *Huygen's principle*, which likewise operates with envelopes of spherical waves.

FLOW WITH GIVEN BOUNDARIES

29. Flow Past a Plate

In this and the following articles we restrict ourselves to two-dimensional flow and may, therefore, make use of the powerful tool of the theory of complex functions.

We take up again our discussion of elliptical coordinates ξ, η in 19. The connection with Cartesian coordinates x, y is given by Eq. (19.15)

$$(1) \quad x + iy = c \cosh (\xi + i\eta).$$

Here c is half the focal distance (cf. Fig. 27) and, at the same time, half the length of the projection of our plate in the x, y -plane; the plate is supposed to be infinite in $\pm z$ -direction.

In elliptic coordinates the front and back sides of the plate are simultaneously given by

$$\xi = 0, \quad -\pi < \eta < +\pi,$$

corresponding to the infinitely narrow ellipse of Fig. 27. For these values the right member of Eq. (1) is real, hence $y = 0$ and $-c < x < c$.

On the other hand, for

$$\eta = \pm \frac{\pi}{2}, \quad 0 < \xi < \infty$$

the right member of (1) becomes a positive or negative purely imaginary number, hence $x = 0$ and $y \gtrless 0$, as can also be seen from Fig. 27.

Consider now the analytic function

$$(2) \quad \Phi + i\Psi = \text{const} \sinh (\xi + i\eta)$$

already anticipated in (19.20). Let the constant in (2) be equal to iqc , where q is a real quantity having the dimension of a velocity:

$$(2a) \quad \text{const} = iqc; \quad q, c > 0.$$

Now, $\Phi + i\Psi$ is not only an analytic function of $\xi + i\eta$, but through (1) an analytic function of $x + iy$ likewise. Consequently Φ and Ψ may be interpreted as *velocity potential* and *streamfunction* (19) provided they satisfy such boundary conditions in the x, y -plane as are required by the

problem under consideration. The values of Φ and Ψ along certain lines of the x, y -plane have been listed in the following table, the auxiliary variables ξ and η playing the role of parameters:

ξ, η -plane	x, y -plane	values of Φ, Ψ
$\xi = 0$	$x = c \cos \eta, \quad y = 0$	$\Psi = 0, \quad \Phi = -qc \sin \eta$
$\eta = -\frac{\pi}{2}$	$x = 0, \quad y = -c \sinh \xi < 0$	$\Psi = 0, \quad \Phi = qc \cosh \xi > 0$
$\eta = +\frac{\pi}{2}$	$x = 0, \quad y = +c \sinh \xi > 0$	$\Psi = 0, \quad \Phi = -qc \cosh \xi < 0$
$\xi \rightarrow \infty$	$x^2 + y^2 = \frac{c^2}{4} e^{2\xi} \rightarrow \infty$	$\Psi = +qx, \quad \Phi = -qy$

The first three lines of this table are explained in our previous remarks, if one takes into account that the right member of (2) becomes real for $\xi = 0$ as well as for $\eta = \pm\pi/2$ because of (2a). As regards the last line, the effect of the limit $\xi \rightarrow \infty$ on x, y , etc. can be easily seen if one separates real and imaginary parts in (1) and (2). One obtains for large ξ

$$(3) \quad \begin{aligned} x &= \frac{c}{2} e^{\xi} \cos \eta, & y &= \frac{c}{2} e^{\xi} \sin \eta, \\ \Phi &= -\frac{qc}{2} e^{\xi} \sin \eta, & \Psi &= \frac{qc}{2} e^{\xi} \cos \eta, \end{aligned}$$

in agreement with the last line of the table.

According to the Φ and Ψ values in the last line, the flow field in the neighborhood of the point infinity of the x, y -plane is uniform, viz.

$$(3a) \quad v_y = -\frac{\partial \Phi}{\partial y} = q, \quad v_x = -\frac{\partial \Phi}{\partial x} = 0.$$

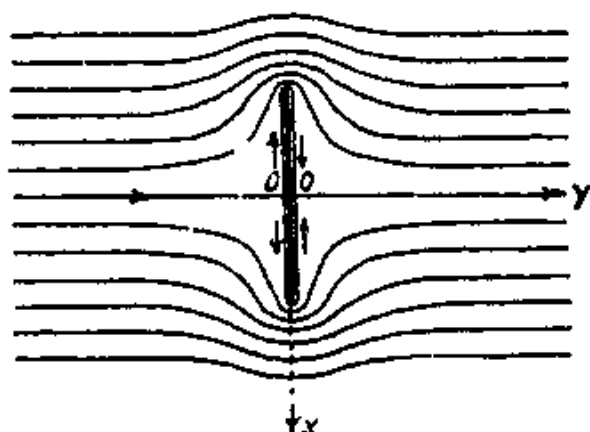
The first three lines of the table contribute the following additional information about the flow: the streamline $\Psi = 0$ coincides with the negative y -axis (line 2) and with the positive y -axis (line 3). According to line 1, a singularity appears at $y = 0$.

In its total extension, $\Psi = 0$ may be described as follows: the branch $\eta = -\pi/2$ corresponding to the negative y -axis splits into two branches at $y = 0$ (center of the front side of the plate). They follow the surface of the plate and are obtained according to the first line, when η passes from $-\pi/2$ to $\pi/2$ either by adding or subtracting π ($-3\pi/2$ being equivalent to $\pi/2$). The two branches join at the center of the back side and, from

there on, $\Psi = 0$ coincides with the positive y -axis.¹ This streamline together with the known flow pattern at infinity forms a frame for the family of streamlines $\Psi = \text{const}$, that makes the diagram of Fig. 46 acceptable.

The point or, rather, the points O on the front and back side of the plate are of particular interest. They are called *points of bifurcation* with

FIG. 46. Plate at right angle to flow. The theoretical flow pattern downstream is a mirror image of the upstream pattern.



reference to the division of the streamline $\Psi = 0$, or *branch points* with reference to the mapping function. Physically speaking, they are *stagnation points* since the velocity of flow vanishes there ($v_x = v_y = 0$). According to Bernoulli's theorem they are points of *maximum pressure*: From (11.9) we have for steady flow and in the absence of external forces ($U = \text{const}$)

$$(4) \quad p = \text{const} - \frac{\rho}{2} v^2.$$

If we put the pressure at infinity equal to zero, so that p is the "hydrodynamic" pressure due to the acceleration of the fluid in the presence of the plate, then $\text{const} = \rho/2 q^2$ [q is the flow velocity at infinity according to (3a)] and

$$(4a) \quad p = \frac{\rho}{2} (q^2 - v^2).$$

This shows immediately that p_{max} is larger than any other possible p -value.

The pressure distribution along the plate is found from our formulas as follows: From

$$\Phi = -qc \sin \eta, \quad x = c \cos \eta,$$

¹If the ξ, η -strip in Fig. 27a which is in one-to-one correspondence with the x, y -plane is shifted downward by $\pi/2$, the streamline $\Psi = 0$ is represented by an H -shaped figure, the upper and lower horizontal of which correspond to the same ray $y > 0, x = 0$.

we find
$$v = - \frac{\partial \Phi}{\partial \eta} \frac{d\eta}{dx} = q \frac{\cos \eta}{\sin \eta},$$

and by (4a)
$$p = \frac{\rho}{2} q^2 \left(1 - \frac{\cos^2 \eta}{\sin^2 \eta} \right).$$

Here the front side is given by $-\pi < \eta < 0$ and the back side by $0 < \eta < \pi/2$ and $-3\pi/2 < \eta < -\pi$, or simply, $0 < \eta < \pi$. The pressure becomes then

$$p = -\infty, \quad \text{for } \eta = \begin{cases} 0, & x = +c \\ \pm\pi, & x = -c \end{cases} \text{ at the edges of the plate}$$

$$p = 0, \quad \text{for } \eta = \begin{cases} \pm \frac{\pi}{4}, & x = +\frac{c}{\sqrt{2}} \\ \pm \frac{3\pi}{4}, & x = -\frac{c}{\sqrt{2}} \end{cases} \text{ at a distance } 0.29c \text{ from} \\ \text{either edge toward the center}$$

and, as we have found before $p = p_{\max} = \rho q^2/2$ for $\eta = \pm\pi/2$, $x = 0$, center of plate.

This pressure distribution is impossible in a perfect fluid because of the occurrence of a negative pressure (which at the edges becomes even infinite); the fluid would separate under a negative pressure.²

A second consequence which is no less paradoxical, is this: The resultant pressure force exerted by the stream upon the plate is zero, (the resultant moment vanishes, too) since the pressure at any surface element on the front side is canceled by the corresponding pressure on the back side. In other words, *if the fluid is transformed to rest at infinity* (by superposition of the constant velocity field $\Phi = qy$), *so that the plate moves through the fluid with uniform velocity* q no work would have to be expended in such a process. Once the plate together with the surrounding fluid has been set in motion, it will keep on moving with constant velocity.

This paradoxical assertion in a generalized form, in which it applies to rigid bodies of arbitrary shape, is attributed to d'Alembert or Euler.³

²Here the emphasis lies on the fact that the pressure becomes *arbitrarily* strongly negative; if it were only negative, but would stay bounded, we could make good for that by choosing the pressure at ∞ sufficiently positive.

³In the introduction to his book, *Hydrodynamik*, Leipzig, Akademische Verlagsgesellschaft 1927, C. W. Oseen quotes Spinoza as an even earlier authority. One could say, in a sense, that Oseen's book has as its aim the analysis of the paradox. For instructive detail observations in two dimension see G. Hamel, *Z. angew. Math. u. Mech.* 15 (1935) 52

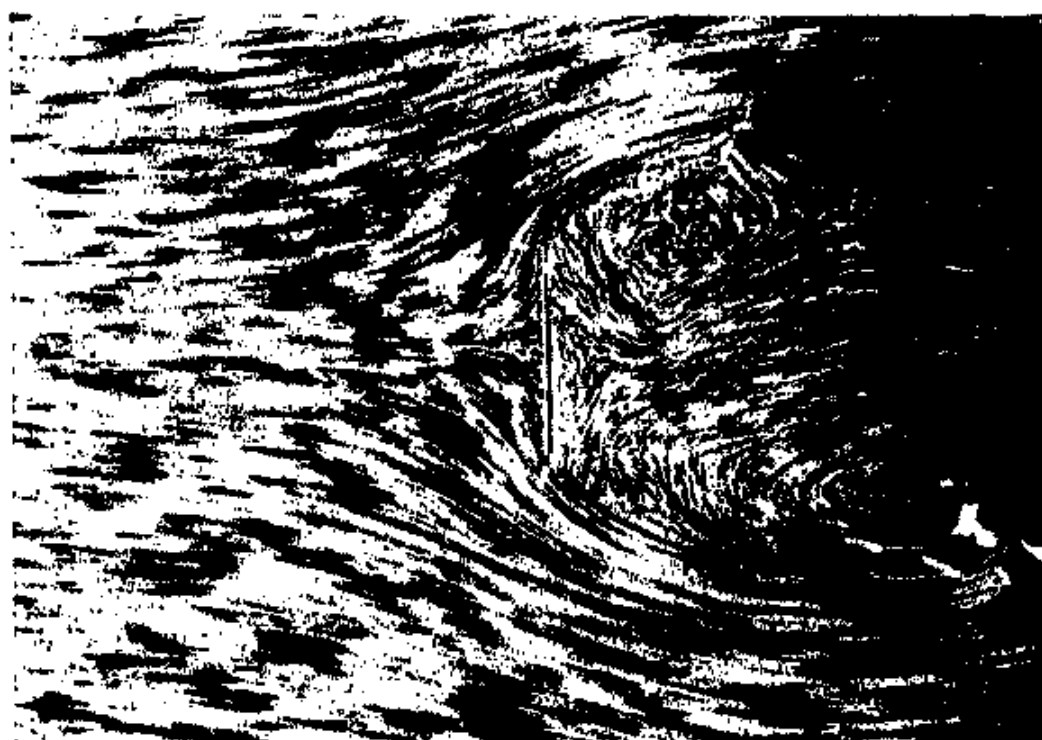


FIG. 47a. The vertical plate in real flow. Stagnation point at the upstream center of the plate.



FIG. 47b. The oblique plate in real flow. The stagnation point is shifted toward the upstream edge of the plate.

Kirchhoff, Lord Kelvin and, later, Minkowski* investigated the general conditions for a steady motion like the one in our example from the point of view of abstract hydrodynamics. It was found that the direction of the momentum would have to coincide with the axis of the moment of momentum to make the motion possible. In the case under consideration, however, the occurrence of arbitrary large negative pressures makes the motion physically impossible.

Let us now compare Fig. 46 with the photograph of a real flow past a plate in Fig. 47a.⁴ The two Figs. 46 and 47a are in very good agreement upstream of the plate but have nothing in common downstream. In the picture of the real flow eddies occur behind the plate that are generated in the process of passing the edges of the plate and move downstream with the fluid. We shall investigate this phenomenon in 30 and 32.

For the present, however, we wish to maintain our theoretical viewpoint incomplete though it is, and investigate the flow past an *oblique plate*. We rotate the x, y -system with respect to the ξ, η -system by the angle $\gamma < \pi/2$ and change Eqs. (1) and (2) into

$$(5) \quad x + iy = ce^{i\gamma} \cosh (\xi + i\eta),$$

$$(6) \quad \Phi + i\Psi = iqc \sinh (\xi + i\eta + i\gamma).$$

The table of p. 208 is now replaced by:

ξ, η -plane	x, y -plane	values of Φ, Ψ
$\xi = 0$	$\begin{cases} x = c \cos \eta \cos \gamma \\ y = c \cos \eta \sin \gamma \end{cases}$	$\Psi = 0, \quad \Phi = -qc \sin (\eta + \gamma)$
$\eta = \mp \frac{\pi}{2}$	$\begin{cases} x = \pm c \sinh \xi \sin \gamma \\ y = \mp c \sinh \xi \cos \gamma \end{cases}$	$\begin{cases} \Psi = \pm qc \sinh \xi \sin \gamma \\ \Phi = \pm qc \cosh \xi \cos \gamma \end{cases}$
$\xi \rightarrow \infty$	$x^2 + y^2 = \frac{c^2}{4} e^{2\xi} \rightarrow \infty$	$\Psi = qx, \quad \Phi = -qy$

*Preuss. Akad. Ber., 1888, 1095.

⁴Figs. 39a, b and 47a, b are reproductions of photographs taken by the firm of Ahlborn in Hamburg.

According to the table, the front and back sides of the plate represented by $\xi = 0$, $-\pi < \eta < 0$, and $0 < \eta < \pi$ are still the streamline $\Psi = 0$. In the neighborhood of the point infinity of the x, y -plane, that is for $\xi \rightarrow \infty$, we have a uniform field of flow with constant velocity q as before; the streamlines in this region are the lines $x = \text{const}$, but the lines $\eta = \mp\pi/2$ are no longer streamlines. Geometrically, they represent a pair of rays that coincide with the two halves of the bisectrix of the plate; the center column of our table gives for these lines the equation $y = -x \cot \gamma$. Altogether, there are no rectilinear streamlines in the finite domain.

In order to analyze the flow pattern we first determine the *stagnation points*. They are the branch points of the function that maps the $\Phi + i\Psi$ -plane on the $x + iy$ -plane. Now, the general condition for a branch point is:

$$(7) \quad \frac{d(\Phi + i\Psi)}{d(x + iy)} = 0.$$

The differential quotient is calculated from (5) and (6) as follows:

$$(7a) \quad \frac{d(\Phi + i\Psi)}{d(\xi + i\eta)} \bigg/ \frac{d(x + iy)}{d(\xi + i\eta)} = \frac{iq \cosh(\xi + i\eta + i\gamma)}{e^{i\gamma} \sinh(\xi + i\eta)}.$$

Condition (7) leads therefore to

$$(7b) \quad \cosh(\xi + i\eta + i\gamma) = 0,$$

or, on separating into real and imaginary parts, to

$$(7c) \quad \cosh \xi \cos(\eta + \gamma) = 0, \quad \sinh \xi \sin(\eta + \gamma) = 0.$$

Since $\cosh \xi$ is never zero, and $\sinh \xi$ vanishes only for $\xi = 0$, the solution of (7c) is

$$(8) \quad \eta = \pm \frac{\pi}{2} - \gamma, \quad \xi = 0.$$

Thus there are again two stagnation points one of which is in the front, the other in the back of the plate:

$$(8a) \quad \xi = 0, \quad \eta = -\frac{\pi}{2} - \gamma \quad \text{and} \quad \xi = 0, \quad \eta = +\frac{\pi}{2} - \gamma.$$

They are not in the center of the plate, but shifted up and down by the same amount $|x + iy| = c \sin \gamma$, as follows from (5). In Fig. 48 they have been denoted by O_1 and O_2 . At these points we have again $v_x = v_y = 0$ and the pressure is, according to Bernoulli's equation (4a), a maximum: $p_{\text{stagn}} = p_{\text{max}} = \rho q^2/2$. From our derivation it follows that the stagnation points are points of bifurcation as before. The streamline that

meets the plate at O_1 and divides there into two branches is normal to the plate, and so is the stream line issuing from O_2 .

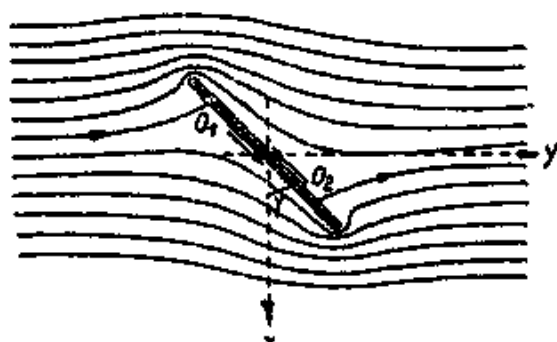


FIG. 48. Flow past an oblique plate. The theoretical flow pattern downstream is obtained by rotating the upstream pattern about the center of the plate by 180° .

The differential quotient computed in (7a) yields quite generally the velocity of flow in the complex form

$$(9) \quad \frac{d(\Phi + i\Psi)}{d(x + iy)} = -v_x + iv_y.$$

This is a direct consequence of the form of $f'(z)$ in (19.4a) together with the Cauchy-Riemann equations (19.5). If we take the absolute value in (9) and substitute (7a), we obtain

$$(10) \quad |v| = q \left| \frac{\cosh(\xi + i\eta + i\gamma)}{\sinh(\xi + i\eta)} \right|.$$

In particular for $\xi = 0$, that is, on the plate itself, we have on either side

$$(10a) \quad |v| = q \left| \frac{\cos(\eta + \gamma)}{\sin \eta} \right|.$$

The pressure distribution is found from (10a) by substitution in Bernoulli's equation (4a):

$$(11) \quad p = \frac{\rho}{2} q^2 \left(1 - \frac{\cos^2(\eta + \gamma)}{\sin^2 \eta} \right).$$

This leads again to

$$p = -\infty \quad \text{for} \quad \eta = \begin{cases} 0 \\ \pm\pi \end{cases},$$

that is at the edges of the plate, and the point where the pressure vanishes is found from $p = 0$ as

$$(12) \quad \cos^2(\eta + \gamma) = \sin^2 \eta$$

which can be written $\cos(\eta + \gamma) = \pm \cos(\pi/2 - \eta)$, and therefore

$$\text{either} \quad \eta + \gamma = \frac{\pi}{2} - \eta, \quad \eta = \eta_1 = \frac{\pi}{4} - \frac{\gamma}{2},$$

$$\text{or} \quad \eta + \gamma = -\frac{\pi}{2} - \eta, \quad \eta = \eta_2 = -\frac{\pi}{4} - \frac{\gamma}{2}.$$

The value η_1 is positive, the corresponding point lies therefore on the backside; the converse is true for η_2 . The zone of negative pressure is on the back side between $\eta = 0$ and $\eta = \eta_1$, on the front side between $\eta = -\pi$ and $\eta = \eta_2$. Since arbitrarily large negative pressures are not permissible, our solution in the case of the oblique plate has as little physical reality as in the special case of a normal plate.

The comparison of Fig. 48 with the photograph of a real flow in 47b illustrates what we mean. Ahead of the plate the two stream patterns are rather similar, in particular, there is a stagnation point in front which has the expected position. The downstream flow, however, is entirely different. Instead of an antisymmetric repetition of the upstream pattern we meet again the characteristic eddies in alternating positions.

In this article we have only dealt with the *steady* flow past a plate. A plate problem occurs also in acoustics, when a light circular disk is suspended in the *non-steady* field of a sound beam. This so-called Rayleigh disk plays a significant part in the measurement of sound intensities; the analytic methods discussed so far and developed in the following are of importance for the theory of this instrument, as Lord Rayleigh⁵ has shown; for this reason our statement about the unrealistic character of the obtained solutions should be restricted to the case of steady flow.

30. The Problem of the Wake; Surfaces of Discontinuity

"There is nothing in the nature of a liquid that should prevent two adjacent fluid layers from sliding past each other with finite relative velocity, provided we consider the fluidity as perfect, that is, exclude all friction. Our assertion is certainly true for those properties of the liquid which are considered in the hydrodynamic equations, viz. the conservation of mass in each volume element and the equality of the pressure in all directions; these two factors do not preclude the possibility of a finite difference of the tangential velocities on either side of a surface in the interior of the fluid. The components of the velocity normal to the surface and the pressure, however, must be equal on both sides of such a surface".⁶

⁵Lamb, *Hydrodynamics*, Cambridge, 5th ed., Chap. IV, Art. 77.

⁶Helmholtz, *Preuss. Akad. Ber.*, 1868.

Helmholtz further points out that a sharp edged obstacle always produces discontinuity surfaces in the flow and that in this way the negative pressures that would otherwise be present are avoided. He also observes that the discontinuity surface which originally is conceived as a geometrical surface curls up under the influence of friction and takes the form of a sequence of eddies (Helmholtz speaks of vortex filaments). All this will be discussed in 32; for the present we maintain the assumption of a perfect inviscid fluid.

With this in mind we again take up the two-dimensional problem of the plate, assuming the plate normal to the flow for the sake of simplicity, as at the beginning of 29. Two lines of discontinuity—if we now use two-dimensional terminology—issue from the end points of the segment that represents our plate. They are symmetric to the y -axis, but of unknown shape (cf. Fig. 49, where the edges of the plate are denoted by A and B).

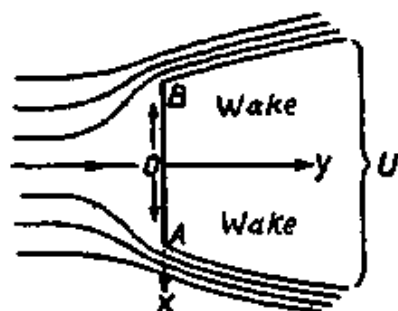


FIG. 49. Helmholtz flow past a vertical plate AB . The wake is delimited by the discontinuity lines A' and B' that originate at the edges of the plate.

The wake is enclosed between the two discontinuity lines and the plate, and we imagine the state of the wake to be dead water in the literal sense: its velocity is assumed as zero, and the pressure is everywhere constant, say, equal to p_0 . According to Helmholtz, we then have also for the fluid that flows along the discontinuity lines the following boundary conditions for the pressure and the normal component of the flow:

$$(1) \quad p = p_0, \quad v_n = 0.$$

The tangential component v_t follows now from Bernoulli's equation (29.4a) which yields because of (1)

$$(2) \quad |\mathbf{v}|^2 = v_t^2 = q^2 - \frac{2}{\rho} p_0.$$

The tangential component is thus constant along the lines of discontinuity. Infinite velocities and negative pressures no longer occur.

Kirchhoff⁷ worked out Helmholtz's idea in detail and applied it among other things to the plate problem. He considers in addition to the two complex variables

⁷G. Kirchhoff, Crelles Journ. 70 (1860) and Vorlesungen über Mechanik, Chap. XXII.

$$z = x + iy \quad \text{and} \quad f = \Phi + i\Psi$$

a third variable

$$(3) \quad \zeta = \frac{1}{-v_x + iv_y}$$

connected with z and f by the relation

$$(4) \quad \zeta = \frac{dz}{df}$$

as in (29.9), and introduces the assumption that *the wake reaches to infinity* (we have tried to indicate that in Fig. 49). The pressure p_0 then equals the pressure at infinity which we defined as zero in (29.4a), so that now p_0 in Eq. (2) must also be put equal to zero. From (2) we obtain

$$(5) \quad |-v_x + iv_y| = v_x = q$$

all along the lines of discontinuity. Consequently we have for ζ the boundary condition

$$(5a) \quad |\zeta| = \frac{1}{q}.$$

We wish to emphasize, however, that the assumption of discontinuity lines reaching to infinity is too restricted. Indications for this will be found at the end of this article.

Let us now look for the image of the discontinuity line, which separates the moving fluid from the wake in the plane of the complex variable ζ (cf. Fig. 50).



FIG. 50. Mapping of the z -plane on the ζ -plane.

Starting with the point infinity of the z -plane, we have there

$$v_y = q, \quad v_x = 0, \quad \text{and by (3)} \quad \zeta = -\frac{i}{q}.$$

The image point U in the ζ -plane is therefore on the negative imaginary axis. The images of the two discontinuity lines must pass through the point U , in agreement with (5a). We draw now a semi-circle of radius $1/q$ and center zero; it passes through U and terminates at the real axis in the points $\zeta = \pm 1/q$, which are the images of the points A and B in Fig. 49 (edges of the plate). All along the plate one has $v_y = 0$ and therefore ζ real; this holds also at the edges.

We finally consider the stagnation point O in Fig. 49. There

$$v_x = v_y = 0, \quad \text{and by (3)} \quad \zeta = \infty.$$

The image of O is the point ∞ of the ζ -plane. This determines completely the mapping of the two halves of the plate: Since ζ is real along the plate, as was just pointed out, one half of the plate maps on the segment OA , the other on the segment OB of the real axis in Fig. 50.

In describing the circuit $UAOB$ in Fig. 49, the region of the "live" flow⁸ remains to the left, and the region of the dead water to the right. Accordingly, the marked region in Fig. 50 to the left of the image circuit corresponds to the region of flow, the unmarked region to the wake. Yet that does not determine the analytic solution of our mapping problem since the shape of the lines of discontinuity is still unknown.

This part of the problem calls for a discussion of the mapping of the z -plane on the f -plane, the characteristic points of which are listed in the following table. As before we assign the equation $\Psi = 0$ to the "symmetric" stream line which separates at the stagnation point, whereupon the entire boundary of the dead water becomes real in the f -plane. The zero level of the velocity potential Φ is now assigned to the stagnation point O ; this makes Φ negative along OA and along OB . We denote the unknown value of Φ at A and B by f_0 . (Note that f_0 is at the same time the value of f at A and B since $\Psi = 0$; Φ must decrease from O to A or from O to B , hence is negative along OA and OB .) f_0 is negative and given by

$$(6) \quad f_0 = - \int_0^{+\infty} v_x dx = - \int_0^{-\infty} v_x dx.$$

We have thus found a correspondence between the total boundary of the wake and the negative real axis of the f -plane, but it is not a one-to-one



FIG. 51. Mapping of the z -plane on the f -plane.

correspondence, since, for instance, the plate front AOB is mapped on the double segment $f_0 - O - f_0$, cf. Fig. 51. This can be helped by making a cut along the negative real axis. The two borders of the cut form the image of the wake boundary, while the positive real axis has already been

⁸If we had used the reciprocal variable $\zeta' = 1/\zeta$, instead of ζ , the region of flow would map on the *interior* of the semi-circle and the circuit $UAOB$ on the closed boundary of the semi-circle. We have retained Kirchhoff's original definition of ζ in (3) to make a comparison with the original paper easier, although the map of the flow in the plane ζ' , being simply the *hodograph* of the flow, has a stronger physical appeal.

in one-to-one correspondence with the stream line $\Psi = 0$. The marking of Fig. 51 indicates that the cut f -plane is the image of the flow field in Fig. 49. The wake is mapped on a second sheet of the f -plane connected with the sheet of Fig. 51 across the branch cut (the two sheets form one Riemann surface).

	z	ζ	f	angular change
A	$+c$	$-\frac{1}{q}$	$f_0 < 0$	$\frac{1}{2}$
B	$-c$	$+\frac{1}{q}$	$f_0 < 0$	$\frac{1}{2}$
O	0	$\pm \infty$	0	$\frac{1}{2}$
U	∞	$-\frac{i}{q}$	∞	

Let us now try to find the analytical form of the mapping of the ζ -plane on the f -plane, omitting the z -plane and comparing the Figs. 50 and 51. (It will be noticed that this is feasible without knowing the discontinuity lines, since the plane of the physical flow matters no longer.) If the mapping were everywhere conformal, it would be given by a bilinear relation between ζ and f . There are, however, points where the conformality breaks down, though not in the interior of the marked regions of the two figures, but on the boundary; they have been indicated in the last column of our table. The number $\frac{1}{2}$ going with A and B means that the angle π occurring at the boundary points A and B in the f -plane appears as the angle $\pi/2$ in the ζ -plane; the number $\frac{1}{2}$ going with O should likewise tell us that the angle 2π at O in the f -plane appears in the ζ -plane as π . From this we infer that the relation between ζ and f must contain

$$f \text{ in the form } \sqrt{f}, \quad \zeta \text{ in the form } \left(\zeta \pm \frac{1}{q}\right)^2.$$

We therefore consider the following bilinear relation of these quantities

$$\frac{\left(\zeta + \frac{1}{q}\right)^2}{\left(\zeta - \frac{1}{q}\right)^2} = \frac{\sqrt{f_0} - \sqrt{f}}{\sqrt{f_0} + \sqrt{f}}$$

or, written more conveniently,

$$(7) \quad \frac{(q\zeta + 1)^2}{(q\zeta - 1)^2} = \frac{1 - \sqrt{\frac{f}{f_0}}}{1 + \sqrt{\frac{f}{f_0}}}.$$

This function fulfills the requirements of the table as can be shown by expanding (7) in power series at the points A , B , O . We shall come back to this in the next article in a more systematic way.

Let us now compute ζ explicitly. Eq. (7) can be rearranged in the form

$$(7a) \quad (q\zeta)^2 + 2\sqrt{\frac{f_0}{f}} q\zeta + 1 = 0;$$

the roots of this equation are

$$(8) \quad \zeta = -\frac{1}{q} \left(\sqrt{\frac{f_0}{f}} \pm \sqrt{\frac{f_0}{f} - 1} \right).$$

The salient point in this analysis is now that it is possible to eliminate the auxiliary variable ζ by (4) and to obtain a *differential equation between the two original variables z and f* , viz.

$$(9) \quad \frac{dz}{df} = -\frac{1}{q} \left(\sqrt{\frac{f_0}{f}} \pm \sqrt{\frac{f_0}{f} - 1} \right).$$

Its integration determines the mapping function.

It is convenient for this purpose to introduce the parameter α by

$$(10) \quad \begin{aligned} \sqrt{\frac{f}{f_0}} &= \sin \alpha, & df &= 2f_0 \sin \alpha \cos \alpha d\alpha. \\ \sqrt{\frac{f_0}{f}} &= \frac{1}{\sin \alpha}, & \sqrt{\frac{f_0}{f} - 1} &= \frac{\cos \alpha}{\sin \alpha}. \end{aligned}$$

Choosing the upper sign in (9), we obtain

$$dz = -\frac{2f_0}{q} (\cos \alpha + \cos^2 \alpha) d\alpha$$

which gives integrated

$$(11) \quad z = -\frac{2f_0}{q} \left\{ \sin \alpha + \frac{1}{2} (\sin \alpha \cos \alpha + \alpha) \right\}.$$

In this integration the constant must be set equal to zero, since $f = 0$ for $\alpha = 0$ according to (10), and z is supposed to be zero if $f = 0$ according to our table.

For $\alpha = \pi/2$, f equals f_0 by (10); according to our table this characterizes the edge of the plate A , or $z = c$. Thus we have

$$(12) \quad c = -\frac{2f_0}{q} \left(1 + \frac{\pi}{4}\right), \quad \frac{f_0}{q} = \frac{-2c}{\pi + 4}$$

which determines the hitherto unknown quantity f_0 . We substitute this value in (11) and obtain by separation of real and imaginary parts the following parametric representation of the plate:

$$(13) \quad \begin{aligned} x &= \frac{4c}{\pi + 4} \left\{ \sin \alpha + \frac{1}{2} (\sin \alpha \cos \alpha + \alpha) \right\}, \\ y &= 0, \end{aligned}$$

traced by the point x, y when α varies according to

$$(13a) \quad -\frac{\pi}{2} < \alpha < +\frac{\pi}{2}.$$

Now we allow α to take complex values, starting from $\pi/2$:

$$(13b) \quad \alpha = \frac{\pi}{2} - i\beta, \quad 0 < \beta < \infty.$$

Then one has

$$(13c) \quad \sin \alpha = \cosh \beta, \quad \cos \alpha = +i \sinh \beta.$$

If α changes according to (13b), f remains real by (10) and (12), hence (13b) represents the continuation of the stream line $\Psi = 0$ beyond the end point A , that is to say, the lower branch of the discontinuity line in Fig. 49. Its parametric representation is obtained from (11) by separation of the real and imaginary part in the following form:

$$(14) \quad \begin{aligned} x &= \frac{4c}{\pi + 4} \left(\cosh \beta + \frac{\pi}{4} \right), \\ y &= \frac{2c}{\pi + 4} (\cosh \beta \sinh \beta - \beta). \end{aligned}$$

This is a transcendental curve; its numerical determination makes no difficulty. For large β -values it can be approximated by a parabola.

Eqs. (10), (11) and (12) contain the general solution of our flow problem. When α takes arbitrary complex values we obtain

$$(15) \quad f = \Phi + i\Psi = -\frac{2qc}{\pi + 4} \sin^2 \alpha,$$

$$(16) \quad z = x + iy = \frac{4c}{\pi + 4} \left\{ \sin \alpha + \frac{1}{2} (\sin \alpha \cos \alpha + \alpha) \right\}.$$

An explicit representation of Φ and Ψ in terms of x and y such as would be obtained by elimination of the parameter α is neither feasible nor necessary.

To check our results we may compare the value of f_0 in (12) with its original definition in (6). According to (6) we ought to have

$$(17) \quad f_0 = - \int_0^{\epsilon} v_x dx = \int_0^{\epsilon} \frac{\partial \Phi}{\partial x} dx = \Phi_{\epsilon} - \Phi_0.$$

By (15) we have the Φ -values

$$\begin{aligned} \Phi_{\epsilon} &= - \frac{2c\eta}{\pi + 4} & \text{for } \alpha &= \frac{\pi}{2}, \\ \Phi_0 &= 0 & \text{for } \alpha &= 0. \end{aligned}$$

Thus (17) and (12) are in agreement.

Contrary to 29, the results of which led us to the d'Alembert-Euler paradox, our present theory gives a reasonable value for the *total pressure* P acting on the plate per unit of length under the influence of the streaming fluid.

From Bernoulli's theorem (29.4a) we have

$$(18) \quad P = \int_{-\epsilon}^{+\epsilon} p dx = \frac{\rho}{2} q^2 \int_{-\epsilon}^{+\epsilon} \left(1 - \frac{v^2}{q^2} \right) dx.$$

Along the front of the plate v equals v_x , hence $v^2 = 1/\zeta^2$ by (3), and (18) takes the form

$$(19) \quad P = \frac{\rho}{2} q^2 \int_{-\epsilon}^{+\epsilon} \left(1 - \frac{1}{q^2 \zeta^2} \right) dx.$$

This integral can be evaluated in the following elegant manner: along the plate, dx equals $d\zeta$; thus we obtain from (4)

$$(19a) \quad dx = \zeta df, \quad P = \frac{\rho}{2} q \int \left(q\zeta - \frac{1}{q\zeta} \right) df.$$

From (8), where we choose again the upper sign, we have

$$- q\zeta = \sqrt{\frac{f_0}{f}} + \sqrt{\frac{f_0}{f} - 1}.$$

The reciprocal value of this expression may be written

$$- \frac{1}{q\zeta} = \sqrt{\frac{f}{f_0}} - \sqrt{\frac{f}{f_0} - 1},$$

so that

$$q\xi - \frac{1}{q\xi} = -2\sqrt{\frac{f_0}{f}} - 1,$$

a result that could have been read from the quadratic equation (7a). Upon introducing the last expression in (19a), we obtain

$$(20) \quad P = -2\rho q \int_0^{f_0} \sqrt{\frac{f_0}{f}} - 1 \, df.$$

As regards the limits of the integral and the factor 2, note that the pressure distribution is symmetric with respect to the center, and therefore the contributions of the upper and the lower half equal.

If we now introduce the parameter α from (10) we obtain

$$(21) \quad P = -4\rho f_0 q \int_0^{\pi/2} \cos^2 \alpha \, d\alpha = -\rho f_0 q \pi$$

and, on substitution of f_0 from (12),

$$(22) \quad \frac{P}{2c} = \frac{\pi}{\pi + 4} \rho q^2 = 0.88 \frac{\rho}{2} q^2.$$

The pressure per unit of surface is thus proportional to the square of the flow velocity at infinity in agreement with the general hydraulic experience (cf., e.g., p. 80); we could formulate our result also in this way: the pressure is proportional to the kinetic energy per unit of volume of the undisturbed flow, or, in other words, proportional to the stagnation pressure at 0. The numerical coefficient, however, does not agree too well with the results of careful measurements. In the case of the infinitely long plate the actual coefficient is about 2.0 according to Prandtl (*Strömungslehre*, p. 165)⁹ instead of our value 0.88. The reason is—also according to Prandtl¹⁰—that the real wake does not reach to infinity and the two separate regions of the flow join at some distance behind the plate. Besides, the pressure downstream of the plate is smaller than the pressure of the undisturbed flow because of the eddies present in the wake. This leads to a “suction” on the back side that adds to the pressure on the front side.

The last remarks throw a good deal of doubt on the basic assumption of Kirchhoff's wake theory which has been widely accepted in literature. An essential feature of the theory is the assumption $p_0 = 0$ introduced in connection with Eq. (2). It is identical with the hypothesis of an infinitely extended wake as explained on p. 217. Attempts at a general treatment

⁹Cf. also Hunsaker and Rightmire, *op. cit.*, p. 198.

¹⁰See C. Schmieden, *Z. für Luftfahrtforschung* 17, (1940) 37 and M. Kolscher, *ibid.* p. 154.

of the problem which drop this assumption (see, e.g., footnote p. 223) are too complicated to be presented in this book.

Let us finally have a look at the interesting question of stability; suppose the plate is not kept rigidly fixed, but is allowed to revolve about an axis through its center, what will be its position relative to the flow? If the preferred orientation were parallel to the flow, the total pressure P would vanish. The experiment shows, however, that the plate orients itself normal to the flow.

A complete investigation of this question would require knowledge of the flow pattern and of the discontinuity surfaces also in the case of the oblique plate. We can do without that if we trust the stream pattern in Fig. 48 as far as the qualitative position of the stagnation point on the front side of the plate is concerned which, in Fig. 47b, is confirmed by the experiment. We take it then that the stagnation point travels toward the leading edge of the plate whenever the plate turns from the normal to an oblique position. The pressure maximum then travels in the same direction. When the plate, as in Fig. 48, is turned counterclockwise out of the normal position, the pressure produces a clockwise moment which tends to restore the normal position. *This is therefore the stable position of the plate.* The same argument applied to the *parallel position* shows that *it is unstable.*

A weather vane likewise, would not turn into the wind but normal to it, if its axis were centrally located (actually the rear face is much larger than the front face). In fact, the Rayleigh disk (cf. p. 215) which is centrally suspended places itself normal to the sound beam.

31. The Problem of the Free Jet Solved by Conformal Mapping

It might appear that the mapping function given in (30.7) is the result of a lucky guess rather than of rigorous deduction. These doubts will be dispelled by the systematic derivation of (30.7) that follows here.

Let it be required to map the positive quadrant¹¹ of a t -plane on the upper half of an s -plane; the simplest way to do this is by the relation

$$(1) \quad t = s^{\frac{1}{2}}.$$

The mapping is conformal everywhere in the finite domain with the exception of the origin of the two planes, where the straight angle π formed by the real axis at $s = 0$ is mapped into the angle $\pi/2$ between the positive

¹¹Our quadrant can be considered as a closed polygon, bounded by two straight lines and possessing two right angles at $t = 0$ and $t = \infty$ as one sees by stereographic projection. More generally, any polygon with arbitrary angles can be mapped on the half plane. There exists for this purpose a general integral formula of H. A. Schwarz and E. B. Christoffel, but we do not have to use it here.

real and the positive imaginary axes at $t = 0$. We consider now the ζ -plane connected with the t -plane by the bilinear transformation

$$(2) \quad t = \frac{\zeta + a}{\zeta - a}, \quad \zeta = a \frac{t + 1}{t - 1},$$

where a is real and positive. By (2) the positive-imaginary t -axis is mapped on the semicircular arc $|\zeta| = a$ which passes from $\zeta = -a$ through $\zeta = -ia$ to $\zeta = +a$ (bilinear transformations are known to map straight lines into circles or straight lines), while the positive real axis of the t -plane is mapped into the two parts of the real ζ -axis $a < \zeta < \infty$ and $-\infty < \zeta < -a$.

We now substitute (2) in (1) and obtain the mapping function

$$(3) \quad \frac{(\zeta + a)^2}{(\zeta - a)^2} = s,$$

which maps the upper half plane of the variable s on the region of the ζ -plane marked in Fig. 50.

Let us turn to the relation between s and f : here we wish to map the upper half plane of s on the f -plane cut along the negative real axis as indicated in Fig. 51. The angle π at $s = 0$ must be doubled in the mapping, so that the angle at $f = 0$ becomes 2π . The simple relation $s = \sqrt{f}$ of Eq. (1) would do this. We wish, however, to achieve at the same time the coordination of f -values and ζ -values prescribed by our table on p. 219, and therefore subject \sqrt{f} first to a bilinear transformation which we write

$$(3a) \quad s = \frac{\alpha \sqrt{f} + \beta}{\gamma \sqrt{f} + \delta}.$$

Combination with (3) and replacement of α by $1/q$ gives

$$(4) \quad \frac{(q\zeta + 1)^2}{(q\zeta - 1)^2} = \frac{\alpha \sqrt{f} + \beta}{\gamma \sqrt{f} + \delta}.$$

The coefficients $\alpha, \beta, \gamma, \delta$, determined¹² according to that table are

$q\zeta = -1,$	$f = f_0$	$\alpha \sqrt{f_0} + \beta = 0,$
$q\zeta = +1,$	$f = f_0$	$-\gamma \sqrt{f_0} + \delta = 0,$
$q\zeta = \infty,$	$f = 0$	$\beta = \delta,$
$q\zeta = -i,$	$f = \infty$	$\alpha = -\gamma.$

¹²The negative sign on γ in the second line of the table is obtained from Fig. 51 in the following way: \sqrt{f} changes its sign in a transition from the upper to the lower border of the cut, if this transition is carried out without leaving the hatched sheet in Fig. 51, that is without crossing the cut. The value $\sqrt{f_0}$ in the first line that corresponds to the point A is thus to be replaced by $-\sqrt{f_0}$ in the second line that corresponds to the point B .

Putting $\alpha = 1$, we obtain for the other constants

$$\gamma = -1, \quad \beta = \delta = -\sqrt{f_0},$$

and Eq. (4) assumes the form

$$(4a) \quad \frac{(q_1^2 + 1)^2}{(q_1^2 - 1)^2} = \frac{\sqrt{f} - \sqrt{f_0}}{-\sqrt{f} - \sqrt{f_0}} = \frac{1 - \sqrt{\frac{f}{f_0}}}{1 + \sqrt{\frac{f}{f_0}}}.$$

Thus we have achieved a more cogent derivation of Eq. (30.7) based on the known mapping properties of the bilinear function and the square root.

We now apply the same method to the classical problem connected with Torricelli's name, viz. the outflow from a large reservoir, already treated in 11, p. 87. At that time we took it for granted that the fluid leaves the orifice in the form of a jet. Now we wish to follow Helmholtz who was the first to realize that the problem of jet formation should be made the object of hydrodynamic investigation. In the paper quoted on p. 215 he actually gives the present problem as the only example to illustrate the idea of the discontinuity surface.

We shall treat here the two-dimensional equivalent of the problem of jet formation. This means replacing the orifice in the tank wall by an infinitely long slit of width $2c$ in a plane wall perpendicular to the drawing of Fig. 52. The action of gravity on the emerging jet will be disregarded.

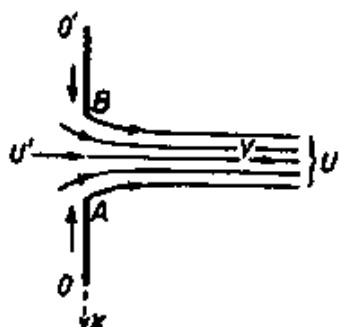


FIG. 52. Two dimensional scheme of jet contraction.

The center line of the jet is taken as y -axis, the wall as x -axis. The two discontinuity lines that issue from the sharp edges of the slit will approach each other since the water enters the orifice also from the sides; the cross-section of the jet decreases therefore after its emergence from the orifice (vena contracta). The principal aim of our investigation is to determine the degree of contraction, that is to say, the ratio of the jet cross-section at large distance to that in the immediate neighborhood of the orifice. The details of the calculation need not be presented since the problem is mathematically identical with that of 30.

Fig. 52 has been drawn in the plane $z = x + iy$. In addition we consider as on p. 217 and p. 218 the planes

$$(5) \quad \zeta = \frac{1}{-v_x + iv_y} \quad \text{and} \quad f = \Phi + i\Psi.$$

Along the two discontinuity lines the pressure is constant; it is equal to the atmospheric pressure which we take as the zero level for p . From Bernoulli's equation (29.4) we then have $v^2 = \text{const}$ along the discontinuity lines, therefore

$$|\mathbf{v}| = \text{const} = q \quad \text{and from (30.3)} \quad |\zeta| = \frac{1}{q}.$$

Here q represents the magnitude of $|\mathbf{v}|$ at the points A and B , or, as we may also put it, the value of the velocity at sufficient distance from the orifice, as it is given by Torricelli's Eq. (11.10).

At the infinitely distant point U of the jet, where the side component v_x evidently vanishes, we obtain from (5)

$$(5a) \quad \zeta = -\frac{i}{q}.$$

Along the inside of the wall where v_x vanishes, ζ becomes real; for, we have from Fig. 52 and Eq. (5)

$$(5b) \quad \begin{aligned} +\infty > \zeta > \frac{1}{q} & \quad \text{at the wall } OA, \\ -\infty < \zeta < -\frac{1}{q} & \quad \text{at the wall } O'B. \end{aligned}$$

The region of flow in the ζ -plane of Fig. 53 has therefore the same shape as in Fig. 50. In addition, Fig. 53 shows the image of the infinitely distant point U' on the upstream prolongation of the center line of the jet. (In approaching the point U' from the orifice we have to assume $v_x = 0$, $v_y \rightarrow 0$, therefore $\text{Im}(\zeta) \rightarrow -\infty$.)

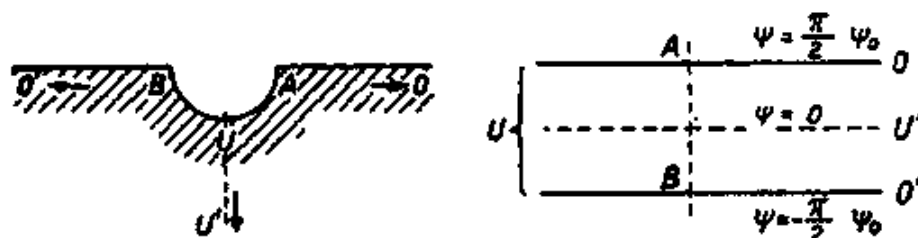


FIG. 53 (left). Mapping of the z -plane on the ζ -plane.

FIG. 54 (right). Mapping of the z -plane on the f -plane.

Fig. 54 shows the flow region of the f -plane. Let the stream line $\Psi = 0$ coincide with the axis of symmetry of the flow UU' (real axis of the f -

plane). Φ increases along this line in the direction from U to U' , since $v = v_x$ has here the direction from U' to U . We assign to the streamlines that run along the walls of the reservoir and the discontinuity lines the value¹³ of the stream function

$$\Psi = \pm \frac{\pi}{2} \Psi_0.$$

The entire region of flow is mapped on the strip of the f -plane bounded by these two Ψ -lines. If we now assign the zero level of the velocity potential Φ to the equipotential line that passes through the edges A and B of the slit, then the images A and B in Fig. 54 are located on the imaginary axis of the f -plane. The "lower" and "upper" tank walls are represented by the segments AO and BO' of the boundaries of the strip, while UA and UB are the images of the boundaries of the jet.

As before we have now to find the analytical relation between ζ and f , and use again a table that incorporates the correlations already indicated in Figs. 52-54.

	z	ζ	f
A	$+c$	$\frac{1}{q}$	$i \frac{\pi}{2} \Psi_0$
B	$-c$	$-\frac{1}{q}$	$-i \frac{\pi}{2} \Psi_0$
O	$+\infty$	$+\infty$	$\infty + \frac{i\pi}{2} \Psi_0$
O'	$-\infty$	$-\infty$	$\infty - \frac{i\pi}{2} \Psi_0$
U	$+i\infty$	$-\frac{i}{q}$	$-\infty$
U'	$-i\infty$	$-i\infty$	$+\infty$

In the previous problem we had to map the cut f -plane in its full extension on a specified part of the ζ -plane which we were able to do by using the square root as a mapping function. In the present case we want to map a mere strip of the f -plane on the same part of the ζ -plane as before. This can be achieved by the exponential function. We use Kirch-

¹³The factor $\pi/2$ of Ψ_0 simplifies some of the following formulas.

hoff's notation e'' where the variable w is proportional to f with a scale factor¹⁴ that depends on the choice of Ψ_0 :

$$(6) \quad w = \frac{f}{\Psi_0}.$$

The mapping function (4) takes now the following form

$$\frac{(q\zeta + 1)^2}{(q\zeta - 1)^2} = \frac{ie'' - 1}{ie'' + 1}.$$

Here $\alpha, \beta, \gamma, \delta$ have been determined according to the foregoing table, e.g., by satisfying the conditions required for the points B, A and U ; the conditions at the remaining points are then automatically fulfilled. In analogy to (30.7a) we obtain for q the quadratic equation

$$(q\zeta)^2 + 2ie''q\zeta + 1 = 0$$

with the solution

$$q\zeta = -i(e'' + \sqrt{e^{2''} + 1}).$$

Because of (30.4), where df is to be replaced by $\Psi_0 dw$, this yields the differential equation

$$(7) \quad dz = -\frac{i\Psi_0}{q}(e'' + \sqrt{e^{2''} + 1})dw.$$

Let us now consider the three straight lines of Fig. 54

$$\Psi = 0, \quad \Psi = \pm \frac{\pi}{2} \Psi_0.$$

The quantities f and w are real if $\Psi = 0$; the left member of (7) is then purely imaginary. The point z travels along the y -axis from $-\infty$ to $+\infty$ as it is expected to do (Fig. 52). Along the two other lines e'' becomes purely imaginary. On putting

$$e'' = \pm is, \quad dw = \frac{ds}{s},$$

Eq. (7) appears in the following form:

$$(8) \quad dz = \frac{\Psi_0}{q}(\pm s + \sqrt{s^2 - 1})\frac{ds}{s}.$$

¹⁴Note that Kirchhoff in his treatise *Vorlesungen über Mechanik*, Chap. XXII, 3, defines the velocity potential with reversed sign, chooses the coordinate axes not symmetrical to the jet, and does not set in evidence physical parameters of the problem such as q . This procedure, which amounts to putting q equal to 1, has been generally accepted in the literature. We are of the opinion, however, that a better grasp of Kirchhoff's ingenious method becomes possible by the introduction of definite units from the start.

The square root is real for $\infty > s > 1$; for these values of s , z is real and travels along the "upper" and "lower" tank wall $y = 0$, $x > c$ and $x < -c$.

As we are mainly interested in the *free boundaries of the jet* that issue from the slit, we must consider the range $s < 1$. This makes dz in (8) complex, as it has to be in the case of curved jet boundaries. We choose the upper sign in (8) which corresponds to the lower jet boundary in Fig. 52 and put

$$s = \sin \alpha, \quad \frac{\pi}{2} > \alpha > 0.$$

Eq. (8) takes now the form

$$dz = \frac{\Psi_0}{q} (\sin \alpha + i \cos \alpha) \frac{\cos \alpha}{\sin \alpha} d\alpha = \frac{i\Psi_0}{q} e^{-i\alpha} \frac{\cos \alpha}{\sin \alpha} d\alpha.$$

This yields after integration the result

$$(9) \quad z = i \frac{\Psi_0}{q} \left(e^{-i\alpha} - \frac{1}{2} \log \frac{1 + \cos \alpha}{1 - \cos \alpha} \right) + \text{const},$$

which may be checked by differentiation. The constant can be determined by requiring $z = c$ for $\alpha = \pi/2$, that is, at the point A . Then we obtain

$$\text{const} = c - \frac{\Psi_0}{q}, \text{ and}$$

$$(9a) \quad z = c - \frac{\Psi_0}{q} \left(1 - ie^{-i\alpha} + \frac{i}{2} \log \frac{1 + \cos \alpha}{1 - \cos \alpha} \right).$$

Separation into real and imaginary parts gives:

$$(10) \quad x = c - \frac{\Psi_0}{q} (1 - \sin \alpha),$$

$$y = \frac{\Psi_0}{q} \left(\cos \alpha - \frac{1}{2} \log \frac{1 + \cos \alpha}{1 - \cos \alpha} \right).$$

These equations are the parametric representation of one of the two jet boundaries; the other is obtained from (10) by changing the sign of x .

For $\alpha = 0$ we obtain from (10)

$$x = c - \frac{\Psi_0}{q}, \quad y = \infty,$$

that is, the behavior at the infinitely distant point U in Fig. 52. Half the width of the jet at U is therefore

$$(11) \quad b = c - \frac{\Psi_0}{q}.$$

Finally, we have to find the physical meaning of the constant Ψ_0 which is of course connected with the velocity q . We write down the second Cauchy-Riemann equation (19.5)

$$(12) \quad \frac{\partial \Psi}{\partial x} = - \frac{\partial \Phi}{\partial y}.$$

At the cross-section U , that is, at large distance from the orifice, we find the velocity uniformly equal to q and parallel to the y -axis; the right member of (12) is therefore constant. Integration of (12) between the limits $x = 0$ and $x = b$ which correspond to $\Psi = 0$ and $\Psi = (\pi/2)\Psi_0$ yields

$$(12a) \quad \frac{\pi}{2} \Psi_0 = qb.$$

In combination with Eq. (11) one has

$$b = c - \frac{2}{\pi} b, \quad \frac{2 + \pi}{\pi} b = c$$

so that the relative contraction of the jet is

$$\frac{b}{c} = \frac{\pi}{2 + \pi} = 0.611.$$

The actual contraction coefficients used in hydraulic practice are close to our theoretical result; they serve to predict the efflux through a given orifice from its geometry.

32. Kármán's Vortex Street

The analysis of a given physical situation usually requires a certain amount of idealization, so that the tools of mathematics can be successfully applied; to find workable idealizations is the art of the mathematical physicist. Bearing this in mind, let us investigate a problem already presented by Helmholtz (cf. p. 216): What is the nature of the mechanism that leads to the curling up of the discontinuity surface into a sequence of vortex filaments? In the case considered in 30, the vortices originate at the sharp edges of the plate; in general vortices may be generated by any obstacle that is in the way of the flow. The photographs 47 a, b give a good idea of them. Loosely speaking, the vortices operate like ball or roller bearings distributed along the discontinuity surface that separates the streaming fluid from the dead water. This visualization gives at least the

correct sense of rotation of the vortices on either side of the wake. There is however this difference: the vortices do not run along in the same way as the balls in the race of a bearing. They would move with the stream, but they influence each other; actually the mutual influence produces an *additional velocity* which is superimposed on the *principal velocity* of the flow and has in the simplest case the opposite direction as we shall see.

Kármán¹⁵ idealized the sequence of vortices produced at an obstacle by assuming an *infinitely long straight vortex street* and disregards the origin of the vortices by transferring it to infinity. The vortices on either side of the street are equidistantly arranged: this characterizes the steady state which has developed in the (infinitely) long time that passed after they had been generated. On either side of the street the sense of rotation is different; the "upper" vortices are clockwise, the "lower" vortices counterclockwise. The principal velocity points to the right, corresponding to that of a free flow that rolls on "bearings" along a wake at rest. Kármán restricts himself to the case in which the additional velocity is parallel to the street; we shall start our analysis with the same assumption.

The first consequence is, as we shall show, that the two rows of vortices must be either opposite as in Fig. 55 or alternating as in Fig. 56. (Note

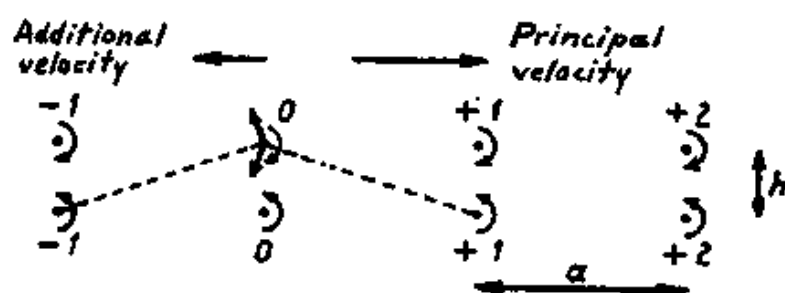


FIG. 55. Kármán's vortex street consisting of two rows of vortices of different orientation. The vortices are in opposite position.

that "vortex" means concentrated vortex filament, indicated in the figure by a dot with semicircular arrow.)

Consider Fig. 55; number the vortices from $-\infty$ to $+\infty$ and choose the vortex 0 of the upper street as "field" vortex, that is, consider this vortex as subjected to the induction of all the other vortices as in 21. There are *no contributions to its velocity* from the neighboring vortices ± 1 on the *same* side of the street, since the induced velocity is normal to the line connecting with the inducing vortex and points in the sense of its rotation. The contributions of the pair $\pm k$ cancel for the same reason. The contribution of the opposite vortex 0 points in the *negative direction*

¹⁵Th. v. Kármán, Über den Mechanismus des Flüssigkeits- und Luftwiderstandes, Göttinger Nachr. 1911 und 1912. Cf. also Kármán and H. Rubach, Physikal. Z. 13 (1912) 49.

of the street. Also the contributions of the pair ± 1 on the other side, which, individually, are inclined to the street, give a resultant in the negative direction of the street, and the same is true for the pair $\pm k$.

When the vortices are not exactly opposite to each other, the same construction will give a total induction at our field vortex that is *oblique to the street* with the one exception of the alternating case of Fig. 56. Here we compound the inductions due to the vortices 0 and 1 on the opposite side (broken lines in Fig. 56), and generally the inductions of the pairs $(-k, k+1)$ and obtain again an additional velocity at the field vortex, pointing in the negative direction of the street. Fig. 56 gives in addition an idea of the presumable path of the fluid particles that wind in and out between the vortices, in reasonable agreement with the photographs in Fig. 47a and b.

These photographs suggest at the same time that the alternating arrangement of Fig. 56 rather than the one of Fig. 55 occurs in nature. In

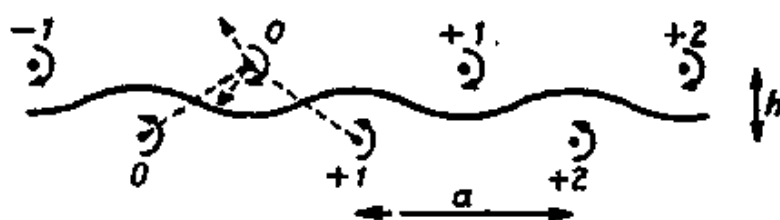


Fig. 56. Kármán's vortex street; the vortices are in alternating position.

order to explain this preference Kármán investigates the *stability* of both arrangements.

The concept of stability applied to a problem such as ours needs some comment. In Figs. 55 and 56 the locations of the vortices

$$z_k = x_k + iy_k$$

are accurately given by the positive or negative integer number k so that the additional velocity can be calculated according to well known complex methods. To investigate the stability we have to impart small displacements or disturbances

$$\bar{z}_k = \bar{x}_k + i\bar{y}_k$$

to the vortices.¹⁶ The introduction of such disturbances changes also the

¹⁶This variation is actually too restricted since it occurs only in the x, y -plane while in the z -direction the displaced vortex filaments are still supposed to be straight as they originally were. A more general stability investigation which introduces variations in z -direction has been carried out by K. Schlayer in his Munich thesis [Z. angew. Mathem. und Mech. 8, (1928) 352]. One can defend the special variations admitted by Kármán, observing that "infinitely long" straight vortex filaments can at any rate only be produced in a flow between parallel plates at a small distance (see p. 160). The

additional velocity. For instance, the statement that the total induction of all vortices on the same side as the field vortex vanishes is no longer true. In the disturbed state the velocity of the field vortex depends on all \bar{z}_k regardless on which side of the street they are. We assume the dependence as linear neglecting the higher powers of \bar{z}_k . The variation of the induction produced by the variation of the \bar{z}_k determines the time rate of change $\dot{\bar{z}}_0$ of the field vortex as a linear function of all the other \bar{z}_k . Thus we obtain first an infinite system of linear differential equations of first order for the \bar{z}_k which can be reduced, however, to a system of only two differential equations of the same character (see later). According to the general theory of small oscillations^{16a}, one determines first the principal modes of the system, that is, one finds solutions with the time dependence $e^{\lambda t}$. This procedure was carried out in a very simple case in Vol. I, 3.24 and prob. IV, 2. If λ is purely imaginary the vortex street is considered stable since oscillations of constant amplitude would be attenuated by friction which has been neglected so far. But the occurrence of even one complex λ -value with a positive real part makes the vortex street unstable, since a system can only be stable if all principal modes are stable. Kármán¹⁷ has shown that an arrangement of vortices as in Fig. 55 is unstable for any value of h different from zero; the alternating arrangement of Fig. 56 is stable if and only if

$$(1) \quad \sinh \frac{h}{a} \pi = 1, \quad \text{hence} \quad \frac{h}{a} = 0.281.$$

Our Figs. 55 and 56 have been drawn with about this numerical value of h/a .

The complete representation of the stability investigation, which is rather an involved mathematical problem, would take up too much space here. We shall indicate only the main points of the calculation. In doing so we shall at the same time broaden the basis of our investigation by following a paper of Maue.¹⁸ We shall assume that one row of the street

stability of such an experimental arrangement is indeed demonstrable by means of the special variations used in the text, provided the plate distance is sufficiently small. According to Schlayer, there is, however, always a weak instability toward general variations, and it increases with increasing plate distance.

^{16a}For a discussion of the method of small oscillations see, for instance, Kármán, *Th. v. and Biot*, M. A. Mathematical methods in Engineering, New York, 1940, Chapters V and VI.

¹⁷Kármán's investigation has been supplemented by Bl. Dolaptschiew who considered terms of second order. [Z. angew. Mathem. u. Mech. 17 (1937) 313, 18 (1938) 203]

¹⁸A. W. Maue, Z. angew. Mathem. u. Mech. 20 (1940) 130. Eqs. (6) to (8) that go beyond Kármán's original investigation have been taken from this paper. Mr. Maue kindly contributed the appendix to the present article.

is shifted relative to the other by the amount $d < a$ (in Fig. 55, $d = 0$, in Fig. 56, $d = a/2$). This makes no formal difficulties if the origin of the z -plane is located at the center of the pair 0, 0 and if the points are numbered in such a way that the points $\pm k$ on the one side are diametrically opposite to the points $\mp k$ on the other side of the street, viz:

$$(2) \quad z_k^2 = -z_k^1, \quad z_k^1 = z_0 + ka, \quad z_0 = \frac{1}{2}(d + ih).$$

(The superscripts 1 and 2 refer to the upper and lower rows of the street respectively, z_0 indicates the location of the 0-vortex belonging to the upper side of the street.)

First, we give a very simple expression for the complex potential $f = \Phi + i\Psi$ valid for arbitrary d , as it depends on d only by z_0 . Our starting point is Eq. (19.10) which describes the combined action of a positive and a negative vortex; the numerator and also the denominator occurring there vanish respectively at the center of the positive and negative vortex. The generalization of this equation for our vortex street is simply

$$(3) \quad f = -\frac{i\mu}{\pi} \log \frac{\sin \frac{(z - z_0)\pi}{a}}{\sin \frac{(z + z_0)\pi}{a}}.$$

In fact, the numerator vanishes now for all points

$$z = z_0 + ka = z_k^1,$$

and the denominator for all points

$$z = -(z_0 + ka) = z_k^2$$

as required by (2). The periodicity of the sine makes each vortex act with the same strength. The factor A in (19.10) that characterizes the vortex strength has been changed in accordance with (19.8a) and the signs have been adjusted in such a way as to correspond to the orientation of the vortices in Fig. 56.

By differentiation of f with respect to z one obtains as in the familiar relations (30.3) and (30.4)

$$(4) \quad \frac{df}{dz} = -v^* \begin{cases} v = v_x + iv_y, \\ v^* = v_x - iv_y. \end{cases}$$

Here v is the velocity of a fluid particle characterized by its complex coordinate z . If we wish to apply this formula to one of the vortex points, e.g., to the field vortex $z = z_0$ itself, then we must keep in mind that its

induction upon itself is zero and has to be omitted in the expression for f . This can be done by adding to the right member of (3)

$$+ \frac{i\mu}{\pi} \log(z - z_0).$$

According to this, Eq. (4) yields, when we introduce $\mathbf{u} = u_x + iu_y$ to denote the velocity of the field vortex z_0 ,

$$\mathbf{u}^* = \frac{i\mu}{\pi} \frac{d}{dz} \left(\log \frac{\sin \frac{(z - z_0)\pi}{a}}{\sin \frac{(z + z_0)\pi}{a}} - \log(z - z_0) \right);$$

the differential quotient (4) is to be taken at $z = z_0$. We therefore introduce

$$(z - z_0) \frac{\pi}{a} = \epsilon, \quad (z + z_0) \frac{\pi}{a} = 2z_0 \frac{\pi}{a} + \epsilon,$$

and find in the limit $\epsilon \rightarrow 0$

$$(5) \quad \mathbf{u}^* = - \frac{i\mu}{a} \operatorname{ctg} \frac{2\pi z_0}{a}.$$

This formula is valid not only for the vortex $z = z_0$, but, because of the periodicity of the street, also for any other vortex. The condition for the "additional velocity" \mathbf{u} to point in the direction of the street is simply that (5) be real; consequently the expression

$$\operatorname{ctg} \frac{2\pi z_0}{a}, \quad \text{or because of (2)} \quad \operatorname{ctg} \frac{\pi}{a} (d + ih)$$

must be *purely imaginary*. This occurs only if $d = 0$ or $d = a/2$. In both cases we have from (5)

$$(5a) \quad d = 0, \quad \mathbf{u}^* = u_x = - \frac{\mu}{a} \coth \frac{h}{a} \pi,$$

$$d = \frac{a}{2}, \quad \mathbf{u}^* = u_x = - \frac{\mu}{a} \tanh \frac{h}{a} \pi.$$

By means of the stability criterion stated in (1), the second line of (5a) can be written in the form

$$(5b) \quad d = \frac{a}{2}, \quad \mathbf{u}^* = u_x = - \frac{1}{\sqrt{2}} \frac{\mu}{a}.$$

Thus the additional velocities constructed in Figs. 55 and 56 have been found. The principal velocity indicated in the same figures—it could be

denoted by q as in the plate problem—has the value zero for the particular frame of reference we have chosen (center of the pair 0, 0). This can be seen from Eq. (3): If z is allowed to go to infinity along some specified path lying outside the vortex street, then the quotient of the sines tends to a constant limit; thus f is constant and $q = 0$.

In all other cases \mathbf{u} is *oblique* relative to the vortex street. The angle of inclination γ follows from (5) by an elementary calculation:

$$(6) \quad \operatorname{tg} \gamma = \frac{u_y}{u_x} = \frac{\sin 2\pi \frac{d}{a}}{\sinh 2\pi \frac{h}{a}}.$$

The absolute value u of \mathbf{u} is also obtained from (5):

$$(6a) \quad u = \frac{\mu}{a} \sqrt{\frac{\cosh^2 \pi \frac{h}{a} - \sin^2 \pi \frac{d}{a}}{\cosh^2 \pi \frac{h}{a} - \cos^2 \pi \frac{d}{a}}}.$$

This formula is the generalization of u_x in Kármán's result, which is our u_x in Eq. (5a).

If we let the vortex street be generated by a flow past a plate, as at the beginning of this article, Eq. (6) must obviously be associated with the case of an *oblique plate*. This should not imply, however, that the angle γ in (6) is equal to the angle γ in Fig. 48 (the former is actually smaller than the latter). Unfortunately, the photograph 47b does not show clearly the obliquity of the street produced.

We have still to discuss the central part of our problem, viz. the investigation of stability. Following Maue, we choose a slightly modified approach: instead of prescribing displacements \bar{z}_k for the vortex points, we modify the sine functions occurring in the complex potential (3) in the following way:

$$(7) \quad \sin(z - z_0) \frac{\pi}{a} = \zeta_1(z) \quad \text{and} \quad \sin(z + z_0) \frac{\pi}{a} = \zeta_2(z).$$

The perturbations ζ_1, ζ_2 are supposed to be small. They are connected to the arbitrary variations \bar{z}_k by the requirement that the zeros of the functions (7) should occur at $z_k^1 + \bar{z}_k^1$ and $z_k^2 + \bar{z}_k^2$ respectively. This is achieved by putting

$$\zeta_1(z_0 + ka) = (-1)^k \frac{\pi}{a} \bar{z}_k^1, \quad \zeta_2(-z_0 + ka) = (-1)^k \frac{\pi}{a} \bar{z}_k^2.$$

In this way the functions ζ have been fixed except for a function of the period a . The velocity of the vortex points after perturbation is now found as in Eq. (4) by differentiation of the modified function f . Upon expressing the same velocity by the time derivatives of the perturbations one obtains two simultaneous linear differential equations between ζ_1, ζ_2 , their first and second derivatives with respect to z , and $\dot{\zeta}_1, \dot{\zeta}_2$. The discussion of these equations which we omit here yields the stability criterion

$$(8) \quad \sinh \frac{h}{a} \pi = \sin \frac{d}{a} \pi.$$

This is the direct generalization of Kármán's criterion (1). Thus there exists also in the case of a vortex street that advances in oblique direction, a certain value h/a for any given value $d \leq a$ such that the vortex street maintains its arrangement after perturbation except for small oscillations that do not increase.

Fig. 57 illustrates this result and at the same time Kármán's special

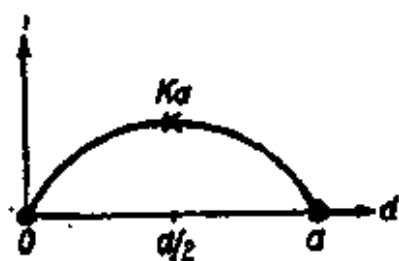


FIG. 57. Illustrating the generalized stability criterion. The width h of the vortex street as a function of the mutual displacement of the two rows of vortices.

case [Eq. (1)]. It is an h versus d plot for all possible stable vortex streets. If the \sinh in (8) were approximated by its tangent at the origin, the h versus d -curve would become an ordinary sine curve. Since the hyperbolic sine increases monotonically with its argument, the actual h, d -dependence is a slight distortion of the sine curve appearing on the right side of (8). The maximum ordinate corresponds to Kármán's case $d = a/2$. For $d = 0$ one has $h = 0$, just as in the geometrically identical case $d = a$. In both cases the two rows of vortex points coincide point by point and cancel each other: the end points of the stability curve have no physical significance.

One can expect that an experimental investigation such as was carried out by Kármán and Rubach for straight vortex streets (cf. footnote on p. 232) will also confirm the generalized stability criterion (8) for oblique vortex streets.

Appendix

The Drag Problem

Kármán's real aim in his investigation of the vortex street was to establish an analytic drag formula. For this purpose he considers a body moving through the fluid in the rear of which a vortex street has developed that reaches to infinity. The calculation is performed in the z -plane of the street, x being the direction of motion of the body. The body itself is considered as infinitely long in the direction normal to the z -plane. One delimits in the plane a sufficiently large rectangular region by a control surface, which includes the body and a sufficiently long stretch of the vortex street, and investigates the balance of momentum inside the control surface. The terms to be considered are: the pressure interaction between fluid and body, the increase of momentum inside the control surface due to the generation of new vortices, and the flux of momentum through the control surface. The first of these three terms is to be taken negative; its real part represents the drag, its imaginary part the side force normal to the direction of motion.

Even in the case where the direction of the vortex street coincides with the direction of motion of the body the calculation is rather involved. The more general case studied by Maue where the directions of the vortex street and that of the motion of the body form an angle with each other will probably always occur, unless there is a special relation of symmetry between the body and the direction of its motion. The drag formula is not given in Maue's paper quoted on p. 234 but has been kindly contributed by the author for use in the following discussion. It contains Kármán's formula as a particular case.

The notations used here conform to those of 30 in the plate problem. The drag P corresponds to the integral of the pressure over the width of the plate in (30.22). The effective width of the body normal to the direction of motion in the $x + iy$ -plane is now denoted by $2c$, while previously $2c$ was directly the width of the plate. In the present notation q is the velocity of the body relative to the fluid which is at rest at infinity; the previous q was the flow velocity at infinity while the plate was at rest. The dimensionless drag coefficient is denoted by λ (previously, 0.88). The distance of two neighbored vortices is a , h is the width of the street, and u its velocity of progression as calculated in (5).

According to Maue, the drag formula that replaces (30.22) is

$$(9) \quad \frac{P}{2c} = \lambda \frac{\rho}{2} q^2$$

where λ is given by

$$(10) \quad \lambda = \frac{a}{c} \left\{ A\eta - B\eta^2 \sqrt{1 - C\eta^2} - 2C\eta^3 \right\}.$$

Here, A , B , C stand for

$$(11) \quad A = \frac{2\sqrt{2}}{\pi} \beta \sinh \beta, \quad B = \frac{4}{\pi} \sinh \beta (\beta \cosh \beta - \sinh \beta),$$

$$C = \frac{1}{2} (1 - \sinh^2 \beta);$$

the numbers β and η are defined by

$$(12) \quad \beta = \pi \frac{h}{a}, \quad \eta = \frac{u}{q}.$$

The drag coefficient (10) thus depends on three ratios, viz. the two quantities appearing in (12) and a/c that appears in (10).

In Kármán's case $d = a/2$; hence, with $\sinh \beta = 1$ by (8), (11) gives

$$(13) \quad A = 0.794, \quad B = 0.31, \quad C = 0.$$

From (10) Kármán's drag coefficient is now obtained in the form

$$(14) \quad \lambda = \frac{a}{c} \left(0.794 \frac{u}{q} - 0.314 \left(\frac{u}{q} \right)^3 \right),$$

depending only on the ratios a/c and u/q . In the papers quoted on p. 232 these ratios have been obtained from experiments carried out with a flat plate:

$$(15) \quad a/2c = 5.5, \quad u/q = 0.20.$$

With these numerical values (14) gives

$$(16) \quad \lambda = 1.6,$$

a considerably larger value than Kirchhoff's coefficient 0.88 in (30.22). The latter value is too small as already mentioned in connection with the results of Prandtl and his coworkers.

W. Heisenberg¹⁹ has made an attempt to determine the quantities (15) theoretically from certain conservation theorems that should hold in the generation of vortices. He finds, in excellent agreement with Kármán, $a/2c = 5.5$, $u/q = 0.23$, which would yield, by (14), $\lambda = 1.8$.

¹⁹Physik. Z. 23 (1922), 363 with an additional remark of Prandtl.

33. Prandtl's Boundary Layer

It seems unsatisfactory that in all examples of potential flows³⁰ discussed in this chapter the fluid was supposed to be inviscid. However, in the case of an *incompressible* fluid in two- or three-dimensional potential flow the disregard of viscosity has no bearing on the *differential equations*, although it is of decisive importance for the *boundary conditions*.

This can be seen by a glance at the Navier-Stokes equations (16.1). If $\text{curl } \mathbf{v} = 0$ (potential flow) and $\text{div } \mathbf{v} = 0$ (incompressible fluid), the friction term $\mu \nabla^2 \mathbf{v}$ can be dropped entirely from Eqs. (16.1) because of the relation

$$\nabla^2 \mathbf{v} = \text{grad div } \mathbf{v} - \text{curl curl } \mathbf{v}.$$

Hence irrotational solutions of the Euler equations are always solutions of the Navier-Stokes equations: *the viscous forces have no dynamic effect in the interior of the fluid if the flow is irrotational.*

This is no longer true at the boundary between the fluid and a solid wall. While the boundary conditions for a perfect fluid are for this case

$$(1a) \quad v_n = 0, v_t \text{ arbitrary},$$

they read for a viscous fluid according to (10.9) (condition of adherence),

$$(1b) \quad v_n = 0, \quad v_t = 0.$$

It is only in the proximity of the wall that the viscous forces become dynamically important as a consequence of the modified boundary condition (1b). For a flow with large Reynolds numbers we may assume, however, that the modification of the perfect fluid stream pattern due to viscosity does not extend very far into the interior of the fluid.

The existence of the two different types of boundary conditions (1a) and (1b) has its *mathematical* basis in the occurrence of second (spatial) derivatives in the general Navier-Stokes equations in contrast to the Euler equations, that contain only first order derivatives. In order to single out a solution one therefore needs *two* vectorial boundary conditions in the first case and only *one* such condition in the second case. On the other hand, to reconcile *physically* the contradictory conditions (1a) and (1b), a fluid layer is inserted between the presumable region of validity of the potential flow and the wall. In passing through this layer, the thickness of which becomes arbitrarily small with decreasing μ , the tangential velocity v_t drops rapidly to zero. For small but finite μ (air, water), the

³⁰Kármán's vortex street is of course also a potential flow; its vorticity is concentrated in the vortex filaments, that is, in the singularities of the potential. Also the wave motions in Chap. V were treated as potential flows (see p. 168).

layer will still be reasonably thin, provided the overall dimensions of the wetted wall and the immersed bodies are large and the flow is sufficiently fast. The principal part of the flow lies then outside the layer and behaves like the potential flow of a perfect fluid. This idea forms the basis of the boundary layer theory first proposed by Prandtl in 1904.

The actual calculations by Prandtl²¹ and his coworkers have the character of an approximate theory.²² In the calculations which are mostly two-dimensional, inside the boundary layer the influence of the small velocity differences parallel to the layer on the friction terms is neglected. For small μ the thickness of the boundary layer is proportional to $\sqrt{\mu}$. Expressed in terms of the overall Reynolds number of the flow, R , it is proportional to $1/\sqrt{R}$. The pressure gradient normal to the boundary layer is neglected on account of its small width, and the pressure inside the layer therefore identified with the pressure of the adjacent potential flow which impresses, as it were, its own pressure on the whole boundary layer beneath. For the flow immediately adjacent to the boundary layer Bernoulli's equation applies, thus we have for a line element ds in the zone where the boundary layer passes over into the free stream

$$\frac{dp}{ds} = - \rho v_s \frac{dv_s}{ds}.$$

Note that the fluid begins to flow against the pressure gradient if dv_s/ds changes sign from plus to minus at some point.

Further calculation shows that $\partial^2 v_s / \partial n^2$ at $n = 0$ is negative in the region downstream from such a point. If the pressure gradient is large enough $(\partial v_s / \partial n)_{n=0}$ can also become negative, which means that the flow in the immediate neighborhood of the wall is reversed. This leads to the generation of vortices so that downstream from this point the laminar flow is replaced by flow processes of a different character.

The accuracy of this theory is quite high in spite of its approximate character, because the Reynolds numbers that occur in practical cases are usually large. In a given case the theory serves to decide whether the flow will everywhere stick to the rigid boundary and thus essentially remain a potential flow except for the thin boundary layer, or whether separation of the flow will occur, and, consequently, formation of eddies.

For very large v , (corresponding to very large R -values) the assumption of a laminar flow within the boundary layer is no longer correct; the flow

²¹Verh. des III. Int. Math. Kongresses in Heidelberg, 1904, Leipzig, 1905. Detailed representation in Prandtl-Tietjens, *Hydro- and Aerodynamics*, New York, 1934, Vol. II. and in *Modern Developments in Fluid Dynamics*, ed. S. Goldstein, Oxford, 1938.

²²The discussion in the text follows a report of Th. v. Kármán in *Vorträge aus dem Gebiet der Hydro- und Aerodynamik* (Innsbruck 1922), Berlin 1924, Springer.

becomes turbulent even inside the narrow boundary layer. This is of importance for the overall flow pattern because the separation point may be shifted downstream quite considerably, if the boundary layer becomes turbulent. An approximate prediction of the behavior of turbulent boundary layers based on a semi-empirical theory, has likewise become feasible.

The flow past a wing is one of the problems which have been solved analytically in this sense, that is, by the application of potential flow theory. The two-dimensional wing theory, the beginnings of which go back to Kutta²³ and Joukowski,²⁴ represents the flow as a plane potential motion with non-vanishing circulation and uses freely the methods of conformal mapping. The mathematical form of the three-dimensional wing theory is due to Prandtl. Here the flow is represented as a three-dimensional potential flow with a discontinuity surface, that may be visualized as a two-dimensional system of vortex filaments. The theory furnishes not only the lift distribution over the span but also the (induced) drag, that is, the dynamic equivalent of the kinetic energy that remains in the flow field behind the wing.

A. Betz and his coworkers developed the theory of the finite wing into a tool for the aeronautical engineer so that today pressure distributions for wings of any plan form (also with deflected ailerons, etc.) can be computed.

²³Sitzungsber. Bayer. Akad. 1910 and 1911.

²⁴Z. f. Flugtechnik u. Motorluftschiffahrt, 1 (1910).

SUPPLEMENTARY NOTES ON SELECTED HYDRODYNAMIC PROBLEMS

34. Lagrange's Equations of Motion

Lagrange's equations¹ have been referred to this supplement since they cannot compete with Euler's equations in importance as far as the setting up of hydrodynamic problems is concerned. Before we occupy ourselves with them, we wish to emphasize that Euler's equations, rather than Lagrange's, are the "unusual" equations. In ordinary mechanics we consider a mass point m , or some well defined part of a mechanical system, give it a certain tag q (coordinate) and determine the time rate of change of momentum p , corresponding to q , in accordance with the given forces and constraints. Now, in the case of Euler's equations we consider a *volume element* $d\tau$ and identify it rather than the mass element $\rho d\tau$, by coordinates; we again determine the change of momentum in accordance with the given forces and constraints, but we do this at the *space-fixed* element $d\tau$ (note that the pressure plays the role of a constraint, cf. 12). The mass element that occupies $d\tau$ at the given instant is lost sight of thereafter.

On the other hand, when setting up Lagrange's equations we follow the usual procedure and ascribe to the mass element dm certain tags a, b, c , e.g. its coordinates x_0, y_0, z_0 at the time $t = 0$. We ask then for the location x, y, z of this element dm at the time t . Thus x, y, z and the pressure p are the dependent variables, while a, b, c and the time t are the independent variables of Lagrange's equations.

No difference has now to be made between local acceleration $\partial v / \partial t$ and material acceleration dv / dt . The material acceleration which is the decisive parameter in the momentum balance, is now directly given by²

$$\frac{\partial^2 x}{\partial t^2}, \quad \frac{\partial^2 y}{\partial t^2}, \quad \frac{\partial^2 z}{\partial t^2},$$

¹The usual terminology is historically not correct, as it so often happens: "Lagrange's equation" occurs actually in the papers of Euler quoted on p. 84.

²This notation seems to be more consistent than the notation d^2x/dt^2 used by Kirchhoff and sometimes found in text books.

The equations that determine the acceleration vector are:

$$(1) \quad \frac{\partial^2 x}{\partial t^2} \frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2} \frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2} \frac{\partial z}{\partial a} = - \frac{\partial}{\partial a} \frac{U + p}{\rho}$$

and two corresponding equations to be formed with respect to b and c , instead of a . Eq. (1) follows from (11.2) if the component equations of (11.2) are multiplied by $\partial x/\partial a$, $\partial y/\partial a$, $\partial z/\partial a$, respectively and added, provided the fluid is incompressible ($\rho = \text{const}$) and the external force has a potential U ($\mathbf{F} = -\text{grad } U$). Equations (1) are non-linear, but so are Euler's equations; if Lagrange's equations were linear they would indeed present a tremendous advantage.

The derivative $\partial/\partial a$ means of course that b , c and also the fourth variable t are to be kept constant; we have e.g.

$$\frac{\partial U}{\partial a} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial a} + \frac{\partial U}{\partial y} \frac{\partial y}{\partial a} + \frac{\partial U}{\partial z} \frac{\partial z}{\partial a}.$$

Note, that (1) contains no other differentiations than with respect to the four independent variables a , b , c , t and that they apply only to the four dependent variables x , y , z , p (and to the given function U).

The three equations (1) must be supplemented by the condition of incompressibility. Now the ratio of the instantaneous volume $d\tau$ to the initial volume $d\tau_0$ is (as in any point transformation) given by the functional determinant (Jacobian)

$$(2) \quad \frac{d\tau}{d\tau_0} = \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{vmatrix}$$

or, with the usual abbreviation, by

$$(2a) \quad \frac{d\tau}{d\tau_0} = \frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)}.$$

The x_0 , y_0 , z_0 can be expressed in terms of the tags a , b , c ; according to the multiplication theorem for determinants the right member of (2) then becomes the product of the two functional determinants

$$\frac{\partial(x, y, z)}{\partial(a, b, c)} \quad \text{and} \quad \frac{\partial(a, b, c)}{\partial(x_0, y_0, z_0)}.$$

The last determinant is evidently independent of the time t . The invariability of (2) in course of time, i.e. the condition of incompressibility, can thus be written in the form:

$$(3) \quad \frac{\partial D}{\partial t} = 0, \quad D = \frac{\partial(x, y, z)}{\partial(a, b, c)}.$$

It contains only differentiations of the three dependent variables x, y, z with respect to the four independent variables a, b, c, t .

The set (1) and Eq. (3) are Lagrange's equations for incompressible fluids.

The corresponding relations for compressible fluids are obtained from (1), if one replaces

$$(3a) \quad \frac{p}{\rho} \quad \text{by} \quad \phi = \int \frac{dp}{\rho} \quad \text{and} \quad \frac{U}{\rho} \quad \text{by} \quad V,$$

where V is the potential per unit of mass of the external force. In addition, (3) must be replaced by

$$(3b) \quad \frac{\partial}{\partial t} (\rho D) = 0.$$

Kirchhoff* used Lagrange's equations in a rigorous proof of Helmholtz's theorems on vortex conservation and was therefore able to avoid the use of formula (18.8), which we had to use, since our proof starts from Euler's equations.

35. Stokes's Resistance Law

On p. 78 we discussed the very slow ("creeping") motion of a viscous incompressible fluid as an example for a flow the dynamics of which is almost exclusively determined by viscous forces; the next step would be neglecting the inertia forces altogether. According to the value of the Reynolds number (16.8)

$$R = \frac{va}{\nu}, \quad \nu = \frac{\mu}{\rho},$$

we decide when omission of the inertia terms is justifiable, the condition being

$$(1) \quad R \ll 1.$$

The problem which interests us here,—it is usually connected with the name of Stokes—is the steady motion of a sphere through a liquid. Let

*Vorlesungen über Mechanik, Chap. XIV, 1.

a be the radius of the sphere and v its velocity which has become uniform after a short initial period. The motion proceeds under the influence of an external force such as the difference of gravity and buoyancy, possibly an electric field, etc. Condition (1) can not only be satisfied by a large value of the kinematic viscosity ν , but also by sufficiently small values of a or v . Instead of moving the sphere through the fluid one may as well keep the sphere at rest and have the fluid flow past the sphere (inverse flow); the resultant pressure force acting on the sphere is then equal to the resistance force encountered by the sphere in the direct flow.

Stokes resistance law plays an important role in the evaluation of certain fundamental experiments in physics. Einstein made use of it in his theory of the Brownian motion which leads to a determination of Loschmidt's number (cf. p. 53). Later it has become important again in Millikan's determination of the electronic charge.

We consider the steady state of the motion. The weight of the sphere is then balanced by the resultant friction forces and cannot appear in the equations or boundary conditions. Adopting now the viewpoint of the inverse motion, we take as a starting point the *equilibrium conditions* for a viscous fluid (10.8a, b), that is, the Navier-Stokes equations with omission of the inertia terms. The body force F of gravity acting on the fluid can be disregarded. In other words, we have now a kinematic problem defined by a system of equations, which reads in Cartesian coordinates:

$$(2) \quad \frac{\partial p}{\partial x} = \mu \nabla^2 u, \quad \frac{\partial p}{\partial y} = \mu \nabla^2 v, \quad \frac{\partial p}{\partial z} = \mu \nabla^2 w;$$

it is to be supplemented by the condition of incompressibility

$$(2a) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

and by the boundary conditions of the problem. Let the origin coincide with the center of the sphere, the z -axis pointing in the direction opposite to the motion of the sphere (the fluid flows past the sphere in the direction of the positive z -axis). From (2) and (2a) we have directly

$$(3) \quad \nabla^2 p = 0.$$

We now introduce spherical coordinates r, ϑ, φ referring again to the center of the sphere, so that e.g.

$$(4) \quad \cos \vartheta = \frac{z}{r}.$$

The spherically symmetric solution of (3) is, according to (19.22),

$$(5) \quad p = \frac{A_0}{r}.$$

but our problem is not spherically symmetric, the z -axis being the preferred direction in which the fluid moves toward the sphere. We consider therefore the expression

$$(5a) \quad p = A_1 \frac{\partial}{\partial z} \frac{1}{r},$$

which obviously is also a solution of (3). On carrying out the differentiation in (5a) and replacing A_1 by A , we obtain, because of (4),

$$(5b) \quad p = -\frac{A}{r^3} \cos \vartheta.$$

A much more general solution of (3) is obtained in the form of the series

$$(5c) \quad p = \sum_{n=0}^{\infty} A_n \frac{\partial^n}{\partial z^n} \frac{1}{r},$$

the first two terms of which coincide with (5) and (5a) respectively. Eq. (5c) represents an expansion in terms of negative powers of r , the coefficients being the spherical harmonics $P_n(\cos \vartheta)$. Without giving here the general definition of spherical harmonics, we note that the factors 1 and $\cos \vartheta$ appearing in (5) and (5a) are spherical harmonics in the usual normalization: $P_0(\cos \vartheta) = 1$, $P_1(\cos \vartheta) = \cos \vartheta$, and that the spherical harmonics with higher subscripts are polynomials in $\cos \vartheta$ of higher order. (Legendre's polynomials). Since in our boundary conditions [(12a, b), see below] only the first power of $\cos \vartheta$ appears, we conclude that in (5c) only the term with $n = 1$ lends itself to our purpose; when we put $A_0 = A_2 = A_3 = \dots = 0$, the series (5c) reduces to our trial solution (5b).

We first have to write equations (2) in their legitimate vectorial form. This is done by means of Eq. (3.10a) which supplies, because of $\text{div } \mathbf{v} = 0$,

$$(6) \quad \mu \text{curl curl } \mathbf{v} = -\text{grad } p$$

instead of (2). It is now easy to pass over to spherical coordinates. We denote the components of the velocity vector \mathbf{v} taken in direction r, ϑ, φ by $v_r, v_\vartheta, v_\varphi$, and observe that the φ -component must vanish because of the axial symmetry of the problem; also, all differential quotients with respect to φ must vanish, in other words,

$$(7) \quad v_\varphi = 0, \quad \frac{\partial}{\partial \varphi} = 0.$$

The r -component of Eq. (6) is, according to the solution of problem I.3, or, according to Appendix III and IV,

$$(8) \quad \frac{\mu}{r^2} \left[\frac{\partial^2 (r^2 v_r)}{\partial r^2} + \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial v_r}{\partial \vartheta} \right) \right] = \frac{\partial p}{\partial r}.$$

Here we assume the r and ϑ dependence of v_r in the form

$$(8a) \quad v_r = R \cos \vartheta,$$

where R denotes a function dependent on r alone. Substituting in (8) and expressing p according to (5a), we obtain, after dropping the factor $\cos \vartheta$, the ordinary differential equation for R

$$\frac{\mu}{r^3} \left(\frac{d^2}{dr^2} (rR) - \frac{2}{r} R \right) = \frac{2A}{r^3},$$

which may also be written

$$(9) \quad r^3 R'' + 2rR' - 2R = \frac{2A}{\mu}.$$

A particular integral of (9) is $R = -A/\mu$ and the general integral of the associated homogeneous equation

$$R = Br + \frac{C}{r^2},$$

hence the general integral of (9) is according to a well known theorem

$$R = \frac{-A}{\mu} + Br + \frac{C}{r^2}$$

and the velocity component v_r is found

$$(10) \quad v_r = \left(-\frac{A}{\mu r} + B + \frac{C}{r^3} \right) \cos \vartheta.$$

The component v_ϑ may now be found from the condition of incompressibility (see solution of problem I.3) the form of which suggests a trial separation $v_\vartheta = R_1 \sin \vartheta$, where again R_1 depends on r alone. One finds easily

$$R_1 = \frac{A}{2\mu r} - B + \frac{C}{2r^3},$$

hence

$$(11) \quad v_\vartheta = \left(\frac{A}{2\mu r} - B + \frac{C}{2r^3} \right) \sin \vartheta.$$

The constants of integration A, B, C must be determined so as to satisfy the boundary conditions of the problem. If q denotes the undisturbed flow velocity at infinity, the boundary conditions are

$$(12a) \quad r = a : v_r = v_\vartheta = 0$$

$$(12b) \quad r = \infty : v_r = q \cos \vartheta, \quad v_\vartheta = -q \sin \vartheta.$$

Eq. (12b) is fulfilled if we put $B = q$; then Eqs. (12a) can be solved for A and C and give

$$\frac{A}{\mu} = \frac{3q}{2} a, \quad C = \frac{q}{2} a^3;$$

hence we obtain finally

$$\begin{aligned} v_r &= q \left(1 - \frac{3}{2} \frac{a}{r} + \frac{1}{2} \frac{a^3}{r^3} \right) \cos \vartheta \\ (13) \quad v_\vartheta &= q \left(-1 + \frac{3}{4} \frac{a}{r} + \frac{1}{4} \frac{a^3}{r^3} \right) \sin \vartheta \end{aligned}$$

and by (5b)

$$(13a) \quad p = -\frac{3}{2} \mu q a \frac{\cos \vartheta}{r^2}.$$

The result shows that the choice of the special solution (5b) in place of the more general one (5c) was in order. It is true that we have not proved that (13) and (13a) are the only possible solutions of our problem. On the other hand we are convinced that every problem of mathematical physics that has been correctly formulated can have only *one* solution.

We have yet to derive the formula for the *resistance* experienced by the *moving* sphere or, which is the same, for the pressure force acting on the sphere at *rest*; it is composed of the following two parts: the component of the hydrodynamic pressure force in the direction of the positive z -axis and the contribution of the viscous pressure force in the same direction.

The *hydrodynamic* pressure acts *radially*. According to (13a) its component in the positive z -direction is at the surface of the sphere, that is for $r = a$

$$-p \cos \vartheta = \frac{3}{2} \mu \frac{q}{a} \cos^2 \vartheta.$$

Integration over the sphere whose surface element is $2\pi a^2 \sin \vartheta d\vartheta$ gives

$$(14) \quad 3\pi\mu a q \int_0^\pi \cos^2 \vartheta \sin \vartheta d\vartheta = 3\pi\mu a q \int_{-1}^{+1} x^2 dx = 2\pi\mu a q.$$

To calculate the *viscous pressures*, we start from Eq. (10.2) which correlates the tensors p_{ik} and $\dot{\epsilon}_{ik}$ and is valid in any orthogonal coordinate system, hence also in our polar coordinates. In the present notation we have

$$(15) \quad p_{rr} = -2\mu \dot{\epsilon}_{rr}, \quad p_{r\vartheta} = -2\mu \dot{\epsilon}_{r\vartheta}.$$

The displacements ϵ have been defined in curvilinear coordinates in Eqs. (4.26) and (4.28). The rates of displacement $\dot{\epsilon}$ are obtained from these equations if the $\delta q_1, \delta q_2, \delta q_3$ are replaced by the components of the velocity

$$v_r, v_\vartheta, v_\varphi, \quad (v_\varphi = 0)$$

and the p_1, p_2, p_3 by r, ϑ, φ . Since $ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2$, one has $g_1 = 1, g_2 = r, g_3 = r \sin \vartheta$. Hence (4.26) yields

$$(16) \quad \dot{\epsilon}_{rr} = \frac{\partial v_r}{\partial r}$$

and (4.28)

$$(17) \quad 2\dot{\epsilon}_{r\vartheta} = \frac{1}{r} \frac{\partial v_r}{\partial \vartheta} + \frac{\partial v_\vartheta}{\partial r} - \frac{1}{r} v_\vartheta.$$

When we substitute for the velocity components according to (13), only the middle term of (17) gives a non-vanishing contribution for $r = a$, so that

$$\dot{\epsilon}_{rr} = 0, \quad \dot{\epsilon}_{r\vartheta} = -\frac{3}{4} \frac{q}{a} \sin \vartheta.$$

According to (15), the friction pressures are then

$$(18) \quad p_{rr} = 0, \quad p_{r\vartheta} = \frac{3}{2} \frac{\mu q}{a} \sin \vartheta.$$

The component of $p_{r\vartheta}$ in the positive z -direction is

$$p_{r\vartheta} \sin \vartheta = \frac{3}{2} \frac{\mu q}{a} \sin^2 \vartheta$$

which must be integrated over the sphere:

$$(19) \quad 3\pi\mu a q \int_0^\pi \sin^3 \vartheta d\vartheta = 3\pi\mu a q \int_{-1}^{+1} (1-x^2) dx = 4\pi\mu a q.$$

This together with the contribution (19) gives Stokes's famous resistance formula

$$(20) \quad D = 6\pi\mu a q.$$

Analogous to the problem of the sphere in uniform translatory motion q is the problem of a sphere rotating uniformly, with angular velocity ω . In this case the opposing moment⁴ (see problem VII.4) is

$$(21) \quad M = 8\pi\mu a^3 \omega.$$

⁴Cf. Kirchhoff, Vorlesungen über Mechanik, Chap. XXVI, 3.

These results are valid for liquids and gases at ordinary pressures. For gases at low pressures one has to add a correction term to Eq. (20) which is due to Cunningham. In the corrected formula the right member of (20) appears with the denominator

$$(22) \quad 1 + \alpha \frac{l}{a},$$

where l is the *mean free path* of the gas molecules, which increases with decreasing pressure; the physical interpretation of the numerical coefficient α is somewhat problematical.⁴ Cunningham's correction was of great importance in the determination of the electronic charge according to Millikan's method.

To introduce the concept of the mean free path means to leave behind the framework of a continuum theory and enter the domain of statistical mechanics. The continuum theory which has led us to Stokes' formula is only valid on a scale that is large in comparison with l . In our case, the scale is set by the radius a . Cunningham's correction term is subject to the same condition if it is to be considered a correction: this is no longer the case if $l \sim a$, as one sees directly from (22), and a comprehensive treatment of the problem is then no longer feasible.

Another limitation of our considerations is that they are only good for *small* Reynolds numbers [cf. inequality (1)]. When R increases, the inertia terms that have been neglected so far gain in importance; the laminar character of the flow, however, is still maintained, disregarding some vortices in the wake. The resistance law, which was linear in q for Stokes's case now assumes by and by a quadratic character. Full turbulence is only reached at a certain $R = R_{crit}$, at which, in addition to the wake, the entire flow field in the neighborhood of the sphere becomes turbulent. At this point the resistance law changes its character suddenly.

Let us now have a critical look at the omission of the inertia terms in the preceding analysis. According to (5b), we have the following asymptotic behavior at $r \rightarrow \infty$:

$$p \sim \frac{1}{r^2}, \quad \text{hence } \text{grad } p \sim \frac{1}{r^3}.$$

The terms $\nabla^2 u$, $\nabla^2 v$, $\nabla^2 w$ retained in Eq. (2) have the same order of magnitude as $\text{grad } p$. The inertia terms which we neglected vanish at $r \rightarrow \infty$ partly with the third, but partly with a smaller power of $1/r$. The latter happens in the case of those inertia terms which read in Cartesian components u , v , w

⁴Th. Sexl, Zur gaskinetischen Begründung der Stokes-Cunningham'schen Formel, Ann. Physik. 81 (1926) 855.

$$(23) \quad \rho w \frac{\partial u}{\partial z}, \quad \rho w \frac{\partial v}{\partial z}, \quad \rho w \frac{\partial w}{\partial z}.$$

When these terms are calculated from (13) they are found to vanish only as $1/r^2$. It is true that the Navier-Stokes equations can be rewritten in such a way that the Reynolds number $R = qa/\nu$ appears in these terms (introduce the dimensionless coordinates x/a , etc. and suitably rearrange the factors ρ , μ , and a , cf. 16). But although R is supposed to be small, this cannot change the fact that the ratio of the neglected terms (23) to the terms retained becomes arbitrarily large if $r \rightarrow \infty$.

On account of this Oseen devised a treatment of the problem that retains the terms (23) in principle, but replaces them by their first order approximation⁶

$$(23a) \quad \rho q \frac{\partial u}{\partial z}, \quad \rho q \frac{\partial v}{\partial z}, \quad \rho q \frac{\partial w}{\partial z}.$$

The integration of the equations so augmented is by no means simple and forms one of the principal parts of Oseen's book quoted on p. 210. In the light of his result our preceding formulas appear as the first term of a power series expansion in terms of the Reynolds number R . In particular, the resistance law (20) has the form

$$(24) \quad W = 6\pi\mu a q \left(1 + \frac{3}{8}R\right),$$

when the first order term is retained.

Fortunately the validity of Stokes result which has found such wide acceptance in experimental and theoretical physics is not perceptibly affected by this correction; this is due to condition (1), the fulfillment of which we have assumed throughout our discussion of laminar motion. On account of Eq. (24) one can now comprehend why the transition from the linear to the quadratic resistance law mentioned before does not proceed suddenly but gradually. The onset of full turbulence, however, at $R = R_{crit}$, cannot be understood in this way.

36. The Hydrodynamic Theory of Lubrication

Lubrication today is essentially a problem of "creeping" motion. In older text books of applied mechanics the friction between journal and bearing was treated as a problem of *dry* friction. The arguments put forth had nothing to do with hydrodynamics and are only mentioned here to point out the contrast to the later developments.

⁶Thus an error is introduced at $r = a$, where actually $w = 0$ and not $= q$, and the error for $r \rightarrow \infty$ is diminished but that should not imply that Oseen's calculation leaves anything to be desired in the way of mathematical rigor.

In order to apply the ordinary concepts of friction theory developed in Vol. I, 14⁷ one must assume direct contact between journal and bearing. In Fig. 58 the journal pressure P , which is referred to the unit of length of the bearing, has been drawn vertically downward. This should not imply that it is actually vertical. In general it is composed of the gravity load, possible components of the driving force and inertia effects; it will therefore be inclined to the vertical.

At the point of contact B the acting forces are the friction R , which is opposite to the sense of rotation, and the normal pressure N . The re-

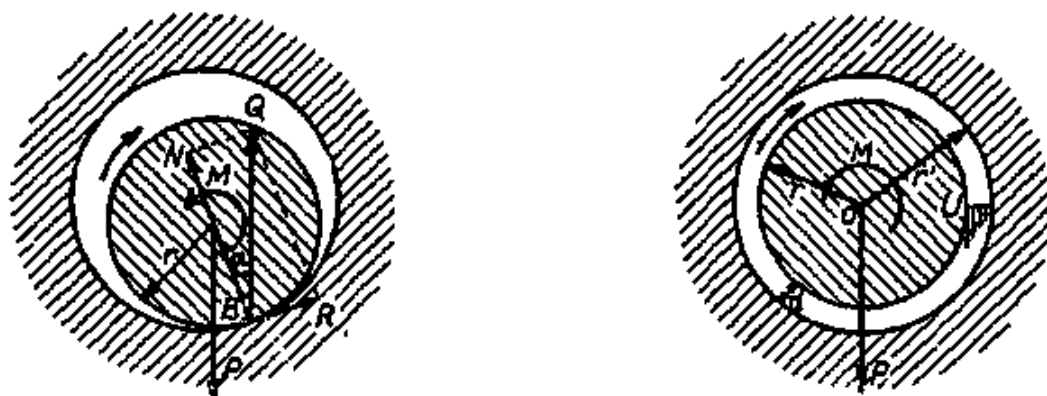


FIG. 58 (left). Relative position of journal and bearing according to the assumption of dry friction.

FIG. 59 (right). Fluid friction in the case of the centered journal.

sultant of the two, Q , forms the angle of friction α with N (α means here the kinetic angle of friction, defined by the coefficient of kinetic friction f according to $f = \tan \alpha$). The equilibrium condition requires that Q and P are antiparallel and equal. The remaining couple QP then represents the moment of friction about the centerline of the journal which is balanced by an opposite driving moment. If r denotes the radius of the journal and $r \sin \alpha$ the arm of the couple, we have

$$(1) \quad M = Pr \sin \alpha.$$

According to this the moment of friction should be *independent of the circumferential velocity U and proportional to the journal pressure P* ; the point of contact B should be situated *behind* the vector of the journal pressure, in reference to the sense of rotation.

The Russian engineer and officer N. Petroff was the first to look at this problem from the point of view of the viscosity of the lubricant.* In

⁷See e.g. Synge and Griffith, *op. cit.*, Sec. 3.2.

*Cf. Petroff, N. Friction in Machines and the Effect of the Lubricant (in Russian), Eng. Jour., St. Petersburg. 1883, a German translation in Ostwalds Klassiker, No. 218 (Leipzig). In the same volume also the following papers: Reynolds, On the Theory of Lubrication etc., Phil. Trans. Roy. Soc. Vol. 177 pt. I, 1886, or Papers, Cambridge 1901,

his conception the journal is exactly centered in the bearing so that the surrounding lubricant forms a layer of uniform thickness d (see Fig. 59). Throughout this layer the velocity drops from its value U at the journal to the value zero at the bearing in an approximately linear way. The tangential friction pressure at the bearing is then $\mu U/d$. By summation over the circumference one obtains the moment of friction per unit of length

$$(2) \quad M = 2\pi\mu \frac{U}{d} r^2.$$

This dependence is inverse to that of (1), since we have now *proportionality with U and independence of P* ; also, the factor μ points to the lubricant and not to the materials of journal and bearing like $\sin \alpha$ in (1).

The equilibrium conditions of statics, however, are not satisfied by (2) since the journal pressure P can never be balanced by the friction forces. Reynolds came therefore to the conclusion that the actual position of the journal in the bearing is not the central one; it could only be central for infinitely large circumferential velocity U . In addition Reynolds showed that the point of smallest layer thickness h is not behind P but *in front* of it. Unfortunately, Reynolds formulas are difficult to understand, since his calculations refer to bearings with only partially encased journals, as of railroad cars.

Fully encased journals with complete peripheral lubrication, the most desirable type for high speed electrical machinery, lend themselves to a comprehensive analysis, (cf. the third of the papers quoted in the footnote. Only a little later and under the influence of this paper Mitchell carried out his construction of the particular type of step bearing that has been named after him and has become important for heavy thrust bearings in turbines with vertical shafts.)

The essential point is that the journal is displaced in the direction normal to the journal pressure while it runs (see Fig. 60). The foundation of the theory of the *bearing* is obtained by an approximate calculation of the flow pattern of the *fluid wedge* in Fig. 61, where we neglect the y -component v of the velocity in comparison with the x -component u ; w is exactly zero, since the problem is considered as two-dimensional. According to the fundamental equations of "creeping" motion (35.2) and (35.2a) we then have

Vol. II; Sommerfeld, Zur hydrodyn. Theorie der Schmiermittelreibung, Z. Math. Phys., 50, 1904. Mitchell, A. G. M. The Lubrication of Plane Surfaces, Z. Math. Phys., 52, 1905; Sommerfeld, Zur Theorie der Schmiermittelreibung, Z. techn. Phys., 2, 1921.—More recent literature e.g. in Hersey, M. D., Theory of Lubrication, New York 1938, Wiley.

$$p = p(x), \quad u = u(y), \quad \frac{dp}{dx} = \mu \frac{d^2 u}{dy^2},$$

or

$$u = \frac{1}{2\mu} \frac{dp}{dx} y^2 + ay + b;$$

this yields because of the boundary condition $u = U$ for $y = 0$ (journal) and $u = 0$ for $y = h$ (bearing)

$$(3) \quad u = U \left(1 - \frac{y}{h}\right) - \frac{1}{2\mu} \frac{dp}{dx} y h \left(1 - \frac{y}{h}\right).$$

The flux through the cross-section is

$$\int_0^h u \, dy = \frac{h}{2} U - \frac{h^3}{12\mu} \frac{dp}{dx}.$$

Although h is variable in Fig. 61, the flux must be the same for all cross-sections (condition of incompressibility!). Thus, if h_0 denotes an assumed cross-section at which $dp/dx = 0$, we have

$$(4) \quad \frac{h^3}{6\mu} \frac{dp}{dx} = (h - h_0) U.$$

From (3) and (4) the friction pressure per unit of area of the boundary $y = 0$ is found

$$(4a) \quad p_{zs} = \mu \frac{du}{dy} = -\mu \frac{4h - 3h_0}{h^3} U.$$

Now transfer these results from Fig. 61 to Fig. 60, introducing the following notations: O , r and O' , r' are center and radius of journal and bearing, respectively, the excentricity $e = \text{distance } OO'$, $d = r' - r$ is

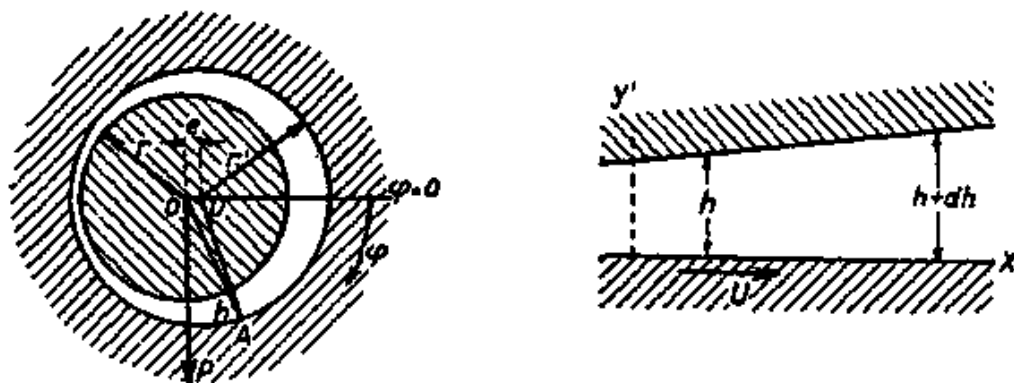


FIG. 60 (left). The journal is shifted normal to the direction of the journal pressure.

FIG. 61 (right). Simplified scheme of a lubricant film of variable thickness.

the play, A is a variable point at the bearing surface, characterized by the angle φ whose vertex is at O . We have correspondingly

$$OA = r + h, \quad O'A = r', \quad r + h = e \cos \varphi + r'.$$

The last (approximate) equation, obtained by projecting the segments OO' and $O'A$ on the line OA , can be written in the form

$$(5) \quad h = d + e \cos \varphi.$$

This meaning of h is now introduced into Eqs. (4) and (4a) in order to adapt⁹ them to the conditions of Fig. 60:

$$(6) \quad \frac{1}{r} \frac{dp}{d\varphi} = 6\mu U \left(\frac{1}{h^2} - \frac{h_0}{h^3} \right),$$

$$(6a) \quad p_{,r} = -\mu U \left(\frac{4}{h} - \frac{3h_0}{h^2} \right).$$

By integration with respect to φ between the limits 0 and 2π one has

$$(7) \quad p(2\pi) - p(0) = 6\mu r U (J_2 - h_0 J_3),$$

$$(7a) \quad M = -\mu r^2 U (4J_1 - 3h_0 J_2).$$

As before, M stands for the friction moment transferred to the journal per unit of length; the J 's serve as abbreviations for

$$(8) \quad \begin{aligned} J_1 &= \int_0^{2\pi} \frac{d\varphi}{d + e \cos \varphi} = 2\pi(d^2 - e^2)^{-1/2}, \\ J_2 &= \int_0^{2\pi} \frac{d\varphi}{(d + e \cos \varphi)^2} = 2\pi d(d^2 - e^2)^{-3/2}, \\ J_3 &= \int_0^{2\pi} \frac{d\varphi}{(d + e \cos \varphi)^3} = 2\pi \left(d^2 + \frac{1}{2} e^2 \right) (d^2 - e^2)^{-5/2}. \end{aligned}$$

If the value given for the integral J_1 is known, the values of J_2 and J_3 can be found according to

$$J_2 = -\frac{\partial J_1}{\partial d}, \quad J_3 = -\frac{1}{2} \frac{\partial J_2}{\partial d}.$$

Since p must have the period 2π , one concludes from (7) and (8)

$$(9) \quad h_0 = \frac{J_2}{J_3} = d \frac{d^2 - e^2}{d^2 + \frac{1}{2} e^2}.$$

⁹It would not have been difficult to derive these equations directly in polar coordinates instead of in coordinates x, y, z as we did it.

With this value of h_0 the moment becomes

$$(10) \quad M = - \frac{2\pi\mu r^2 U}{\sqrt{d^2 - e^2}} \frac{d^2 + 2e^2}{d^2 + \frac{1}{2}e^2}.$$

Putting in analogy to (5)

$$(11) \quad h_0 = d + e \cos \varphi_0,$$

one obtains from (9)

$$(11a) \quad \cos \varphi_0 = - \frac{3}{2} \frac{de}{d^2 + \frac{1}{2}e^2}.$$

If $e = 0$ (centered position of the bearing), we have

$$\cos \varphi_0 = 0, \quad \varphi_0 = \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2};$$

on the other hand, we obtain in the case of the largest possible eccentricity $e = d$

$$\cos \varphi_0 = -1, \quad \varphi_0 = \pi.$$

Now, the definition of h_0 according to (6) has been

$$\frac{dp}{d\varphi} = 0 \quad \text{for} \quad h = h_0,$$

therefore the two values φ_0 belonging to h_0 determine the points of minimum and maximum pressure. In the centered position these points are diametrically opposite to each other. At maximum eccentricity, that is to say, if contact is made between journal and bearing, they coincide at the point of contact.

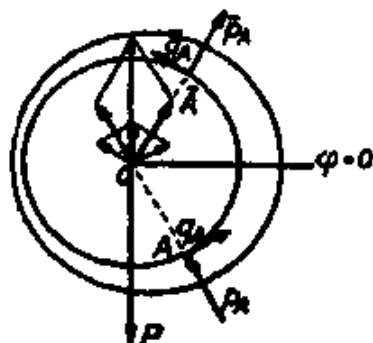


FIG. 62. Equilibrium between the hydrodynamic pressures p , q , and the bearing pressure P .

Our next task is to relate the pressure distribution to the direction of the vector of the journal pressure.

We notice that the pressure gradient $dp/d\varphi$ depends only on h [see (6)], and is therefore an *even* function of φ , as h is itself, when we count φ from

the zero line OO' . The pressure itself is consequently an *odd* function of φ , say p_1 , plus an integration constant, say p_0 . In forming the sum total of the pressure forces, the contribution of the *constant* pressures p_0 obviously vanishes. Denoting by p_A and $p_{\bar{A}}$ the pressure values p_1 at two symmetrically situated points relative to $\varphi = 0$, we find the resultant action on the journal indicated in the larger of the two force parallelograms in Fig. 62: it is normal to OO' .

Applying the same reasoning to Eq. (6a), we see that the *friction pressures* (shear stresses) at the journal surface are *even* functions of φ ; they are indicated by q_A , $q_{\bar{A}}$ in the figure (note, however, their tangential direction opposite to the sense of increasing φ). By construction of the corresponding parallelogram of forces at O we see that the friction pressures at A and A' give also a resultant *normal* to OO' . Thus the overall resultant of the hydrodynamic forces acting on the journal is normal to OO' . Since this resultant balances the journal pressure P , it must be opposite to P , in other words, *the displacement of the journal is normal to the journal pressure P* , a result anticipated in Fig. 60.

The *value* of the displacement e can now be determined. One has only to form the sum of the resultants of p and q and balance it against the prescribed value of P :

$$(12) \quad \frac{P}{r} = \int p \sin \varphi \, d\varphi - \int q \cos \varphi \, d\varphi.$$

The first term can be transformed by partial integration; one of the two resulting terms, $[-p \cos \varphi]_0^{2\pi}$, vanishes because of the periodicity. Altogether (12) becomes

$$(12a) \quad \frac{P}{r} = \int \left(\frac{dp}{d\varphi} - q \right) \cos \varphi \, d\varphi.$$

Here the term q may be omitted since $dp/d\varphi$ outweighs $q = p_{r\varphi}$ by a full order of magnitude of r/h [cf. Eqs. (6) and (6a)]. Hence we obtain the simpler relation

$$(12b) \quad \frac{P}{r} = 6\mu r U (J_4 - h_0 J_6)$$

where

$$J_4 = \int \frac{\cos \varphi \, d\varphi}{(d + e \cos \varphi)^3}, \quad J_6 = \int \frac{\cos \varphi \, d\varphi}{(d + e \cos \varphi)^5}.$$

From (8) we find easily

$$J_4 = \frac{1}{e} (J_1 - d \cdot J_2), \quad J_6 = \frac{1}{e} (J_2 - d \cdot J_3).$$

If this is substituted in (12b) and the value of h_0 taken from (9), an elementary calculation gives the result

$$(13) \quad \frac{P}{2\pi\mu U} \frac{d^2}{r^2} = \frac{3e}{\sqrt{d^2 - e^2}} \frac{d^2}{d^2 + \frac{1}{2}e^2}.$$

The excentricity e or rather the ratio e/d is here determined as a function of the left member of (13). This value of e , substituted in (10), leads to the determination of the moment M . In what follows we are only interested in the magnitude and not in the sign of the moment. Note, that a particularly simple expression for M is obtained, if (10) is divided by (13):

$$(13a) \quad \frac{M}{P} = \frac{d^2 + 2e^2}{3e}.$$

In discussing this result, we first consider two limiting cases:

a) $U \rightarrow \infty$. From (13) we have $e = 0$, that is, the journal is centered; Eq. (10) yields immediately

$$(14) \quad M = 2\pi\mu r^2 \frac{U}{d},$$

which is Petroff's equation.

b) $U \rightarrow 0$. In this case, (13) requires $e = d$, that is, contact between journal and bearing. Eq. (13a) gives accordingly $M = Pd$ which we may also write in the form

$$(15) \quad M = rP \cdot \frac{d}{r}.$$

This becomes identical with our formula (1) for dry friction if we define the angle of friction in the present case by $\sin \alpha = d/r$.

c) We are also interested in the case where M is a minimum for given journal pressure P and given play d . When the differential quotient of the right member of (13a) with respect to e is set equal to zero, one obtains

$$-\frac{d^2}{e^3} + 2 = 0, \quad e = \frac{d}{\sqrt{2}}.$$

With this value of e Eq. (13a) gives for the moment of friction

$$(16) \quad M = \frac{2\sqrt{2}}{3} Pd = 0.94 Pd.$$

that is, only 6% less than the limit of M for $U \rightarrow 0$, which has been determined in (15).

These results are collected in the diagram of Fig. 63. Here the dimensionless quantity $y = M/Pd$ has been plotted as ordinate over the abscissa z , which is the reciprocal of the left side of (13), viz.

$$(17) \quad z = \frac{2\pi\mu U}{P} \frac{r^2}{d^2}^{10}$$

The solid curve represents the content of the two Eqs. (10) and (13), that is to say, the result of the elimination of the displacement e . It is best computed from the following parametric representation which is identical with (13a) and (13) or (10) and (13):

$$(18) \quad y = \frac{1 + 2\beta^2}{3\beta}, \quad z = \sqrt{1 - \beta^2} \frac{1 + \frac{1}{2}\beta^2}{3\beta}, \quad \beta = \frac{e}{d}.$$

In the same figure formulas (1) and (2) have been plotted as broken lines; in the present variables they simply read $y = 1$ and $y = z$, the tangents

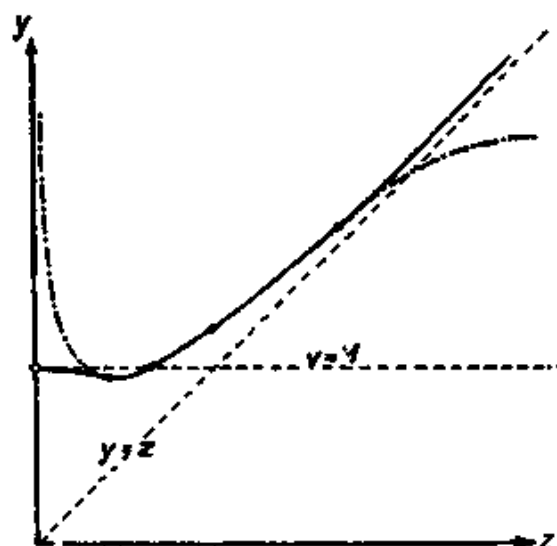


FIG. 63. Comparison between theory and experiment in terms of the dimensionless variables $y = M/Pd$ and $z = 2\pi\mu U r^2 / Pd^2$.

of the solid curve at $z = 0$ and $z = \infty$. The fact that our curve depends solely on the dimensionless quantity z expresses a *similarity law* which is valid in the hydrodynamic theory of lubrication: No matter what particular values of the dimensions r and d , the lubricant μ , the speed U , and the load P combine to give a certain z , it will always lead to the same $\beta = e/d$ and thus to the same value $y = M/Pd$ by (18).

When we now ask how well this theoretical result is confirmed by measurements on actual bearings, the answer is this: in the neighborhood of the minimum and to the right of it for not too large z -values we have not only qualitative but numerical agreement between theoretical and experimental values. One would then conclude that in this range the phenomenon of lubrication is determined entirely by the coefficient of viscosity μ , and that the laminar character of the motion, which we have assumed throughout this analysis, actually occurs. But for large and for very small z -values deviations from the theoretical curve occur, as indicated by the

¹⁰This expression is sometimes called the Sommerfeld variable in lubrication literature.

dotted branches in the figure. Since in these regions the similarity law is no longer valid, the proper representation should consist of a family of curves y vs. z depending on the variation of one of the parameters P , U etc.

The empirical curve does not rise nearly so steeply for large z -values as the theoretical curve, which has been interpreted to indicate that the motion of the lubricant is no longer laminar but turbulent.

For very small z -values the actual rise is very much steeper than it should be according to our theory. Here, qualities of the lubricant other than its viscosity become prevalent; they have been given the name of *oiliness* and refer to the adhesion of the lubricant to the metallic boundaries of the bearing. It is highly probable that the strength of adhesion be connected with the formation of monomolecular layers as studied by Langmuir, which originate in the parallel orientation of the chain molecules along the surface of the oil. However, a discussion of the physical problems of "boundary lubrication" would lead us far beyond the limits of classical hydrodynamics.

So far we have treated our problem as two-dimensional, assuming an *infinitely long* bearing. Introducing a finite bearing length means to impose new boundary conditions on pressure and velocity. For the designer the practical problem arises to regulate the inflow of the lubricant and its outflow at the ends of the bearing in such a way that the free bearing space is filled by the lubricant all along the bearing. There is also the difficulty of negative pressures. The variable part of p , denoted by p_1 , is an *odd* function of the angle φ , as we have seen. While p_1 rises in the narrowing sector between the journal and bearing wall, it must drop again to negative values when the point of proximity is passed. There is thus the danger that the layer breaks in the region of minimum p_1 , unless the constant part of p_0 of the total pressure p is large enough to make $p = p_0 + p_1$ everywhere positive. The quantity p_0 which depends on the inflow and outflow conditions of the lubricant is in our theory an arbitrary parameter; its proper choice is left to the skill of the designer.

One cannot expect an idealizing theory to do better than to establish general rules in an eminently practical problem like ours. Yet these rules may be quite important for the practical design as is born out by the success of Mitchell's segmental bearings.

37. Riemann's Shock Waves. General Integration of Euler's Equations for a Compressible Fluid in One-dimensional Flow

We have so far successfully avoided dealing with the quadratic terms in the equations of hydrodynamics assuming either infinitesimal amplitudes (wave motion) or "creeping" flow, in which case those terms were dropped

as part of the effect of inertia. In the present article, however, we shall finally attack the quadratic terms directly. Riemann¹¹ was the first to brave the challenge of this problem. Hugoniot, Hadamard and many others followed in his steps.

Riemann's method consists in transforming the *non-linear* hydrodynamic equations by an interchange of the dependent and independent variables; this leads to *linear* equations in the case of one-dimensional states. Riemann then shows how to integrate the resulting linear differential equations, and his method has become the model for the general integration problem of linear partial differential equations of second order of the corresponding (hyperbolic) type. This will be taken up in Vol. VI.

We first present a preliminary review of the results of Riemann's investigation: two waves proceed in opposite directions from each point of the initial disturbance in a similar way as in d'Alembert's solution of the corresponding problem for infinitesimal amplitudes (see Fig. 16, 13). Let the pressure be given as function of the density, $p = \varphi(\rho)$, for example in the polytropic form of (7.1); then we have according to Riemann:¹² "The velocity of propagation of both waves relative to the gas is $\sqrt{\varphi'(\rho)}$, but in space it is increased by the velocity of the gas flow measured in the direction of the propagation of the wave. If it is supposed that $\varphi'(\rho)$ does not decrease with increasing ρ , an assumption that holds in reality, stronger compressions must advance with larger velocities; it follows that the waves of compression will diminish in breadth and must finally become compression shocks".

So much for an introduction. We start now with the transition from the non-linear to the linear equations. Disregarding gravity or other external forces we choose the x -axis parallel to the direction of motion. If differentiations are indicated by the usual subscript notation the original hydrodynamic equations for a one-dimensional gas read:

$$(1) \quad \rho_t + u\rho_x + \rho u_x = 0,$$

$$(2) \quad u_t + uu_x + \frac{1}{\rho} p_x = 0.$$

In (2) p_x can be expressed by ρ_x since according to the assumption $p = \varphi(\rho)$

$$(3) \quad p_x = c^2 \rho_x, \quad c^2 = \frac{dp}{d\rho} = \varphi'(\rho).$$

As in (13.5) c is again the *sound velocity*; in the case of infinitesimal amplitude c could be considered as a constant, but now c is a function of ρ .

¹¹Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite. Abh. d. Gött. Ges. d. Wiss. 1860. Also Ges. Werke, p. 156.

¹²From Riemann's abstract of his paper in Göttinger Nachr. 1859, No. 19.

Eq. (2) can be rewritten by the use of (3):

$$(4) \quad u_t + uu_x + \frac{c^2}{\rho} \rho_x = 0.$$

Eqs. (1) and (4) are linear and homogeneous in the differentials of the independent variables x and t , while these variables themselves do not occur in the equations. On the other hand, the quantities ρ and u and their differentials occur in non-linear form. We therefore introduce with Riemann the pair x, t as dependent and the pair ρ, u as independent variables by putting

$$x = x(\rho, u), \quad t = t(\rho, u)$$

$$dx = x_\rho d\rho + x_u du, \quad dt = t_\rho d\rho + t_u du.$$

If these expressions for dx and dt are introduced in the differentials

$$du = u_x dx + u_t dt \quad \text{and} \quad d\rho = \rho_x dx + \rho_t dt,$$

the following identities result:

$$(5) \quad du = (u_x x_\rho + u_t t_\rho) d\rho + (u_x x_u + u_t t_u) du$$

$$(6) \quad d\rho = (\rho_x x_\rho + \rho_t t_\rho) d\rho + (\rho_x x_u + \rho_t t_u) du.$$

Since $d\rho$ and du are now independent we conclude from (5)

$$(7) \quad \begin{aligned} u_x x_\rho + u_t t_\rho &= 0 \\ u_x x_u + u_t t_u &= 1 \end{aligned} \quad \text{or} \quad u_x = \frac{t_\rho}{\Delta}, \quad u_t = -\frac{x_\rho}{\Delta},$$

where we assume that the functional determinant

$$(7a) \quad \Delta = \begin{vmatrix} x_u & t_u \\ x_\rho & t_\rho \end{vmatrix}$$

does not vanish. In the same way we conclude from (6)

$$(8) \quad \begin{aligned} \rho_x x_\rho + \rho_t t_\rho &= 1 \\ \rho_x x_u + \rho_t t_u &= 0 \end{aligned} \quad \text{or} \quad \rho_x = -\frac{t_u}{\Delta}, \quad \rho_t = \frac{x_u}{\Delta}.$$

Eqs. (7) and (8) represent the formulas for u_x, u_t, ρ_x, ρ_t , to be used in transforming (1) and (4). Substituting in (1) and (4), we obtain the system

$$(9) \quad x_u - ut_u + \rho t_\rho = 0, \quad x_\rho - ut_\rho + \frac{c^2}{\rho} t_u = 0.$$

The second equation can be transformed as follows:

$$\frac{\partial}{\partial \rho} (x - ut) = \frac{-c^2}{\rho} \frac{\partial}{\partial u} t = \frac{\partial^2 V}{\partial \rho \partial u},$$

where we have introduced a new dependent variable $V = V(\rho, u)$. The variables x and t are connected with V in the following way (c depends only on ρ):

$$(10) \quad x - ut = V_u, \quad t = -\frac{\rho}{c^2} V_\rho,$$

so that the second equation (9) is identically fulfilled. The new variable V is now introduced¹⁹ in the first equation (9); this gives

$$(11) \quad V_{uu} - \frac{\partial}{\partial \rho} \left(\frac{\rho^2}{c^2} V_\rho \right) = 0,$$

which is a *linear partial differential equation of second order* for the single unknown V .

This equation is simplified if we introduce instead of ρ the new independent variable

$$(12) \quad \xi = \int_0^\rho \frac{c}{\rho} d\rho; \quad \frac{d\xi}{d\rho} = \frac{c}{\rho}.$$

Writing at the same time η instead of u , we have the following relations:

$$\frac{\rho^2}{c^2} V_\rho = \frac{\rho^2}{c^2} V_\xi \frac{d\xi}{d\rho} = \frac{\rho}{c} V_\xi,$$

$$\frac{\partial}{\partial \rho} \left(\frac{\rho^2}{c^2} V_\rho \right) = \frac{c}{\rho} \frac{\partial}{\partial \xi} \left(\frac{\rho}{c} V_\xi \right) = V_{\xi\xi} + \frac{c}{\rho} V_\xi \frac{d}{d\xi} \frac{\rho}{c} = V_{\xi\xi} + V_\xi \frac{d}{d\rho} \frac{\rho}{c}.$$

We now assume the polytropic p, ρ -relation as in (7.1), such that

$$p = a\rho^n, \quad \frac{dp}{d\rho} = c^2 = na\rho^{n-1}, \quad \frac{c}{\rho} = \sqrt{na} \rho^{(n-3)/2}.$$

The relation between ξ and ρ and the expression $d(\rho/c)/d\rho$ is then

$$(12a) \quad \xi = \frac{2\sqrt{na}}{n-1} \rho^{(n-1)/2}, \quad \frac{d}{d\rho} \frac{\rho}{c} = \frac{3-n}{n-1} \frac{1}{\xi},$$

and our differential equation (11) reads in the new variables

$$(13) \quad V_{\xi\xi} - V_{\eta\eta} + \frac{k}{\xi} V_\xi = 0,$$

¹⁹From the first and second equation (10) one has respectively $x_u - ut_u - t = V_{uu}$ and $\rho t_\rho + t = -\partial/\partial \rho (\rho^2 V_\rho / c^2)$.

where for brevity we have put

$$(14) \quad k = \frac{3-n}{n-1}.$$

In the case $k = 0$, Eq. (13) reduces to the equation of the vibrating string, and we have d'Alembert's solution (13.11)

$$(15) \quad V = V_0 = F(\xi + \eta) + G(\xi - \eta),$$

where F and G are *arbitrary* functions that have only to meet certain continuity conditions. More generally, we have the following lemma due to Bechert:¹⁴ If W is a solution of the equation

$$(16) \quad W_{\xi\xi} - W_{\eta\eta} + \frac{l}{\xi} W_{\xi} = 0$$

then

$$(16a) \quad V = \frac{1}{\xi} \frac{\partial W}{\partial \xi}$$

is a solution of the equation

$$(16b) \quad V_{\xi\xi} - V_{\eta\eta} + \frac{l+2}{\xi} V_{\xi} = 0.$$

This can be shown simply by differentiating (16) partially with respect to ξ and putting $W_{\xi} = \xi V$ according to (16a). One obtains then

$$\frac{\partial^2}{\partial \xi^2} (\xi V) - \frac{\partial^2}{\partial \eta^2} (\xi V) + l V_{\xi} = 0$$

which becomes identical with (16b) on carrying out the differentiations and dropping the factor ξ .

Since the general solution (15) of Eq. (13) is known in the case $k = 0$, we have the general solution in the cases $k = 2, 4, 6 \dots$ in the following form:

$$(17) \quad V_2 = \frac{1}{\xi} \frac{\partial V_0}{\partial \xi}, \quad V_4 = \frac{1}{\xi} \frac{\partial}{\partial \xi} \frac{1}{\xi} \frac{\partial V_0}{\partial \xi}, \quad V_6 = \frac{1}{\xi} \frac{\partial}{\partial \xi} \frac{1}{\xi} \frac{\partial}{\partial \xi} \frac{1}{\xi} \frac{\partial V_0}{\partial \xi}, \quad \dots$$

These elementary integrable cases may be written comprehensively by setting k in (14) equal to an arbitrary even integer $2h$. The polytropic exponent (in the sense of p. 49) associated with h is found from (14) as

$$(18) \quad n = 1 + \frac{2}{2h+1}.$$

¹⁴K. Bechert, Zur Theorie ebener Störungen in reibungsfreien Gasen. Ann. d. Phys. 37 (1940) 89; 38 (1940) 1.

It is this rational form of n which makes Bechert's result so interesting since it discloses a surprising thermodynamic side view.

For an adiabatic change of state of a gas we have according to p. 50

$$(18a) \quad n = 1 + \frac{2}{f},$$

where f is the number of degrees of freedom of the molecule. For diatomic gases (e.g. for air) f equals 5, as already mentioned on p. 50. For monatomic gases f equals 3. In this case there are only the three degrees of freedom of *translation* since those of rotation cannot be excited. In present day physics this follows directly from a quantum-theoretical argument, while Boltzmann had still to make the rather artificial assumption that the monatomic molecules were spheres and the diatomic molecules were ellipsoids of revolution, in order to understand the cases $f = 3$ and $f = 5$. In these two cases there is

$$f = 2h + 1 \begin{cases} h = 1 & \text{for monatomic gases} \\ h = 2 & \text{for diatomic gases} \end{cases}$$

Thus it turns out that just the physically most interesting cases are distinguished in that they can be integrated¹⁸ by the elementary expressions V_2 and V_4 of Eq. (17).

It is difficult not to be impressed by what appears to be a "pre-established harmony" between mathematics and physics. An idea suggests itself that the performance of mathematics is adapted in some miraculous way to the exigencies of physics. Yet, the scope of the miracle is rather a limited one: For molecules consisting of three and more atoms such as H_2O or NH_3 , the number $f = 6$: three translatory degrees of freedom plus *three* of rotation. These gases do not belong to the integrable cases in this sense. On the other hand, the case $f = 1$, $k = 0$, which is the simplest integrable case, is not realized in nature; one would have to imagine a strictly "one-dimensional gas" whose atoms can only move in one direction and cannot rotate.

It would be a mistake to think that the integration becomes impossible unless n is one of the rational numbers defined in (18). To show this we¹⁹ write the sequence of equations in (17) once more in general form:

$$(19) \quad V_{2h} = \left(\frac{\partial}{\partial \xi} \right)^h V_0 \quad \text{with} \quad \xi = \frac{\xi^2}{2}, \quad \xi = \sqrt{2\zeta}.$$

¹⁸Mr. Bechert kindly informed me that this peculiar behavior of the cases (18) had been noticed before by A. E. H. Love and F. B. Pidduck. Cf. Trans. R. Soc. London 222, 167 (1922).

¹⁹Similarly as in Bechert's second paper quoted on p. 266.

According to a well known extension of Cauchy's integral formula in the theory of complex functions it is possible to express the h -fold differential quotient in (19) by a complex integral:

$$(20) \quad V_{2h} = \frac{1}{2\pi i} \frac{(-1)^h}{h!} \oint \frac{V_0(\sqrt{2w}, \eta)}{(w - \zeta)^{h+1}} dw.$$

Here h is supposed to be an integer number, and integration refers to a closed path in the w -plane, surrounding the point $w = \zeta$. The argument of V_0 as it is written in (20) indicates that the original dependence on ξ is to be replaced by that on $\sqrt{2w}$ while the dependence on η remains unchanged. However, in the representation (20) the fact that h is an integer is unessential. One may replace $2h$ by the arbitrary quantity k the specialization of which led us to $2h$, if one replaces at the same time

$$h + 1 \quad \text{by} \quad \frac{k}{2} + 1, \quad \frac{(-1)^h}{h!} \quad \text{by} \quad \frac{\exp\left(\frac{i\pi}{2} k\right)}{\Gamma\left(\frac{k}{2} + 1\right)}.$$

In this way one has instead of (20)¹⁷

$$(21) \quad V_k = \frac{1}{2\pi i} \frac{\exp\left(\frac{i\pi}{2} k\right)}{\Gamma\left(\frac{k}{2} + 1\right)} \oint \frac{V_0(\sqrt{2w}, \eta)}{(w - \zeta)^{(k/2)+1}} dw.$$

Evidently (21) satisfies the differential equation in the same way as (20); it represents the general solution since it contains two arbitrary functions F and G in the expression for V_0 under the integral sign. Our statement that just the physically interesting cases of the monatomic and diatomic gases are *integrable*, should now be put more precisely: they are distinguished from the other cases by the elementary form of their general integral.

We have limited this discussion to Riemann's case of *one* spatial coordinate. The generalization to two or three variables¹⁸ presents great difficulties. The differential equation which is the analogue of (13) in the general case has so far resisted all attempts at a general solution.

¹⁷The sign \oint does not mean here a *single* loop about the point $w = \zeta$, since this would not be a closed path because of the branch points of $(w - \zeta)^{(k/2)+1}$ at ζ and at infinity; the path must now be a double loop about $w = \zeta$ and $w = \infty$. Such paths occur in theory of functions at various occasions.

¹⁸K. Bechert, Ann. Phys. 39 (1941) 169 and 357, 40 (1941) 207. In the first of these papers a class of cylindrical and spherical waves is treated, in the second friction and heat conduction is taken into account.

In the foregoing we have only dealt with the formal side of the problem. Although this presents enough interest, it is now time to turn to the *physical aspect* of the motion as it was described above in Riemann's summary. For this purpose we do not need the integral V in its completeness, but go back to the definition of the variables ξ and η [see Eq. (12) and after]. We put with Riemann

$$(22) \quad r = \frac{1}{2} (\xi + \eta) = \frac{1}{2} \left(\int_0^p \frac{c}{\rho} d\rho + u \right),$$

$$s = \frac{1}{2} (\xi - \eta) = \frac{1}{2} \left(\int_0^p \frac{c}{\rho} d\rho - u \right),$$

and note that the assumption of a polytropic p, ρ -relation is immaterial for what follows. The quantities r and s represent in our particular case the *characteristics* of the differential equation (13) and play a decisive part in the general theory of differential equations of this type. (For further details see Vol. VI).

From (22) we calculate the derivatives

$$(23) \quad r_t = \frac{1}{2} \left(\frac{c}{\rho} \rho_t + u_t \right), \quad r_x = \frac{1}{2} \left(\frac{c}{\rho} \rho_x + u_x \right).$$

(differentiations with respect to the original variables x and t are again represented by subscripts.) A glance at the expression

$$(24) \quad r_t + (u + c)r_x = \frac{c}{2\rho} \left\{ \rho_t + (u + c)\rho_x + \frac{\rho}{c} u_t + \frac{\rho}{c} (u + c)u_x \right\}$$

shows that the right member vanishes because of the equations of motion (1) and (4). If we compute in the same way s_t and s_x from (22) and form the expression $s_t + (u - c)s_x$ we find again the right member zero as before. Both results may be indicated in the following way:

$$(25) \quad \frac{\partial}{\partial t} \left\{ \begin{matrix} r \\ s \end{matrix} \right\} + (u \pm c) \frac{\partial}{\partial x} \left\{ \begin{matrix} r \\ s \end{matrix} \right\} = 0.$$

If we form the total differentials dr and ds with respect to x and t and express the partial derivatives r_t and s_t by r_x and s_x respectively according to (25), we obtain

$$dr = r_x dx - (u + c)r_x dt \quad \text{and} \quad ds = s_x dx - (u - c)s_x dt.$$

These are differential identities between the original independent variables x and t and the variables r and s , which are valid if u and ρ (of which c is a known function) fulfill the differential equations (1) and (4). Consider

now the point x varying in course of time. The total derivatives of r and s with respect to t , if $x = x(t)$, are

$$(26) \quad \frac{dr}{dt} = \left(\frac{dx}{dt} - u - c \right) r_x, \quad \frac{ds}{dt} = \left(\frac{dx}{dt} - u + c \right) s_x.$$

These relations can be interpreted as follows: at a point (*not a particle*) that travels in such a way that

$$(27) \quad \text{either} \quad \frac{dx}{dt} = u + c \quad \text{or} \quad \frac{dx}{dt} = u - c$$

the quantities r or s keep their values. In other words, a certain value of r respectively s travels with the variable velocity $u + c$ respectively $u - c$. As long as $|u| < c$, the first moves in the positive the second in the negative x -direction.

As an example take the case of an initial pressure disturbance restricted to a short interval of the x -axis. Compression waves issue in both directions from the interval if we assume an initial disturbance in the form of a pressure *increase*. However, the two wave profiles do not preserve their shape as in Fig. 16; the wave front becomes steeper since larger densities travel faster according to (27). Hence it is understood that the compression waves finally become compression *shocks*. At this point our calculation which disregards friction and heat conduction, ceases to be valid. The extensions of the theory that are necessary in the case of shocks have been investigated by Hugoniot and Hadamard.¹⁰

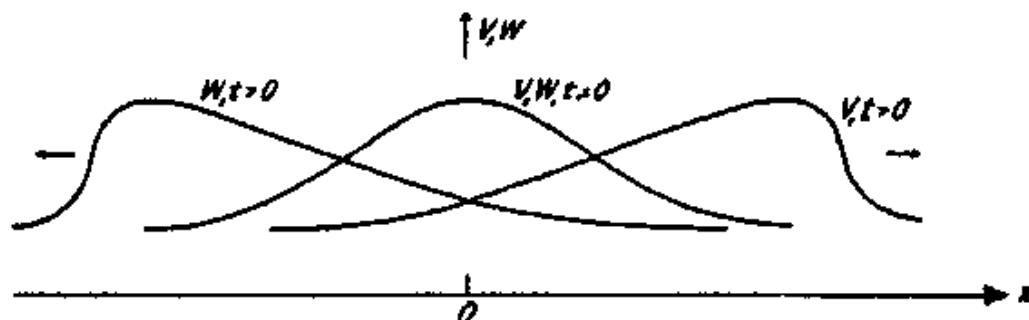


FIG. 64. The waves caused by an initial disturbance progress in both directions and are deformed in this process. The notations V and W of the figure have the same meaning as the parameters of the characteristics, r and s , as functions of x and t . In comparing Figs. 64 and 16, note that the central curve in the present figure represents the disturbances V and W *individually*, while the corresponding *sum* is represented in Fig. 16.

This may suffice for an exposition of Riemann's results which we quoted on p. 263. Fig. 64 has been taken from Bechert's first paper quoted

¹⁰Cf. the comprehensive representation by G. Zemplén in Enc. d. Math. Wiss., Vol. IV.3, Art. 19, which is to be supplemented by a more recent paper of R. Becker, Z. Phys. 8 (1922) 321.

on p. 266. It refers to the simplest case of a strictly "one-dimensional" gas (see p. 267) which shows, as Bechert points out, the characteristic traits of wave propagation in spite of the extreme idealization.

38. On Turbulence

With the present report which is divided in a number of sections we enter the most difficult part of hydrodynamics. Under (A) we have collected a series of *half empirical, half theoretical statements* about the properties of turbulent flow. Inasmuch as they are of a theoretical nature, they are based on the Stokes-Navier equations and the diffusion of momentum which these equations can be considered to express (see p. 76).

Under (B) we approach the fundamental question whether it is at all possible to explain the observed facts on the basis of the Navier-Stokes equations. In particular, one asks whether these equations give account of the instability of the laminar motion that sets in at a certain Reynolds number. Earlier attempts to settle this question by the method of small oscillations failed; it had been impossible to find a critical limit, and the laminar motion appeared to be stable at any value of R . The method of small oscillations had been applied to the fully developed laminar flow, that is to the Hagen-Poiseuille flow and to the Couette flow with the aim of investigating the stability of a flow in a pipe and between plates, respectively.

Considering these negative results, such experts in the field of turbulence as Th. v. Kármán and Sir Geoffrey Taylor have occasionally in their papers favored the idea that turbulence like gas theory can only be understood by statistical methods. The underlying physical fact is that a fluid is a system of very many (in principle infinitely many) degrees of freedom, all of which come into play at large Reynolds numbers. The motion becomes then so complicated that one can expect simple answers only to those questions that concern the *average* behavior of the moving particles. This is especially true for the limiting case of the so-called isotropic turbulence; some preliminary remarks according to hitherto unpublished work of C. F. v. Weizsäcker and W. Heisenberg will be made on this topic under (D) where we are concerned with what one may call fully developed *turbulent equilibrium*.

Previous to that we shall briefly discuss under (C) investigations of *the process of generation of laminar flow in its relation to turbulence*: the processes at the inlet in pipe flow experiments and the formation of Prandtl's boundary layer when the plate is set in motion. This was Prandtl's own view of the problem, and was soon taken up successfully by Tollmien, Tietjens, Görtler, and Schlichting, all of whom belong to Prandtl's school.

In these investigations the method of small oscillations is not applied to the velocity profile of the developed laminar flow, but to the intermediate profiles that occur while the laminar flow comes into existence. The significance of the velocity profile for the stability of the flow had been anticipated in much older investigations of Lord Rayleigh which, however, dealt only with inviscid fluids. The more recent investigations of Prandtl's group start from the complete Navier-Stokes equations (i.e. in their abbreviated form for *small* oscillations) and actually result in stability limits, which, to be sure, do not refer to the final laminar motion but to the way of its generation.

Under (E) we shall bring a mathematical example of a system of differential equations that lead to phenomena similar to those of turbulence in fluids. It is a non-linear system that resembles the Navier-Stokes equations in structure, but is much simpler than those. This example, which is due to I. M. Burgers, may strengthen our belief that turbulence does belong to classical hydrodynamics, although to a part of it that presents great difficulties to the mathematical attack.

A. Some Properties of Turbulent Flow

Empirical data on the turbulent state in pipes and channels are very extensive, having been compiled by hydraulic engineers over a long time. In organizing this material two principles have to be observed: The flow must be referred to equal Reynolds numbers and the wall roughness must be taken into account. As to the second requirement we shall restrict ourselves to *smooth* walls so that we do not have to digress too far into technological details. "Smooth" is, of course, meant in the sense of the limit: wall roughness $\rightarrow 0$. Under these conditions one obtains the general result: *In the fully developed turbulent state the velocity distribution in pipes with geometrically similar cross-sections is similar*, or, the ratio of the velocity at a certain point to the velocity average over the entire cross-section is independent of the Reynolds number.

Fig. 65 shows this velocity distribution for a circular cross-section in

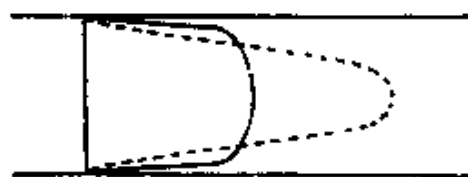


FIG. 65. The velocity profile in a circular pipe for the turbulent and laminar states of flow (solid and dotted lines respectively).

comparison with that of the Hagen-Poiseuille flow. In the latter case we had a parabolic velocity profile; the velocity at the center being equal to twice the average velocity. In the turbulent case the velocity at the axis of the tube is only a little larger than the average velocity, which has

been assumed the same for both cases in the diagram. Only in the proximity of the wall the velocity drops quickly to zero; a narrow wall zone develops in this region, where the flow is more or less laminar, and the viscosity the important quantity.

The term velocity as it is used here with reference to the turbulent state is to be understood in the sense of the *time average of the velocity*. The individual particle velocity at a given point is by no means constant in time. The *spectrum* of its temporal and spatial variability covers a wide range; it has been possible to obtain more detailed information about it in a characteristic limiting case [see section (D)]. We denote the time average of the velocity by U , a velocity in the direction of the axis of the pipe (x -axis); the additional components of the individual velocities are denoted by u, v, w . Likewise, P stands for the time average of the pressure and p is the individual instantaneous pressure difference relative to P .

In addition to the law of turbulent velocity distribution we have the law of the turbulent *pressure drop*. In the laminar case the pressure gradient was proportional to the velocity, in our case it is proportional to the square or in better approximation to the $7/4^{\text{th}}$ power of the velocity (Blasius Law, see p. 120). When this law is expressed according to (16.11) and (16.13) by means of the similarity invariants R and S , it reads

$$\frac{\Pi}{l} = \frac{\rho v^3}{a} S, \quad S = \lambda R^{-1/4},$$

where a is the radius of the pipe and λ a numerical coefficient. Written in our present notation, it takes the form

$$(1) \quad -\frac{dP}{dx} = \frac{\lambda \rho U^2}{a} R^{-1/4} = \frac{\lambda \rho}{a} \left(\frac{\nu}{a}\right)^{1/4} U^{7/4}.$$

This equation shows that the coefficient of viscosity is no longer of decisive importance for the state of the flow; its influence expresses itself by the factor $\nu^{1/4}$ and is rather weak. Another kind of energy loss prevails which is connected with inertia effects and becomes evident in the factor ρ , or rather $\rho^{3/4}$ if the dependence of the kinematic viscosity ν on ρ is also considered.

Prandtl and Kármán were able to deduce the velocity distribution in the wall zone from (1), using certain additional assumptions. Consider a fluid element of thickness dx extending over the pipe cross-section: The difference of the average pressures P acting on the two faces of the element must be balanced by the shear forces acting between fluid and pipe wall. Thus we have

$$-\frac{dP}{dx} dx \pi a^2 = \tau_0 dx 2\pi a,$$

or, according to (1),

$$(2) \quad \tau_0 = -\frac{dP}{dx} \frac{a}{2} = \frac{\lambda \rho}{2} \left(\frac{y}{a}\right)^{1/4} U^{7/4}.$$

Let now $y = a - r$ be the distance between a point inside the wall zone and the wall. According to Prandtl and Kármán there is only one way to construct a dimensionally correct and physically meaningful expression that approximates²⁰ the shear stress τ at the point y , viz.

$$(2a) \quad \tau = \text{const } \rho \left(\frac{y}{y_0}\right)^{1/4} U^{7/4}.$$

From (2a) one obtains directly

$$(3) \quad U = \text{const} \left(\frac{y}{y_0}\right)^{1/7} \left(\frac{\tau}{\rho}\right)^{4/7}.$$

Hence the velocity inside the wall zone increases from zero at the wall with the $1/7^{\text{th}}$ power of the distance from the wall. This agrees well with experimental results. Kármán was also able to develop from the same assumptions a formula for the heat conduction in the wall zone, which is in good agreement with the observations.

We shall now discuss the original approach to the turbulence problem made by Reynolds²¹ and, in connection with his work, by H. A. Lorentz.²² We restrict ourselves to a steady main flow along the x -direction with a pressure gradient $-dP/dx$ on which a non-steady quasiperiodic additional motion is superimposed. Let the state be two-dimensional, i.e. depending only on x and y . The main flow depends then only on y , the turbulent additional motion has the components u and v that depend on t , x and y . We shall consider the time averages of the total velocity \mathbf{v} and of the total pressure Q , that is of the quantities

$$(4) \quad v_x = U + u, \quad v_y = v, \quad Q = P + p$$

for which we stipulate

$$(5) \quad \bar{v}_x = U, \bar{v}_y = 0, \bar{Q} = P, \text{ hence } \bar{u} = 0, \bar{v} = 0, \bar{p} = 0.$$

²⁰Note that (2a) does not give (2) for $y = 0$, but becomes infinite. According to Kármán (Z. angew. Mathem. u. Mech. 1, 1921, p. 233) the reason is that (2a) is only asymptotically valid (i.e., for infinitely large Reynolds numbers).

²¹In the paper quoted in footnote 18 on p. 114.

²²Akad. Amsterdam, Zittingsverlag, 6. p. 28, 1897 and Ges. Abhandlungen, Teubner 1907, p. 43.

This implies

$$\begin{aligned} \overline{v_x v_x} &= U^2 + 2U\bar{u} + \bar{uu} = U^2 + \bar{uu} \\ (5a) \quad \overline{v_x v_y} &= U\bar{v} + \bar{uv} = \bar{uv} \end{aligned}$$

and

$$(5b) \quad \frac{\partial}{\partial x} \overline{v_x v_x} = \frac{\partial}{\partial x} \bar{uu}, \quad \frac{\partial}{\partial y} \overline{v_x v_y} = \frac{\partial}{\partial y} \bar{uv}.$$

Furthermore, in a quasiperiodic motion the averages of the time derivatives such as $\partial u / \partial t$ vanish:

$$(5c) \quad \overline{\frac{\partial u}{\partial t}} = 0;$$

and because of (4) and (5)

$$(5d) \quad \frac{\partial U}{\partial t} = \frac{\partial U}{\partial x} = 0, \quad \overline{\nabla^2 u} = 0, \quad \overline{\nabla^2 v} = 0, \quad \overline{\frac{\partial p}{\partial x}} = 0.$$

We now write the equation of motion in the abbreviated vectorial form (16.1) and set $\mathbf{F} = 0$:

$$(6) \quad \rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \text{ grad})\mathbf{v} = -\text{grad } Q + \mu \nabla^2 \mathbf{v}.$$

As a consequence of the condition $\text{div } \mathbf{v} = 0$ we have the identity

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{\partial}{\partial x} v_x v_x + \frac{\partial}{\partial y} v_x v_y$$

which permits us to write the x -component of (6) in the form

$$(6a) \quad \rho \frac{\partial v_x}{\partial t} + \rho \frac{\partial}{\partial x} v_x v_x + \rho \frac{\partial}{\partial y} v_x v_y = -\frac{\partial Q}{\partial x} + \mu \nabla^2 v_x.$$

Forming the time average of this equation we obtain with the use of (4), (5), and (5a, b, c, d)

$$(7) \quad \frac{\partial}{\partial x} \rho \bar{uu} + \frac{\partial}{\partial y} \rho \bar{uv} = -\frac{\partial P}{\partial x} + \mu \nabla^2 U.$$

The quantities on the left side are gradients of *friction pressures* which appear here alongside with the gradient of the normal pressure P . To be in accordance with our system of notations (10.4) these quantities ought to be denoted as follows:

$$(7a) \quad \frac{\partial}{\partial x} p_{xx} + \frac{\partial}{\partial y} p_{xy} \quad \text{where} \quad p_{xx} = \rho \bar{uu}, \quad p_{xy} = \rho \bar{uv}.$$

Now it should be remembered that the term $\mu \nabla^2 U$ to the right is also due to friction pressures; this was pointed out in Eqs. (10.6)-(10.8b). But the previous p_{ix} originate in the *molecular* disorder of fluid particles on a microscopic scale, while the present p_{xx} , p_{xy} , in (7a) are caused by the *molar* disorder of fluid domains of macroscopic size engaged in turbulent motion. In either case the physical origin of the friction pressures is the *transfer of momentum*. Take as an example the molar friction pressure p_{xy} : momentum is transferred from the motion in x -direction to the side motion y , a process that cannot remain without influence on the momentum balance of the main flow U . In this sense we should interpret the appearance of the terms (7a) on the left side of Eq. (7) of the main flow. But the average pressures (7a) do not lend themselves to the ordinary methods of differential analysis; this is the reason why in this approach to the turbulence problem recourse is taken to statistical methods.

Of the two friction pressures p_{xy} and p_{xx} the first is the important one, hence we shall indicate the contribution of the second only by \dots . Let us also adopt in what follows the usual terminology in this branch of hydrodynamics and speak of frictional *shear stress* rather than of frictional pressure; we write accordingly

$$(8) \quad \tau = \sigma_{xy} = -p_{xy} = -\rho \overline{uv}$$

Instead of Eq. (7) we have then

$$(9) \quad \frac{\partial P}{\partial x} = \mu \nabla^2 U + \frac{\partial \tau}{\partial y} + \dots,$$

For $\nabla^2 U$ we may also write $d^2 U/dy^2$ since U is a function of y alone. Eq. (9) is the *determinative equation of the main flow*.

We obtain the equation of the x -component of the turbulent additional motion by introducing (4) into (6) and by subsequent reduction with the help of (9) and (5d):

$$(10) \quad \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + v \frac{\partial U}{\partial y} = -\frac{1}{\rho} \left(\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \right) + \frac{\mu}{\rho} \nabla^2 u + \dots$$

Here the sign \dots covers also the terms of second order in u and v . There is a corresponding equation for the y -component v of the turbulent motion and a second equation (9) for $\partial P/\partial y$.

Quantitative information about the character of the turbulence can hardly be expected from these very involved equations. We have mentioned them here mainly for the purpose of later reference in section (E) and to point out the significance of the molar friction stress τ . The latter concept leads to the idea of a length l characteristic for the turbulence, viz. the *mixing length*, introduced by Prandtl in 1926; it corresponds to the notion of the mean free path in gases:

Prandtl's argument²³ is this: Suppose a "fluid lump" of average velocity $U(y)$ is shifted from y to $y + l$; in terms of the additional motion, this means a fluctuation of approximately

$$u = l \frac{\partial U}{\partial y}.$$

It is a reasonable assumption that the fluctuation of the side motion v in the mixing process will be of the same order of magnitude. This admitted, one obtains for τ according to (8) an expression of the following form

$$(11) \quad \tau = \rho l^2 \left| \frac{\partial U}{\partial y} \right| \frac{\partial U}{\partial y}.$$

If this is introduced in (9), the result is a differential equation that depends only on the quantities of the main flow and can be used for its analysis. However, the results obtainable in this way require continual experimental checking because of the hypothetical character of the relation (11),

B. Older Attempts at a Mathematical Theory of Reynolds' Turbulence Criterion

Even before Reynolds started his theoretical investigations, Lord Kelvin²⁴ had treated the problem of flow stability by the method of small oscillations, but did not obtain definite results. Twenty years later the problem was attacked again; the object at this time was to study the stability of the form of the Couette flow²⁵ represented in Eq. (16.14), which is mathematically so very simple. If a small disturbance (u, v) with stream function ψ is superimposed on the main flow U , which has a rectilinear velocity profile, the resulting differential equation of fourth order

$$(12) \quad \frac{\partial \nabla^2 \psi}{\partial t} + U \frac{\partial \nabla^2 \psi}{\partial x} - \frac{d^2 U}{dy^2} \frac{\partial \psi}{\partial x} = \nu \nabla^2 \nabla^2 \psi$$

that describes the problem in the general case, splits up, because of $d^2 U/dy^2 = 0$, into

$$(12a) \quad \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \nu \nabla^2 \right) \chi = 0, \quad \nabla^2 \psi = \chi.$$

One resolves now ψ and χ in Fourier components by putting

$$(12b) \quad \psi = f(y)e^{i(\alpha x - \beta t)}, \quad \chi = F(y)e^{i(\alpha x - \beta t)}.$$

²³Strömungslehre, Vieweg 1942, p. 105.

²⁴Phil. Mag. 24, 188 and 272 (1887).

²⁵W. M. F. Orr, Proc. Irish Acad. 27 (1907). — A. Sommerfeld, Internat. Mathem. Congress, Rome, 1908, Vol. III, p. 116.

The resulting ordinary differential equations for f and F are of second order and can be solved by known functions over the whole range $0 < y < h$ (Bessel functions of order $\frac{1}{2}$ and $\frac{3}{2}$). The boundary conditions at $y = 0$ and $y = h$ give a *transcendental equation* for β in the form of a certain second order determinant that must vanish if non trivial values of the integration constants are to be obtained. If for a given real α this equation has a solution with positive-imaginary part, then *instability* prevails in the sense that disturbances of the wave number α increase in time. The *critical Reynolds number* would thus be determined by that α -value for which the imaginary part of β vanishes for the first time.

However, Mises²⁶ was able to deduce from the nature of the transcendental equation that such a value of β cannot exist, or that the straight Couette flow is seemingly stable toward all disturbances α . This conclusion has been confirmed by Hopf²⁷ in a more direct way by a detailed discussion of the corresponding types of oscillations. Moreover, it seems that according to Noether²⁸ the stability proof can be extended to all flows between two plane parallel plates (unless special discontinuities occur in the initial velocity profile).

Detailed calculations in the sense of Eqs. (12) and (12b) have been carried out by Sexl²⁹ not only for the flow between plane plates but also for the Couette flow proper³⁰ between coaxial cylinders, and likewise for the Hagen-Poiseuille flow. In every case the transcendental equation associated with the problem leads to the conclusion that the flow is stable.

Thus we have striking contradiction between theory and experiment. What conclusions should we draw? Should we suspect the method of small oscillations that has proved reliable in all other domains of mechanics including astronomy? Should we rather assume that for such a proof of instability *finite* instead of (infinitely) small disturbances ought to be resorted to just because the object of investigation is the laminar motion?

Or should we blame the Navier-Stokes equations as inadequate for our problem? This does not seem justifiable either, particularly since in the last analysis they form the foundation for all previous theoretical statements, including Reynolds's laws of similarity.

²⁶R. v. Mises, Heinrich Weber-Festschrift, 1912, p. 112 (Teubner), Jahresbericht d. D. Mathem. Vereinigung 21 (1912) 241.

²⁷L. Hopf, Ann. d. Phys. 43 (1914) 1.

²⁸F. Noether, Z. angew. Mathem. u. Mech. 6 (1926) 232.

²⁹Th. Sexl, Ann. d. Phys. 83 (1927) 835, 84 (1927) 807, 87 (1928) 570.

³⁰In the more refined experiments on Couette flow performed by Sir G. Taylor (cf. p. 121) the deviation from the laminar motion occurring beyond the stability limit turned out to be helical (hence three-dimensional) rather than axially-symmetric.

C. Reformulation of the Turbulence Problem; the Origin of the Turbulence

It was possible to find a way out of these difficulties when investigators directed their attention, as Prandtl had done it, to the *manner in which turbulence comes into being* in a given flow arrangement (cf. also our discussion of pipe flow experiments on p. 118). Here again the Couette flow, in which the flow field is bounded by a plane plate moving in x -direction, proved simpler than the pipe flow. At the beginning of the motion a boundary layer develops along the plate; the other plate bounding the flow field may be supposed at infinite distance, for the time being. The fluid that fills the space between the boundary layer and that plate is considered at rest or in laminar motion according to a preassigned law $U(y)$. The decisive factor for the stability is the velocity profile of the boundary layer, as was first recognized by Tollmien.³¹ Tollmien's stability criterion (Gött. Nachr. 1935) states that velocity profiles with inflexion points, such as occur in a boundary layer with positive pressure gradient, are extremely likely to become unstable. To characterize the investigations that aim in this direction (determination of the eigenvalues and approximate calculation of the eigenfunctions in the strip between the plates) we quote a paper³² of Schlichting: "The main flow is not assumed to have a fully developed linear velocity distribution; one rather investigates one by one the velocity profiles that occur between two parallel plane walls when one wall is suddenly set in motion, changing from a state of rest to one of constant velocity. A boundary layer then forms along the moving wall, the thickness of which increases with time, and if the main flow stays laminar, its velocity profile approaches asymptotically a straight line. In contrast to this, the earliest among these transient profiles have finite critical Reynolds numbers whose magnitudes depend on the shape of the profile. The actual Reynolds number of the Couette flow is then obtained as the lowest stability limit during the initial period".

If the problem is formulated in this way, its solution is of course much more difficult than the problem contained in our Eqs. (12). It can only be solved by the use of step by step approximations, carried out by numerical and graphical methods, which we can only mention here. They lead to a definite, practically meaningful value of the stability limit which depends, however, on how the turbulent state was produced.

³¹W. Tollmien, Über die Entstehung der Turbulenz, Göttinger Nachr. 1929.

³²H. Schlichting, Ann. Phys. 14 (1932) 905. He refers to previous work done by other authors in Prandtl's group: O. Tietjens, Z. angew. Mathem. Mech. 5 (1925) 200, W. Tollmien, l. c., cf. also later papers of Schlichting in Gött. Nachr. 1932, 1933, 1935.

To illustrate once more the changed view point we could perhaps say: Not the instability of the laminar flow as such is proved as it was tried under (B), but it is shown that the road toward it is blocked by unstable states. Thus the instability is not an inherent quality of the final state but of its *history*.

D. The Limiting Case of Isotropic Turbulence

In view of this complicated state of the problem one can ask whether there exist ideal flow conditions that are approached by the turbulent fluid in the limit of infinitely increasing Reynolds number; this final state could be called turbulent equilibrium. We suppose that no spatial direction is preferred and thus speak of *isotropic turbulence*. In the state of isotropic turbulence the space of the fluid is divided into turbulence elements or turbulence cells of ever changing size that are arranged in some statistical way. The eddies in the cells follow the Navier-Stokes equations, but are too complicated for complete mathematical treatment. Thus the equilibrium between the different turbulence elements can only be described statistically. One asks for the probability that a certain size of the turbulence elements occurs. This brings us back to the search for the spectrum of the turbulence elements which was already mentioned on p. 271.

The kinetic theory of gases may serve as an approximate model of this theory. Although the mathematically exact state of the gas (position and velocity of the gas molecules) is entirely determined by the history of the gas (i.e., the preceding states), one is only interested in the *statistically most probable distribution* (sometimes also in the probability of a given deviation from that distribution) which is given by the Maxwell-Boltzmann law. Similarly in the turbulence problem, where one is interested in the *statistically most probable distribution* of the vortices, which can be described in the form of a statement about the frequency of vortices of a certain size and a certain rotational speed.

Weizsäcker obtains the velocity spectrum of the isotropic turbulence by a similarity consideration in which turbulence cells of different sizes are compared under the assumption of a quasisteady state (i.e., its change in time is small compared with the time needed to establish statistical equilibrium). A further assumption is the random nature of the state. As a result the following spectral law is obtained: The mean velocity of a particle relative to its surrounding neighbors varies as *the third root of its diameter*. As a consequence, the energy density in function of the wave number is proportional to the $-5/3$ power of the wave number. However, these laws are only valid in the middle part of the spectrum, which part is very extended in the case of large Reynolds numbers. For *small wave*

numbers (large turbulence elements that are comparable in size with the container) the statistical treatment must be replaced by the consideration of the individual hydrodynamic boundary value problem; for very large wave numbers (small turbulence cells) the Navier-Stokes friction causes the spectral intensity to drop to zero very quickly.

Heisenberg substantiates the same results by the use of Fourier analysis and supplements them by the calculation of the factors of proportionality, by a more exact investigation of the limiting cases of small and large wave numbers, and by the determination of the pressure fluctuations.

For all details and in particular, for the comparison with the experiment we refer to the papers of the two authors.*

E. A Mathematical Model to Illustrate the Turbulence Problem

To J. M. Burgers who had been trying to explain the facts of turbulence on an statistical basis, we owe a mathematical model³³ from which properties analogous to those observed in actual turbulence can be deduced with mathematical rigor. The simplified system of equations which Burgers puts tentatively in place of preceding Eqs. (9) and (10), is

$$\frac{dU}{dt} = P - u^2 - \nu U, \quad (13)$$

$$\frac{du}{dt} = Uu - \nu u.$$

These equations are non-linear like the actual equations of turbulence (u^2 occurs in the first and Uu in the second equation), each equation contains a dissipative friction term [νU in the first and νu in the second equation, corresponding to the terms $\mu \nabla^2 U$ in (9) and $(\mu/\rho) \nabla^2 u$ in (10)]. The constant P in (13) replaces the pressure gradient dP/dx in (9) and, like it, represents a driving force. The independent variables are U and u as before; U is the main motion, u the turbulent additional motion. However, Eqs. (9) and (10) refer to a domain of three independent variables x, y, t , while there is only *one independent variable* t in (13).

*C. F. v. Weizsäcker, Das Spektrum der Turbulenz bei grossen Reynoldsschen Zahlen; W. Heisenberg, Zur statistischen Theorie der Turbulenz, Z. f. Phys., 124, (1948) p. 608-657. See also W. Heisenberg, On the theory of statistical and isotropic turbulence, Proc. R. Soc., 195, (1948) p. 402, and Th. v. Kármán and C. C. Lin, The statistical theory of turbulence, Advances in Applied Mechanics, II, Academic Press, New York, 1950.

³³Mathematical Examples illustrating relations occurring in the theory of turbulent fluid motion, Akad. Amsterdam 17 (1939)1.

The non-linear terms in (13) have been chosen in such a way that they drop out when the expression for the "energy" is formed; for we require the permanence of the general rule, according to which two equations of the form (13) should be added after multiplication with U and u respectively, and so would yield the energy expression. In our case one obtains

$$(13a) \quad \frac{1}{2} \frac{d}{dt} (U^2 + u^2) = PU - \nu(U^2 + u^2).$$

In the right member, the first term is the work per unit of time performed by the driving force P which, supposedly, acts only upon the main motion; the second term is the energy dissipated by friction.

There exists a "laminar" solution of (13), viz.

$$(14) \quad U = U_0 = \frac{P}{\nu}, \quad u = 0.$$

It is stable as long as

$$(14a) \quad P < \nu^2.$$

This can be seen directly if the method of small oscillations is applied. One puts

$$U = U_0 + \xi, \quad u = \eta$$

and neglects higher powers of ξ and η . Then, by (13), one has

$$(14b) \quad \frac{d\xi}{dt} = -\nu\xi$$

$$\frac{d\eta}{dt} = (U_0 - \nu)\eta.$$

The first of these equations represents an exponentially decreasing disturbance; this is also true for the second equation but only if $U_0 < \nu$; hence by (14) we have condition (14a). For $U_0 > \nu$ and, therefore, $P > \nu^2$, Eq. (14) is still a solution of (13), but it is unstable because of the second Eq. (14b).

There exist, however, two solutions for $P > \nu^2$ which we characterize as turbulent, since they include a contribution of the additional motion. They are

$$(15) \quad U = \nu, \quad u = u_0 = \pm \sqrt{P - \nu^2}.$$

Both solutions are stable for $P > \nu^2$, for, if one puts

$$U = \nu + \xi, \quad u = u_0 + \eta$$

and again neglects higher powers of ξ and η in accordance with the method of small oscillations, (13) becomes

$$(15a) \quad \frac{d\xi}{dt} = -\nu\xi - 2u_0\eta,$$

$$\frac{d\eta}{dt} = u_0\xi.$$

If one sets up a trial solution in the usual form

$$(15b) \quad \xi = Ae^{\lambda t}, \quad \eta = Be^{\lambda t},$$

one obtains the following conditions which determine λ and B/A :

$$\lambda A = -\nu A - 2u_0 B,$$

$$\lambda B = u_0 A;$$

they yield on elimination of B/A

$$(15c) \quad \lambda^2 + \nu\lambda = -2u_0^2$$

$$\lambda = -\frac{\nu}{2} \pm \sqrt{\frac{\nu^2}{4} - 2u_0^2} = -\frac{\nu}{2} \pm \sqrt{\frac{9}{4}\nu^2 - 2P}.$$

The roots λ are *negative real* for

$$\nu^2 < P < \frac{9}{8}\nu^2$$

and turn complex with *negative real part* when

$$P > \frac{9}{8}\nu^2.$$

The disturbance (15b) thus attenuates exponentially for any $P > \nu^2$, which means that the solution (15) is *stable* for these values of P . The stability limit lies at $P = \nu^2$ where one of the roots (15c) vanishes. The same value of P is the stability limit of the laminar solution and thus reflects Reynolds criterion of stability. It is true that, in contrast to the hydrodynamic reality, in our case both the laminar and the turbulent motion are steady, i.e., independent of t , but the example should be taken as a model and does not attempt to explain the turbulence. (There is also this difference that in the model *two* forms of turbulent motion are equally possible [see (15)], while in reality the laws of turbulent motion appear to be *uniquely* determined by the Reynolds number.)

Following Burgers, we give a graphic representation of the results by drawing in Figs. 66 and 67 a u , U -diagram of the two curves

$$(16) \quad \frac{dU}{dt} = 0 \quad \text{and} \quad \frac{du}{dt} = 0.$$

The equations are given by (13), viz.

$$(16a) \quad u^2 + \nu U = P \quad \text{and} \quad (16b) \quad u = 0, U = \nu \text{ respectively}$$

Eq. (16a) is a parabola, Eq. (16b) a pair of lines consisting of the U -axis and a parallel line to the u -axis. Since we have got two steady solutions in (14) and (15), their representations in our diagram are the intersection

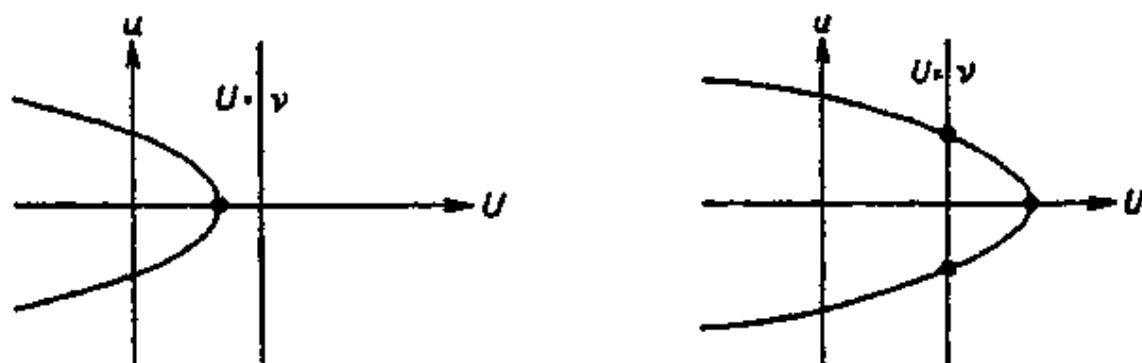


FIG. 66 (left). Diagram of Burger's turbulence model in the "laminar" case $P < \nu^2$.

FIG. 67 (right). Diagram of Burger's turbulence model in the "turbulent" case $P > \nu^2$.

points of (16a) and (16b). In Fig. 66 the parabola lies entirely to the left of the line $U = \nu$, hence there is only one intersection point; it belongs to the solution (14) and is unstable. There are three intersection points in Fig. 67. One of them lies on the U -axis. It belongs to solution (14) that is now unstable; the two others belong to the stable solution (15).

Burgers³⁴ extended his method of "mathematical models" in several steps in the attempt to bring it closer to the actual behavior of a turbulent fluid. He considers a system of three simultaneous differential equations in which the main "flow" U is coupled with two additional motions, u and v , and permits these motions to depend not only on time but also on a spatial coordinate. This we shall not discuss here since the simple model of Eq. (13) in our opinion illustrates the mathematical situation sufficiently: *By the non-linear character of the equations the occurrence of (two) entirely different types of motion is made possible.* These types, which we again distinguish by the terms "laminar" and "turbulent" are in some ways analogous to those of the turbulence problem; they branch off from one another at a certain stability limit, which we can compare with Reynolds stability criterion.

³⁴Advances in Applied Mechanics, ed. R. v. Mises and Th. v. Kármán, Vol. I., Academic Press, New York, 1948.

SUPPLEMENTS TO THE THEORY OF ELASTICITY

39. *Elastic Limit, Proportional Limit, Yield Point, Plasticity, and Strength*

Up to this point we have assumed the solid body to be homogeneous and isotropic, and only permitted small deformations. In the following discussion we shall consider very briefly the principal deviations which appear under more general conditions.

Let us consider a cylindrical steel bar which is clamped at one end and subject to a tensile force P acting at the other end. We shall first refer the tensile stress σ to the unloaded cross section of the bar F_0 , as is customary in engineering practice. Thus

$$(1) \quad \sigma = \frac{P}{F_0}$$

or the stress is proportional to the load P .

If we plot the measured strains ϵ as abscissas to the tensile stresses σ calculated according to (1) as ordinates, we obtain for loads that produce very small deformations ϵ a curve which is practically an exact straight

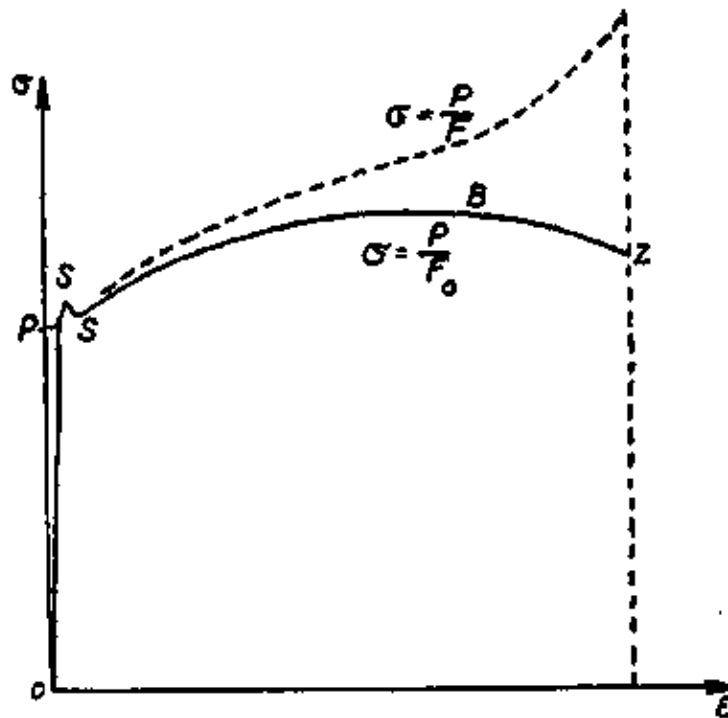


FIG. 68. Stress-strain diagram of a tensile test with a tough material (steel).

line (see solid curve in Fig. 68). This is valid up to a point P that is called the proportional limit.

The elastic limit designates a point on the diagram, below which a decrease or removal of the load produces a decrease or removal of strain. For steel, the elastic limit lies slightly below the proportional limit.

Beyond the proportional limit, the stress-strain line becomes curved and reaches a maximum, which is called the upper yield point (also flow or plasticity limit) S_u . The curve drops now to the lower yield point S_l ; following the point S_l the curve remains in the mean parallel to the ϵ -axis over a short distance, usually presenting a few small waves in this section, the behavior of a given sample depending upon the composition and previous treatment of the steel.

This transition region is followed by the plastic or flow region in which permanent deformations occur. A small change of load causes a rapid change of strain, so that in this range the curve first rises slowly with little curvature only, turning its concave side towards the ϵ -axis. The maximum ordinate B corresponds to the ultimate load or ultimate stress, but fracture occurs only at the point Z , where the stress-strain curve ends abruptly. Note in addition that

1. the bar undergoes lateral contraction everywhere and that
2. in the neighborhood of the place where fracture is to occur the reduction of the cross-section is particularly large; the constriction or "neck" formed there becomes quite noticeable once the point B in the diagram is passed. If we refer the load to the (variable) smallest cross-section F , that is, if we put

$$(2) \quad \sigma = \frac{P}{F}$$

and plot the tensile stress so calculated against the measured strain, we obtain the dashed curve in the plastic region. Within the elastic region, the two curves obviously coincide.

Steel is the most important example of *tough* materials, while cast iron is typical for *brittle* materials. There is no plastic range in the latter case and no constriction either. Fracture follows almost immediately at the end of the sharply limited elastic range. According to earlier experiments, Hooke's law did not seem to hold for cast iron altogether, but Grüneisen¹ has shown that for sufficiently small strains proportionality exists between stress and strain.

Materials that can be formed into a given shape like loam, clay, and lead show hardly any elastic properties, but exhibit plastic behavior throughout.

¹E. Grüneisen, Ber. d. Deutsch. Phys. Ges. 1906.

Similar results as in the tension test described above are obtained in compression, bending, and torsion tests. It should be noted, however, that pure compression causes fracture only in brittle but not in tough materials. It is further assumed that in all these experiments the load is applied slowly, since the deformation depends not only upon the load, but also upon the duration of the application of the load.

In connection with this last point we mention the *elastic aftereffect*. Even in the elastic range the deformation connected with a particular load assumes its final value only after a considerable time. We speak of the *creep* of the material in question.

In analogy to the behavior of ferro-magnetic bodies we find also an *elastic hysteresis*. As in the magnetic case, the area enclosed by the hysteresis loop in the ϵ , σ -plane represents the input of deformation energy per cycle.

In the previous discussion the load was assumed constant or slowly varying. Of more importance, especially in machine design, is the case of periodic loading, where the stress oscillates above and below a mean value. In such cases fracture occurs at a much lower loading than in the case of stationary load; one speaks of the *fatigue* of the material.

When we plot the number of load cycles until fracture, z , against the average stress σ_m as ordinate, we obtain a monotonically decreasing curve (so-called Wöhler diagram), which approaches a horizontal asymptote σ_{∞} as z increases. This indicates that the average stress must be kept smaller than the value of σ_{∞} (fatigue limit) in order to prevent a fatigue failure with certainty.

When the load that finally causes the fracture has been applied *slowly*, the cross section at the fracture will exhibit coarse grained structure; in the case of failure under oscillating load, however, a portion of the broken cross-section will be fine grained, possibly showing the spread of an imperfection (see below). The remaining portion of the section has the same appearance as in an ordinary fracture. We may conclude from this that the first part contributes only little to load bearing capacity, so that the remainder simply fails in a fracture by overload. Moreover, we know that the number of cycles z depends greatly upon the quality of the surface of the specimen. Small invisible cracks reduce z considerably. It may be mentioned in this connection that Voigt's measurements of the breaking strength of sodium chloride crystals gave only 1/400 of the value calculated from the lattice theory. In this case, too, small cracks and surface irregularities are the cause of the discrepancy, since tests made under water, where the crystal surface is smoothed out in the process of dissolution, gave 3/4 of the theoretical value of the breaking strength.

Solid bodies used in engineering are mainly metals and metal alloys

composed of small crystal grains. Fine grained layers of foreign substances are interspersed between the crystals of the basic material. These intermediate layers can transfer considerable cohesive forces and give the solid body a certain degree of homogeneity. But in the same layers are also found places of lesser strength, imperfections of the structure from which a fracture develops. Organic materials like wood and leather, are composed of chains of crystal-like cells, in which other materials are imbedded. Glass, on the other hand, is an example of an amorphous material.

Concerning the crystalline structure of metals we know today that every crystal is a spatial lattice. The elementary units of the crystal lattice are the atoms of the metal, which are arranged in lattice planes and, within these planes, in lattice lines. The lattice planes can be arranged in three families of parallel planes which are not necessarily orthogonal to each other.

At small loads in the proportional range, the space lattice suffers a continuous deformation without essential changes in shape; the deformation disappears as the load is removed. Within the plastic range, the crystal deforms with a change of shape. The lattice planes of that family which contains the most densely occupied lattice lines, slide in the direction of these lattice lines. In this process the intermediate layers are also distorted.

40. Crystal Elasticity

The entire field of physical phenomena in anisotropic bodies has been expertly surveyed in "Lehrbuch der Kristallphysik" by Waldemar Voigt.² In this work the most general relations are developed that are compatible with the symmetry properties of the crystal and with the geometrical characteristic of the physical phenomenon under consideration, which may be a scalar field, a vector field or a tensor field of second or higher order (cf. p. 60). In crystal elasticity we are concerned with the relation between stress and strain tensor. Since both are symmetrical tensors of second order and have six components, the most general linear stress-strain relation will depend upon $6 \times 6 = 36$ coefficients.

This number will reduce, however, to

$$(1) \quad 6 + \frac{6 \times 5}{1 \times 2} = 21,$$

when the strain energy W is a *function of state*, as Green first postulated.

²Taubner 1910, second impression 1928. Beside dielectric and conductive properties such phenomena as pyroelectricity and piezoelectricity are treated; the latter has gained special interest in the meantime. Crystal optics, however, has been considered beyond the scope of this work.

We have pointed out (p. 73) that this assumption is justified both in the cases of an isothermal process (problems of equilibrium) and of an adiabatic change (rapid oscillations). In these cases then, which alone will interest us in the following dW is a total differential. Replacing for the present our general notation σ_{ik} and ϵ_{ik} by a single subscript notation we set³

$$(2) \quad \begin{aligned} \sigma_{11} &= \sigma_1, & \sigma_{22} &= \sigma_2, & \sigma_{33} &= \sigma_3, & \sigma_{23} &= \sigma_4, & \sigma_{31} &= \sigma_5, & \sigma_{12} &= \sigma_6 \\ \epsilon_{11} &= \epsilon_1, & \epsilon_{22} &= \epsilon_2, & \epsilon_{33} &= \epsilon_3, & 2\epsilon_{23} &= \epsilon_4, & 2\epsilon_{31} &= \epsilon_5, & 2\epsilon_{12} &= \epsilon_6. \end{aligned}$$

Then the double sum of Eq. (9.24) becomes

$$(3) \quad dW = \sum_{k=1}^6 \sigma_k d\epsilon_k$$

and the general stress-strain relation appears in the form

$$(4) \quad \sigma_k = \sum_{i=1}^6 s_{ki} \epsilon_i.$$

where the s_{ik} are now the moduli of elasticity.⁴ If (3) is to be a total differential, we have

$$\frac{\partial \sigma_k}{\partial \epsilon_i} = \frac{\partial \sigma_i}{\partial \epsilon_k},$$

which according to (4) means

$$(5) \quad s_{ki} = s_{ik}.$$

When we write the coefficients s in a quadratic array of 6 rows, it follows that they are *symmetrical with respect to the diagonal* s_{11}, \dots, s_{66} . This immediately leads us to the count in Eq. (1). The same is of course true for the elasticity constants c .

Let us insert here the remark that the older molecular theory of Navier, Cauchy, and Poisson based on the idea of central forces between molecules led to a further reduction of that number to 15. We know, however, that this theory is too narrow in the case of the isotropic body. It would assign the universal value $\frac{1}{2}$ to the Poisson ratio ν in equation (9.9) and in this way reduce the required number of 2 elastic constants for

³The strains $\epsilon_1, \epsilon_2, \epsilon_3$ introduced in (2) are defined in the same way as Kirchhoff's y_1, y_2, y_3 (cf. the table on p. 600). At this point the omission of the factor $\frac{1}{2}$ which was introduced in our definition of the ϵ_{ik} with a view to the general structure of tensor algebra proves advantageous.

⁴This notation is due to Voigt who uses it throughout (cf. footnote on p. 64). The coefficients in the strain-stress relations obtained by solving Eqs. (4) for the quantities ϵ are denoted by c_{ik} according to Voigt (constants of elasticity).

isotropic bodies to 1. Moreover, modern crystal lattice theory, which was already mentioned at the end of the previous article, has shown that the cause of this difficulty lay not so much in the molecular theory as such, as in its too narrow formulation: since the crystal in general consists of several superposed lattices which can move with respect to each other in the elastic deformation, the concept of pure central forces becomes inadequate.

The number 21 mentioned previously refers to a crystal without symmetry, that is to a triclinic crystal. The state of symmetry of a physical process acting on a crystal is determined according to the following principle (Voigt, L.d.K., 53): *The symmetry of the physical process is superimposed on the symmetry of the crystal.* Now, the elastic deformation has a center of symmetry since the strain in any one direction is the same as in the opposite direction. On the other hand, there are two possible cases of symmetry relations in a triclinic crystal, one with, and one without center of symmetry. In the latter case, however, the missing center of symmetry is supplied, as one might say, by the center of symmetry of the strain. Both sub-groups of the triclinic system therefore behave in the same way as far as elasticity is concerned.

There exists a total of 32 groups of symmetry^{4a} which make up the 7 crystal systems, viz. the triclinic, monoclinic, orthorhombic, tetragonal, trigonal, hexagonal, and cubic systems. The principle quoted before implies in our case that the first three systems and the last two are uniform in regard to their elastic constants while the tetragonal as well as the trigonal systems must be divided each into two subclasses with different schemes of elastic constants. We shall give a few examples below of these $5 + 2 \times 2 = 9$ different schemes.

The crystal axes are generally designated by a, b, c . In the triclinic system the axes have different lengths and arbitrary directions. In the monoclinic system, one of the axes is perpendicular to the other two and represents an axis of symmetry for rotations through 180° . If we choose this axis as z -axis of an orthogonal coordinate system x, y, z , and call the elastic displacements ξ, η, ζ , a rotation through 180° about the z -axis will cause the changes

$$(a) \quad x, y, z \rightarrow -x, -y, z, \quad \xi, \eta, \zeta \rightarrow -\xi, -\eta, \zeta;$$

the associated changes in the strains are according to (2) on the one hand

$$(b) \quad \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_6 \rightarrow \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_6$$

^{4a}As is well known, they are distinguished from each other according to their symmetry relative to a point. The more basic classification according to the spatial symmetry character that leads to 230 groups need not interest us here.

on the other hand

$$(c) \quad \epsilon_4, \epsilon_5 \rightarrow -\epsilon_4, -\epsilon_5;$$

One verifies easily that the expressions

$$\epsilon_4 = \frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y} \quad \text{and} \quad \epsilon_5 = \frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}$$

change their signs in the operation (a), while for example the expression

$$\epsilon_6 = \frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}$$

keeps its sign. Since the stresses σ behave in the same way as the strains ϵ , they must follow the same commutation rules (b) and (c). We now rewrite the stress-strain relations (4), marking the terms whose signs change in the operation (a) by \pm and indicating the remaining terms on the right side by \dots . In this way we obtain for $k = 1, 2, 3, 6$

$$(d) \quad \sigma_k = \dots \pm s_{k4} \epsilon_4 \pm s_{k5} \epsilon_5 \dots,$$

but for $k = 4, 5$

$$(e) \quad \pm \sigma_k = \dots \pm s_{k4} \epsilon_4 \pm s_{k5} \epsilon_5 + \dots.$$

For fixed k , (d) represents two equations which, on subtraction, lead to the following relation:

$$(f) \quad s_{k4} \epsilon_4 + s_{k5} \epsilon_5 = 0 \quad \text{for} \quad k = 1, 2, 3, 6.$$

Similarly, the pair (e) gives on addition

$$(g) \quad s_{k1} \epsilon_1 + s_{k2} \epsilon_2 + s_{k3} \epsilon_3 + s_{k6} \epsilon_6 = 0 \quad \text{for} \quad k = 4, 5.$$

From (f) we conclude

$$(6) \quad 0 = s_{14} = s_{24} = s_{34} = s_{64} = s_{15} = s_{25} = s_{35} = s_{65},$$

and the same result follows from (g) because of the relation $s_{ik} = s_{ki}$.

On the basis of (6) the behavior of the moduli of elasticity in the transition from the triclinic to the monoclinic system has now been established:

Triclinic system

no symmetry axis

21 constants

s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}
	s_{22}	s_{23}	s_{24}	s_{25}	s_{26}
		s_{33}	s_{34}	s_{35}	s_{36}
			s_{44}	s_{45}	s_{46}
				s_{55}	s_{56}
					s_{66}

Monoclinic system

one digonal axis of symmetry A_z^2	s_{11}	s_{12}	s_{13}	0	0	s_{16}
13 constants		s_{22}	s_{23}	0	0	s_{26}
			s_{33}	0	0	s_{36}
				s_{44}	s_{45}	0
					s_{55}	0
						s_{66}

In the orthorhombic system where a , b , and c are mutually orthogonal another digonal axis of symmetry A_z^2 is added to A_x^2 ; the axis A_y^2 which is also present can be considered as a consequence of the two other axes and must not be taken into separate account.

By repeating the argument that leads to equation (6) one obtains in the present case

Orthorhombic system

two digonal axes of symmetry	s_{11}	s_{12}	s_{13}	0	0	0
A_x^2 and A_z^2 , 9 constants		s_{22}	s_{23}	0	0	0
			s_{33}	0	0	0
				s_{44}	0	0
					s_{55}	0
						s_{66}

Similarly, one has for the

Cubic system

three interchangeable tetragonal	s_{11}	s_{12}	s_{12}	0	0	0
axes of symmetry, 3 constants		s_{11}	s_{12}	0	0	0
			s_{11}	0	0	0
				s_{44}	0	0
					s_{44}	0
						s_{44}

From the last scheme we conclude that the cubic single crystal has an elastic behavior *different from the isotropic body*. The former possesses only

two independent moduli of elasticity, for example, our previous μ and λ . In order to adapt the above scheme to that of equation (9.6), we would have to specialize the three constants as follows:

$$s_{11} = 2\mu + \lambda, \quad s_{12} = \lambda, \quad s_{44} = \mu,$$

that is, we should add the following condition which is not fulfilled in the cubic single crystal:

$$s_{11} = s_{12} + 2s_{44}.$$

From a physical point of view this is connected with the fact that the so called isotropic solid body has no uniform structure, but is a polycrystal. The elastic behavior which we actually observe is simpler than that of the single crystal, but it is only the result of averaging the elastic reactions of a very large number of single crystals that are in random positions with respect to each other. For an example we refer to the metals discussed in the previous article.

Quite different is the answer to the corresponding question in the field of optics. Cubic single crystals of fluorite or rock salt are frequently preferred as lense materials for optical instruments in certain ranges of the spectrum, where they are superior in transmissivity to ordinary isotropic (amorphous) glass lenses. This is possible, because the cubic single crystal is optically isotropic. In the framework of Voigt's general system the cause for this different behavior must be seen in the following distinction: in elasticity the basic relation is a tensor-tensor relation (stress-strain), in optics it is a vector-vector relation (electric field- dielectric displacement).

We cannot further pursue the interesting details of crystal elasticity. Our principal aim was to characterize the elastic behavior of isotropic bodies as contrasted with that of the uniform anisotropic bodies.

41. The Bending of Beams

Loaded and bent beams are among the principal elements of all building constructions. In planning a new structure bending loads are calculated and must be legally approved. The theory employed for this purpose is simple and has been in use for 200 years; its origin dates back to Daniel Bernoulli. We shall discuss below how this theory can be substantially justified through more rigorous methods (St.-Venant), but we shall first follow the historical development.

We limit ourselves to the simplest case of a *straight* beam subject to slight bending. The cross-section of the beam is arbitrary (one may think of a *T* or *I* section), but it is assumed as constant over the length of the beam.

Let us first disregard the weight of the beam. At one end the beam supports a vertical load, while the other end is clamped rigidly; thus the upper fibers of the beam are stretched and the lower ones compressed.

Bernoulli assumes that each plane cross-section of the beam remains plane also after bending. Navier concludes from this that the normal stresses transmitted by a cross-section are distributed according to a *linear law*. We consider two neighboring sections originally separated by the distance dx (x is taken from right to left as indicated in Fig. 69a). The

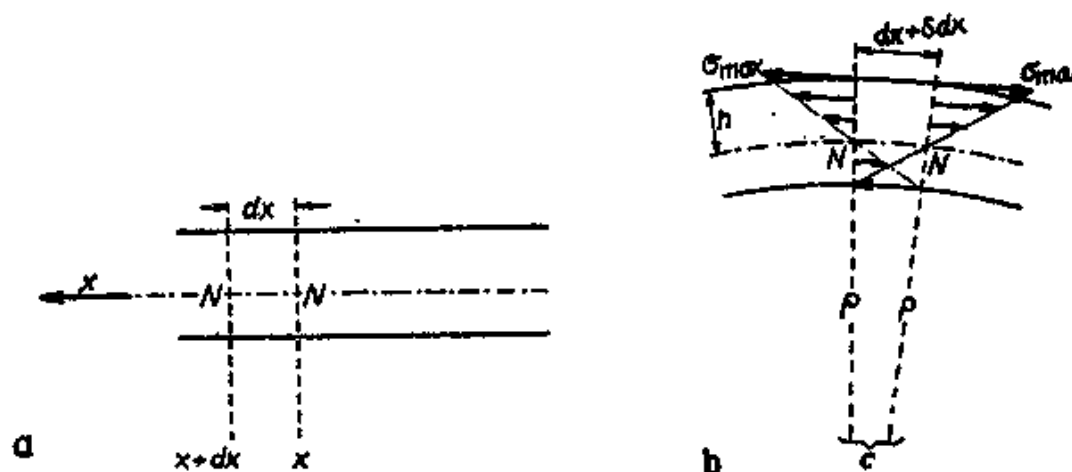


FIG. 69. Illustrating the elementary bending theory. An element of a beam (a) in the original state and (b) in the bent state.

fibers running between the two cross-sections have the length $dx + \delta dx$ after bending. Due to the assumed planeness of both cross sections, the extension of the fibers decreases linearly from the upper face of the beam downward and becomes zero at some particular layer NN ; δdx now stays negative down to the lower face of the beam and represents the contraction of the fibers. The strains are distributed in the same linear way.

$$\epsilon = \frac{\delta dx}{dx},$$

and so are the stresses [cf. (9.8)]:

$$(1) \quad \sigma = E\epsilon.$$

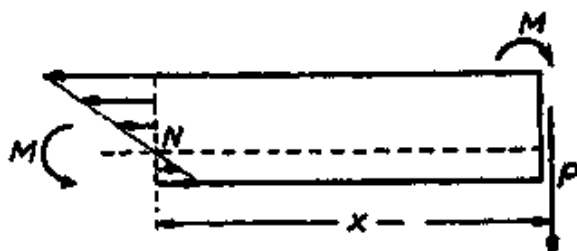
When σ_{\max} is the stress in the (upper or lower) boundary fiber and h its distance from NN , (cf. Fig. 69b), the linear law states that

$$(2) \quad \sigma = \frac{\sigma_{\max}}{h} y \cdots y = \text{distance between fiber with stress } \sigma \text{ and } NN$$

In order to determine the maximum stress σ_{\max} , we shall investigate the equilibrium of forces acting on an isolated beam section of length x (cf. Fig. 70); in doing so we may disregard the deflection of the beam. We

are mainly concerned with the *equilibrium of moments* taken about an axis normal to the plane of Fig. 70 and containing the point N at the cross-section x , but we shall also consider the *equilibrium of force components in the horizontal x -direction*. Equilibrium of forces in the vertical y -direction will be considered later.

FIG. 70. Equilibrium between the bending moment M and the moment of the normal stresses σ .



For the particular loading here assumed the moment of the external forces, usually designated as *bending moment* M , is

$$(3) \quad M = Px.$$

According to Eq. (2) the forces σdf have the moment

$$(4) \quad \int y \sigma df = \frac{\sigma_{\max}}{h} I, \quad I = \int y^2 df.$$

The quantity I is called the *moment of inertia of the cross-section*. In Vol. I the moment of inertia of a plane mass distribution has already been introduced and a distinction made between *polar* and *equatorial* moments of inertia. Here, however, we are dealing with a plane area and not with a plane mass distribution, hence the dimension of I is not gm^2 but rather cm^4 . Since the axis through N for which I is computed lies within the cross-section, I is to be considered as an *equatorial moment of inertia*. When the opposite direction of the moment arrows shown in Fig. 70 is considered, the moment equilibrium requires according to (3) and (4)

$$(5) \quad \sigma_{\max} = \frac{hM}{I}.$$

This relation together with (2) determines the stress σ for any other point of the cross-section.

The engineer is particularly interested in σ_{\max} since it is his task to keep the maximum occurring stress below the *legally admissible limit*, σ_{adm} . He has to find the cross-section with the largest loading, that is, with the largest bending moment, and proportions the beam in such a way that in this cross-section the stress at the boundary face $\sigma_{\max} < \sigma_{\text{adm}}$. For beam profiles used in engineering, the values of I and I/h are tabulated in tables of moments of inertia and section moduli. For a circular cross-section of radius a we have⁶ (cf. Vol. I, Problem IV. 1)

⁶Cf. Synge and Griffith, *op. cit.*, p. 220.

$$(6) \quad I = \frac{1}{4} P a^2 = \frac{\pi}{4} a^4;$$

for a rectangular cross-section of width b and depth $2h$

$$(6a) \quad I = \frac{2}{3} b h^3.$$

In both cases we have anticipated that the axis $y = 0$ passes through the *centroid* of the area of the circle or rectangle, being parallel to b in the latter case.

To show this in general we have to use the *equilibrium condition for force components in x -direction*. The external forces do not contribute anything to the equilibrium equation, neither in our particular case where the external force P is vertical, nor in general, since otherwise the problem would not be one of pure bending but of combined bending and tension or compression. The condition is then simply

$$\int \sigma df = 0,$$

or, because of (2),

$$\int y df = 0.$$

On the other hand, the *centroid* of the cross-sectional area is defined by the fact that the first moment⁶ of the area with respect to its centroid is zero, equivalent to $\int y df = 0$. The fiber passing through the consecutive centroids of all cross-sections of the beam is called its *neutral fiber* or *neutral line*. The horizontal plane containing the points $y = 0$, is known as the *neutral plane*. This has already been indicated by the notation NN .

We now determine the shape of the neutral line in its deflected position. For this purpose we have to set up the *differential equation of the elastic line*, which can be derived on the basis of Fig. 69b: Here the center of curvature C of the elastic line NN has been indicated as the intersection of the prolonged traces of the cross-sections at x and $x + dx$. The sides of the triangle CNN are ρ , ρ , and dx where ρ is the radius of curvature. If we draw through one of the points N a parallel to the trace of the other cross-section we obtain a triangle similar to CNN the third side of which is the face of the beam. The sides of this triangle are h , h , $\delta dx_{\max} = \epsilon_{\max} dx$. Hence the following proportion holds:

$$\frac{dx}{\rho} = \frac{\epsilon_{\max} dx}{h}.$$

⁶Sometimes the term static moment is used which is not a very appropriate expression. Likewise, second moment should be preferred to moment of inertia.

By the use of (1) and (5), it may also be written

$$(7) \quad \frac{1}{\rho} = \frac{M}{EI}.$$

As to the choice of the sign in this equation we refer to Fig. 69b for the sake of brevity and omit the introduction of sign rules for ρ . The right hand side is a known function of x or, more generally, a *known point function along line NN*. (The second formulation refers to finite deflections where the abscissa must be replaced by the arc length s of the elastic line.) The left hand side of (7) is the curvature, a differential invariant of the curve; thus equation (7) is the differential equation required.

For sufficiently small deflections we know that approximately

$$\frac{1}{\rho} = \frac{d^2y}{dx^2},$$

and consequently

$$(8) \quad \frac{d^2y}{dx^2} = \frac{M(x)}{EI}.$$

Referring to the particular loading (3), we obtain

$$(9) \quad \frac{d^2y}{dx^2} = \frac{P}{EI} x$$

and after integration

$$(9a) \quad y = \frac{P}{EI} \left\{ \frac{x^3}{6} + Ax + B \right\},$$

which is a cubic parabola. The integration constants A and B result from the conditions at the point of support:

$$(9b) \quad \text{for } x = l \left\{ \begin{array}{l} y = 0 \\ \frac{dy}{dx} = 0 \end{array} \right\} \cdots A = -\frac{l^2}{2}, \quad B = \frac{l^3}{3}.$$

The maximum deflection occurs at $x = 0$:

$$(9c) \quad y_0 = \frac{1}{3} \frac{P}{EI} l^3.$$

We consider now a beam simply supported at each end by a pair of uprights at distance l from each other, and loaded only by its own weight. The (constant) weight per unit length being p , the reactions at each support will be $-pl/2$, where the minus sign indicates the upward direc-

tion of these forces. The bending moment at any point is composed of the moment of the weight and the moment of one of the end reactions:

$$(10) \quad M(x) = \frac{p}{2} x(x - l).$$

Integration of (8) then gives

$$(10a) \quad y = \frac{p}{2EI} \left\{ \frac{x^4}{12} - \frac{x^3 l}{6} + Ax + B \right\},$$

a parabola of the fourth order.

Instead of the conditions (9b) we have now

$$\left. \begin{array}{l} \text{for } x = 0 \\ \text{for } x = l \end{array} \right\} y = 0 \dots \quad B = 0; \quad A = \frac{p}{12}.$$

The largest deflection occurs, of course, at the central cross-section and amounts to

$$(10b) \quad y_{\max} = \frac{5}{16 \times 24} \frac{p}{EI} l^4.$$

Of more interest for the following is the case where a *couple* acts at the end of the beam instead of a single force as in Fig. 70. This could be physically realized by two equal forces of opposite direction, acting in close proximity to each other and having such magnitude that their moment has a finite value. Then M can be considered as constant and according to (8)

$$(11) \quad y = \frac{M}{EI} \left\{ \frac{x^2}{2} + Ax + B \right\}.$$

If the beam is clamped at the other end ($x = l$) we have the same conditions as in (9b), and it follows that $A = -l$, $B = \frac{1}{2}l^2$. The deflection at $x = 0$ is then

$$(11a) \quad y = \frac{1}{2} \frac{M}{EI} l^2.$$

In this case the shape of the elastic axis can be more accurately described than by the parabola (11): according to (7) it is a circular arc, since its curvature is constant.

This last case is the only one in which we are led to a clear-cut result by our equilibrium conditions. The *third equilibrium condition* which we have to stipulate, viz.

$$(12) \quad \text{Sum of all vertical components} = 0$$

is here satisfied, whereas this condition requires in general that at every cross-section x a vertical shear force opposes the external loads. This shear force results from the combined action of all shear stresses of the section under consideration. If we take the beam of Fig. 70 as an example, shear stresses σ_{xy} would have to occur in addition to the normal stresses σ_{xx} , so that for every cross-section

$$(13) \quad \int \sigma_{xy} df = P.$$

This, however, would imply the occurrence of angular changes γ_{xy} , causing the originally plane cross-section to warp. In other words, Bernoulli's hypothesis, in asserting the permanent planeness of the cross-sections, contradicts the general fundamentals of statics. *Thus it would seem that Navier's law of linear variation of the normal stress which is based on Bernoulli's hypothesis, is also invalidated.*

Before we show a way to overcome this difficulty, we wish to determine the magnitude of the required shear stresses. The integral condition (13) is not suitable for this purpose, but we can make use of the differential relation (8.11). In the absence of external body forces and with σ_{xx} evidently vanishing if for instance a rectangular cross-section is assumed, this relation may be written

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0.$$

Notwithstanding the previous critical remarks, the statements of equations (2), (5) and (3) may be considered as approximations. We have then

$$(14) \quad \frac{\partial \sigma_{xx}}{\partial y} = \frac{-Py}{I}, \quad \sigma_{xy} = \frac{P}{2I} (A - y^2) = \frac{P}{2I} (h^2 - y^2).$$

The integration constant A in the last equation must equal h^2 , so that $\sigma_{xy} = 0$ for $y = \pm h$. This must be required since the value of σ_{xy} at the upper or lower edge of the cross-section determines the shear stress σ_{yx} acting within the face of the beam; the face, however, must be completely free of stress. It is not difficult to determine the warping of the rectangular cross-section that results from the σ_{xy} and the associated shear angle γ_{xy} .

We now wish to point out briefly how the bending theory can be freed from the internal contradiction revealed above. St.-Venant⁷ accomplished this in 1855. First of all, it must be emphasized that the relations between stress and external forces used so far are not sufficient to describe the

⁷A complete representation is found in de Saint-Venant's translation of Clebsch's lectures: *Théorie de l'élasticité des corps solides*, trad. par Saint-Venant and Flamant, Paris, 1883.

elastic state completely. One must make recourse to the elastic displacements ξ , η , ζ , as they occur in the fundamental elastic equations (9.18). Then we note that nowhere in the preceding argument we have mentioned the stresses σ_{yy} , σ_{zz} , σ_{yz} . In fact, they do not enter into the consideration of the equilibrium of a bent beam and can therefore be set equal to zero. If we express them in terms of ξ , η , ζ , we obtain simple differential relations between these quantities by the use of which the equations (9.18) can be greatly simplified and finally come out as:⁸

$$(15) \quad \left\{ \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial y \partial z} \right\} \frac{\partial \xi}{\partial x} = 0.$$

From this we conclude first: the strain $\partial \xi / \partial x$ of the x fibers and consequently also their stress $\sigma = \sigma_{xx}$ is a linear function of the coordinates in the cross-section y and z . The linear distribution law for σ_{xx} is thus a necessary consequence of the fundamental laws of elasticity and Bernoulli's hypothesis is not required for its derivation.

Eq. (15) shows further that the extension of the x -fibers is also a linear function of x ; this implies together with Eq. (5) that the bending moment M should be a linear function of x , which is actually the case for one or several single loads or single moments (but not for the continuously distributed load of the weight of the beam itself). The following restriction should be noticed, however: the single load ought to be applied in such a way that it can be balanced by the shear forces σ_{xy} that act across the cross-section at which the load acts; in other words, the load should be distributed over the cross-section in the same way as previously computed for σ_{xy} . This is, of course, never fulfilled in reality. We may, however, take the following view: whether we have a lumped or a distributed load can only make a difference in the neighborhood of the cross-section under consideration, provided, of course, the loads in both cases are statically equivalent. This can be demonstrated by suitably chosen experiments as well as by theoretical examples and is often formulated as a general theorem under the name of *Saint-Venant's principle*.

Thus the simple method of computing beams set forth at the beginning of this article that has given satisfactory results in innumerable cases has been vindicated from a more rigorous point of view.

42. Torsion

We first consider the simplest case of the torsion of a rod with circular cross-section, which, in a way, is also the most important case. We

⁸Cf. for this calculation A. Föppl, Vorl. üb. techn. Mechanik, 4th ed. Leipzig, 1909, Vol. III, 73.

think of the fact that every engine shaft is subject to torsional loading: a torque about the axis of the shaft is produced at one end by the driving mechanism, at the other end an opposite torque acts which originates in the mechanism that receives the power; in steady motion both torques have the same magnitude. This example shows the general characteristic of torsional loading. For a straight rod of arbitrary cross-section it consists in a torque that acts about the rod axis at one end; whether the other end is kept fixed or subject to the opposite moment makes no difference; an additional tensile loading, e.g., in the direction of the axis of the rod would mean that we deal no longer with pure torsion.

It is easy to imagine the kind of deformation that is established by twisting a rod of circular cross-section whose one end is kept fixed, but we shall have to show that the deformation we think of is theoretically admissible. We assume (cf. Fig. 71) that every cross-section of the rod is

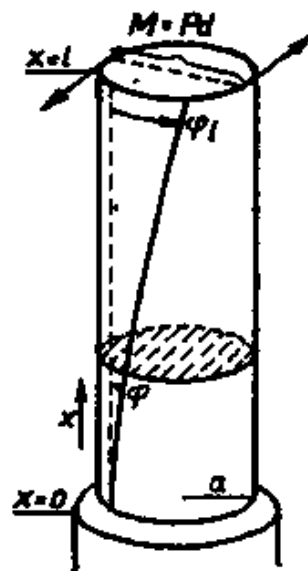


FIG. 71. The torsion of a rod of circular cross-section. The base is rigidly clamped, a torsional moment M acts on the free end.

rotated through a certain angle φ which is proportional to the x -coordinate. This means

$$(1) \quad \varphi = \alpha x, \quad \alpha = \text{Const.}$$

The originally straight generatrices of the rod are transformed into circular helices; the components of the displacement vector \mathbf{s} are according to (1) in cylindrical polar coordinates r, φ, x

$$(2) \quad s_r = 0, \quad s_\varphi = \alpha x, \quad s_x = 0.$$

In order to obtain the strains we have simply to introduce the quantities (2) in Eqs. (4.26) and (4.28) for $\delta q_1, \delta q_2, \delta q_3$, putting, because of $ds^2 = dr^2 + r^2 d\varphi^2 + dx^2$,

$$g_1 = 1, \quad g_2 = r, \quad g_3 = 1.$$

This leads to

$$(3) \quad \begin{aligned} \epsilon_{rr} = \epsilon_{\varphi\varphi} = \epsilon_{zz} &= 0 \\ \epsilon_{r\varphi} &= 0, \quad 2\epsilon_{\varphi z} = \alpha r, \quad \epsilon_{zr} = 0. \end{aligned}$$

The associated stresses σ are now found according to Eqs. (9.7), which are relations between tensors and therefore valid in arbitrary orthogonal coordinates:

$$(4) \quad \begin{aligned} \sigma_{rr} = \sigma_{\varphi\varphi} = \sigma_{zz} = \sigma_{zr} = \sigma_{r\varphi} &= 0 \\ \sigma_{z\varphi} = \tau = \mu\alpha r. \end{aligned}$$

In agreement with common engineering usage the notation τ is here applied to the only shear stress that acts in the cross-section $x = \text{const.}$

In the state of stress described by (4) we have again a *linear* stress distribution law as we had it in bending; in fact, we can write (4) in the form

$$(4a) \quad \tau = \frac{r}{a} \tau_{\max} \quad \text{where} \quad \tau_{\max} = \mu\alpha a.$$

In this simplest case of torsion the planeness of the cross-sections is strictly preserved, as already anticipated in the assumption $\epsilon_z = 0$ in (2). We have previously seen that this is not true in bending; in torsion it is only correct if the cross-section is circular, as we shall see.

We now prove that our system of stresses is in agreement with the boundary conditions of the problem. This is obviously true for the curved surface of the rod, $r = a$, which must be free of stress. There ought to be

$$\sigma_{rr} = \sigma_{r\varphi} = \sigma_{rz} = 0,$$

which, according to (4), is in fact fulfilled. Across the base the condition of vanishing stress must be replaced by the condition of vanishing displacements. This is satisfied by (2), if $x = 0$. For the free end cross-section we should at least stipulate as a *necessary* condition that

$$(5) \quad M = \int r\tau \, df,$$

where M is the moment of the forces P that produce the torsion (cf. Fig. 71). By (4a) this may be rewritten in the following form:

$$(5a) \quad M = \frac{\tau_{\max}}{a} \int r^2 \, df = \frac{\tau_{\max}}{a} J_p = \mu\alpha J_p, \quad J_p = \frac{\pi}{2} a^4,$$

where J_p stands for the polar moment of inertia of the cross-section which

in the case of a circular area is twice the equatorial moment of inertia given in (41.6).

Strictly speaking, this is not a *sufficient* condition. The external moment M should not only be equal to the total moment of the shear forces about the axis of the rod, but the distribution of M over the cross-section should be point by point identical with the distribution of τdf . The couple $M = Pd$ as it appears in the figure obviously does not satisfy this condition. This difficulty is again taken care of by the Saint-Venant principle (see p. 300), according to which the local disturbance of equilibrium produced by the dissimilarity between the actual and the theoretical load is limited to the proximity of the end face and does not influence the elastic equilibrium anywhere else in the rod.

To be on the safe side we would have to show that the differential equations of elasticity, which the displacements must obey, are satisfied by our special solution (2). These relations written in Cartesian displacement components are given by Eqs. (9.18); in their place we should have to use Eqs. (9.20) which refer to general orthogonal coordinates. A glance at the quantities Θ and P_i that appear in these equations shows that in our case

$$\Theta = 0, \quad P_1 = -\alpha x, \quad P_2 = 0, \quad P_3 = 2\alpha x.$$

This is easily checked with the aid of Eqs. (9.20a), and it follows immediately that Eqs. (9.20) are satisfied (F is of course equal to zero).

We have now to study in greater detail Eq. (5a) which determines the only remaining unknown in our problem, viz. the angular torsion per unit of rod length. We may as well determine the total *torsion angle* of the rod which becomes according to (1)

$$\varphi_1 = \alpha l, \quad \text{where } l = \text{length of rod.}$$

From (5a) we obtain

$$(6) \quad \varphi_1 = \frac{Ml}{\mu J_p}.$$

This equation is quite evident as regards the proportionality with the torsion moment M and the length l and also as to the inverse proportionality with the torsion modulus μ . More important is what Eq. (6) says about the dependence on the *dimensions* of the cross-section. This expresses itself in the factor J_p of the denominator. Note, however, that the dependence on the polar moment J_p that occurs here does not hold for arbitrary cross-sections, like the moment I in the bending problem; if the cross-section is not a circle this dependence is by no means expressible through the polar moment J_p . This applies e.g. to the elliptic cross-section as will be seen below.

For cross-sections of *unspecified shape* we have no preferred coordinate system, such as the cylindrical polar coordinates in the previous case, and we must take recourse to Cartesian coordinates and displacements $x, y, z; \xi, \eta, \zeta$, the equilibrium conditions now being given by Eq. (9.18) with $\mathbf{F} = 0$.

In general we shall now have to admit the possibility of warping of the cross-section. Taking the axis of the rod as x -axis, there is then no longer $\xi = 0$ [corresponding to the previous $s_x = 0$ in (2)]. As a consequence of the prismatic shape of the rod we may, however, assume that the warping is independent of the location of the cross-section, that is to say, the warping function ξ is independent of the x -coordinate:

$$(7) \quad \xi = \Phi(y, z), \quad \frac{\partial \xi}{\partial x} = 0.$$

We further assume that the deformation, as in the case of the circular cross-section, is free of dilatation. This would mean $\Theta = 0$; let us, however, introduce the more restricted conditions

$$(8) \quad \frac{\partial \eta}{\partial y} = \frac{\partial \zeta}{\partial z} = 0,$$

and finally assume that, similarly to $\epsilon_{r,\varphi} = 0$ in Eq. (3),

$$(9) \quad 2\epsilon_{yz} = \frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y} = 0.$$

These various assumptions will have to be justified in the course of our investigation.

By (7) and (8), the differential equations (9.18) take the following simple form

$$(10) \quad \begin{aligned} \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} &= 0 \\ \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial z^2} &= 0 \\ \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} &= 0. \end{aligned}$$

The first of these equations states that the warping function ξ introduced in (7) satisfies the *two-dimensional potential equation*.

$$(11) \quad \nabla^2 \Phi = 0,$$

The two remaining equations (10) can be fulfilled together with (8) if η and ζ are taken as bilinear functions as follows:

$$(12) \quad \eta = b_0 + b_1x + b_2z + b_3xz,$$

$$\zeta = c_0 + c_1x + c_2y + c_3xy.$$

According to (9), the following relation must hold for the constants b and c identically in x :

$$b_2 + b_3x + c_2 + c_3x = 0;$$

or

$$(13) \quad b_2 = -c_2, \quad b_3 = -c_3.$$

As in Fig. 71, the base of the rod is kept fixed, in the sense that lateral displacements are excluded. For $x = 0$ we have therefore identically in y and z

$$b_0 + b_2z = 0, \quad c_0 + c_2y = 0,$$

or

$$(13a) \quad b_0 = c_0 = b_2 = c_2 = 0.$$

If we finally wish to retain the right angle between x -axis and end face, we have for $x = 0$

$$(13b) \quad \frac{\partial \eta}{\partial x} = b_1 = 0, \quad \frac{\partial \zeta}{\partial x} = c_1 = 0.$$

Because of (13) and (13a, b) the Eqs. (12) take the form

$$(14) \quad \eta = -\alpha xz, \quad \zeta = +\alpha xy$$

where α has been written instead of c_3 .

The elastic differential equations and the boundary conditions at the end face are certainly satisfied by the assumptions about ξ , η , ζ , expressed in (7), (11), and (14), but the question remains whether *the boundary conditions for the curved surface of the rod* can be satisfied in this way. If n denotes the normal at a point P of the curved surface and s the direction tangential to the contour of the cross-section at P , the conditions are

$$(15) \quad \sigma_{nn} = 0 \quad \text{and} \quad \sigma_{ns} = 0.$$

The first of these equations can certainly be satisfied: We may draw the y -axis which so far has not been fixed parallel to n so that the z -axis becomes parallel to s and σ_{nn} coincides with σ_{ss} at the point P . But since $\epsilon_{ss} = 0$ by (9), it follows that σ_{ss} vanishes also at P . This argument can

obviously be applied to each point P of the curved surface. As to the second condition (15), we base our argument, on the general tensor relation (8.5) which becomes in our case

$$(16) \quad \begin{aligned} \sigma_{nz} &= \beta_1 \sigma_{yz} + \gamma_1 \sigma_{xz} = 2\mu(\beta_1 \epsilon_{yz} + \gamma_1 \epsilon_{xz}) \\ &= \mu \left(\beta_1 \left\{ \frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x} \right\} + \gamma_1 \left\{ \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x} \right\} \right). \end{aligned}$$

Here β_1 and γ_1 are the cosines of angles between the n direction and the y and z directions respectively. If s is oriented in such a way that the sense of rotation $s \rightarrow n$ coincides with $y \rightarrow z$, we have

$$(17) \quad \begin{aligned} \beta_1 &= \cos(n, y) = -\cos(s, z) = -\frac{dz}{ds}, \\ \gamma_1 &= \cos(n, z) = +\cos(s, y) = +\frac{dy}{ds}. \end{aligned}$$

Using (16), (17), (7), and (14) the second condition (15) can be written in the following form:

$$(18) \quad 0 = -\left(\frac{\partial \Phi}{\partial y} - \alpha z\right) dz + \left(\frac{\partial \Phi}{\partial z} + \alpha y\right) dy.$$

This equation must be satisfied at each point of the contour. It can be simplified by introducing the *conjugate* of the potential function Φ (corresponding to the *stream function* in hydrodynamics), which is connected with Φ by the Cauchy-Riemann equations (19.5). In our present coordinates (y, z instead of x, y) we have therefore

$$(18a) \quad \frac{\partial \Phi}{\partial y} = \frac{\partial \Psi}{\partial z}, \quad \frac{\partial \Phi}{\partial z} = -\frac{\partial \Psi}{\partial y}.$$

Eq. (18) thus transforms into

$$\left(\frac{\partial \Psi}{\partial z} - \alpha z\right) dz + \left(\frac{\partial \Psi}{\partial y} - \alpha y\right) dy = 0,$$

or

$$d\Psi = \alpha(y dy + z dz) = \frac{\alpha}{2} d(y^2 + z^2)$$

or, integrated,

$$(19) \quad \Psi = \frac{\alpha}{2} (y^2 + z^2) + C \dots \quad (\text{boundary condition}).$$

The conjugate potential Ψ satisfies also the differential Eq. (11):

$$(19a) \quad \nabla^2 \Psi = 0.$$

By (19) and (19a) the potential function Ψ is uniquely determined being the (always existing) solution of a simple potential problem. We have therefore proved that our boundary condition (18) can be fulfilled. At the same time the special assumptions made in (8), (9), and (12) have been justified.

Let us now consider once more the circular cross-section. The contour is given by $y^2 + z^2 = a^2$, the origin coinciding with the center of the circle. Eq. (19) becomes therefore

$$\Psi = \text{const.}$$

along the contour and, according to (19a), also in the interior. From (18a) we have then

$$\Phi = \text{Const} \quad \text{and by} \quad (7) \quad \xi = \text{Const.}$$

which means: *the cross-sections remain plane*. Conversely, if, for a certain shape of the cross-section, ξ vanishes along the rod, we have $\Phi = 0$ by (7) and therefore $\Psi = \text{const}$ by (18a). From (19) the contour is obtained in the form $y^2 + z^2 = \text{const}$: *The cross-section is circular*.

Take now as a less trivial example the elliptical cross-section, the contour of which may be given as

$$(20) \quad \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

Since the general solution of (19a) is

$$\Psi = \text{Re } f(y + iz), \quad (\text{Re} = \text{real part of})$$

we shall assume f as the second power of the argument $y + iz$ so that

$$(21) \quad \Psi = A(y^2 - z^2).$$

Along the contour we have according to (19)

$$A(y^2 - z^2) = \frac{\alpha}{2}(y^2 + z^2) + C.$$

On substituting z^2 from (20), an identity in y results from which A follows as

$$(21a) \quad A = \frac{\alpha}{2} \frac{a^2 - b^2}{a^2 + b^2}.$$

From (18a) and (21) we can now calculate

$$\frac{\partial \Phi}{\partial y} = -\alpha \frac{a^2 - b^2}{a^2 + b^2} z, \quad \frac{\partial \Phi}{\partial z} = -\alpha \frac{a^2 - b^2}{a^2 + b^2} y$$

and obtain the shear stresses according to (7) and (14):

$$\begin{aligned} \sigma_{xy} &= 2\mu\epsilon_{xy} = \mu\left(\frac{\partial \Phi}{\partial y} + \frac{\partial \eta}{\partial x}\right) = -\mu\alpha\left(\frac{a^2 - b^2}{a^2 + b^2} + 1\right)z = -2\mu\frac{\alpha a^2}{a^2 + b^2}z, \\ (22) \quad \sigma_{xz} &= 2\mu\epsilon_{xz} = \mu\left(\frac{\partial \Phi}{\partial z} + \frac{\partial \xi}{\partial x}\right) = -\mu\alpha\left(\frac{a^2 - b^2}{a^2 + b^2} - 1\right)y = 2\mu\frac{\alpha b^2}{a^2 + b^2}y. \end{aligned}$$

The shear stress τ resulting from σ_{xy} and σ_{xz} is at any point of the cross-section parallel to the tangent that passes through the intersection point of the contour with the prolonged radius vector. Along the radius vector, τ follows a linear law. The moment of the shear stresses about the x -axis results from (22) in the form of the following formula which is a generalization of our previous formula (5):

$$(23) \quad M = \iint (y\sigma_{xz} - z\sigma_{xy}) dy dz = \frac{2\mu\alpha}{a^2 + b^2} \iint (b^2 y^2 + a^2 z^2) dy dz.$$

If the integration is carried out one obtains instead of the factor J_p , which appears in (5a) as the coefficient of $\mu\alpha$, the expression

$$(24) \quad \pi \frac{a^3 b^3}{a^2 + b^2},$$

which is not equal to the polar moment of inertia of the area of an ellipse (cf. problem VIII.4).

The simultaneous consideration of the potential function Φ and the stream function Ψ suggests a hydrodynamic analogy which has proved useful in the approximate solution of more complicated problems such as the torsion of an I -beam.

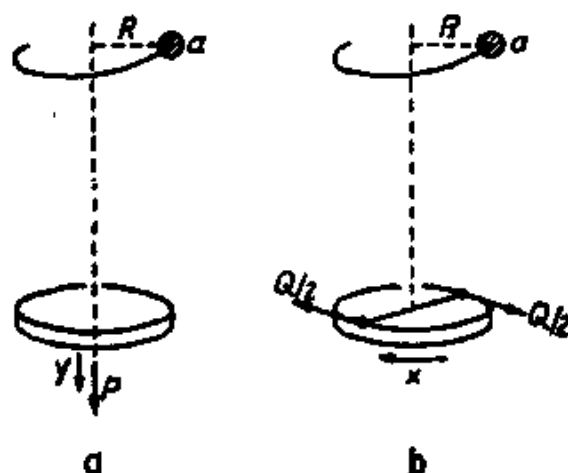
43. Torsion and Bending of a Helical Spring

Imagine a coil consisting of a single layer of thin wire wound on a circular cylinder. On removing the mandrel, we obtain a helical spring which we fasten at one end in some way, loading the other end with a disk, the weight of which is supposed to be small enough so that the resulting pitch of the spring is *small* compared to the radius of the cylinder. If we now put additional weight on the disk so as to have a small load acting along the axis of the spring, this will result in a slightly increased pitch and a corresponding lengthening of the spring. *This kind of loading is accompanied by a torsion of the wire.* If, on the other hand, a couple

acts on the disk (e.g., when the disk is turned in its own plane) the pitch remains essentially unchanged and the spring yields to the torque by a slight diminution of the cylinder radius. *In this case the wire is bent.*

Both facts seem paradoxical at first, but can be easily clarified: In *torsion* the individual cross-sections of the wire are loaded by a torque about the *polar axis* while in *bending* the moment acts about an *equatorial axis*, that is an axis in the plane of the cross-section. With this in mind let us examine in Figs. 72a and 72b what happens in the planes that pass in either case through the cylinder axis and the cross-sections marked in the respective figures. The load P acting along the cylinder axis in Fig. 72a has the moment PR about the center of the cross-section; we call it

FIG. 72a, b. A helical spring subject (a) to torsion (force P , vertical displacement y) and (b) to bending (moment QR , rotational displacement x).



the *torsional moment* M_T since its axial vector is normal to the cross-section. In Fig. 72b the couple QR acts upon the disk whose diameter has been assumed as $2R$ so that the two single forces of the couple can be denoted by $Q/2$. This moment has the character of a *bending moment* M_B since its axial vector coincides with the cylinder axis and can be transferred parallel to itself to the center of the marked cross-section. We denote its position by NN indicating in this way that it coincides with the trace of the neutral layer in the bent cross-section. In the loading according to Fig. 72a there will be *shear stresses* in the plane of the cross-section; they increase proportionally to the distance from the wire axis and will be denoted by τ as in the previous article. We have therefore

$$(1) \quad \int r \tau \, df = M_T .$$

In the loading according to Fig. 72b *normal stresses* σ act upon the cross-section. They are proportional to the distance y from the equatorial axis NN . Again we have

$$(2) \quad \int y \sigma \, df = M_B .$$

Eqs. (1) and (2) may be evaluated according to (42.4a) and (42.5a), or (41.5):

$$(3) \quad \frac{\tau_{\max}}{a} J_p = M_\tau \quad \text{respectively} \quad \frac{\sigma_{\max}}{a} I = M_B .$$

Apart from the maximum stresses τ_{\max} and σ_{\max} we should like to know the displacement of the spring in vertical direction y and horizontal direction x for the two respective kinds of loading. For this purpose we should have to know the shape of the elastic line of an *originally curved* rod in torsion and bending. There is, however, a simpler way to answer our question when use is made of the *energy principle*: one calculates the work of deformation expended in the torsion or bending of the spring and compares it with the work of the external forces P or Q . This, by the way, is a welcome opportunity to come back to the topic of energy of deformation which is highly important in engineering applications.⁹

So far we have only derived the general expression (9.26b) for the energy of deformation. In the case of pure shear, where

$$\sigma_{12} = \sigma_{21} = \tau, \quad \epsilon_{12} = \epsilon_{21} = \frac{\tau}{2\mu},$$

or in the case of pure extension and compression (bending), where

$$\sigma_{11} = \sigma, \quad \epsilon_{11} = \frac{\sigma}{E},$$

(9.26b) reduces to

$$(4) \quad W = \frac{1}{2} (\sigma_{12}\epsilon_{12} + \sigma_{21}\epsilon_{21}) = \frac{1}{2} \frac{\tau^2}{\mu}$$

or to

$$(5) \quad W = \frac{1}{2} \sigma_{11}\epsilon_{11} = \frac{1}{2} \frac{\sigma^2}{E},$$

respectively. These expressions refer to the unit volume and must be applied in the case of locally variable stresses to each single volume element $df \, ds$, where df is the area element of the cross section and ds the length element of the axis of the rod. Thus we obtain from (4) in the case of torsion

$$(6) \quad dW = \frac{1}{2} \frac{\tau^2}{\mu} df \, ds, \quad \tau = \frac{r}{a} \tau_{\max} \quad \text{from (42.4a).}$$

⁹In all statically indeterminate systems, in particular in the theory of trusses, Castigliano's theorems about the *minimum of the energy of deformation* play an important role, and so does *Maxwell's theorem of the reciprocity of the displacements*.

Integration over the cross-section yields

$$(6a) \quad \int dW = \frac{1}{2\mu} \frac{\tau_{\max}^2}{a^2} \int r^2 df ds = \frac{\tau_{\max}^2}{2\mu a^2} J, ds,$$

and, by (3),

$$(6b) \quad \int dW = \frac{M_T^2}{2\mu J} ds.$$

The entire torsional energy W_T follows by integration over the length l of the rod (in our case the length of the spring):

$$(7) \quad W_T = \frac{M_T^2 l}{2\mu J} = \frac{P^2 R^2 l}{2\mu J}.$$

Likewise one obtains in the case of bending from (5)

$$(8) \quad dW = \frac{1}{2} \frac{\sigma^2}{E} df ds, \quad \sigma = \frac{y}{a} \sigma_{\max} \quad \text{according to (41.2),}$$

and by integration over the cross-section

$$(8a) \quad \int dW = \frac{1}{2E} \frac{\sigma_{\max}^2}{a^2} \int y^2 df ds = \frac{\sigma_{\max}^2}{2E a^2} I ds.$$

The entire bending energy contained in the spring is obtained from (8a) and (3) as

$$(9) \quad W_B = \frac{M_B^2 l}{2EI} = \frac{Q^2 R^2 l}{2EI}.$$

Expressions (7) and (9) represent the energy intake of the spring in an *elastic* deformation in which the spring passes through a sequence of states of equilibrium. It must be compared with the *mechanical* work performed along the paths y (or x) when the load P (or the couple QR) increases *gradually* from zero to its final value. The work is therefore

$$(10) \quad W_T = \frac{Py}{2}, \quad \text{respectively} \quad W_B = 2 \frac{(Q/2)x}{2} = \frac{Qx}{2}.$$

Comparison with (7) and (9) gives

$$(11) \quad y = \frac{R^2 l}{\mu J} P, \quad x = \frac{R^2 l}{EI} Q.$$

Thus the use of the energy principle leads indeed to a very simple determination of the displacements x and y . Note that Eqs. (11) can also

be obtained from the first principle of Castigliano in the following way:¹⁰

$$(11a) \quad W(P, Q) = W_T + W_B, \quad y = \frac{\partial W}{\partial P}, \quad x = \frac{\partial W}{\partial Q}.$$

From these equations we can easily pass to the *free oscillations* of the spring when we replace P and Q by the inertia forces

$$(12) \quad P = -M \frac{d^2 y}{dt^2}, \quad Q = -M_{\text{red}} \frac{d^2 x}{dt^2}.$$

Here the masses M and M_{red} have to be given the appropriate physical meaning. M is essentially the mass M_0 of the disk that oscillates up and down, but since the mass m of the spring partakes also in the oscillation, (its lower portion more strongly than its upper one) a fraction of this mass must be added as a correction to the mass of the disk. According to Lord Rayleigh this fraction is $\frac{1}{3}$, hence M is to be interpreted as $M_0 + m/3$. On the other hand, the inertia force governing the oscillations about the cylinder axis is $-\Theta \, d^2 \varphi / dt^2$, where $\varphi = x/R$ is the angular displacement of the disk according to the meaning of x in Fig. 72b. Again Θ is essentially the moment of inertia Θ_0 of the disk augmented by $\frac{1}{3}$ of the moment of inertia of the coil mR^2 . Consequently M_{red} is to be defined as

$$M_{\text{red}} R^2 = \Theta = \Theta_0 + \frac{1}{3} m R^2.$$

The equations of the free torsional and bending oscillations are according to (12) and (11)

$$(13) \quad \frac{d^2 y}{dt^2} + \frac{\mu J_p}{MR^2 l} y = 0, \quad \frac{d^2 x}{dt^2} + \frac{EI}{M_{\text{red}} R^2 l} x = 0$$

and the associated circular frequencies

$$(14) \quad \omega_T = \sqrt{\frac{\mu J_p}{MR^2 l}}, \quad \omega_B = \sqrt{\frac{EI}{M_{\text{red}} R^2 l}}.$$

In general they are different from each other.¹¹ If they are equal, the two

¹⁰The rule (11a) presupposes that y , x and P , Q are linearly connected so that $W(P, Q)$ is a quadratic expression in P and Q as in (7) and (9). In the general case one would have to replace W by the "modified" expression for the energy $U(P, Q) = yP + xQ - W(x, y)$. Cf. about this transformation Vol. I. 42.7 and E. L. Ince, *Ordinary Differential Equations*, Dover Publ., New York, 1944, Chapter 2.5 (Legendre's transformation).

¹¹This can be enforced in an actual experiment by loading the spring with an additional weight that acts along its axis and has a small moment of inertia about it (cf. Vol. I, Fig. 37). Thus the mass M that takes part in the torsional oscillation is increased and the frequency ω_T reduced. M_{red} and ω_B remain essentially unchanged, however.

oscillations are in *resonance* and the behavior of the spring becomes very sensitive to small changes of the parameters involved. Then the present theory is no longer sufficient: we have to take into account that the two modes of oscillation are coupled through the finite pitch of the spring which so far has been neglected.

For finite pitch h the slope α of the helix is given by

$$\sin \alpha = \frac{h}{2\pi R}.$$

This is also the angle of inclination that the cross-section of the wire forms with the vertical. The two loading moments PR and QR may then be decomposed as follows:

	normal to the cross-section	parallel to the cross-section
PR	$\cos \alpha \cdot PR$	$-\sin \alpha \cdot PR$
QR	$\sin \alpha \cdot QR$	$\cos \alpha \cdot QR$

The contributions to the torsional moment are on the left side, those to the bending moment on the right side, therefore

$$M_T = \cos \alpha PR + \sin \alpha QR, \quad M_B = \cos \alpha QR - \sin \alpha PR.$$

The total bending energy becomes

$$\begin{aligned} W &= W_T + W_B = \frac{M_T^2 l}{2\mu J_p} + \frac{M_B^2 l}{2EI} \\ (15a) \quad &= \frac{R^2 l}{2} \left(\frac{(\cos \alpha P + \sin \alpha Q)^2}{\mu J_p} + \frac{(\cos \alpha Q - \sin \alpha P)^2}{EI} \right), \end{aligned}$$

and the displacements can be determined according to the rule (11a):

$$\begin{aligned} y &= \frac{\partial W}{\partial P} = \frac{R^2 l}{\mu J_p} (\cos^2 \alpha P + \sin \alpha \cos \alpha Q) \\ &\quad + \frac{R^2 l}{EI} (-\sin \alpha \cos \alpha Q + \sin^2 \alpha P), \\ (15b) \quad x &= \frac{\partial W}{\partial Q} = \frac{R^2 l}{\mu J_p} (\sin \alpha \cos \alpha P + \sin^2 \alpha Q) \\ &\quad + \frac{R^2 l}{EI} (\cos^2 \alpha Q - \sin \alpha \cos \alpha P). \end{aligned}$$

These equations now replace Eqs. (11). Upon replacing the inertia forces (12) by P and Q , one obtains the following system of equations which describes the coupled oscillations of the spring:

$$(16) \quad \frac{d^2 y}{dt^2} + \omega_T^2 (y - kx) = 0, \quad \frac{d^2 x}{dt^2} + \omega_B^2 (x - hy) = 0$$

where ω_T , ω_B , k , and h are

$$(16a) \quad \omega_T^2 = \frac{\mu J_p}{MR^2 l} (1 + \nu \sin^2 \alpha), \quad \omega_B^2 = \frac{\mu J_p}{M_{rod} R^2 l} (1 + \nu \cos^2 \alpha),$$

$$(16b) \quad k = \frac{\nu \sin \alpha \cos \alpha}{1 + \nu \sin^2 \alpha}, \quad h = \frac{\nu \sin \alpha \cos \alpha}{1 + \nu \cos^2 \alpha}$$

and ν is Poisson's ratio. The coupling coefficients k and h vanish if $\alpha = 0$ so that the frequencies ω_T and ω_B reassume the previous values (14) and Eqs. (16) become identical with Eqs. (13).¹²

The further discussion of Eq. (16) is no longer a problem of theory of elasticity. For the case of resonance, $\omega_T = \omega_B$, the behavior of the spring has already been described in Vol. I 20: the energy passes back and forth between the originally excited mode and the mode that is excited consequential to it, to the extent that the two modes are alternately extinguished. The beat period N expressed in time units of single oscillations can be very accurately determined by counting the number of oscillations in a beat. It becomes a maximum in the case of resonance. The large number N_{\max} and the small angle α permit a precise determination¹³ of Poisson's ratio ν , as shown by the author.

The resonance phenomena discussed here have first been described and explained by Wilberforce,¹⁴ but his determination of Poisson's ratio is founded on the relations (14) rather than on the more elegant resonance method.

44. The Elastic Energy Content of a Rectangular Parallelepiped

In 14 we have studied elastic waves in an *infinitely extended* isotropic solid; in the following two articles we shall discuss the elastic oscillations in *finite* bodies, dealing first with a rectangular parallelepiped with sides a, b, c . Let us assume that the faces which are situated at $x = 0, a; y = 0, b;$

¹²In comparing the values of ω_B in (16a) and (14), use relation $J_p = 2I$ for circular cross-sections and relation (9.11), $E = 2\mu(1 + \nu)$, which has already been used in the transition from (15) to (16).

¹³Wüllner-Festschrift, Teubner, Leipzig 1905, p. 162.

¹⁴Phil. Mag. 38 (1894) 386.

$z = 0, c$, are free, in which case we must stipulate that the corresponding normal and shear stresses vanish. The oscillations possible under these conditions are called the *proper oscillations* of the body; they are executed without any *exchange of energy* with the surroundings.

The problem as it has just been outlined presents great mathematical difficulties.¹⁶ A particular case of the general problem, the vibrations of a rectangular plate, has withstood all attempts at a rigorous solution although Chladni's beautiful figures have been known for 150 years and are still a challenge for the best mathematicians. The most successful step so far was taken by Ritz;¹⁸ it does not, however, give a direct solution of the differential equations with their boundary conditions, but establishes a minimum principle by which a stepwise approximation of the solution becomes feasible.

It is, however, possible to prevent the exchange of energy with the surroundings by other sets of boundary conditions. Consider the work performed per unit of x -surface in the displacement with components ξ, η, ζ :

$$(1) \quad \delta A = \sigma_{xx}\xi + \sigma_{xy}\eta + \sigma_{xz}\zeta.$$

It is not only zero if

$$(2) \quad \sigma_{xx} = \sigma_{xy} = \sigma_{xz} = 0 \quad (\text{surface free of stress})$$

or if

$$(2a) \quad \xi = \eta = \zeta = 0 \quad (\text{surface kept immovable}),$$

but also if for example

$$(3) \quad \xi = 0, \quad \sigma_{xy} = \sigma_{xz} = 0$$

or

$$(3a) \quad \sigma_{xx} = 0, \quad \eta = \zeta = 0.$$

The assumption of *mixed* boundary conditions as in (3) or (3a) makes an elementary treatment of our problem possible as C. Somigliana¹⁷ showed in 1902 for the case of static loading; they are, on the other hand, equiva-

¹⁶The special case of the *cube* which is usually studied in similar problems of electromagnetic radiation is essentially not simpler than the more general case considered here. One can see that from the final result of Eq. (23), which refers only to the size of the body V , but not to its shape.

¹⁸W. Ritz, Theorie der Transversalschwingungen einer quadratischen Platte mit freien Rändern, Ann. Physik 28 (1909) 737. The circular plate can be directly integrated, as was shown by Kirchhoff.

¹⁷See the comprehensive article of O. Tedone in Encykl. d. Mathem. Wiss. Vol. IV, 4, Art. 25, No. 2.

lent to the uniform conditions (2) or (2a) as regards the physical applications which we have in mind.

We therefore base our calculation on the boundary conditions (3) and the corresponding conditions for y and z -surfaces, obtained from (3) by rotating the letters. Since the first Eq. (3) implies

$$\frac{\partial \xi}{\partial y} = \frac{\partial \xi}{\partial z} = 0, \quad 2\epsilon_{xy} = \frac{\partial \eta}{\partial x}, \quad 2\epsilon_{xz} = \frac{\partial \zeta}{\partial x},$$

the boundary conditions can be written:

$$(4) \quad \begin{aligned} \xi &= 0, \quad \frac{\partial \eta}{\partial x} = \frac{\partial \zeta}{\partial x} = 0 & \text{for } x &= \begin{cases} 0 \\ a \end{cases}, \\ \eta &= 0, \quad \frac{\partial \zeta}{\partial y} = \frac{\partial \xi}{\partial y} = 0 & \text{for } y &= \begin{cases} 0 \\ b \end{cases}, \\ \zeta &= 0, \quad \frac{\partial \xi}{\partial z} = \frac{\partial \eta}{\partial z} = 0 & \text{for } z &= \begin{cases} 0 \\ c \end{cases}. \end{aligned}$$

Our differential equations are the Eqs. (14.1b) which because of $\mathbf{F} = 0$ now read

$$(5) \quad \begin{aligned} \rho \frac{\partial^2 \xi}{\partial t^2} &= \mu \nabla^2 \xi + (\lambda + \mu) \frac{\partial \Theta}{\partial x}, \\ \rho \frac{\partial^2 \eta}{\partial t^2} &= \mu \nabla^2 \eta + (\lambda + \mu) \frac{\partial \Theta}{\partial y}, \\ \rho \frac{\partial^2 \zeta}{\partial t^2} &= \mu \nabla^2 \zeta + (\lambda + \mu) \frac{\partial \Theta}{\partial z}. \end{aligned}$$

Eqs. (4) can be satisfied by the following set of expressions for ξ , η , ζ :

$$(6) \quad \begin{aligned} \xi &= A \sin \frac{a\pi x}{a} \cos \frac{b\pi y}{b} \cos \frac{c\pi z}{c} e^{-i\omega t}, \\ \eta &= B \cos \frac{a\pi x}{a} \sin \frac{b\pi y}{b} \cos \frac{c\pi z}{c} e^{-i\omega t}, \\ \zeta &= C \cos \frac{a\pi x}{a} \cos \frac{b\pi y}{b} \sin \frac{c\pi z}{c} e^{-i\omega t}. \end{aligned}$$

where a, b, c are arbitrary integer numbers. A, B, C are constants whose ratios are to be determined; also the circular frequency ω must be found.

On introducing the expressions (6) in (5), there appear, in each of the three terms of the equations (5), the same trigonometrical factors as in ξ, η, ζ . One has for instance

$$\rho \frac{\partial^2 \xi}{\partial t^2} = -\rho \omega^2 A \sin \dots \cos \dots \cos \dots e^{-i\omega t},$$

$$\nabla^2 \xi = -A \left(\frac{a^2}{a^2} + \frac{b^2}{b^2} + \frac{c^2}{c^2} \right) \pi^2 \sin \dots \cos \dots \cos \dots e^{-i\omega t},$$

$$\frac{\partial \Theta}{\partial x} = - \left(A \frac{a^2}{a^2} + B \frac{ab}{ab} + C \frac{ac}{ac} \right) \pi^2 \sin \dots \cos \dots \cos \dots e^{-i\omega t}.$$

On omission of these factors, the differential equations (5) give the following linear system of equations for A, B, C

$$\begin{aligned} (7) \quad & A \left(\Omega + \frac{aa}{a^2} \right) + B \frac{ab}{ab} + C \frac{ac}{ac} = 0, \\ & A \frac{ba}{ba} + B \left(\Omega + \frac{bb}{b^2} \right) + C \frac{bc}{bc} = 0, \\ & A \frac{ca}{ca} + B \frac{cb}{cb} + C \left(\Omega + \frac{cc}{c^2} \right) = 0. \end{aligned}$$

Here Ω is defined by

$$(7a) \quad (\lambda + \mu) \Omega = \mu s - \frac{\rho \omega^2}{\pi^2}, \quad s = \frac{a^2}{a^2} + \frac{b^2}{b^2} + \frac{c^2}{c^2}.$$

The linear equations (7) have solutions different from zero only, if the determinant

$$(8) \quad \Delta = \begin{vmatrix} \Omega + \frac{aa}{a^2}, & \frac{ab}{ab}, & \frac{ac}{ac} \\ \frac{ba}{ba}, & \Omega + \frac{bb}{b^2}, & \frac{bc}{bc} \\ \frac{ca}{ca}, & \frac{cb}{cb}, & \Omega + \frac{cc}{c^2} \end{vmatrix} = 0$$

On expanding (8) in powers of Ω , one sees easily that the factors of Ω^0

and Ω^1 vanish while those of Ω^2 and Ω^3 are s and 1. Eq. (8) can thus be rewritten in the form

$$(9) \quad \Omega^3 + s\Omega^2 = \Omega^2(\Omega + s) = 0,$$

an equation that has the double root $\Omega_1 = \Omega_2 = 0$ and the simple root $\Omega_3 = -s$. From (7a) we obtain the corresponding frequency values

$$(10) \quad \omega_1^2 = \omega_2^2 = \pi^2 \frac{\mu}{\rho} s, \quad \omega_3^2 = \pi^2 \frac{\lambda + 2\mu}{\rho} s.$$

The ratios $A : B : C$ associated with these values of ω are found from (7) so that the set (6) has now been determined except for a common amplitude factor. Further arbitrary elements are the numbers a, b, c .

What is the physical meaning of this arbitrariness? By the numbers a, b, c the sides a, b, c are subdivided into segments which must be interpreted as wave lengths:

$$(11) \quad \lambda_a = \frac{a}{\alpha}, \quad \lambda_b = \frac{b}{\beta}, \quad \lambda_c = \frac{c}{\gamma}.$$

They just fit the boundaries of the parallelepiped. The associated wave numbers are

$$(11a) \quad k_a = \frac{2\pi}{\lambda_a} = 2\pi \frac{\alpha}{a}, \quad k_b = \frac{2\pi}{\lambda_b} = 2\pi \frac{\beta}{b}, \quad k_c = \frac{2\pi}{\lambda_c} = 2\pi \frac{\gamma}{c}.$$

With a view to Eqs. (6) and to Eqs. (11) we can write

$$(12) \quad \xi = A \sin \pi \frac{x}{\lambda_a} e^{-i\omega t}, \quad \eta = \zeta = 0,$$

if we consider a particular solution depending on x only. Eq. (12) represents a standing wave which has the boundaries $x = 0$ and $x = a = \alpha\lambda_a$ as nodal surfaces. Altogether, the Eqs. (6) represent the entirety of standing waves that can be accommodated by the parallelepiped. Unlike the example (12), however, they are in general not plane waves but are composed of waves in the three directions of the edges according to the particular rule set by Eqs. (6).

The structure of Eqs. (10) teaches further that the waves are partly of *transverse*, partly of *longitudinal* nature. According to Eqs. (14.6) and (14.3) we have

$$c_{\text{trans}} = \sqrt{\frac{\mu}{\rho}}, \quad c_{\text{long}} = \sqrt{\frac{\lambda + 2\mu}{\rho}},$$

where now c_{trans} and c_{long} stand for the previous notations b and a for

the sound velocities. If we further introduce a wave number k_r , resulting from the three wave numbers k_a, k_b, k_c , by

$$k_r = \sqrt{k_a^2 + k_b^2 + k_c^2},$$

we obtain from (11a) and (7a)

$$(13) \quad k_r = 2\pi \sqrt{\frac{a^2}{a^3} + \frac{b^2}{b^3} + \frac{c^2}{c^3}} = 2\pi \sqrt{s}.$$

Eqs. (10) are then simply

$$(14) \quad \omega_1 = \omega_2 = \frac{1}{2} c_{\text{trans}} k_r, \quad \omega_3 = \frac{1}{2} c_{\text{long}} k_r.$$

The fact that the transverse waves are counted twice points to the physical possibility of polarization in two mutually orthogonal directions.

We have thus found a system of infinitely many elastic waves which can be excited in an adiabatically insulated parallelepiped. The system is *complete*, that is, the *only* standing waves that obey our boundary conditions are those of the system (6). The proof of completeness consists in showing that an arbitrarily given initial state can be expanded in terms of the functions (6) for $t = 0$.¹⁸

Let us now determine the number z of all waves whose frequency is smaller than a given $\nu = \omega/2\pi$, which can be done by geometrical reasoning. We first count the longitudinal waves.

By marking in the rectangular system k_a, k_b, k_c all points that belong to integer values a, b, c , we obtain a three-dimensional orthorhombic lattice, whose elementary cell has according to (11a) the sides

$$\frac{2\pi}{a}, \quad \frac{2\pi}{b}, \quad \frac{2\pi}{c}.$$

Consider now the lattice points inside the spherical surface $k_r = \text{const}$, which corresponds to the frequency ν . According to (14) the radius of the sphere $k_r = 4\pi\nu/c_{\text{long}}$. Because of $a, b, c \geq 0$, we are only interested in the lattice points that lie in the positive octant, whose volume is

$$\frac{4\pi}{3} \frac{k_r^3}{8} = \frac{4\pi}{3} \left(\frac{2\pi\nu}{c_{\text{long}}} \right)^3.$$

¹⁸Cf. for such a proof and for a generalization to anisotropic bodies: R. Ortqvist, Ann. Physik. 42 (1913), 745. A treatment of the problem from the point of view of group theory was presented by H. Wierzejewski in his Breslau thesis suggested by E. Fues: Elastische Eigenschwingungen von Kristallen bei gemischten Randbedingungen, Z. f. Kristallographie, Vol. 101, 1939.

On dividing by the volume of the elementary cell

$$(15) \quad \frac{(2\pi)^3}{abc},$$

we obtain the number of elementary cells or the number of lattice points in the spherical octant which, at the same time, is the number of longitudinal waves with frequencies below ν :

$$(16) \quad \mathfrak{z}_{\text{long}} = \frac{4\pi}{3} \frac{abc\nu^3}{c_{\text{long}}^3}.$$

In exactly the same way one has for either state of polarization of the transversal waves

$$(16a) \quad \mathfrak{z}_{\text{trans}} = \frac{4\pi}{3} \frac{abc\nu^3}{c_{\text{trans}}^3};$$

thus the total number of vibrational states below ν becomes

$$(16b) \quad \mathfrak{z} = \mathfrak{z}_{\text{long}} + 2\mathfrak{z}_{\text{trans}} = \frac{4\pi}{3} abc\nu^3 \left(\frac{2}{c_{\text{trans}}^3} + \frac{1}{c_{\text{long}}^3} \right).$$

The number \mathfrak{z} just obtained is accurate except for an error which is due to the fact that the spherical surface and the boundaries of the enclosed elementary cells do not coincide. The error is of the order of magnitude $1/abc$ [cf. (15)] and thus negligible for a large parallelepiped. By differentiating (16b) with respect to ν we can now determine the number of vibrational states between ν and $\nu + d\nu$:

$$(17) \quad d\mathfrak{z} = 4\pi V \left(\frac{2}{c_{\text{trans}}^3} + \frac{1}{c_{\text{long}}^3} \right) \nu^2 d\nu$$

where V is the volume abc of the parallelepiped.

We are now ready to attack the problem which we formulated at the outset. We want to calculate *the energy content of the adiabatically insulated solid body*, which, according to the definition in the older literature, is *identical with the vibrational energy of its molecules*. It is preferable, however, to follow Debye¹⁹ who defines the energy content as that of

¹⁹P. Debye, Ann. Physik., 39 (1912) 789. In this fundamental paper Debye calculates the proper vibrations of a sufficiently large *sphere*, the surface of which is considered *free of stresses*, that is, he uses the boundary conditions corresponding to our equations (2). As the circular shape is the only case in which the plate problem can be directly integrated (cf. footnote, p. 315), so the sphere is the only elastic three-dimensional problem whose integration leads in the case of "uniform boundary conditions" to known functions (Bessel functions). But even in this case the solution is very involved and becomes only possible by the use of asymptotic approximations of Bessel functions for large argument and index that are due to Debye. The simplification that can be

the energy of the elastic vibrations of the body. The molecules are strongly coupled, hence it is impossible to excite oscillations of individual molecules; on the other hand, the waves considered in (6) are independent from each other and satisfy individually the differential equations of the elastic body.

Each of these vibrational states may therefore be quantized individually according to the quantum rules first established by Planck in the discovery of his radiation law:

$$(18) \quad \epsilon_n = nh\nu \begin{cases} n = \text{integer number} \\ h = \text{Planck's constant} \end{cases}$$

The quantum rules are here applied without further explanation; they will be discussed in Vol. V. Also the refinement of the rules due to modern quantum mechanics according to which the integer number n in (18) should be replaced by $n + \frac{1}{2}$ is here disregarded as irrelevant for our purpose.

In addition, we require for the following the Boltzmann factor which was introduced in (7.15c), though without proof. In the present case the Boltzmann factor has this physical meaning: Given N oscillating systems in thermodynamical equilibrium at the temperature T , the number N_n of those systems whose energy equals ϵ_n is then

$$(19) \quad N_n = A \exp\left(-\frac{\epsilon_n}{kT}\right) = A\alpha^n, \quad \alpha = \exp\left(-\frac{h\nu}{kT}\right);$$

the constant A can be determined, when the total number N of all oscillating systems is written as the sum of the N_n :

$$(20) \quad N = \sum_{n=0}^{\infty} N_n = A(1 + \alpha + \alpha^2 + \cdots) = \frac{A}{1 - \alpha}.$$

In our case, N is to be identified with the (large!) number d_3 from (17). We then obtain for A

$$(20a) \quad A = (1 - \alpha)d_3.$$

achieved by "mixed boundary conditions" was first stated by the author in a course on quantum theory in 1912 and then left to Mr. Ortqvist for further development and publication (cf. the preceding footnote). It should be noticed that Debye had to take the radius of the sphere sufficiently large to enforce independence of the distribution of the eigenfunctions from the shape of the body. For the same reason the dimensions of the parallelepiped in the present case must be sufficiently large as was already assumed in the text preceding Eq. (17) in order to make the "surface error" negligibly small.

Let us denote by dU that fraction of the total energy U which is contained in these $d\lambda$ oscillations. According to (19) and (20a), one has

$$(21) \quad dU = \sum_{n=0}^{\infty} N_n \epsilon_n = Ah\nu \sum n\alpha^n = (1 - \alpha)h\nu d\lambda \sum_{n=0}^{\infty} n\alpha^n.$$

The series occurring here can be summed easily (by differentiating the geometrical series in (20) with respect to α), and equals $\alpha/(1 - \alpha)^2$. Eq. (21) yields together with the definition of α in (19)

$$(22) \quad dU = h\nu \frac{\alpha}{1 - \alpha} d\lambda = \frac{h\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1} d\lambda.$$

The denominator in this formula will reoccur in Planck's radiation law and, in a more general way, in the discussion of the Bose statistics in Vol. V.

We now substitute for $d\lambda$ according to (17) and integrate over ν from $\nu = 0$ up to an upper limit ν_s :

$$(23) \quad U = 4\pi Vh \left(\frac{2}{c_{\text{trans}}^3} + \frac{1}{c_{\text{long}}^3} \right) \int_0^{\nu_s} \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}.$$

Our first task is the determination of the limit ν_s , which we find according to Debye by identifying the total number z_s of oscillations with frequencies smaller than ν_s with the *number of degrees of freedom of the body*. Since a single mass point has three degrees of freedom the number in question equals three times the number of molecules. If in particular the body is one mole of a uniform substance, the number of molecules, according to p. 53, is Loschmidt's number L . The number z_s thus equals $3L$. From Eq. (16b) we then obtain for the determination of the upper limit ν_s the relation

$$(24) \quad 3L = \frac{4\pi}{3} V_{\text{mol}} \nu_s^3 \left(\frac{2}{c_{\text{trans}}^3} + \frac{1}{c_{\text{long}}^3} \right),$$

where the molar volume V_{mol} has been written instead of abc . This equation that defines ν_s may also be used to simplify (23) which now takes the form

$$(25) \quad U_{\text{mol}} = \frac{9Lh}{\nu_s^3} \int_0^{\nu_s} \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}.$$

With the abbreviations

$$(26) \quad y = \frac{h\nu}{kT}, \quad x = \frac{h\nu_0}{kT} = \frac{\Theta}{T}, \quad \Theta = \frac{h}{k} \nu_0,$$

we obtain finally

$$(27) \quad U_{\text{mol}} = 9Lk \frac{T^4}{\Theta^3} \int_0^x \frac{y^3 dy}{e^y - 1}.$$

The quantity Θ here introduced is known as *Debye's characteristic temperature*; for hard materials such as diamond and quartz it is approximately 100°K , for soft metals it amounts only to a few degrees. By and large, Eq. (27) is a good representation of the large number of data collected by Nernst and his co-workers. One can, of course, not expect that such details as depend on the specific structure of the crystal or the molecule should be covered by the general theory here developed.

We examine now Eq. (27) in the case of $T \gg \Theta$. This makes the upper limit $x \ll 1$, and the same is true for the integration variable y . Expanding the denominator

$$e^y - 1 = y \left(1 + \frac{y}{2} \dots \right),$$

we obtain for the integral in (27) with sufficient accuracy

$$\int_0^x y^3 dy = \frac{x^3}{3} = \frac{1}{3} \frac{\Theta^3}{T^3}.$$

From this we infer

$$(28) \quad U_{\text{mol}} = 3RT,$$

where the gas constant per mole, R , has been written for kL according to Eq. (7.15b).

Relation (28) contains the law of *Dulong-Petit*, since

$$(29) \quad c_{\text{mol}} = \frac{dU_{\text{mol}}}{dT} = 3R \cong 6 \text{ cal},$$

where c_{mol} denotes the specific heat per mole of the substance. Eq. (29) should be universally valid for all solids as an asymptotic law for high temperatures, provided one can disregard the relative vibrations of the components of the individual molecule. This restriction is necessary, since in counting the degrees of freedom we have assumed rigid molecules and disregarded the possible internal degrees of freedom. In fact, the rule of Dulong and Petit is particularly well satisfied for monatomic metals where the restriction is irrelevant. (In this case, c_{mol} is the gram atomic heat capacity).

The converse case, $T \ll \theta$, presents greater physical interest. The integral in (27) has now a very large upper limit ($x \gg 1$) and can be approximated by

$$(30) \quad \int_0^{\infty} \frac{y^3 e^{-y}}{1 - e^{-y}} dy = \int_0^{\infty} y^3 e^{-y} (1 + e^{-y} + e^{-2y} + \dots) dy.$$

Upon interchanging the order of summation and integration and introducing $z = ny$ in the n^{th} integral, one obtains instead of (30)

$$(30a) \quad \sum_{n=1}^{\infty} \frac{1}{n^4} \int_0^{\infty} z^3 e^{-z} dz = 3! \sum_{n=1}^{\infty} \frac{1}{n^4} = 3! \frac{\pi^4}{90}.$$

(The summation formula $\sum_{n=1}^{\infty} n^{-4} = \pi^4/90$ will be derived in Vol. VI in the chapter on Fourier series.)

With the use of the results (30) and (30a), Eq. (27) can be rewritten as

$$(31) \quad U_{\text{mol}} = \frac{3\pi^4}{5} \frac{RT^4}{\theta^3},$$

and differentiation with respect to T yields

$$(32) \quad c_{\text{mol}} = \frac{12\pi^4}{5} R \left(\frac{T}{\theta} \right)^3.$$

This is *Debye's third power law for the molar specific heat of a solid body at low temperature.*

Fig. 73 represents the general dependence of the specific heat on the temperature, as it is obtained by differentiation of Debye's equation (27); it also shows the two asymptotic laws for large T , i.e. the horizontal

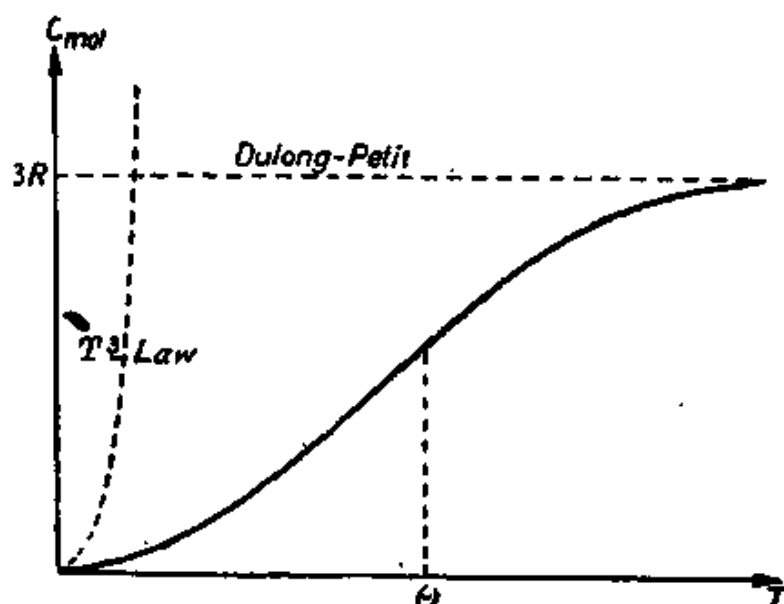


FIG. 73. The specific heat per mole of a solid according to Debye, cf. Eq. (26) for the meaning of θ

straight line of the Dulong-Petit law, and for small T the T^3 -parabola of Debye. The latter law is a useful approximation only for extremely small values of T . Furthermore, there is a general deviation from this law in the proximity of absolute zero for metallic substances, which is caused by the degrees of freedom of the conduction electrons in this temperature region.

We do not want to leave this topic without mentioning the results that are obtained in the corresponding electromagnetic problem. In that case we are concerned with the energy confined within a cavity containing electromagnetic radiation but entirely devoid of matter. The free oscillations of such a cavity can be described and counted in a fashion similar to the case of the elastic parallelepiped. (Actually, the electromagnetic case is much simpler.) Methods to deal with this problem have been developed by Lord Rayleigh and Jeans, including the procedure of counting lattice points. These investigations have served as a pattern for the elastic theory developed in this article. In the electromagnetic case the result is equivalent to our Eq. (23) with the following simplifications:

1. Longitudinal waves do not occur in the electromagnetic field, so that the term $1/c^3_{\text{long}}$ does not occur.

2. Transverse waves travel with the velocity of light c ; hence the term $2/c^3_{\text{trans}}$ is simply $2/c^3$.

3. The number of degrees of freedom of the electromagnetic field is, as far as we know, unlimited (at least, it is not restricted by a molecular structure); ν_s becomes thus infinitely large.

4. The energy content per unit of volume (energy density) is obtained, if $V = 1$ in (23). In this case, one usually writes u instead of U and obtains from (23)

$$(33) \quad u = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}.$$

Using again the substitution (26), and the formulas (30) and (30a), one has

$$(34) \quad u = \frac{8\pi h}{c^3} \frac{\pi^4}{15} \left(\frac{kT}{h}\right)^4 = \frac{8\pi^5}{15c^3} \frac{k^4}{h^3} T^4.$$

This is the law for the black body or cavity radiation in the definitive form due to Planck. *The T^4 -radiation law of Stefan and Boltzmann has the same physical origin as Debye's T^4 -law for the energy content of a solid and Debye's T^3 -law for the specific heat of a solid near absolute zero.*

45. The Surface Waves of the Elastic Half-Space.

We consider a homogeneous elastic solid having a plane boundary $y = 0$ and extending to infinity throughout the half-space $y > 0$, and want to study the simplest types of waves that can be propagated in the half-space. Along the boundary plane it is assumed that

$$(1) \quad \sigma_{yy} = \sigma_{yz} = \sigma_{yx} = 0;$$

then $y = 0$ is a stress-free boundary, as it would be if vacuum (or air) were on the other side.

A. Reflection of a Plane Transverse Spatial Wave

In the discussion of three-dimensional waves in 14 we distinguished between transverse and longitudinal waves; in the first case, the displacement vector is solenoidal (free of sources and sinks) and in the second case irrotational (cf. 20). Also the terms torsional and compressional waves are in use (cf. 14). The present discussion is restricted to the simplest case, that is, to *plane* waves of either sort.

Let us first consider a plane transverse wave whose direction of propagation forms the angle of incidence α with the positive y -axis and is "polarized" in the x, y -plane. This means that of the three displacements only ξ and η are different from zero. In complex notation such a wave is given in the following way:

$$(2) \quad \left. \begin{matrix} \xi_i \\ \eta_i \end{matrix} \right\} = \left. \begin{matrix} \cos \alpha \\ \sin \alpha \end{matrix} \right\} A \exp [ik(x \sin \alpha - y \cos \alpha)]$$

where the time-dependent factor $e^{-i\omega t}$ has been omitted for brevity [cf. the remarks to Eq. (13.15) and (13.15a)]; k is the wave number, and the subscript i characterizes the wave as incident. By Eq. (2) the condition of vanishing divergence is automatically fulfilled:

$$\Theta = \frac{\partial \xi_i}{\partial x} + \frac{\partial \eta_i}{\partial y} = 0.$$

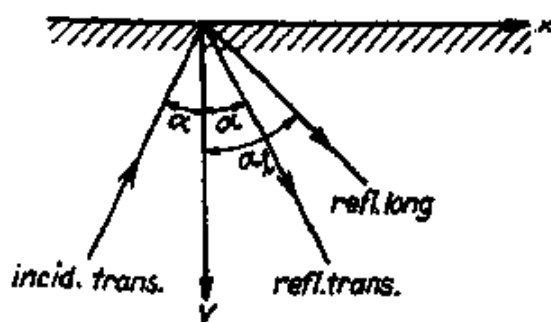
Also, the planes of constant phase

$$(3) \quad x \sin \alpha - y \cos \alpha = \text{Const},$$

are normal to the direction of propagation defined by the angle α , as they have to be, and the displacement vector lies in this plane. The signs of the trigonometric functions in (2) follow from Fig. 74 and appear in the

first section of the following table, which gives the cosines of direction of the propagation and of the displacement vector relative to the positive x and y axes.

FIG. 74. A longitudinal wave is created in the reflexion of a transverse wave polarized in the plane of incidence.



	x	y	x	y	x	y
dir. of prop.	$\sin \alpha$	$-\cos \alpha$	$\sin \alpha$	$\cos \alpha$	$\sin \alpha_1$	$\cos \alpha_1$
dir. of displ.	$\cos \alpha$	$\sin \alpha$	$\cos \alpha$	$-\sin \alpha$	$\sin \alpha_1$	$\cos \alpha_1$
	incid. transv.		refl. transv.		refl. longit.	

The differential equations of elasticity, which in the case of a transverse wave (because of $\Theta = 0$) have the simple form

$$(4) \quad \rho \frac{\partial^2 \xi}{\partial t^2} = \mu \nabla^2 \xi, \quad \rho \frac{\partial^2 \eta}{\partial t^2} = \mu \nabla^2 \eta,$$

are likewise satisfied by the set (2) provided that k for a given ω is determined from the well known equation

$$(5) \quad \frac{\omega^2}{k^2} = \frac{\mu}{\rho} = c_{\text{trans}}^2 = b^2$$

(b is the notation used for c_{trans} in 14).

The set (2) as it stands does not, of course, satisfy the boundary conditions (1). One might think that this can be achieved by superimposing a *reflected transversal* wave ξ_r, η_r

$$(6) \quad \left. \begin{array}{l} \xi_r \\ \eta_r \end{array} \right\} = \left. \begin{array}{l} \cos \alpha \\ -\sin \alpha \end{array} \right\} B \exp [ik(x \sin \alpha + y \cos \alpha)].$$

(the signs of the trigonometric functions now chosen as in the second section of the preceding table). This, however, cannot be done as we shall see now.

The quantities k and α in (6) must be identical with the corresponding quantities in (2). This follows for k from the differential equations (4), and for α as in elementary optics from the boundary conditions. For, the combination of an incident and a reflected wave

$$(7) \quad \xi = \xi_i + \xi_r, \quad \eta = \eta_i + \eta_r$$

is subject to the boundary conditions

$$\begin{aligned} 0 = \sigma_{yy} &= 2\mu\epsilon_{yy} = 2\mu \frac{\partial \eta}{\partial y} = -2\mu ik(A + B) \sin \alpha \cos \alpha e^{ikx \sin \alpha} \\ (8) \quad 0 = \sigma_{xy} &= 2\mu\epsilon_{xy} = \mu \left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x} \right) = -\mu ik(A - B)(\cos^2 \alpha - \sin^2 \alpha) e^{ikx \sin \alpha} \end{aligned}$$

The third condition $\sigma_{yy} = 0$ is automatically fulfilled. But the two preceding conditions contradict each other, since the one requires $A = -B$ and the other $A = B$. Choosing another angle of reflexion ($\alpha_r \neq \alpha_i$) would make the contradiction even worse, since it would introduce an infinity of contradictory conditions instead of two, viz. a different one for every single x -value.

There is only one way to avoid the dilemma: assume that in the process of reflexion of a transverse wave there is generated a *longitudinal* wave in addition to the reflected transverse wave, so as to make available a further amplitude constant C . With a view to the third section of the table we put

$$(9) \quad \left. \begin{matrix} \xi_i \\ \eta_i \end{matrix} \right\} = \left. \begin{matrix} \sin \alpha_i \\ \cos \alpha_i \end{matrix} \right\} C e^{ik_i(x \sin \alpha_i + y \cos \alpha_i)}$$

where k_i is now *different* from the previous k . The elastic differential equations (14.1b) require in the present case, because of the term with $\text{grad } \theta$, that

$$(10) \quad \frac{\omega^2}{k_i^2} = \frac{2\mu + \lambda}{\rho} = c_{\text{long}}^2 = a^2,$$

where $a = c_{\text{long}}$ is the notation already used in 14; the prescribed ω is of course the same as in (5). From (5) and (10) we have

$$(11) \quad \frac{k_i}{k} = n,$$

where we have put

$$(11a) \quad n = \frac{b}{a} = \left(2 + \frac{\lambda}{\mu} \right)^{-1/2}$$

But the angle of reflexion α_i is now *different* from the previous angle of incidence α . In fact, the *reflected longitudinal* wave obeys a law of the same structure as the *refracted transversal* wave in elementary optics, namely

$$(12) \quad \sin \alpha = n \sin \alpha_i.$$

Let us call this the refraction formula, although it refers in the present case to a reflexion rather than to a refraction, and designate the coefficient n as an index of refraction in the same sense.

In order to prove (12), we first calculate from (9)

$$\Theta = \frac{\partial \xi_i}{\partial x} + \frac{\partial \eta_i}{\partial y} = ik_i C e^{ik_i x \sin \alpha_i}$$

for $y = 0$. In setting up the new boundary conditions the right member of the first equation (8) must be augmented by

$$(12a) \quad 2\mu \frac{\partial \eta_i}{\partial y} + \lambda \Theta = ik_i (2\mu \cos^2 \alpha_i + \lambda) C e^{ik_i x \sin \alpha_i},$$

and the second Eq. (8) by

$$(12b) \quad \mu \left(\frac{\partial \xi_i}{\partial y} + \frac{\partial \eta_i}{\partial x} \right) = ik_i 2\mu \sin \alpha_i \cos \alpha_i C e^{ik_i x \sin \alpha_i}.$$

The modified Eqs. (8) should now establish conditions for the A , B , C that are independent of x . This is possible only in the case that the same exponential function of x occurs in (8) and (12). *The law of refraction (12) must therefore be valid.*

Eqs. (8) modified according to (12) yield now the linear homogeneous system

$$(13) \quad \begin{aligned} -2\mu k \sin \alpha \cos \alpha (A + B) + k_i (2\mu \cos^2 \alpha_i + \lambda) C &= 0 \\ -\mu k (\cos^2 \alpha - \sin^2 \alpha) (A - B) + k_i 2\mu \sin \alpha_i \cos \alpha_i C &= 0 \end{aligned}$$

which determines in a unique way the ratios B/A and C/A , that is, the reflected fractions of the incident amplitude A ; the latter may, of course, be arbitrarily prescribed. One obtains from (13)

$$(14) \quad \begin{aligned} \frac{C}{A} &= \frac{n \sin 4\alpha}{\cos^2 2\alpha + n^2 \sin 2\alpha \sin 2\alpha_i}, \\ \frac{B}{A} &= \frac{\cos^2 2\alpha - n^2 \sin 2\alpha \sin 2\alpha_i}{\cos^2 2\alpha + n^2 \sin 2\alpha \sin 2\alpha_i}. \end{aligned}$$

This solves the problem of the reflexion of a transverse wave polarized in the plane of incidence. If the incident transverse wave is polarized normal to the plane of incidence, no longitudinal component occurs in addition to the regularly reflected wave. When, on the other hand, an incident longitudinal wave is reflected there appears beside the regularly reflected longitudinal wave a transverse wave which is polarized in the plane of incidence. This will be taken up in problem VIII.5.

In the present discussion we are not so much interested in the *elastic* but in the *optical* interpretation of the mathematical formalism. As is well known, the central idea of 19th century optics is an elastic ether that carries the optical phenomena; this is true for the period of development that leads from Fresnel and Fraunhofer to F. Neumann and ends with H. Hertz. We shall now briefly indicate why the endeavors of these men to build an elastic theory of light resulted in insurmountable contradictions.

Once the polarizability of light had been recognized, the only waves admissible in optics had to be transverse. It is true that Röntgen, under the influence of one of Helmholtz's last papers, was considering the possibility that his newly discovered X-rays might be longitudinal waves. However, experiments on polarization of X-rays, first carried out by Barkla, precluded this possibility.

Now we have seen that even the simplest process of reflexion is elastically impossible without the occurrence of longitudinal waves. Starting with purely *transverse waves*, we have found that *longitudinal waves*—essentially foreign to optics—are generated in the process of reflexion.

The root of this dilemma is in the last analysis an arithmetical one. To see that, let us consider the more general case of reflexion and refraction that occurs in the transition of a wave from a medium 1 into a medium 2. Here we have altogether *six* elastic boundary conditions: three equations for the stresses and three equations that express the continuity of the displacements ξ , η , ζ . For an incident transverse light wave, however, we have only four unknowns at our disposal, viz. two pairs of amplitude ratios for the two modes of polarization in the reflected and the refracted wave. This is a contradiction that can not be resolved.

When we now replace the elastic ether by the quasi-elastic ether of 15, a different set of conditions appears. The stress is no longer a symmetric but an antisymmetric tensor which can be represented by a vector. The quantities σ_{ii} are zero and the number of stress conditions to be fulfilled is reduced from 3 to 2; the same is true for the displacement conditions. We then arrive at boundary conditions that are identical with those of the *electromagnetic theory of light*, and the intrinsic difficulties of the elastic theory of light are most beautifully resolved.

B. Elastic Surface Waves

In dealing with this problem we follow closely the treatment of the analogous hydrodynamic problem in Chapter V. We therefore consider plane waves that progress in the positive x -direction with prescribed circular frequency ω ; the wave number k is to be determined. When speak-

ing of plane waves, we mean that the excitation is independent of the third coordinate z . The wave is composed of a longitudinal and a transversal component.

The longitudinal component is again *irrotational* as in hydrodynamics, and can be derived from a potential Φ . Accordingly we put

$$(15) \quad \xi_i = -\frac{\partial \Phi}{\partial x_i}, \quad \eta_i = -\frac{\partial \Phi}{\partial y_i}, \quad \zeta_i = 0,$$

but in the present case Φ does not satisfy the condition of incompressibility $\nabla^2 \Phi = 0$ as in hydrodynamics; Φ obeys, however, the relation

$$(16) \quad \rho \frac{\partial^2 \Phi}{\partial t^2} = (2\mu + \lambda) \nabla^2 \Phi,$$

which follows from the elastic differential equations for ξ and η . We try

$$(17) \quad \Phi = e^{i(kx - \omega t)} u(y)$$

and obtain from (16) for the unknown function u the equation

$$-\rho \omega^2 u = (2\mu + \lambda) \left(-k^2 u + \frac{d^2 u}{dy^2} \right),$$

or

$$(18) \quad \frac{d^2 u}{dy^2} = p^2 u, \quad \text{where} \quad p^2 = k^2 - \frac{\omega^2}{a^2},$$

$$a^2 = \frac{2\mu + \lambda}{\rho}.$$

Let us assume that p is real and take the positive sign of the square root in the determination of p . Of the two particular integrals of (18)

$$u = e^{-py} \quad \text{and} \quad u = e^{+py}$$

the second must be excluded because of its behavior at $y = \infty$. Thus (17) becomes (if the time-dependent term is omitted as usual)

$$\Phi = Ae^{ikx - py},$$

which yields by (15):

$$(19) \quad \xi_i = -ikAe^{ikx - py},$$

$$\eta_i = +pAe^{ikx - py}.$$

The transverse component of the wave to which we now turn is *free of divergence*. We may therefore introduce a function Ψ in analogy to the hydrodynamic stream function and put

$$(20) \quad \xi_i = -\frac{\partial \Psi}{\partial y}, \quad \eta_i = +\frac{\partial \Psi}{\partial x}.$$

The function Ψ satisfies the same differential equations as ξ_i and η_i , viz:

$$(21) \quad \rho \frac{\partial^2 \Psi}{\partial t^2} = \mu \nabla^2 \Psi.$$

Again we try

$$(22) \quad \Psi = B e^{i(kx - \omega t)} v(y)$$

and obtain from (21) the following differential equation for v :

$$(23) \quad \frac{d^2 v}{dy^2} = q^2 v \begin{cases} q^2 = k^2 - \frac{\omega^2}{b^2}, \\ b^2 = \frac{\mu}{\rho}. \end{cases}$$

With the only admissible solution

$$v = e^{-qv}$$

(where q is supposed to be real and positive) we have from (20) and (22)

$$(24) \quad \begin{aligned} \xi_i &= q B e^{i k x - q y}, \\ \eta_i &= i k B e^{i k x - q y}, \end{aligned}$$

where again the time-dependent term has been omitted. By superposition of (19) and (24) we obtain for the total displacement, ξ and η , expressions in which *two* constants are to be determined: the wave number k and the amplitude ratio A/B . They are to be found from the *two* boundary conditions (1) $\sigma_{yy} = \sigma_{xz} = 0$; the third condition (1) is automatically fulfilled because of the planeness of the wave. If one calculates the strains ϵ_{yy} , ϵ_{xy} and Θ from the displacements ξ , η , and the stresses σ_{yy} , σ_{xz} from the strains, the first two boundary conditions (1) can be written in the form

$$-2\mu(p^2 A + ikqB) + \lambda(k^2 - p^2)A = 0$$

$$2ikpA - (q^2 + k^2)B = 0.$$

If one substitutes for p and q by (18) and (23) and expresses the elastic constants λ and μ by the wave velocities a and b , one obtains

$$(25) \quad \left(2k^2 - \frac{\omega^2}{b^2}\right)A + 2ik \sqrt{k^2 - \frac{\omega^2}{b^2}} B = 0,$$

$$2ik \sqrt{k^2 - \frac{\omega^2}{a^2}} A - \left(2k^2 - \frac{\omega^2}{b^2}\right)B = 0.$$

The determinant of this homogeneous system must vanish, hence

$$(26) \quad \left(k^2 - \frac{\omega^2}{2b^2}\right)^2 = k^2 \sqrt{k^2 - \frac{\omega^2}{a^2}} \sqrt{k^2 - \frac{\omega^2}{b^2}},$$

which is a cubic equation for k^2 [the term containing k^6 cancels if (26) is squared]. The ratio A/B follows from (25), once (26) has been solved.

We wish to discuss (26) in some detail only in the limiting case $a \rightarrow \infty$, that is, in the case of an almost incompressible medium. Eq. (26) is then transformed into

$$(27) \quad (1 - x)^2 = \sqrt{1 - 2x}, \quad x = \frac{\omega^2}{2b^2 k^2}.$$

Here we are interested in the root close to $x = \frac{1}{2}$. Putting $x = \frac{1}{2} - \delta$, we obtain by expansion of (27)

$$(28) \quad \left(\frac{1}{2} + \delta\right)^2 = 2\delta, \quad \frac{1}{16} + \frac{4}{8}\delta + \dots = 2\delta, \quad \delta \cong \frac{1}{24}.$$

Hence the root x of (27) is approximately

$$(29) \quad x = \frac{\omega^2}{2b^2 k^2} = \frac{1}{2} \left(1 - \frac{1}{12}\right), \quad \frac{\omega}{k} = b \left(1 - \frac{1}{24}\right),$$

where ω/k is the velocity of propagation of the surface waves. It is slightly smaller than the velocity b of the transversal three-dimensional waves. According to our assumption, the velocity a of the longitudinal three-dimensional waves is much larger.

We finally ask for the depth at which the disturbance due to the surface waves is no longer noticeable. This question is answered by the exponential factors e^{-uy} in (24) and e^{-vy} in (19). Characterizing the attenuation in y -direction by the depth at which the amplitude decreases to $1/e$ of its surface value, we obtain the depths $y = 1/q$ resp. $y = 1/p$. From (23) and (29) we find for the first of these quantities

$$(30) \quad \frac{1}{q} = \frac{1}{k} \frac{1}{\sqrt{1 - 2x}} = \frac{\sqrt{12}}{k} = \frac{\sqrt{12}}{2\pi} l,$$

where l is the wave length; for the second we have from (18) with $a \rightarrow \infty$

$$(30a) \quad \frac{1}{p} = \frac{1}{k} = \frac{l}{2\pi}.$$

The depth of penetration is in either case only a fraction of the wave length l with which the wave progresses along the x -axis; l is, of course, the same for the two components of the wave since the boundary conditions require the same coordination of the two components everywhere along the surface. The penetrations of the two components are nevertheless different.

The results (30) and (30a) show that we deal here with an elastic process that is restricted to the proximity of the surface and is made possible only by the discontinuity of the elasticity constants at the surface. The particular value of the velocity of the surface waves should be interpreted in the same sense. The elastic reaction at the surface is milder, since yielding is easier than in the interior, whence the smaller velocity of the surface waves in comparison with the spatial waves. (The smaller velocity of the spatial transverse waves as compared with the spatial longitudinal waves can be understood along similar lines; in the second case the material is subject to compressive loading which raises stronger reactive forces than the shear loading that permits yielding in the transverse direction.)

In 1886, when the problem of elastic surface waves was first treated, Lord Rayleigh voiced the belief that they might play an important part in earthquakes. In the subsequent development of seismology, which is due in a large measure to the theoretical and experimental efforts of E. Wiechert, Lord Rayleigh's conjecture was fully confirmed. The relative importance of the two-dimensional waves that spread along the surface of the earth increases with increasing distance from the center of the disturbance in comparison with the spatial waves, the energy of which disperses in three dimensions through the interior of the earth. Hence the surface waves form the *principal part* of the observable seismic effect. Furthermore they are also temporally separate from the spatial waves, since they arrive at the point of observation somewhat later than the transverse spatial waves, partly because of their smaller velocity, partly because of their longer path (arc versus cord). The *longitudinal* spatial waves arrive at the seismograph still earlier than the transverse spatial waves, and form the first part of the *primary wave*.

The seismic signals bring us information not only about the average elastic behavior of the interior of the earth but also about its possible discontinuities. One knows that the analysis of these signals together with other geophysical and astronomical data led Wiechert to the idea of a

“solid” core of the earth that consists essentially of iron and is surrounded by lighter layers of sulfides and silicates. This theory is based not only on propagation processes but also on processes of reflexion along successive discontinuity surfaces in the interior of the earth, the simplest form of which has been studied in section A of this article.

PROBLEMS

Chapter I

I.1. *The definition of curl \mathbf{A} by a linear vector function compared with its definition by a differential operator.* Show that the two definitions of the curl

$$\mathbf{B} = \text{curl } \mathbf{A} \quad \text{in (2.6)}$$

and

$$2(\mathbf{A} - \mathbf{A}_0) = \mathbf{B} \times \mathbf{r} \quad \text{in (2.7a)}$$

are equivalent, provided that \mathbf{r} is considered infinitesimally small as in 1.—It will be noticed that the field vector \mathbf{A} of Eq. (2.7a) of the text vanishes for $\mathbf{r} = 0$ since it corresponds to the rotatory part ∇_1 of the general motion in 1. To avoid this restriction, we have written $\mathbf{A} - \mathbf{A}_0$ instead of \mathbf{A} , so that \mathbf{A}_0 is the value of the new field vector at $\mathbf{r} = 0$. Of course, \mathbf{A}_0 corresponds to a velocity \mathbf{v}_0 associated by $\mathbf{s}_0 = \mathbf{v}_0 \Delta t$ with the translatory part \mathbf{s}_0 of the general motion.

I.2. *The vector character of curl \mathbf{A} .* Investigate the vector character of the curl by applying the transformation formulas (2.2) to the differential definition

$$\mathbf{B} = \text{curl } \mathbf{A}$$

(Note that the components of \mathbf{A} and the “differential operators” $\partial/\partial x_i$ must be transformed.)

I.3. *A table of the most important vector analytic operations in polar coordinates.* Set up the expression for grad, div, curl, ∇^2 and D in cylindrical and spherical coordinates.

I.4. *The symbol $\nabla^2 \mathbf{A}$.* Calculate the expression

$$\text{grad div } \mathbf{A} - \text{curl curl } \mathbf{A}$$

in cylindrical coordinates and compare the result with what is obtained when the ∇^2 operator calculated in I.3 is formally applied to the components A_r , A_ϕ , A_z . The two results are identical only in the case of the “Cartesian” component A_z [cf. the remarks following Eq. (3.10).]

I.5. *Concerning the so-called second boundary value problem of potential theory.* Show that a potential problem in which $\partial U/\partial n$ instead of U is prescribed along the boundary of the region has a unique solution except for an additional constant.

I.6. *The geometrical meaning of the tensor invariants.* True geometrical properties of a geometrical object are those that do not change in an orthogonal transformation; conversely, an orthogonal invariant must always allow a geometrical interpretation. The geometrical correlative associated with the tensor concept is the tensor quadric which, for convenience, may be assumed as an ellipsoid; hence it must be possible to interpret the three tensor invariants (4.10) as geometrical quantities connected with that ellipsoid.

Interpret the three tensor invariants as geometrical properties of the tensor ellipsoid.

I.7. *Condition that a given pattern of field lines belongs to a potential field.* If the field vector \mathbf{F} is everywhere given, the condition for the existence of a potential U is $\text{curl } \mathbf{F} = 0$ according to Eq. (3.7), but if the field lines are given, we know only the ratios $F_x : F_y : F_z$, and a scalar multiplier λ is left undetermined. Is it possible to choose λ in such a way that \mathbf{F} has a potential? Show that this can not be done unless

$$\mathbf{F}_0 \perp \text{curl } \mathbf{F}_0,$$

which is therefore a necessary condition. Here \mathbf{F}_0 is any vector that satisfies the given ratio condition, e.g. a unit vector that is everywhere parallel to the field lines.

Chapter II

II.1. *The altitude of the polytropic atmosphere.* a) Determine the altitude h of the polytropic atmosphere for the following values of the polytropic exponent: $n = 1.4$ (adiabatic case), $n = 1.2$, and $n = 1$ (isothermal case). b) Find the pressure as a function of the distance from the surface of the earth, the surface being considered spherical rather than plane as in 7. Show that under this condition the pressure in the isothermal atmosphere does not vanish at any finite distance from the earth nor, contrary to what one might expect, at infinity.

II.2. *Separation of a gas mixture in a centrifugal field.* Determine the speed of revolution required to produce a pressure difference equivalent to a water column of 40cm height between the axis and the circumference of a rotating drum that has a diameter of 60cm and contains normal air at 0°C (21% O_2 and 79% N_2 by volume), and find the composition of the air at the circumference of the drum.

II.3. a) *The two-dimensional analogue to the flow in a capillary tube.* A fluid flows between two parallel plates at a distance $2h$ from each other, the straight streamlines being parallel to the x -axis as in Fig. 19. Determine the velocity profile in the x,y -plane and the pressure loss Π between

two cross-sections in the distance l . Determine also the force required to keep the plates at rest against the reactive forces of the flow.

b) *River in laminar flow* (cf. Fig. 19a and the remarks on p. 120 about the experimental realization). Show that the velocity profile is identical with the lower half of the velocity profile in part (a) of this problem. (Assume both river depth and angle of inclination α of the bottom constant, and consider the boundary condition at $y = 0$.) Here the external force of gravity replaces the driving pressure of case (a).

II.4. *Straight and cylindrical Couette-flow*. According to (16.14) the so-called Couette-flow in straight streamlines (cf. Fig. 19b) is characterized by the velocity law

$$(1) \quad u = \frac{y}{h} U, \quad v = w = 0.$$

The equilibrium conditions (10.8a, b) are fulfilled if $p = \text{const}$; no driving pressure is required since the upper plate is kept in motion by the force $\mu U/h$.

Likewise in the case of the Couette flow proper, which takes place between two coaxial cylinders of radii r_1 and r_2 , no driving pressure, but only the moments M_1 and M_2 are required, the one to maintain the motion of the external cylinder and the other to keep the internal cylinder at rest. Set up and integrate the differential equations of the flow and compare the result with Eq. (1) in the limiting case $r_2 \rightarrow \infty$. Calculate also the moments M_1 and M_2 and note that they are not equal.

II.5. *The problem of Boussinesq*. An elastic body bounded by a horizontal plane, but otherwise infinite, is subject to a vertical force P applied at a single point this plane, all other points of which are free of stress. Calculate the state of stress in the body, and in particular the hoop stresses, that is to say, the principal stresses which are tangential to the circles about the direction of P .

Chapter III

III.1. *The convective terms of the hydrodynamic equations in polar coordinates*. Show that there is a difference between the correct calculation of the convective terms according to (11.6) and the result obtained from the "misinterpreted" symbol $(\nabla \text{ grad})$. Take cylindrical coordinates as an example, where one might be tempted to give the following meaning to the symbol $(\nabla \text{ grad})$

$$v_r \frac{\partial}{\partial r}, \quad v_\varphi \frac{1}{r} \frac{\partial}{\partial \varphi}, \quad v_z \frac{\partial}{\partial z}.$$

Verify that such a difference occurs in the r - and φ -component but not in the z -component. Relate the missing terms to the *centripetal* and the *Coriolis accelerations*.

III.2. *Similarity considerations in pipe flow*. Let the connection between the pressure gradient per unit of length Π/l and the average velocity v be given by

$$(1) \quad \frac{\Pi}{l} = Cv^n.$$

For the laminar state we have $n = 1$, for the turbulent state $n = 1.722$ according to Reynolds, and $n = 1.75$ according to Hagen (cf. the schematic representation in Fig. 18).

The quantity C can depend on the density, the kinematic viscosity ν , and the radius of the pipe a .

a) Following Kármán [Physik. Z.12, (1911) 283], assume C proportional to a product of powers of these quantities and find the exponents by dimensional reasoning.

b) It must also be possible to represent the relation (1) by the dimensionless quantities R and S and the quotient l/a in the general form

$$(2) \quad S = f\left(\frac{l}{a}, R\right).$$

In particular, if the function f is again assumed as a product of powers and the exponent of l/a is taken as 1, one obtains instead of (2)

$$(3) \quad S = \lambda \cdot \frac{l}{a} R^\delta,$$

where λ is a dimensionless constant; δ can be easily expressed by the exponent n occurring in (1).

c) Show finally that the relation $S = \lambda R^{-1/4}$ in (16.13) is in accordance with Hagen's value $n = 1.75$.

III.3. *The free surface of a fluid in the presence of external forces*. Using the principle of virtual work, find the differential equation that determines the shape of the free surface of a liquid subject to capillary forces and external (body) forces that have a potential. The fact that the total volume of the liquid mass does not change in the displacement will have to be used.

III.4. *Rise of a liquid in a capillary tube*. The result of the foregoing problem can be applied to find the elevation (or depression) of a wetting (or non wetting) liquid in a narrow capillary tube. Consider for this purpose the surface of the liquid inside the tube as spherical; outside the tube the surface may be considered everywhere as plane with sufficient accuracy.

Chapter IV

IV.1. *Calculations in bipolar coordinates.* The bipolar coordinates ρ and φ have been defined by Eqs. (19.10a, b). Show that the line element in these coordinates has the form

$$(1) \quad ds^2 = g^2(d\rho^2 + d\varphi^2), \quad g = \frac{c}{\cosh \rho - \cos \varphi}.$$

Set up the expression for ∇^2 and the equation of the vibrating membrane (17.8) in these coordinates. Note that the latter is not separable in ρ and φ , that is, there is no solution of the form

$$(2) \quad u = f_1(\rho)f_2(\varphi).$$

IV.2. *The field of flow about two vortices of equal strength and opposite orientation.* Eq. (19.10), which is equivalent to Eq. (19.11), yields the velocity field

$$\mathbf{v} = -\text{grad } \Phi = A \text{ grad } \varphi, \quad v_\rho = 0, \quad v_\varphi = \frac{A}{g}$$

where the value of g is to be taken from Eq. (1) of the foregoing problem. Show by a geometrical argument that this representation of \mathbf{v} is equivalent to Eq. (19.14).

IV.3. *Calculations in elliptic coordinates.* Set up the line element ds and the operator ∇^2 in elliptic coordinates ξ, η . The equation of the vibrating membrane becomes separable in these coordinates, that is, a trial solution of the form (2) in problem IV.1

$$u = f_1(\xi)f_2(\eta)$$

makes the integration possible.

IV.4. *Transition from bipolar to toroidal and "spindle" coordinates.* By rotating the plane of the bipolar coordinates ρ and φ about the axis $\rho = 0$ (cf. Fig. 26) one obtains a system of toroidal coordinates; by rotation about the axis $\varphi = 0$ another spatial coordinate system results for which the name "spindle" coordinates seems appropriate. Find the line element in these coordinates, set up the operator ∇^2 , and discuss the question of separability.

Chapters V and VI

V.1. *The kinetic energy of water waves.* As a supplement to the topic of 26, p. 189, compute for gravity waves in deep water the kinetic energy contained in a column of cross-section $\Delta x \cdot \Delta y = \lambda \cdot 1$, bounded by the

bottom and the free surface; compare this with the amount of potential energy contained in the same column.

Carry out the computations of p. 189 and p. 190 for moderately deep water and show that the value of S/E agrees with the value of U/N in Eq. (26.13).

VI.1. *The drift current.* Winds blowing along the surface of the ocean generate water currents known as drift currents. The surface layer which is exposed to the immediate action of the wind imparts its motion to the adjacent layers by the friction pressures. Consider the ocean infinitely extended and its density and viscosity everywhere constant; introduce a coordinate system ξ, η, ζ rigidly connected with a surface point of the earth, the ξ and η axes being tangential to the meridian and the parallel, respectively, and the ζ axis coinciding with the outward surface normal, so that ξ, η, ζ form a right hand system and the ξ axis points south (cf. Fig. 49, Vol. I). The Navier-Stokes equations must now be augmented by the Coriolis terms of Vol. I. 30.2 or 30.5.¹ For an unlimited and homogeneous ocean the components v_ξ and v_η , as well as p are independent of ξ and η ; the component v_ζ can be neglected.

Investigate the steady state that results if one assumes the wind velocity vector constant in the ξ, η -system.

VI.2. *Instability of a plane discontinuity surface separating two flow fields with different velocities.* In 30 and 32, pp. 216, 232, it was pointed out that discontinuity surfaces are unstable since they curl into vortices under the influence of friction. However, this instability is also present *without friction*; this is to be shown for the simplest case of a plane discontinuity surface $y = 0$, the fluid being the same on both sides.

Assume the fluid flows in x -direction with the constant velocities U_1 and U_2 for $y > 0$ and for $y < 0$ respectively. The problem is thus restricted to two-dimensions and remains so if a disturbance of the surface is introduced in form of a sinusoidal cylinder with generatrices parallel to the z -axis. External forces may be neglected.

Use the velocity potentials

$$\left. \begin{array}{l} \Phi_1 = -U_1 x \\ \varphi_1 \end{array} \right\} y > 0 \quad \left. \begin{array}{l} \Phi_2 = -U_2 x \\ \varphi_2 \end{array} \right\} y < 0 \quad \begin{array}{l} \text{(principal motion)} \\ \text{(disturbance)} \end{array}$$

and apply Bernoulli's theorem in the form (11.15) valid for non-steady flow. The discussion of the equations of motion so obtained shows that the principal motion is unstable for all wave lengths of the disturbance, and, since every disturbance can be compounded from sinusoidal com-

¹Cf. Synge and Griffith, *op. cit.*, Sec. 12.3.

ponents according to the Fourier integral theorem, the flow is unstable for any disturbance.

According to Helmholtz (*Über atmosphärische Bewegungen*, Akad. Ber. Berlin, 1898 and 1899) the formation of high cirrocumulus clouds is connected with the instability of the boundary between two air currents of equal directions but different velocities. The clouds are generated by the condensation of water vapor in form of droplets or ice crystals in regions of diminished pressure.—The instability of the discontinuity surface to be shown here for incompressible fluids occurs doubtlessly in compressible fluids, too, as long as all velocities are small compared to the sound velocity (cf. appendix to 13).

VI.3. *Stability of a horizontal water surface for varying wind velocities.* Proceeding as in the foregoing problem, investigate the stability of a water surface at rest under the influence of an air current moving parallel to it. The density of the air must not be neglected as was done in Chap. V., although it is small relative to water (0.00129:1). Gravity must be considered in this problem.

Chapter VII

VII.1. *The shape of the free surface of a rotating liquid derived according to Lagrange's equations.* The free surface of an incompressible fluid rotating with constant angular velocity was determined in 6 p. 44 by the methods of statics and is now to be found by the use of Lagrange's equations. The axis of the drum is vertical as before.

VII.2. *Transition from Euler's to Lagrange's equations in the one-dimensional compressible case.* For the special case of a one-dimensional flow of a compressible fluid in the absence of external forces, derive Lagrange's form of the hydrodynamic equations directly from Euler's equations and the law of conservation of mass.

VII.3. *Simplified determination of the linear differential equations for Riemann's shock wave problem according to Hadamard.* The one-dimensional Lagrange equation derived in the preceding problem permits us, according to J. Hadamard (*Leçons sur la propagation des ondes*, Paris, 1903), to shorten the transition to the linear differential equation of second order in 37. Assume that the compressible fluid has initially constant density and use the Legendre transformation of Vol. I. Eq. (42.8), cf. also footnote 10 on p. 312.

VII.4. *Stokes' formula for a rotating sphere.* In analogy to the translatory motion of a sphere analyzed in 35, investigate the uniform rotation of a sphere of radius a . Assume again laminar motion and an incompressible fluid, and show that the moment $M = 8\pi\mu a^3\omega$ is required to maintain uniform rotation with angular velocity.

Chapter VIII

VIII.1. *The bending of a beam on two supports under a single load.* A beam of length $l = a + b$ rests with its ends A and B freely on two supports of equal height; a vertical force P acts at a distance a from A . Make a diagram of the shear force and the bending moment along the beam and determine the elastic line.

VIII.2. *The vibrations of a rod.* a) A vertically suspended prismatic rod is rigidly clamped at the upper end and carries at the lower end a mass M that is large compared with the mass of the rod m . The system is free to oscillate in the direction of the axis of the rod. Determine the circular frequency of the longitudinal oscillations neglecting m as small relative to M , and the elasticity of the body M as unimportant relative to that of the rod.

b) The system a) is now allowed to oscillate about the axis of the rod. In calculating the frequency of the torsional oscillations the weight of M that acts in the axis of the rod may be disregarded, since it is balanced by tensile stresses that are constant in time. The appropriate assumption is now that the moment of inertia Θ corresponding to the mass M is large compared to that of the rod.

c) We finally consider a horizontal rod that is either rigidly clamped at one end and loaded at the other end by a mass M , or supported at both ends A and B as in prob. VIII.1 and loaded at a distance a from A (respectively b from B) by a mass M . The equilibrium of the rod is disturbed by a vertical impulse. Determine the circular frequency of the oscillations normal to the axis of the rod (bending oscillations), if again $M \gg m$.

VIII.3. *Rod subject to an impulsive load.* Assume that the mass M of the rod of prob. VIII. 2a is not a steady load, but was put in contact with the rod at $t = 0$, and that at that instant the velocity of M was v_0 . Calculate the maximum displacement ξ_{\max} of the mass M and the maximum stress σ_{\max} at the end cross-section of the rod in general and show in particular:

a) for $v_0 = 0$, σ_{\max} is twice its static value.

b) for large values of v_0 , σ_{\max} increases with the kinetic energy of M at the instant of contact, and depends on the individual dimensions of the rod only through its volume.

VIII.4. *A supplement to the torsion problem for an elliptic rod.* Determine the torsional moment M and the angle of torsion φ , for a rod with elliptic cross-section by carrying out the integrations in (42.23), and compare the factor that expresses the influence of the geometry of the rod with the polar moment of inertia of the area of the ellipse.

VIII.5. *The reflexion of a plane transverse and plane longitudinal wave*

within an elastic solid having a plane boundary. As a supplement to 45 show that a transverse wave oscillating perpendicularly to the plane of incidence is reflected regularly at a plane surface that is free of forces, and no longitudinal wave is generated simultaneously. Show on the other hand, that a plane longitudinal wave generates, in addition to the regularly reflected longitudinal wave, a transverse wave that oscillates in the plane of incidence.

ANSWERS AND COMMENTS

I.1. a) Differentiate the z -component of

$$2(\mathbf{A} - \mathbf{A}_0) = \mathbf{B} \times \mathbf{r}$$

partially with respect to y , the y -component partially with respect to z and subtract the results. On disregarding all terms that vanish simultaneously with r , the usual expression for the x -component of

$$\mathbf{B} = \text{curl } \mathbf{A}$$

is obtained.

b) Note that the omitted terms have the form

$$(\mathbf{r} \text{ grad})\mathbf{B}_0 - x \text{ div } \mathbf{B}.$$

This points to the connection with the general formula

$$\text{curl}(\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \text{ grad})\mathbf{a} - (\mathbf{a} \text{ grad})\mathbf{b} + \mathbf{a} \text{ div } \mathbf{b} - \mathbf{b} \text{ div } \mathbf{a},$$

valid for arbitrary vectors \mathbf{a} , \mathbf{b} but restricted to Cartesian components.

I.2. From (2.2) one has

$$x'_i = \sum_k \alpha_{ik} x_k, \quad x_k = \sum_i \alpha_{ik} x'_i, \quad A'_i = \sum_i \alpha_{ik} A_k$$

and finds by differentiation

$$\frac{\partial}{\partial x'_i} = \sum_k \frac{\partial x_k}{\partial x'_i} \frac{\partial}{\partial x_k} = \sum_k \alpha_{ik} \frac{\partial}{\partial x_k}, \quad \frac{\partial A'_l}{\partial x'_i} = \dots, \quad \frac{\partial A'_m}{\partial x'_i} = \dots$$

The difference of the derivatives $\partial A'_m / \partial x'_i - \partial A'_l / \partial x'_m$ gives the 1, 2, 3 components of $\text{curl}' A'$, when the pair (l, m) takes the values (2,3), (3,1), (1,2). Each derivative consists of 9 terms. When the difference is formed, 6 of the 18 terms cancel. The remaining 12 terms can be arranged in 6 pairs according to the 6 different $\partial A_k / \partial x_i$ and finally rearranged in 3 terms containing the three differences $\partial A_k / \partial x_i - \partial A_i / \partial x_k$. Hence each component of the transformed curl is a linear function of the components of the original curl; the coefficients are the cofactors of the elements of the transformation determinant as in (2.5). This points again to the axial vector character of the curl.

The treatment of this problem in Eqs. (2.10)-(2.15) of the text deserves preference.

I.3. It is, of course, possible to obtain the required expressions from

their definitions in Cartesian coordinates by applying the corresponding orthogonal transformations of coordinates, but this is a highly cumbersome procedure. A more efficient way is to specialize the general formulas (2.24), (2.25), (2.26), (3.9a), (3.9b) for the required type of coordinates. The results of this computation are collected for later use in the following table, which contains also the answers to problems I.4 and III.1 for cylindrical and spherical coordinates. As for the correct meaning of the symbols $\nabla^2 A$ and $(A \text{ grad})A$, the reader is again referred to Eqs. (3.10a) and (11.6).

It should be noticed that the sign of the curl depends on the orientation of the coordinate system. We have here assumed that the three coordinates in the sequence r, φ, z (and in the sequence r, ϑ, φ) form dexteral systems. Only under this condition the use of formula (2.26) is legitimate. In the case of spherical coordinates the usual definition of ϑ is such that ϑ increases from north to south. If this is adopted, φ runs contrary to the geographical longitude coordinate. If, however, φ is required to coincide with the geographical longitude, then the sign of the curl components would be reversed in accordance with Eq. (2.16), since the curl is an *axial* vector. The same thing happens when the order r, ϑ, φ is changed into r, φ, ϑ .

Cylindrical coordinates:

$$r, \varphi, z, \quad ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$$

$$g_1 = 1, \quad g_2 = r, \quad g_3 = 1$$

$$\text{grad } U = \frac{\partial U}{\partial r}, \quad \frac{1}{r} \frac{\partial U}{\partial \varphi}, \quad \frac{\partial U}{\partial z}$$

$$\text{div } A = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\text{curl}_r A = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}$$

$$\text{curl}_\varphi A = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$\text{curl}_z A = \frac{1}{r} \frac{\partial (r A_\varphi)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \varphi}$$

$$\nabla^2 U = \frac{1}{r} \left\{ \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{r} \frac{\partial U}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left(r \frac{\partial U}{\partial z} \right) \right\}$$

$$= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2}$$

$$= \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2}$$

$$DU = \left(\frac{\partial U}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial U}{\partial \varphi} \right)^2 + \left(\frac{\partial U}{\partial z} \right)^2,$$

$$(\mathbf{A} \cdot \text{grad}) \mathbf{A} = \begin{cases} \mathbf{A} \cdot \text{grad } A_r - \frac{A_\varphi^2}{r} \\ \mathbf{A} \cdot \text{grad } A_\varphi + \frac{A_r A_\varphi}{r} \\ \mathbf{A} \cdot \text{grad } A_z \end{cases}$$

$$\nabla^2 \mathbf{A} = \begin{cases} \nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2} \frac{\partial A_\varphi}{\partial \varphi} \\ \nabla^2 A_\varphi - \frac{A_\varphi}{r^2} + \frac{2}{r^2} \frac{\partial A_r}{\partial \varphi} \\ \nabla^2 A_z \end{cases}$$

Spherical coordinates:

$$r, \vartheta, \varphi, \quad ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2$$

$$g_1 = 1, \quad g_2 = r, \quad g_3 = r \sin \vartheta$$

$$\text{grad } U = \frac{\partial U}{\partial r}, \quad \frac{1}{r} \frac{\partial U}{\partial \vartheta}, \quad \frac{1}{r \sin \vartheta} \frac{\partial U}{\partial \varphi}$$

$$\text{div } \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin \vartheta A_\vartheta) + \frac{1}{r \sin \vartheta} \frac{\partial A_\varphi}{\partial \varphi}$$

$$\text{curl}_r \mathbf{A} = \frac{1}{r \sin \vartheta} \left(\frac{\partial (\sin \vartheta A_\varphi)}{\partial \vartheta} - \frac{\partial A_\vartheta}{\partial \varphi} \right)$$

$$\text{curl}_\vartheta \mathbf{A} = \frac{1}{r \sin \vartheta} \frac{\partial A_r}{\partial \varphi} - \frac{1}{r} \frac{\partial (r A_\varphi)}{\partial r}$$

$$\text{curl}_\varphi \mathbf{A} = \frac{1}{r} \left(\frac{\partial (r A_\vartheta)}{\partial r} - \frac{\partial A_r}{\partial \vartheta} \right)$$

$$\begin{aligned}
\nabla^2 U &= \frac{1}{r^2 \sin \vartheta} \left\{ \frac{\partial}{\partial r} \left(r^2 \sin \vartheta \frac{\partial U}{\partial r} \right) + \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial U}{\partial \vartheta} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{\sin \vartheta} \frac{\partial U}{\partial \varphi} \right) \right\} \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial U}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 U}{\partial \varphi^2} \\
&= \frac{\partial^2 U}{\partial r^2} + \frac{2}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \vartheta^2} + \frac{1}{r^2} \operatorname{ctg} \vartheta \cdot \frac{\partial U}{\partial \vartheta} + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 U}{\partial \varphi^2} \\
DU &= \left(\frac{\partial U}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial U}{\partial \vartheta} \right)^2 + \frac{1}{r^2 \sin^2 \vartheta} \left(\frac{\partial U}{\partial \varphi} \right)^2
\end{aligned}$$

$$(\mathbf{A} \operatorname{grad}) \mathbf{A} = \begin{cases} \nabla^2 A_r - \frac{2}{r^2} \left[A_r + \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin \vartheta A_\vartheta) + \frac{1}{\sin \vartheta} \frac{\partial A_\varphi}{\partial \varphi} \right] \\ \nabla^2 A_\vartheta + \frac{2}{r^2} \left[\frac{\partial A_r}{\partial \vartheta} - \frac{A_\vartheta}{2 \sin^2 \vartheta} - \frac{\operatorname{ctg} \vartheta}{\sin \vartheta} \frac{\partial A_\varphi}{\partial \varphi} \right] \\ \nabla^2 A_\varphi + \frac{2}{r^2 \sin \vartheta} \left[\frac{\partial A_r}{\partial \varphi} + \operatorname{ctg} \vartheta \frac{\partial A_\vartheta}{\partial \varphi} - \frac{A_\varphi}{2 \sin \vartheta} \right] \end{cases}$$

I.4. The r -component of $\operatorname{curl} \operatorname{curl} \mathbf{A}$ is according to the preceding table

$$\begin{aligned}
\operatorname{curl}_r \operatorname{curl} \mathbf{A} &= \frac{1}{r} \frac{\partial}{\partial \varphi} \operatorname{curl}_\vartheta \mathbf{A} - \frac{\partial}{\partial z} \operatorname{curl}_\varphi \mathbf{A} \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} r \frac{\partial A_\varphi}{\partial \varphi} - \frac{1}{r^2} \frac{\partial^2 A_r}{\partial \varphi^2} - \frac{\partial^2 A_r}{\partial z^2} + \frac{\partial^2 A_z}{\partial r \partial z},
\end{aligned}$$

and, likewise,

$$\operatorname{grad}_r \operatorname{div} \mathbf{A} = \frac{\partial^2 A_r}{\partial r^2} + \frac{\partial}{\partial r} \frac{A_r}{r} + \frac{1}{r} \frac{\partial^2 A_\varphi}{\partial r \partial \varphi} - \frac{1}{r^2} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial^2 A_z}{\partial r \partial z}.$$

We obtain therefore for the difference

$$\operatorname{grad}_r \operatorname{div} \mathbf{A} - \operatorname{curl}_r \operatorname{curl} \mathbf{A} = \dots - \frac{A_r}{r^2} - \frac{2}{r^2} \frac{\partial A_\varphi}{\partial \varphi}.$$

The terms fully written on the right hand side are those that are missed if the expression $\nabla^2 A_r$ is calculated by a schematical application of the ∇^2 -expression in cylindrical coordinates. One finds in the same way

$$\operatorname{grad}_\vartheta \operatorname{div} \mathbf{A} - \operatorname{curl}_\vartheta \operatorname{curl} \mathbf{A} = \dots - \frac{A_\vartheta}{r^2} + \frac{2}{r^2} \frac{\partial A_r}{\partial \varphi}.$$

I.5. For the difference $U = U_2 - U_1$ of two solutions U_1 and U_2 the Laplacian $\nabla^2 U$ vanishes in the interior, and the normal derivative $\partial U / \partial n$ vanishes on the boundary. From Green's theorem (3.16a) it follows that $U = \text{const}$ as in the first boundary value problem so that $U_2 = U_1 + \text{const}$. To conclude that $U_2 = U_1$ is, of course, no longer possible.

I.6. Let the strain ellipsoid with the semi-axes a, b, c be given by the equation

$$(1) \quad \sum \epsilon_{ik} x_i x_k = 1,$$

referred to a system x_1, x_2, x_3 which is connected with the system of principal axes by some rotation.

a) $\Theta = \Theta'$: For $x_2 = x_3 = 0$ we obtain $\epsilon_{11} = 1/\bar{x}_1^2$ where $|\bar{x}_1|$ is the radius vector from O to the point at which the x_1 -axis cuts the ellipsoid. The lengths $|\bar{x}_2|$ and $|\bar{x}_3|$ being correspondingly defined, we obtain for Θ

$$(2) \quad \Theta = \frac{1}{\bar{x}_1^2} + \frac{1}{\bar{x}_2^2} + \frac{1}{\bar{x}_3^2}.$$

The invariance of Θ permits the following geometric interpretation. *If an ellipsoid is cut by an orthogonal triad whose apex coincides with the center of the ellipsoid, the sum of the reciprocal squares of the intercepts is independent of the position of the triad; in particular, it is equal to*

$$\Theta' = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

b) $\Delta = \Delta'$: The plane $x_3 = 0$ intersects the ellipsoid (1) in the ellipse

$$\epsilon_{11}x_1^2 + 2\epsilon_{12}x_1x_2 + \epsilon_{22}x_2^2 = 1,$$

whose principal semi-axes are $1/\sqrt{\lambda_1}$ and $1/\sqrt{\lambda_2}$, λ_1 and λ_2 being the roots of the equation

$$(3) \quad \begin{vmatrix} \epsilon_{11} - \lambda & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} - \lambda \end{vmatrix} = 0.$$

The area of the ellipse is

$$(4) \quad F_2 = \frac{\pi}{\sqrt{\lambda_1 \lambda_2}}.$$

From (3) and (4) we conclude

$$\lambda_1 \lambda_2 = \epsilon_{11}\epsilon_{22} - \epsilon_{12}^2 = \frac{\pi^2}{F_2^2},$$

and, therefore, (F_1 , F_2 are defined correspondingly)

$$(5) \quad \Delta = \pi^2 \left(\frac{1}{F_1^2} + \frac{1}{F_2^2} + \frac{1}{F_3^2} \right).$$

Hence, if an ellipsoid is cut by an orthogonal trihedron whose apex coincides with the center of the ellipsoid, the sum of the reciprocal squares of the areas cut out by the trihedron is independent of its position; in particular it is equal to

$$\Delta' = \frac{1}{b^2 c^2} + \frac{1}{c^2 a^2} + \frac{1}{a^2 b^2}.$$

c) $D = D'$: The volume of an ellipsoid with the semi-axes a , b , c is given by

$$(6) \quad V = \frac{4\pi}{3} abc.$$

When the determinant (4.7) is set up in principal coordinates, it reduces to the product of the terms $1/a^2$, $1/b^2$, $1/c^2$ in the main diagonal so that

$$(7) \quad D' = \frac{1}{a^2 b^2 c^2} = \frac{16\pi^2}{9V^2}.$$

On the other hand, by an orthogonal transformation D' is transformed into D [cf. (4.7)]. We have therefore once more

$$(8) \quad D = \frac{16\pi^2}{9V^2}.$$

The invariance of D has its geometrical equivalent in the independence of the volume V from the coordinate system.

I.7. The relation $\mathbf{F} = \lambda \mathbf{F}_0$ implies

$$\text{curl } \mathbf{F} = \lambda \text{curl } \mathbf{F}_0 + \text{grad } \lambda \times \mathbf{F}_0.$$

By scalar multiplication with \mathbf{F}_0 and application of the vector formula

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}),$$

one obtains an equation from which one concludes that $\mathbf{F}_0 \perp \text{curl } \mathbf{F}_0$, provided $\text{curl } \mathbf{F} = 0$.

If U is a potential function that belongs to the pattern of field lines given by \mathbf{F}_0 so that $\lambda \mathbf{F}_0 = \text{grad } U$, then any function V of U is also a potential and the associated multiplier $\lambda' = \lambda dV/dU$. Thus the problem under consideration has an infinity of solutions, if it has a solution at all.

II.1. a) With the values of p_0 and ρ_0 of p. 99 one obtains from (7.9b) the values of h for $n = 1.4$, 1.2 and 1 , given on p. 52. The last value is in

accordance with the fact that the exponential function in (7.14) vanishes only for $z = \infty$.

In the adiabatic case ($n = 1.4$) the lapse rate of the temperature can be determined from Eq. (7.11). If we assume a ground temperature of $0^\circ\text{C} = 273^\circ\text{K}$, the lapse rate is just about 1°C per 100m. This same adiabatic lapse rate for dry air is also found in the descending air current of a foehn² between two points of *the same stream line*, while in vertical direction, that is, in the transition to a different streamline, only half this value is found (corresponding to the average temperature gradient in the atmosphere).

b) We substitute in the second equation (7.7)

$$V = - \frac{GM}{r},$$

where G = constant of Newton's law of gravitation, M = mass of the earth, r = distance from the center of the earth. The constant in (7.7) is determined by the condition: $p = p_0$ for $r = R$, R being the radius of the earth. In this way one obtains from (6a) and (7)

$$(1) \quad p = p_0 \left[1 - \frac{n-1}{n} \frac{\rho_0}{p_0} GM \left(\frac{1}{R} - \frac{1}{r} \right) \right]^{n/(n-1)}.$$

At the surface of the earth we have

$$(2) \quad \frac{GM}{R^2} = g.$$

The limit of the atmosphere is obtained if we let $p = 0$; with $r = R + h$ we obtain from (1)

$$(3) \quad 1 = \frac{n-1}{n} \frac{\rho_0}{p_0} gh \frac{R}{R+h},$$

in accordance with (7.9b), provided $h \ll R$ and $n > 1$.

For the isothermal case, $n = 1$, the two results no longer agree. If we substitute $G = R^2 g / M$ in (1) and pass to the limit $n \rightarrow 1$ we obtain

$$(4) \quad p = p_0 \exp \left[- \frac{\rho_0}{p_0} \cdot g R \left(1 - \frac{R}{r} \right) \right], \quad r > R.$$

In this case there is no upper limit of the atmosphere that would be definable by $p = 0$: for $r \rightarrow \infty$ we have the paradoxical result of a small but finite pressure value

²This is a not uncommon meteorological phenomenon in the European Alps, its characteristic feature being the sliding down of air masses along mountain slopes.

$$(5) \quad p = p_0 \exp \left(- \frac{\rho_0}{p_0} gR \right).$$

II.2. The partial pressures of N_2 and O_2 are originally ($\omega = 0$)

$$(1) \quad p_{01} = p_0 \frac{21}{100}, \quad p_{02} = p_0 \frac{79}{100}, \quad \text{with } p_0 = 1033 \times 981 \frac{\text{gr}}{\text{cm sec}^2}.$$

From (7.18) we have the partial pressures at the circumference r

$$(2) \quad \begin{aligned} p_1 &= p_{01} \exp \left(\frac{\mu_1}{2} \frac{r^2 \omega^2}{RT} \right) \cdots \mu_1 = 32, \\ p_2 &= p_{02} \exp \left(\frac{\mu_2}{2} \frac{r^2 \omega^2}{RT} \right) \cdots \mu_2 = 28. \end{aligned}$$

The pressure difference $(p_1 + p_2) - p_0$ is prescribed as $40 \times 981 \text{ gr cm}^{-1} \text{ sec}^{-2}$ and equals by (1) and (2)

$$p_1 + p_2 - p_0 = p_{01} \left[\exp \left(\frac{\mu_1}{2} \frac{r^2 \omega^2}{RT} \right) - 1 \right] + p_{02} \left[\exp \left(\frac{\mu_2}{2} \frac{r^2 \omega^2}{RT} \right) - 1 \right].$$

The arguments of the exponentials being small, we obtain a good approximation by expanding the exponentials and retaining the linear terms. With $R = 8.31 \times 10^7 \text{ erg/grad}$ and $T = 273^\circ \text{ K}$, this leads to

$$40 = 10.33(21 \times 32 + 79 \times 28) \frac{30^2 \omega^2}{2 \times 8.31 \times 10^7 \times 273},$$

whence $\omega = 260 \text{ sec}^{-1}$ and $n = 30 \omega / \pi = 2480 \text{ rpm}$. The pressure (or density) ratio at the circumference is found as

$$\frac{p_1}{p_2} = \frac{\rho_1}{\rho_2} = \frac{p_{01}}{p_{02}} \cdot \frac{1 + \frac{\mu_1}{2} \frac{r^2 \omega^2}{RT}}{1 + \frac{\mu_2}{2} \frac{r^2 \omega^2}{RT}} = \frac{21}{79} \cdot \frac{1.043}{1.038},$$

or only 0.5% larger than at the center, hence the centrifuge is not an efficient means to separate mixtures of gases or isotopes. Even the ultracentrifuge mentioned on p. 55 separates only aggregates of colloids or, at best, the giant molecules of protein mixtures.

II.3. a) In close analogy with Eqs. (10.11) to (10.17) one obtains

$$(1) \quad u = \frac{A}{2\mu} (h^2 - y^2) \quad u_m = \frac{Q}{2h} = A \frac{h^2}{3\mu} \quad \Pi = Al = 3\mu l \frac{u_m}{h^2},$$

that is, a parabolic velocity profile as in Fig. 14 and a pressure loss proportional to the first power of the average velocity and to the minus

second power of the half width h , which corresponds to the radius in Eq. (10.17). The reactive force acting on the upper or lower plate in the direction of the flow is $3\mu u_m/h$ per unit of area.

b) The component of gravity in the direction of the x -axis of Fig. 19a equals

$$(2) \quad F_x = \rho g \sin \alpha;$$

the component directed toward the bottom $y = -h$ need not concern us here. At the free surface the pressure equals the atmospheric pressure ($p = 0$); no pressure gradient in x -direction is required since its role is taken over by the body force (2). The differential equation (10.8a) reads under these conditions

$$-\mu \frac{\partial^2 u}{\partial y^2} = \rho g \sin \alpha.$$

The boundary conditions at $y = 0$ require both $p = 0$ and $p_{,yy} = -\mu \partial u / \partial y = 0$, but these conditions are satisfied in (a) because of the symmetry of the flow relative to the plane $y = 0$. The velocity profile in the present case is therefore identical with the lower half of the profile in a), the constant A being modified as required.

II.4. We refer to the formulas for cylindrical coordinates developed in the answer to problem I.3 and put

$$v = v_\varphi, \quad v_r = v_z = 0.$$

The condition of incompressibility

$$\dots \frac{1}{r} \frac{\partial v}{\partial \varphi} + \dots = 0$$

requires that v be independent of φ , so that $v = v(r)$. The same is true for the pressure p .

The expression for $\nabla^2 v$, which we find again in the table I.3, yields the differential equation

$$\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \frac{v}{r^2} = 0.$$

The general solution is a linear combination of the particular integrals $v = r$ and $v = 1/r$; the coefficients follow from the boundary conditions

$$v = 0 \quad \text{for} \quad r = r_i, \quad v = U \quad \text{for} \quad r = r_e.$$

In this way we obtain the solution

$$v = U \frac{r_e}{r} \frac{r^2 - r_i^2}{r_e^2 - r_i^2}$$

which becomes identical with the solution (16.14) for the straight Couette flow if we pass to the limit infinity with r_i and use the notations $r_o = r_i + h$ and $r = r_i + y$.

The reactive moments (per unit of length of the cylinder axis) are given by

$$M_o = 2\pi\mu Ur_o \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2}, \quad M_i = 2\pi\mu Ur_o \frac{2r_i^2}{r_o^2 - r_i^2}.$$

Their difference per unit of circumferential length of the external cylinder equals μU ; the dimension of this quantity is that of a torque per unit of area. It is in fact equal to the torque per unit of area which acts in the case of the straight Couette flow upon the two boundary plates (stress $\mu U/h \times \text{arm } h$).

II.5. This problem shows in a typical way how the fundamental equations of elasticity are applied. The solution requires a somewhat lengthy calculation, the single phases of which we shall clearly indicate. It is left to the reader to supply the connecting steps.

Let the direction of the force P coincide with the z -axis and its point of application be the origin of a system of cylindrical coordinates r, φ, z . On denoting the components of the displacement vector by

$$\rho = \sqrt{\xi^2 + \eta^2}, \quad 0, \quad \zeta,$$

the dilatation Θ is obtained according to the table I.3 in the form

$$(1) \quad \Theta = \frac{1}{r} \frac{\partial}{\partial r} (r\rho) + \frac{\partial \zeta}{\partial z}$$

since the problem is symmetrical about the z -axis.

a) The differential equation for Θ is found from (9.18) similarly as in (14.2):

$$(2) \quad \nabla^2 \Theta = 0.$$

Solutions that have the required symmetry are to be chosen from the following (cf. p. 248):

$$(3) \quad \Theta = \frac{1}{R}, \frac{\partial}{\partial z} \frac{1}{R}, \frac{\partial^2}{\partial z^2} \frac{1}{R}, \dots \quad R^2 = r^2 + z^2.$$

We select the second solution and put therefore

$$(4) \quad \Theta = -A \frac{\partial}{\partial z} \frac{1}{R} = A \frac{z}{R^3}.$$

b) The differential equation for the Cartesian component ζ is according to (9.18) and the table I.3

$$(5) \quad \frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r} \frac{\partial \zeta}{\partial r} + \frac{\partial^2 \zeta}{\partial z^2} = - \frac{\mu + \lambda}{\mu} A \left(\frac{1}{R^3} - \frac{3z^2}{R^5} \right).$$

A particular solution of this equation is

$$(6) \quad \zeta = B \frac{z^2}{R^3}, \quad B = - \frac{\mu + \lambda}{\mu} \frac{A}{2},$$

to which we may add solutions of the form (3). Choosing the first solution (3), we assume ζ in the more general form

$$(7) \quad \zeta = \frac{C}{R} - \frac{\mu + \lambda}{\mu} \frac{A}{2} \frac{z^2}{R^3}.$$

c) The differential equation for ρ follows easily from Eq. (1), viz.

$$(8) \quad \frac{1}{r} \frac{\partial}{\partial r} (r\rho) = \Theta - \frac{\partial \zeta}{\partial z} = A \frac{z}{R^3} + C \frac{z}{R^3} + \frac{\mu + \lambda}{\mu} A \frac{z}{R^3} \left(1 - \frac{3}{2} \frac{z^2}{R^2} \right).$$

This may be transformed into

$$(9) \quad \frac{\partial}{\partial r} (r\rho) = - \left(\frac{2\mu + \lambda}{\mu} A + C \right) z \frac{\partial}{\partial r} \frac{1}{R} + \frac{\mu + \lambda}{\mu} \cdot \frac{A}{2} z^3 \frac{\partial}{\partial r} \frac{1}{R^3}$$

and can be integrated to give

$$(10) \quad r\rho = - \left(\frac{2\mu + \lambda}{\mu} A + C \right) \frac{z}{R} + \frac{\mu + \lambda}{\mu} \frac{A}{2} \frac{z^3}{R^3} + f(z),$$

where $f(z)$ is the integration constant. For $r = 0$ and all positive values of z we must require $r\rho = 0$; since then $R = z$, we have

$$f(z) = C + \frac{3\mu + \lambda}{\mu} \frac{A}{2},$$

and Eq. (10) yields

$$(11) \quad \rho = \left(C + \frac{3\mu + \lambda}{\mu} \frac{A}{2} \right) \frac{1}{r} - \left(C + \frac{2\mu + \lambda}{\mu} A \right) \frac{z}{rR} + \frac{\mu + \lambda}{\mu} \frac{A}{2} \frac{z^3}{rR^3}.$$

d) The boundary conditions at the surface $z = 0$ are for $r > 0$

$$(12) \quad \sigma_{rr} = 0, \quad \sigma_{rz} = 0, \quad \sigma_{zz} = 0.$$

The first condition requires that

$$(13) \quad \frac{\partial \zeta}{\partial r} + \frac{\partial \rho}{\partial z} = 0.$$

The second condition is automatically fulfilled for reasons of symmetry,

not only at the surface $z = 0$, but everywhere inside the elastic body (see below under *f*). The third condition is equivalent to

$$\frac{\partial \zeta}{\partial z} = 0$$

since $\Theta = 0$ for $z = 0$ and $r > 0$; according to (7) it is also automatically fulfilled. We now substitute in (13) according to (7) and (11) for $z = 0$

$$\frac{\partial \zeta}{\partial r} = -\frac{C}{r^2}, \quad \frac{\partial \rho}{\partial z} = -\left(C + \frac{2\mu + \lambda}{\mu} A\right) \frac{1}{r^2}$$

and obtain

$$(14) \quad C = -\frac{2\mu + \lambda}{\mu} \frac{A}{2}.$$

e) The state of deformation can thus be described by the following relations

$$(15) \quad \Theta = A \frac{z}{R^3}, \quad \zeta = -\frac{A}{2} \left(\frac{2\mu + \lambda}{\mu} \frac{1}{R} + \frac{\mu + \lambda}{\mu} \frac{z^2}{R^3} \right)$$

$$\rho = \frac{A}{2r} \left(1 - \frac{2\mu + \lambda}{\mu} \frac{z}{R} + \frac{\mu + \lambda}{\mu} \frac{z^3}{R^3} \right).$$

It follows that

$$(16) \quad \sigma_{zz} = 2\mu \frac{\partial \zeta}{\partial z} + \lambda \Theta = A(\mu + \lambda) \frac{3z^3}{R^5}.$$

Only the integration constant A remains to be determined; this must be done by relating it to the external force P . If the body is cut in two at an arbitrary depth z_0 along the plane $z = z_0$, the following condition of equilibrium must hold:

$$(17) \quad P = -2\pi \int_{r=0}^{\infty} \sigma_{zz} r dr \quad \text{when} \quad z = z_0.$$

On substituting for σ_{zz} according to (16) and observing that

$$\int_{r=0}^{\infty} \frac{r dr}{R^5} = -\frac{1}{3R^3} \Big|_{r=0}^{\infty} = \frac{1}{3z^3},$$

one obtains

$$(18) \quad P = -2\pi A(\mu + \lambda).$$

f) Of the three shear stresses $\sigma_{r\theta}$, $\sigma_{\theta z}$, σ_{rz} the first two obviously vanish everywhere within the elastic body. This implies that the normal

stress $\sigma_{\varphi\varphi}$ belonging to any point of the surface $\varphi = \text{const}$ is a *principal stress* usually called the *hoop stress*. The associated stress trajectories are circles about the axis given by the forces P . In every torus or hoop of infinitesimal cross-section whose axis is a circular stress trajectory, there acts a definite stress $\sigma_{\varphi\varphi}$ that is characteristic of the hoop. The same hoop stresses occur in pipes and boilers subject to internal pressure, and their magnitude is the determining factor for the safety of the construction. In the present case the action of the external force P on the "hoops", of which the semi infinite body can be thought to consist, may be compared with that of an internal pressure. Since ρ denotes the increment of the hoop radius, the change of its length is $2\pi\rho$ and its extension $2\pi\rho/2\pi r = \rho/r$. It follows therefore that

$$(19) \quad \epsilon_{\varphi\varphi} = \frac{\rho}{r}, \quad \sigma_{\varphi\varphi} = 2\mu\epsilon_{\varphi\varphi} + \lambda\theta = 2\mu\frac{\rho}{r} + \lambda A \frac{z}{R^3},$$

or, because of (15),

$$\sigma_{\varphi\varphi} = \frac{A}{r^3} \left(\mu - (2\mu + \lambda) \frac{z}{R} + (\mu + \lambda) \frac{z^3}{R^3} \right) + \lambda A \frac{z}{R^3}.$$

This may be rewritten in the following form

$$(20) \quad \sigma_{\varphi\varphi} = \frac{A\mu}{r^3} \left(1 - 2 \frac{z}{R} + \frac{z^3}{R^3} \right).$$

The expression stays finite for $r = 0$ since the factor in parentheses vanishes.

The justification for the use of the particular solutions (4) and (7) must be seen in the fact that they can be made to fit all continuity requirements and boundary conditions of the problem. The vanishing of the displacements and stresses at infinity should also be counted among the boundary conditions. It can be verified directly from Eqs. (15), (16), and (20).

The assumption of a load concentrated at one point is physically not justifiable, but greatly simplifies the mathematics of the problem. The general case of a prescribed load distribution along the surface $z = 0$ can be easily constructed from the foregoing formulas by integration.

Boussinesq published his solution in 1879 and gave a complete representation of it in his book "Applications des potentiels directes, inverses, logarithmiques" Paris 1885.

III.1. The misinterpretation of the operator ($\nabla \text{ grad}$) would lead to the following expression for the r component

$$(1) \quad (\nabla \text{ grad})v_r = v_r \frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r} \frac{\partial v_r}{\partial \varphi} + v_z \frac{\partial v_r}{\partial z}.$$

Instead of this one obtains by (11.6)

$$v_r \frac{\partial v_r}{\partial r} + v_\varphi \frac{\partial v_\varphi}{\partial r} + v_z \frac{\partial v_z}{\partial r} - v_\varphi \operatorname{curl}_\varphi v + v_z \operatorname{curl}_z v.$$

With the values for $\operatorname{curl}_\varphi A$ and $\operatorname{curl}_z A$ according to table I.3, the last equation yields after cancellation of two pairs of terms

$$(2) \quad v_r \frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r} \frac{\partial v_r}{\partial \varphi} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\varphi^2}{r}.$$

The last term is the *centripetal acceleration* [missing in (1)]. The corresponding computation for the φ -component of $(\nabla \operatorname{grad}) v$ gives the additional term

$$\frac{v_r v_\varphi}{r} = v_r \omega, \quad \omega = \frac{v_\varphi}{r}.$$

When we here interpret ω as the amount of the axial vector ω parallel to the z -axis that represents the angular velocity associated with the linear velocity v_φ , then the term $v_r \omega = (\omega \times \mathbf{v})_\varphi$ becomes one half of a *Coriolis acceleration* [cf. Vol. I Eq. (29.4a)]. One finds, however, that the two results are identical in the case of the z -component. The formulas have already been tabulated in I.3 for cylindrical and spherical coordinates.

III.2 a) We put in Eq. (1) of the problem

$$C = A \rho^\alpha v^\beta a^\gamma,$$

where the α, β, γ are to be determined and A is dimensionless. (On substituting the dimensions of $v, \rho, \Pi/l$ and v , the following dimensional identity is obtained

$$\text{gr cm}^{-2} \text{ sec}^{-2} = \text{gr}^\alpha \text{ cm}^{-3\alpha+2\beta+\gamma+n} \text{ sec}^{-\beta-n},$$

which yields $\alpha = 1, \beta = 2 - n$, and $\gamma = n - 3$, whence

$$C = A \rho v^{2-n} a^{n-3}.$$

b) On substituting in Eq. (3) of the problem the expressions (16.9) and (16.8) for S and R , one obtains

$$\frac{\Pi}{\rho v^3} = \lambda \frac{l}{a} \left(\frac{va}{v} \right)^\delta.$$

v must occur here in the n^{th} power since this relation should be equivalent to Eq. (1) of the problem. It follows that $\delta = n - 2$ and therefore

$$\Pi = \lambda \rho \frac{l}{a} v^2 R^{n-2}.$$

c) On using Hagen's value $n = 1.75$ in this equation, one obtains

$$\Pi = \lambda \rho \frac{l}{a} v^2 R^{-1/4},$$

which agrees with the first form of Eq. (16.13), the exponent κ having Blasius's value of $\frac{1}{4}$.

III.3. Let the virtual displacement of the liquid be a motion of the surface elements dF in the direction of their normals by δn . Since the volume is conserved both positive and negative displacements δn (outward and inward displacements) must occur. The virtual work is composed of the work of the capillary forces and that of the external forces; the first contribution follows from (17.11) by multiplication with δn and integration over the surface, and the second contribution equals $\int U \delta n dF$ where U is the potential energy per unit of volume.

The constancy of the volume τ as a whole requires that the variation $\delta\tau$ vanish:

$$(1) \quad \delta\tau = \int \delta n dF = 0.$$

This restriction on δn can be taken care of by a Lagrangian multiplier λ , that is, by adding the term $\lambda \delta\tau$ to the left side of the virtual work and setting the total equal to zero.

The displacements δn being now arbitrary, one obtains

$$(2) \quad T \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + U + \lambda = 0.$$

If the mean curvature is here replaced by the appropriate expressions from differential geometry, one has in (2) a partial differential equation for the determination of the shape of the surface. The hitherto unknown multiplier λ is fixed by requiring the surface to pass through a given point with given mean curvature (cf. the next problem).

III.4. Let the z -axis coincide with the axis of the vertical capillary tube and count z positive upward from the external liquid level. Then

$$(1) \quad U = \rho g z.$$

For a wetting liquid the radius of the concave spherical surface is negative. Eq. (2) of the preceding problem then gives

$$(2) \quad -T \frac{2}{R} + \rho g z + \lambda = 0.$$

Since for the external surface $z = 0$ and the radius $R = \infty$, the multiplier

λ must be set equal to zero. The radius of the sphere R can be expressed by the radius of the tube r if the angle of contact is known:

$$(3) \quad R = r / \cos \vartheta.$$

Defining the elevation h by a suitably chosen mean value of the z -ordinates of the meniscus, one obtains from (2) and (3)

$$(4) \quad h = \frac{2T}{\rho g r} \cos \vartheta$$

where the value of $\cos \vartheta$ for wetting liquids is nearly 1. For non-wetting liquids R is positive and the right member in Eq. (4) takes therefore a minus sign; h denotes now the capillary depression.

IV.1. From (19.10a, b) one calculates $z = x + iy$ as a function of $\rho + i\varphi$ and obtains by differentiation the differential $dx + idy$ in terms of $d\rho + i d\varphi$. The square of the line element ds^2 is found as the absolute value of $dx + idy$ which leads directly to relation (1).

The fact that the factors of $d\rho^2$ and $d\varphi^2$ [denoted by g_1^2 and g_2^2 in (2.22)] are here equal means that the network of the two families of curves $\rho = \text{const}$, and $\varphi = \text{const}$ consists of infinitesimal squares (provided the increments $d\rho$ and $d\varphi$ are equal). In other words, the bipolar coordinates are isometric in the sense of p. 139. Note that the same can be achieved with polar coordinates in the plane if one replaces $z = re^{i\varphi}$ by $z = e^{\rho+i\varphi}$, where $\rho = \log r$.

In setting up the operator ∇^2 the three dimensional scheme (3.9b) must be specialized for *two* coordinates $p_1 = \rho$, $p_2 = \varphi$ which amounts to putting

$$g_1 = g_2 = g, \quad g_3 = 1, \quad \frac{\partial}{\partial p_3} = 0.$$

In the case of the *potential equation* $\nabla^2 U = 0$ the additional factor $1/g^3$ is immaterial, but its presence destroys the separability in the case of the *wave equation*. With the assumption of a periodic time dependence $U = u e^{-i\omega t}$ the wave equation $\nabla^2 U + k^2 U = 0$ reads

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial \varphi^2} + k^2 g^2 u = 0, \quad k = \frac{\omega}{c}.$$

On substituting here the trial solution (2) and dividing by $f_1 f_2$, one is led to a contradiction if one differentiates once more with respect to ρ or φ .

IV.2. We start from Eq. (19.11) which yields

$$\mathbf{v} = A \text{ grad } \varphi.$$

For an arbitrary function of the bipolar coordinates $f(\rho, \varphi)$ we have

$$\text{grad}_\rho f = \frac{1}{g} \frac{df}{d\rho}, \quad \text{grad}_\varphi f = \frac{1}{g} \frac{df}{d\varphi}.$$

If in particular $f = \varphi$, then $v_\rho = 0$ and $v_\varphi = 1/g$, as already mentioned in the text of the problem. With $A = \mu/\pi$ as in Eq. (19.8a) we have therefore

$$(1) \quad |\mathbf{v}| = v_\varphi = \frac{\mu}{\pi c} (\cosh \rho - \cos \varphi).$$

This should be compared with the value of $|\mathbf{v}|$ that follows from (19.14)

$$(2) \quad |\mathbf{v}| = \frac{\mu}{\pi} \left| \frac{\hat{\mathbf{e}}_1}{r_1} - \frac{\hat{\mathbf{e}}_2}{r_2} \right|.$$

The vector φ_1/r_1 (φ_2/r_2) has the amount $1/r_1$ ($1/r_2$) and is perpendicular to the radius vector from $z = +c$ ($z = -c$) to the field point. The angle subtended by the two radius vectors $\mathbf{r}_1, \mathbf{r}_2$ is therefore $\varphi = \varphi_1 - \varphi_2$. Vectorial addition then yields

$$\begin{aligned} \sqrt{\frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{2}{r_1 r_2} \cos \varphi} &= \sqrt{\frac{2}{r_1 r_2}} \sqrt{\frac{1}{2} \left(\frac{r_2}{r_1} + \frac{r_1}{r_2} \right) - \cos \varphi} \\ &= \sqrt{\frac{2}{r_1 r_2}} \sqrt{\cosh \rho - \cos \varphi}. \end{aligned}$$

Relations (1) and (2) are equivalent when

$$\frac{1}{c} \sqrt{\cosh \rho - \cos \varphi} = \sqrt{\frac{2}{r_1 r_2}},$$

which is in fact the case, as one sees by applying the law of cosines to the triangle whose corners are $z = \pm c$ and the field point.

IV.3. From the definition of elliptic coordinates in (19.15) we derive similarly as in problem IV.1

$$ds^2 = g^2(d\xi^2 + d\eta^2), \quad g = \frac{c}{\sqrt{2}} \sqrt{\cosh 2\xi - \cos 2\eta}.$$

The equation of the vibrating membrane has the same form as in IV.1, but the meaning of g is different:

$$(1) \quad \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + k^2 g^2 u = 0.$$

Substitution of the trial solution and division by $f_1 f_2$ leads to

$$(2) \quad \frac{f_1''}{f_1} + \frac{c^2 k^2}{2} \cosh 2\xi = -\frac{f_2''}{f_2} + \frac{c^2 k^2}{2} \cos 2\eta = \lambda.$$

Now the quantity λ (the "separation parameter") is independent of ξ and η ; to show this differentiate (2) with respect to η or ξ . (In other words, λ is a function of ξ alone according to the first, and of η alone according to the second member of Eq. (2) and is therefore constant.) The partial differential equation (2) then yields two ordinary differential equations

$$(3) \quad f_1'' = \left(\lambda - \frac{c^2 k^2}{2} \cosh 2\xi \right) f_1, \quad f_2'' = \left(-\lambda + \frac{c^2 k^2}{2} \cos 2\eta \right) f_2.$$

The solutions of (3) are called Mathieu's functions.

IV.4a. Toroidal coordinates first occur in a posthumous paper of Riemann "About the potential of a torus", Ges. Werke XXIV, p. 431. We introduce the notations ρ, φ, ψ and restrict the ranges as follows:

$$(1) \quad 0 \leq \rho < \infty, \quad 0 \leq \varphi < 2\pi, \quad 0 \leq \psi < 2\pi.$$

The geometrical meaning of ρ and φ follows from Fig. 26, the right half of which need only be considered (ρ positive). The angle ψ is the angle of rotation about the axis $\rho = 0$ (azimuth); the surfaces $\psi = \text{const}$ are half planes containing this axis. In the rotation the circles $\rho = \text{const}$ yield a system of coaxial tori about the axis $\rho = 0$, and the circular arcs $\varphi = \text{const}$ yield spherical segments.

Let us now change the notation of the complex variable z of Eq. (19.10a) into $\bar{z} = \bar{x} + i\bar{y}$; then by (19.10a, b)

$$(2) \quad \bar{x} = -\frac{c \sinh \rho}{\cosh \rho - \cos \varphi}, \quad \bar{y} = \frac{c \sin \varphi}{\cosh \rho - \cos \varphi}.$$

If we introduce a spatial Cartesian system x, y, z such that the z -axis coincides with $\rho = 0$, the origin with the center of Fig. 26, and the positive x -axis has the azimuth $\psi = 0$, the following relations hold between the coordinates x, y, z and the coordinates \bar{x}, \bar{y}, ψ of the same point:

$$(3) \quad x = \bar{x} \cos \psi, \quad y = \bar{x} \sin \psi, \quad z = \bar{y}.$$

The square of the line element follows now by differentiation (cf. prob. IV.1),

$$(4) \quad ds^2 = dx^2 + dy^2 + dz^2 = g^2(d\rho^2 + d\varphi^2) + g_3^2 d\psi^2,$$

where

$$(5) \quad g = g_1 = g_2 = \frac{c}{\cosh \rho - \cos \varphi}, \quad g_3 = -\bar{x} = \frac{c \sinh \rho}{\cosh \rho - \cos \varphi}.$$

The Laplacian operator follows by (3.9b):

$$(6) \quad \nabla^2 U = \frac{1}{g^2 g_3} \left\{ \frac{\partial}{\partial \rho} \left(g_3 \frac{\partial U}{\partial \rho} \right) + \frac{\partial}{\partial \varphi} \left(g_3 \frac{\partial U}{\partial \varphi} \right) + \frac{g^2}{g_3} \frac{\partial^2 U}{\partial \psi^2} \right\}.$$

One assumes U in the form

$$(7) \quad U = \sqrt{\cosh \rho - \cos \varphi} V(\rho, \varphi) e^{im\psi},$$

and obtains the following form of $\nabla^2 U$ after several steps

$$(8) \quad c^2 \nabla^2 U = (\cosh \rho - \cos \varphi)^{5/2} \cdot \left\{ \frac{\partial^2 V}{\partial \rho^2} + \coth \rho \frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial \varphi^2} + \left(\frac{1}{4} - \frac{m^2}{\sinh^2 \rho} \right) V \right\} e^{im\psi}.$$

When we are interested in the potential equation $\nabla^2 U = 0$, the resulting differential equation for V can be separated by setting

$$(9) \quad V = f(\rho) e^{in\varphi}.$$

The ordinary differential equation that results for f is

$$(10) \quad f'' + \coth \rho f' + \left(\frac{1}{4} - n^2 - \frac{m^2}{\sinh^2 \rho} \right) f = 0,$$

the solutions of which are generalized spherical harmonics.

In the case of the wave equation, however, the differential equation for V , obtained from (7), is

$$(11) \quad \frac{\partial^2 V}{\partial \rho^2} + \coth \rho \frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial \varphi^2} + \left(\frac{1}{4} - \frac{m^2}{\sinh^2 \rho} + \frac{k^2 c^2}{(\cosh \rho - \cos \varphi)^{3/2}} \right) V = 0.$$

It is not separable in the coordinates ρ and φ .

IV.4b. If the diagram in Fig. 26 is made to rotate about the axis $\varphi = 0$ rather than $\rho = 0$, the circles $\rho = \text{const}$ generate spheres and the circular arcs $\varphi = \text{const}$ generate "spindles" (the latter name is appropriate only for the surfaces $\varphi < \pi/2$). The third coordinate is the angle of rotation ψ about the axis, hence the surfaces $\psi = \text{const}$ are half-planes as before. The ranges for the coordinates are now

$$(1') \quad -\infty < \rho < +\infty, \quad 0 \leq \varphi < \pi, \quad 0 \leq \psi < 2\pi.$$

Introduce again Cartesian coordinates x, y, z , so that the z -axis coincides with the axis of rotation and the origin with the center of Fig. 26.

With \bar{x} and \bar{y} having the same meaning as before, we have now instead of (3)

$$(3') \quad x = \bar{y} \cos \psi, \quad y = \bar{y} \sin \psi, \quad z = \bar{x}.$$

By differentiation, a line element of the same form and also with the same value of g as in (4) is obtained, but the value of g_3 is now

$$(5') \quad g_3 = \bar{y} = \frac{c \sin \varphi}{\cosh \rho - \cos \varphi}.$$

The expression (6) for $\nabla^2 U$, which is still correct, yields when the form (7) of U and the present value of g_3 are used

$$(8') \quad c^2 \nabla^2 U = (\cosh \rho - \cos \varphi)^{5/2} \cdot \left\{ \frac{\partial^2 V}{\partial \rho^2} + \cot \varphi \frac{\partial V}{\partial \varphi} + \frac{\partial^2 V}{\partial \varphi^2} - \left(\frac{1}{4} + \frac{m^2}{\sin^2 \varphi} \right) V \right\} e^{im\psi}.$$

When we are interested in the potential equation $\nabla^2 U = 0$, the resulting differential equation for V becomes separable if one puts

$$(9') \quad V = f(\varphi) e^{im\psi}.$$

The equation for f is

$$(10') \quad f'' + \cot \varphi f' - \left(\frac{1}{4} - \alpha^2 + \frac{m^2}{\sin^2 \varphi} \right) f = 0,$$

which can be solved by generalized spherical harmonics like (10).

The wave equation $\nabla^2 U + k^2 U = 0$, however, is also not separable in these coordinates.

V.1. The kinetic energy to be computed is given by

$$(1) \quad E_{kin} = \frac{\rho}{2} \int_0^\lambda dx \int_\eta^\infty dy \left\{ \left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 + \left(\frac{\partial \Phi}{\partial z} \right)^2 \right\}.$$

On substituting for Φ according to (26.6) one obtains without difficulty

$$(2) \quad E_{kin} = \rho k \pi A^2 \int_\eta^\infty e^{-2ky} dy = \frac{\rho}{2} \pi A^2 e^{-2k\eta}.$$

This agrees with the expression (26.7a) for E_{pot} , since the amplitude A , which occurs in the expression for η [cf. Eq. (26.6)], is considered as a first order infinitesimal, and $e^{-2k\eta}$ may therefore be replaced by 1. Since E_{kin} is zero if the fluid is at rest, the expression (2) is at the same time the difference in kinetic energy between states of motion and rest for the spatial domain in question.

In the second part of the problem

$$\Phi = C \cos(kx - \omega t) \cosh k(h - y),$$

and one has by (24.3) and (24.8)

$$\eta = -\frac{\omega}{g} \sin(kx - \omega t) \cosh kh;$$

this yields

$$S = \frac{\rho}{4} \pi C^2 (\sinh 2kh + 2kh), \quad E_{kin} = E_{pot} = \frac{\rho}{4} \pi C^2 \sinh 2kh.$$

VI.1. The Navier-Stokes equations (10.8a, b) augmented by the Coriolis term read

$$(1) \quad 2\rho \mathbf{v} \times \boldsymbol{\omega} + \mu \nabla^2 \mathbf{v} - \text{grad } p + \mathbf{F} = 0.$$

With the simplification indicated in the problem text the ξ and η components of Eq. (1) yield for a point on the northern hemisphere

$$(2) \quad \begin{aligned} 2\omega \sin \varphi v_\eta + \nu \frac{d^2 v_\xi}{d\zeta^2} &= 0, \\ -2\omega \sin \varphi v_\xi + \nu \frac{d^2 v_\eta}{d\zeta^2} &= 0, \end{aligned}$$

where φ is the geographical latitude. (Note that the gravity vector \mathbf{F} is perpendicular to ξ and η .) Since $v_\zeta = 0$ and the vertical component of the Coriolis acceleration is negligibly small in comparison with g , the ζ -component of Eq. (1) gives simply the hydrostatic pressure gradient in ζ direction

$$-\frac{dp}{d\zeta} = \rho g.$$

With the simplifications introduced, the condition of incompressibility is automatically fulfilled. The two Eqs. (2) can be conveniently written as one complex equation, similarly as in the theory of Foucault's pendulum (cf. Vol. I. 31.5). We put

$$V = v_\xi + i v_\eta$$

and obtain instead of (2)

$$(3) \quad \frac{d^2 V}{d\zeta^2} - 2i\lambda^2 V = 0, \quad \lambda^2 = \frac{\omega}{\nu} \sin \varphi.$$

The integration yields

$$(4) \quad V = Ae^{\lambda(1+i)z},$$

if due regard is given to the orientation of ζ . The constant A must be determined from the excitation due to the wind pressure, whose components have to balance the friction pressures $p_{\tau\tau}$ and $p_{\tau\eta}$ everywhere on the surface $\zeta = 0$. If one introduces a complex friction pressure P by

$$P = p_{\tau\tau} + ip_{\tau\eta} = -\mu \left(\frac{dv_{\tau}}{d\zeta} + i \frac{dv_{\eta}}{d\zeta} \right) = -\mu \frac{dV}{d\zeta},$$

where v_{τ} has again been neglected, one obtains from (4) for $\zeta = 0$

$$(5) \quad -P_0 = \mu A \lambda (1 + i).$$

This is the value of the wind pressure on the surface $\zeta = 0$ in complex notation; it must be considered as a known vector. We introduce the notations

$$P_0 = |P_0| e^{i\delta}, \quad v_0 = -\frac{|P_0|}{\mu \lambda \sqrt{2}}, \quad \zeta = -z,$$

and obtain the velocities v_{τ} and v_{η} when we take A from (5), substitute it in (4), and separate real and imaginary parts:

$$(6) \quad \begin{aligned} v_{\tau} &= v_0 e^{-\lambda z} \cos \left(\lambda z + \frac{\pi}{4} - \delta \right), \\ v_{\eta} &= -v_0 e^{-\lambda z} \sin \left(\lambda z + \frac{\pi}{4} - \delta \right). \end{aligned}$$

This result permits the following interpretation: *The drift current on the surface makes an angle of -45° with the wind direction. The velocity of the current $|v|$ attenuates with increasing depth and changes its direction in such a way that at the depth*

$$(7) \quad z_E = \frac{\pi}{\lambda} = \frac{\pi}{\sqrt{\frac{\omega}{\nu}} \sin \varphi}$$

the current is opposite to the surface current. At that depth $|v|$ is equal to $v_0 \cdot e^{-\pi}$ (Ekman's friction-depth).

For a justification of our simplifying assumptions and for further discussion see: V. W. Ekman, "Dynamische Gesetze der Meeresströmungen" in the Innsbruck lectures quoted on p. 242. The fact that the actual motion is properly a turbulent one can be accounted for by giving

μ a larger value than usual; this takes care of the increased internal friction due to turbulence.

VI.2. When higher powers of φ_1 and φ_2 are neglected and the constant terms $U_1^2/2$ and $U_2^2/2$ of Bernoulli's equation are absorbed in the constants C_1 and C_2 , the following equations are obtained from (11.15) for the disturbance potentials φ_1 and φ_2

$$\frac{p_1}{\rho} = \frac{\partial \varphi_1}{\partial t} + U_1 \frac{\partial \varphi_1}{\partial x} - C_1 \cdots y > 0,$$

$$\frac{p_2}{\rho} = \frac{\partial \varphi_2}{\partial t} + U_2 \frac{\partial \varphi_2}{\partial x} - C_2 \cdots y < 0.$$

Since the pressure on both sides of the discontinuity surface ($y = 0$) must be the same, one obtains

$$(1) \quad \frac{\partial \varphi_1}{\partial t} + U_1 \frac{\partial \varphi_1}{\partial x} = \frac{\partial \varphi_2}{\partial t} + U_2 \frac{\partial \varphi_2}{\partial x} + c \cdots y = 0,$$

and the constant $c = C_1 - C_2$ must be zero because of the assumed periodicity of φ_1 and φ_2 .

We write the sinusoidal disturbance of the surface in the convenient complex form

$$(2) \quad \eta = Ae^{ikx}, \quad k = \frac{2\pi}{\lambda}, \quad A = \text{function of } t$$

The y -velocity $-\partial\varphi_1/\partial y$ of a particle close to the discontinuity surface is, except for second order terms, given by

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + U_1 \frac{\partial \eta}{\partial x}.$$

We obtain therefore

$$(3) \quad \frac{\partial \eta}{\partial t} + U_1 \frac{\partial \eta}{\partial x} = - \frac{\partial \varphi_1}{\partial y}$$

and, correspondingly,

$$(4) \quad \frac{\partial \eta}{\partial t} + U_2 \frac{\partial \eta}{\partial x} = - \frac{\partial \varphi_2}{\partial y}.$$

Since the problem is two-dimensional, we put

$$(5) \quad \varphi_1 = A_1 e^{ik(x+iy)}, \quad \varphi_2 = A_2 e^{ik(x-iy)}, \quad \left. \begin{matrix} A_1 \\ A_2 \end{matrix} \right\} = \text{functions of } t$$

Note that (5) fulfills the condition of incompressibility and φ_1 (φ_2) stays finite for $y = +\infty$ ($y = -\infty$). The problem has a solution only if the wave number k in (5) has the same value as in (2).

Upon substituting (2) and (5) in Eqs. (1), (3), and (4) the amplitudes A , A_1 , A_2 are found to obey the following linear system of differential equations

$$(6) \quad \frac{dA_1}{dt} + ikU_1A_1 = \frac{dA_2}{dt} + ikU_2A_2$$

$$(7) \quad \frac{dA}{dt} + ikU_1A = kA_1$$

$$(8) \quad \frac{dA}{dt} + ikU_2A = -kA_2.$$

The solution can be set up in the form

$$(9) \quad A = ae^{\alpha t}, \quad A_1 = a_1e^{\alpha t}, \quad A_2 = a_2e^{\alpha t}.$$

If one substitutes for A_1 and A_2 according to (9), one can compute a_1 and a_2 from (7) and (8), and obtains a quadratic equation for α by introducing these values into (6). The two roots are

$$(10) \quad \alpha = \frac{k}{2} [\pm(U_1 - U_2) - i(U_1 + U_2)].$$

Unless U_1 equals U_2 , one of the two roots has a positive real part which implies a continual increase of the amplitudes A , A_1 , A_2 . At any wave number k there are thus sinusoidal initial disturbances that will grow in the course of time.

VI.3. In the first pair of equations of VI.2, replace U_1 by U (wind velocity) and set $U_2 = 0$ (water at rest), add the action of gravity $-g\eta$ (η positive upward), and introduce the two densities ρ_1 and ρ_2 instead of ρ . With $s = \rho_1/\rho_2$ and $c = 0$ as before, these equations yield

$$(1') \quad s\left(\frac{\partial\varphi_1}{\partial t} - g\eta + U\frac{\partial\varphi_1}{\partial x}\right) = \frac{\partial\varphi_2}{\partial t} - g\eta.$$

Eqs. (2) and (5) of VI.2 remain unchanged, and so do (3) and (4) except for the specified values of U_1 and U_2 . One then has the same form of a solution as (9) for the differential equations that correspond to Eqs. (6), (7), (8) in VI.2, and expresses again a_1 and a_2 by a . In this way one obtains for α the quadratic equation

$$(10') \quad s(\alpha + ikU)^2 + \alpha^2 = -gk(1 - s).$$

One of the two roots has a positive real part if

$$(11') \quad U^2 > \frac{g}{k} \frac{1-s^2}{s}.$$

Now, $\sqrt{g/k}$ is, according to (23.16), the phase velocity V of the deep water waves if the inertia of the air is neglected (the value of V is only insignificantly smaller if the inertia of the air is considered, as can be shown by suitably expanding the calculations of 23); hence the condition of instability (11') can be written as

$$U > V \sqrt{\frac{1-s^2}{s}}.$$

In order to excite disturbances of wave length λ , the wind velocity U must be considerably larger than the phase velocity V associated with this wave length. The plane water surface is therefore stable with regard to this wave length for all wind velocities

$$U < \frac{V}{\sqrt{0.0013}} = 28V.$$

If in addition to gravity capillarity is considered, the stabilizing tendency of gravity is strengthened and the plane water surface is seen to be absolutely stable for all wave lengths as long as

$$U < 28 V_{\min}; \quad V_{\min} = 23.2 \text{ cm/sec.}$$

The phase velocity V_{\min} belongs to the wave length $\lambda_{\min} = 1.73 \text{ cm}$ as calculated on p. 184. A wind velocity smaller than $28 \times 23.2 \text{ cm/sec} \sim 6.5 \text{ m/sec}$ (wind strength 2-3 Beaufort) would not be able to disturb the plane surface of the water. This result assumes perfect homogeneity of the wind, which in reality is always turbulent.

VII.1. Let a, b, c be the coordinates of a fluid element at the time $t = 0$; in cylindrical coordinates r, φ, z

$$(1) \quad \begin{aligned} a &= x_0 = r \cos \varphi, \\ b &= y_0 = r \sin \varphi, \\ c &= z_0. \end{aligned}$$

Because of the uniform rotation the coordinates of the same element at the time t are obtained if we replace φ by $\varphi + \omega t$, where $\omega =$ angular velocity. Hence

$$\begin{aligned}
 x &= a \cos \omega t - b \sin \omega t, \\
 (2) \quad y &= b \cos \omega t + a \sin \omega t, \\
 z &= c.
 \end{aligned}$$

It is easy to verify that the condition of incompressibility (34.3) is fulfilled. Lagrange's equations (34.1) give in the present case (gravity being the only external force, and the z -axis being counted positive downward as in Fig. 7)

$$\begin{aligned}
 (3) \quad & \frac{\partial^2 x}{\partial t^2} \cos \omega t + \frac{\partial^2 y}{\partial t^2} \sin \omega t + \frac{1}{\rho} \frac{\partial p}{\partial a} = 0, \\
 & -\frac{\partial^2 x}{\partial t^2} \sin \omega t + \frac{\partial^2 y}{\partial t^2} \cos \omega t + \frac{1}{\rho} \frac{\partial p}{\partial b} = 0, \\
 & \frac{\partial^2 z}{\partial t^2} - g + \frac{1}{\rho} \frac{\partial p}{\partial c} = 0.
 \end{aligned}$$

If the accelerations in (3) are computed according to (2), one obtains for p the system of differential equations

$$(4) \quad \frac{1}{\rho} \frac{\partial p}{\partial a} = \omega^2 a, \quad \frac{1}{\rho} \frac{\partial p}{\partial b} = \omega^2 b, \quad \frac{1}{\rho} \frac{\partial p}{\partial c} = g.$$

The value of p obtained by integration of the system (4) is identical with Eq. (6.10) if (2) is taken into account.

To carry out the same calculation for a compressible gas is quite instructive; the continuity equation gives in this case $\partial \rho / \partial t = 0$ and the equation of motion becomes $\mathcal{O} = gz + r^2 \omega^2 / 2$, which becomes identical with Eq. (7.19) by applying (7.18) (the orientation of the z -axis is now different).

VII.2. One starts from the one-dimensional Euler equation (13.1) (no external force)

$$(1) \quad \frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

and introduces t and a as independent variables, where $a = x_0$ is the initial coordinate of a volume element whose coordinate at any later time is $x(t)$. The mass contained in this volume element equals

$$\text{for } t = 0 : \rho_0 da, \quad \text{for } t > 0 : \rho dx = \rho \frac{\partial x}{\partial a} da.$$

Conservation of mass demands

$$(2) \quad \rho \frac{\partial x}{\partial a} = \rho_0.$$

It follows that

$$(3) \quad \frac{\partial}{\partial t} \left(\rho \frac{\partial x}{\partial a} \right) = 0,$$

which is identical with Eq. (34.3b).

According to the remarks preceding Eq. (34.1), we can express the material acceleration in (1) by

$$(4) \quad u = \frac{\partial x}{\partial t} \quad \text{as} \quad \frac{du}{dt} = \frac{\partial^2 x}{\partial t^2};$$

the pressure gradient may be written

$$(5) \quad \frac{\partial p}{\partial x} = \frac{\partial p}{\partial a} \frac{\partial a}{\partial x} = \frac{\partial p}{\partial a} / \frac{\partial x}{\partial a}.$$

Substitution of these derivatives in (1) yields

$$(6) \quad \frac{\partial^2 x}{\partial t^2} \frac{\partial x}{\partial a} = - \frac{1}{\rho} \frac{\partial p}{\partial a}.$$

This agrees with (34.1) if and only if one takes into account the modification of Lagrange's equations for compressible fluids stated in (34.3a).

VII.3. If the compressible fluid has initially the constant density ρ_0 one concludes from Eq. (2) of answer VII.2 by differentiation with respect to the independent variable a

$$(1) \quad \frac{\partial}{\partial a} \left(\rho \frac{\partial x}{\partial a} \right) = 0,$$

or

$$(2) \quad \frac{1}{\rho} \frac{\partial \rho}{\partial a} = - \frac{1}{\omega} \frac{\partial^2 x}{\partial a^2},$$

where we have used Hadamard's abbreviation

$$(3) \quad \omega = \frac{\partial x}{\partial a}.$$

If now a p, ρ -relation $p = \varphi(\rho)$ is assumed one obtains, using (2),

$$(4) \quad \frac{1}{\rho} \frac{\partial p}{\partial a} = \frac{\varphi'(\rho)}{\rho} \frac{\partial \rho}{\partial a} = - \frac{\varphi'(\rho)}{\omega} \frac{\partial^2 x}{\partial a^2}.$$

Eq. (6) of the preceding answer then takes the form

$$(5) \quad \frac{\partial^2 x}{\partial t^2} = \Omega \frac{\partial^2 x}{\partial a^2}, \quad \Omega = \frac{\varphi'(\rho)}{\omega^2}.$$

The quantity Ω is a function of ω alone since, by Eq. (2) of the preceding answer, ρ is inversely proportional to ω . This suggests the introduction of ω as an *independent variable*. A second independent variable, suited for our purpose, is

$$(6) \quad u = \frac{\partial x}{\partial t}.$$

By (3) and (6), one finds

$$(7) \quad dx = \omega da + u dt.$$

If one introduces in accordance with the rule of Legendre's transformation the dependent variable

$$(8) \quad z(u, \omega) = ut + \omega a - x(t, a)$$

and forms its total differential, then one finds

$$(9) \quad dz = t du + a d\omega.$$

From (8) and (9) one has

$$(10) \quad \frac{\partial^2 x}{\partial a^2} = \omega_a, \quad \frac{\partial^2 x}{\partial t^2} = u_t;$$

$$(11) \quad \frac{\partial^2 z}{\partial \omega^2} = a_{\omega}, \quad \frac{\partial^2 z}{\partial u^2} = t_u,$$

and from (5) and (10) follows

$$(12) \quad u_t = \Omega \omega_a.$$

Using a well known transformation rule we conclude from (12)

$$(13) \quad a_{\omega} = \Omega t_u.$$

Because of (11) this is a *linear differential equation of second order* for z since the coefficient Ω depends only on the *independent variable* ω . Writing (13) in the variable z , we have

$$(14) \quad \frac{\partial^2 z}{\partial \omega^2} = \Omega \frac{\partial^2 z}{\partial u^2},$$

which is the basis of Hadamard's discussion of Riemann's shock waves. The fact that Eq. (14) is not identical with (37.11) is of course due to the different meanings of the new dependent and independent variables.

VII.4. Introduce spherical coordinates r, ϑ, φ , so that $r = 0$ coincides with the center of the sphere and $\vartheta = 0$ with the axis of rotation; let the constant angular velocity be $\dot{\vartheta} = \omega$. Assume that the fluid moves in circular streamlines about the axis in the sense of increasing φ , and show that this very simple assumption is permissible.—The first two of the velocity components $v_r, v_\vartheta, v_\varphi = v$ vanish and, because of the assumed incompressibility,

$$\frac{\partial v}{\partial \varphi} = 0,$$

while $\partial v / \partial \vartheta$ and $\partial v / \partial r$ are different from zero. Eq. (35.6) is again our starting point; developing the expressions for curl curl with the aid of answer I.3 we obtain

$$(1) \quad \text{curl}_r \text{curl} \mathbf{v} = \text{curl}_\vartheta \text{curl} \mathbf{v} = 0$$

$$(2) \quad \text{curl}_\vartheta \text{curl} \mathbf{v} = -\frac{1}{r} \frac{\partial^3(rv)}{\partial r^3} - \frac{1}{r} \frac{\partial}{\partial \vartheta} \frac{1}{r \sin \vartheta} \frac{\partial(\sin \vartheta v)}{\partial \vartheta}.$$

From (35.6) and (1) we conclude first

$$\text{grad}_r p = \text{grad}_\vartheta p = 0,$$

and since for reasons of symmetry $\text{grad}_\varphi p = 0$, we have altogether

$$(3) \quad p = \text{const.}$$

From (2), the differential equation for v is obtained:

$$(4) \quad \frac{\partial^3 v}{\partial r^3} + \frac{2}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \left(\frac{\partial^3 v}{\partial \vartheta^3} + \cot \vartheta \frac{\partial v}{\partial \vartheta} - \frac{v}{\sin^2 \vartheta} \right) = 0.$$

On substituting $v = C \cdot f(r)g(\vartheta)$, one obtains the two equations

$$(5) \quad f'' + \frac{2}{r} f' - \frac{\lambda}{r^2} f = 0, \quad g'' + \cot \vartheta g' + \left(\lambda - \frac{1}{\sin^2 \vartheta} \right) g = 0,$$

λ being the parameter of separation. A solution² of this equation that vanishes at infinity and remains finite for all values $0 < \vartheta < \pi$ is

$$(6) \quad f = \frac{1}{r^2}, \quad g = \sin \vartheta, \quad \lambda = 2.$$

²A more general solution is

$$f = \frac{1}{r^n}, \quad g = P_n^1(\cos \vartheta), \quad \lambda = n(n+1)$$

where P_n^1 is Legendre's associated function of the first kind, cf. Vol. VI, Chap. IV.

The velocity law is obtained as

$$(7) \quad v = \omega \frac{a^3}{r^3} \sin \vartheta$$

where the factor C was chosen so as to make $v = \omega a \sin \vartheta$ on the surface of the sphere. Thus, since we can fulfill the boundary conditions, the special choice $\lambda = 2$ in (6) is justified, and it is seen that the more general solutions mentioned in the footnote would be unsuited for our problem.

In order to pass from (7) to the friction pressures, one calculates similarly as on p. 36 from (4.26) and (4.28) with $\delta \dot{q}_1 = \delta \dot{q}_2 = 0$, $\delta \dot{q}_3 = v = \omega \sin \vartheta a^3/r^3$:

$$\dot{e}_{rr} = \dot{e}_{\vartheta\vartheta} = \dot{e}_{\varphi\varphi} = \dot{e}_{r\varphi} = \dot{e}_{r\vartheta} = 0$$

$$\dot{e}_{r\varphi} = \frac{1}{2g_1} \frac{\partial v}{\partial r} - \frac{1}{2g_3 g_1} v \frac{\partial g_3}{\partial r} = \frac{\omega}{2} \sin \vartheta \left(-2 \frac{a^3}{r^3} - \frac{a^3}{r^3} \right) = -\frac{3}{2} \omega \sin \vartheta \frac{a^3}{r^3}.$$

Hence all friction pressures vanish except

$$(8) \quad p_{r\varphi} = -2\mu \dot{e}_{r\varphi} = 3\mu\omega \frac{a^3}{r^3} \sin \vartheta.$$

The moment can now be computed:

$$(9) \quad -M = 2\pi a^3 \int p_{r\varphi} \sin^2 \vartheta d\vartheta = 6\pi\mu\omega a^3 \int_0^\pi \sin^3 \vartheta d\vartheta = 8\pi\mu\omega a^3$$

in agreement with (35.21). The moment required for the maintenance of uniform rotation is opposite to the moment of the friction pressures and acts therefore in the positive φ -direction as expected.

Kirchhoff treats the same problem in a more elegant though less concise manner using Cartesian coordinates, cf. footnote on p. 251.

VIII.1. We denote the reactions at the points A and B by these same letters and have

$$A = P \frac{b}{l}, \quad B = P \frac{a}{l}.$$

These reactions determine the shear force introduced in the context of Eq. (41.12) in the regions I (between A and P) and II (between P and B) as

$$(1) \quad S_I = A, \quad S_{II} = -B, \text{ respectively}$$

Note that the positive x -direction is from A to B and that both shears S_{II} and S_I refer to positive x -surfaces in the sense of Fig. 6. The shear force is constant throughout either one of the regions I and II, but jumps at the load point by the amount P ; this discontinuity is, of course, due

to the unrealistic assumption of a concentrated load. In a cross-section at a distance x from A (region I) the bending moment equals

$$(2) \quad M_I = Ax = P \frac{b}{l} x.$$

For the region II, in a cross-section at a distance ξ from B , the bending moment is

$$(3) \quad M_{II} = B\xi = P \frac{a}{l} \xi.$$

M_{II} refers to a positive ξ -surface which evidently is equivalent to a negative x -surface. A graph of M consists therefore of two straight lines that intersect at $x = a$ (or $\xi = b$). At this point, $M = M_{\max} = Pab/l$ according to both (2) and (3).

The deflection of y_I is found for the region I from

$$(4) \quad \frac{d^2 y_I}{dx^2} = \frac{P}{EJ} \frac{b}{l} x,$$

and for the region II from

$$(5) \quad \frac{d^2 y_{II}}{d\xi^2} = \frac{P}{EJ} \frac{a}{l} \xi.$$

The constants occurring in the solutions of (4) and (5) must be determined from the following boundary and continuity conditions:

- a) The deflection y_I must vanish at A , and y_{II} at B .
- b) At the point of application of the load both curves must join continuously and with common tangent. These conditions yield

$$(6) \quad \begin{aligned} y_I &= \frac{P}{EJ} \frac{a^2 b^2}{6l} \left(\frac{x^3}{a^2 b} - 2 \frac{x}{a} - \frac{x}{b} \right), \\ y_{II} &= \frac{P}{EJ} \frac{a^2 b^2}{6l} \left(\frac{\xi^3}{ab^2} - 2 \frac{\xi}{b} - \frac{\xi}{a} \right). \end{aligned}$$

The deflection at the load point is found from both equations as

$$(7) \quad y = y_{\max} = \frac{P}{EJ} \frac{a^2 b^2}{3l}.$$

The shear forces (1) and the bending moments (2) and (3) are always interconnected by the relation

$$(8) \quad S = \frac{dM}{dx}.$$

If the beam is loaded by several vertical forces at given points, shear force, bending moment, and deflection can be easily obtained by superposition of the corresponding expressions for single loads.

VIII.2. a) Let F be the cross-section of the rod, and count the displacement ξ of M (which is also that of the lower end cross-section) positive downward. M is subject to the downward action of gravity and the elastic force of the rod $\sigma \cdot F = FE \cdot \xi/l$ that acts upward. The equation for the displacement ξ reads

$$(1) \quad M\ddot{\xi} = Mg - k\xi, \quad k = \frac{FE}{l},$$

and gives, when integrated,

$$(2) \quad \xi = \frac{g}{\omega_0^2} + A \cos \omega_0 t + B \sin \omega_0 t,$$

where

$$(3) \quad \omega_0 = \sqrt{\frac{k}{M}} = \sqrt{\frac{EF}{Ml}};$$

the constants A and B are to be found from the given initial conditions. One sees that ξ oscillates about the constant value $\xi_0 = g/\omega_0^2$ which is nothing else but the static extension obtained from (1) for $\xi = 0$.

b) With the notations J_p = polar moment of inertia of the cross-section of the rod (assumed as circular), μ = torsion modulus, φ = angle of torsion of the mass M about the axis of the rod, we have by (42.6)

$$(5) \quad \Theta \ddot{\varphi} = \frac{-\mu J_p}{l} \varphi,$$

and

$$(6) \quad \omega_0 = \sqrt{\frac{\mu J_p}{\Theta l}}.$$

c) The motion of M can be described in good approximation as a vertical displacement (it is exactly that for a rod on two supports if $a = b = l/2$), hence Eq. (1) may be used again, but ξ means now the transverse displacement of M .

The "spring constant" k is

for the rod clamped at one end: $k = 3 EI/l^3$

for the rod supported at both ends: $k = 3 EI/l^3$

where I is the equatorial moment of inertia of the cross-section. This leads to the frequencies

$$(7) \quad \omega_0 = \sqrt{\frac{3EI}{Ml^3}}, \quad (8) \quad \omega_0 = \sqrt{\frac{3EI}{Ma^3b^3}} \quad \text{respectively.}$$

In a more accurate theory of extensional, torsional, and bending oscillations the mass of the rod itself would have to be considered; this would lead to partial differential equations of second order in place of the ordinary equations (1) and (5). In addition to the fundamental oscillation of frequency ω , which was determined in the foregoing, one would obtain an infinite sequence of harmonic oscillations according to the infinitely many degrees of freedom of the elastic rod that are now active.

VIII.3. For $t = 0$, one has $\xi = 0$, $\dot{\xi} = v_0$; then, according to Eq. (2) of the preceding answer, one obtains

$$(1) \quad \xi = \frac{g}{\omega_0^2} (1 - \cos \omega_0 t) + \frac{v_0}{\omega_0} \sin \omega_0 t,$$

hence

$$(2) \quad \xi_{\max} = \frac{g}{\omega_0^2} + \sqrt{\left(\frac{g}{\omega_0^2}\right)^2 + \left(\frac{v_0}{\omega_0}\right)^2}.$$

The maximum tension is found according to Hooke's law as

$$(3) \quad \sigma_{\max} = \frac{E}{l} \xi_{\max}.$$

Assume now

a) $v_0 = 0$,

then

$$\xi_{\max} = 2 \frac{g}{\omega_0^2} = \frac{2gMl}{EF}, \quad \sigma_{\max} = 2 \frac{gM}{F};$$

b) for large v_0

$$\xi_{\max} = \frac{v_0}{\omega_0} = \sqrt{\frac{Mv_0^2 l}{EF}}, \quad \sigma_{\max} = \sqrt{\frac{Mv_0^2 E}{Fl}}.$$

Both results verify the statements a) and b) made in the text of the problem.

VIII.4. The equations

$$y = a \cos u, \quad z = b \sin u, \quad 0 < u < 2\pi;$$

constitute a parametric representation of the elliptic circumference of the cross-section; a family of similar concentric ellipses is given by

$$y = a\lambda \cos u, \quad z = b\lambda \sin u, \quad 0 < \lambda < 1.$$

The parameters λ and u occurring here are suitable coordinates for the calculation of the integral (42.23). The Jacobian of the transformation $y, z \rightarrow \lambda, u$ is

$$D = \begin{vmatrix} \frac{\partial y}{\partial \lambda} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial \lambda} & \frac{\partial z}{\partial u} \end{vmatrix} = ab\lambda.$$

One obtains therefore

$$M = \mu\alpha \frac{\pi a^3 b^3}{a^2 + b^2}, \quad \varphi_1 = \alpha l = \frac{Ml}{\mu} \frac{a^2 + b^2}{\pi a^3 b^3}.$$

On the other hand, the polar moment of inertia J_z of the area of the ellipse is conveniently calculated by adding the two equatorial moments of inertia I_x and I_y , which, in turn, can be obtained from the equatorial moment of inertia of a circular area of radius a , $I = a^2\pi/4$, by "homogeneous compression". This means that I_x and I_y take up the factors b^3/a^3 and b/a respectively. The resulting polar moment of inertia is

$$J_z = \left(\frac{b^3}{a^3} + \frac{b}{a}\right)I = \frac{\pi}{4} ab(a^2 + b^2).$$

It is different from the geometrical factor occurring in M and φ_1 , viz.

$$\frac{\pi a^3 b^3}{a^2 + b^2},$$

the two factors being identical only for a circular cross-section where $a = b$.

VIII.5. Let the plane of incidence be the x, y -plane as in 45. Polarization perpendicular to this plane means that of the three displacements ξ, η, ζ , only ζ is excited. According to the table on p. 327 a *transverse wave* incident at the angle α_1 and reflected at the angle α_2 , is given by

$$\zeta_i = A \exp [ik(x \sin \alpha_1 - y \cos \alpha_1)],$$

$$\zeta_r = B \exp [ik(x \sin \alpha_2 + y \cos \alpha_2)].$$

Since the frequency ω of both waves is the same this is also true for the wave number k , cf. (45.5). The three conditions (45.1) (boundary $y = 0$ free of forces) reduce to one condition

$$\sigma_{yz} = 2\mu e_{yz} = \mu \frac{\partial \zeta}{\partial y} = \mu \left(\frac{\partial \zeta_i}{\partial y} + \frac{\partial \zeta_r}{\partial y} \right) = 0,$$

which can be fulfilled, without assuming an additional longitudinal wave, simply by setting

$$(1) \quad \alpha_1 = \alpha_2 \quad \text{and} \quad B = A.$$

This is a regular reflexion with constant amplitude.

If on the other hand, a *longitudinal* wave strikes at the angle α_1 and is reflected at the angle α_2 , we have to deal with the two components ξ, η given by

$$(2) \quad \left. \begin{matrix} \xi_i \\ \eta_i \end{matrix} \right\} = \left. \begin{matrix} \sin \alpha_1 \\ -\cos \alpha_1 \end{matrix} \right\} A \exp [ik(x \sin \alpha_1 - y \cos \alpha_1)],$$

$$(3) \quad \left. \begin{matrix} \xi_r \\ \eta_r \end{matrix} \right\} = \left. \begin{matrix} \sin \alpha_2 \\ \cos \alpha_2 \end{matrix} \right\} B \exp [ik(x \sin \alpha_2 + y \cos \alpha_2)].$$

Since ω is again the same for both waves [it is given by (45.10)], the wave number is the same too.

Of the three conditions (45.1) only $\sigma_{xx} = 0$ is now identically fulfilled. The two others

$$(4) \quad \sigma_{xx} = 0, \quad \sigma_{xy} = 0$$

call again for regular reflexion, that is, $\alpha_2 = \alpha_1 = \alpha$, but they cannot be satisfied by adjusting the one available constant B/A . Again a reflected *transverse* wave polarized parallel to the plane of incidence must be added; this wave can be written according to the central column of the table on p. 327 in the following form

$$(5) \quad \left. \begin{matrix} \xi_t \\ \eta_t \end{matrix} \right\} = \left. \begin{matrix} \cos \alpha_t \\ -\sin \alpha_t \end{matrix} \right\} C \exp [ik_t(x \sin \alpha_t + y \cos \alpha_t)],$$

The two angles α_t and α are connected by the "law of refraction" (45.12)

$$\sin \alpha = n \sin \alpha_t, \quad n = \frac{k_t}{k} = \left(2 + \frac{\lambda}{\mu}\right)^{1/2}.$$

One computes from (2), (3), and (5) the total displacements.

$$\xi = \xi_i + \xi_r + \xi_t, \quad \eta = \eta_i + \eta_r + \eta_t$$

and finds from the two equations (4) two conditions for the two ratios $A : B : C$ [cf. the similar equations (45.13) and (45.14)].

THE COMPONENTS OF THE THREE-DIMENSIONAL STRAIN TENSOR IN ORTHOGONAL CURVILINEAR COORDINATES

The formulas (4.26) and (4.28) for the strains ϵ_{ii} and ϵ_{ik} simplify considerably if instead of the displacements δq , which have the dimensions of a length, the associated variations of the coordinates δp are introduced; the latter do not, in general, have uniform dimensions; e.g. in the case of polar coordinates, the quantities δp are (in part) angular changes. On substituting in (4.26) and (4.28) for the δp_k according to Eq. (4.19) $\delta q_k = g_k \delta p_k$ and carrying out the differentiations $\partial \delta q_i / \partial p_i$ one obtains directly

$$(1) \quad \epsilon_{ii} = \frac{\partial \delta p_i}{\partial p_i} + \frac{\delta p_\alpha}{g_i} \frac{\partial g_i}{\partial p_\alpha},$$

$$(2) \quad 2\epsilon_{ik} = \frac{g_i}{g_k} \frac{\partial \delta p_i}{\partial p_k} + \frac{g_k}{g_i} \frac{\partial \delta p_k}{\partial p_i}.$$

The second term in the right member of Eq. (1) symbolizes the sum of the terms that are obtained by putting $\alpha = 1, 2, \dots, n$, in accordance with the following summation convention introduced by Einstein: a Greek¹ character that occurs twice in one term is to be considered as a summation index. Since we operate in three-dimensional space, the second term on the right side of Eq. (1) stands for

$$\sum_{\alpha=1}^3 \frac{\delta p_\alpha}{g_i} \frac{\partial g_i}{\partial p_\alpha}.$$

To simplify the notation further we shall denote the variation of the coordinate, δp_i , by ξ^i . This we do in accordance with the general conventions of tensor calculus, where superscripts indicate contravariant and subscripts covariant components of a vector. With these changes Eq. (1) reads

$$(1a) \quad \epsilon_{ii} = \frac{\partial \xi^i}{\partial p_i} + \frac{\xi^\alpha}{g_i} \frac{\partial g_i}{\partial p_\alpha} = \xi^i_{;i}.$$

¹By restricting the indices of summation or dummy indices to Greek letters one avoids confusing ϵ_{ii} or the later occurring $\xi^i_{;i}$ with the first scalar or spur of the corresponding tensors. Without this or a similar restriction the notations of tensor calculus are likely to mislead the student, however concise or convenient they may be.

The *subscript* that occurs here in the symbol $\xi^i_{;}$ is used to denote the *covariant derivative* of the contravariant vector ξ^i with respect to the coordinate p_i (note that the position of the subscript indicating the differentiation is shifted to the right; in this way a definite order of the indices is established). Vector components are called *contravariant*² if they transform in the same way as the coordinate differentials dp_i ; covariant vector components transform conversely to the differentials in the sense that the matrix of the transformation $\xi^i \rightarrow \xi_i$ is the inverse of the matrix of the transformation $dp_i \rightarrow dp'_i$. The differential quotients of a scalar point function with respect to the coordinates form examples of covariant vector components (gradient of a scalar). The same is *not* true for the differential quotients of a vector like $\partial \xi^i / \partial p_i$. The latter quantity must be augmented by the sum over α that appears in (1a) in order to become a covariant derivative.

We shall first verify that the factor that goes with ξ^α in (1a) is a special case of what is generally called a Christoffel symbol³ (of the second kind).

The general definition of this symbol (for arbitrary coordinates x_1, x_2, x_3) is

$$(3) \quad \{\mu\nu, \sigma\} = \frac{1}{2} g^{\sigma\lambda} \left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda} \right),$$

where $g_{\alpha\lambda}$ are the covariant components of the metric tensor. In the case of orthogonal coordinates to which we restrict ourselves in the following, we have

$$g_{ik} = \delta_{ik} g_i g_k, \quad g^{ik} = \frac{\delta_{ik}}{g_i g_k}$$

(for the definition of δ_{ik} cf. p. 10). It is now easy to verify that in this case

$$(3a) \quad \{\alpha i, i\} = \frac{1}{g_i} \frac{\partial g_i}{\partial p_\alpha},$$

²The terms contravariant and covariant were introduced by Einstein to replace the terms *cogredient* and *contragredient* (respectively) which had been used before. It will be noticed that for reasons of consistency the coordinates and their differentials should also be written with superscripts, that is, p_i and dp_i should be replaced by p^i and dp^i respectively; this notation has actually been adopted by some authors.---The differential operation $\xi^i_{;}$ is covariant in the sense defined before, but we shall not prove that here, nor shall we prove formula (4) or the rule concerning the raising or lowering of indices. For these matters the reader is referred to the very careful although condensed representation by A. S. Eddington in his *Mathematical Theory of Relativity*, (Cambridge, 1923) and to *The absolute differential Calculus* by T. Levi-Civita, (engl. transl. London, 1927). Our Eqs. (3b) and (4) are derived in 20, Eq. (4) of Eddington's book.

³Instead of $\{\mu\nu, \sigma\}$ one often finds $\left\{ \begin{smallmatrix} \mu\nu \\ \sigma \end{smallmatrix} \right\}$; note also Einstein's symbol $\Gamma^{\sigma}_{\mu\nu}$.

which permits us to write instead of (1a)

$$(3b) \quad \xi^i_{;i} = \frac{\partial \xi^i}{\partial p_i} + \{\alpha i, i\} \xi^\alpha.$$

We supplement this by the general expression for the covariant derivative

$$(4) \quad \xi^i_{;k} = \frac{\partial \xi^i}{\partial p_k} + \{\alpha k, i\} \xi^\alpha.$$

In the present case $\{\alpha k, i\}$ is only different from zero for $\alpha = i$ and $\alpha = k$, therefore

$$(5) \quad \xi^i_{;k} = \frac{\partial \xi^i}{\partial p_k} + \frac{1}{g_i} \frac{\partial g_i}{\partial p_k} \xi^i - \frac{g_k}{g_i^2} \frac{\partial g_k}{\partial p_i} \xi^k.$$

One might think that the last expression represents the strain ϵ_{ik} [Eq. (2)], but this cannot be correct, since it is not symmetrical in the indices i and k . To satisfy the symmetry condition we have to consider the covariant vector ξ_i and its contravariant derivative $\xi^{i;k}$. The latter is obtained by lowering the superscript i and raising the subscript k in $\xi^i_{;k}$. In the case of an orthogonal line element these operations require only a multiplication by g_i^2 and division by g_k^2 . In this way we obtain from (5)

$$\xi^{i;k} = \frac{g_i^2}{g_k^2} \frac{\partial \xi^i}{\partial p_k} + \frac{g_i}{g_k^2} \frac{\partial g_i}{\partial p_k} \xi^i - \frac{1}{g_k} \frac{\partial g_k}{\partial p_i} \xi^k.$$

We finally interchange i and k , which is entirely a matter of notation, and obtain,

$$(6) \quad \xi^{k;i} = \frac{g_k^2}{g_i^2} \frac{\partial \xi^k}{\partial p_i} + \frac{g_k}{g_i^2} \frac{\partial g_k}{\partial p_i} \xi^k - \frac{1}{g_i} \frac{\partial g_i}{\partial p_k} \xi^i.$$

The last two terms in (6) are opposite in sign to the corresponding terms in (5), hence we obtain by adding (5) and (6)

$$\xi^i_{;k} + \xi^{k;i} = \frac{\partial \xi^i}{\partial p_k} + \frac{g_k^2}{g_i^2} \frac{\partial \xi^k}{\partial p_i}$$

or, after multiplication with g_i/g_k ,

$$(7) \quad \frac{g_i}{g_k} (\xi^i_{;k} + \xi^{k;i}) = \frac{g_i}{g_k} \frac{\partial \xi^i}{\partial p_k} + \frac{g_k}{g_i} \frac{\partial \xi^k}{\partial p_i}.$$

The right member of this equation agrees with the right member of (2) if one returns to the original notation $\delta p_i = \xi^i$, $\delta p_k = \xi^k$. In the notation of tensor calculus the strains are thus given by the representation

$$(8) \quad 2\epsilon_{ik} = \frac{g_i}{g_k} (\xi^i_{;k} + \xi^{k;i}),$$

which includes the representation (1a) when $i = k$:

$$(8a) \quad \epsilon_{ii} = \xi^i_{,i}$$

Note that the last two equations are in complete analogy with the original representation (1.11) in rectangular coordinates. This is in accordance with the general aim of tensor calculus in which one tries to introduce symbols and operational rules of such a nature that the equations written in curvilinear coordinates correspond as closely as possible to those written in Cartesian coordinates.

APPENDIX II

THE CONVECTIVE TERMS OF THE ACCELERATION VECTOR IN TENSOR NOTATION

From the contravariant displacement vector $\delta p_i = \xi^i$ (Appendix I) we pass now to the contravariant velocity vector $\delta p_i = v^i$; the general form of the contravariant vector of the material acceleration is

$$(1) \quad \frac{dv^i}{dt} = \frac{\partial v^i}{\partial t} + v^\alpha v^i_{,\alpha},$$

where the Greek index again indicates the summation with respect to α . This formula corresponds to the elementary vector equation

$$(2) \quad \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \text{ grad})\mathbf{v};$$

but (2) is only correct in Cartesian coordinates, while (1) is correct for any form of the line element and yields automatically the additional inertia terms that were computed in Problem III.1 in an indirect way for the case of cylindrical coordinates.

Let us take up this example once more to examine the structure of the convective terms $v^\alpha v^i_{,\alpha}$ for $i = 1, 2, 3$. With $p_1 = r$, $p_2 = \varphi$, $p_3 = z$ and $g_1 = 1$, $g_2 = r$, $g_3 = 1$ we have

$$(3) \quad v^1 = \frac{dr}{dt} = v_r, \quad v^2 = \frac{d\varphi}{dt} = \frac{v_\varphi}{r}, \quad v^3 = \frac{dz}{dt} = v_z.$$

According to Eqs. (3a, b) of Appendix I we calculate the first line and according to Eq. (5) the second and the third lines of the following table:

$$\begin{aligned}
 v^1_{,1} &= \frac{\partial v_r}{\partial r}, & v^2_{,2} &= \frac{\partial}{\partial \varphi} \frac{v_\varphi}{r} + \frac{v_r}{r}, & v^3_{,3} &= \frac{\partial v_z}{\partial z}, \\
 (4) \quad v^1_{,2} &= \frac{\partial v_r}{\partial \varphi} - v_\varphi, & v^2_{,1} &= \frac{\partial}{\partial r} \frac{v_\varphi}{r} + \frac{v_\varphi}{r^2}, & v^3_{,1} &= \frac{\partial v_z}{\partial r}, \\
 v^1_{,3} &= \frac{\partial v_r}{\partial z}, & v^2_{,3} &= \frac{\partial}{\partial z} \frac{v_\varphi}{r}, & v^3_{,2} &= \frac{\partial v_z}{\partial \varphi}.
 \end{aligned}$$

Multiplication of the terms of the $1^{st}(2^d, 3^d)$ column with those terms of (3) that have the same index α , and summation over α yield the $1^{st}(2^d, 3^d)$ component of the convective acceleration:

$$(5) \quad v^\alpha v^\alpha_{,\alpha} = \begin{cases} v_r \frac{\partial v_r}{\partial r} + v_\varphi \frac{\partial v_r}{r \partial \varphi} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\varphi^2}{r} \\ \frac{1}{r} \left(v_r \frac{\partial v_\varphi}{\partial r} + v_\varphi \frac{\partial v_\varphi}{r \partial \varphi} + v_z \frac{\partial v_\varphi}{\partial z} + \frac{v_r v_\varphi}{r} \right) \\ v_r \frac{\partial v_z}{\partial r} + v_\varphi \frac{\partial v_z}{r \partial \varphi} + v_z \frac{\partial v_z}{\partial z} + * \end{cases}$$

In the second line the common denominator of all terms, $1/r$, has been factored out in preparation for the final equation of motion in Appendix IV.

Note again the "inertia terms" that have been placed in the fourth column:

$$-\frac{v_\varphi^2}{r}, \quad \frac{v_r v_\varphi}{r}, \quad *;$$

they are identical with those of Prob. III.1.

While only the first three terms are correctly obtained by applying the pseudo-vectorial operation $(\nabla \text{ grad})\nabla$ (cf. also the table on p. 347 where these terms have been indicated in the form $\mathbf{A} \cdot \text{grad } \mathbf{A}$, etc.), the present computation yields all four terms at once without discrimination. The correspondence between the "additional" terms on the one hand, the centripetal and half of the Coriolis acceleration on the other hand, was already pointed out.

If the same calculation is made in spherical coordinates r, ϑ, φ where $g_1 = 1, g_2 = r, g_3 = r \sin \vartheta$, one obtains instead of (3), (4), (5)

$$(3a) \quad v^1 = \frac{dr}{dt} = v_r, \quad v^2 = \frac{d\vartheta}{dt} = \frac{v_\vartheta}{r}, \quad v^3 = \frac{d\varphi}{dt} = \frac{v_\varphi}{r \sin \vartheta},$$

$$(4a) \quad \begin{cases} v^1_{,1} = \frac{\partial v_r}{\partial r}, & v^2_{,2} = \frac{\partial v_\vartheta}{r \partial \vartheta} + \frac{v_r}{r}, & v^3_{,3} = \frac{\partial v_\varphi}{r \sin \vartheta \partial \varphi} + \frac{v_r}{r} + \frac{\cos \vartheta v_\vartheta}{r \sin \vartheta} \\ v^1_{,2} = \frac{\partial v_r}{\partial \vartheta} - v_\vartheta, & v^2_{,1} = \frac{\partial v_\vartheta}{r \partial r}, & v^3_{,1} = \frac{\partial v_\varphi}{r \sin \vartheta \partial r}, \\ v^1_{,3} = \frac{\partial v_r}{\partial \varphi} - \sin \vartheta v_\varphi, & v^2_{,3} = \frac{\partial v_\varphi}{r \partial \varphi} - \frac{\cos \vartheta}{r} v_\vartheta, & v^3_{,2} = \frac{\partial v_\varphi}{r \sin \vartheta \partial \vartheta}. \end{cases}$$

$$(5a) \quad v^\alpha v_{;\alpha} = \begin{cases} \mathbf{v} \cdot \text{grad } v_r = \frac{v_\vartheta^2}{r} - \frac{v_\varphi^2}{r} \\ \frac{1}{r} \left\{ \mathbf{v} \cdot \text{grad } v_\vartheta + \frac{v_r v_\vartheta}{r} - \frac{\cos \vartheta}{r \sin \vartheta} \frac{v_\varphi^2}{r} \right\} \\ \frac{1}{r \sin \vartheta} \left\{ \mathbf{v} \cdot \text{grad } v_\varphi + \frac{v_r v_\varphi}{r} + \frac{\cos \vartheta v_\vartheta v_\varphi}{r \sin \vartheta} \right\}. \end{cases}$$

With $1/r$ and $1/r \sin \vartheta$ factored out (in the second and third line), the result shows the presence of the terms corresponding to the elementary vector formula $\mathbf{v} \cdot \text{grad } \mathbf{v}$. Let us again consider the additional "inertia terms": in the first line we find the ordinary centripetal accelerations $-v_\vartheta^2/r$ and $-v_\varphi^2/r$. The full value of the latter term that corresponds to the motion along the parallel of radius $r \sin \vartheta$ is $-v_\varphi^2/r \sin \vartheta$, but the component in r -direction is obtained by multiplying with $\sin \vartheta$ so that this factor cancels. In the second line we find the other component, which has the factor $\cos \vartheta$. We further recognize in the middle term of the second line and also in the last two terms of the third line one half of a Coriolis acceleration. We need not attempt to visualize these terms in a direct kinematic way, nor do we need the geometrical arguments that single out the components, since we trust the validity of the general formulations of tensor calculus.—It is, of course, possible to obtain the same result by vector analytic methods (cf. the table on p. 348).

APPENDIX III

THE TENSOR OF FRICTION PRESSURES AND ITS DIVERGENCE

The connection between the tensor of the friction pressure and the rate of strain is given by Eq. (10.2), $p_{ik} = 2\mu \dot{\epsilon}_{ik}$, when we restrict our

argument to incompressible fluids. To obtain i_{ik} , we differentiate Eq. (8) of Appendix I with respect to t and obtain

$$(1) \quad i_k = -\mu \frac{g_i}{g_k} (v_{,k}^i + v_k^i).$$

This equation expresses the friction pressure in terms of general tensor analysis (Eqs. (4) and (4a) of Appendix II can again be used for the particular cases of cyl. and sph. coordinates). In the equations of motion, however, the friction pressures do not occur individually, but as arguments of the "vector divergence" [cf. Eq. (10.5)]. In Cartesian coordinates, the latter could be transformed into the pseudo-vector $-\mu \nabla^2 \mathbf{v}$ of Eq. (10.8) (see problem I.4 for the correct form of $\nabla^2 \mathbf{v}$ in cylindrical coordinates). In order to carry out the corresponding computation in terms of general tensor calculus, we have to replace $\nabla^2 \mathbf{v}$ by the differentiation and subsequent contraction of the tensor $v_{,k}^i + v_k^i$ of Eq. (1), in other words, we have to carry out the operation

$$(2) \quad v_{,a}^{i;a} + v_a^{i;a}.$$

The dummy index α indicates here the summation as well as contravariant differentiation. The second term in (2) vanishes, however, when the summation is carried out as a consequence of the assumed incompressibility. The i^{th} component of the vector so obtained (we denote it in the following simply by $\text{Div}^i p$) is therefore

$$(3) \quad \text{Div}^i p = -\mu v_{,a}^{i;a}.$$

This calculation requires the contravariant differentiation of a tensor⁴ rather than a vector, which is dealt with in Eddington's book, loc. cit. §30. In the case of cylindrical coordinates one obtains on the basis of Eq. (4) Appendix II

$$(4) \quad v_{,a}^{i;a} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r^i}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r^i}{\partial \varphi^2} + \frac{\partial^2 v_r^i}{\partial z^2} - \frac{v_r^i}{r^2} - 2 \frac{\partial v_r^i}{r^2 \partial \varphi}.$$

The first three terms of the right member represent the ordinary Laplacian of Prob. I.3, p. 346f applied to v_r and may be denoted by $\nabla^2 v_r$;

⁴The formula in question is

$$v_{,kl}^i = \frac{Dv_{,k}^i}{Dx_l} = \frac{\partial v_{,k}^i}{\partial x_l} - \{kl, \beta\} v_{,\beta}^i + \{\beta l, i\} v_{,\beta}^i,$$

and, of course, $v_{,k}^{i;k} = g^{ik} v_{,k}^i$.

the last two terms are due to the fact that v_r is not a scalar but a vector component. *These terms and the corresponding terms in $v_{,a}^{1;\alpha}$ agree with those of Prob. I.4 as expected.* In the "Cartesian" component $v_{,a}^{3;\alpha}$ no such terms occur. Applying the same ∇^2 -operator to the other components we can write the result as

$$(5) \quad \begin{aligned} v_{,a}^{1;\alpha} &= \nabla^2 v_r - \frac{v_r}{r^2} - 2 \frac{\partial v_\varphi}{r^2 \partial \varphi}, & v_{,a}^{2;\alpha} &= \frac{1}{r} \left(\nabla^2 v_\varphi - \frac{v_\varphi}{r^2} + 2 \frac{\partial v_r}{r^2 \partial \varphi} \right), \\ v_{,a}^{3;\alpha} &= \nabla^2 v_z. \end{aligned}$$

Note that in the second equation $1/r$ has been factored out as in Appendix II.

In the case of spherical coordinates one obtains in the same way instead of (4)

$$(4a) \quad \begin{aligned} v_{,a}^{1;\alpha} &= \frac{1}{r^3} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_r}{\partial r} \right) + \frac{1}{r^3 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial v_r}{\partial \vartheta} \right) + \frac{1}{r^3 \sin^2 \vartheta} \frac{\partial^2 v_r}{\partial \varphi^2} \\ &\quad - \frac{2}{r^3} \left(v_r + \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin \vartheta v_\vartheta) + \frac{1}{\sin \vartheta} \frac{\partial v_\varphi}{\partial \varphi} \right). \end{aligned}$$

The first line is again the Laplacian $\nabla^2 v_r$ in the form valid for the present coordinates. The additional terms in the second line can be reduced if the condition of incompressibility is used. Giving $\nabla^2 v_\vartheta$ and $\nabla^2 v_\varphi$ the appropriate meaning and factoring out $1/r$ and $1/r \sin \vartheta$ as in Appendix II, Eq. (5a), we obtain

$$(5a) \quad \begin{aligned} v_{,a}^{1;\alpha} &= \nabla^2 v_r + \frac{2}{r^3} \frac{\partial}{\partial r} (r v_r), \\ v_{,a}^{2;\alpha} &= \frac{1}{r} \left[\nabla^2 v_\vartheta + \frac{1}{r^2} \left(2 \frac{\partial v_r}{\partial \vartheta} - \frac{v_\vartheta}{\sin^2 \vartheta} - \frac{2 \cos \vartheta}{\sin^2 \vartheta} \frac{\partial v_\varphi}{\partial \varphi} \right) \right], \\ v_{,a}^{3;\alpha} &= \frac{1}{r \sin \vartheta} \left[\nabla^2 v_\varphi + \frac{1}{r^2 \sin \vartheta} \left(2 \frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{\sin \vartheta} + 2 \frac{\cos \vartheta}{\sin^2 \vartheta} \frac{\partial v_\vartheta}{\partial \varphi} \right) \right]. \end{aligned}$$

APPENDIX IV

THE EQUATIONS OF MOTION OF A VISCOUS INCOMPRESSIBLE FLUID IN CYLINDRICAL AND SPHERICAL COORDINATES

The general form of the Navier-Stokes equations for arbitrary coordinates is:

$$(1) \quad \rho \left(\frac{\partial v^i}{\partial t} + v^\alpha v_{;\alpha}^i \right) - \mu v_{;\alpha}^{i;\alpha} + (p + V)^i = 0.$$

The individual terms correspond exactly to the vector equation (16.1), provided we write the external force \mathbf{F} as the negative gradient of the potential energy V and collect the latter together with the pressure in one term. The notation $(p + V)^i$ in (1) means contravariant differentiation with respect to the i^{th} coordinate.

Eq. (1) presents an opportunity to point out a homogeneity law of tensor calculus that refers to the degree in which the individual indices occur. One sees that all terms of Eq. (1) are of degree $+1$ in the index i and of degree 0 in the index α , provided we count the occurrence of a certain index positive, if it is a superscript and negative if it is a subscript. It is this circumstance that requires the last term in (1) to be a contravariant differentiation, and is responsible for the factor $1/g_i$ that will occur later, when we pass over to the ordinary gradient operation. (Remember here that the derivative of a *scalar* with respect to x_i is in itself a covariant vector, as pointed out in Appendix I.) It should be kept in mind that v^i is the rate of change of the i^{th} coordinate, but not of the corresponding component of the path; likewise $\partial v^i / \partial t$ is the local acceleration of the coordinate which differs from the corresponding acceleration along the path by the factor $1/g_i$. The condition of incompressibility, $\text{div } \mathbf{v} = 0$, appears now in the form $v_{;\alpha}^\alpha = 0$; it has already been used in setting up the friction term in (1), cf. the simplification of Eq. (2) in Appendix III.

Eq. (1) is perfectly general and does not presume an orthogonal system of coordinates; the simplicity is lost, however, when Eq. (1) is spelled out for any particular system of non-Cartesian coordinates. Even if we assume an orthogonal line element, we should have to introduce the quantities g_i on which the Christoffel symbols depend that are implicitly present in (1). Thus it is hardly worthwhile to develop Eq. (1) in terms of general

orthogonal coordinates, and we shall give only the results for our two special cases:

Cylindrical coordinates.

We substitute expressions (5) of Appendix II for the convective terms and expressions (4) of Appendix III for the friction terms, and multiply the second component of Eq. (1) with the factor r that appears in the denominators of the second line of III.4 and II.5. The first and third terms in the second component of Eq. (1), viz.

$$\frac{\partial v^2}{\partial t} = \frac{\partial}{\partial t} \frac{v_\varphi}{r} \quad \text{and} \quad (p + V)^2 = \frac{1}{r^2} \frac{\partial(p + V)}{\partial \varphi}$$

are then transformed into the ordinary acceleration and the ordinary φ component of the gradient

$$\frac{\partial v_\varphi}{\partial t} \quad \text{and} \quad \frac{\partial(p + V)}{r \partial \varphi}.$$

Using again the appropriate (scalar) ∇^2 operating on the components of \mathbf{v} and the notation $\mathbf{v} \cdot \text{grad } v_i$ we obtain

$$\begin{aligned} & \rho \left(\frac{\partial v_r}{\partial t} + \mathbf{v} \cdot \text{grad } v_r - \frac{v_\varphi^2}{r} \right) \\ & - \mu \left(\nabla^2 v_r - \frac{v_r}{r^2} - 2 \frac{\partial v_\varphi}{r \partial \varphi} \right) + \frac{\partial(p + V)}{\partial r} = 0, \\ (2) \quad & \rho \left(\frac{\partial v_\varphi}{\partial t} + \mathbf{v} \cdot \text{grad } v_\varphi + \frac{v_r v_\varphi}{r} \right) \\ & - \mu \left(\nabla^2 v_\varphi - \frac{v_\varphi}{r^2} + 2 \frac{\partial v_r}{r \partial \varphi} \right) + \frac{\partial(p + V)}{r \partial \varphi} = 0, \\ & \rho \left(\frac{\partial v_z}{\partial t} + \mathbf{v} \cdot \text{grad } v_z \right) - \mu \nabla^2 v_z + \frac{\partial(p + V)}{\partial z} = 0. \end{aligned}$$

This must be supplemented by the condition of incompressibility $v_{,a}^a = 0$, viz.

$$(3) \quad \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial v_\varphi}{r \partial \varphi} + \frac{\partial v_z}{\partial z} = 0.$$

The corresponding calculation in *spherical coordinates* uses (II.5a) and (III.4a). We again have to multiply with the denominators factored out

in the second and third lines of these equations and obtain with the symbols ∇^2 and $\mathbf{v} \cdot \text{grad } v_i$, the meaning of which is now different, the following relation:

$$\begin{aligned} & \rho \left(\frac{\partial v_r}{\partial t} + \mathbf{v} \cdot \text{grad } v_r - \frac{v_\vartheta^2}{r} - \frac{v_\varphi^2}{r} \right) \\ & \quad - \mu \left(\nabla^2 v_r + \frac{2}{r^2} \frac{\partial}{\partial r} (rv_r) \right) + \frac{\partial(p+V)}{\partial r} = 0, \\ (2a) \quad & \rho \left(\frac{\partial v_\vartheta}{\partial t} + \mathbf{v} \cdot \text{grad } v_\vartheta + \frac{v_r v_\vartheta}{r} - \frac{\cos \vartheta}{r \sin \vartheta} \frac{v_\varphi^2}{r} \right) \\ & \quad - \mu \left(\nabla^2 v_\vartheta + \frac{2}{r^2} \left[\frac{\partial v_r}{\partial \vartheta} - \frac{v_\vartheta}{2 \sin^2 \vartheta} - \frac{\cos \vartheta}{\sin^2 \vartheta} \frac{\partial v_\varphi}{\partial \varphi} \right] \right) + \frac{\partial(p+V)}{r \partial \vartheta} = 0, \\ & \rho \left(\frac{\partial v_\varphi}{\partial t} + \mathbf{v} \cdot \text{grad } v_\varphi + \frac{v_r v_\varphi}{r} + \frac{\cos \vartheta}{r \sin \vartheta} \frac{v_\vartheta v_\varphi}{r} \right) \\ & \quad - \mu \left(\nabla^2 v_\varphi + \frac{2}{r^2 \sin \vartheta} \left[\frac{\partial v_r}{\partial \varphi} + \frac{\cos \vartheta}{\sin \vartheta} \frac{\partial v_\vartheta}{\partial \varphi} - \frac{v_\varphi}{2 \sin \vartheta} \right] \right) + \frac{\partial(p+V)}{r \sin \vartheta \partial \varphi} = 0. \end{aligned}$$

We add the condition of incompressibility $v^a_{;a} = 0$, viz.

$$(3a) \quad \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin \vartheta v_\vartheta) + \frac{1}{r \sin \vartheta} \frac{\partial v_\varphi}{\partial \varphi} = 0.$$

The appearance of these equations is rather unsymmetric, although we have tried to write the inertia terms that accompany the ∇^2 expressions as uniformly as possible. The only perfectly symmetric form that can compete with the form of the Stokes-Navier equations in Cartesian coordinates is Eq. (1) itself.—It cannot be denied that the calculation of the foregoing results by tensor methods is rather involved; the elementary, vector methods of Prob. I.4 and III.1 yield the same results more quickly. In more general cases, however, such as the oblique four-dimensional coordinates indispensable in the general theory of relativity, there is no other method available but that of tensor calculus.

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Electrodynamics

Lectures on Theoretical Physics, Vol. III

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PREFACE

Heinrich Hertz's great paper on the "Fundamental Equations of Electrodynamics for Bodies at Rest" has served as model for my lectures on electrodynamics ever since my student days (see §1). Following this example I proceed in Part I from Maxwell's equations as an axiomatic basis, expressed not as with Hertz, in the coordinates and in differential form, but in vectorial integral form. In Part II the several classes of phenomena, in static, stationary, quasistationary, and rapidly variable fields, are derived from these equations, as in Hertz's paper. After I had heard Hermann Minkowski's lecture on "Space and Time" in 1909 in Cologne, I carefully developed the four-dimensional form of electrodynamics as an apotheosis of Maxwell's theory and at the same time as the simplest introduction to the theory of relativity; in return, this has always met with an enthusiastic reception on the part of my audience. This four-dimensional electrodynamics is presented in Part III. Its title "Theory of Relativity and Electron Theory" requires the comment that it is limited to the *special* theory of relativity on the one hand and to the theory of the *individual* electron on the other. The statistics of electrons in metals and electrons in insulators belong in Vols. IV* and V* of the Lectures. Schwarzschild's principle of action, which establishes the fundamental relationship between Maxwell's theory and the dynamics of the individual electron (or individual electrons), is presented at the end of Part III with certain modifications which appear necessary from our point of view. In Part IV is developed the electrodynamics of moving media, again following Minkowski rather closely. As the most important application the fields of unipolar induction are discussed and are calculated as an exercise for a particularly simple example.

Part II constitutes the main portion of the Lectures. I distinguish between summation and boundary-value problems in electrostatics and magnetostatics. The computation of the electric potential for given charge distribution and the calculation of the magnetic potential for given magnetization are examples of the first; the theory of the permanent magnet, insofar as it falls within the competence of Maxwell's theory rather than atomic theory, becomes simple and clear from this standpoint. On the other hand, the solution of the electric and magnetic boundary-value problems properly belongs in Vol. VI; only the most important cases are treated in the present volume. The calculation of stationary fields for a given distribution of the current density, either by the method of the vector potential in §15 or that of the magnetic shell in §16, is also a simple summation prob-

* See p. xii for list of *Lectures on Theoretical Physics*.

lem. Among the rapidly variable fields those of the wire-wave type are treated with some completeness. The principal wave on a single wire in §22 (symmetric electrical type) serves as primary example; however, because of their recent practical applications (theory of wave guides in §24) and their utilization in the theory of the Lecher system, the magnetic type and the asymmetric secondary waves as well as wire waves on nonconductors are also dealt with in §23. As conclusion of Part II the Lecher system is treated fully for arbitrary separation and dimensions of the two parallel wires, employing bipolar coordinates for the exterior of the wires and ordinary polar coordinates for their interior. It is only assumed that the two wires are rather good conductors.

The dimensional character of the field entities is taken seriously throughout. We do not accept Planck's position, according to which the question of the real dimension of a physical entity is meaningless; Planck states in §7 of his *Lectures on Electrodynamics* that this question has no more meaning than that of the "real" name of an object. Instead, we derive from the basic Maxwell equations the fundamental distinction between entities of *intensity* and entities of *quantity*, which has heretofore been applied consistently in the excellent textbooks of G. Mie. The Faraday-Maxwell induction equation shows that the magnetic induction \mathbf{B} is an entity of intensity along with the electric field strength \mathbf{E} ; \mathbf{B} , rather than \mathbf{H} , deserves the name *magnetic field strength*. \mathbf{H} , like \mathbf{D} , is best designated as "excitation." $\text{div } \mathbf{H}$ represents the magnetic density, just as $\text{div } \mathbf{D}$ represents the electric charge density. Hertz's distinction between "true" and "free" electricity becomes pointless, since $\text{div } \mathbf{E}$ is, dimensionally, not a charge, but a divergence of lines of force. The same applies to the distinction between "true" and "free" magnetism, particularly since $\text{div } \mathbf{B}$ is everywhere zero. The current density \mathbf{J} , the electric polarization \mathbf{P} , and the magnetization \mathbf{M} , are entities of quantity like \mathbf{H} and \mathbf{D} . Energy quantities always take the form of products of an entity of quantity and an entity of intensity, e.g. $\frac{1}{2}\mathbf{D} \cdot \mathbf{E}$, $\frac{1}{2}\mathbf{H} \cdot \mathbf{B}$, $\mathbf{J} \cdot \mathbf{E}$, $\mathbf{E} \times \mathbf{H}$. The fact that \mathbf{B} and \mathbf{E} , and \mathbf{H} and \mathbf{D} , belong together follows unambiguously from the theory of relativity, in which the quantities $c\mathbf{B}$ and $-i\mathbf{E}$, and \mathbf{H} and $-ic\mathbf{D}$, respectively, are coupled together in a six-vector (antisymmetric tensor). We call the first the field tensor F , the second, the excitation tensor f .

The introduction of a *fourth electric unit*, independent of the mechanical units, is decisive for the fruitfulness of these dimensional considerations. We choose for this the unit of charge Q , which, as a matter of convenience we may identify with the coulomb if we wish. In this manner we avoid the "bed of Procrustes" of the cgs-units, in which the electromagnetic quantities are forced to take on the well known unnatural dimensions. Since we must definitely give up the hope of a mechanical interpretation of electrical quantities, we must regard the charge as a basic, irreducible entity which can claim a dimension of its own. We shall refer to the "electrostatically"

or "electromagnetically" measured charge only in passing, and exclusively for historical reasons. With the particular unit of charge $Q = 1$ coulomb the electric current has the customary unit amperes $= Q/\text{sec}$.

As mechanical units, following the suggestion of G. Giorgi, we shall employ the units meter M , kilogram (mass) K , and second S . The unit of energy then becomes 1 joule (without a power of ten as factor!) and that of power 1 joule $S^{-1} = 1$ watt. Furthermore, the powers of ten disappear also for the electric units volt, ohm, farad, and henry; we have 1 volt $= 1 \text{ joule}/Q$, 1 ohm $= 1 \text{ joule } S/Q^2$, 1 farad $= 1 Q^2/\text{joule}$ and 1 henry $= 1 \text{ joule } S^2/Q^2$.

On the other hand powers of ten must appear as factors when the units of the magnetic field strength B and the magnetic excitation H are expressed in terms of the gauss and the oersted, respectively, which as we shall see in §8, have been adapted to the cgs-system. As expected, the unit Q automatically drops out of the energy densities $\frac{1}{2}D \cdot E$, $\frac{1}{2}H \cdot B$, and $J \cdot E$ referred to above; their dimension becomes joule/ M^3 directly, whereas that of the energy flux $E \times H$ is joule/(M^2S).

With our dimensional differentiation between entities of intensity and entities of quantity the dielectric constant and the permeability evidently become *dimensional* quantities and therefore cannot be set equal to 1 in vacuum. Their choice, in which we accept electrical engineering practice, happily permits us to meet the demand for "rational units" without difficulty. It is only necessary to set,

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{QS}{M}$$

in accord with international conventions, and to derive ϵ_0 from the relation $\epsilon_0\mu_0 = 1/c^2$, verified by Hertz's experiments. With this choice the 4π 's disappear wherever they do not belong, as in Poisson's equation and the energy expressions in Poynting's theorem, and appear where they belong, as in Coulomb's law and for the spherical condenser. We thus avoid the desperate expedient by which Lorentz achieves rationalization in his articles in the Enzyklopaedie, namely the introduction of the factor $\sqrt{4\pi}$ in the definition of the charge and of magnetism.

At the same time, with this choice of ϵ_0 and μ_0 , the square root of the ratio of μ_0 and ϵ_0 evidently becomes a resistance, namely the so-called "wave resistance of vacuum." This quantity occurs in Part II as a factor wherever the wave fields E and H enter into formulas of the same dimensions. It occurs again in the theory of relativity in the relation between the excitation tensor f and the field tensor F , which in vacuum assumes the simple form for all six components

$$f = \sqrt{\frac{\mu_0}{\epsilon_0}} F.$$

These questions of units, dimensions, and rationalization, often discussed to excess in recent years, are disposed of as briefly as possible in the lectures; however, the reader is repeatedly urged in them to convince himself of the dimensional logic of formulas. In numerical computations our MKSQ system of units is found convenient throughout since it is adapted to the practical and legal units, volt, ampere, etc. We leave the question open as to whether it is also appropriate for atomic physics. So as to permit an effortless transition to the Gaussian system ($\epsilon_0 = \mu_0 = 1$), which is customary in this case, we explain Cohn's system in §9, which in our interpretation is based on the five units MKSQP (P = magnetic unit pole).

The wonderful simplicity and beauty of the Maxwell equations, which is most striking in their relativistic formulation for vacuum, lead to the conviction that these equations, along with the equations of gravitation (§38), are the demonstration of an all-inclusive world geometry. Approaches to this by no means resolved problem are summarily discussed in §37. The amazingly simple representation of the general theory of relativity in §38 is based on a derivation of Schwarzschild's line element kindly made available to me by W. Lenz. In this manner the three tests of the theory open to astronomical observation may be treated without tensor calculus.

This volume is based on lecture notes prepared by H. Welker in the winter semester of 1933/34, at which time I first abandoned the cgs system and passed over to the more general system of the four units. In the final formulation of Parts I and II I have had the benefit of the constant advice of Professor J. Jaumann, whose electrotechnical experience and point of view have been of great advantage to this volume. I am grateful to Messrs. P. Mann and E. Gora and to my colleague F. Bopp for critical remarks and suggestions for improvements. Dr. W. Becker has kindly assisted me in reading the proof of this, as of preceding volumes.

Munich, April 1948

Arnold Sommerfeld

TRANSLATOR'S NOTE

A minimum number of changes has been made in this translation of Sommerfeld's "Elektrodynamik" (the third volume of the Lectures on Theoretical Physics) to adapt it for use in English-speaking countries. As far as possible, the same conventions regarding notation are employed as in G. Kuerti's translation of Volume II, "Mechanics of Deformable Bodies." Thus vectors are represented by bold-face letters, vector components and scalars (as well as tensors and their components) by italics; this in spite of the fact that the Gothic letters employed in the original text for both vectors and vector components were used even in Maxwell's Treatise. To avoid confusion a few additional changes of symbols were required in consequence of this major change.

E. G. R.

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PART I

FUNDAMENTALS AND BASIC PRINCIPLES OF MAXWELL'S ELECTRODYNAMICS

§1. *Historical Review. Action at a Distance and Action by a Field*

I can best give you an idea of the sweeping changes in viewpoint brought about by the theory of Faraday and Maxwell by telling you of the time I spent as a student, 1887-1891.

My native city, Königsberg, was the earliest fountainhead of mathematical physics in Germany, thanks to the activity of the revered Franz Neumann, 1798-1894. At the University of Königsberg he taught, in addition to crystallography, theoretical physics which was not at the time given elsewhere in Germany. His students, of whom Gustav Kirchhoff of Königsberg was the most prominent, spread the teachings of the master to the other German universities. Through the seminar in mathematical physics, founded by him and C. G. J. Jacobi, he also saw to it that the East Prussian secondary-school teachers received a particularly thorough preparation. This may bear some relation to the fact that the Gymnasium in the Altstadt graduated the mathematician Hermann Minkowski and the physicists Max and Willy Wien shortly before my final examination, while at the same time the only slightly older David Hilbert and Emil Wiechert were attending other Königsberg schools. Neumann's greatest successes in research were achieved in the elastic theory of light and in the physics of crystals; his mathematical formulation of the induction currents discovered by Faraday will be discussed in §15. Simultaneously with Neumann and Jacobi, and almost outshining them, F. W. Bessel taught in Königsberg.

My time of study coincided with the period of Hertz's experiments. At first, however, electrodynamics was still presented to us in the old manner—in addition to Coulomb and Biot-Savart, Ampère's law of the mutual action of two elements of current and its competitors, the laws of Grassmann, Gauss, Riemann, and Clausius, and as a culmination the law of Wilhelm Weber, all of which were based on the Newtonian concept of action at a distance. The total picture of electrodynamics thus presented to us was awkward, incoherent, and by no means self-contained. Teachers and students made a great effort to familiarize themselves with Hertz's

experiments step by step as they became known and to explain them with the aid of the difficult original presentation¹ in Maxwell's Treatise.

It was as though scales fell from my eyes when I read Hertz's great paper:² "Über die Grundgleichungen der Elektrodynamik für ruhende Körper." Here Maxwell's equations, purified by Heaviside and Hertz, were made the axioms and the starting point of the theory. The totality of electromagnetic phenomena is derived from them systematically by deduction. Coulomb's law, which formerly provided the basis, now appears as a necessary consequence of the all-inclusive theory. Electric currents are always closed. Current elements arise only as mathematical increments of line integrals. All effects are transmitted by the electromagnetic field, which may be represented by force-line models. *Action at a distance* gives way to field action,³ the "constructable representation" of a space-time propagation postulated already by Gauss.⁴

I have held to the order of Hertz's paper in all my lectures on Maxwell's theory. In this presentation, too, we shall not begin with electrostatics, as is done so commonly and also in Maxwell's Treatise, but treat it merely as an extreme simplification of the general field theory. We shall deviate from Hertz only insofar as we shall start not from Maxwell's equations in differential form, but in integral form. It goes without saying that we shall replace the rather extensive coordinate calculations of Hertz by vector algebra, which is perfectly suited to the electromagnetic field. We shall see that this algebra, extended to four dimensions, leads directly to the special theory of relativity. The latter will provide an approach to the electrodynamics of moving bodies, which Hertz unsuccessfully sought to master in the second paper cited. In agreement with Hertz we see in Maxwell's *equations* the essence of his theory. We need not discuss the *mechanical pictures*, which guided Maxwell in the setting up of his equations. We have discussed one such picture in Vol. II, §15 of these lectures.

¹ The great student of electrolysis, Wilhelm Hittorf, who had heard much of the new theory of electricity, in advanced years attempted to study the Treatise, but was unable to find his way through the unfamiliar mass of equations and concepts. He was thus led into a state of deep depression. His colleagues in Münster persuaded him to take a vacation trip to the Harz Mountains. However when just before his departure they checked his luggage they found in it—the two volumes of the Treatise on Electricity and Magnetism by James Clerk Maxwell. (As told by A. Heidweiller.)

² Göttinger Nachr. March 1890 and Ann. Physik, Vol. 40; continued in Ann. Physik, Vol. 41: "Über die Grundgleichungen der Elektrodynamik für bewegte Körper."

³ We avoid the alternative term "near action" which signifies merely action at a small distance, and by our notation direct attention to the medium transmitting the effect, namely the field.

⁴ In a letter to Wilhelm Weber, of 1845. See Collected Works, Vol. V, p. 627.

Biographical Notes

MICHAEL FARADAY, 1791-1867

He was born as son of a blacksmith in impecunious circumstances. The family belonged to the pious sect of the Sandemanians, to which Faraday remained faithful to his death. His high ethical concept of life and human kindness derived from the religious spirit of his family. He was first newspaper carrier, then bookbinder. In science and letters he was entirely self-taught. The lectures of Sir Humphry Davy at the Royal Institution were decisive for his career; he wrote them up carefully and found an opportunity to present them to the great chemist. He became his laboratory assistant in the Royal Institution. His first important work was "the rotation of a current about a magnet and the rotation of a magnet about a current," and also the liquefaction of chlorine. This work brought about his election as Fellow of the Royal Society and later the indirect succession to Davy at the Royal Institution. In 1832 he began the publication of the "Experimental Researches." His discoveries recorded in these extend to the most diverse fields of physics, electrochemistry, and the study of materials. We mention as most significant for us: The discoveries of the law of electromagnetic induction in 1831, the dielectric constant, para- and diamagnetic behavior, and the picture of electric and magnetic lines of force. His magneto-optical discoveries are discussed in Vol. IV. The failing of his memory forced many pauses in his work, as well as the repetition of experiments made at an earlier date. It is uncertain whether this is to be attributed to mental overexertion or, as is commonly assumed today, to mercury poisoning in the poorly ventilated basement rooms of the Royal Institution. Certainly his purely intuitive method of working, devoid of any mathematical aid, required tremendous mental concentration. In his last years a restful summer retreat in the royal palace, Hampton Court, was made available to him at the suggestion of the Prince Consort, Albert. At his death there were found ninety-five honorary diplomas of learned societies, bound with his own hand.

JAMES CLERK MAXWELL, 1831-1879

He came from a prominent Scottish family (the father's name was Clerk, the added name Maxwell being derived from his mother) and was given the best in education that his time offered, both in the field of letters and that of science and mathematics. Thus, at an early date, he could translate Faraday's pictures of lines of force into a mathematical form which could be generally understood. See his paper of 1855 "On Faraday's Lines of Force" (translated into German by Boltzmann in Ostwald's *Klassiker* Nr. 69). In the preface to his Treatise he states: "Faraday, in his mind's eye, saw lines of force traversing all space where the mathematicians

(from the preceding discussion it is apparent that he refers particularly to Gauss, Wilhelm Weber, Riemann, Franz and Carl Neumann) saw centres of force attracting at a distance: Faraday saw a medium where they saw nothing but distance: Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids. When I had translated what I considered to be Faraday's ideas into a mathematical form, I found that in general the results of the two methods coincided, . . . but that . . . several of the most fertile methods of research discovered by the mathematicians could be expressed much better in terms of ideas derived from Faraday than in their original form."

The *Traité* appeared in 1873. Its greatest achievement is the unification of optics and electrodynamics. The simplified form of the Maxwell equations, later rediscovered by Heaviside and Hertz, is to be found already in Part III of his paper for the Royal Society of 1864. Almost as important as his electromagnetic papers are those on the kinetic theory of gases (Maxwellian velocity distribution) and on general statistics, to which belongs also his theory of the rings of Saturn. He is also the author of purely mathematical papers (on cycloidal surfaces, the theory of the top, and the determination of magnitudes in Helmholtz's color triangle) and of an important paper on lattice structures (see Vol. II of these Lectures, p. 310).

After a brief teaching engagement in Aberdeen he became the first director of the newly founded Cavendish Laboratory in Cambridge; he died there at an early age.

ANDRÉ MARIE AMPÈRE, 1775-1836

We shall add a biographical note on Ampère not on account of the fundamental law already mentioned, nor because of the classical experiments, which enabled him to derive it with the simplest possible means, but for his discovery of the general relationship between the magnetic field and electric currents.

Born in Lyon, as a precocious boy he occupied himself with philological and mathematical studies. His father was a victim of the Revolution. Because of his mathematical papers he was named professor at the *École Polytechnique* in Paris in 1804. Here he soon directed his attention to chemistry, where he was able to compete with Avogadro in the field of atomism. There follow five years in which he is concerned primarily with psychology and metaphysics, though accepted as a mathematician into the Academy of Sciences of Paris. His interest in physics is not awakened until 1820, when he hears of Oersted's discovery. In a few weeks he verifies his belief that electricity in motion, and not electricity at rest, has a mag-

netic effect. The years 1820–1826 he spent elaborating his concept of the connection between the magnetic field and the electric current, which is equivalent to half of Maxwell's equations provided that the concept of the electric current is extended by the addition of Maxwell's displacement current. We shall hence denote this portion of the Maxwell equations (in integral form) in §3 directly as Ampère's law. From this point of departure Ampère recognized the equivalence of a solenoid traversed by current to a permanent magnet. The strengthening of the magnetic field by a soft-iron core placed in the solenoid is also to be attributed to him. Ampère may thus be regarded as the father of the "electromagnet." We may mention in addition Ampère's molecular currents and the elegant method of the magnetic sheet.

When, in 1826, however, Ampère obtained a professorship in physics at the Collège de France his interests changed once more: he returned to philosophy and logic and devoted himself finally to biology and comparative anatomy. Altogether a scientific career of extraordinary breadth and depth, of intensity and versatility! (This material has been taken from an essay by Louis de Broglie in his book *Continu et Discontinu*, Paris, 1941.)

HEINRICH HERTZ, 1857–1894

He was born in Hamburg the son of a respected merchant family; his father was in later years Senator of the Free City. Initially his great modesty prevented Heinrich Hertz from entering upon the career of a scholar; instead, he turned to engineering at the Technische Hochschule in Munich. Soon, however, he begged his father to permit him to transfer to pure physics. He studied first in Munich, then in Berlin, and became the favorite student and assistant of Helmholtz. The relationship between teacher and student was the closest imaginable and finds touching expression in the memorial addressed to him by Helmholtz (reprinted in Vol. I of Hertz's *Collected Works*). A prize problem set up by Helmholtz directed him to the testing of Maxwell's theory. After a short term as Privatdozent in Kiel he was called to the Technische Hochschule in Karlsruhe.

Even the earliest papers of Hertz show his mastery in relating theory and experiment. Several of them received the warm recognition of his colleagues, as his quantitative determination of hardness among engineers, and his description of the condensation processes in rising air currents among meteorologists. His years in Karlsruhe, from 1885 to 1889, represent the high point in his creative activity. We mention in particular his paper of 1888: "Forces of electrical oscillations treated by Maxwell's theory." It provides the characteristic solution now generally designated as the Hertzian vector and shows the familiar force-line pictures of the

Hertzian dipole. It is amazing how much of the later development of radio telegraphy has been anticipated in this paper. We should also point out the great paper on "Rays of electric force." The theoretical papers (basic equations of electrodynamics) have already been discussed.

The discovery of the photoelectric effect also falls into this period. With his last experimental paper of 1891 "On the passage of cathode rays through thin metal films" he reached beyond the problems set by Maxwell's theory and without knowing it, blazed the path to the electron theory. The very thin metal films later designated as "Lenard windows" are described already in this paper.

In 1889 he was called to Bonn. Here he prepared his last work, "Principles of Mechanics," which we have discussed in Vol. I, §39. The introduction of non-holonomic auxiliary conditions, the polydimensional treatment of mechanical systems of many degrees of freedom, the principle of the straightest path attest the keen logic and the geometric intuition of their author. Increasing illness prevented experimental work. He died on January 1, 1894, 37 years of age.

§2. *Introduction to the Basic Concepts of the Electromagnetic Field*

We regard the existence of electric charges as an established fact, whether we produce them by rubbing a piece of amber, the godparent of electricity, or recognize them from the spark when connecting the poles of a battery. We interpret the observed attraction, repulsion, and heat generation as the result of charges which have been produced. We take care not to define the charge verbally or to ascribe a derived dimension to it by some arbitrary procedure. Instead we regard it as having its own dimension, as an entity beyond the range of mechanics. We call this quantity Q . We could choose as unit of charge, whether negative or positive, the familiar universal charge of the electron. We prefer however to let Q stand for the coulomb, the accepted unit in the practical system, in terms of which the electron charge is expressed by $e = 1.60 \cdot 10^{-19}$ coulomb. We assume that electrometer apparatus is available with which we can compare different charges with each other and with the coulomb as unit of charge. The atomistic nature of charge is disregarded in the Maxwell theory proper. The charge of the atoms and elementary particles is to a much higher degree an absolute constant than the mass (see §27J).

In addition to the electric unit Q we normally employ as mechanical units of length, mass, and time the Giorgi units M (meter), K (kilogram mass), and S (second), which have been established internationally by the decision of the appropriate commissions. As already pointed out in Vol. I, p. 8, there is the advantage that in this system the units of energy and power correspond exactly (without multiplying powers of ten) to the joule and watt introduced previously in the cgs system. We designate them as

$$1 \text{ joule} = 1 \text{ M}^2\text{KS}^{-2} = 10^7 \text{ cm}^2 \cdot \text{g} \cdot \text{sec}^{-2} = 10^7 \text{ erg}$$

$$1 \text{ joule/S} = 1 \text{ M}^2\text{KS}^{-2} = 10^7 \text{ cm}^2 \cdot \text{g} \cdot \text{sec}^{-3} = 10^7 \text{ erg/sec} = 1 \text{ watt}$$

and define correspondingly

$$1 \text{ newton} = 1 \text{ MKS}^{-2} = 10^5 \text{ cm} \cdot \text{g} \cdot \text{sec}^{-2} = 10^5 \text{ dynes}$$

This unit of force "newton" is seen to be conveniently comparable in size with the practical unit of force, the "kilogram" = $9.81 \cdot 10^5$ dynes.

We will show presently that the annoying powers of ten vanish also for the practical units of the volt and ohm when the MKSQ system is employed.

We now proceed to examine in sequence the basic electromagnetic concepts. In most cases we shall be concerned with a dimensional description rather than with a complete definition; the latter will be derived from their interrelation through the basic equations of the theory, which can be tested by experiment. In the succeeding section we will follow directly the present day enumeration of the basic concepts.

We begin with the *electric fieldstrength*, for which a true definition is possible and is generally conventional. Let this quantity be denoted by E .¹ We define it as the mechanical force exerted in an electric field on an (infinitesimally small) test body, divided by the charge of the test body. E is therefore a vector with the dimension²

$$E = \frac{\text{Force}}{\text{Charge}} = \frac{\text{newton}}{Q}; \quad (1)$$

within the field it varies from point to point in direction and magnitude. In following everywhere the direction of E we describe an electric line of force.

We now consider the *line integral*

$$\int_A^B E_s ds = \int_A^B \mathbf{E} \cdot d\mathbf{s} \quad (2)$$

between two points A and B . E_s is the perpendicular projection of E on the direction of the line element ds and ds is the line element regarded as vector; $\mathbf{E} \cdot d\mathbf{s}$ denotes, as usual, the scalar product. We call this line in-

¹ Maxwell employed gothic letters (rather than bold-face letters) for the vectors of the electromagnetic field (see Vol. II of the Treatise, art. 618). Except for this distinction we use the symbols here given. J will denote the electric current density, I the total current in a wire.

² Here, and at many other points, we use the equality sign to indicate equality of dimension. Where, as in Eq. 2a, we wish to distinguish between actual numerical equality and mere dimensional equality, we write $= \dots$, i.e., "equal except for a numerical factor."

tegral the "voltage" V :

$$\begin{aligned} V &= \int_A^B \mathbf{E} \cdot d\mathbf{s} = \dots \frac{\text{newton} \cdot \text{M}}{Q} = \dots 10^7 \frac{\text{dyne} \cdot \text{cm}}{Q} \\ &= \dots 10^7 \frac{\text{erg}}{Q}. \end{aligned} \quad (2a)$$

The conversion of the dimension from the MKSQ- to the cgs-system shows that our unit of voltage is identical with

$$1 \text{ volt} = 10^8 \text{ cgs units} \quad (2b)$$

if, as decided above, we fix Q at

$$1 \text{ coulomb} = \frac{1}{10} \text{ cgs unit}. \quad (2c)$$

For the definition of the voltage it is necessary that in addition to the *terminal points* A, B , the path between them be prescribed. Only in lamellar fields (see Vol. I, Eq. 6.16, and Vol. II, p. 137) is the independence of the line integral with respect to the path guaranteed by Stokes' law (see Vol. II, Eq. 3.6). In place of voltage we may then speak of difference of potential between the two points A and B , designated by V_{AB} .

We introduce as companion to the fieldstrength \mathbf{E} a second electric vector \mathbf{D} . We shall call this preferably *electric "excitation,"* but shall also frequently employ, particularly in the first part of these Lectures, the customary term "*dielectric displacement*" (Maxwell's designation).

We make the introduction of \mathbf{D} comprehensible by the following consideration: Charge, in its historical origin, is a concept based on the notion of action at a distance. To adapt it to the viewpoint of action by a field it is necessary to imagine an excitation of the surrounding medium proceeding from the charge centers, which excitation will be described by the vector \mathbf{D} . For a single point charge e we imagine " \mathbf{D} lines" leaving e uniformly in all directions, with such density that the " \mathbf{D} -flux" becomes

$$\oint D_n d\sigma = e. \quad (3)$$

$d\sigma$ is an element of an arbitrary surface surrounding e . If, in particular, we choose a spherical surface of radius r , we find

$$4\pi r^2 D = e. \quad (3a)$$

For arbitrarily, including continuously, distributed charges, Eq. (3) is replaced by

$$\oint D_n d\sigma = \bar{e}, \quad \bar{e} = \sum e \quad (3b)$$

where $\bar{\epsilon}$ indicates the total charge within σ , the algebraic sum of positive and negative charges. We will see in §4 that this description of \mathbf{D} , for an arbitrary choice of σ , is selfconsistent, but does not suffice for a unique definition of \mathbf{D} . We will also see there that in the simplest case (isotropic medium, linear relation between \mathbf{D} and \mathbf{E}) the "D-lines" are identical with the lines of force defined by the \mathbf{E} vector.

From the preceding equations the dimension of \mathbf{D} is seen to be

$$\mathbf{D} = \frac{\text{charge}}{\text{area}} = \frac{Q}{M^2} \quad (4)$$

This dimension is entirely different from the dimension of the fieldstrength \mathbf{E} , given by Eq. (1). With regard to Maxwell's designation "dielectric displacement," we note that it fits strictly not the vector \mathbf{D} itself, but only that fraction of \mathbf{D} which arises from the presence of ponderable matter and which will later (see §11C) be designated as the *polarization* \mathbf{P} . Thus this portion \mathbf{P} vanishes for vacuum, the medium which is of greatest importance to us. Nevertheless the "displacement" \mathbf{D} retains its individual meaning, distinct from \mathbf{E} , in this case also.

We compare Eq. (4) with the dimension of the *electric current density* \mathbf{J} . One knows that this is to be defined as the quantity of electricity traversing unit area in unit time in a conductor. Its dimension is therefore

$$\mathbf{J} = \frac{\text{charge}}{\text{area} \cdot \text{time}} = \frac{Q}{M^2S} \quad (4a)$$

Depending on whether the unit area is placed perpendicular to the direction of the current or at an angle thereto, the absolute magnitude of \mathbf{J} or a component of it is obtained. \mathbf{J} is thus a vector similar in character to \mathbf{D} . Dimensionally, however, not \mathbf{D} , but the time rate of change of \mathbf{D} , the so-called *displacement current* $\dot{\mathbf{D}}$, corresponds to \mathbf{J} .

We have here assumed a sharp distinction between conductor and non-conductor (dielectric medium). Actually, no perfect insulator exists since even the best nonconductor conducts to some extent, e.g. under the influence of cosmic radiation. Maxwell therefore supplements the displacement current to form the total current

$$\mathbf{C} = \dot{\mathbf{D}} + \mathbf{J}; \quad (5)$$

the designation \mathbf{C} (current) was introduced by Maxwell. This notion of the equivalence of $\dot{\mathbf{D}}$ and \mathbf{J} is a basically new idea of Maxwell, which is a prerequisite for the unified representation of electromagnetic phenomena. Similarly, he supplements in the metallic conductor the conduction current \mathbf{J} by the addition of a hypothetical displacement current $\dot{\mathbf{D}}$, although here the first term completely outweighs the second.

We now pass to the *magnetic field*. This quantity exerts a mechanical

force on a magnetic pole P , which, to begin with, may be thought of as isolated. With the same letter P we designate also the strength of the magnetic pole and with P the as yet undetermined dimension "pole strength." The mechanical force divided by P we should most properly call the *magnetic field strength*. We will however, at least in the beginning, adhere to custom and call this quantity the *magnetic induction* B :

$$B \equiv \frac{\text{force}}{\text{pole strength}} = \frac{\text{newton}}{P}. \quad (6)$$

We shall even go a step further in our adherence to customary notions and utilize the relation between current and magnetism elaborated by Ampère, whose systematic description must however be postponed until §17. Thus, for example, the magnetic field of a plane circulating current I about the area F is, at a great distance from I , equal to the field of a bar magnet placed normal to F at I , with the moment

$$m = IF. \quad (6a)$$

This relation, which in the conventional cgs system serves to measure the current I "magnetically," we shall here employ to define the pole strength P in terms of our electric unit of charge Q . We set

$$m = \text{pole strength} \cdot \text{pole separation} = Pl \quad (6b)$$

and obtain from Eq. (6a)

$$P = I \cdot \frac{F}{l} = \frac{Q}{S} \cdot \frac{M^2}{M} = Q \frac{M}{S}. \quad (7)$$

Our dimensional equation (6) thus becomes

$$B = \frac{\text{newton}}{Q} \frac{S}{M}. \quad (8)$$

A complete description of the magnetic field also requires in addition to B a second vector which we shall designate with H . We cannot however adhere to the customary notation "magnetic field strength," which, as we have seen, rightfully belongs to the vector B , but will call H the *magnetic excitation*. We follow here the carefully thought-out representation of electrodynamics of Mie.¹ With the name magnetic excitation we place H in parallel with the "electric excitation" D . Corresponding to Eq. (4) we therefore define H dimensionally by

$$H = \frac{\text{pole strength}}{\text{area}} = \frac{P}{M^2}, \quad (9)$$

¹ Gustav Mie, *Lehrbuch der Elektrizität und des Magnetismus*, 2nd Ed., Enke, Stuttgart, 1941, and *Handbuch der Experimentalphysik* Vol. XI, Part 1, *Elektrodynamik*.

which, in view of Eq. (7), we may write

$$H = \frac{Q}{MS}. \quad (9a)$$

This representation also justifies a designation which is commonly employed in engineering and which, though rather awkward, is more appropriate than the unfortunate name "magnetic field strength," namely, the designation "ampere turns per unit length." For further details see the end of §4.

The direction of the field vector \mathbf{B} , varying from point to point, is represented by the form of the *magnetic lines of force*. As is well known, these are made evident by the automatic alignment of iron filings which are brought into the neighborhood of the magnet and were known long before the corresponding *electric* lines of force. Their expressive appearance still contributes greatly to the understanding of the field concept. Because of the equality of direction of \mathbf{H} and \mathbf{B} in air or any other isotropic medium the line patterns corresponding to the \mathbf{H} vector are identical with the lines of force of the \mathbf{B} vector.

We may indicate finally a subdivision of physical entities into entities of intensity and entities of quantity. \mathbf{E} and \mathbf{B} belong to the first class, \mathbf{D} and \mathbf{H} , to the second. The entities of the first class are answers to the question "how strong," those of the second class, to the question "how much." In the theory of elasticity, for example, the stress is an entity of intensity, the corresponding strain, one of quantity; in the theory of gases pressure and volume form a corresponding pair of entities. In \mathbf{D} the quantity character is clearly evident as the quantity of electricity that has passed through; in \mathbf{H} the situation is slightly obscured by the fact that there are no isolated magnetic poles (see §3). We are in general inclined to regard the entities of intensity as cause, the corresponding entities of quantity as their effect.

§3. Maxwell's Equations in Integral Form

After this very incomplete preparation we pass to the *axiomatic foundation of Maxwell's theory*. The axioms of electrodynamics, just as the Newtonian axioms of mechanics, rest on experience—more exactly on the ordering of the totality of experience into a simplified and idealized form. Thus the law of inertia of mechanics appears very different from what is observed in a particular case for terrestrial bodies. Similarly, our electromagnetic axioms are much more abstract and mathematically generalized than what is measured with coils, wires, and pointer instruments. Nevertheless like the mechanical axioms, they are simply a summary of diverse observations.

To begin with we set up two principal axioms which we shall then supple-

ment by secondary axioms. One of these we shall call *Faraday's law of induction*. We shall state it, as far as practicable, in Faraday's own line-of-force language. The other axiom we shall name after Ampère, since he was the first to formulate the relationship between current and magnetic fields. The fact that *Ampère's law* also rests on experience has been emphasized by its author.¹ We shall, however, state both axioms in the universal form whose possibility was first realized by Maxwell.

For this purpose we consider an arbitrary surface σ with the boundary curve s . We provide the latter with a pointer indicating sense of travel and shall define that direction of the normal to the surface as positive which forms a *right-handed screw* with the s -pointer. We compute the surface integrals

$$\int B_n d\sigma \quad \text{and} \quad \int C_n d\sigma \quad (1)$$

extended over σ and shall call them *magnetic flux* and *electric current flux*. *Number of lines of force* and *number of lines of current* is another common designation. This notation is of course audacious since these bundles of lines are not countable. It is first necessary to group them in "tubes," just as in Vol. II (p. 136) the lines of turbulence were grouped in tubes of turbulence. The tubes must be constructed so that their cross section becomes inversely proportional to the magnitude of B and C , respectively, at the point in question. The counting of the tubes of force or current traversing our surface then amounts to the same as the evaluation of the integrals (1).

Next we compute the following line integrals extended over the boundary curve s :

$$\oint \mathbf{E} \cdot d\mathbf{s} \quad \text{and} \quad \oint \mathbf{H} \cdot d\mathbf{s}. \quad (2)$$

We call these the *electric* and *magnetic loop tension*. The first has also for a long time been called E.M.F. or electromotive force; included in this designation, it is true, are also other "electromotive" causes, such as differences in temperature and chemical effects. The word "force" is here used in its antiquated meaning of energy.

The remarkable thing in Maxwell's point of view is that the E.M.F., which to the experimenter had had meaning only for closed metallic circuits, is here defined for arbitrary loops, whether they pass through conductors, nonconductors, or through parts of both. The same geometric freedom then exists also for the magnetic loop tension or magnetomotive force. We now write down the two principal axioms which relate the

¹ In the title of his comprehensive paper: *La théorie analytique des phénomènes électrodynamiques, uniquement déduite de l'expérience.*

quantities defined in Eqs. (1) and (2) in this completely general sense. They are:

$$\frac{d}{dt} \int B_n d\sigma = - \oint \mathbf{E} \cdot d\mathbf{s}, \quad (3)$$

$$\int C_n d\sigma = \oint \mathbf{H} \cdot d\mathbf{s} \quad (4)$$

In words: *Every change in the number of magnetic lines of force which traverse a given surface σ produces in its boundary s an electric loop tension which is numerically equal to the rate of change, but opposite in sign (Faraday's law of induction) and*

The number of electric current lines, which traverse an arbitrary surface σ is accompanied by a magnetic loop tension in the bounding curve s of σ which is equal to it in both magnitude and direction (Ampère's law relating magnetic field and electric current).

Let us convince ourselves first that this equating of electric and magnetic quantities is dimensionally proper. The two surface integrals defined in (1) (in spite of their dimensionally incorrect designation as *numbers* of force lines or current lines) have, according to (2.8) and 2.4a), the dimensions

$$\frac{\text{newton MS}}{Q} = \frac{\text{joule S}}{Q} \quad \text{and} \quad \frac{Q}{S}, \text{ respectively.}$$

According to (2.9a) the latter dimension agrees with the dimension of the line integral in (4). The time rate of change of the first expression yields joule/ Q , i.e. the dimension of an electric tension (expressible in volts), in agreement with the right side of Eq. (3). From this dimensional check our fundamentally different conception of \mathbf{B} and \mathbf{H} becomes apparent, and it is clear that our special introduction of the symbol Q for the dimension of charge is unavoidable.

Next we concern ourselves with the signs in Eqs. (3) and (4). They correspond to the rules of Lenz and Ampère. Ampère's rule is simply the right-handed screw rule, by which we correlated the positive normal of the surface σ with the sense of travel along the boundary s . The various rules of thumb commonly given in textbooks are merely specializations of our righthanded screw convention. To check Lenz's rule we imagine in Eq. (3) the boundary curve s to be realized by a wire loop, and the magnetic flux traversing the surface σ in the direction n as lines of force proceeding from the positive pole P of a bar magnet, the negative pole being assumed to be sufficiently far away. We bring (see Fig. 1) the bar magnet near to the wire loop and thus increase the magnetic flux, so that the left side of (3) becomes *positive*. Then, as shown by the equation, the line integral on the right side must become *negative*. The E.M.F. and the corresponding cur-

rent induced in the wire loop then form a left-handed screw with the direction of motion of the bar magnet. The magnetic field corresponding to the induced current is, on the other hand, represented, according to our right-hand screw rule, by the arrow P' in Fig. 1. The positive pole of this magnetic field thus points in the direction from which the positive pole P of the magnet approaches the loop: The two poles repel each other or the induced current *inhibits* the motion of the inducing magnet. This is the meaning of Lenz's rule: The appearance of the induced current opposes the disturbance of equilibrium produced by the motion of the bar magnet.

We emphasized above that the bounding curve s may be fixed quite arbitrarily; the same remark applies also for fixed boundary to the surface σ . If two different surfaces σ_1 and σ_2 are passed through the same curve s , the left sides of Eqs. (3) and (4) computed for σ_1 and σ_2 , must turn out to

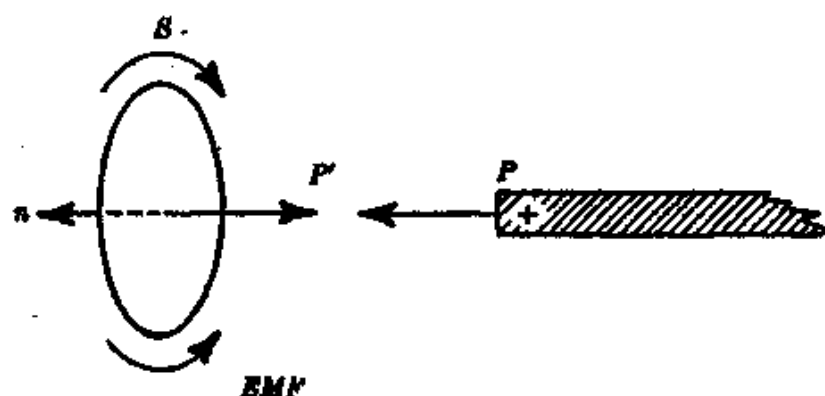


FIG. 1. Lenz's rule.

be equal. This is equivalent to stating that they must vanish for the closed surface formed by σ_1 and σ_2 if the positive normal (n always pointing outward) is defined in a uniform manner. We realize this fact also from the following: We consider a closed surface σ with a boundary curve which has contracted to a point. This does not contribute to the line integrals in Eqs. (3) and (4). If we indicate the integration over the now unbounded surface with \oint , we thus obtain

$$\frac{d}{dt} \oint B_n d\sigma = 0 \quad \text{and} \quad \oint C_n d\sigma = 0; \quad (5)$$

by Eq. (2.5) the second equation may also be written

$$\oint J_n d\sigma + \frac{d}{dt} \oint D_n d\sigma = 0. \quad (5a)$$

More particularly, if the surface σ lies entirely in nonconducting material and is hence traversed by no conduction currents,

$$\frac{d}{dt} \oint D_n d\sigma = 0. \quad (5b)$$

The first equation (5) and Eq. (5b) state, in integrated form,

$$\oint B_n d\sigma = \text{const.}, \quad \oint D_n d\sigma = \text{const.}, \quad (6)$$

while the second Eq. (5) and Eq. (5a) show that the total electric current is always *closed* in Maxwell's theory: the quantities entering and leaving just compensate each other; the current lines traversing our surface σ form closed loops somewhere outside of it. Furthermore, the *magnetic* lines of force also are always closed. If a magnet (or electromagnet) is subdivided, north poles and south poles, which compensate each other as far as the total magnetic flux is concerned, are formed anew on every part. It follows that the constant in the first Eq. (6) *must be zero*, while in the second equation this constant is the algebraic sum \bar{e} of the charges e enveloped by the surface σ . According to the above this must be a *constant in time for a nonconductor*:

$$\oint B_n d\sigma = 0, \quad \oint D_n d\sigma = \bar{e}, \quad \bar{e} = \sum e = \text{const.} \quad (6a)$$

The first Eq. (6a) is a *supplementary axiom*, an addition to our principal axioms required by experience. The second Eq. (6a) agrees with our earlier Eq. (2.3b) and states the constancy in time of the charge in nonconductors. The D-lines and the E-lines coinciding with them geometrically originate at points of positive charge and end at points of negative charge. Eq. (5a) generalizing the second Eq. (6a), may be designated in hydrodynamic terminology as the *continuity equation of electricity*. If the definition of \bar{e} in Eq. (6a) is employed it takes on the form

$$\frac{d\bar{e}}{dt} + \oint J_n d\sigma = 0. \quad (6b)$$

This expresses the fact that the electricity within a surface σ may decrease as the result of flowing off through metallic portions of σ .

The first Eq. (6a) may be expressed, with Heriz, in the form: *There is no true magnetism*. In this statement one proceeds from the assumption, formerly regarded as obvious, that \mathbf{B} is the magnetic analogue of \mathbf{D} . From our standpoint, however, this analogue is \mathbf{H} , and not \mathbf{B} . We shall hence have to relate the definition of "magnetism," in particular of the pole strength P (see §7), not to \mathbf{B} but to \mathbf{H} .

We now apply the first Eq. (6a) to the neighborhood of the *boundary surface between two bodies of different magnetic properties* such as iron and air. Let the closed surface σ be the surface of a very flat prism (Fig. 2), whose height Δh is very small compared to the base Δf , and let this base lie for example in iron, while the parallel top side is in air. Eq. (6a) then

demands, with arbitrary accuracy in view of the arbitrary smallness of Δh ,

$$(B'_{n'} + B_n) \Delta f = 0. \quad (7)$$

Let B' refer, for example, to iron, B , to air. The normal (n' in iron, n in air) points outward on both surfaces Δf . Then, in view of Eq. (7),

$$B'_{n'} = -B_n \quad \text{and hence also} \quad B'_{n'} = B_n,$$

provided that now n denotes the same direction in both media. We have thus obtained a first *boundary condition* for the magnetic field: At the transition between two magnetically different media the *normal component of the induction is continuous*.

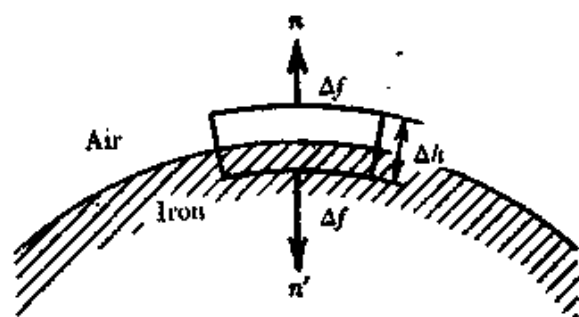


FIG. 2. Derivation of the continuity of B_n at the boundary between two media from the relation $\oint B_n d\sigma = 0$.

We will show that the same applies to the *tangential component of the excitation H*. For this purpose we consider a very small rectangular loop s (Fig. 3), with the height Δh normal to the boundary surface and the side length Δs parallel to it. Here we assume that $\Delta h \ll \Delta s$ so that in the limit $\Delta h \rightarrow 0$ the area $\Delta\sigma = \Delta h \Delta s$ vanishes. With the assumption that the current density parallel to the boundary surface, referred to in Eq. (4), does not become infinitely large¹ we obtain from Eq. (4):

$$0 = (H'_{s'} + H_s) \Delta s \quad (8)$$

so that

$$H'_{s'} = -H_s \quad \text{and hence also} \quad H_s = H'_{s'} \quad (8a)$$

where again s denotes the same direction in the two media.

From exactly the same figure and the same consideration for two electrically different media we are led from Faraday's law of induction to the conclusion that the *tangential components of the electric field strength E* are continuous along the boundary of the two media:

$$E'_{s'} = E_s. \quad (9)$$

¹ This limiting case is the normal one for good conductors at high frequencies. - Then H_s becomes discontinuous and B_n vanishingly small.

Nothing has been said regarding the *normal component* of \mathbf{E} . Furthermore, the continuity of the normal component of \mathbf{D} (unlike that of \mathbf{B}) is not required by Eq. (6a). For, if D_n has a discontinuity at the boundary of two electrically different media (e.g. glass and air) or at any other surface, we say that a surface charge is present on the surface. If we call this surface charge ω (dimension Q/M^2), the charge present in the prism in Fig. 2 for the transition to the limit $\Delta h \rightarrow 0$ is,

$$\bar{e} = \omega \Delta f. \quad (10)$$

thus by the consideration leading to Eqs. (7) and (7a), the second Eq. (6a) demands

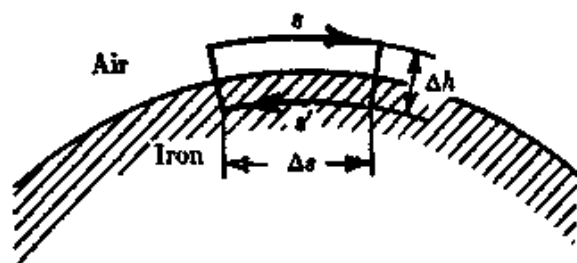
$$(D'_n + D_n) \Delta f = \omega \Delta f, \quad (10a)$$

or, employing the same direction of the normal n :

$$D_n - D'_n = \omega. \quad (11)$$

Discontinuous behavior of the normal component of \mathbf{D} signifies that the boundary surface considered carries a surface charge; the magnitude of the discontinuity indicates the surface charge directly.

FIG. 3. Derivation of the continuity of H_s at the boundary between two media from the relation $\oint \mathbf{H} \cdot d\mathbf{s} = 0$.



Finally, we obtain from Eq. (6b) for the boundary surface between a conductor and a nonconductor by utilizing Fig. 2 and Eq. (10),

$$\frac{d\omega}{dt} + J_n = 0, \quad (12)$$

that is, a loss of surface charge if electric current is possible in the conductor. In electrostatics, where the interior of conductors is fieldfree ($\mathbf{D} = 0$, $\mathbf{J} = 0$), Eq. (12) is fulfilled identically and Eq. (11) takes on the special form

$$\omega = D_n. \quad (12a)$$

In the static field conductors bear a surface charge varying from point to point and given by the normal component of \mathbf{D} .

§4. The Maxwell Equations in Differential Form and the Material Constants of the Theory

We pass from the integral to the differential form by allowing the loops s in the integral form, and hence also the surfaces σ passed through them, to become arbitrarily small. If we call the latter $\Delta\sigma$ we can write in the limit:

$$\int B_n d\sigma = \Delta\sigma B_n, \quad \int C_n d\sigma = \Delta\sigma C_n. \quad (1)$$

Furthermore we recall the definition of the vector operation "curl" by the transition to the limit of a loop integral (Vol. II, Eq. 2.21). For our infinitesimal loops this leads to

$$\oint E_s ds = \Delta\sigma \text{curl}_n E, \quad \oint H_s ds = \Delta\sigma \text{curl}_n H. \quad (2)$$

We must form the time derivative of the first Eq. (1). We will here imagine the surface $\Delta\sigma$ to remain fixed, which obviously applies to media *at rest*, to which we shall confine ourselves initially. We then obtain

$$\frac{d}{dt} \int B_n d\sigma = \Delta\sigma \dot{B}_n, \quad \dot{B} = \frac{\partial B}{\partial t}. \quad (3a)$$

At the same time, using Eq. (2.5), we write Eq. (1) in the form

$$\int C_n d\sigma = \Delta\sigma (J_n + \dot{D}_n), \quad \dot{D} = \frac{\partial D}{\partial t}. \quad (3b)$$

With Eqs. (2) and (3a, b), cancelling the factor $\Delta\sigma$ which is common to all terms, as well as omitting the common index n , the principal axioms (3.3) and (3.4) lead to the two *vectorial differential equations*:¹

$$\dot{B} = - \text{curl } E \quad (4)$$

$$\dot{D} + J = \text{curl } H.$$

The universal importance and impressive beauty of these equations led Boltzmann² to quote: "Was it a god who wrote these lines . . ."

¹ Our second equation (4) is usually called the first set of Maxwell's equations, our first Eq. (4), the second set. We prefer the sequence of the text since in our presentation the intensity entities E and B were introduced first as being more readily interpreted. We can also point to §7, where electrostatics will result from the specialization of the first, magnetostatics, from the specialization of the second Eq. (4), in support of our order. Since it would be improper to treat magnetostatics ahead of the simpler electrostatics the numbering of the Maxwell equations which differs from ours appears unsuitable.

² Motto of the second volume of his "Vorlesungen über Maxwells Theorie der Elektrizität und des Lichtes," München 1893. Our formulation, which deviates slightly from Boltzmann's (vector in place of coordinate notation), clearly only serves to enhance the beauty and simplicity of the equations.

We complete them by the supplementary axiom (3.6a) for \mathbf{B} , and the relation between \mathbf{D} and the charge, contained in the same equation. We shall now regard the latter as continuously distributed in accord with our differential point of view. Thus, we shall not speak of point charges e , but of finite densities in space ρ , so that the infinitesimal charge

$$\Delta e = \rho \Delta \tau$$

is contained in an element of volume $\Delta \tau$ which approaches zero in magnitude. At the same time we recall the vector operation "divergence" and its representation (in Vol. II, Eq. 2.20) by the limit of a volume integral.¹ For our present purposes we write this representation

$$\lim \frac{1}{\Delta \tau} \oint B_n d\sigma = \text{div } \mathbf{B}, \quad \lim \frac{1}{\Delta \tau} \oint D_n d\sigma = \text{div } \mathbf{D}$$

and obtain for Eqs. (3.6a, b), omitting the factor $\Delta \tau$, their differential form:

$$\text{div } \mathbf{B} = 0, \quad (4a)$$

$$\text{div } \mathbf{D} = \rho, \quad (4b)$$

$$\frac{\partial \rho}{\partial t} + \text{div } \mathbf{J} = 0. \quad (4c)$$

Our Eqs. (4) and (4a, b, c) set up the framework into which the phenomena of electrodynamics must be fitted. But this framework is still *too wide*. Five vectors \mathbf{E} , \mathbf{D} , \mathbf{J} , \mathbf{B} , and \mathbf{H} occur in our equations, or altogether 15 unknown functions of time and space. (The scalar ρ is referred back to the vectors \mathbf{D} and \mathbf{J} by the Eqs. (4b) and (4c) respectively.) For their determination we have two vector equations (4), i.e., altogether only six differential equations. We must narrow down the framework to be able to fill it out with a unified electrodynamic model. The electromagnetic

¹ We contrast the volume divergence here introduced with the term *surface divergence*. Referring to Fig. 2 and the integration there carried out over a prism with a base Δf and vanishing height, we understand by this the result of the integration divided by Δf . According to Eq. 3.7 and with the meaning of the normals \mathbf{n} and \mathbf{n}' there given, the surface divergence of an arbitrary vector \mathbf{A} is:

$$A_{n'} + A_n; \quad (4d)$$

Eqs. 3.7a and 3.10a then state simply: *The surface divergence of \mathbf{B} vanishes, that of \mathbf{D} equals the surface charge.*

Similarly, we can contrast the volume curl with the *surface curl*. Referring to Fig. 3 and the integration over a rectangle of base Δs and vanishing height carried out in Eq. 3.8, we understand by the surface curl the result of the integration divided by Δs . The surface curl of an arbitrary vector \mathbf{A} is hence, according to Eq. 3.8,

$$A_{s'} + A_s; \quad (4e)$$

it represents the discontinuity of \mathbf{A} at the surface in question.

material constants serve this purpose. We shall discuss them in the sequence conductivity, dielectric constant, permeability.

1. Conductivity and Ohm's Law

The electric current density \mathbf{J} depends on the electric field strength \mathbf{E} within the conductor. We assume a linear dependence

$$\mathbf{J} = \sigma \mathbf{E} \quad (5)$$

and call the real positive constant σ the *electric conductivity*. Eq. (5) expresses *Ohm's law* for unit length of a wire carrying a stationary current. To recognize this, we replace \mathbf{J} by the total current $I = qJ$ (q = cross section of the wire) and multiply Eq. (5) with the length of the wire. We obtain

$$RI = V \cdot \begin{cases} R = \frac{1}{q\sigma} \\ V = lE = \int_0^l E ds = \text{voltage} \end{cases} \quad (5a)$$

The concept of voltage had been created already by Volta, while the concept of resistance was first introduced by Georg Simon Ohm in 1827. For us Ohm's law signifies the introduction of the material constant σ . According to Eq. (5) its dimension is:

$$\sigma = \frac{Q^2}{M^2 S \text{ newton}} = \frac{Q^2}{MS \text{ joule}} \quad (5b)$$

According to Eq. (5a) σ may also be designated as the reciprocal of the *specific electric resistance*, i.e. the resistance of a prism of the length $l = 1 M$ and of the cross section $q = 1 M^2$. The dimension of the resistance is by Eqs. (5a, b):

$$R = \frac{\text{joule S}}{Q^2} \quad (5c)$$

The unit of resistance in the practical system of units is the Ω (pronounced "ohm") = 10^9 cgs units. It is identical with the unit in our MKSQ system provided that we choose, according to our convention, Q equal to 1 coulomb = $\frac{1}{10}$ cgs unit. We then obtain

$$1 \frac{\text{joule S}}{Q^2} = 10^7 \frac{\text{erg sec}}{Q^2} = 10^9 \text{ cgs units} = 1\Omega. \quad (5d)$$

Ohm's law applies only to macrophysical events, not to Ampère's molecular currents, electron paths in atoms, Larmor precessions; cathode rays in vacuum tubes are also resistance-free electric currents.

2. Dielectric Constant

The displacement \mathbf{D} depends on the electric field strength \mathbf{E} at the point in question. We assume the dependence to be linear:

$$\mathbf{D} = \epsilon \mathbf{E} \quad (6)$$

and call the real positive constant ϵ the dielectric constant. Its dimension is, by Eqs. (2.4) and (2.1),

$$\epsilon = \frac{Q^2}{M \text{ joule}} \quad (6a)$$

We denote the dielectric constant of vacuum by ϵ_0 . It also is a definite quantity of the dimension (6a). The relation

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \quad (6b)$$

valid for vacuum, was pointed out already in §2. Invariably $\epsilon > \epsilon_0$.

3. Permeability

A relation also exists between the two magnetic vectors \mathbf{H} and \mathbf{B} , which, as a first approximation, we shall also assume to be linear. We would like to write it in the form

$$\mathbf{H} = \mu' \mathbf{B},$$

since we regard \mathbf{H} as analogue of \mathbf{D} and \mathbf{B} as analogue of \mathbf{E} . However, we are unfortunately obliged to follow general usage and choose the form

$$\mathbf{B} = \mu \mathbf{H}. \quad (7)$$

The material constant μ is called *permeability* and has, according to Eqs. (2.8) and (2.9a), the dimension

$$\mu = \frac{\text{joule S}^2}{Q^2 M}. \quad (7a)$$

This introduction of μ , which is illogical in view of Eq. (6), leads to the obvious consequence that in later formulas, such as Coulomb's law, not μ , but its reciprocal μ' will take the place of ϵ . For vacuum we write

$$\mathbf{B} = \mu_0 \mathbf{H}; \quad (7b)$$

μ_0 also obviously has the dimension given in Eq. (7a). For *paramagnetic* bodies $\mu > \mu_0$, for *diamagnetic* bodies, $\mu < \mu_0$. Our formulas (5), (6), and (7) do not have the same degree of certainty and general validity as Maxwell's equations (4). This has long been known for the *ferromagnetic materials*, where a general functional relationship

$$\mathbf{B} = \mathbf{B}(H, T), \quad T = \text{absolute temperature}$$

takes the place of the linear relation (7). Rochelle salts¹ show a dielectric behavior similar to that of the ferromagnetic materials, exhibiting, like the latter, both saturation and hysteresis phenomena.

For paramagnetic materials deviations from linearity occur only at extremely high field strengths or extremely low temperatures. Deviations from the linearity of Ohm's law have been expected at very high field strengths; the failure of this law for superconductors is obvious. Furthermore the simple proportionality between corresponding vectors expressed by Eqs. (5), (6), and (7) is true only for isotropic bodies. In *crystals* the dependence is expressed instead quite generally by a *linear vector function* (see Vol. II, Eq. 1.10). The varied and interesting phenomena of crystal optics, which we shall treat in Vol. IV, rest on this fact.

On the other hand, the general field equations (4) apply also for anisotropic bodies. Beyond this, they appear to hold true even in the face of all new proposals of a *generalized* electrodynamics, proposals which are concerned with extremely strong fields (such as must occur, for example, close to an electron), but intrinsically amount merely to a replacement of the linear relation (6) by a generalized variation of \mathbf{D} with \mathbf{E} (see the final section of this volume). The deeper reason for the remarkable vitality of the form of equation discovered by Maxwell will be found to rest in its invariance properties, which will not, however, be taken up until Part III.

We can now undertake the required contraction of our electrodynamic framework. If, in particular, we employ for this purpose our simple linear relations and treat σ , ϵ , and μ as quantities independent of t (restriction to media *at rest*), we obtain by substituting Eqs. (5), (6), and (7) in Eq. (4):

$$\begin{aligned}\mu \frac{\partial \mathbf{H}}{\partial t} &= - \text{curl } \mathbf{E}, \\ \left(\epsilon \frac{\partial}{\partial t} + \sigma \right) \mathbf{E} &= \text{curl } \mathbf{H},\end{aligned}\tag{8}$$

i.e., six simultaneous differential equations of the first order for six unknowns, the 2 · 3 components of \mathbf{E} and \mathbf{H} . Thus we find ourselves presented with a well-defined mathematical problem.²

¹ Also known as Seignette salts. Seignette was the name of a pharmacist in the French fortress La Rochelle. We are here concerned with hydrated sodium potassium tartrate:



² We could of course also have written Eqs. (8) as relations between \mathbf{E} and \mathbf{B} , or also between \mathbf{D} and \mathbf{H} . However, the form in the text is the customary one and, in general, also the most convenient one.

At the same time the conditions (4a, b, c) take on the form

$$\operatorname{div}(\mu \mathbf{H}) = 0, \quad \operatorname{div}(\epsilon \mathbf{E}) = \rho, \quad (8a, b)$$

$$\operatorname{div} \left\{ \left(\epsilon \frac{\partial}{\partial t} + \sigma \right) \mathbf{E} \right\} = 0. \quad (8c)$$

Eq. (8a) is to be regarded as a restrictive supplementary condition on Maxwell's equations, Eq. (8b), as defining equation for ρ . Eq. (8c) is obtained by forming the divergence of the second equation (8). Its simplest solution results from setting the parenthesis $\{\}$ equal to zero; it is represented by the exponential function

$$\mathbf{E} = \mathbf{E}_0 \exp \left(-\frac{\sigma}{\epsilon} t \right), \quad \mathbf{E}_0 = \text{arbitrary function of space.} \quad (9)$$

We set

$$\frac{\epsilon}{\sigma} = T, \quad (9a)$$

and call T the *relaxation time of the conductor*. Its dimension is the second by Eq. (5a) and (6a), as must be the case, its magnitude for good conductors a very small fraction of a second. The field decays within the conductor everywhere in accord with this relaxation time and is known, provided that \mathbf{E}_0 is given.

We might continue with the already discussed conditions at the boundary between two electromagnetically different media. To use the differential form of the Maxwell equations, however, it would be necessary to regard the transition between the two media as continuous, i.e., to speak of a "boundary layer" rather than a "boundary surface." We will carry out this procedure in problem I.1, where we shall find that the derivation becomes less straightforward than in the Eqs. 3.7 to 3.12, which followed from the integral form of Maxwell's equations.

The same conclusion is reached in other problems distinguished by a particular symmetry: *The general development of Maxwell's theory must proceed from its differential form; for special problems the integral form may, however, be more advantageous.*

The following two fundamental problems, which will be treated also by the differential method in problem I.2 and I.3, are examples of this:

1. An infinitely long wire in the form of a circular cylinder is traversed by current distributed uniformly over its cross section. The return of the current may take place through a similarly traversed hollow cylinder which is coaxial with the wire. The magnetic excitation is to be determined within the wire, within the hollow cylinder, and in the region between them.

2. An infinitely long, tightly wound coil is traversed similarly by stationary current. The magnetic excitation is to be determined at any point within the coil.

Regarding 1: Let a be the radius of the wire, b and c , the inner and outer radius of the cylindrical return conductor. We introduce a right-handed coordinate system about the center line of the wire as z -axis. Let the current density have the direction of the positive z -axis in the wire, that of the negative z -axis in the return conductor. Let the total current be I and $-I$ respectively:

$$I = \pi a^2 J_z, \quad -I = \pi(c^2 - b^2) J_{-z}.$$

The symmetry of the problem indicates that \mathbf{H} is independent of φ and has the direction of increasing φ . We write $H_\varphi = H$ and carry out the line

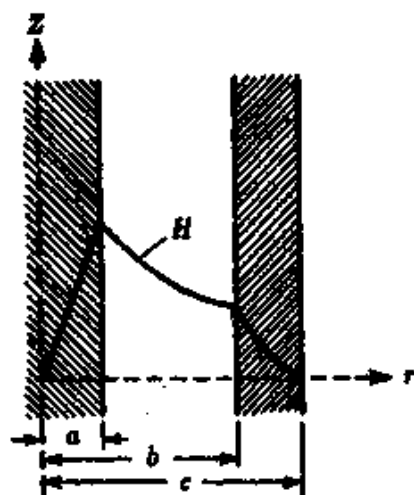


FIG. 4. A straight wire carrying a stationary current and a hollow cylinder surrounding it as return conductor. The magnetic excitation $H_\varphi = H$ within the wire, in the air space between the two conductors, and in the return conductor.

integral of \mathbf{H} about any circle $r = \text{const}$ in any cross-section plane of the wire. Since the displacement current vanishes everywhere in view of the assumed stationary condition, we obtain

$$0 < r < a: \quad 2\pi r H = \pi r^2 J_z = \frac{r^2}{a^2} I, \quad H = \frac{r}{a} \frac{I}{2\pi a}, \quad (10)$$

$$a < r < b: \quad 2\pi r H = I, \quad H = \frac{I}{2\pi r} \quad (11)$$

$$b < r < c: \quad 2\pi r H = I + \pi(r^2 - b^2) J_{-z} = I \left(1 - \frac{r^2 - b^2}{c^2 - b^2} \right), \quad (12)$$

$$H = \frac{I}{2\pi r} \frac{c^2 - r^2}{c^2 - b^2}$$

$$c < r: \quad 2\pi r H = I - I, \quad H = 0. \quad (13)$$

The boundary conditions for \mathbf{H} at the surface of the wire $r = a$ and at the cylinder surfaces $r = b, c$ are satisfied automatically by Eqs. 10 to 13. The variation of H is plotted in Fig. 4.

Regarding 2: We use a right-handed system r, φ, z which has the center line of the coil as z -axis. For sufficient length of the coil and sufficiently close winding no magnetic lines of force penetrate to the exterior of the coil; the current I has the direction of increasing φ , the excitation \mathbf{H} that of increasing z . We shall show that $H_z = H$ is constant within the coil.

For this purpose we consider the rectangular loop, of length l in the z -direction, shown in Fig. 5. Its plane intersects the coil in $N_1 l$ points, where N_1 is the number of turns per unit length of the coil. Since $H_r = 0$ both

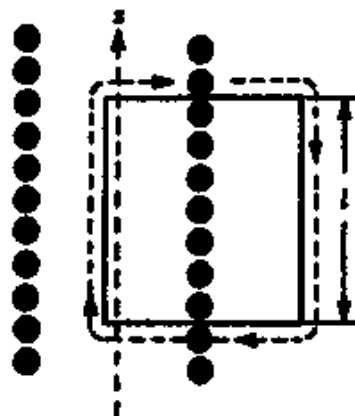


FIG. 5. The magnetic excitation \mathbf{H} within an infinitely long coil.

within and outside of the coil and $H_r = 0$ outside of the coil, only one side of the loop contributes to the line integral. We find

$$Hl = N_1 l I, \quad H = N_1 I. \quad (14)$$

The magnetic excitation within the coil is given by the "number of ampere turns per unit length" $N_1 I$. This explains the designation of H customary in engineering practice which was introduced on p. 12. The value of H given by Eq. 14 is independent of r , i.e. the same throughout the interior of the coil.

§5. Law of Conservation of Energy and Poynting Vector

Starting from Eqs. (4.4) we carry out a scalar multiplication of the first with \mathbf{H} , a scalar multiplication of the second with \mathbf{E} . We obtain as the sum of the two:

$$\mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{E} \cdot \mathbf{J} = \mathbf{E} \cdot \text{curl } \mathbf{H} - \mathbf{H} \cdot \text{curl } \mathbf{E}. \quad (1)$$

On the right-hand side we apply the transformation, valid for arbitrary vectors \mathbf{U}, \mathbf{V} :

$$\mathbf{V} \cdot \text{curl } \mathbf{U} - \mathbf{U} \cdot \text{curl } \mathbf{V} = \text{div}(\mathbf{U} \times \mathbf{V}).$$

We prove this relation most readily by utilizing the symbol "nabla operator"

$$\nabla = \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$



(see Vol. II, footnote 1 on p. 23) and interpreting the divergence as scalar multiplication, the curl as vector multiplication with this vector:

$$\text{div}(\mathbf{U} \times \mathbf{V}) = \nabla \cdot (\mathbf{U} \times \mathbf{V}) = \nabla_U \cdot (\mathbf{U} \times \mathbf{V}) + \nabla_V \cdot (\mathbf{U} \times \mathbf{V}), \quad (2a)$$

$$\text{curl } \mathbf{U} = \nabla \times \mathbf{U}, \quad \text{curl } \mathbf{V} = \nabla \times \mathbf{V}. \quad (2b)$$

In Eq. (2a) the subscripts U, V indicate that the ∇ -differentiations are to be carried out only on the vectors \mathbf{U} and \mathbf{V} , respectively. Since the sequence of the vectors may be cyclically interchanged in the double products, Eq. (2a) may also be written

$$\begin{aligned} \text{div}(\mathbf{U} \times \mathbf{V}) &= \mathbf{V} \cdot (\nabla \times \mathbf{U}) + \mathbf{U} \cdot (\nabla \times \mathbf{V}) \\ &= \mathbf{V} \cdot (\nabla \times \mathbf{U}) - \mathbf{U} \cdot (\nabla \times \mathbf{V}). \end{aligned} \quad (2c)$$

Here the right side, in view of Eq. (2b), is the same as the left side of Eq. (2), so that Eqs. (2c) and (2) become identical. This proof of Eq. (2) is only an abbreviated form for the direct, but much more involved, calculation with rectangular coordinates x, y, z .

Let us now set $\mathbf{V} = \mathbf{E}$ and $\mathbf{U} = \mathbf{H}$ in Eq. (2) and introduce the abbreviation

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \quad (3)$$

Then Eq. (1) becomes

$$\mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{E} \cdot \mathbf{J} + \text{div } \mathbf{S} = 0. \quad (4)$$

Eq. (4) is *Poynting's theorem*, \mathbf{S} , the *Poynting vector*. We shall show that \mathbf{S} is the *energy flux vector*.

We consider first the dimension of the individual terms of Eq. (4). The first two terms have, according to Eqs. (2.9a) and (2.8), and (2.1) and (2.4), respectively, the dimension

$$\frac{\text{newton}}{\text{M}^3\text{S}} = \frac{\text{joule}}{\text{M}^3\text{S}} = \text{energy per unit volume and unit time}. \quad (4a)$$

The third term has, as must be the case, the same dimension (see Eqs. (2.1) and (2.4a)). The dimension of Eq. (3) is, by Eqs. (2.1) and (2.9a),

$$\frac{\text{joule}}{\text{M}^2\text{S}} = \text{energy per unit area and unit time}. \quad (4b)$$

The operation div , which indicates a differentiation with respect to the space coordinates, yields for the dimension of the fourth term in Eq. (4) the same result.

We see that our *electrical unit* Q does not occur in (4a, b). It has *discreetly withdrawn from the company of the mechanical units MKS*. The same will be noted in many later dimensional considerations in which

we are dealing with purely mechanical quantities, which are independent of the choice of the electrical unit.

We pass to the physical interpretation of the individual terms in Eq. (4). It is simplest for the third term: this signifies the work done by the electric field on moving electric charge per unit volume and per unit time. It is generally converted into heat and is known as *Joule heat*. We designate it W_J , transferring the symbol W (work), which Maxwell generally employs for total energy, to *energy density*. Thus we obtain

$$W_J = \mathbf{E} \cdot \mathbf{J}. \quad (5)$$

We shall see right away that the two first terms of (4) are the time rate of change of the *magnetic* and *electric energy densities*; the latter are defined, in accord with Maxwell, by

$$W_m = \frac{1}{2} \mathbf{H} \cdot \mathbf{B}, \quad W_e = \frac{1}{2} \mathbf{E} \cdot \mathbf{D}. \quad (6)$$

By this definition the energy is *localized* in the field; a definite electric and magnetic energy content $W_e d\tau$ and $W_m d\tau$ is ascribed to every element of volume $d\tau$. This constitutes a first step in the adaptation of the energy concept to the ideas of field theory.

The factor 1/2 in the two defining equation (6) evidently points to a continuous generation of energy, comparable with the stretching of a spring. In accord with the pattern force \times increase in path length = intensity entity \times change in quantity entity, we obtain

$$W_e = \int \mathbf{E} \cdot d\mathbf{D},$$

which, for a linear relationship between \mathbf{E} and \mathbf{D} , reverts, in fact, to (6). The situation is slightly different for the magnetic energy. Here Poynting's theorem (4) directs us to start from

$$W_m = \int \mathbf{H} \cdot \dot{\mathbf{B}} dt = \int \mathbf{H} \cdot d\mathbf{B}. \quad (6a)$$

From the point of view of our general system (\mathbf{B} = intensity entity, \mathbf{H} = quantity entity) it would have seemed more reasonable to represent the energy density not by (6a), but by

$$W_m = \int \mathbf{B} \cdot d\mathbf{H}. \quad (6b)$$

For a linear relationship between \mathbf{H} and \mathbf{B} this of course leads again to Eq. (6); for a nonlinear variation, on the other hand, it leads to a result which differs from $\int \mathbf{H} \cdot d\mathbf{B}$, and is therefore *incorrect* by Poynting's theorem. From this we learn that work need not be expressible in the form intensity entity \times change in quantity entity.

G. Mie, who takes the same standpoint throughout in respect to the meaning of \mathbf{B} and \mathbf{H} as we do, on p. 467 of his excellent textbook cited in §2, points to the following mechanical analogue: a moving body carries with it, in unit volume, the momentum (intensity entity) \mathbf{p} . For its acceleration the force per unit volume $d\mathbf{p}/dt$ is required, and hence the work

$$\frac{d\mathbf{p}}{dt} \cdot d\mathbf{s} = d\mathbf{p} \cdot \mathbf{v} = \mathbf{v} \cdot d\mathbf{p};$$

This is a product of the type $\mathbf{H} \cdot d\mathbf{B}$, i.e. quantity entity \times change in an intensity entity.¹

In this representation the magnetic energy parallels the *kinetic* energy of mechanics. We shall meet the same correspondence in electron theory. Also in Helmholtz's analogy between vortices in fluids and electric currents the magnetic energy corresponds to the kinetic energy of the fluid. The same applies for our semielastic ether model in Vol. II, §15.

If we should refer to Maxwell in connection with Eqs. (6), we would find that in Maxwell's work the factor $1/2$ is replaced by $1/(8\pi)$, which from Maxwell has passed over into the major portion of the literature. It evidently lacks the simple logical basis of the factor $1/2$ and has only historical justification.

We must now belatedly give the proof that the quantities $\mathbf{H} \cdot \dot{\mathbf{B}}$ and $\mathbf{E} \cdot \dot{\mathbf{D}}$ occurring in (4) are identical with the time rates of change of the energy densities given by (6). For this purpose we deduce from (6)

$$\dot{W}_e = \frac{1}{2} \mathbf{E} \cdot \dot{\mathbf{D}} + \frac{1}{2} \dot{\mathbf{E}} \cdot \mathbf{D}. \quad (6c)$$

The two terms on the right are equal, to begin with, in an *isotropic* medium, where $\mathbf{D} = \epsilon \mathbf{E}$. They are also equal in an *anisotropic crystal*, where a "linear vector function" replaces the simple proportionality (see p. 22):

$$D_i = \sum_k \epsilon_{ik} E_k.$$

From this we calculate for the two expressions on the right side of (6)

$$\begin{aligned} \sum_i E_i \dot{D}_i &= \sum_i E_i \sum_k \epsilon_{ik} \dot{E}_k, & \sum_k \dot{E}_k D_k &= \sum_k \dot{E}_k \sum_i \epsilon_{ki} E_i \\ & & &= \sum_i E_i \sum_k \epsilon_{ki} \dot{E}_k \end{aligned}$$

¹ For the elementary relationship between \mathbf{p} and \mathbf{v} , i.e. $\mathbf{p} = M\mathbf{v}$, and for constant mass we have again $d\mathbf{p} \cdot \mathbf{v} = \mathbf{p} \cdot d\mathbf{v}$. However, for a mass varying with time, in particular, the velocity-dependent mass of relativity theory, this is not the case. Then the form $d\mathbf{p} \cdot \mathbf{v}$ of the text expresses the energy change uniquely.

The two expressions are equal to each other since, irrespective of the crystal symmetry¹

$$\epsilon_{ik} = \epsilon_{ki} \quad (6d)$$

It follows from (6c) that for the anisotropic case, as for the isotropic case,

$$\dot{W}_e = \mathbf{E} \cdot \dot{\mathbf{D}}. \quad (6e)$$

The same applies for the magnetic energy density both for the isotropic medium (proportionality between \mathbf{H} and \mathbf{B}) and for the magnetic crystal (linear vector function with $\mu_{ik} = \mu_{ki}$). Here also

$$\dot{W}_m = \frac{1}{2} \mathbf{H} \cdot \dot{\mathbf{B}} + \frac{1}{2} \dot{\mathbf{H}} \cdot \mathbf{B} = \mathbf{H} \cdot \dot{\mathbf{B}}. \quad (6f)$$

In view of (5) and (6e, f), (4) yields

$$\dot{W} + \text{div } \mathbf{S} = -W_J, \quad W = W_e + W_m. \quad (7)$$

In this form Poynting's theorem expresses the *energy balance* in the electromagnetic field. The Joule heat is recorded as a loss on the right side of the equation. The left side corresponds to the energy exchange between the volume element $d\tau$ in question and neighboring elements. This becomes even clearer if Eq. (7) is integrated over a given volume; then the application of Gauss's theorem leads to

$$\frac{\partial}{\partial t} \int W d\tau + \int S_n d\sigma = - \int W_J d\tau. \quad (7a)$$

The significance of \mathbf{S} as energy flux through the surface of the volume considered is now evident.

With the introduction of this concept Poynting passes beyond Maxwell's localization of the energy. We now learn not merely how much energy exists at any place, but also where it will go or (for the opposite sign of \mathbf{S}) from where it has come.

¹ This restriction on the otherwise arbitrary coefficients ϵ_{ik} is necessary in order that the work done on an element of volume, $\mathbf{E} \cdot d\mathbf{D}$, may be a complete differential. Otherwise the electric energy density would not be a characteristic function of the state, as we postulate for ideal solid bodies. (It is true that for certain known crystals hysteresis phenomena occur which make the notion of a quantity characteristic of this state illusory). Compare the quite analogous situation in the case of the elastic body, Vol. II, p. 72 and p. 288.

In the crystal W_e is a general positive form of the second order in the E_i , not a simple sum of squares as for the isotropic case. The notation in the text as scalar product is in any case conceptually preferable, particularly since it becomes necessary for \dot{W}_e , where \mathbf{E} and $\dot{\mathbf{D}}$ need not have the same direction even in the isotropic case.

In contrast with W_e and W_m , W_J is not a state function. The condition $\sigma_{ik} = \sigma_{ki}$ should hence apply, in the crystalline conductor, only for a particular crystal symmetry.

In the ideal nonconductor the right side of (7) vanishes, so that (7) takes on the form of the *hydrodynamic equation of continuity* (see Vol. II, Eq. (5.4)): W replaces the hydrodynamic density ρ , S replaces ρv . Continuing with this hydrodynamic analogy, we may say that even in the insulator the energy flows not like an incompressible but like a compressible fluid. In a conductor it is absorbed furthermore, in the measure in which heat is generated in any element of volume.

In optics S plays a dominant role as *ray vector*; the emission and irradiation of a given surface element $d\sigma$ is distinguished by the positive and the negative sign of S .

We know from mechanics that the law of conservation of energy is not only of fundamental importance physically, but is also highly useful mathematically as a first integral of the equations of motion. Something similar applies for our electrodynamic law of conservation of energy: From it may be derived the *uniqueness* of the integration of the Maxwell equations for given *initial condition* and suitably prescribed *boundary conditions* on the boundaries of the region considered.

As usual, the proof is indirect: We assume the existence of two solutions, form their difference, and deduce therefrom a contradiction.

Let the two solutions be E_1, H_1 and E_2, H_2 (by §4 the corresponding vectors D, B are then also known).

We put

$$E = E_1 - E_2, \quad H = H_1 - H_2. \quad (8)$$

In view of the *linearity* of Maxwell's Eqs. (4.8), E and H are solutions as well as E_1, H_1 and E_2, H_2 . Hence Poynting's theorem, e.g. in the form (7a), applies formally also for them. However, the quantities W, S, W_s , because of their *quadratic* character, are composed not merely of the corresponding quantities of the individual fields 1 and 2, but also of *mixed* terms involving 1 and 2. We show this for the quantity W , as example, assuming isotropy for the sake of brevity.

$$W_s = \frac{1}{2} E \cdot D = \frac{\epsilon}{2} E^2 = \frac{\epsilon}{2} (E_1 - E_2)^2, \quad (9)$$

or expanded

$$W_s = \frac{\epsilon}{2} E_1^2 + \frac{\epsilon}{2} E_2^2 - \epsilon E_1 \cdot E_2. \quad (9a)$$

The last expression on the right is the mixed term mentioned above, while the first two terms denote the electric energy of the individual fields 1 and 2. However, we shall not need this expanded form and shall refer below to the representation in (9). Now, including the quantity W_m and the case of anisotropic media in our consideration, we can say: The quan-

tity W in (7a) represents a *definitely positive* quadratic form, formed with the components of the difference field \mathbf{E} , \mathbf{H} . The same applies for the quantity W_j . Finally the quantity \mathbf{S} is (irrespective of the difference terms arising in its calculation) the vector product $\mathbf{E} \times \mathbf{H}$ formed by the difference fields.

Let the domain over which (7a) is integrated be composed of partial domains $a, b, \dots j, \dots$ with, in general, different material constants ϵ, μ, σ . We indicate this by replacing W and W_j by $\sum_j W$ and $\sum_j W_j$, which, according to the preceding, can, just like the individual W , *never become negative*. Consider now the term

$$\sum_j \int S_n d\sigma_j \quad (10)$$

which arises from (7a) in the same manner. Pairs of terms which refer to the same *inner* boundary surface cancel here because for them the S_n are equal and opposite—*opposite* because of the opposite direction of the normal n , *equal* because of the boundary conditions for the tangential components of the fields $\mathbf{E}_1, \mathbf{E}_2$ and $\mathbf{H}_1, \mathbf{H}_2$, from which follows the equality of the tangential components of the difference fields \mathbf{E}, \mathbf{H} and of the component of \mathbf{S} normal to the boundary surface. The sum (10) becomes, therefore, simply equal to the surface integral over the outer boundary of the region of integration

$$\int S_n d\sigma. \quad (10a)$$

Let the boundary condition to be prescribed for this outer boundary simply consist in the tangential component of either the electric or the magnetic field being given everywhere on it. For the difference field (8) this signifies that the tangential components of either \mathbf{E} or of \mathbf{H} vanish. In either case the vector product \mathbf{S} formed with them and, hence, the integral (10a) vanish also.

Now (7a) applied to our case takes on the form

$$\frac{\partial}{\partial t} \sum_j \int W d\tau_j = - \sum_j \int W_j d\tau_j, \quad (11)$$

or, integrated with respect to t :

$$\sum_j \int W d\tau_j \Big|_0^t = - \int_0^t dt \sum_j \int W_j d\tau_j. \quad (11a)$$

Here the right side is less than or at most equal to zero. The left side vanishes at the lower limit $t = 0$, since for prescribed initial values of the fields 1 and 2 $\mathbf{E} = 0$ and $\mathbf{H} = 0$ in every one of the domains j , so that $W = 0$ also. At the upper limit t , on the other hand, the left side of (11a) is, in

view of the meaning of W , certainly not negative; its least value is zero. Only then the inconsistency with the right side is resolved. For this value we must have for all $t > 0$

$$\mathbf{E} = 0, \quad \mathbf{H} = 0,$$

so that, by (8),

$$\mathbf{E}_1 = \mathbf{E}_2, \quad \mathbf{H}_1 = \mathbf{H}_2.$$

This proof of uniqueness satisfies any demand for rigor. An unrigorous proof may be deduced directly from the form of Eqs. (4.8). For these equations permit the determination of the change with *time* of \mathbf{E} and \mathbf{H} if their distribution in *space* is known at any one moment. This means in a sense: the values of \mathbf{E} and \mathbf{H} at the time $t + dt$ can be calculated from their values at the time t . This calculation is *unique* since the Maxwell equations are linear in \mathbf{E} and \mathbf{H} .

In the preceding we have confined ourselves to a finite closed domain. Physically the unlimited domain is of course of greater interest. The uniqueness of the integration problem can be proved here for the static case as in §10D. We will consider the significance of the Poynting vector for the unique formulation of the problem of waves along wires in §22.

§6. The Role of the Velocity of Light in Electrodynamics

It appears reasonable to eliminate \mathbf{H} from Eqs. (4.8) and to obtain in this manner a single vector equation for \mathbf{E} . For this purpose the operation curl is applied to the first Eq. (4.8), the operation $\mu\partial/\partial t$, to the second. Adding the two equations yields

$$\epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = - \text{curl curl } \mathbf{E}, \quad (1)$$

i.e., a linear differential equation of the second order in four coordinates of space and time.

We will convert this expression to a form which is more familiar to the mathematician. For this we utilize the general transformation (3.10) of Vol. II:

$$\text{curl curl } \mathbf{E} = \text{grad div } \mathbf{E} - \Delta \mathbf{E}. \quad (2)$$

As indicated there, this equation is to be applied with caution, since the Laplace operator Δ can, by its definition as div grad , only be applied to scalar quantities. Incidentally, (2) may also be derived from the well-known vector formula

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \quad (2a)$$

by symbolic calculation with the nabla operator (see the beginning of §5), where it takes the form

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - (\nabla \cdot \nabla)\mathbf{E}. \quad (2b)$$

This is identical with Eq. (2), term for term. We consider similar vector formulas in Problem I.4.

Equation (1) is valid in any coordinates, curvilinear as well as Cartesian. On the other hand, Eq. (2), according to the above, is restricted to the Cartesian coordinates x, y, z and the components E_x, E_y, E_z , since only these may be treated as scalar quantities. With this restriction we find from (1) and (2)

$$\varepsilon_\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma_\mu \frac{\partial \mathbf{E}}{\partial t} = \Delta \mathbf{E} - \text{grad div } \mathbf{E}. \quad (3)$$

This can be further simplified if we specify that \mathbf{E} represents a solution for a medium of uniform dielectric constant and *free of charge*. Then Eq. (4.4b), with $\varepsilon = \text{const}$ and $\rho = 0$, becomes $\text{div } \mathbf{D} = \varepsilon \text{ div } \mathbf{E} = 0$. Thus the last term on the righthand side of Eq. (3) vanishes and Eq. (3) assumes the form of the wave equation:

$$\varepsilon_\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma_\mu \frac{\partial \mathbf{E}}{\partial t} = \Delta \mathbf{E}. \quad (4)$$

The same equation evidently applies, under similar restricting conditions, also for \mathbf{H} (as well as for \mathbf{D} and \mathbf{B}).

The first coefficient in (4) is, as can be read directly out of Eq. (4), the reciprocal square of a velocity. Correspondingly, we find from Eqs. (4.6a) and (4.7a):

$$\varepsilon_\mu = \frac{Q^2}{M \text{ joule}} \cdot \frac{\text{joule S}^2}{Q^2 M} = \left(\frac{M}{S} \right)^{-2}. \quad (5)$$

What is the meaning of this velocity? Maxwell's answer is: *It is the velocity of propagation of electromagnetic waves, which in vacuum is identical with that of light:*

$$(\varepsilon_0 \mu_0)^{-1} = c = (2.9978 \pm 0.0002) 10^8 \frac{M}{S} \sim 3 \cdot 10^8 \frac{M}{S}. \quad (6)$$

Even at an early date the velocity of light c , then denoted as "critical velocity," maintained an elusive existence in electrodynamics, as in the theorem of Wilhelm Weber and the numerous measurements of the ratio of an "electromagnetically" and "electrostatically" determined charge on a condenser (§16D). However the role of c in electrodynamics was first clarified by Maxwell's theory of light and Hertz's experiments.

If we pass from vacuum to an arbitrary electromagnetic medium, the velocity $(\epsilon\mu)^{-1/2}$ appearing in (5) signifies, according to Maxwell, the velocity of light (more precisely, the "phase velocity of the light") in a ponderable body characterized by ϵ and μ

$$(\epsilon\mu)^{-1/2} = v, \quad \frac{c}{v} = n = \text{refractive index.} \quad (7)$$

It is true that the statement (7) has by no means the same certainty as statement (6). For it does not account for dispersion phenomena and hence cannot even explain the prismatic colors. We will learn in Vol. IV how these are to be fitted into electromagnetic optics.

Eq. (6) is evidently a supplementation of Maxwell's theory derived from experiment, which establishes a relationship between the two material constants ϵ_0, μ_0 of vacuum. In the following section we will discuss how the constants are to be determined individually.

We now turn to the integration of Eq. (4) specialized for vacuum

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \Delta \mathbf{E} \quad (8)$$

with the auxiliary condition already made use of

$$\text{div } \mathbf{E} = 0. \quad (8a)$$

We seek, in particular, solutions of (8) which are independent of y and z . For purely periodic time dependence these represent monochromatic plane waves which advance along the x -axis. We shall show that they are necessarily transverse. In view of the assumed independence of y and z of the function Eq. (8a) reduces to

$$\frac{\partial E_x}{\partial x} = 0.$$

Equation (8) yields accordingly:

$$\frac{\partial^2 E_x}{\partial t^2} = 0. \quad (8b)$$

E_x would thus be a linear function of t , which is inconsistent with the periodic dependence on t . Hence $E_x = 0$. Thus we already note a decided advantage of electromagnetic optics over the old elastic optics. As we saw in Vol. II, §45, the latter could never get rid of the longitudinal component of the plane wave: Even if it was originally absent, a reflection or refraction would cause its appearance along with the transverse component. In contrast to this we have proved that the plane wave of the electromagnetic theory of light is necessarily transverse. We can designate Eq. (8a) as the condition of transversality.

If the wave has a single electrical component or, in the usual terminology, is plane polarized, we can take its direction of vibration¹ as the y -axis, so that, in addition to $E_z = 0$, also $E_x = 0$. Eq. (8) then becomes

$$\frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = \frac{\partial^2 E_y}{\partial x^2}. \quad (9)$$

The solution which is purely periodic in time is

$$E_y = a \cos(kx - \omega t + \alpha). \quad (10)$$

According to (9) the wave number k introduced here and the angular frequency ω are related by

$$\frac{\omega}{k} = c; \quad (10a)$$

in terms of the wave length λ and the period τ we have

$$k = \frac{2\pi}{\lambda}, \quad \omega = \frac{2\pi}{\tau}. \quad (10b)$$

Omitting the sign Re , denoting "real part of," we shall write (10) in a form which will prove more convenient for what follows:

$$E_y = A e^{ikx - i\omega t}, \quad A = ae^{i\alpha}. \quad (11)$$

This is permissible as long as we are dealing with linear relations, such as the Maxwell differential equations; in dealing with energetic quantities which are quadratic in the field components we must obviously return to real expressions such as (10).

We next investigate the magnetic component of the plane wave. It may be derived from the first vector equation (4.8), specialized for vacuum:

$$\mu_0 \frac{\partial \mathbf{H}}{\partial t} = - \text{curl } \mathbf{E}.$$

Since $E_z = E_x = 0$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$, this leads to

$$H_z = H_y = 0$$

and furnishes the following equation for H_x :

$$\mu_0 \frac{\partial H_x}{\partial t} = - \frac{\partial E_y}{\partial x} = - ikA e^{ikx - i\omega t}. \quad (12)$$

¹ It should be noted that we are here dealing with the direction of vibration of the electric field, not with the direction of any material displacement.

For purely periodic time dependence its integration with respect to t is carried out simply by dividing the right side by $-i\omega$. Accordingly,

$$\mu_0 H_z = \frac{k}{\omega} A e^{ikx - i\omega t} = \frac{1}{c} A e^{ikx - i\omega t},$$

and, in view of (6),

$$H_z = \sqrt{\frac{\epsilon_0}{\mu_0}} A e^{ikx - i\omega t}. \quad (13)$$

The dimension of the coefficient $(\epsilon_0/\mu_0)^{1/2}$ is that of a reciprocal resistance, i.e. Ω^{-1} . For, by (4.6a), (4.7a), and (4.5c),

$$\frac{\epsilon}{\mu} = \frac{Q^2}{\text{joule M}} / \frac{\text{joule S}^2}{Q^2 M} = \left(\frac{Q^2}{\text{joule S}} \right)^2 = \frac{1}{\Omega^2}. \quad (14)$$

$(\mu_0/\epsilon_0)^{1/2}$ is designated as "wave resistance of vacuum." We shall see in §18D that this quantity actually assumes the role of a resistance (voltage/current) in the telegraph equation.

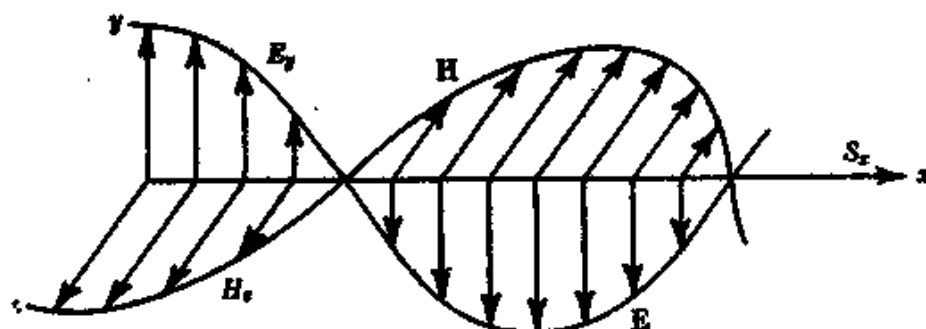


FIG. 6. The relative orientation of \mathbf{E} , \mathbf{H} , and \mathbf{S} for a plane wave progressing in the x -direction.

Fig. 6 shows the orientation of \mathbf{E} and \mathbf{H} relative to each other and relative to the Poynting vector \mathbf{S} at a given instant. In this sequence they form a right-handed system. With increasing t the figure is displaced with the velocity of light in the direction of the positive x -axis. It may not be superfluous to point out that \mathbf{E} and \mathbf{H} become zero at the same point and attain their maxima at the same point. The situation differs from that of pendulum vibrations in mechanics, where the energy appears in turn in its kinetic and in its potential form.

For the experiments of Hertz and many optical experiments air and our vacuum are equivalent. A distinction between air and vacuum need only be made in high-precision wave-length determinations.

We have continually employed the term vacuum in preference to the term "ether" ("light ether"), which is frequently used elsewhere. This negative term appears to have more significance than the latter scholastic

word, which gives rise to false notions that cannot be reconciled with the theory of relativity.

We can indicate the material constants of ponderable bodies by their relative values referred to vacuum instead of by ϵ , μ , setting

$$\epsilon = \epsilon_{\text{rel}}\epsilon_0, \quad \mu = \mu_{\text{rel}}\mu_0. \quad (15)$$

ϵ_{rel} and μ_{rel} then are pure numbers, which in general do not differ greatly from 1. In a ponderable nonconductor Eq. (10a) must of course be replaced by

$$\frac{\omega}{k} = v = \frac{c}{n} \quad (16)$$

and Eq. (13) by

$$H_x = \sqrt{\frac{\epsilon}{\mu}} A e^{ikx - i\omega t} \quad (17)$$

Plane transverse waves are possible also in an absorbing medium ($\sigma \neq 0$). The general wave equation (4) is satisfied by the form (11), for given ω , by subjecting k to the condition generalizing Eq. (16):

$$k^2 = \epsilon\mu\omega^2 + i\sigma\mu\omega, \quad k = \sqrt{\epsilon'\mu}\omega, \quad \epsilon' = \epsilon + \frac{i\sigma}{\omega}. \quad (18)$$

ϵ' is the "complex dielectric constant" frequently employed in the optics of absorbing media. If the relaxation time introduced in (4.9a) is employed, we obtain

$$\frac{\epsilon'}{\epsilon} = 1 + \frac{i\sigma}{\epsilon\omega} = 1 + \frac{i}{2\pi} \frac{\tau}{T_r}. \quad (18a)$$

If $T_r \gg \tau$ the added imaginary term of k is merely a correction term; if $T_r \ll \tau$ the real and imaginary parts of k become equal (because $\sqrt{i} = (1+i)/\sqrt{2}$). In both cases the wave is damped exponentially as it progresses along the positive x -axis.

§7. The Coulomb Field and the Fundamental Constants of Vacuum. Rational and Conventional Units

On the basis of their time dependence we classify fields as *static*, *stationary*, *quasistationary*, and *rapidly varying fields*.

In *static fields* not only field and density variations, but also currents of electricity and energy are to be zero. Hence we demand

$$\dot{\mathbf{B}} = 0, \quad \dot{\mathbf{D}} = 0, \quad \dot{\rho} = 0, \quad \mathbf{J} = 0, \quad \mathbf{S} = 0.$$

According to Eqs. (4.4) and the succeeding equations these conditions are fulfilled if we set:

A. Electrostatics

$$\begin{aligned}\text{curl } \mathbf{E} &= 0, & \text{div } \mathbf{D} &= \rho \text{ in nonconductors,} \\ \mathbf{D} &= \mathbf{E} = 0 \text{ in conductors,} \\ \mathbf{H} &= 0 \text{ in all cases.}\end{aligned}\quad (1)$$

B. Magnetostatics

$$\begin{aligned}\text{curl } \mathbf{H} &= 0, & \text{div } \mathbf{B} &= 0 \text{ always,} & \text{but eventually } \text{div } \mathbf{H} &= \rho_m, \\ \mathbf{E} &= 0 \text{ in all cases.}\end{aligned}\quad (2)$$

An explanation of the "magnetic density" ρ_m here introduced will be given in connection with Eq. (9a) below

In *stationary fields* we retain the conditions $\dot{\mathbf{E}} = 0$, $\dot{\mathbf{D}} = 0$, $\dot{\rho} = 0$, but prescribe current fields \mathbf{J} in the conductors, which according to Eq. (4.4c) must be free of sources. The electric field must still satisfy, both within and outside the currents, $\text{curl } \mathbf{E} = 0$; on the other hand, $\text{curl } \mathbf{H} = 0$ only outside the currents.

In *quasistationary fields* we shall determine the fields as in the stationary case, but take account of their time dependence in the first approximation. The system of the Maxwell equations is fully utilized only for *rapidly varying fields*.

A. Electrostatics

We defer all problems requiring the use of the theory of functions. These are the *boundary-value problems* with conductors or nonconductors of different dielectric constant present in the field. We shall therefore deal first of all only with a *uniform dielectric*, so that we may set $\epsilon = \text{const.}$ In this case we are faced with a simple *summation problem* instead of a boundary-value problem.

Eqs. (1) then take the simpler form

$$\text{curl } \mathbf{E} = 0, \quad (3) \quad \text{div } \mathbf{E} = \frac{\rho}{\epsilon}. \quad (3a)$$

Eq. (3) states that \mathbf{E} may be treated as gradient of a *scalar potential*

$$\mathbf{E} = - \text{grad } \Psi, \quad (4)$$

which evidently brings about a substantial simplification of the problem of integration.

According to (3a) this potential must satisfy the *Poisson equation*

$$\Delta \Psi = - \frac{\rho}{\epsilon}. \quad (4a)$$

Lamellar field ($\text{curl } \mathbf{E} = 0$) and *potential field* ($\mathbf{E} = -\text{grad } \Psi$) have the same meaning; the surfaces $\Psi = \text{const.}$ divide the field into layers (lamellae), to which the lines of force are orthogonal. The line integral of the field strength

$$\int_A^B \mathbf{E} \cdot d\mathbf{s} = \Psi_A - \Psi_B \quad (4b)$$

is independent of the path; carried out over any closed path ($B = A$) it vanishes. The voltage V_{ab} is identical with the *potential difference*

$$\Psi_A - \Psi_B.$$

The summation problem mentioned above consists in the integration of Eq. (4a) and can be carried out directly with the aid of *Green's theorem*, for which we refer to Vol. II, §20, Nr. 1a. We obtain

$$4\pi\epsilon\Psi = \int \frac{\rho}{r} d\tau, \quad r = r_{PQ}. \quad (5)$$

P is the point at which Ψ is to be calculated, Q is the point of integration. The left side results from the integration over a small sphere surrounding the point $r = 0$, $Q = P$; the integral over the sphere bounding the region of integration externally vanishes provided that the total charge enclosed by this sphere is finite.

If the charge is not distributed in space, but concentrated on a surface or on a line, the mathematical method employed in (5) leads to

$$4\pi\epsilon\Psi = \int \frac{\omega}{r} d\sigma, \quad (5a)$$

$$4\pi\epsilon\Psi = \int \frac{\lambda}{r} ds; \quad (5b)$$

ω is the surface density, λ the line density (charge per unit length). A final step in this series leads us to the charge e concentrated in a point:

$$4\pi\epsilon\Psi = \frac{e}{r}, \quad (6) \quad \mathbf{E} = E_r = -\frac{\partial\Psi}{\partial r} = \frac{e}{4\pi\epsilon r^2}. \quad (6a)$$

This is the *Coulomb field*. We could also have read it off directly from Eq. (3a), which, using Gauss's theorem, we could integrate over a sphere of radius r described about the charge e . We obtain then directly

$$\oint \mathbf{E}_n d\sigma = \int \frac{\rho}{\epsilon} d\tau = \frac{e}{\epsilon}. \quad (6b)$$

In view of the spherical symmetry we must put $E_n = E_r = \text{const.}$ on the left, whereupon (6b) becomes in fact identical with (6a).

The Coulomb force F , with which two equal charges e at the distance r repel each other, follows from (6) according to our definition of field strength:

$$F = F_r = eE_r = \frac{e^2}{4\pi\epsilon r^2}. \quad (7)$$

It has been customary in the past to write instead, for vacuum conditions,

$$F = f \cdot \frac{e^2}{r^2} \quad \text{with} \quad f = 1. \quad (8)$$

Here F is supposed to be measured in dynes, r in centimeters. However, in this manner the whole structure of our system of dimensions is cast aside; we pass from our former system of dimensions to the so-called *electrostatic*¹ system of cgs units. The charge e would then, according to (8), take on the unnatural and ungainly dimensions

$$e = \sqrt{\text{dyn cm}^2} = \text{cm}^{3/2} \text{ g}^{1/2} \text{ sec}^{-1}. \quad (8a)$$

Furthermore the charge e would be 1 if two equal charges e at a distance of 1 cm would repel each other in air with a force of 1 dyne.

We must reject, on dimensional grounds, Hertz's distinction between "true charge density" expressed by $\text{div } \mathbf{D}$ and "free charge density" expressed by $\text{div } \mathbf{E}$. We shall designate the latter quantity correctly as "divergence of the lines of force"; in the preceding we have avoided it by writing ρ/e for it.

B. Magnetostatics

Although in most treatments the analogy between electrostatics and magnetostatics is emphasized, our approach compels us to point out clearly the differences as well.

As we saw in (2), it is not the *intensity* \mathbf{B} , but the *quantity* \mathbf{H} that is lamellar. We again denote the corresponding scalar potential by Ψ , distinguishing where necessary between Ψ_e and Ψ_m , and find

$$\mathbf{H} = - \text{grad } \Psi. \quad (9)$$

We shall call the quantity ρ_m defined above in Eq. (2) simply "magnetic density"; in view of the correspondence of \mathbf{H} and \mathbf{D} we can regard it as the direct analog of the electric density ρ .

How can we reconcile its existence, i.e. the Eq. $\text{div } \mathbf{H} \neq 0$ with the universally valid Eq. $\text{div } \mathbf{B} = 0$? This is only possible at points of local varia-

¹ The term *electrical system* seems to us to be preferable in principle to the customary term *electrostatic system*, since its application is not limited to equilibrium conditions, but may also be extended to electrodynamic processes. See §16D.

tion of permeability, as is shown by the following lines:

$$\begin{aligned}\operatorname{div} \mathbf{B} &= \mu \operatorname{div} \mathbf{H} + \mathbf{H} \cdot \operatorname{grad} \mu = 0 \\ \rho_m &= \operatorname{div} \mathbf{H} = \mathbf{H} \cdot \operatorname{grad} \log (\mu_0/\mu).\end{aligned}\tag{9a}$$

We will give a physical interpretation of this rather formal explanation of the concept of magnetic quantity and density by introducing the "magnetization" \mathbf{M} in §12. We have no reason for a distinction between "true" and "free" magnetism, such as was also given by Hertz. For, since we have already interpreted $\operatorname{div} \mathbf{H}$ as magnetic density, the differently dimensioned quantity $\operatorname{div} \mathbf{B}$ is not a magnetic density. Furthermore it is everywhere equal to zero.

We now return to (9) and form the divergence of the vectors on the left and the right. We then obtain the Poisson equation of magnetostatics, i.e.

$$\Delta \Psi = -\rho_m \tag{9b}$$

If ρ_m is given throughout this is integrated, in analogy to (5), by

$$4\pi\Psi = \int \frac{\rho_m}{r} d\tau, \tag{10}$$

or, in analogy to (5a), for given surface charge ω_m , by

$$4\pi\Psi = \int \frac{\omega_m}{r} d\sigma. \tag{10a}$$

If the density is concentrated on a point pole and if we call

$$p = \int \rho_m d\tau$$

the pole strength (an opposite pole is imagined to lie at infinity), (10) leads to

$$4\pi\Psi = \frac{p}{r}, \tag{10b}$$

$$\mathbf{H} = \mathbf{H}_r = -\frac{\partial \Psi}{\partial r} = \frac{p}{4\pi r^2}. \tag{10c}$$

This is the *Coulomb field* of the isolated magnetic pole. The Coulomb force, with which two poles of equal magnitude and the same sign repel each other, is, however, not $p\mathbf{H}$, but, according to our definition of the intensity \mathbf{B}

$$\mathbf{F} = \mathbf{F}_r = p\mathbf{B}_r = p\mu\mathbf{H}_r = \frac{p^2\mu}{4\pi r^2}. \tag{11}$$

The fact that μ appears here in the numerator, although ϵ , in (7) appears in the denominator, results evidently from the inconsistency, pointed out

in connection with Eq. (4.7); in the introduction of μ as compared with that of ϵ . (We would have liked to define the reciprocal of μ as the magnetic constant at that point.) In (11) μ evidently signifies the permeability of the surroundings of the magnetic pole p ; for air (vacuum) we put $\mu = \mu_0$.

Just as in connection with (8) we took cognizance of an electrostatic system of units and a unit of charge corresponding to this system, so we can introduce, on the basis of (11), a magnetic system of units and a corresponding unit of pole strength. To this end (11) is replaced, for vacuum in particular, (we follow the pattern of Eq. (7) and what follows literally) by:

$$F = f \frac{p^2}{r^2} \quad (12)$$

and f is put equal to 1; F is supposed to be measured in dynes, r in centimeters. Our former system of units is once more cast aside, and we pass over to the *Gaussian magnetic cgs system*.¹ In this system the pole strength p has, according to (12), the same unsatisfactory dimension as the charge e in the electrical system (8a). Unity pole strength would correspond to a repulsion with a force of 1 dyne of two poles of equal sign and magnitude separated by 1 cm (in air as surrounding medium).

C. Rational and Conventional Units

We must now deal with the factor 4π in Coulomb's law. It is true that this is much less fundamental than the question of dimensions and bears to the latter only a historical relationship. Historically the forms (8) and (12) of Coulomb's law result from an effort to approach as closely as possible the customary form of Newton's law. We shall denote the suppression of the numerical factor 4π in Coulomb's law as *conventional*, our retention of it as *rational*. It is in fact evident that in a problem with spherical symmetry, such as the Coulomb problem, the factor 4π is appropriate (this follows in particular from our argument in (6b)). If we wish to avoid this factor, we must rewrite Poisson's equation (4a) as well as the second of Eqs. (1) as follows:

$$\Delta\psi = -4\pi \frac{\rho}{\epsilon}, \quad \text{div } \mathbf{D} = 4\pi\rho. \quad (13)$$

The factor 4π would thus be improperly introduced into the fundamental equations of the Maxwell theory. Furthermore, the transparent expression (5.6) for the energy density would be distorted into

$$W_e = \frac{1}{8\pi} \mathbf{E} \cdot \mathbf{D}. \quad (14)$$

¹ The fact that Gauss employed mm instead of cm as unit of length is a superficial distinction.

Heaviside fought a life-long battle for the rational units. In this connection he pointed also to the expression for the capacity of a condenser (for details see §10, where the relationship with the expression for the energy density is also indicated): The *plate condenser* (area F , plate separation a) has, in rational and conventional units respectively, the capacity

$$K = \frac{F\epsilon}{a} \quad \text{and} \quad \frac{F\epsilon}{4\pi a}, \quad (15)$$

the *spherical condenser* (radius of sphere r , outer sphere imagined at infinity), the capacity

$$K = 4\pi\epsilon r \quad \text{and} \quad \epsilon r. \quad (15a)$$

We see that, with rational units, the factor 4π appears for the sphere, where it belongs; with conventional units it is missing for the sphere and appears for the plane condenser, where it does not belong.

Heaviside makes the following striking comparison: In passing from the measurement of distance to the measurement of area one might define as unit of area the area of a circle of radius 1. This would be logically possible. It would however lead to the strange result that a square with the side 1 would have the area $1/\pi$. Everyone would then say that π was at the wrong place. We said the same of the factor 4π in the formulas to the right in (15) and (15a).

D. Final Determination of the Fundamental Constants ϵ_0 , μ_0 in the MKSQ System

The viewpoint of the rational units together with the requirement of meaningful dimensions and adaptation to the legal units leads to a quite definite choice of the fundamental constant μ_0 of vacuum. For we can obtain agreement between Eq. (11), which is dimensionally correct in our sense, and Eq. (12) by requiring

$$\left[\frac{\mu_0}{4\pi} \right] \frac{\text{joule S}^2}{\text{Q}^2 \text{M}} = [f] \text{ cgs.} \quad (16)$$

$[f]$ is the numerical value of f in the cgs-system, which we wished to set equal to 1. The brackets on the left are the numerical value of the quantity $\mu_0/(4\pi)$ in our MKSQ-system; its dimension (see e.g. (4.7a)) is indicated. The conversion of these dimensions into the cgs-system follows from

$$\text{Q} = 1 \text{ Coulomb} = \frac{1}{300} \text{ cgs}, \quad \text{M} = 10^3 \text{ cm}, \quad \text{joule} = 10^7 \text{ erg.}$$

Accordingly

$$1 \frac{\text{joule S}^2}{\text{Q}^2 \text{M}} = 10^7 \text{ cgs.}$$

With $[f] = 1$ we thus obtain, after cancelling the dimensional factor cgs on both sides of (16)

$$\left[\frac{\mu_0}{4\pi} \right] = 10^{-7}.$$

We obtain hence, entering the dimensions:

$$\mu_0 = 4\pi 10^{-7} \frac{\text{joule S}^2}{\text{Q}^2 \text{M}} = 4\pi 10^{-7} \frac{\Omega \text{S}}{\text{M}}. \quad (16a)$$

Regarding the unit of resistance, $\Omega = \text{"ohm,"}$ here employed, see Eq. (4.5d).

We have thus determined one of the two fundamental constants of vacuum in such a manner that the demands stated above are satisfied. The numerical value of μ_0 so obtained, which is accurate to an arbitrary number of digits, shows clearly that our determination is not established by direct measurements, but by our choice of the unit of Q and is equivalent to the latter. The other fundamental constant ϵ_0 of vacuum follows then from the relation (6.6) which is supported by the sum total of the Hertzian experiments:

$$\epsilon_0 = \frac{1}{\mu_0 c^2} = \frac{10^7}{4\pi c^2} \frac{\text{M}}{\Omega \text{S}}. \quad (17)$$

If we substitute for c the approximate value $c = 3 \cdot 10^8 \text{ M/S}$ we find

$$\epsilon_0 \sim \frac{10^{-9}}{36\pi} \frac{\text{S}}{\Omega \text{M}}. \quad (18)$$

We can also write (17) in the form

$$4\pi c^2 \epsilon_0 = 10^7 \frac{\text{M}}{\Omega \text{S}}. \quad (18a)$$

Division of (16a) by (18) and taking the square root leads to the following value for the "wave resistance of vacuum," introduced with (6.14), in terms of the unit Ω :

$$\sqrt{\frac{\mu_0}{\epsilon_0}} \cong 120\pi\Omega \cong 377\Omega. \quad (19)$$

In the preceding we have disregarded the small differences between the velocity of light c and its approximate value $3 \cdot 10^8 \text{ M/S}$, as well as the difference between the "international" and the "absolute," i.e., the ideal, ohm. These differences, which concern only the higher decimals, are of course of great importance in precision measurements and have called forth, in the determination of the Ω in relation to the old Siemens unit,¹

¹ Resistance of a mercury thread 1 m long and 1 mm² in cross section at 0° C = 0.937 Ω .

the competitive efforts of the best experimenters (Kirchhoff, Lord Rayleigh, F. Kohlrausch etc.). They play no role, however, in the general theory.

In summary: Our form of the Maxwell equations is adapted to the rational choice of the units MKSQ, with the value (16a) for μ_0 taken over from the conventional Gaussian magnetic units. Below we shall use the numerical values (16a), (18), (19) only in specific numerical computations and not introduce them, as often happens in engineering literature, into the general theory. Instead we shall always take account of the dimensions of all quantities, also those of ϵ_0 , μ_0 , and thus make ourselves independent of the particular choice of $Q = 1$ Coulomb.

§8. Four, Five, or Three Fundamental Units?

A. Supplementary Note on Our System of Four Units

Our four units MKSQ are simply intended to translate Giorgi's idea (introduction of a separate electrical unit) into a form which is particularly convenient for the theory. It is basically indifferent whether the unit of charge Q is employed or a standard resistance R , as Giorgi has occasionally advocated, for reasons of convenience of measurement. In view of the relationship $Q = \text{ampere-second}$ we would of course also be content with the ampere as fourth unit. We take less kindly to the designation of Giorgi's system by the units MKSVA. In view of

$$VA = \text{watt} = \text{joule/sec}$$

these units are not independent of each other. We can well understand that the long-employed quantities V and A appear more convenient in use than our unit of charge Q . Nevertheless, of the two dimensions

$$E = \frac{\text{newton}}{Q} \quad \text{and} \quad E = \frac{\text{Volt}}{M}$$

the first appears to be the more natural one. Kalantaroff's system of the four units MSQO (magnetic flux) is selfconsistent, but seems, by the elimination of the unit of mass, somewhat too artificial for general use in physics.

It is to be welcomed, from our point of view, that, by international agreement, separate designations gauss and oersted have been introduced for the two magnetic vectors B and H . Historically, the name gauss also seems proper for B , since Gauss' methods of determining magnetic moment rest on measurements of force and hence refer to B and not to H . The unhappy term "magnetic field" for H should be avoided as far as possible. It seems to us that this term has led into error none less than Maxwell himself, who, in art. 625 of the Treatise puts the force exerted by the field on a magnetic pole m equal to mH .

We have repeatedly stressed as an advantage of our system that it avoids the annoying powers of ten of the cgs-system. This applies to the electrical as well as to the mechanical quantities. The converse is true, however, for the *magnetic* quantities. The unit gauss of the magnetic induction B in the cgs-system is, by definition, equal to the unit in the cgs-system. Hence, transferred to our system of units, it acquires a power of ten. We determine the latter as follows:

Let $[B]$ be the numerical magnitude of a given field in our system of units, so that

$$B = [B] \frac{\text{newton S}}{QM} = [B] \frac{\text{joule S}}{QM^2} \quad (1)$$

We substitute again

$$\text{joule} = 10^7 \text{ erg}, \quad M = 10^2 \text{ cm}, \quad Q = \frac{1}{10} \text{ cgs-units}.$$

If, in particular, we set $[B] = 1$, we find from (1) as corresponding value of this quantity B in cgs-units:

$$B = 1 \frac{10^7}{10^2} \text{ cgs-units} = 10^5 \text{ gauss}, \quad (2)$$

and conversely

$$1 \text{ gauss} = 10^{-5} \frac{\text{joule S}}{QM^2} = 10^{-5} \frac{VS}{M^2}. \quad (3)$$

It may be mentioned in favor of this choice of unit that the gauss is inconveniently small for practical purposes, so that not only in engineering, but even in pure physics (except for terrestrial magnetism) the kilogauss must generally be employed (e.g. in the Zeeman effect). Hence our 10,000 times greater unit is to be preferred in practice.

In order to express the oersted in our system of units as well, we proceed from the relationship between H and B :

$$H = \frac{B}{\mu_0}. \quad (4)$$

We now set $H = 1$ oersted, $B = 1$ gauss, so that by (3)

$$B = 10^{-5} \frac{\text{joule S}}{QM^2}$$

and by (7.16a)

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{\text{joule S}^2}{Q^2 M}.$$

We then obtain from (4)

$$1 \text{ oersted} = 10^{-4} \frac{\text{joule S}}{\text{QM}^2} / 4\pi \cdot 10^{-7} \frac{\text{joule S}^2}{\text{Q}^2 \text{M}}; \quad (5)$$

so that

$$1 \text{ oersted} = \frac{1}{4\pi} 10^3 \frac{\text{Q}}{\text{SM}} = \frac{1}{4\pi} 10^3 \frac{\text{amp}}{\text{M}} \quad (5a)$$

$$H_{\text{amp/M}} = 4\pi \cdot 10^{-3} H_{\text{oersted}}. \quad (5b)$$

B. The Five Units MKSQP

It may be stated generally: A dimensional analysis will be more successful¹ in the degree in which more independent units are at its disposal. Our four units are hence more informative than the three units of the "absolute" system, in which the dimensional character of the fundamental electromagnetic vectors is obscured. The five independent units considered below are of even greater value from a general theoretical point of view.

We introduced the magnetic pole strength P as a dimension in §2, but expressed it immediately in (2.7) in terms of the charge Q , in accord with Ampère's hypothesis. Is this hypothesis binding even today, after the discovery of the neutron, a nuclear particle as basic and universal as the proton? The neutron has a magnetic moment which is not associated with any charge, unlike the electron and proton which, though endowed with *equal* charge of opposite sign, have magnetic moments of entirely *different* magnitude. Certainly an attempt to abandon Ampère's hypothesis and to introduce P as *independent fifth dimension* is justified and instructive. We shall, for the present, refrain from fixing the magnitude of P .

We write down the following sets of dimensional relations, which now show a complete correspondence:

$$\begin{array}{ll} E = \frac{\text{newton}}{Q} & B = \frac{\text{newton}}{P} \\ D = \frac{Q}{M^2} & H = \frac{P}{M^2} \\ \epsilon = \frac{D}{E} = \frac{Q^2}{\text{joule M}} & \frac{1}{\mu} = \frac{H}{B} = \frac{P^2}{\text{joule M}} \\ E \cdot D = \frac{\text{newton}}{M^2} = \frac{\text{joule}}{M^2} & B \cdot H = \frac{\text{newton}}{M^2} = \frac{\text{joule}}{M^2} \end{array} \quad (6)$$

¹ E. Fues, Z. Phys. 107, 662, 1937, indicates an upper limit to the useful number of dimensions.

The entries for E , D , B , and H are identical with the original formulas in (2.1) to (2.9.) In accord with the note accompanying Eq. (4.7) we have entered the reciprocal of μ as analog of ϵ in our table.

The last line, which is *independent* of Q and P , has the dimension of energy density. On the other hand the (scalar or vector) product of \mathbf{E} and \mathbf{H} has a dimension which *depends* on P and Q :

$$\mathbf{E}\mathbf{H} = \frac{P}{Q} \frac{\text{newton}}{M^2} = \frac{P}{Q} \frac{S}{M} \frac{\text{joule}}{M^2 S}. \quad (7)$$

The last factor of the last expression has the dimension of energy flux (radiation vector). Let the factor multiplying it be $1/\Gamma$. We thus put

$$\Gamma = \frac{Q}{P} \frac{M}{S} \quad (8)$$

and write (7) in the form

$$\Gamma \mathbf{E}\mathbf{H} = \frac{\text{joule}}{M^2 S}. \quad (8a)$$

This dimensional equation suggests that the energy flux \mathbf{S} is now to be defined as $\Gamma \mathbf{E} \times \mathbf{H}$. We furthermore compute the product $\epsilon\mu$ from (6) and find

$$\epsilon\mu = \frac{Q^2}{P^2} = \Gamma^2 \left(\frac{S}{M} \right)^2. \quad (9)$$

We suspect from this that the velocity of light c is no longer given by $(\epsilon_0\mu_0)^{-1/2}$, but by $\Gamma(\epsilon_0\mu_0)^{-1/2}$.

The same factor Γ occurs now also in Maxwell's equations. We assert that these should be written:

$$\dot{\mathbf{B}} = -\Gamma \text{curl } \mathbf{E}, \quad \dot{\mathbf{D}} + \mathbf{J} = \Gamma \text{curl } \mathbf{H}. \quad (10)$$

If, as in §5, we proceed to Poynting's theorem (scalar multiplication of the first equation with \mathbf{H} , of the second equation with \mathbf{E}), we obtain

$$\mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{E} \cdot \mathbf{J} + \Gamma \text{div } \mathbf{E} \times \mathbf{H} = 0;$$

with the former definitions of the energy densities and of the Joule heat in (5.6) and (5.5) and with the definition of the energy flux suggested by (8a) this expresses the law of conservation of energy:

$$\dot{W}_m + \dot{W}_e + W_J + \text{div } \mathbf{S} = 0. \quad (11)$$

If, on the other hand, just as in §6, we integrate Eq. (10) for the case of the plane wave in vacuum propagated in the x -direction, we obtain the wave equation in the form

$$\epsilon\mu \ddot{\mathbf{E}} = -\Gamma^2 \text{curl curl } \mathbf{E} = \Gamma^2 \Delta \mathbf{E}. \quad (12)$$

Since this is supposed to represent a process with the velocity of propagation c , our expectation suggested by (9) is confirmed:

$$\frac{\Gamma}{\sqrt{\epsilon_0 \mu_0}} = c, \quad \sqrt{\epsilon_0 \mu_0} = \frac{\Gamma}{c}. \quad (12a)$$

The general form (10) of the Maxwell equations is not new. It was introduced by Emil Cohn, the friend and fellow student of Heinrich Hertz, and forms the basis of his important book¹ "Das elektromagnetische Feld." We have avoided Cohn's notation V , taking the place of our Γ , since we have otherwise disposed of V . It is true that Cohn does not work out the relationship of this constant with our unit P of pole strength, nor does he place dimensional considerations in the foreground as has been done here. Students of Cohn, in particular J. Zenneck, have used Cohn's system by preference.

H. A. Lorentz clearly recognized the advantages of Cohn's standpoint when, in 1902, he wrote his two great articles on Maxwell's theory and electron theory for the *Enzyklopädie der mathematischen Wissenschaften*. He wrote:² "Cohn's system has the advantage of easy transition to other systems, by specific choice of the values of V , ϵ_0 , and μ_0 . Eventual later advances in the understanding of the phenomena could be utilized for the ultimate determination of the units. On the other hand we could not bring ourselves to introduce indeterminate quantities into the formulas which are complex to begin with."

The "eventual later advances in the understanding of the phenomena" contemplated by Lorentz can only be expected when we have a *theory of the elementary particles* which now constitutes the greatest problem on the program of atomic physics; this would have to explain not only the magnetic moments, but also the possible masses and charges of the elementary particles. However, we can even now benefit by the flexibility of Cohn's system.

C. The Gaussian System of Only Three Units

We evidently return to our system with the four units MKSQ and our former form (4.4) of Maxwell's equations if we set

$$\Gamma = 1. \quad (13)$$

Then P has, according to (8), the dimension

$$P = Q \times \text{velocity}, \quad (13a)$$

¹ First Edition, Leipzig 1900, Second Edition, 1927.

² Vol. V, second part, p. 87.

in agreement with Ampère's hypothesis in (2.7). Furthermore, our specific choice of μ_0 and ϵ_0 in (7.16a) and (7.17) is evidently consistent with Eq. (12a) for these values of Γ and P .

We obtain another, also very simple, form of Maxwell's equations if we set

$$\Gamma = c. \quad (14)$$

Then, by (12a), the product $\epsilon_0\mu_0$ must be a pure number. It is tempting to make ϵ_0 and μ_0 separately pure numbers and to set

$$\mu_0 = 1, \quad \epsilon_0 = 1. \quad (14a)$$

In this manner we pass over to the *Gaussian system of units*. In view of (14) the Maxwell equations then become (we confine ourselves first to *nonconductors*):

$$\frac{1}{c} \dot{\mathbf{B}} = - \text{curl } \mathbf{E}, \quad \frac{1}{c} \dot{\mathbf{D}} = \text{curl } \mathbf{H}. \quad (15)$$

In view of (8) P and Q have now the same dimension. Hence, by Table (6), the dimensions of \mathbf{E} and \mathbf{B} , as well as those of \mathbf{D} and \mathbf{H} , also become mutually identical. (The same follows also from the form of Eqs. (15).) Furthermore the dimensions of the two pairs become the same, since now ϵ and μ , just as ϵ_0 and μ_0 in (14a), become pure numbers, equal to the pure numbers ϵ_{rel} and μ_{rel} introduced in (6.15). *The Gaussian system obscures the dimensional character of the four fundamental vectors \mathbf{E} , \mathbf{D} , \mathbf{B} , \mathbf{H} completely, while Cohn's system expresses it most clearly.*

The two Coulomb laws (7.8) and (7.12) were written in the *conventional form* (with the factor 4π suppressed). This has the result that the 4π do not occur in the Maxwell Eqs. (15) for nonconductors but *arise once more* in their integration. For, taking the divergence and integrating Eqs. (15) with respect to t leads to:

$$\text{div } \mathbf{B} = \text{const.}, \quad \text{div } \mathbf{D} = \text{const.}$$

The first constant is, of course, equal to zero; the second must now not be set equal to ρ , but equal to $4\pi\rho$:

$$\text{div } \mathbf{D} = 4\pi\rho, \quad (15a)$$

in order that the factors 4π cancel each other on the two sides of the equation if it is applied to a point charge $e = \int \rho \, d\tau$ and is integrated over a sphere about e . Only in this manner is the field strength $E_r = e/(er^2)$, obtained corresponding to the conventional form of the Coulomb force \mathbf{F} . From this follows also as the expression for the corresponding electrostatic potential $\Psi = e/(er)$, unlike Eq. (7.6), where the appropriate factor

4π appears on the left. Hence also (7.5) and (7.4a) must now be replaced by the less appropriate expressions

$$\epsilon\Psi = \int \frac{\rho}{r} d\tau, \quad \Delta\Psi = -4\pi\rho/\epsilon. \quad (15b)$$

The same follows from the form (7.12) of Coulomb's law for the magnetic density ρ_m and the magnetic potential Ψ_m :

$$\text{div } \mathbf{H} = 4\pi\rho_m, \quad \Psi_m = \int \frac{\rho_m}{r} d\tau, \quad \Delta\Psi_m = -4\pi\rho_m. \quad (15c)$$

We now extend Eq. (15) to the case of a *conductor*. Here we must temporarily multiply the conduction current \mathbf{J} , which is to be added to $\dot{\mathbf{D}}$, with a numerical factor γ which we shall determine in a moment. Hence we write in place of (15)

$$\frac{1}{c} \dot{\mathbf{B}} = -\text{curl } \mathbf{E}, \quad \frac{1}{c} (\dot{\mathbf{D}} + \gamma\mathbf{J}) = \text{curl } \mathbf{H}. \quad (16)$$

Taking the divergence of the second of these equations, as in (15a), and utilizing the definition of ρ given there leads to

$$4\pi \frac{\partial \rho}{\partial t} + \gamma \text{div } \mathbf{J} = 0. \quad (16a)$$

We must set $\gamma = 4\pi$ in order that this equation may express the conservation of charge, or in other words, the absence of sources of the total current \mathbf{C} ; only then does (16a) become the analogue (4.4c) or (3.6b) of the hydrodynamic equation of continuity.

Having entered this value of γ in (16), we seek the expression for the *Poynting theorem* by the procedure followed at the beginning of §5. Multiplying the two equations (16) scalarly with \mathbf{H} and \mathbf{E} respectively, and utilizing the transformation (5.2) we obtain

$$\frac{1}{c} \mathbf{H} \cdot \dot{\mathbf{B}} + \frac{1}{c} \mathbf{E} \cdot \dot{\mathbf{D}} + \frac{4\pi}{c} \mathbf{E} \cdot \mathbf{J} + \text{div } \mathbf{E} \times \mathbf{H} = 0. \quad (17)$$

We compare this with the earlier form (5.7) of the same theorem:

$$\dot{W}_m + \dot{W}_e + \text{div } \mathbf{S} = -W_J. \quad (17a)$$

Since we cannot disturb Ohm's law W_J is still given by the product $\mathbf{E} \cdot \mathbf{J}$. We must hence divide (17) by $4\pi/c$ in order that (17) may correspond with (17a). Then a comparison of the terms of (17) and (17a) leads to

$$\dot{W}_m = \frac{1}{4\pi} \mathbf{H} \cdot \dot{\mathbf{B}}, \quad \dot{W}_e = \frac{1}{4\pi} \mathbf{E} \cdot \dot{\mathbf{D}}, \quad (17b)$$

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}, \text{ and } W_J = \mathbf{E} \cdot \mathbf{J} = \sigma \mathbf{E}^2 \text{ as before.} \quad (17c)$$

Integration of (17b) with respect to t , as on p. 27, yields (for isotropic and anisotropic media):

$$W_m = \frac{1}{8\pi} \mathbf{H} \cdot \mathbf{B}, \quad W_e = \frac{1}{8\pi} \mathbf{E} \cdot \mathbf{D}; \quad (17d)$$

They express the localization of energy in conventional units. Already in connection with Eq. (7.14) we pointed out the unsuitable form of the denominator 8π , as compared with the denominator 2 in our rational notation (5.6). The same applies for the factor $c/(4\pi)$ in the present expression (17c) for the energy flux. Even in the Maxwell equations (16) the suppression of 4π , carried out at the wrong place, avenges itself: These equations, in the form appropriate for both conductors and nonconductors, become:

$$\frac{1}{c} \dot{\mathbf{B}} = - \text{curl } \mathbf{E}, \quad \frac{1}{c} \dot{\mathbf{D}} + \frac{4\pi}{c} \mathbf{J} = \text{curl } \mathbf{H} \quad (18)$$

with the supplementary conditions, applying specifically for isotropic media

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mathbf{J} = \sigma \mathbf{E}. \quad (18a)$$

We hope that by this summary we have facilitated for the reader the laborious transition between our two systems of units

$$\text{MKSQ (rational)} \rightleftharpoons \text{cgs (Gauss, conventional)}$$

as far as possible. We have discussed the historical source of this annoyance at the end of §7. It is unavoidable in view of the present status of the question of units in electrical engineering, experimental physics, and theoretical physics. The following remarks may serve to clarify the situation.

H. A. Lorentz, when writing his articles for the *Enzyklopädie* in 1902, like Hertz, utilized the Gaussian system, postulating: *Electrical quantities* (including the electric current) *are measured electrically* (electrostatically), *magnetic quantities, magnetically*. Contrary to his original intention he decided, in the course of composing the articles, to convert the Gaussian system (unlike Gauss and Hertz) into *rational* units. In this manner the theoretical relationships became clearer and the 4π 's were eliminated from the Maxwell equations. Lorentz set for vacuum $\epsilon_0 = \mu_0 = 1$, as in our Eqs. (14a). In order to retain the rational form of the Coulomb force law he then had to introduce the 4π 's appearing in it into the definition of the unit charge and the unit pole strength, respectively. This somewhat artificial conversion of units¹ has not found wide acceptance, in spite of the authority of Lorentz.

¹ See table on p. 87 of Vol. 5, part 2, of the *Enzyklopädie*.

We have here—also against our original intention—arrived at the decision to write the Gaussian system, insofar as we shall use it, in *conventional units*. The reason is the following: Since the year 1902 *atomic physics* has come to be the most important branch of our science. It deals always with conventional units, e.g. with the electron charge $e = 4.80 \cdot 10^{-10}$ (electrostatic cgs-units) and with the electric potential, e.g. in the hydrogen atom, $\Psi = e/r$ (not $\Psi = e/(4\pi r)$). We consider it inadvisable to overturn this whole formalism anew by passing over to the rational form of the Gaussian system or even to our system of four units.

On the other hand Giorgi's system of the units MKSQ, freed of 4π 's, is most suitable for the macrophysical problems of this lecture. We are here in agreement with the international conventions, with the practice of engineering, and, in particular, with the textbooks of Mie (quoted on p. 10) and Pohl.¹ We regard the dogma of the scientific superiority of the *three* purely mechanical units cm, g, sec, which for example is supported in Kohlrausch, *Praktische Physik*, as outmoded.

D. Supplement Regarding Other Systems of Units

In the restriction to these two systems of units, the Gaussian system in conventional form and the MKSQ system, we follow the practice of the excellent textbook of Joos.² The Gaussian system (whether in rational or conventional form) is a *mixed* system, consisting of electrical (electrostatic) and magnetic cgs-units. There are however, as is well known, also a *purely electrical* and a *purely magnetic* system of units, of which the latter is particularly important, since the legal units volt, ampere, ohm, etc. are based on it.

The reason for setting up a separate *electrical system of units* rests on a certain quantitative difference between electrostatics and electrokinetics: Electrostatics deals with large voltages and small quantities of electricity, electrokinetics, with moderate voltages and large quantities of electricity. To give a comparison from hydrodynamics, the electric spark of a condenser discharge corresponds to a waterfall (great height, small quantity), the electric current, to a river (small grade, great flow), as is indicated in Fig. 7. Thus, for electrostatics, a small unit of charge and a large unit of field strength are suitable. The *electrostatic system* based on the electrical Coulomb law provides such units. In terms of this small unit of charge the charge of the electron (see above) has the relatively large value $4.80 \cdot 10^{-10}$ cgs-unit. The unit of charge in the *electromagnetic system* (equal to 10 coulombs), on the other hand, is larger by the factor c ; in terms of it, the charge of the electron appears smaller by a factor c , i.e. equal to $1.60 \cdot 10^{-20}$ cgs = $1.60 \cdot 10^{-19}$ coulomb (see p. 43). On the other hand, the unit of *field strength* in the electromagnetic system, according to the definition

¹ R. W. Pohl, *Elektrizitätslehre*, Springer, 8th and 9th Edition, 1943.

² G. Joos, *Theoretical Physics*, 2nd Ed., G. E. Stechert and Co., New York, 1950.

of the volt, is 10^{-8} volts/cm; that in the electrostatic system is c times as large, or 300 volts/cm. We shall return to this in §16D.

We have frightened generations of students with these two sets of values for charge and field strength (their number would be increased to 4 if, in addition to the usual conventional units, rational units would also be considered). It is, in our opinion, a special advantage of the introduction of our fourth unit of charge, Q , which is independent of all other units, that we need deal only with quite definite charges, expressed as multiples of Q .

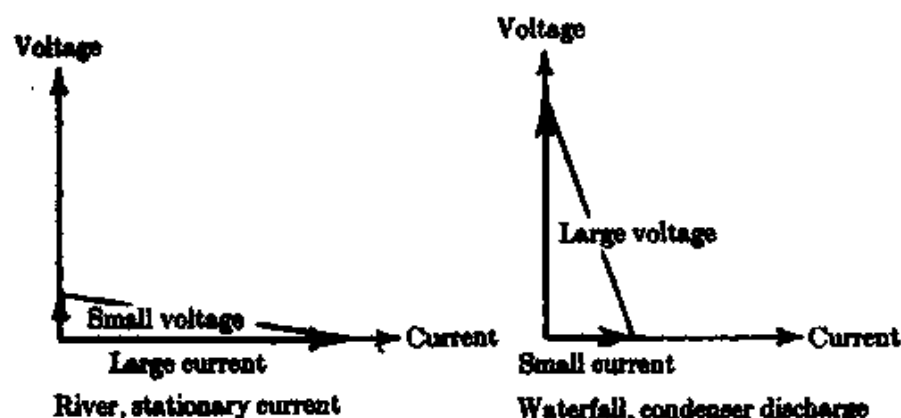


FIG. 7. The hydrodynamic representation of a stationary electric current and of a condenser discharge.

We quote finally an informative analog to the double (electrostatic and electromagnetic) measure of charge which, like so many other clarifications in the question of units, we owe to J. Wallot.¹ Suppose that someone had decided to describe mechanical processes in terms of only *two* independent units, cm and sec. He eliminates the gram as unit by setting either the density δ or the modulus of elasticity E of some standard material such as copper arbitrarily equal to 1. He can then express the mass m of a given copper rod in two ways, either by a measurement of volume according to the formula

$$\delta = \frac{m}{V}, \text{ which, because of } \delta = 1, \text{ leads to: } m = m_1 = V$$

or by a vibration experiment with longitudinal waves according to the formula

$$c^2 = \frac{E}{\delta} = \frac{EV}{m}, \text{ which, because of } E = 1, \text{ leads to: } m = m_2 = V/c^2.$$

If he now divides one of the two values of m so found by the other he obtains—perhaps to his surprise—the square of the velocity of propagation c of elastic waves in copper. The analogy to electrodynamics is striking and requires no further explanation.

¹ J. Wallot, *Elektrotechn. Z.*, Vol. 43, Nr. 44 (1922), section 28 of the paper "Physical and Engineering Units." The latest relevant publication is *Phys. Z.* 44, p. 17, 1943.

PART II

DERIVATION OF THE PHENOMENA FROM THE MAXWELL EQUATIONS

§9. *The Simplest Boundary-Value Problems of Electrostatics*

We have set up the fundamental equations of electrostatics in the beginning of §7 and have dealt with the resulting *summation problem* for a uniform medium in Eq. (7.5). We now turn to the *boundary-value problems* arising from the presence of *conductors* or *nonconductors* of different dielectric constant.

We think of the simplest electrostatic experiments: Let a metallic conductor of arbitrary shape, originally insulated, A, be connected to a source of potential V (with respect to ground) or B, be given a known charge (e.g. by a piezoquartz, see p. 78). We wish to know the field outside of the conductor. We describe this field by the potential Ψ associated with the field strength $\mathbf{E} = -\text{grad } \Psi$. Let Ψ be set equal to zero at infinity in both cases, A and B. In both cases $\Delta\Psi = 0$ outside of the conductor; on the surface, as well as in the interior of the conductor, we have $\Psi = \Psi_L = \text{const.}$

A. *Charging Problems*

In case A, $\Psi_L = V$ is given, in case B, Ψ_L must be found. According to (3.12a) the surface charge density at any point $d\sigma$ of the surface of L is given by

$$\omega = D_n = \epsilon E_n = -\epsilon \left(\frac{\partial \Psi}{\partial n} \right)_L. \quad (1)$$

ϵ is the dielectric constant outside of the conductor, n the outward normal to the surface of L . According to (1) the total charge on L is

$$q = \int \omega d\sigma = -\epsilon \int \frac{\partial \Psi}{\partial n} d\sigma. \quad (2)$$

In case A, q is sought; in case B, where q is given, (2) determines Ψ_L .

For the case of a *sphere*, of radius a , the appropriate solution of the differential equation $\Delta\Psi = 0$ may be written down immediately, in the form

$$\Psi = \frac{a}{r} \Psi_L. \quad (3)$$

This yields for A, since $\Psi_L = V$,

$$\Psi = \frac{a}{r} V. \quad (3a)$$

In case B, (2) and (3) lead to

$$q = 4\pi a^2 \frac{\epsilon \Psi_L}{a}, \Psi_L = \frac{q}{4\pi \epsilon a}, 4\pi \epsilon \Psi = \frac{q}{r}. \quad (3b)$$

The charge q , uniformly distributed over the spherical surface, thus acts at a distance like a point charge concentrated at the center.

It is also possible to guess the field of a conductor of the shape of a prolate spheroid (ellipsoid of revolution with long axis as axis of symmetry). For this it is merely necessary to stretch, so to speak, the center of the sphere, which appears in the last Eq. (3b) as locus of the total charge, into the connecting line of the two focal points of the generating ellipse and to distribute the charge q uniformly over this line. If we call the distance of the two focal points from the center of the ellipsoid c , the linearly distributed charge density becomes $q/(2c)$ and its potential becomes, by Eq. (7.5b):

$$\begin{aligned} 4\pi \epsilon \Psi &= \frac{q}{2c} \int_{-c}^{+c} \frac{d\zeta}{\sqrt{x^2 + y^2 + (z - \zeta)^2}} \\ &= \frac{q}{2c} \log \frac{z + c + \sqrt{x^2 + y^2 + (z + c)^2}}{z - c + \sqrt{x^2 + y^2 + (z - c)^2}} \end{aligned} \quad (4)$$

In Problem II.1 we will show that this expression assumes a constant value $\Psi = \Psi_L$ on each of the confocal ellipsoids pertaining to the given separation of foci and that therefore it solves our potential problem for each one of these ellipsoids. Since only the separation c of the focal points occurs in (4), this formula applies for all confocal ellipsoids of the family, in the sense that all of the equipotential surfaces of the confocal ellipsoids with the semiaxes $a_2 > a_1$, $b_2 > b_1$ are included among the equipotential surfaces of the ellipsoid with the semiaxes a_1 , b_1 . The ellipsoid $a = c$, $b = 0$, which degenerates to a straight line of length $2c$, also belongs to this family. In II.2 the limiting case of a paraboloid of revolution, and its degeneration, the field of a semi-infinite glass rod uniformly charged by friction, are studied from this point of view.

B. Induction Problems and Method of Reciprocal Radii

The "induction problem," which we shall specialize to an inducing point charge, is more complex than the "charging problem" treated thus far. Here also we can distinguish between two cases: A. The (otherwise arbitrarily shaped) conductor is grounded and B., it is insulated. The generally

accepted meaning of "grounding" is a conducting connection with an infinitely distant surface at the potential $\Psi = 0$. "Insulation" signifies, for an originally uncharged conductor, that even after induction the total charge q continues to be zero.

Problem A is solved by *Green's function* $G(P, Q)$ —more exactly, "Green's function of the potential equation for the exterior of the conductor L ." Q is the "source point," which will be assumed to represent a "unit source," P , the "reference point." G is defined by the following conditions:

$$\begin{aligned} \Delta G &= 0 \text{ for all } P \neq Q \text{ outside of } L, \\ G &\rightarrow 1/(4\pi r_{PQ}) \text{ for } P \rightarrow Q \text{ (definition of unit source),} \\ G &= 0 \text{ on the surface of } L, \\ G &\rightarrow 0 \text{ for } P \rightarrow \infty. \end{aligned} \quad (5)$$

Green's function plays a central role not only in potential theory, but generally in the theory of linear differential equations, to be treated in Vol. VI. Here we shall merely point out its significance for our special problem. If Q represents the position of the inducing charge e , the solution of problem A is given by

$$\Psi(P, Q) = \frac{e}{\epsilon} G(P, Q) \quad (6)$$

and that of B by

$$\Psi(P, Q) = \frac{e}{\epsilon} G(P, Q) + \alpha \Psi; \quad (6a)$$

here Ψ is the solution of our "charging problem" A for the same conductor L , α , a parameter which, according to Eq. (2), is determined by the condition

$$\int \frac{\partial \Psi(P, Q)}{\partial n} d\sigma = 0. \quad (6b)$$

As special case we consider once more a sphere of radius a . Its Green's function can be written down in closed form by the ingenious method of the young William Thomson, later Lord Kelvin, which will be treated in detail in Vol. VI, §23. With Q as source point and Q' as "electrical image of Q with reference to the sphere of radius a " this solution is:

$$4\pi G(P, Q) = \frac{1}{r_{PQ}} - \frac{a}{\rho} \frac{1}{r_{PQ'}}. \quad (7)$$

$\rho = OQ$ is the separation of the source point Q from the center of the sphere, $\rho' = OQ'$, that of its image point Q' . They are related by the condition of "reciprocal radii":

$$\rho\rho' = a^2, \quad (8)$$

from which Thomson's procedure has been given the name "method of reciprocal radii." It is immediately evident that our formula (7) satisfies the first, second, and last condition (5); the fulfilment of the third condition (5) can be demonstrated by elementary geometry.

According to (6), Eq. (7) yields for induction on the grounded sphere:

$$4\pi\epsilon\Psi(P, Q) = \frac{e}{r_{PQ}} - \frac{e'}{r_{PQ'}}, e' = \frac{a}{\rho} e \quad (8a)$$

and for the insulated sphere, by (6a, b) and (3b):

$$4\pi\epsilon\Psi(P, Q) = \frac{e}{r_{PQ}} - \frac{e'}{r_{PQ'}} + \frac{e'}{r_{PO}}, e' = \frac{a}{\rho} e. \quad (8b)$$

The last term in this formula corresponds to the added term $\alpha\Psi$ in (6a). It has the effect of raising the potential of our insulated sphere to the

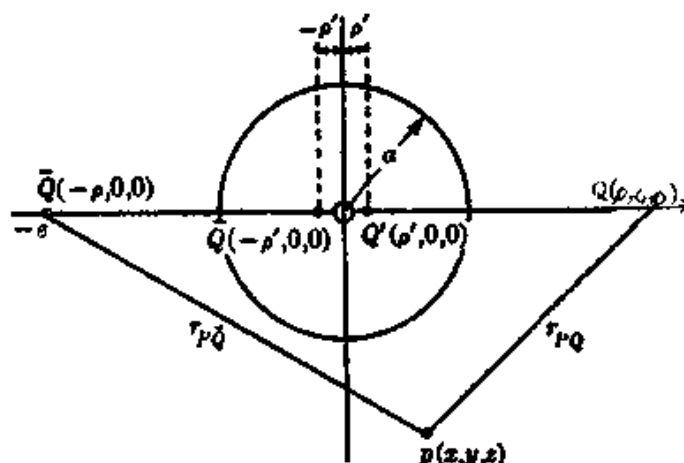


FIG. 8. Two charges $\pm e$ moving toward infinity and their electric images at a conducting sphere of radius a produce a uniform electric field and an electric dipole at the center of the sphere.

value $V = e'/(4\pi\epsilon a)$ and of making the total charge on its surface $q = 0$, as it should be.

C. Conducting Sphere in a Uniform Field

For the practically unlimited possibilities of application of spherical images in potential theory we refer to the portion of Vol. VI cited above. Here we shall treat only the simple case of the sphere in a *uniform field*, whose lines of force may, for example, be parallel to the x -axis. In the absence of the conducting sphere the uniform field is given by

$$\Psi = -Fx, E_x = -\frac{\partial\Psi}{\partial x} = F, E_y = E_z = 0. \quad (9)$$

We imagine this field as resulting from the superposition of two fields, originating in the infinitely distant source points Q, \bar{Q} (see Fig. 8) on the

x -axis, with the charges $\pm e$; their coordinates are $x = \pm \rho$ ($\rho \rightarrow \infty$), $y = z = 0$. The superposition leads to the potential

$$\begin{aligned} 4\pi\epsilon\Psi &= \frac{e}{r_{PQ}} - \frac{e}{r_{P\bar{Q}}} = e\{(\rho - x)^2 + y^2 + z^2\}^{-1/2} \\ &\quad - e\{(\rho + x)^2 + y^2 + z^2\}^{-1/2} = e\{\rho^2 + x^2 + y^2 + z^2\}^{-1/2} \quad (9a) \\ &\quad \cdot \left(1 + \frac{\rho x}{\rho^2 + \dots} - 1 + \frac{\rho x}{\rho^2 + \dots}\right) \rightarrow \frac{2ex}{\rho^2}. \end{aligned}$$

We have thus in fact a uniform field of the same form as (9) provided that we let e become infinite as ρ^2 . To obtain quantitative agreement with (9) we must put

$$\frac{2e}{\rho^2} \rightarrow -4\pi\epsilon F. \quad (9b)$$

In Fig. 8 we have also shown the sphere of radius a and the appropriate spherical images of the source points Q, \bar{Q} constructed for it:

$$Q' = \rho', 0, 0 \text{ and } \bar{Q}' = -\rho', 0, 0.$$

They approach each other as the charges $\pm e$ move apart and form in the limit an electric dipole with the moment

$$M = 2\rho'e'. \quad (10)$$

Here we substitute from Eq. (8) $\rho' = a^2/\rho$ and from Eq. (8a, b) $e' = ea/\rho$. In view of (9b), Eq. (10) then states that the moment M assumes in the limit for $\rho \rightarrow \infty$ the finite value

$$M = 2 \frac{a^2}{\rho} \frac{ea}{\rho} = -4\pi\epsilon F a^3. \quad (10a)$$

We conclude therefore that the boundary value problem for the homogeneous field is solved by placing a virtual electric dipole of the finite moment M at the center of the sphere. The homogeneous field (9) is then replaced by the inhomogeneous field¹

$$\Psi = -Fx + \frac{M}{4\pi\epsilon} \frac{\partial}{\partial x} \frac{1}{r}, \quad (11)$$

which contains the distortion created by the dipole. Since, now, r represents the distance from the center of the sphere,

$$r = (x^2 + y^2 + z^2)^{1/2}, \text{ so that } \frac{\partial}{\partial x} \frac{1}{r} = -\frac{x}{r^3}.$$

Eq. (11) hence becomes

$$\Psi = -Fx \left(1 + \frac{M}{4\pi\epsilon F} \frac{1}{r^3}\right). \quad (11a)$$

¹ The denominator $4\pi\epsilon$ is to be added here for the same reason as the factor $4\pi\epsilon$ on the left side of (9a).

If we substitute here the value of M from (10a), we obtain

$$\Psi = -Fx \left(1 - \frac{a^3}{r^3}\right). \quad (11b)$$

On the surface of the sphere, $r = a$, Ψ assumes the constant value $\Psi_L = 0$. The value of M found in (10a) by the method of reciprocal radii is thus confirmed for the *conducting sphere*.

D. Dielectric Sphere in a Uniform Field

We shall now show that the formula (11), with the value of M undetermined, has a much greater range of validity, i.e. will also fulfill the boundary conditions for a *non-conducting sphere of arbitrary dielectric constant*. If we distinguish the exterior and the interior of the sphere by the indices 1 and 2 (see Fig. 9), these conditions require that

$$\Psi_1 = \Psi_2, \quad \epsilon_1 \frac{\partial \Psi_1}{\partial n} = \epsilon_2 \frac{\partial \Psi_2}{\partial n} \quad \text{for } r = a. \quad (12)$$

The first of these guarantees the continuity of the *tangential* components of \mathbf{E} ; the second indicates the continuity of the *normal* component of \mathbf{D} , which, for an originally uncharged, nonconducting sphere, is equivalent to the absence of surface charge, $\omega = 0$ (Eq. (3.11)). We assert that both equations may be satisfied if for Ψ_1 (exterior of sphere, $r > a$) we use formula (11a) and for Ψ_2 (interior of sphere, $r < a$) we assume a *homogeneous field*, in the same direction as, but differing in strength from, the exterior primary field.

With $x = r \cos \theta$ (θ = geographic latitude on the sphere, measured from the field direction) we write tentatively

$$\Psi_1 = -F \left(r + \frac{M}{4\pi\epsilon_1 F} \frac{1}{r^2} \right) \cos \theta, \quad \Psi_2 = -F_2 r \cos \theta \quad (13)$$

and, according to (12), we must demand that for $r = a$ ($\cos \theta$ cancels in both Eqs. (12)):

$$\begin{aligned} F \left(1 + \frac{M}{4\pi\epsilon_1 F} \frac{1}{a^3} \right) &= F_2, \\ F \left(1 - \frac{2M}{4\pi\epsilon_1 F} \frac{1}{a^3} \right) &= \frac{\epsilon_2}{\epsilon_1} F_2. \end{aligned} \quad (13a)$$

From this we find, with $\epsilon = \epsilon_2/\epsilon_1$ as *relative dielectric constant*:

$$\frac{F_2}{F} = \frac{3}{\epsilon + 2}, \quad \frac{M}{4\pi\epsilon_1 F} = -\frac{\epsilon - 1}{\epsilon + 2} a^3. \quad (14)$$

Thus our assumption of a homogeneous field within the sphere has proved adequate. This field F_2 is weaker than the primary external field F if

$\epsilon > 1$, as, for example, if the sphere is in air. The lines of force penetrate the interior of the sphere (see Fig. 9); although they are curved outside by the action of the (virtual) dipole moment they are straight and parallel to the x -axis within the sphere.

To understand this figure properly it should be noted that it does not represent the lines of force E , but the lines of induction D . The two systems of lines have the same direction but different density both inside and outside of the sphere (see the remarks on magnetic lines of force or better tubes of force on p. 11), and hence behave differently at the surface. The D -lines are *source-free* not only within and outside of the sphere but also at its surface, because of the vanishing surface divergence (D_n is continuous); this does not apply to the E -lines (E_n is discontinuous). The fact that Fig. 9 represents the D -lines is evident from the fact that just one line passes through every point of the spherical surface. In the

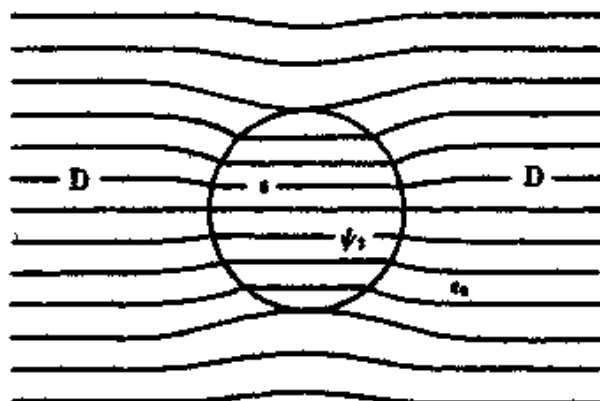


FIG. 9. A dielectric sphere in a uniform electric field. The excitation lines within and outside of the sphere.

case of the E -lines, more lines would arrive at the surface on the outside than leave it on the inside.

We consider two limiting cases, $\epsilon \rightarrow \infty$ and $\epsilon \rightarrow 0$. The first proves to be identical with that of the *conducting* sphere. Eqs. (14) yields then

$$M = -4\pi\epsilon_1 Fa^3, \quad F_2 = 0, \quad (14a)$$

in agreement with Eq. (10a) and with the fact that the interior of the sphere is field-free. This appears to contradict Fig. 9a, which shows a finite field inside of the sphere. We must note again, however, that this figure, as limiting case of Fig. 9, represents the D -field and that " $D = \epsilon E = \text{finite}$ " is consistent with passing to the limit $\epsilon \rightarrow \infty$, $E \rightarrow 0$.

The limiting case $\epsilon \rightarrow 0$ cannot be realized electrostatically,¹ but only

¹ Or only by assuming $\epsilon_2 \ll \epsilon_1$, i.e. considering a spherical cavity in a medium of very high dielectric constant. Then indeed the interior of the sphere is *relatively* free of D -lines, as is shown in Fig. 9b.

by placing a nonconductor in the stationary current field of a conductor. The magnetic analog would be a superconductor; hydrodynamically it represents the case of a rigid sphere immersed in a liquid whose flow is nonturbulent, incompressible and in parallel lines at infinity. In contrast with (14a) we have now

$$M = 2\pi\epsilon_1 F a^3, F_2 = \frac{3}{2} F. \quad (14b)$$

In spite of the finite value of F_2 the magnetic induction lines in the case of the superconductor and the hydrodynamic flow lines do not penetrate into

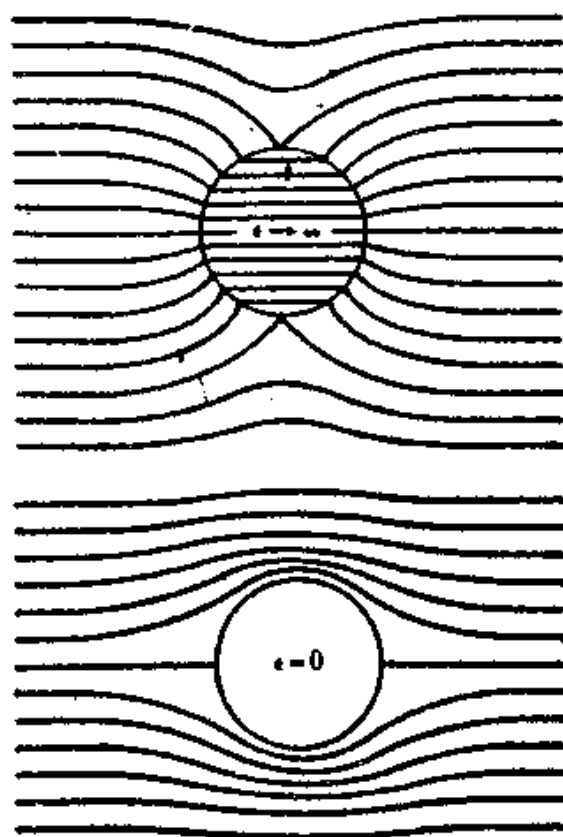


FIG. 9a. A conducting sphere in a uniform electric field. Representation of the excitation lines.

FIG. 9b. Stream lines about a rigid sphere. At the same time, magnetic lines of force about a superconducting sphere.

the interior of the sphere, as shown in Fig. 9b; they are pushed out of the interior since they must run tangential to the surface.

In the other limiting case, $\epsilon \rightarrow \infty$, the lines of force are perpendicular to the surface,¹ as for a conductor. In Problem II.3 we shall indicate the close relationship between Figs. 9 and 9a. We shall return to these important formulas and figures in §11.

¹ The force lines at the upper and lower pole of the sphere (in three dimensions at the diametral plane passing through the poles) form an exception. They make an angle of 45° with the surface of the sphere (see figure), an angle of 90° with each other. This may for example be seen from the fact that the Taylor expansion of Ψ (Eq. (11b)) begins with a term of the second order. Maxwell calls such a point a "point of equilibrium" (see art. 112 of the Treatise).

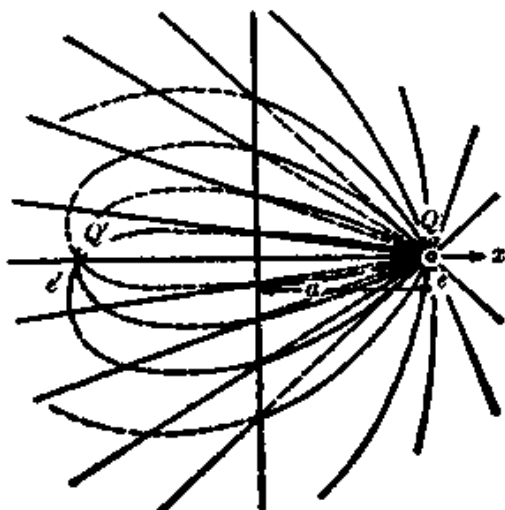
E. Reflection and Refraction of Lines of Force at the Boundary of a Semi-infinite Dielectric

For the sake of completeness a rather trivial problem will be dealt with here, namely induction in a dielectric bounded by a plane (see Fig. 10): Let a unit electric charge Q be at the point $x = a$ in the right halfspace, $x > 0$; it brings about a state of induction in the left halfspace, $x < 0$; ϵ is the *relative* dielectric constant of the left halfspace referred to the right halfspace. As in (12), the boundary conditions are

$$\Psi_1 = \Psi_2, \quad \frac{\partial \Psi_1}{\partial x} = \epsilon \frac{\partial \Psi_2}{\partial x} \quad \text{for} \quad x = 0. \quad (15)$$

A solution may be obtained with the aid of the simple reflection method familiar from optics: a virtual charge Q' of opposite sign is imagined at the point $x = -a$, which has an effect in medium 1, but has no effect in medium

FIG. 10. In the right-hand halfspace (air) at point Q is a point charge producing induction in the left halfspace (dielectric). Representation of the excitation lines. In the right halfspace they are curved; their continuations (dotted in the figure) pass through the image point Q' of Q . In the left halfspace they are straight lines whose (dotted) continuations pass through Q .



2, in view of the finiteness of the field throughout this medium; here all effects appear to proceed from the primary charge. Introducing the two disposable parameters e'/e and e''/e we write tentatively:

$$4\pi\epsilon_0\Psi_1 = \frac{e}{r} + \frac{e'}{r'}, \quad 4\pi\epsilon_0\Psi_2 = \frac{e''}{r}. \quad (16)$$

In the boundary plane $x = 0$, $r = r_{PQ}$ and $r' = r_{PQ'}$ are identical; at the same time

$$\frac{\partial}{\partial x} \frac{1}{r} = -\frac{x-a}{r^3}, \quad \text{and} \quad \frac{\partial}{\partial x} \frac{1}{r'} = -\frac{x+a}{r^3}$$

are equal and opposite, that is, are respectively equal to $\pm a/r^3$. Hence conditions (15) require

$$\begin{aligned} e + e' &= e'' \\ e - e' &= \epsilon e' \end{aligned} \quad \begin{aligned} e' &= \frac{1-\epsilon}{1+\epsilon} e, & e'' &= \frac{2}{1+\epsilon} e. \end{aligned} \quad (16a)$$

Employing optical terminology, e'' might be called the "refracted" charge; the "reflected" charge e' of course becomes zero for $\epsilon = 1$ (uniform dielectric, no discontinuity at $x = 0$). In the limiting case $\epsilon \rightarrow \infty$ (conductor) $e' = -e$, $e'' = 0$, so that $\Psi_2 = 0$; this corresponds to the requirement of a constant potential for $x < 0$. For $\epsilon \rightarrow 0$ the second condition (15) becomes $\partial\Psi_1/\partial x = 0$ and leads to $e' = +e$. See in this connection Fig. 10 (arbitrary $\epsilon > 1$), Fig. 10a ($\epsilon \rightarrow \infty$), and Fig. 10b ($\epsilon \rightarrow 0$, i.e. $\epsilon_{\text{right}} \gg \epsilon_{\text{left}}$).

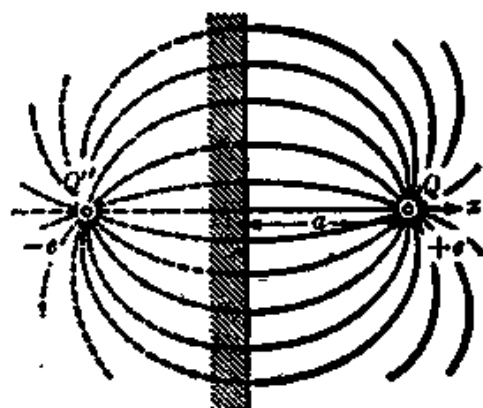


FIG. 10a. Limiting case $\epsilon \rightarrow \infty$: the left halfspace is a conductor.

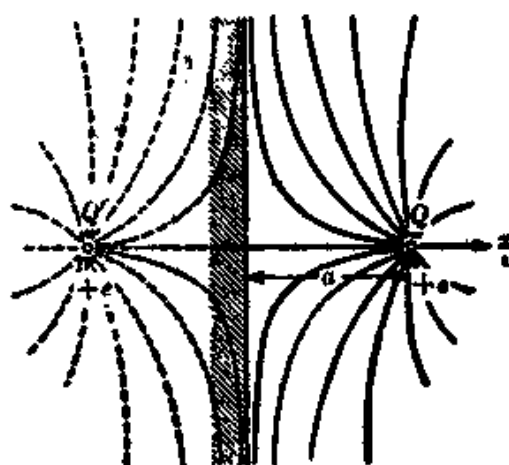


FIG. 10b. Limiting case $\epsilon \rightarrow 0$; dielectric constant of the right halfspace very large compared with that of the left halfspace.

§10. Capacity and its Connection with Field Energy

We consider two conductors of arbitrary shape L_1 and L_2 and give them charges $+q$ and $-q$ respectively. Such a system is called a *condenser* because the field between them is concentrated and limited to their neighborhood. The lines of force pass from L_1 to L_2 without diverging to infinity. Let the potential Ψ have the constant values Ψ_1 and Ψ_2 on L_1 and L_2 . The potential difference between them is then

$$V = \Psi_1 - \Psi_2 = \int_{L_1}^{L_2} \mathbf{E} \cdot d\mathbf{s}. \quad (1)$$

In this line integral we are permitted to leave the shape of the path indefinite; every path from L_1 to L_2 (it need by no means be a line of force) yields, as we know, the same value of the potential difference for a lamellar electrostatic field.

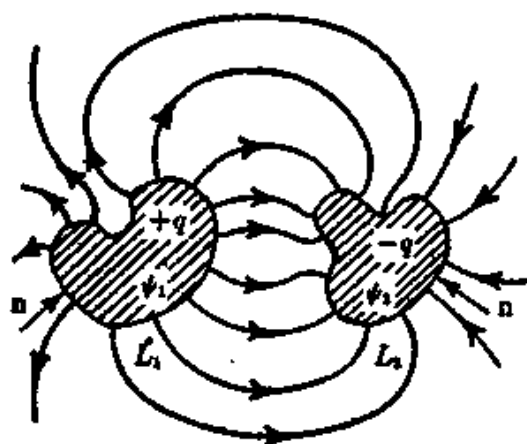
The ratio q/V is called the *capacity*, i.e. the ability of the system to take up charge. We employ for it the letter K (instead of the often-employed symbol C , which we use so often with the meaning "constant"):

$$K = \frac{q}{V}. \quad (2)$$

The unit of capacity is the "farad":

$$1 \text{ farad} = 1 \frac{\text{coulomb}}{\text{volt}} = 1 \frac{Q^2}{\text{joule}}; \quad (3)$$

FIG. 11. The pair of conductors L_1, L_2 with the charges $\pm q$ and the potentials ψ_1, ψ_2 form an electric condenser.



which is thus defined in our system without the appearance of inconvenient powers of ten. On the other hand, in electromagnetic cgs-units, according to (3),

$$1 \text{ farad} = \frac{10^{-1}}{10^9} \text{ cgs} = 10^{-9} \text{ cgs}; \quad (3a)$$

thus the so-called absolute cgs-unit of capacity would be $=10^9$ farad. The microfarad ($=10^{-6}$ farad) is employed more often than the farad.

According to (3) and (4.6a) the dielectric constant of vacuum has the dimension farad/ M and, according to (7.18), it has the magnitude

$$\epsilon_0 = \frac{10^{-9}}{36\pi} \frac{\text{farad}}{M} = \frac{10^{-9}}{36\pi} \frac{\text{microfarad}}{M}. \quad (3b)$$

However, if one chooses to give the capacity in "absolute electrostatic units" not only is the numerical value of the capacity different but also its dimensions (here cm!) are changed and a very confusing state of affairs results.

We shall consider, as simplest example,

A. The Plate Condenser

This condenser is to consist of two conducting plates with (large) area F , which are placed parallel and facing each other with (small) separation

a. We imagine their charges $\pm q$ to be distributed over F with uniform surface density ω . Hence we have for $x = \pm a/2$

$$\omega = \mp \frac{q}{F}, \quad \text{and also } \omega = D_n \text{ (Eq. (3.12a))}. \quad (4)$$

We regard the field between the plates as homogeneous, neglecting as we already have in (4), the perturbation at the edge zones. The lines of force are then normal to the plates *throughout*, and we find, with the value of D given by (4)

$$E_x = \frac{\omega}{\epsilon} = \frac{q}{F\epsilon}. \quad (4a)$$

It follows hence that

$$V = \int_{-a/2}^{+a/2} E_x dx = \frac{qa}{F\epsilon} \quad (4b)$$

and thus, from (2)

$$K = \frac{F\epsilon}{a}. \quad (5)$$

This is the "rational" value of our capacity, previously given in (7.15).

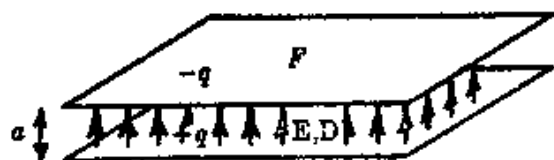


FIG. 12. The plate condenser with the field E , D between plates considered as uniform.

If, for example, we consider $F = 20 \cdot 20 \text{ cm}^2$, $a = 1 \text{ mm}$, and $\epsilon = 2\epsilon_0$ (paraffin filling) we find, with the value (3b) of ϵ_0 in our system of units,

$$K = \frac{80}{36\pi} 10^{-9} \text{ farad} = \frac{2}{9\pi} \cdot 10^{-8} \text{ farad} = \frac{2}{9\pi} \cdot 10^{-2} \text{ microfarad}. \quad (5a)$$

Thus, to obtain a capacity comparable with a microfarad, a very large number of condensers of the type considered have to be added together.

The neglected edge correction and the resulting inhomogeneity of the field will be treated in Problem II.4.

B. Spherical Condenser

This is to consist of an inner sphere of radius r_1 and an outer sphere of radius r_2 . The spheres need not be conductors throughout; it suffices if the outer surface of the inner sphere, and the inner surface of the outer sphere are "coated with tin foil." (The same remark applies for the plate condenser.) Let the inner sphere have the charge $+q$, the outer sphere,

the charge $-q$. The field is spherically symmetrical— E and D depend only on r and are directed radially. Hence for every r between r_1 and r_2

$$\int D, d\sigma = 4\pi r^2 D = q, \quad (6)$$

$$E_r = \frac{D}{\epsilon} = \frac{q}{4\pi\epsilon r^2}.$$

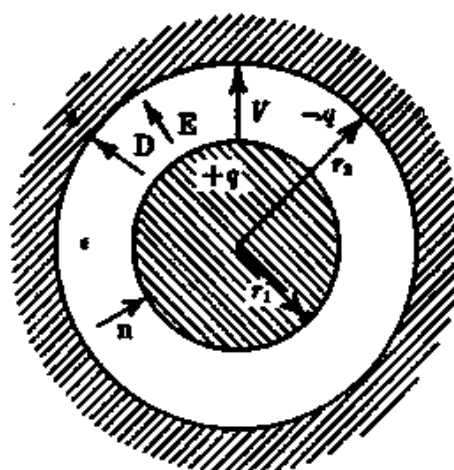
From E_r we obtain

$$V = \int_{r_1}^{r_2} E_r dr = \frac{q}{4\pi\epsilon} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) = \frac{q}{4\pi\epsilon} \frac{r_2 - r_1}{r_1 r_2}. \quad (7)$$

This leads to

$$K = \frac{q}{V} = 4\pi\epsilon \frac{r_1 r_2}{r_2 - r_1}. \quad (8)$$

FIG. 13. The spherical condenser with its radial field E , D and a voltage $V = \Psi_1 - \Psi_2$.



If we allow r_2 to become infinite we obtain

$$K = 4\pi\epsilon r_1, \quad (8a)$$

i.e. the value for the capacity for this limiting form of spherical condenser which was designated as "rational" in Eq. (7.15a). If, on the other hand, both radii in (8) are permitted to approach infinity while yet their difference $r_2 - r_1 = a$ remains finite, and if only a finite segment F of the two spherical surfaces $4\pi r_1^2$ and $4\pi r_2^2$ is considered, the capacity of this segment becomes according to (8)

$$K = \frac{F\epsilon}{a} \quad (8b)$$

i.e., identical, as it should be, with that of the plate condenser in Eq. (5).

The cylindrical condenser (Leyden jar¹) will be treated in Problem II.5.

C. Capacity of an Ellipsoid of Revolution and of a Straight Piece of Wire

In Eq. (9.4) we have given the potential Ψ of an ellipsoid with the charge q . If we substitute for x, y, z the coordinates of any point of its surface, e.g. one of the two endpoints of its major axis $x = y = 0, z = a$, Ψ assumes the value V , where V is its potential relative to an infinitely distant ground ($\Psi = 0$). If, at the same time, we introduce the minor axis b in place of half the focal separation c , we obtain $c = \sqrt{a^2 - b^2}$. Eq. (9.4) thus becomes²

$$\begin{aligned} \frac{V}{q} = \frac{1}{K} &= \frac{1}{8\pi\epsilon\sqrt{a^2 - b^2}} \log \frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}} \\ &= \frac{1}{4\pi\epsilon\sqrt{a^2 - b^2}} \log \frac{a + \sqrt{a^2 - b^2}}{b}. \end{aligned} \quad (9)$$

For $b \rightarrow a$, in the limit, the capacity (8a) of the sphere is obtained. In a similar manner the capacity of the axially symmetric *ellipsoidal condenser*, consisting of an inner and an outer ellipsoid of the confocal family, may be derived.

On the other hand, if we let $b \rightarrow 0$, our ellipsoid degenerates into a *straight segment* of length $l = 2c$ (focal separation). This may be thought of as representing a straight wire³ of radius $b \rightarrow 0$. Its capacity becomes, according to (9)

$$K = 2\pi\epsilon l / \log \frac{l}{b}. \quad (9a)$$

We will obtain a similar logarithmic formula for the selfinduction of a straight piece of wire (see §15).

D. Energetic Definition of Capacity

The usual elementary definition (2) of capacity may appear rather arbitrary and formal. We attain a physically more significant understanding of it by considering the *energy of the electrostatic field*.

¹ Also known as "Kleist jar," since it was built by the pastor of Kleist in Cammin (Pomerania) and first demonstrated at the Danzig Scientific Society in December 1745. At the beginning of the next year it was demonstrated in Leyden and has hence become internationally famous. In this manner it acquired its present customary name, which is thus purely accidental in origin.

² Compare Kohlrausch, *Praktische Physik*, 12th Ed., p. 631. The factor 4π is missing here (conventional system of units). Obviously the last formula in Eq. (9) results from the preceding one by rationalization of the denominator.

³ Not, it is true, a cylindrical wire, but one getting thinner toward the ends.

We utilize Green's theorem in the form (3.16) of Vol. II:

$$\int \text{grad } U \cdot \text{grad } V \, d\tau + \int U \Delta V \, d\tau = \int U \frac{\partial V}{\partial n} \, d\sigma. \quad (10)$$

U and V are two arbitrary continuous functions. The integration on the left side is extended over an arbitrary region of space, the integration on the right, over its surface; n is the normal to the surface of the space, pointing outward. We put $U = V = \Psi$ and obtain, because of $\Delta \Psi = 0$, $\mathbf{E} = -\text{grad } \Psi$, from (10)

$$\int \text{grad } \Psi \cdot \text{grad } \Psi \, d\tau = \int \mathbf{E}^2 \, d\tau = \int \Psi \frac{\partial \Psi}{\partial n} \, d\sigma. \quad (10a)$$

Let the volume integration on the left be carried out through the region exterior to the two conductors L_1 and L_2 in Fig. 11, the surface integration on the right, over the two conductor surfaces and a sphere K with the very large radius $r = R$. The integral over K vanishes.¹ (10a) then leads to

$$\int \mathbf{E}^2 \, d\tau = \Psi_1 \int_{L_1} \frac{\partial \Psi}{\partial n} \, d\sigma + \Psi_2 \int_{L_2} \frac{\partial \Psi}{\partial n} \, d\sigma. \quad (10b)$$

After multiplication with $\epsilon/2$ we can write instead

$$\frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} \, d\tau = -\frac{1}{2} \left(\Psi_1 \int_{L_1} D_n \, d\sigma + \Psi_2 \int_{L_2} D_n \, d\sigma \right) = \frac{q}{2} (\Psi_1 - \Psi_2) = \frac{1}{2} qV.$$

At the left end of this multiple equation we have the total energy of the volume considered, which we shall call W :

$$W = \int W_e \, d\tau.$$

We then find, with the definition (2) of K ,

$$W = \frac{1}{2} qV = \frac{K}{2} V^2 = \frac{q^2}{2K}. \quad (11)$$

¹ We have, by Gauss's theorem, for an arbitrary system of n conductors L_1, L_2, \dots, L_n , since $\text{div } \mathbf{D} = 0$,

$$0 = \int_K D_n \, d\sigma + \int_{L_1} D_n \, d\sigma + \dots + \int_{L_n} D_n \, d\sigma = \int_K D_n \, d\sigma - \sum_{i=1}^n q_i.$$

If Ψ on K is put equal to Ψ_∞ , it follows that

$$\int_K \Psi \frac{\partial \Psi}{\partial n} \, d\sigma = -\frac{\Psi_\infty}{\epsilon} \int_K D_n \, d\sigma = -\frac{\Psi_\infty}{\epsilon} \sum q_i.$$

In the present case this vanishes because of $q_1 = q, q_2 = -q$. In general, with $\sum q_i \neq 0$, it is only necessary to put $\Psi_\infty = 0$, i.e. to refer the potentials Ψ_i to the zero level at infinity, to make it possible to carry over the following formulas also to unneutral systems. This is to be remembered for section E.

This fundamental relation is reminiscent of the expression for the kinetic energy of a particle in rectilinear motion in terms of its velocity v and its momentum $p = mv$:

$$W = \frac{1}{2} vp = \frac{p^2}{2m} = \frac{m}{2} v^2. \quad (11a)$$

In both cases the energy is factored (see p. 11) into the product of an entity of quantity and one of intensity or expressed by the square of one of these two quantities. The quantity is in one case q , in the other, v , the intensity, V and p , respectively. Comparison of (11) and (11a) shows that the capacity K corresponds to the reciprocal mass $1/m$, which we might call "compliance," in contrast with the "inertia" m . However, the analogy is not very profound and will have to be modified in §33.

We can call (11) an energetic definition of capacity, just as (11a) may serve as energetic definition of inertia.

E. The Capacities of an Arbitrary System of Conductors

If we pass from two conductors L_1, L_2 with charges $\pm q$ to an arbitrary number of conductors L_1, L_2, \dots, L_n with charges q_1, q_2, \dots, q_n , where once more the total charge $\sum_{i=1}^n q_i$ is assumed to be zero, Green's theorem (10a, b) shows directly that the total energy W of the system is the sum of n terms, according to the formula

$$W = \frac{1}{2} \sum_{i=1}^n \Psi_i q_i. \quad (12)$$

Here Ψ_i denotes the constant value of the potential on the conductor L_i . Now, however, Ψ_i depends not only on q_i , but depends *linearly* on all the q_j as well. This follows from the general representation (7.5) of the potential.

In order to perceive this, we rewrite (7.5) in terms of the surface charge ω_j , whose distribution on every conductor L_j we can assume as known, in place of the volume charge ρ , and put $\omega_j = q_j \omega'_j$, where ω'_j is the distribution of unit charge on L_j (in the presence of the remaining conductors!). Then (7.5) becomes

$$\Psi_i = \sum_{j=1}^n q_j H_{ij}. \quad (13)$$

The coefficients which appear here,

$$H_{ij} = \frac{1}{4\pi\epsilon} \int \frac{\omega'_j}{r_{ij}} d\sigma_j, \quad (13a)$$

are purely geometrical quantities, which depend only on the location of the L_j relative to each other and relative to L_i ; they are independent

of the choice of the origin $r_{ij} = 0$ on L_i , since Ψ_i has the same value for every choice of this point. Eqs. (13) and (13a) thus confirm the linear relation between the Ψ_i and q_i .

The solution of the system (13) of n equations for the n charges q_i yields

$$q_i = \sum_{j=1}^n K_{ij} \Psi_j \quad \text{with} \quad K_{ij} = \frac{\Delta_{ij}}{\Delta}. \quad (14)$$

Δ_{ij} is the sub-determinant of the $n \cdot n$ row determinant Δ of the H_{ij} associated with the term i, j . Maxwell calls the H_{ij} "potential coefficients" of the system in art. 87 of his Treatise and the K_{ij} , "capacity coefficients."

Substituting the relations (13) and (14) in (12) yields the *multiple equation*

$$W = \frac{1}{2} \sum \Psi_i q_i = \frac{1}{2} \sum \sum H_{ij} q_i q_j = \frac{1}{2} \sum \sum K_{ij} \Psi_i \Psi_j. \quad (15)$$

which generalizes our earlier Eq. (11). Since W is a quantity determined by the state of the system (the work done in charging the system must be independent of the "path," i.e. the sequence of the individual processes), the K and H fulfill the reciprocity relations

$$K_{ij} = K_{ji}, \quad H_{ij} = H_{ji}. \quad (15a)$$

Capacity and potential coefficients play a role in communications, where complicated systems of interacting conductors are of frequent occurrence. Their theoretical calculation is difficult since it presupposes the solution of the potential problem of the multiconductor system in question. As we have seen, even for two conductors the solution is possible only for particularly simple shapes of the conductors (plane, sphere, ellipsoid). In general approximations are required.

In Problem II.6 we shall discuss the (somewhat complicated) relation between these coefficients and the elementary definition of capacity in Eq. (2).

§11. General Considerations on the Electric Field

The following statements and concepts apply not only to electrostatic, but also to arbitrarily varying fields.

A. The Law of Refraction for the Lines of Force

The boundary conditions applying at the interface between two insulators of different dielectric constant,

$$E_{\text{tang}} \text{ continuous and } D_{\text{norm}} \text{ continuous} \quad (1)$$

(the latter in the absence of surface charge, Eq. (3.11)) show directly that the "angle of incidence" α_1 and the "angle of refraction" α_2 , both measured with respect to the normal to the interface and given by

$$\tan \alpha_1 = \left(\frac{E_{\text{tang}}}{E_{\text{norm}}} \right)_1, \quad \tan \alpha_2 = \left(\frac{E_{\text{tang}}}{E_{\text{norm}}} \right)_2,$$

are related by

$$\frac{\tan \alpha_1}{\epsilon_1} = \frac{\tan \alpha_2}{\epsilon_2}. \quad (2)$$

This "law of refraction of the electric lines of force" deviates from the optical law of refraction not only in the appearance of the tangent instead of the sine, but also in the direction of refraction: *In entering into the electrically denser medium a line of force is refracted away from the normal to the interface.* We here describe the medium with the higher dielectric constant as "electrically denser." If this is medium 2, then it follows from Eq. (2) that

$$\tan \alpha_2 > \tan \alpha_1.$$

Examples of this phenomenon are shown in Figs. 10 (refraction at a plane) and 9 (refraction at a sphere). The conductor (limiting case $\epsilon_2/\epsilon_1 = \infty$) satisfies this law of refraction inasmuch as here generally $\alpha_1 = 0$ (lines of force normal to surface of conductor).

B. On the Definition of the Vectors \mathbf{E} and \mathbf{D}

Now we do not accept the "positivistic" standpoint, according to which only observables may be employed in theoretical physics, but instead are of the opinion that the introduction of not directly observable quantities is justified whenever the resulting conclusions agree with experiment (as in the kinetic theory of gases). Nevertheless we demand that the concepts introduced in a hypothesis may be based at least on an imaginary experiment, i.e. an observational method, even if it cannot be carried out in practice.

In §2 we defined the electric field strength dynamically as the force on a unit test charge. This force can, however, only be measured in air (more generally, in a fluid) by the motion produced by it. Thus the definition fails in the solid body.

In order to define the component of \mathbf{E} at a given point in a given direction s within the solid body we proceed as follows: We drill a *tube* with the direction s in the body at the point in question. The tube is so narrow and so short that it does not appreciably disturb the field elsewhere; it remains empty or is filled with air. According to the boundary condition (1) the field-strength component E_s is the same within it as in the surrounding solid body and as it was originally in the tube. Thus E_s can be measured within the tube on a test body which has been introduced, and by varying the direction of the tube all three components of the vector \mathbf{E} can be ob-

tained. If the field is not stationary, but variable, we must measure more rapidly than the field changes.

Our definition of D in §2 requires supplementation in even greater degree. We described, in Eq. (4), D as the quantity of electricity which, at a given point, has passed through an area F during the excitation of the field, divided by the magnitude of F . More precisely, we obtain in this manner the component D_n of D in the direction of the normal n to F . This explanation is unsatisfying since it does not contain specific directions for measurement. We can, however, obtain such directions by the following imaginary experiment:

We place at the point in question a plate condenser,¹ whose surfaces F are made normal to the n -direction; the space between the plates is to be filled with the surrounding dielectric (if we are dealing with a solid body a slit must be cut into it into which the condenser fits exactly). In view of the boundary condition (1) the value of D_n in the condenser is equal to that in its surroundings and hence also equal to the value of D_n which prevailed before the introduction of the condenser at the point in question. We can now measure D_n on our condenser directly as the surface charge ω on that coating toward which the prescribed direction n points.

In this manner the displacement D also becomes, in a sense, an "observable quantity."

C. The Concept of Electric Polarization; the Clausius-Mossotti Formula

We give up temporarily the purely phenomenological point of view of Maxwell's theory and attempt to construct a molecular model of the dielectric. A molecule consists of positive and negative charges (protons and electrons), but acts as a neutral entity in the absence of a field. With the application of the field the charges are separated and form a dipole.² The induced moment m is proportional to the external field and is a characteristic of the molecule.

Such a moment has the dimension charge · lever arm = QM . If we pass from the single molecule to the sum of the molecular moments "per unit volume" the dimension

$$\frac{QM}{M^3} = \frac{Q}{M^2}$$

¹ If the field is inhomogeneous the condenser must be made adequately small. Its metallic coatings distort the field, but do not disturb the measurement of D_n between the plates.

² We think here of *nonpolar molecules*. The *polar molecules*, which have been studied with great success, both experimentally and theoretically, by Debye, have a dipole even in the absence of a field. The formulas of the text would have to be altered for polar molecules and would show a dependence on temperature. For details see Debye: *Polar Molecules*, Dover, New York, 1945. In analogy to paramagnetism, polar molecules may be called *paraelectric*.

is obtained. It corresponds to the dimension (2.4) of \mathbf{D} . We shall designate this sum $\Sigma \mathbf{m}$ divided by the volume with \mathbf{P} and call it the *polarization*.¹

We shall divide \mathbf{D} into one part which is present even in the absence of the molecules, and another part which is produced by the molecules. The first corresponds to the case of vacuum and is $\mathbf{D}_0 = \epsilon_0 \mathbf{E}$, where \mathbf{E} is the applied, macroscopically measurable field; the second is our polarization \mathbf{P} . Hence we write

$$\mathbf{D} = \mathbf{D}_0 + \mathbf{P} = \epsilon_0 \mathbf{E} + \mathbf{P}. \quad (3)$$

\mathbf{D} is the macroscopically measurable excitation and is hence equal to $\epsilon \mathbf{E}$. Accordingly (3) leads to

$$\mathbf{P} = (\epsilon - \epsilon_0) \mathbf{E}. \quad (4)$$

We also wish to determine \mathbf{P} from the behavior of the molecules in the electric field. In agreement with the notation of Eqs. (9.9)ff. we call this field \mathbf{F} and indicate by this that it differs from the macroscopic field \mathbf{E} . The difference between them results from the effect of the polarized molecules according to the formula

$$\mathbf{F} = \mathbf{E} + \frac{1}{3} \frac{\mathbf{P}}{\epsilon_0}. \quad (5)$$

In order not to interrupt our train of thought we defer proof of this expression to section *D*. The moment \mathbf{m} acquired by the individual molecule is proportional to this \mathbf{F} . We put

$$\mathbf{m} = \alpha \epsilon_0 \mathbf{F}, \quad (6)$$

where α is a constant characteristic of the molecule. Here the molecule is assumed to be isotropic; otherwise \mathbf{m} and \mathbf{F} would not have to have the same direction.

If N is the number of molecules per unit volume, we obtain from (6) and (5):

$$\mathbf{P} = \sum_{\text{unit volume}} \mathbf{m} = N \alpha \epsilon_0 \mathbf{F} = N \alpha \left(\epsilon_0 \mathbf{E} + \frac{1}{3} \mathbf{P} \right). \quad (7)$$

If we substitute here expression (4) for \mathbf{P} and cancel the common factor \mathbf{E} , we find

$$\epsilon - \epsilon_0 = N \alpha \left(\epsilon_0 + \frac{\epsilon - \epsilon_0}{3} \right) = \frac{N \alpha}{3} (\epsilon + 2 \epsilon_0)$$

or, if we pass to the relative dielectric constant ϵ/ϵ_0 :

¹ Every individual moment has the direction of its lever arm as axis. Hence the direction of \mathbf{P} is obtained by the geometric addition of all the \mathbf{m} in the volume considered and passing to the limit of a sufficiently small volume.

$$\frac{\epsilon_{\text{rel}} - 1}{\epsilon_{\text{rel}} + 2} = \frac{N\alpha}{3} \quad (8)$$

This is the Clausius-Mossotti formula. In optics, where ϵ_{rel} is the square of the refractive index, it is known as the Lorenz-Lorentz formula.

To clarify the physical content of Eq. (8), we multiply numerator and denominator with m , the mass of the individual molecule. Thus we obtain, in the product Nm , the mass of unit volume of the dielectric or its *density* and, at the same time, in the quotient α/m , a new constant characteristic of the molecule. Then in Eq. (8) the left side is *proportional to the density*. This assertion can be tested directly on compressed gases for which ϵ_{rel} differs appreciably from 1. For highly diluted gases, where $\epsilon_{\text{rel}} \sim 1$, $\epsilon_{\text{rel}} + 2 \sim 3$, Eq. (8) leads to

$$\epsilon_{\text{rel}} - 1 = N\alpha = \rho \frac{\alpha}{m}. \quad (8a)$$

Historically it may be mentioned that Mossotti, in his paper dating as far back as 1850, treated the molecules as *conducting* spheres, which were assumed to be distributed in some fashion in the imponderable "ether." We know from §9 that an external field F induces a moment M in such a sphere whose magnitude is given by (Eq. (9.10a))

$$M = 4\pi\epsilon_0 a^3 F.$$

Our molecular constant α has then, according to the definition (6), the value

$$\alpha = 4\pi a^3. \quad (9)$$

If this expression is substituted in (8) the right side becomes

$$\frac{4\pi}{3} a^3 N. \quad (9a)$$

This is simply the ratio of the volume of the spheres contained in unit volume to unit volume (dimensionless, as it should be).

We summarize what we have learned about the concept of dielectric displacement which was inadequately explained in §2. D is composed of two parts, a vacuum portion $D_0 = \epsilon_0 E$ and a portion arising from matter P :

$$D = D_0 + P. \quad (10)$$

We call P the polarization of the matter; it is also the electric moment per unit volume of the dielectric. Similarly, the dielectric constant is made up of two parts, its vacuum component ϵ_0 and its component arising from matter $\epsilon_0 \eta$:

$$\epsilon = \epsilon_0(1 + \eta). \quad (11)$$

The material constant η , which is defined as a pure number, is called the *electric susceptibility*. It is pleasant to note that in both Eqs. (10) and (11) the factor 4π , which otherwise occurs in \mathbf{P} and η , is absent as the result of the rational character of our MKSQ-system. In determining \mathbf{P} and η we had to differentiate between the field strength \mathbf{F} acting on the molecule and the macroscopically defined field strength \mathbf{E} ; the difference between them arises from the field of the neighboring molecules.

As already noted on p. 9, the designation "dielectric displacement" for \mathbf{D} really fits only its polarization component \mathbf{P} . In vacuum there is no charge which can be displaced and \mathbf{D} is nevertheless by no means equal to zero. This is the reason why we have preferred (see p. 8) the term "excitation" for \mathbf{D} .

D. Supplement to the Calculation of the Polarization

We are here concerned with the proof of Eq. (5). We consider an arbitrary molecule, surround it with a small sphere whose radius b is nevertheless very great compared to the molecular radius, and remove from the sphere all matter except the one molecule at the center; this molecule is therefore in vacuum. It is thus acted upon by the field strength \mathbf{E} , corresponding to the first term on the right side of (5). The removal of the molecules from the interior of the sphere does not result in a change of the field resulting from the matter present, provided that the molecules were distributed randomly, i.e. that they were not oriented in any way by the molecule under consideration. If we limit ourselves to isotropic dielectrics we can assume this.¹

After exclusion of this sphere we can treat the remaining dielectric as a continuous medium, i.e. neglect its molecular structure and proceed according to Maxwell's phenomenological theory. Hence we shall replace the action of the residual dielectric by charge densities ω on the elements $d\sigma$ of the inner bounding sphere of the cavity of radius b ; here we shall have to determine ω not from the complete \mathbf{D} , but only from its molecular component \mathbf{P} in Eq. (3). Since \mathbf{P} differs from zero only for $r > b$ (in vacuum, for $r < b$, $\mathbf{P} = 0$), ω is not given by the difference of two \mathbf{P} -values (as in (3.11) by that of two \mathbf{D} -values), but directly by $\omega = P_n$. In view of the fact that \mathbf{P} has the same direction as the primary field \mathbf{E} , which shall have the x -direction, we find

$$\omega = P_n = P_x \cos \theta \text{ with } \theta = \text{angle of } n \text{ with respect to } x. \quad (12)$$

According to Coulomb's law (7.7) the contribution of $\omega d\sigma$ to the field strength acting on our molecule at $r = 0$ in the direction of the radius vector is

¹ H. A. Lorentz has proved this also for crystals of cubic structure; for other symmetries, as well as for associating liquids, the assumption in the text is unproved.

$$dF = \frac{\omega d\sigma}{4\pi\epsilon_0 b^2} = \frac{P_z \cos \theta}{4\pi\epsilon_0 b^2} d\sigma \quad (13)$$

and its z -component, with which alone we are concerned,

$$dF_z = \frac{P_z \cos^2 \theta}{4\pi\epsilon_0 b^2} d\sigma. \quad (13a)$$

Integration over the whole sphere, with $d\sigma = b^2 \sin \theta d\theta d\phi$ leads to

$$F_z = \frac{P_z}{4\pi\epsilon_0} \cdot 2\pi \int_0^\pi \cos^2 \theta \sin \theta d\theta = \frac{1}{3} \frac{P}{\epsilon_0}. \quad (14)$$

This corresponds to the second term of the right side of Eq. (5), which is proved herewith.

E. Permanent Polarization

We have assumed so far that the polarization is caused by an external field and vanishes with it. That is not the case in general. We have already noted (p. 73, footnote 2) that permanent electric moments exist on a *molecular* scale. It is true that they compensate each other, particularly in the liquid or gaseous state, because of the thermal disorder in *any finite volume*, so that here also the resulting polarization vanishes with the external field E . However, if a substance made up of such polar molecules (a wax or resin) is liquefied by heating and exposed to a strong electric field, the latter forces the molecular moments largely into its direction. After solidification the substance retains its polarization for a time even if the field is subsequently removed. If the environment could be made completely insulating a substance would be obtained with a macroscopically *permanent electric field*.

Heaviside has christened a substance treated in this manner with the rather forced name "electret," in view of its analogy to the permanent magnet.

The assumption of a completely insulating environment is, however, never satisfied. Even pure, highly diluted air is, because of radioactive emanation and, in particular, because of cosmic radiation, always somewhat ionized and hence conducting. An electret hence tends to lose its effectiveness to the outside in the course of hours or days.

There are however also natural substances with similar properties. We find these among crystals which are asymmetric in structure (crystals with a polar axis). The most familiar example is *tourmalin*. A crystal is in general made up of positively and negatively charged ions which, if there is an imperfect symmetry of structure, have an electric moment in any elementary domain. Depending on the lattice of the crystal, the elementary moments may combine to form a macroscopic moment, which then produces an electric field in its neighborhood.

Such a field can in fact be detected on *fresh fragments* of tourmalin. Since the environment, as noted above, never insulates perfectly, the field decays in the course of a few hours. Surface charges are built up by the conduction currents at the entrance and exit points of the lines of force which then compensate the external field of the interior electric moment. The difference between the electric and magnetic permanent moment consists merely in the fact that there are no such conduction currents in the magnetic field. Hence a steel magnet shows no appreciable change in its field in the course of decades. This difference between electret and magnet is not fundamental, but only quantitative.

The polar asymmetry existing in tourmalin may be produced artificially in other, less asymmetric¹ crystals by subjecting them to a deformation. This distorts the crystal lattice and impresses an electric moment proportional to the deformation. The crystal thus becomes *piezoelectric*. Quartz is the typical representative of this class of substances. It was used by Pierre Curie, as "piezo-quartz," to produce well-defined quantities of electric charge. It is true that here also, because of imperfect insulation, the electric charge decays with a certain finite relaxation time. Of even greater importance in more recent times has been the role of the quartz crystal when vibrating with its characteristic frequency and producing a corresponding oscillatory electric field. Inversely, by applying an alternating field of this frequency it is possible to maintain the characteristic vibration of the quartz at a constant amplitude. In this manner one obtains (Cady) an ideal microscale of time, which plays its well known role in present-day radio engineering.

In all these cases (electret, tourmalin, piezoquartz) the external electric field may be calculated from the inner moment which is assumed to be known. We will omit this, however, since the calculation is quite similar to that of the external field of a permanent magnet, which is carried out below.

§12. *The Field of the Permanent Bar Magnet*

The forces which emanate from certain forms of iron have excited the popular imagination since the earliest times. The Greeks called the carriers of such effects *magnets*.² The Chinese were the first to utilize their interaction with the great magnet "Earth" for geographic orientation on the

¹ The degree of the required asymmetry may be predicted exactly by means of the general rules of Voigt. See Vol. II, §40.

² Apart from *steel* the metals *cobalt* and *nickel*, which are related to iron, show permanent magnetism, similarly the *Heusler alloys*, containing manganese, which adjoins iron in the periodic system. The iron ore $\text{Fe}_3\text{O}_4 \cdot \text{FeO}$ which, crystallizes in a cubic lattice, is known as *magnetite*, the hexagonal FeS (with admixture of Fe_2S_3), as *pyrrhotin* or *magnetic gravel*; both are characterized by permanent magnetism and magnetic anisotropy.

broad expanses of their country. In the 18th Century it was fashionable to attribute all mysterious processes in the human body to "animal magnetism" (Mesmer). Today the importance of the natural or *permanent magnets* is outdistanced by that of the *electromagnets*. In spite of this we shall begin with some consideration of permanent magnets; we shall then be in a position to cover briefly the general properties of the magnetostatic field, in analogy to those of the electrostatic field in §13.

It is true that the nature of the permanent magnets lies outside the range of the Maxwell theory and can be understood only with the aid of atomic physics. It is based on the spin of the electron and its magnetic moment, of which the Maxwell theory is, of course, ignorant. The same remark applies eventually to the electromagnet: the electric currents which produce the electromagnet, unless they are generated electro-dynamically by induction, have their origin in electrochemical processes, which are foreign to the Maxwell theory; only the magnetic fields proceeding from the electromagnet are described by the latter. In similar manner, the fields proceeding from permanent magnets fit into the framework of the Maxwell theory.

We commence with the *magnetization*, which we shall call \mathbf{M} or, to begin with, \mathbf{M}^* , as counterpart to the electric polarization \mathbf{P} ; the analog of our equation of definition (11.3) for \mathbf{P} ,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}. \quad (1)$$

would be, from our point of view,

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} + \mathbf{M}^*. \quad (1a)$$

(Here \mathbf{H} and \mathbf{M}^* are "quantities" like \mathbf{D} and \mathbf{P} , \mathbf{B} is an "intensity" like \mathbf{E} ; $1/\mu_0$ corresponds to ϵ_0 , as emphasized in Eq. (4.7)). Solved for \mathbf{B} , Eq. (1a) yields

$$\mathbf{B} = \mu_0(\mathbf{H} - \mathbf{M}^*). \quad (1b)$$

The customary definition of magnetization is, on the other hand, contained in the equation

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \quad (2)$$

which we shall utilize from here on. We shall return to Eq. (1b) in §13D, in discussing diamagnetism. As defined by (2), \mathbf{M} signifies the *part of the excitation derived from matter* and is at the same time the *sum*, referred to unit volume, of the *moments of elementary magnets*, just as \mathbf{P} was a corresponding sum of electrical elementary moments. In the following we shall imagine the distribution of \mathbf{M} within the magnet to be given arbitrarily and shall calculate from the Maxwell equations the corresponding fields of the vectors \mathbf{B} and \mathbf{H} .

We know from §7 that \mathbf{H} is throughout *lamellar*, \mathbf{B} throughout *solenoidal* (free of sources). In view of the absence of sources of \mathbf{B} Eq. (2) leads to the condition

$$\operatorname{div} \mathbf{H} = -\operatorname{div} \mathbf{M}. \quad (3)$$

At a surface of discontinuity of \mathbf{M} the Eq. $\operatorname{div} \mathbf{B} = 0$, which presupposes that \mathbf{B} is continuous and differentiable, is replaced by the condition that the "surface divergence" of \mathbf{B} vanishes; i.e.

$$B_n + B_{n'} = 0, \quad (3a)$$

where n and n' , as in (3.7), denote the normals of the surface of discontinuity pointing toward opposite sides. Applied to the surface of a magnet, at which \mathbf{M} jumps from the external value $\mathbf{M} = 0$ to a value of \mathbf{M} which, in general, differs from zero, (3a) yields in view of (2)

$$H_n + H_{n'} = -M_n. \quad (3b)$$

If, now, $\mathbf{H} = -\operatorname{grad} \Psi$ is substituted in Eqs. (3) and (3b), the differential equation of the problem

$$\Delta \Psi = \operatorname{div} \mathbf{M}, \quad (4)$$

and the surface condition

$$\frac{\partial \Psi}{\partial n} + \frac{\partial \Psi}{\partial n'} = M_n \quad (4a)$$

are obtained. We add, as boundary condition at infinity

$$\Psi = 0. \quad (4b)$$

Since, in the two Eqs. (4) and (4a), the right sides can be assumed to be known, we are dealing here not with a boundary value problem, but, in the sense of §7, Eqs. (10) and (10a), with a simple summation problem. The solution is

$$4\pi\Psi = -\int \frac{\operatorname{div} \mathbf{M}}{r} d\tau - \int \frac{M_n}{r} d\sigma. \quad (5)$$

The first term sums all magnetic volume densities ρ_m in the *interior* of the magnet, the second all surface densities ω_m on its *boundary*. The negative signs result from the fact that, according to (4) and (4a)

$$\rho_m = -\operatorname{div} \mathbf{M}, \quad \omega_m = -M_n.$$

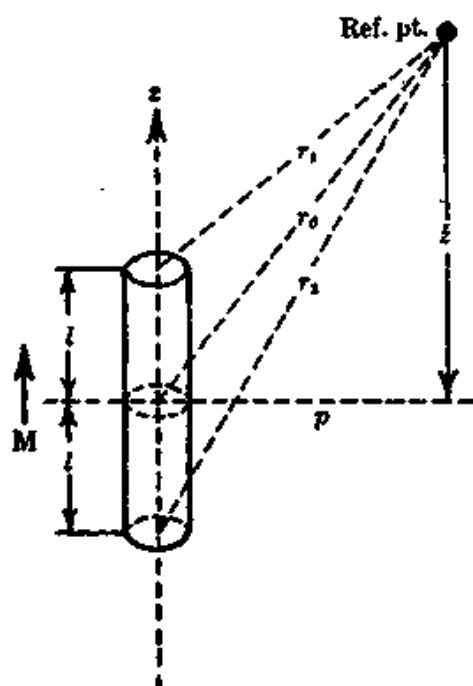
(If we had continued to employ $\mathbf{M}^* = -\mathbf{M}$ in our calculation, the signs would have been positive, as in electrostatics.) We consider two special cases: *a.* homogeneous magnetization parallel to the bar axis, *b.* magnetization increasing from zero toward the center, also parallel to the bar axis.

For *a* the first integral on the right in Eq. (5) vanishes because of $\text{div } \mathbf{M} = 0$, for *b*, the second one because of $M_n = 0$.

a. Only the two end surfaces contribute to the surface integral in (5) since the normal component of \mathbf{M} vanishes, by assumption, on the sides. The pole strengths of the magnet are thus, in a sense, uniformly distributed over the end surfaces; if the cross-section area is F the two total pole strengths are $\pm P = \pm FM$.

Approximate integration of (5) for *large* distance of the reference point from the magnet (see Fig. 14: p , r_1 , r_2 distance of reference point from bar axis and from the centers of the end surfaces, respectively, z , coordinate

FIG. 14. Bar magnet, magnetized longitudinally. Point of reference on the outside.



of the reference point parallel to the bar axis, measured from its center, $2l$, length of bar) leads to

$$4\pi\Psi = -MF \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \text{ with } \begin{aligned} r_1^2 &= (z - l)^2 + p^2 \\ r_2^2 &= (z + l)^2 + p^2 \end{aligned} \quad (6)$$

Series expansion yields

$$\begin{aligned} \frac{1}{r_1} &= \frac{1}{r_0} \left(1 + \frac{1}{2} \frac{2zl + l^2}{r_0^2} + \dots \right), \\ \frac{1}{r_2} &= \frac{1}{r_0} \left(1 - \frac{1}{2} \frac{2zl + l^2}{r_0^2} + \dots \right), \end{aligned}$$

with $r_0 = \sqrt{z^2 + p^2}$ = distance of reference point from center of bar. Hence, according to (6),

$$4\pi\Psi = -2lP \frac{z}{r_0^3} = 2lP \frac{\partial}{\partial z} \frac{1}{r_0}. \quad (6a)$$

As was to be expected, the external action of the magnet for distances $\gg 2l$ is that of a dipole with the lever arm $2l$ and pole strength $P = MF$.

The same representation by the surface integral in Eq. (5) applies for the interior of the bar, but this requires a more careful evaluation. For the sake of brevity we limit ourselves to the center line of the bar and

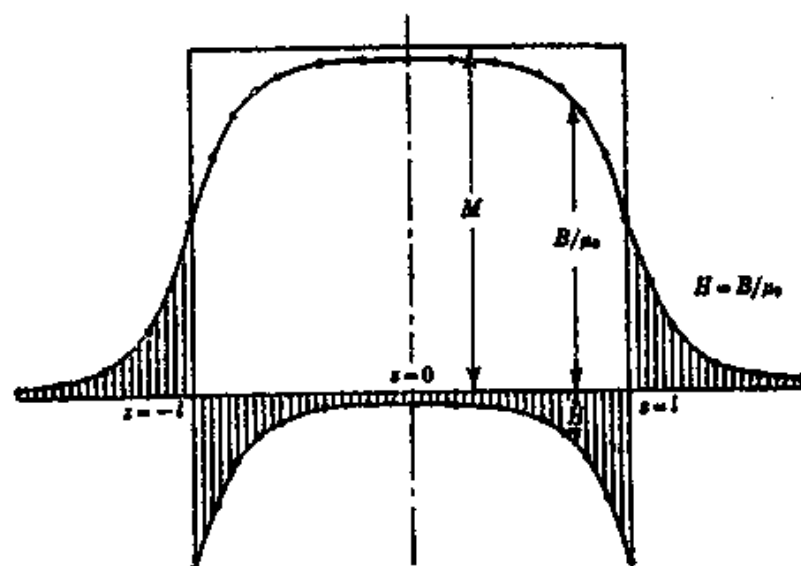


FIG. 15. Demagnetization of a uniformly magnetized bar. In the drawing it is assumed that $a = l/4$.

assume the cross section to be circular (a = radius, ρ = distance of the point of integration from the center of the end surfaces). We then find

$$4\pi\Psi = 2\pi M \left\{ \int_0^a \frac{\rho d\rho}{r_1} - \int_0^a \frac{\rho d\rho}{r_2} \right\}, \quad (7)$$

$$r_1^2 = (l - z)^2 + \rho^2, \quad r_2^2 = (l + z)^2 + \rho^2.$$

The evaluation yields, since $0 \leq |z| \leq l$:

$$\Psi = \frac{M}{2} [\{ (l - z)^2 + a^2 \}^{\frac{1}{2}} - \{ (l + z)^2 + a^2 \}^{\frac{1}{2}} + 2z], \quad (7a)$$

$$H = -\frac{\partial\Psi}{\partial z} = \frac{M}{2} \left[\frac{l - z}{\{ (l - z)^2 + a^2 \}^{\frac{1}{2}}} + \frac{l + z}{\{ (l + z)^2 + a^2 \}^{\frac{1}{2}}} - 2 \right] \quad (7b)$$

Since l is in any case many times larger than a , we obtain

$$H \sim 0, \quad \frac{\partial H}{\partial z} \sim 0 \quad \text{for } z = 0$$

$$H = -\frac{M}{2}, \quad \frac{\partial H}{\partial z} = \mp \frac{M}{2} \frac{1}{a} \quad \text{for } z = \pm l;$$

as shown in Fig. 15, there is a sharp decrease of $-H$ at the two ends of the bar and a vanishing of a high order at the center of the bar. The sign of H

is opposite to that of \mathbf{M} ; \mathbf{H} has what is commonly expressed as a "demagnetizing action." This is seen also in the pattern of \mathbf{B} . Though \mathbf{B} attains almost the full magnitude $\mu_0 \mathbf{M}$ at the center of the bar, it is only half as large at the ends, the same of course, inside and outside of the bar.

b. Let \mathbf{M} be constant in every cross section of the bar, but be dependent on z in such fashion that \mathbf{M} vanishes at the ends $z = \pm l$ and increases parabolically toward the center:

$$M = \frac{C}{2} \left(1 - \frac{z^2}{l^2}\right), \quad \text{div } \mathbf{M} = -\frac{Cz}{l^2}. \quad (8)$$

As already mentioned, the surface integral in (5) vanishes here, and only the volume integral remains to be calculated.

With ξ, ρ, ψ for the coordinates of the point of integration, and z, p, φ for the coordinates of the reference point Eq. (5) yields:

$$4\pi\Psi = \frac{C}{l^2} \iiint \frac{\xi}{r} d\xi d\sigma, \quad (9)$$

$$r^2 = (z - \xi)^2 + p^2 + \rho^2 - 2p\rho \cos(\varphi - \psi), \quad d\sigma = \rho d\rho d\psi.$$

The field at a distance here, just as in case *a*, is that of a dipole. If the average magnetization \bar{M} is computed for the length of the bar and if as in *a*, we put $\bar{M}l = P$, the moment of the dipole field becomes, as in (6a), $2lP$.

Within the bar, in particular on the bar axis $p = 0$, we obtain from (9), by carrying out the integrations with respect to ρ and ψ

$$\begin{aligned} \Psi &= \frac{C}{2l^2} \int_{-l}^{+l} \xi d\xi \left[\{(z - \xi)^2 + a^2\}^{\frac{1}{2}} - |z - \xi| \right] \\ &= \frac{C}{2l^2} \int_{-l}^{+l} \xi d\xi \{(z - \xi)^2 + a^2\}^{\frac{1}{2}} - \frac{C}{2l^2} \left(\frac{1}{3} z^3 - zl^2 \right) \end{aligned} \quad (10)$$

and

$$\begin{aligned} H &= -\frac{\partial \Psi}{\partial z} = \frac{C}{2l^2} \int_{-l}^{+l} \xi d\xi \frac{\partial}{\partial \xi} \{(z - \xi)^2 + a^2\}^{\frac{1}{2}} + \frac{C}{2l^2} (z^2 - l^2) \\ &= \frac{C}{2l} \left[\{(z - l)^2 + a^2\}^{\frac{1}{2}} + \{(z + l)^2 + a^2\}^{\frac{1}{2}} + \frac{z^2 - l^2}{l} \right] \\ &\quad - \frac{C}{2l^2} \int_{-l}^{+l} |(z - \xi)^2 + a^2|^{\frac{1}{2}} d\xi. \end{aligned} \quad (11)$$

The symbol $||$ in the last integral indicates that the sign of this square-root, just as that of the preceding ones, is to be positive.

We will show that, just as in the case of uniform magnetisation, \mathbf{H} is *nearly zero* (of the order a/l) everywhere except at the ends of the bar. Ex-

clusion of the bar ends signifies $a \ll l - |z|$. We may then neglect a in (11) and obtain

$$H = \frac{C}{2l} \left[l - z + l + z + \frac{z^2 - l^2}{l} - \frac{z^2 + l^2}{l} \right], \quad (11a)$$

i.e. zero (more exactly, vanishing to the order Ca/l). For $z = \pm l$, i.e. at the bar ends, H is small to the same order of magnitude. This does not apply, however, to dH/dz . This is given by¹

$$\frac{dH}{dz} = \frac{C}{2l} (1 + \dots). \quad (12)$$

Thus H has *positive* values, appreciably differing from zero, only in a small region near the two ends of the bar. This is illustrated by Fig. 16.

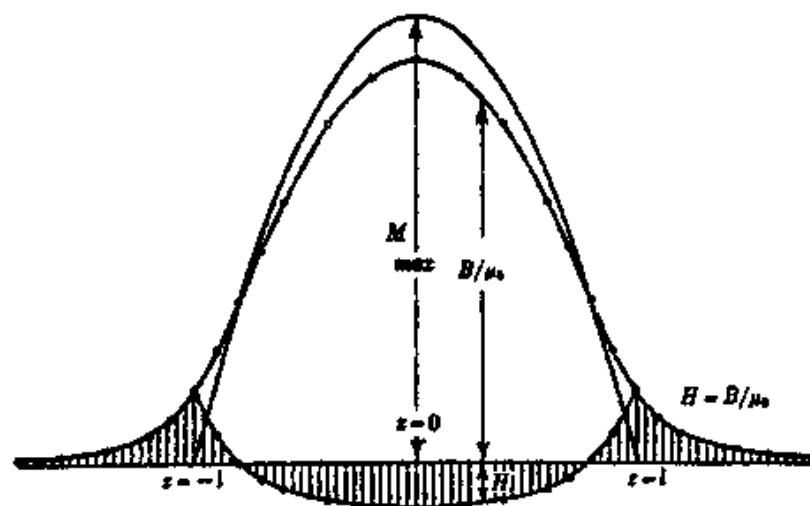


FIG. 16. Demagnetization of a bar magnetized according to the formula $M = \frac{C}{2} \left(1 - \frac{z^2}{l^2} \right)$. As in Fig. 15, $a = l/4$.

Also here H has a "demagnetizing action," i.e. is opposite to the impressed moment M . B/μ_0 approximates the full value of M in the whole middle portion of the bar and deviates from it slightly only at the ends.

The fact that H always acts in the direction of demagnetization (for arbitrary distribution of the magnetization and for any shape of the magnet) may be recognized from the following: The lines of force (B -lines) are closed

¹ Differentiation of (11) leads to the exact elementary formula

$$\frac{dH}{dz} = \frac{C}{2l^2} \left\{ \frac{z(z-l) + a^2}{\sqrt{(z-l)^2 + a^2}} - \frac{z(z+l) + a^2}{\sqrt{(z+l)^2 + a^2}} + 2z \right\}$$

from which (12) is readily derived. The dots in (12) indicate terms of the order a/l . The upper positive sign in (12) refers to $z = +l$ and positive dz ; the gradient of H toward the *interior* of the bar (negative dz) is thus *negative*, just as at the other end of the bar.

because of $\text{div } \mathbf{B} = 0$ and part of their path lies in the interior, and part lies in the region outside of the magnet. We carry out a line integral of \mathbf{H} over such a closed line of force in the positive \mathbf{B} -direction. Then we obtain, because of the lamellar character of \mathbf{H} , as for any closed path,

$$\oint \mathbf{H} \cdot d\mathbf{s} = 0. \quad (12a)$$

The part of the integral over the path *outside* of the magnet, where the directions of \mathbf{H} and \mathbf{B} coincide, is *positive*; hence the part of the integral over the path *within* the magnet must be negative:

$$\int_{\text{inside}} \mathbf{H} \cdot d\mathbf{s} < 0 \quad (12b)$$

On the other hand, the integral of \mathbf{B} over the same part is, by assumption, positive. Eq. (2) shows that this applies even more to the integral of \mathbf{M} :

$$\int_{\text{inside}} \mathbf{M} \cdot d\mathbf{s} = \frac{1}{\mu_0} \int_{\text{inside}} \mathbf{B} \cdot d\mathbf{s} - \int_{\text{inside}} \mathbf{H} \cdot d\mathbf{s} > 0. \quad (12c)$$

The two inequalities (12b, c) for \mathbf{H} and \mathbf{M} show together that \mathbf{H} has within the magnet, along any line of force, on the *average* the *opposite* direction of \mathbf{M} .

We have dealt with the preceding rather arbitrarily selected problem in such detail because most textbooks contain little of a quantitative nature regarding the vectors \mathbf{B} , \mathbf{H} , and \mathbf{M} in the interior of a magnet. We have seen that if \mathbf{M} is known \mathbf{H} may in principle be evaluated, in accord with the rules of potential theory, by a simple summation, whereupon \mathbf{B} is also known. It is true that the assumption of a known distribution of \mathbf{M} is not fulfilled in practice. Bar magnets are therefore unsuited for a practical study of ferromagnetism: we shall return to this later.

Although our calculation was limited to the center line of the bar magnet, Figs. 17 and 18 give quantitative information¹ regarding the shape of the lines of force and lines of excitation throughout the interior of a uniformly magnetized bar magnet: The \mathbf{B} -lines are drawn into the interior, the \mathbf{H} -lines pushed out of it. On the outside the two sets of lines coincide, of course, since $\mathbf{B} = \mu_0 \mathbf{H}$.

The *ring magnet*, provided with a narrow gap, is both simpler and of greater practical importance than the bar magnet. Because of the equivalence of all cross sections the magnetization may here be regarded as uniform, so that $\text{div } \mathbf{M} = 0$, and \mathbf{M} is everywhere parallel to the center line.

¹ These figures were kindly prepared by Prof. J. Jaumann, by a graphical method which was developed by Maxwell for the numerous line-of-force patterns at the end of his *Treatise* and which is widely employed by electrical engineers; see also art. 123 of the *Treatise*.

A "magnetic coating" exists only on the gap faces; between them a magnetic field is formed which is similar in geometry to the electric field in a plate condenser. This applies not only to a permanent ring magnet, but almost identically to the ring-shaped electromagnet with an iron core.

Referring once more to Eq. (12a), we consider the line integral of the excitation H carried out over the center line of the ring. It may be divided into two parts, the short section through the air gap of thickness a , which we shall traverse in the positive direction of the magnetic condenser field

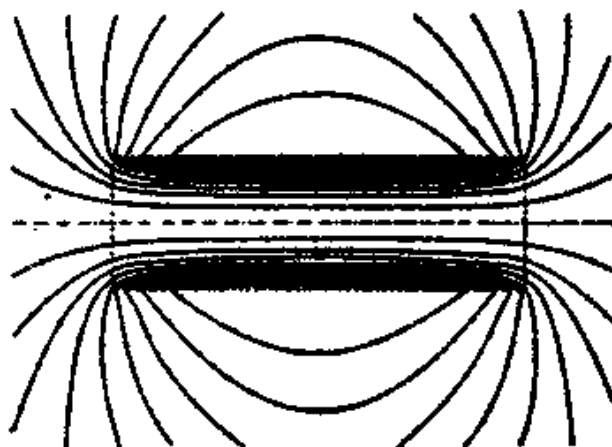


FIG. 17. Lines of force of a uniformly magnetized bar magnet; they are drawn into the interior.

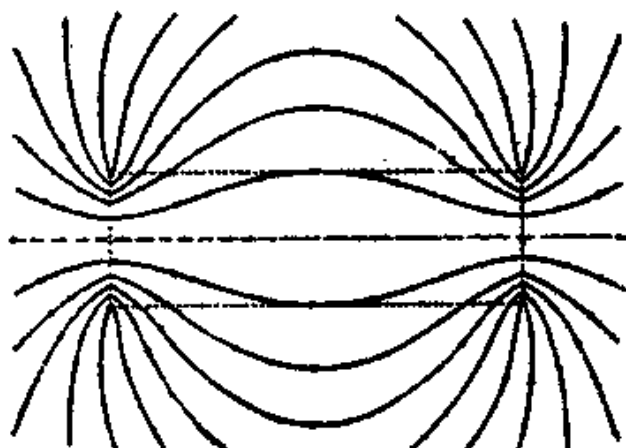


FIG. 18. Lines of excitation of a uniformly magnetized bar magnet; they are pushed out of the interior.

which exists there, and the long part through the ring-shaped iron core of length l , on which the integration is to be carried out in the same sense. We put $H = H_0$ in the gap and $H = H_r$ in the iron core, respectively. Eq. (12a) yields

$$l \cdot H_r = -a \cdot H_0.$$

Thus a "demagnetization" of the iron core goes with the magnetization of the air gap. We have here the same state of affairs as in Figs. 15 and 16, only in a much simpler and more obvious form.

We return once more to the bar magnet and to the definition of its pole strength P . From our point of view it is a *quantity* of magnitude

$$P = \oint H_n d\sigma, \quad (13)$$

where the closed surface σ envelops, starting from the center of the bar, the one or the other half of the bar in arbitrary manner. This definition of P corresponds to the definition (7.2) of the magnetic volume density $\rho_m = \text{div } \mathbf{H}$ and states that P is equal to the sum of all magnetic quantities $\rho_m d\tau$ which are present in the half of the bar in question.

In contrast to this the pole strength is often defined as *intensity* in the literature and described in terms of the magnetic flux. We shall call the pole strength so defined \bar{P} and write

$$\bar{P} = \int B_n d\sigma \quad (14)$$

The surface σ cannot now be closed since otherwise, in view of $\text{div } \mathbf{B} = 0$, $\bar{P} = 0$. Rather, the cross section q passing through the center of the bar must be *excluded* from the integration; or, as an alternative, the integration is carried out *only* over this cross section with reversed sign of the normal n :

$$\bar{P} = \int B_n dq. \quad (14a)$$

It may readily be shown with the aid of (13) and (14) that then, very nearly,

$$\bar{P} = \mu_0 P \quad (15)$$

where μ_0 is the permeability of the surroundings.¹ In fact, we convinced ourselves above that \mathbf{H} very nearly vanishes in the central cross section, so that the closed integration in (13) may be replaced by the open integration in (14), where we may put $H_n = B_n/\mu_0$. The definitions (13) and (14) would thus be practically equivalent in the presence of a plane of symmetry (which, incidentally, exists also for the horseshoe magnet). However, for asymmetric shape or asymmetric magnetization relation (14) fails and only definition (13) remains meaningful. It is also recommended by the fact that it exactly corresponds to the definition of charge:

$$e = \oint D_n d\sigma.$$

¹ While P , according to (13), may be regarded as an *internal* property of the magnet, \bar{P} also depends, according to (15), on its surroundings. This evidently arises from the fact that an environment differing from vacuum contains itself magnetic moments which, quite understandably, are included in \bar{P} . See in this respect Phys. Z. 1935, p. 424.

§13. General Considerations on Magnetostatics and Its Boundary-Value Problems

While, in §12, we have considered only the proper field of permanent magnets, we shall now approach the behavior of *arbitrary bodies* in an *outside field*, arising from either permanent magnets or electromagnets. The laws which apply here closely parallel those of electrostatics; however, contrary to our classification of the field vectors into entities of quantity and intensity, \mathbf{H} here corresponds to \mathbf{E} , \mathbf{B} to \mathbf{D} . This follows from the familiar fundamental equations, e.g. (7.2), (7.9), and (7.9a)

$$\text{curl } \mathbf{H} = 0, \quad \mathbf{H} = -\text{grad } \Psi, \quad (1)$$

$$\text{div } \mathbf{B} = 0, \quad \text{div } \mathbf{H} = \rho_m \quad (2)$$

and the boundary conditions at an interface:

$$\text{continuity of the tang. comp. of } \mathbf{H} \text{ and the normal comp. of } \mathbf{B}. \quad (3)$$

The line integral of the magnetic excitation, which, by (1), is independent of the path, i.e. depends only on the endpoints A, B , we shall call magnetomotive force and designate by U_{AB} , in analogy to V_{AB} in §2, U_{AB} bears following relation to the magnetic potential:

$$U_{AB} = \int_A^B H_s ds = \Psi_A - \Psi_B. \quad (4)$$

The "loop magnetomotive force" is given by

$$U_{A \rightarrow B} = \oint H_s ds = 0, \quad (5)$$

irrespective of the manner in which the closed path is traversed, whether it passes through magnetized material or through air.

We designate the "induction" or the "flux" through an arbitrarily shaped surface σ by the usual symbol Φ :

$$\Phi = \int B_n d\sigma. \quad (6)$$

In view of (2), Φ depends only on the boundary curve of the surface σ , i.e. is identical for all surfaces passing through the same boundary curve. For a closed surface it vanishes, of course:

$$\Phi = \oint B_n d\sigma = 0. \quad (6a)$$

We shall now discuss briefly the magnetic analogs of the topics dealt with in §§11, 10, and 9.

A. The Law of Refraction of the Lines of Magnetic Excitation

The law of refraction (11.2) of the magnetic lines of *force* may be carried over to the lines of magnetic *excitation*, but applies also to the magnetic lines of *force* because of the identical direction of \mathbf{H} and \mathbf{B} . If the angles α_1 and α_2 have the same meaning as before,

$$\frac{\tan \alpha_1}{\mu_1} = \frac{\tan \alpha_2}{\mu_2}. \quad (7)$$

We prefer to speak here of the lines of *force* because they, unlike the lines of *excitation*, are individually continued into the second medium. We may therefore say: Every individual \mathbf{B} -line is refracted away from the normal in entering the more permeable medium (e.g. $\mu_2 > \mu_1$).

B. Definition of the Vectors \mathbf{H} and \mathbf{B} , Particularly in Solid Bodies

To measure the component of \mathbf{H} in a given direction at a given point by means of an imaginary experiment a short narrow tube must be drilled, to measure \mathbf{B} , a thin slit must be cut. Within the cavity (vacuum or filled with air) so prepared a deflection experiment can be carried out and \mathbf{B} or $\mathbf{H} = \mathbf{B}/\mu_0$, respectively, be determined from the force, so measured, acting on a test body.¹

C. The Magnetization \mathbf{M} in Any Non-Ferromagnetic Substance

As in Eq. (12.2) we define \mathbf{M} by

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \quad (8)$$

and set

$$\mathbf{M} = \kappa \mathbf{H}. \quad (8a)$$

κ is a material constant, the *magnetic susceptibility* of the substance. Physically more significant is the *molar susceptibility*

$$\chi = \kappa M / \rho, \quad (8b)$$

where M is the mass of a mole, the so-called molecular weight of the substance, and ρ is its density. (As we know, the proportionality between \mathbf{M} and \mathbf{H} does not in general apply for ferromagnetic materials). $\mathbf{B} = \mu \mathbf{H}$ and Eqs. (8) and (8a) lead to

$$\frac{\mu}{\mu_0} = 1 + \kappa, \quad \kappa = \frac{\mu - \mu_0}{\mu_0}. \quad (9)$$

¹ A magnet needle, a wire traversed by current, or, eventually, a bismuth spiral; from our standpoint such an experiment yields the intensity \mathbf{B} directly; from it is determined, in the tube experiment, the proportional quantity \mathbf{H} . For details see also §11B.

The conventional form of Eqs. (8) and (9) is marred by the appearance of 4π as factor of \mathbf{M} and κ (see our corresponding remark regarding \mathbf{P} and η in connection with Eq. (11.3)). We must hence note that in the use of experimental data employing conventional notation the factor 4π must be added. That is indicated in the small table in the following section D.

D. Dia- and Paramagnetism

Diamagnetism corresponds to *dielectricity*; like the latter it is a general property of ponderable matter and *independent of temperature*. Both owe their origin to the electronic (and nuclear) structure of matter. *Paramagnetism* occurs only for magnetically polar molecules, i.e. molecules which have a magnetic moment of their own. (Electrically polar molecules were discussed on p. 73.) Paramagnetism is temperature-dependent and hence has a statistical origin. This phenomenon, like ferromagnetism, lies outside of Maxwell's theory. The paramagnetic susceptibility obeys the *law of Curie and Langevin*:

$$\chi = \frac{C}{T}, \quad \begin{array}{l} C = \text{Curie constant} \\ T = \text{absolute temperature} \end{array} \quad (10)$$

This dependence on the temperature indicates that increasing thermal agitation interferes with the alignment of the magnetic moments in the field direction, decreasing thermal agitation favors it.

In the diamagnetic case we have

$$\mu < \mu_0, \quad \kappa < 0 \quad (11)$$

This apparent difference from the dielectric case

$$\epsilon > \epsilon_0, \quad \eta > 0 \quad (11a)$$

is explained in the manner already indicated following Eq. (4.7): the true analog of ϵ is not μ , but $1/\mu$. The magnetic parallel of the statement $\epsilon > \epsilon_0$ is thus

$$\frac{1}{\mu} > \frac{1}{\mu_0}, \quad \mu < \mu_0, \quad \kappa < 0$$

corresponding to (11). The negative sign of \mathbf{M}^* in Eq. (12.1b) is also related to this. For, if we put $\mathbf{M}^* = \kappa^* \mathbf{H}$, the susceptibility κ^* so defined becomes, in view of $\mathbf{M}^* = -\mathbf{M} = -\kappa \mathbf{H}$,

$$\kappa^* = -\kappa > 0$$

in the diamagnetic case corresponding to $\eta > 0$ in (11a). The introduction of \mathbf{M}^* , which was suggested previously but immediately given up above, thus corresponds in fact to the inner relationship of dielectricity and diamagnetism.

For paramagnetism, we have, in contrast with (11),

$$\mu > \mu_0, \quad \kappa > 0. \quad (11b)$$

The numerical value of κ is very small for both paramagnetic and diamagnetic materials. The following represent extreme values:

Paramagnetism	Diamagnetism
$\kappa = +4\pi \cdot 1.8 \cdot 10^{-6}$ for O_2	$\kappa = -4\pi \cdot 0.007 \cdot 10^{-6}$ for N_2
$\kappa = +4\pi \cdot 782 \cdot 10^{-6}$ for Pd	$\kappa = -4\pi \cdot 160 \cdot 10^{-6}$ for Bi

The paramagnetic values refer here to 18°C .

The Clausius-Mossotti law, Eq. (11.8), with ϵ replaced by μ , gives the dependence on density.

E. Soft Iron as Analog to the Electric Conductor

With certain restrictions soft iron may be classified with the paramagnetic substances. The initial value of the permeability (referred to the value for vacuum) is several thousandfold, according to the variety of iron; for increasing H , B approaches a saturation value in the neighborhood of 21,000 gauss. The equation $B = \mu H$ must hence be replaced by the functional relationship $B = B(H)$, as was mentioned already on p. 21.

Just as we noted on p. 61, that the electric conductor corresponds, with respect to the electrostatic boundary conditions and boundary-value problems, to the limiting case $\epsilon \rightarrow \infty$ of a dielectric, so we may regard soft iron ($\mu \rightarrow \infty$) as the magnetostatic analog of the electric conductor. Eq. (7) shows, in fact, that the magnetic lines of force are perpendicular to the surface of soft iron ($\alpha_1 \rightarrow 0$ follows from $\mu_2 \rightarrow \infty$). If two such pieces of soft iron Fe_1 and Fe_2 are placed at different magnetic potentials (e.g. if they are placed on the poles of a horseshoe magnet), the H -lines between them are similar to the E -lines in Fig. 11. Thus, in a sense, a magnetic condenser is obtained.

In engineering applications it is also convenient to introduce the concept of "magnetic resistance" and to employ a "magnetic Ohm's law."

F. Specific Boundary-Value Problems

The methods of solution developed in §9 may be taken over directly into magnetostatics; an example is the imaging at a plane, where it makes no difference whether the induction in halfspace 2 in Fig. 10 is produced by a single pole or a dipole in halfspace 1. The same remark applies for the method of reciprocal radii as applied to a sphere (previously regarded as a conductor, here as consisting of soft iron).

We refer in particular to the sphere in a uniform magnetic field. Within there is a uniform field F_2 , while outside the field becomes nonuniform, through the superposition on the original field F of the field of a virtual

magnetic moment M at the center of the sphere, with its axis in the field direction. ("Field" here denotes "excitation field".) The values of F_2 and M are, by (9.14),

$$F_2 = \frac{3}{\mu + 2} F, \quad M = -\frac{\mu - 1}{\mu + 2} a^3 \cdot 4\pi\mu_0 F. \quad (12)$$

Here a = radius of sphere, $\mu = \mu_2/\mu_0$ = relative permeability of the sphere referred to its surroundings (air). F_2 is stronger than F for diamagnetic substances, weaker for paramagnetic materials. In the interior of soft iron $H \cong 0$, just as in the electric conductor.

G. The Uniform Field within an Ellipsoid of Revolution

The solution (9.13) for the sphere, translated into magnetic terms, becomes, after substitution of the values (12)

$$\Psi_1 = -F \left(r - \frac{\mu - 1}{\mu + 2} \frac{a^3}{r^2} \right) \cos \theta, \quad \Psi_2 = -\frac{3}{\mu + 2} F r \cos \theta. \quad (13)$$

This solution satisfied the boundary conditions

$$\Psi_1 = \Psi_2, \quad \frac{\partial \Psi_1}{\partial r} = \mu \frac{\partial \Psi_2}{\partial r} \quad (13a)$$

because the factor $\cos \theta$, which varies over the sphere, factors out of these equations. We will show that a solution of the same form is valid for the ellipsoid.

In passing from the sphere to the ellipsoid we must first replace the spherical polar coordinates r, θ by corresponding elliptical coordinates, which we shall call u, v . We proceed here from the well-known parametric representation of the ellipse (instructions for Problem II.1), in which we write for the principal axes a and b

$$\begin{aligned} a &= c \cosh u, & b &= c \sinh u, \\ c &= \sqrt{a^2 - b^2} = \text{independent of } u. \end{aligned} \quad (14)$$

Rotation about the long axis (z -axis, angle of rotation φ) produces a family of elongated confocal ellipsoids of revolution, corresponding to Eq. (19.17a) in Vol. II:

$$\frac{z^2}{c^2 \cosh^2 u} + \frac{x^2 + y^2}{c^2 \sinh^2 u} = 1. \quad (14a)$$

Let the ellipsoid considered by us be one of these, namely that with the parameter $u = u_0$. The relation between x, y, z and the elliptical coordinates u, v, φ is the following:

$$\begin{aligned}
 z &= c \cosh u \cos v, \\
 x &= c \sinh u \sin v \cos \varphi, \\
 y &= c \sinh u \sin v \sin \varphi.
 \end{aligned}
 \tag{15}$$

The expression for the line element ds consequently is given by

$$\frac{ds^2}{c^2} = (\cosh^2 u - \cos^2 v)(du^2 + dv^2) + \sinh^2 u \sin^2 v d\varphi^2. \tag{15a}$$

According to the rule (3.9b) in Vol. II the potential equation in these coordinates, becomes:

$$\frac{\partial}{\partial u} \left(\sinh u \sin v \frac{\partial \Psi}{\partial u} \right) + \frac{\partial}{\partial v} \left(\sinh u \sin v \frac{\partial \Psi}{\partial v} \right) = 0 \tag{16}$$

if the potential is independent of the cyclic coordinate φ .
One solution is the uniform field, parallel to the major axis,

$$\Psi = \frac{z}{c} = \cosh u \cos v. \tag{17}$$

We seek a second solution of such form that the factor $\cos v$ which varies over the surface of the ellipsoid corresponding to the factor $\cos \theta$ giving geographic latitude on the sphere, is cancelled out in the boundary conditions. We write for this second solution

$$\Psi = f(u) \cos v \tag{17a}$$

and obtain from (16) for the differential equation

$$\frac{d}{du} (\sinh u f'(u)) - 2 \sinh u f(u) = 0 \tag{17b}$$

which is evidently satisfied by the uniform field $f = \cosh u$. Following a general rule,¹ we place the desired second solution equal to the product of the known first solution and an unknown function $U(u)$:

$$f(u) = \cosh u U(u). \tag{18}$$

The resulting differential equation for $U(u)$

$$U'' + \frac{3 \sinh^2 u + 1}{\sinh u \cosh u} U' = 0$$

can be integrated directly and yields, with A and B as integration constants,

¹ It corresponds to the method of solving an algebraic equation with one known root.

$$U'(u) = \frac{A}{\sinh u \cosh^2 u}, \quad U(u) = \frac{A}{2} \log \frac{\cosh u - 1}{\cosh u + 1} + \frac{A}{\cosh u} + B,$$

$$f(u) = A \left(1 + \frac{\cosh u}{2} \log \frac{\cosh u - 1}{\cosh u + 1} \right). \quad (18a)$$

In the last formula we have omitted the term multiplied with B , since this corresponds to our known solution representing a uniform field.

We now complete our expression (17) for the external field by the addition of (18a) and retain the uniform-field expression (17) for the internal field:

$$\Psi_1 = F \cosh u \cos v + A \left(1 + \frac{\cosh u}{2} \log \frac{\cosh u - 1}{\cosh u + 1} \right) \cos v, \quad (19)$$

$$\Psi_2 = F_2 \cosh u \cos v.$$

We regard the constant F of the external field, which has been added, as known; the two constants A and F_2 are to be determined from the boundary conditions. These are identical with (13a), with dr replaced by du (more exactly, by ds_u , the line element in the direction of the normal to the ellipsoid,

$$ds_u = c \sqrt{\cosh^2 u - \cos^2 v} du,$$

where, however, the square root cancels out in the second Eq. (13a)). The boundary conditions demand hence for $u = u_0$

$$F + A \left(\frac{1}{\cosh u_0} + \frac{1}{2} \log \frac{\cosh u_0 - 1}{\cosh u_0 + 1} \right) = F_2, \quad (19a)$$

$$F + A \left(\frac{\cosh u_0}{\sinh^2 u_0} + \frac{1}{2} \log \frac{\cosh u_0 - 1}{\cosh u_0 + 1} \right) = \mu F_2. \quad (19b)$$

Subtraction yields the somewhat simpler relation

$$A = \cosh u_0 \sinh^2 u_0 (\mu - 1) F_2$$

and substitution thereof in (19a)

$$F_2 \left\{ 1 - (\mu - 1) \sinh^2 u_0 \left(1 + \frac{1}{2} \cosh u_0 \log \frac{\cosh u_0 - 1}{\cosh u_0 + 1} \right) \right\} = F. \quad (20)$$

The field strength F_2 inside is thus expressed in terms of the known strength of the original external, homogeneous field.

If we utilize Eqs. (14) and denote by a, b, c the principal axes and focal distance from the center of our ellipsoid $u = u_0$, we may write instead of (20)

$$F_2 \left\{ 1 - (\mu - 1) \frac{b^2}{c^2} \left(1 + \frac{1}{2} \frac{a}{c} \log \frac{a - c}{a + c} \right) \right\} = F \quad (20a)$$

or, in terms of the numerical eccentricity $e = c/a$ and the magnetic susceptibility $\kappa = \mu - 1$:

$$F_z \left\{ 1 + \kappa \frac{1 - e^2}{e^3} \left(\frac{1}{2} \log \frac{1 + e}{1 - e} - e \right) \right\} = F. \quad (21)$$

We note here that:

a. for $\kappa = 0$ we have of course

$$F_z = F; \quad (21a)$$

b. for $e \rightarrow 0$ series expansion of (21) leads to

$$F_z = F / \left(1 + \frac{\kappa}{3} \right). \quad (21b)$$

The field inside is *weaker* than outside for paramagnetic bodies, stronger for diamagnetic bodies.

c. The same applies for $e \rightarrow 1$. If we put $\eta = 1 - e$, (21) yields

$$F_z = F / \left\{ 1 + \kappa \eta \left(\log \frac{2}{\eta} - 2 \right) \right\}. \quad (21c)$$

In the limiting case b the ellipsoid becomes nearly *spherical*; in fact, (21b) is identical with Eq. (12) for the sphere. In the limiting case c the ellipsoid degenerates to a *thin rod*.

The problem treated here is usually related to a famous formula of Dirichlet for the gravitational potential of a triaxial ellipsoid uniformly filled with matter. This procedure is mathematically more elegant than ours, but is rather indirect. We have preferred the direct method of the magnetic boundary-value problem because it appears to give us more profound insight into the physical conditions.

Our solution of the magnetic problem is of course transferable without change to the corresponding electrostatic problem.

H. The So-Called Demagnetization Factor

The ellipsoid and its limiting forms (sphere, rod) is the *standard shape* of the magnetic test body because it alone possesses a uniform and easily calculable internal field when introduced into an originally uniform external field. For other shapes the determination of the internal field leads to a practically insoluble *boundary-value problem*; the internal field is by no means uniform, but varies from point to point.

It is clear, however, that all questions concerned with the magnetic properties of the material depend on the internal field F_i . This field interacts with the molecular components of the material directly, while the *external* field F has no direct effect on them. Accordingly we may express our Eq. (8a) more precisely by

$$M = \kappa F_i. \quad (22)$$

Such questions become particularly important with *ferromagnetic* materials, where the differences between the external and the internal field (F and F_i in our present notation) are extremely large; for para- and diamagnetic materials they are negligible because of the smallness of κ . Since for ferromagnetic, as for paramagnetic, materials $F_i < F$, we write, with P denoting a numerical factor,

$$F_i = F - PM \quad (22a)$$

or, in view of (22),

$$F_i = F - \kappa P F_i, \quad (22b)$$

$$F_i(1 + \kappa P) = F. \quad (22c)$$

The numerical factor P is a measure for the attenuation of the external field by the presence of the magnetizable material and is hence called the *demagnetization factor*. A comparison of (22c) and (21) yields for its value

$$P = \frac{1 - \epsilon^3}{\epsilon^3} \left(\frac{1}{2} \log \frac{1 + \epsilon}{1 - \epsilon} - \epsilon \right). \quad (23)$$

It is purely geometric in character. For the two limiting cases of (21b) and (21c) we have

$$P = 1/3 \text{ for } \epsilon \rightarrow 0, \text{ sphere,}$$

$$P = \eta \left(\log \frac{2}{\eta} - 2 \right) \rightarrow 0 \text{ for } \eta \rightarrow 0, \text{ rod.}$$

Intermediate values are readily calculated from (23) (log always signifies in this book the natural logarithm) and are tabulated for example, by Kohlrausch.¹

It is clear that this factor P has a legitimate meaning only for the ellipsoid and its degenerate forms, since only here we are dealing with the ratio of one uniform field to another. Clearly the boundary-value problem for other body-shapes cannot be circumvented by the employment of a numerical factor which is guessed in some manner. In practice the procedure consists of measuring the value of B experimentally with an induction coil at some characteristic points (e.g. the center of the test body).

§14. Some Remarks on Ferromagnetism

This section does not pretend to be an introduction to the broad field of ferromagnetism, but merely intends to mention certain important features

¹ F. Kohlrausch, *Praktische Physik*, 12th Ed., p. 540. It should be noted that the factor 4π , by which Kohlrausch's Eq. (4) differs from our Eq. (23), is, with us, included in the definition of κ . See the remark at Eq. (9) above.

which, since they lie outside our subject, will be indicated rather than logically derived. As has already been noted at the beginning of §12, ferromagnetism is not based on Maxwell's phenomenological theory, but on the more profound laws of atomic physics and on the statistical behavior of electrons.

A. The Weiss Domains

The sign of the ferromagnetic susceptibility and its temperature dependence indicate that it, like the paramagnetic susceptibility, results from the alignment of elementary magnets in a magnetic field. The fact that it differs from the paramagnetic susceptibility in order of magnitude shows, however, that we are here dealing not with individual, freely mobile, magnets, but with whole groups of them which, perfectly aligned within the group, have different preferential directions. The individual group is "saturated" internally even in the absence of an external field, while in a macroscopic block of ferromagnetic material saturation occurs only at a field excitation of 10 to 1000 oersted.

This concept of the ferromagnetic state is illustrated by a model constructed by Ewing. Magnet needles are arranged in a lattice on a board. With the earth's field compensated by a current loop, they arrange themselves in groups or rows in which they are parallel. This state is stable against external disturbances, such as shaking of the board. A weak external field produces only a slight deflection from the equilibrium position since the internal aligning field is much stronger than the external field. Complete alignment in the direction of the external field, i.e. saturation of the entire system, takes place only at a very much higher field strength.

Pierre Weiss has elaborated this interpretation of ferromagnetism in all directions, both experimentally and theoretically, and has, together with Langevin, given it a thermodynamic basis. The individual groups are known as *Weiss domains*. Their size is estimated at about 10^{-6} cm in linear dimension, corresponding to $5 \cdot 10^8$ Fe-atoms. The smallest ferromagnetically active domains are however certainly very much smaller and contain fewer than 100 Fe-atoms.¹ It is tempting to identify them with the single crystals of which a polycrystal of the material is composed on a microscopic scale. However, it is necessary to assume such a subdivision into Weiss domains even for the macroscopic single crystal, since its behavior is qualitatively similar to that of the polycrystal. (It is true that quantitatively the shape of the hysteresis loop, discussed below, differs from that for the polycrystal; it has a rectangular shape.)

¹ See H. König, *Naturwiss.* 1946, p. 1.

B. The Electron Spin as Elementary Magnet

All ferromagnetic materials are conductors of electricity, i.e. contain *free electrons*.¹ We have very definite reasons for regarding these electrons as the elements whose alignment causes ferromagnetism. These are the so-called "gyromagnetic effects": *magnetization by rotation* of a bar of a material of the iron group (J. S. Barnett 1914) and *rotation by magnetization* of a small ferromagnetic rod suspended on a torsion fiber (Einstein and de Haas 1915). In both cases measurements gave as ratio of the mechanical to the magnetic moment *half the value* to be expected if the effects resulted from the electron *orbits* in the atom. It must therefore be concluded that it is not the charge of the revolving electron and the magnetic field produced by it that are responsible, but the inner structure of the electron itself. The electron possesses, apart from its charge, an inner mechanical moment, a "*spin*," and a *magnetic moment* which is *twice as large* as the magnetic moment which would be assigned classically to its spin. This *magnetic anomaly* of the electron is the general result which may be deduced from the total observed data on the anomalous Zeeman effects (Goudsmit and Uhlenbeck 1925). It explains directly the observed results of the two gyro-magnetic effects and proves at the same time that the electron spin plays the role of the magnetic needles in Ewing's model.

On the basis of this discovery Heisenberg, in 1928, with the aid of modern electron statistics, was able to proceed toward a true physical understanding of ferromagnetism and to calculate qualitatively the extraordinary magnitude of the inner magnetic field in a Weiss domain. We see from this how long the road is from the Maxwell theory to the actual theory of ferromagnetism and it becomes evident that we cannot travel this road.

C. Hysteresis Loop and Reversible Magnetization

The figure which represents the magnetization M of a ferromagnetic material as function of the excitation H with increasing and decreasing H is well known. If the material is originally unmagnetized, the "virginal curve" is first traversed, beginning in the origin $H = 0, M = 0$ and passing over into the horizontal asymptote $M = M_s$ of *saturation* for sufficiently large H . If, from this point, H is permitted to decrease, the characteristic lies above the virginal curve and cuts the ordinate axis in a point $H = 0, M = M_r$, which indicates the *remanent magnetization*. If H is permitted to decrease still further, i.e. is reversed in direction, a region is entered in which B and H have opposite directions. The iron specimen has then become a "permanent magnet". With further decrease of H the hysteresis loop cuts the axis of abscissas in a point $H = -H_c, M = 0$, where the

¹ Atomic theory has not demonstrated fully why just the atoms of the iron group are ferromagnetically active.

remanent magnetization is just nullified. H_c is called the "coercive force". If H is decreased further, the negative saturation $M = -M_s$ is approached. If, now, H is once more increased, the gradually rising characteristic remains below the descending branch and below the virgin curve. It does not pass through the origin, but cuts the axis of abscissas in a point $H = +H_c$, $M = 0$. The ascending branch is symmetrical through the origin to the descending branch and approaches finally once more the positive saturation $M = +M_s$.

As a rule M_R is approximately $\frac{1}{2}M_s$. In order that the magnet may retain its remanence for all opposing fields which occur it is important that H_c may be as large as possible. This is the case for hard steel (tungsten steel has $H_c \sim 70$ oersteds).

The ascending and descending branches form together the *hysteresis loop*; its area is the magnetic work $\oint \mathbf{H} \cdot d\mathbf{B}$, which is performed on the material in a complete cycle. If the increase of H is stopped, in traversing the virgin curve, before reaching saturation, e.g. for $H_1 < H_s$, and H is then permitted to decrease to $H = -H_1$, then to increase to $H = +H_1$, a smaller hysteresis loop is obtained, which lies inside of the one previously described. For very small $H_1 = \delta H$ the loop degenerates into a twice traversed line; its area becomes zero and the process is *reversible*. The ratio $\delta M / \delta H$ defines the *initial susceptibility* κ_0 . It is possible to carry out such a reversible process not only at the origin, but at any arbitrary point of the cycle and to define for every such point a *reversible susceptibility* κ_{rev} .

With reversible magnetization the elementary magnets are *deflected* only slightly out of their original position in the direction of the external field. With irreversible magnetization some *reorientations* take place as well. In both cases changes occur in the boundaries of the Weiss domains, which are described as wall displacements; they are small for reversible, large for irreversible processes. In an induction coil with telephone connection they become acoustically noticeable by noises (clicks) and can be made visible on an oscilloscope as *Barkhausen jumps*. In fact, for sufficient oscillographic magnification, the apparently continuous course of the hysteresis loop resolves itself, particularly in the descending branch, into a sequence of small steps.

The individual processes which go to make up the magnetization curves are thus of varied nature. They depend on the composition of the iron specimen and on its microcrystalline structure; even for the single crystal they depend on the orientation relative to the magnetic field. It is the problem of the metallurgist to find the alloy (permalloy, permivar, cobalt steel) suited for each purpose (transformer laminations, communications engineering).

D. Thermodynamics

Ferromagnetism is even more dependent on temperature than paramagnetism. Above a certain critical temperature ferromagnetism ceases and passes over into ordinary paramagnetism. This critical temperature is designated with θ and is called the *Curie point*. On the centigrade scale ($\theta = \theta_{\text{cent}} + 273$) we have:

	for iron	cobalt	nickel
$\theta_{\text{cent}} =$	770	1120	358

For $T > \theta$ we have in place of Curie's law (13.10) the Curie-Weiss law

$$\chi = \frac{C}{T - \theta}. \quad (1)$$

This suggests that there may be no difference in principle between paramagnetism and ferromagnetism, that, in other words, the Curie point, in the former case, lies close to absolute zero. With this assumption a region of ferromagnetic behavior near $T = 0$ is to be expected also for ordinary paramagnetic materials. It then seems reasonable to transfer Langevin's statistical and thermodynamic theory of paramagnetic materials to the conditions of ferromagnetism.

In fact Weiss in this manner arrived at a representation of the whole complex of ferromagnetic phenomena which, in its main features, is satisfactory. However, this representation utilizes concepts with which we will only be able to deal in Volume V. Also detailed questions of atomic physics play here a role, such as the question of the atomic unit of magnetic moment (Bohr's magneton as compared with the Weiss magneton, which is smaller by a factor of five) and the question to what extent, in addition to the spin moment of the free electrons, the orbital moment of the electrons bound in the atom, which is twice as large, must be considered. The standard textbook¹ of R. Becker and W. Döring gives complete information on all relevant questions.

§15. Stationary Currents and Their Magnetic Field. Method of the Vector Potential

Since the assumption of stationary fields demands $\partial/\partial t = 0$ throughout, the Maxwell equations (4.4) reduce to

$$\text{curl } \mathbf{E} = 0, \quad \mathbf{J} = \text{curl } \mathbf{H}. \quad (1)$$

¹ *Ferromagnetismus*, Springer, Berlin 1939. For a less detailed treatment, see F. Bitter, *Introduction to Ferromagnetism*, McGraw-Hill, New York, 1937.

The second of these leads to

$$\operatorname{div} \mathbf{J} = 0. \quad (2)$$

For the *surface* of a conductor carrying current we have therefore

$$J_n = 0. \quad (2a)$$

(2) states that the electricity within a conductor behaves like an incompressible fluid (increase and decrease of current density for a narrowing and broadening of the conductor, respectively). Also *Kirchhoff's branching laws for linear conductors*, which Kirchhoff worked out as solution to a seminar problem given by F. Neumann (see p. 1), rest in the final analysis on (2) and the existence of the electrical potential (see below).

Eqs. (1) state that \mathbf{E} has *everywhere* a potential, $\mathbf{E} = -\operatorname{grad} \Psi_e$, while \mathbf{H} has a scalar potential, $\mathbf{H} = -\operatorname{grad} \Psi_m$, only *outside* of the current-carrying conductors. We will deal with this scalar potential in §16. Here we shall give a representation of \mathbf{B} (and hence also one of \mathbf{H}) which is valid both *inside* and *outside* of the conductors. We recall here Helmholtz's representation of the velocity field \mathbf{v} for given distribution of turbulence ω in Vol. II, Eq. (20.13). In Helmholtz's analogy the electric current density \mathbf{J} corresponded¹ to the second, the magnetic excitation \mathbf{H} , to the first. Just as there, we introduce a *vector potential* \mathbf{A} , by setting:²

$$\mathbf{B} = \operatorname{curl} \mathbf{A}. \quad (3)$$

The second Eq. (1) then becomes

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} = \mathbf{J}. \quad (4)$$

For constant μ we can write instead

$$\operatorname{curl} \operatorname{curl} \mathbf{A} = \mu \mathbf{J}. \quad (4a)$$

¹ But for a factor $\frac{1}{4\pi}$ whose suppression was justified in Vol. II, p. 15 by the requirements of electrodynamics.

² It is customary to write instead $\mathbf{H} = \operatorname{curl} \mathbf{A}$, which, however, assumes the complete absence of sources of \mathbf{H} , a condition which is not fulfilled in regions of non-vanishing magnetic density ρ_m . Our formula (3) is more satisfactory, since $\operatorname{div} \mathbf{B} = 0$ throughout; in addition, it will generally simplify our formulas, particularly in Part III. Incidentally, both formulas amount to much the same thing if the assumption is made that μ is constant everywhere, which occurs already in Eq. (4a) of the text. For nonconstant μ the summation problem to be solved in (7) would have to be supplemented by a magnetostatic *boundary-value problem* (determination of the discontinuity of B_{tang} at the boundary between media of different permeability for our expression for \mathbf{A} and determination of the magnetic surface densities appearing there for the usual expression for \mathbf{A} , respectively).

We here employ the general transformation (6.2) with the restriction to Cartesian coordinates which is there emphasized and obtain instead of (4a) the form which is more convenient for integration:

$$\Delta A - \text{grad div } A = -\mu J. \quad (4b)$$

This may be simplified by the supplementary condition

$$\text{div } A = 0, \quad (5)$$

which transforms (4b) into

$$\Delta A = -\mu J. \quad (6)$$

The condition (5) may be added since A , for given B , is determined by Eq. (3) with the exception of the gradient of a scalar function. The latter may be utilized to satisfy (5). For, if A_1 is any solution of (3),

$$A = A_1 + \text{grad } f \quad (6a)$$

is similarly a solution; if we now write

$$\text{div grad } f = \Delta f = -\text{div } A_1, \quad (6b)$$

which, according to the well-known integration procedure of Poisson's equation is always possible, we find $\text{div } A = 0$.

This method of integration yields at the same time as the solution of (6):

$$4\pi A_P = \int \frac{\mu J_Q d\tau_Q}{r_{PQ}}. \quad (7)$$

Here the point of integration $Q = \xi, \eta, \zeta$ traverses the entire interior of the conductors; $P = x, y, z$ is the reference point for which the Cartesian components A_x, A_y, A_z are to be calculated. It was shown in Vol. II, §20, Nr. 2a that this representation satisfies (5) provided that J is solenoidal, in accord with (2), and that μ is constant. The integration in Eq. 7 is to be extended over the *closed* current field J (just as in Vol. II over the *closed* vortex rings).

The current density J appearing in (7) may be obtained as solution of a potential problem. Since $J = \sigma E$ the potential equation applies, for constant σ , just as much for J as for E :

$$\Delta J = 0. \quad (8)$$

For varying σ Eq. (8) takes on a somewhat more complicated form. The total current I is obtained from J by integration over any cross section of the conductor:

$$I = \int J_n d\sigma. \quad (9)$$

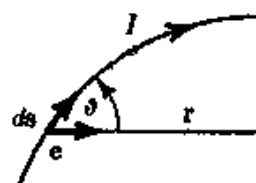
The familiar fact that I has a fixed value independent of position and shape of the cross section follows from integration of Eq. (2) over a segment of the conductor bounded by two arbitrary cross sections as for the analogous spatial law of conservation of vortex theory (see Vol. II, p. 136, Fig. 24).

We shall now give some applications of our representation (7).

A. The Law of Biot-Savart

We subdivide the three-dimensional conductor into current tubes with the cross section dq , normal to the tube axis, and the element of length ds ; the current $J_n dq$ in such a tube element, which has the same dimension as the total current I , we shall also call I , for the present. We then can set

FIG. 19. The law of Biot-Savart, derived from the vector potential of an element of current.



$\mathbf{J} d\tau = I ds$, where the direction of the current flow \mathbf{J} is indicated by the vectorial character of ds . By (7) the contribution of our tube element to \mathbf{A} then becomes

$$4\pi d\mathbf{A} = \frac{\mu I ds}{r}$$

and, by (3), the corresponding contribution to \mathbf{B} is,

$$4\pi d\mathbf{B} = \text{curl} \frac{\mu I ds}{r}. \quad (10)$$

We shorten the remaining calculation by employing the symbolic vector ∇ :

$$\text{curl} \frac{\mu I ds}{r} = \nabla \times \frac{\mu I ds}{r} = \left(\text{grad} \frac{1}{r} \right) \times \mu I ds. \quad (11)$$

For $I ds$ is dependent on x, y, z in direction, but not in magnitude; $1/r$ depends on x, y, z in magnitude, but not in direction (being a scalar). We see furthermore from Fig. 19 that

$$\text{grad} \frac{1}{r} = -\frac{\mathbf{r}}{r^2} = -\frac{\mathbf{e}}{r^2}, \quad (11a)$$

where \mathbf{r} denotes the radius vector from the current element to the reference point and \mathbf{e} the corresponding unit vector. Substitution of (11) and (11a)

in (10) yields

$$4\pi d\mathbf{B} = \frac{-\mu I}{r^2} \mathbf{e} \times d\mathbf{s}, \quad 4\pi |d\mathbf{B}| = \frac{\mu I ds}{r^2} \sin \vartheta. \quad (12)$$

ϑ signifies the angle between the vectors \mathbf{e} and $d\mathbf{s}$; the direction of $d\mathbf{B}$, in view of the negative sign in (12) and the meaning of the vector product, is that of a left-hand screw for the direction of rotation $\mathbf{e} \rightarrow d\mathbf{s}$. Let us imagine a magnetic unit pole at the reference point and let the vector $d\mathbf{B}$ act on it; $d\mathbf{B}$ then represents the *Biot-Savart force* exerted on the unit pole by the current element $I d\mathbf{s}$. The corresponding line of force then surrounds the current element, in a *right-hand* screw direction, as shown in the figure and as expected. Evidently the first Eq. (12) is the more complete one, since it expresses the dependence on direction which is characteristic of the magnetic field in the simplest and most appropriate manner; we have added the second form merely because it is the historically more familiar one.

B. The Magnetic Energy of the Field of Two Conductors

If we denote the energy integrated over space by W we obtain, for the magnetic energy density W , from Eq. (5.6)

$$2W = \int \mathbf{H} \cdot \mathbf{B} d\tau = \int \mathbf{H} \cdot \text{curl } \mathbf{A} d\tau. \quad (13)$$

For the evaluation we utilize the vector formula (5.2), previously derived in connection with the Poynting theorem, which we rewrite in terms of our present symbols (\mathbf{A} , \mathbf{H} in place of \mathbf{U} , \mathbf{V}) as follows:

$$\mathbf{H} \cdot \text{curl } \mathbf{A} = \mathbf{A} \cdot \text{curl } \mathbf{H} + \text{div } \mathbf{A} \times \mathbf{H}. \quad (14)$$

We assert that the second term on the right vanishes in the integration over infinite space. According to Gauss's theorem this term yields

$$\int \text{div } \mathbf{A} \times \mathbf{H} d\tau = \int (\mathbf{A} \times \mathbf{H})_n d\sigma, \quad (14a)$$

where the integration on the right is to be carried out over a surface bounding the region at a great distance, e.g. a sphere of radius R . Let the two conductors, whose total magnetic energy is to be determined, be entirely confined to a finite region. The distance of all their points from the infinitely distant element of area $d\sigma$ of the bounding sphere may then be set equal to the constant value R . By Eq. (7) \mathbf{A} approaches zero on $d\sigma$ as $1/R$ and, by the law of Biot-Savart, \mathbf{H} approaches zero as $1/R^2$. Since $d\sigma = R^2 d\Omega$ ($d\Omega$ = solid angle intercepted by $d\sigma$), the right side of (14a) approaches zero as R^2/R^3 .

In view of (1), (14) and (13) lead to

$$2W = \int \mathbf{A} \cdot \text{curl } \mathbf{H} \, d\tau = \int \mathbf{A} \cdot \mathbf{J} \, d\tau. \quad (15)$$

The integration is now to be carried out only over the conductors 1 and 2, since \mathbf{J} is zero everywhere outside of them. If we designate the point of integration in (15) by P ($d\tau_P$ instead of $d\tau$) and if we substitute $\mathbf{A} = \mathbf{A}_P$ from (7), the simple volume integral is replaced by a double volume integral:

$$\frac{2W}{\mu} = \frac{1}{4\pi} \iint \mathbf{J}_P \cdot \mathbf{J}_Q \frac{d\tau_P \, d\tau_Q}{r_{PQ}}. \quad (16)$$

In evaluating this integral we have to distinguish four cases, depending on the position of P and Q on conductors 1 and 2:

- | | |
|------------------------|------------------------|
| a. P and Q on 1, | b. P and Q on 2, |
| c. P on 1, Q on 2, | d. P on 2, Q on 1; |

the cases c and d are however alike in view of the symmetry of (16) with respect to P and Q . We can write the result in the form

$$W = \frac{1}{2}(L_{11} I_1^2 + L_{22} I_2^2 + 2L_{12} I_1 I_2), \quad (17)$$

$$L_{11} = \frac{\mu}{4\pi} \iint \mathbf{j}_1 \cdot \mathbf{j}_1' \frac{d\tau_1 \, d\tau_1'}{r_{11}'}, \quad L_{22} = \frac{\mu}{4\pi} \iint \mathbf{j}_2 \cdot \mathbf{j}_2' \frac{d\tau_2 \, d\tau_2'}{r_{22}'}, \quad (17a)$$

$$L_{12} = \frac{\mu}{4\pi} \iint \mathbf{j}_1 \cdot \mathbf{j}_2 \frac{d\tau_1 \, d\tau_2}{r_{12}}. \quad (17b)$$

The factor 2 in the product term in (17) results from the equality of cases c and d , which causes the coefficient L_{21} also to be given by (17b), i.e. $L_{21} = L_{12}$. I_1 , I_2 are the total currents in conductors 1 and 2, which, as noted at (9), are independent of the place in the conductor. 1 , $1'$ are two points on the conductor 1; 2 , $2'$ two points on conductor 2. Division of the current densities \mathbf{j}_1 , \mathbf{j}_2 by I_1 , I_2 leads to the purely geometrically defined "current-line density vectors"

$$\mathbf{j}_1 = \frac{\mathbf{J}_1}{I_1}, \quad \mathbf{j}_2 = \frac{\mathbf{J}_2}{I_2}. \quad (17c)$$

The L 's are called *induction coefficients*: L_{11} , L_{22} are the *coefficients of self-induction*, L_{12} is the *coefficient of mutual induction*. Maxwell uses the letter M in place of L_{12} .

The unit of the coefficients L (or M) is the *henry*.¹ In accord with (17)

¹ Joseph Henry, 1792–1878, American physicist, discovered almost simultaneously with Faraday the appearance of an electromotive force in a coil when the magnetic field in its interior is changed.

this unit is fixed in value and dimension by the statement

$$1 \text{ henry} = 1 \frac{\text{joule}}{\text{I}^2} = 1 \frac{\text{joule S}^2}{\text{Q}^2}. \quad (18)$$

Converted into electromagnetic cgs-units we find, since

$$\begin{aligned} \text{Q} &= 10^{-1} \text{ cm}^{\frac{1}{2}} \text{g}^{\frac{1}{2}}, & 1 \text{ joule} &= 10^7 \text{ cm}^2 \text{g sec}^{-2}, \\ 1 \text{ henry} &= 10^9 \text{ cm} = 1 \text{ quadrant of the earth.} \end{aligned} \quad (18a)$$

From our standpoint we can, however, attach no significance to this apparently so simple dimension and this relationship to the earth's circumference, since it rests on the arbitrary assumptions of the electromagnetic system of units. At the same time we are glad to point out the following relation between the henry and the permeability in vacuum given by (7.16a):

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{\text{joule} \cdot \text{S}^2}{\text{Q}^2 \text{M}} = 4\pi \cdot 10^{-7} \frac{\text{henry}}{\text{M}}. \quad (18b)$$

Compare with this the analogous relation between the dielectric constant of vacuum and the farad as given by Eq. (10.3b).

C. Neumann's Potential as the Coefficient of Mutual Induction

In (17b) it is possible to pass to the limiting condition of linear conductors, i.e. infinitely thin wires. In (17a) this is not permitted since r'_{11} (or r'_{22}) would vanish as the two integration points approach each other and the convergence of the integrals would be destroyed.

We write in (17b)

$$d\tau_1 = dq_1 ds_1, \quad d\tau_2 = dq_2 ds_2 \quad (19)$$

and combine dq_1 and j_1 , dq_2 and j_2 . The products $j_1 dq_1$ and $j_2 dq_2$ then have, by (17c), unit magnitude and their scalar product is equal to the cosine of the angle θ_{12} between the two directions of flow ds_1 and ds_2 . Hence:

$$\frac{4\pi}{\mu} L_{12} = \int ds_1 \int ds_2 \frac{\cos \theta_{12}}{r_{12}} = \iint \frac{ds_1 \cdot ds_2}{r_{12}}. \quad (20)$$

This amazingly simple and beautiful representation was discovered¹ by Franz Neumann as early as 1845. It is known as *Neumann's potential*; according to (17) it represents that portion of the magnetic energy which results from the interaction of the two circuits. A relative displacement or rotation of the two circuits with the currents I_1 and I_2 left unaltered hence is accompanied by a change in energy δW in the amount

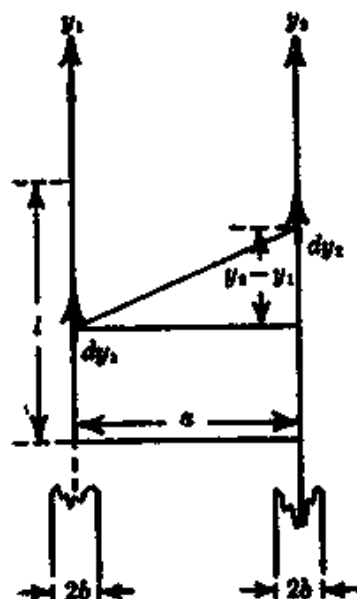
$$\delta W = I_1 I_2 \delta L_{12}. \quad (20a)$$

¹ Abhandl. Preuss. Akad., reprinted in Ostwald's *Klassiker*, Nr. 10.

The *work* which must be done in a displacement or rotation, as well as the *force* or *torque* which one circuit exerts on the other, are related to this.

In spite of the simplicity of expression (20) the actual calculation of the mutual induction coefficient is rather inconvenient. To begin with we work out a formal mathematical example, i.e. two straight parallel wires of length l separated by distance a . The condition of closed circuits stated in Eq. (7) is temporarily not fulfilled here. We shall take due account of it only

FIG. 20. The coefficient of mutual induction of two straight, parallel segments of wire of length l . The finite cross section of the wires is indicated at the bottom of the figure.



when we reach Eq. (24). Referring to Fig. 20, we have ($dy_1 dy_2$ in place of $ds_1 ds_2$, $\cos \theta_{12} = 1$):

$$\frac{4\pi}{\mu} L_{12} = \int_0^l dy_1 \int_0^l \frac{dy_2}{\sqrt{a^2 + (y_2 - y_1)^2}}. \quad (21)$$

The formula of integration already employed in (9.4) yields for the second integral

$$\begin{aligned} \int_0^l \frac{dy_2}{\sqrt{a^2 + (y_2 - y_1)^2}} \\ = \log(l - y_1 + \sqrt{a^2 + (l - y_1)^2}) - \log(-y_1 + \sqrt{a^2 + y_1^2}). \end{aligned} \quad (22)$$

A further formula, which may readily be checked by differentiation,

$$\int^z \log(\xi + \sqrt{a^2 + \xi^2}) d\xi = z \log(z + \sqrt{a^2 + z^2}) - \sqrt{a^2 + z^2} + \text{const}$$

yields for the integration of the first term on the right of (22)

$$\begin{aligned} \int_0^l dy_1 \log(l - y_1 + \sqrt{a^2 + (l - y_1)^2}) \\ = l \log(l + \sqrt{a^2 + l^2}) - \sqrt{a^2 + l^2} + a \end{aligned}$$

and for the integration of the second term

$$\begin{aligned}
 - \int_0^l dy_1 \log(-y_1 + \sqrt{a^2 + y_1^2}) \\
 = -l \log(-l + \sqrt{a^2 + l^2}) - \sqrt{a^2 + l^2} + a.
 \end{aligned}$$

A simple transformation under the logarithm sign yields for the sum of the two

$$\frac{4\pi}{\mu} L_{12} = 2l \log \frac{l + \sqrt{a^2 + l^2}}{a} - 2 \sqrt{a^2 + l^2} + 2a. \quad (23)$$

We assume $l \gg a$, and obtain as a first approximation

$$\frac{4\pi}{\mu} L_{12} = 2l \left(\log \frac{2l}{a} - 1 \right). \quad (24)$$

It is worth noting in this result that we have not obtained simple proportionality to l , since the parenthesis depends on l logarithmically. Accordingly we can not speak of a mutual induction coefficient per unit length of our wires.

The reason for this is the following: Our derivation assumes, as has been stressed repeatedly, two *closed* circuits, while our example deals with two circuit segments. Our result (24) is nevertheless meaningful. For example, it is possible to determine with its aid the mutual inductance of two parallel rectangles such as occur in Ampere's basic experiments. For such a pair of rectangles (the second rectangle is supposed to be obtained from the first by a parallel displacement perpendicular to its plane) only parallel pairs of sides contribute; for mutually perpendicular sides the product $ds_1 \cdot ds_2$ occurring in Eq. (20) is equal to zero. The mutual inductance of two such rectangles becomes equal to the sum of four terms of the form of (24).

D. The Coefficient of Selfinduction

As already noted we cannot in this case pass to the limit of the linear conductor, but must return to the double volume integrals in (17a). We can readily convince ourselves, however, that then any convergence difficulty is avoided. For if, for an arbitrary position of 1, we employ polar coordinates r, ϑ, φ with this point as origin to locate the point 1', $d\tau_1' = r^2 dr \sin \vartheta d\vartheta d\varphi$ and the denominator $r_{11}' = r$ cancels one of the factors r in $d\tau_1'$. However the carrying out of the integrations becomes now, in general, even more awkward than in C.

We therefore limit ourselves to a simple mathematical example, namely a straight wire of the (great) length l and the (small, but finite) cross section q . We consider two current filaments parallel to the axis of the wire

(y -axis) and employ once again Fig. 20, where now the two linear currents are assumed to refer to the same wire of cross section q . Let dq_1, dq_2 be the cross sections of the two current filaments; their separation, formerly denoted by a , will now be called ρ since it is variable, depending on the position of the two current filaments within q . The volume elements are once again given by (19), with $ds_1 = dy_1, ds_2 = dy_2$, and the vectors defined in (17c) become

$$j_1 = j_2 = \frac{l}{q}.$$

The defining equation (17a) for the selfinductance then takes the form

$$\frac{4\pi}{\mu} L = \iint \frac{dq_1 dq_2}{q^2} \int_0^l \int_0^l \frac{dy_1 dy_2}{\sqrt{\rho^2 + (y_2 - y_1)^2}}. \quad (25)$$

The second double integral has exactly the same form as (21). We can utilize the approximate evaluation (24) here also and obtain

$$\begin{aligned} \frac{4\pi}{\mu} L &= \frac{2l}{q^2} \iint dq_1 dq_2 \left(\log \frac{2l}{\rho} - 1 \right) \\ &= \frac{2l}{q^2} \left\{ \iint dq_1 dq_2 (\log 2l - 1) - \iint dq_1 dq_2 \log \rho \right\}. \end{aligned}$$

In the first term on the right the integration with respect to dq_1 and dq_2 can be carried out easily; the second term requires more detailed discussion because of the variability of $\rho = \rho_{12}$ = separation of our two current filaments. We note the preliminary result

$$\frac{4\pi}{\mu} L = 2l \{ \log 2l - 1 - \log \bar{\rho} \}, \quad \log \bar{\rho} = \frac{1}{q^2} \int dq_1 \int dq_2 \log \rho_{12}. \quad (26)$$

Maxwell calls the quantity $\bar{\rho}$ here introduced the *mean geometric separation* of the elements dq_1, dq_2 within the cross section q .¹ It can be determined more elegantly by an electrostatic consideration than by direct calculation.

In terms of polar coordinates the two-dimensional potential equation becomes

$$\Delta \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d\Phi}{d\rho} + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \varphi^2} = 0. \quad (27)$$

¹ Treatise, art. 691 ff. In explanation of the notation we remark: The integral to be evaluated in (26) is the arithmetic mean of all values of $\log \rho$ occurring on our surface q . In view of the relation

$$\sum \log \rho_i = \log \Pi \rho_i$$

this arithmetic mean of the logarithms is at the same time the logarithm of the *geometric* mean of all the ρ_i .

Apart from a multiplying and an additive constant the solution which is independent of φ is known to be the "logarithmic potential"

$$\Phi = \log \rho.$$

This signifies, in two dimensions, a negative charge concentrated at the point $\rho = 0$. If the charge is distributed over the area q with a positive surface density f , and with the surface element of q being designated with dq_1 , Green's theorem yields for its potential at the reference point 1.

$$2\pi\Phi_1 = - \int_q f \log \rho_{12} dq_1. \quad (28)$$

This is the two-dimensional analog to the familiar Eq. (7.5). If, in (28), we put $f = -2\pi$, we obtain the inner integral in (26),

$$\Phi_1 = \int_q \log \rho_{12} dq_1 \quad (28a)$$

and our desired mean geometric separation may be written

$$\log \beta = \frac{1}{q^2} \int \Phi_1 dq_1. \quad (28b)$$

The integral (28b) can be readily evaluated in the case that q is a circle, e.g. of radius b , and the point 1 coincides with the center of the circle. We then have $\rho_{12} = \rho$, i.e. equal to the polar coordinate employed previously, and $dq_1 = \rho d\rho d\varphi$. We denote by Φ_0 the corresponding specific value of Φ_1 . Eq. (28a) then takes the form

$$\Phi_0 = \int_0^{2\pi} d\varphi \int_0^b \log \rho \rho d\rho. \quad (29)$$

The integral with respect to ρ can readily be evaluated by integration by parts; thus

$$\int_0^b \log \rho \rho d\rho = \frac{\rho^2}{2} \log \rho \Big|_0^b - \int_0^b \frac{\rho}{2} d\rho = \frac{b^2}{2} (\log b - \frac{1}{2}).$$

Hence (29) leads to

$$\Phi_0 = \pi b^2 (\log b - \frac{1}{2}). \quad (29a)$$

Furthermore Φ_1 , as the potential for a known surface distribution, can be calculated for arbitrary location of point 1 directly from Poisson's equation, which, in analogy to (7.4a), in two dimensions takes the form

$$\Delta\Phi_1 = -f,$$

where f is the surface density. In our special case ($f = -2\pi$, circle q of radius b , Φ_1 a function of ρ only) it becomes

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d\Phi_1}{d\rho} = 2\pi. \quad (30)$$

Integrated twice this yields

$$\rho \frac{d\Phi_1}{d\rho} = \pi\rho^2 + A, \quad \Phi_1 = \frac{\pi\rho^2}{2} + A \log \rho + B.$$

In order that this expression for Φ_1 may pass over, for $\rho = 0$, into the expression (29a) for Φ_0 , we must set

$$A = 0, \quad B = \Phi_0, \quad \text{hence} \quad \Phi_1 = \pi b^2 \left(\log b - \frac{1}{2} + \frac{1}{2} \frac{\rho^2}{b^2} \right). \quad (31)$$

If we substitute this expression for Φ_1 in (28b) we obtain

$$\begin{aligned} \log \bar{\rho} &= \frac{1}{q} \int dq_1 (\log b - \tfrac{1}{2}) + \frac{1}{2q} \int dq_1 \frac{\rho^2}{b^2} \\ &= \log b - \tfrac{1}{2} + \frac{1}{b^2} \int_0^b \rho^2 d\rho = \log b - \tfrac{1}{2}. \end{aligned} \quad (32)$$

For $\bar{\rho}$ itself we obtain from this the peculiar value

$$\bar{\rho} = b/\sqrt[4]{e}. \quad (32a)$$

Here the proportionality with b is rather obvious in view of the definition of the geometric mean (see last footnote); however the numerical factor is to be found only by detailed analysis, which following Maxwell, we have here based on potential theory. It may be mentioned that Maxwell carried out these considerations even for cross sections of arbitrary shape.

To pursue our real goal, the calculation of the selfinductance L , we return to (26). We then find, utilizing (32)

$$\frac{4\pi}{\mu} L = 2l \left\{ \log 2l - \log b - \frac{3}{4} \right\} = 2l \left\{ \log \frac{2l}{b} - \frac{3}{4} \right\}. \quad (33)$$

This formula confirms our original expectation that the transition to the linear conductor ($b \rightarrow 0$) is not permissible for the selfinductance. Regarding the dependence on l we must make the same remark as at the end of section C: We cannot, by division of l , obtain the *selfinductance per unit length of an infinitely long wire*; however, we can, as for the *mutual inductance*, piece together the selfinductance of any *closed* circuit, made up of straight wires, by adding up terms of the form of (33).

We shall further answer a rather obvious objection which may be raised against the somewhat indirect derivation of (33). The magnetic field of an

infinitely long wire is known from Fig. 4 and the corresponding equations (10) to (13) on p. 24. Cannot the energy and selfinduction of the straight wire be calculated much more directly from them?

With I as current we found (writing now b, c in place of the earlier a, b)

$$r < b: H = \frac{r}{b} \frac{I}{2\pi b}; \quad r > b: H = \frac{I}{2\pi r}.$$

Hence the contribution of the interior of the wire to the energy per unit length is

$$\frac{\mu}{2} 2\pi \int_0^b H^2 r \, dr = \frac{\mu}{4\pi} \frac{I^2}{b^4} \int_0^b r^3 \, dr = \frac{\mu}{16\pi} I^2 \quad (33a)$$

and the contribution of the exterior:

$$\frac{\mu}{2} 2\pi \int_b^c H^2 r \, dr = \frac{\mu}{4\pi} I^2 \int_b^c \frac{dr}{r} = \frac{\mu}{4\pi} I^2 \log \frac{c}{b}. \quad (33b)$$

From this we find for the selfinductance per unit length, from the energetic formula of definition (17),

$$\frac{\mu}{2\pi} \left(\frac{1}{4} + \log \frac{c}{b} \right). \quad (33c)$$

This expression becomes logarithmically infinite if we let $c \rightarrow \infty$, i.e. pass over to the single wire without return conductor. Our intended simplified derivation hence fails—quite understandably—as the result of the unphysical assumptions of the problem, which must lead to an infinite energy content of space for any unit length of the wire as we pass to the limit $c \rightarrow \infty$.

E. Selfinductance of the Two-Wire Line

The system of two straight parallel wires traversed by current in opposite directions plays an important role in electric power transmission and is known, in Hertz's experiments with high frequency waves, as a Lecher system. We shall determine the selfinductance per unit length of such a system (with appropriate restriction to the direct-current case).

So as to be able to utilize Fig. 20, we shall call the (very great) length of the wires l , their separation a , and their radius b . We proceed from the energetic formula (17), where we put

$$I_1 = I, \quad I_2 = -I, \quad L_{11} = L_{22} = L.$$

We then obtain

$$W = \frac{1}{2} L_D I^2, \quad L_D = 2(L - L_{12}). \quad (34)$$

We can designate L_D , introduced here, as inductance of the *line*, regarded as a uniform system.¹ Substitution from (33) and (24) yields:

$$L_D = \frac{\mu}{\pi} l \left(\log \frac{2l}{b} - \frac{3}{4} - \log \frac{2l}{a} + 1 \right).$$

In evaluating the logarithmic terms the two terms $\pm \log 2l$ drop out and the following simple formula remains:

$$\frac{L_D}{l} = \frac{\mu}{\pi} \left(\log \frac{a}{b} + \frac{1}{4} \right). \quad (35)$$

L_D/l is the desired selfinductance per unit length of the two-wire line, which evidently is independent of l . Hence the transition to $l \rightarrow \infty$, which, according to (33) and (24), was inappropriate in the expressions for L/l and L_{12}/l because of the term $\log 2l$, is now feasible. This is obviously related to the fact that the field of a two-wire line traversed by oppositely directed currents corresponds to that of a circuit closed at infinity. On the other hand the transition to the linear two-wire line ($b \rightarrow 0$), which was possible in the expression for the mutual inductance, cannot be carried out even now.

F. General Theorem Regarding Energy Transmission by Stationary Currents

We consider the section of an arbitrarily shaped wire between two cross sections F_1 and F_2 . Let it contain a "load" in which electrical energy is translated into work or some other form of energy, e.g. a light bulb. We ask what *power* is supplied to the load (the Joule heat generated in our section of wire to be counted as part of the load).

We extend the cross sections F_1, F_2 to a closed surface F and calculate the power N as the inward-directed energy flux through this surface. According to Poynting's theorem (5.7a) and the meaning of Joule heat (5.5) we obtain under stationary conditions

$$N = \int_V S_n dF = \int_V \mathbf{E} \cdot \mathbf{J} dV. \quad (36)$$

V is the volume enclosed by F . Under stationary conditions we have everywhere within V

$$\text{curl } \mathbf{E} = 0, \quad \mathbf{E} = -\text{grad } \Psi, \quad \text{hence } \mathbf{E} \cdot \mathbf{J} = -\text{grad } \Psi \cdot \mathbf{J}.$$

We transform this with the aid of the obvious and universally valid identity

$$\text{div}(\Psi \mathbf{J}) = \text{grad } \Psi \cdot \mathbf{J} + \Psi \text{div } \mathbf{J}$$

¹ A similar definition is employed in electrical engineering for multiple conductor systems with the designation "operating selfinductance," if under the conditions of operation all circuit currents are determined by one of them.

where \mathbf{J} is any vector, Ψ , any scalar. For our meaning of \mathbf{J} the last term vanishes. We hence deduce from (36)

$$N = - \int \operatorname{div}(\Psi \mathbf{J}) dV \quad (37)$$

and, by application of Gauss's theorem

$$N = \int_r \Psi J_n dF. \quad (38)$$

This integral need only be carried out over the surfaces of entry and exit, F_1 and F_2 , of the current, since only here J_n differs from zero. It is most convenient to choose F_1 and F_2 as the equipotential surfaces $\Psi = \Psi_1$ and $\Psi = \Psi_2$. We then obtain from (38)

$$N = \Psi_1 \int_{F_1} J_n dF_1 + \Psi_2 \int_{F_2} J_n dF_2. \quad (39)$$

With the convention regarding the direction of \mathbf{n} established above

$$\int J_n dF_1 = - \int J_n dF_2 = I$$

and hence, by (39),

$$N = (\Psi_1 - \Psi_2)I = VI. \quad (40)$$

V is the voltage drop between F_1 and F_2 (in volts) and I , the current entering and leaving (in amperes).

This fundamental formula, which expresses the power directly in watts (joules/S), has here been derived for stationary conditions; in §18 it will be found applicable also to "quasistationary" states. To avoid misunderstandings it should be noted that Eq. (40) makes no statement regarding the brightness of the light bulb or the energy radiated by it. This phenomenon lies outside of the domain of Maxwell's theory and rests on atomic processes which are made possible by the Joule heat supplied to the filament, but whose energy balance has nothing to do with Eq. (40). Our Eq. (40) follows the energy conversion only up to the generation of the Joule heat, not beyond it.

§16. *Ampère's Method of the Magnetic Double Layer*

In the last paragraph we had to introduce the concept of the *vector potential* in order to arrive at a representation of the magnetic field without and within the current-carrying conductor. If we content ourselves with a representation which applies only *outside* of the conductor we can get along with the ordinary *scalar* magnetic potential Ψ .

Outside of the conductors we have, by (15.1),

$$0 = \text{curl } \mathbf{H}, \quad \mathbf{H} = -\text{grad } \Psi. \quad (1)$$

If we add the condition $\text{div } \mathbf{B} = 0$ and assume uniform permeability, e.g. $\mu = \mu_0$, outside, we have furthermore

$$\text{div grad } \Psi = \Delta \Psi = 0. \quad (2)$$

This potential Ψ is *not*, however, a *unique* function of position, as in the magnetostatic case. For every closed loop about a conductor carrying I it changes by the amount I , independently of the shape and length of the path:

$$\Psi_1 - \Psi_2 = \oint \mathbf{H} \cdot d\mathbf{s} = \pm I. \quad (3)$$

Here Ψ_1 and Ψ_2 are the values of Ψ at the starting point 1 and the coinciding endpoint 2 of the loop. The upper or lower sign of I applies depending on whether the loop forms a right or left screw with the direction of the current. On the other hand, for every closed path which does not link such a conductor:

$$\Psi_1 - \Psi_2 = \oint \mathbf{H} \cdot d\mathbf{s} = 0. \quad (3a)$$

The proof of both Eqs. (3) and (3a) follows again from (15.1). An arbitrary surface σ bounded by the path of integration cuts the conductor in question in the first instance; in the second it can always be placed so that it cuts no conductor. If the component of Eq. (15.1) normal to every surface element $d\sigma$ is formed and integrated over all $d\sigma$, we obtain

$$\int_{\sigma} J_n d\sigma = \int_{\sigma} \text{curl}_n \mathbf{H} d\sigma.$$

By (15.9) the left side is the total current traversing the surface σ , i.e. I for Eq. (3) and 0 for (3a). The right side may be transformed by Stokes's theorem into the line integral over s .

For multiple loops about the conductors in one or the other direction a change in Ψ is obtained which is equal to the sum of the changes corresponding to individual circuits:

$$\Psi_1 - \Psi_2 = \sum n_k I_k.$$

Here n_k denotes the number of circuits about the k th conductor. Thus Ψ is infinitely multivalued. Any two "branches" of Ψ differ by a constant which is a sum of the currents I_k multiplied by integer coefficients n_k .

A. The Magnetic Shell for Linear Conductors

To give a prescription for the computation of Ψ which is unique in spite of this multiplicity we must confine ourselves to the limiting case of linear conductors (cross section $\rightarrow 0$); furthermore, it will suffice for the present to consider a single conductor. Let it be Λ . Through Λ as boundary we place an otherwise arbitrarily shaped "branch cut" surface S and forbid passage through S . In this manner we select, from the infinitely many-valued potential, a "function branch." Carrying over the already somewhat daring language of the *Riemannian surfaces* into three dimensions, we could also say: Of the "Riemannian space," whose infinite number of "leaves" have the branch line Λ in common and are joined in the branch cut S , we separate out one leaf as alone physically significant. This leaf has become "singly connected."

The calculation of Ψ may now be carried out by simple application of Green's theorem:

$$\int (u\Delta v - v\Delta u) d\tau = \int \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\sigma. \quad (4)$$

Here we put,

$$u = \Psi, \quad v = \frac{1}{r}, \quad r = r_{PQ} \quad (4a)$$

and carry out the volume integral on the left over all points P of our physical leaf, excluding a sphere of radius $\rho \rightarrow 0$ about the source point Q and a sphere of radius $R \rightarrow \infty$ cutting off the infinite; the latter sphere may have an arbitrary point O as center, which may for example be situated on S . Correspondingly, the surface integral on the right is to be carried out over the two spherical surfaces K_ρ and K_R , as well as the two "sides" of the branch cut S . Since on the sphere K_ρ ,

$$u \frac{\partial v}{\partial n} = -\Psi_Q \left(\frac{d}{dr} \frac{1}{r} \right)_{r=\rho} = \frac{1}{\rho^2} \Psi_Q$$

integration over $d\sigma$ here yields evidently

$$4\pi\Psi_Q. \quad (5)$$

The sphere K_R contributes

$$-\frac{1}{R^2} \int \Psi d\sigma - \frac{1}{R} \int \frac{\partial \Psi}{\partial n} d\sigma.$$

According to the law of Biot-Savart H decreases with increasing R as $1/R^2$. Hence the second of the above integrals is finite and, with its factor, vanishes as $1/R$. The first integral becomes infinite only in proportion to

R and yields similarly a vanishing contribution when multiplied with $1/R^2$. Thus the contribution of K_s is zero. We could confirm this result which is here based on the law of Biot-Savart, i.e. the method of the vector potential, also by the method of this paragraph.

Finally, we must consider the two sides 1 and 2 of the branch cut. In view of the opposite direction of n on the two sides we have

$$\left(\frac{\partial v}{\partial n}\right)_1 = -\left(\frac{\partial v}{\partial n}\right)_2. \quad (6)$$

Since

$$\frac{\partial u}{\partial n} = \frac{\partial \Psi}{\partial n} = -H_n$$

is a physical quantity which has nothing to do with the mathematical fiction of our branch cut we have, in addition to (6),

$$\left(\frac{\partial u}{\partial n}\right)_1 = -\left(\frac{\partial u}{\partial n}\right)_2. \quad (6a)$$

The sum of the contributions of the two sides of the branch cut to the right side of (4) may hence be written

$$\int \left\{ (u_1 - u_2) \left(\frac{\partial v}{\partial n}\right)_1 - (v_1 - v_2) \left(\frac{\partial u}{\partial n}\right)_1 \right\} d\sigma. \quad (6b)$$

Here $v_1 - v_2$ vanishes because of the meaning of $v = 1/r$, but, by Eqs. (3) and (4a), $u_1 - u_2 = \pm I$. Hence (6b) takes the form

$$\pm \int I \frac{\partial}{\partial n_1} \frac{1}{r} d\sigma. \quad (7)$$

To fix the sign, consider Fig. 21. Here 1 is to denote that side of S on which the normal n_1 directed toward S forms a *right screw* with the direction of current flow. The loop from 1 to 2 shown in the figure then forms a *left screw* with the direction of the current. Hence by prescription (3) we must choose the negative sign in formula (7). If we now write n for n_1 and place the factor I , which is a constant for all pairs of points 1, 2, ahead of the integral sign, this becomes

$$-I \int_s \frac{\partial}{\partial n} \frac{1}{r} d\sigma. \quad (7a)$$

Together with (5) we then obtain as the value of the right side of (4):

$$4\pi\Psi_0 - I \int_s \frac{\partial}{\partial n} \frac{1}{r} d\sigma.$$

Since the left side of (4) vanishes because $\Delta u = \Delta v = 0$, we obtain as the final representation of Ψ :

$$4\pi\Psi_Q = I \int_S \frac{\partial}{\partial n} \frac{1}{r} d\sigma. \quad (8)$$

The integral on the right has a very simple geometrical significance: It is the *solid angle* Ω intercepted by the circuit A at the point Q . In fact the integrand in Eq. (8)

$$\frac{\partial}{\partial n} \frac{1}{r} d\sigma = \frac{1}{r^2} \cos(n, r) d\sigma = \frac{d\sigma_n}{r^2}$$

is the surface element df of the unit sphere about Q cut out by the radii directed toward the boundary of $d\sigma$; $d\sigma_n$ is the corresponding surface element of a sphere of radius r . Hence

$$\Omega = \int_S df \quad (8a)$$

is the total area cut out on the unit sphere by the cone of radii directed toward the boundary of S , i.e. the above mentioned solid angle.

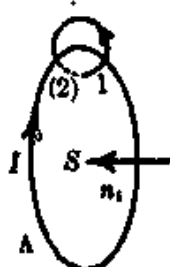


FIG. 21. The magnetic line integral about the conductor A , extended from side 1 to side 2 of the branch cut S .

The potential jump at the branch cut, $\Psi_1 - \Psi_2$, now also acquires a certain simple meaning. For if we place our point Q on side 1 of S the cone of radii degenerates into a flat fan and the solid angle Ω to 2π ; on the other hand, if we place it on side 2, the solid angle becomes $\Omega = -2\pi$. If we form the difference of Eq. (8) for the two cases we find

$$4\pi(\Psi_1 - \Psi_2) = I(2\pi - [-2\pi]),$$

i.e. the potential jump demanded by (3).

We will supplement this *geometrical* interpretation of the expression (8) by a *magnetic* interpretation: We speak of a *double layer* on the branch cut S with the *magnetic surface densities* $\pm \omega_m$, which we think of as distributed parallel to S at a distance dn from each other. We regard the current I as the moment per unit area of this magnetic double layer:

$$I = \omega_m dn. \quad (9)$$

Eq. (8) may then be written

$$4\pi\Psi = \int_s \omega_m dn \frac{\partial}{\partial n} \frac{1}{r} d\sigma. \quad (9a)$$

If we designate by r_{\pm} the distances of the point Q from the positive and the negative layer respectively we have

$$\frac{\partial}{\partial n} \frac{1}{r} dn = \frac{1}{r_+} - \frac{1}{r_-} \quad \text{and} \quad 4\pi\Psi = \int \frac{\omega_m d\sigma}{r_+} - \int \frac{\omega_m d\sigma}{r_-}. \quad (9b)$$

Accordingly Eq. (8) is in fact the potential of a *magnetic double layer* whose moment has the constant value I over its entire surface. Following Ampère we call the carrier of this double layer a *magnetic shell*; the linear conductor A forms the *boundary* of this shell.

In this connection we state a generally valid law of potential theory for simple and double layers: A simple layer of the type (7.5a)

$$4\pi\Psi = \int \frac{\omega}{r} d\sigma$$

leaves the *potential* Ψ *continuous* in passing through the carrier surface σ , but leads to a *discontinuous* normal component of the potential *gradient*, since

$$\left(\frac{\partial\Psi}{\partial n}\right)_1 - \left(\frac{\partial\Psi}{\partial n}\right)_2 = \omega;$$

on the other hand, a double layer of the type (8)

$$4\pi\Psi = \int I \frac{\partial}{\partial n} \frac{1}{r} d\sigma$$

makes the *potential discontinuous*, but leaves its *gradient continuous*. We have here

$$\Psi_1 - \Psi_2 = I, \quad \text{but } H_1 - H_2 = 0.$$

B. Magnetic Energy and Magnetic Flux

The calculation of the magnetic energy of a linear conductor in a medium of uniform permeability, i.e. the carrying out of the integration in the expression defining this energy:

$$W = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{B} d\tau = \frac{\mu}{2} \int \mathbf{H}^2 d\tau, \quad (10)$$

becomes particularly simple by the above method. We utilize here the so-called "second form of Green's theorem" (Vol. II, Eq. 3.16) which

becomes,

$$\int u \Delta u \, d\tau + \int \text{grad } u \cdot \text{grad } u \, d\tau = \int u \frac{\partial u}{\partial n} \, d\sigma \quad (11)$$

if the two functions u and v occurring there are set equal to each other. We set $u = \Psi$ and extend the integration on the left over the entire exterior of our linear conductor, having made the potential Ψ unique with the aid of the branch cut S . The integral on the right is then to be carried out over the two sides 1 and 2 of S . We can omit the integration over the surface bounding the space considered at infinity in view of our knowledge regarding the behavior of Ψ at infinity.

The first term on the left of (11) vanishes since $\Delta\Psi = 0$; the second is identical with $2W/\mu$ since $\text{grad } \Psi = -\mathbf{H}$. Summing over sides 1 and 2 with due regard of the opposite signs of $\partial\Psi/\partial n$, the right side of (11) becomes

$$\int (\Psi_1 - \Psi_2) \frac{\partial \Psi}{\partial n} \, d\sigma = I \int H_n \, d\sigma, \quad (11a)$$

where the established rule regarding the correlation of the sign of the normal n and the direction of the current is to be observed. We thus obtain from (11)

$$W = \frac{\mu}{2} I \int H_n \, d\sigma = \frac{1}{2} I \Phi; \quad (12)$$

Here Φ is the flux of the magnetic induction through our conductor Λ .

A particularly simple definition of the coefficient of selfinduction L may be deduced from (12). For, if we compare (12) with the equation of definition (15.17) specialized for a single conductor,

$$W = \frac{1}{2} LI^2 \quad (13)$$

we find directly

$$\Phi = LI; \quad L = \Phi/I. \quad (14)$$

We now pass from the single linear conductor considered so far to two linear conductors Λ_1 and Λ_2 . We must then make space "singly connected" by means of two branch cuts S_1 and S_2 , which are bounded by the currents I_1 and I_2 . We shall call the normals correlated to I_1 and I_2 by the right screw rule n_1 and n_2 . By superposition the magnetic fields of I_1 and I_2 form the total field $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$. In the integration over S_1 and S_2 , to be carried out as in (11a), there appear the expressions

$$I_1 \int_{S_1} (\mathbf{H}_1 + \mathbf{H}_2)_{n_1} \, d\sigma_1 \text{ and } I_2 \int_{S_2} (\mathbf{H}_1 + \mathbf{H}_2)_{n_2} \, d\sigma_2.$$

Thus (12) is replaced by an energy expression of four terms, which we shall write, as in (15.17),

$$W = \frac{1}{2}(L_{11}I_1^2 + (L_{12} + L_{21})I_1I_2 + L_{22}I_2^2), \quad (15)$$

$$L_{11} = \frac{\mu}{I_1} \int_{S_1} H_{1n_1} d\sigma_1, \quad L_{22} = \frac{\mu}{I_2} \int_{S_2} H_{2n_2} d\sigma_2, \quad (15a)$$

$$L_{12} = \frac{\mu}{I_2} \int_{S_1} H_{2n_1} d\sigma_1, \quad L_{21} = \frac{\mu}{I_1} \int_{S_2} H_{1n_2} d\sigma_2. \quad (15b)$$

If \mathbf{H} is expressed by Ψ , and Ψ by (8), it will be recognized that also with this definition

$$L_{12} = L_{21}.$$

We then obtain

$$4\pi L_{12} = \mu \int_{S_1} d\sigma_1 \int_{S_2} d\sigma_2 \frac{\partial^2}{\partial n_1 \partial n_2} \frac{1}{r_{12}}, \quad (16)$$

where r_{12} is the distance between any one point on S_1 and any second point on S_2 . Since the right side of (16) is symmetrical with respect to the indices 1 and 2 it represents also L_{21} .

By the representation in (15a, b) the magnetic fluxes Φ_1 and Φ_2 through S_1 and S_2 may be written

$$\begin{aligned} \Phi_1 &= \mu \int_{S_1} (\mathbf{H}_1 + \mathbf{H}_2)_{n_1} d\sigma_1 = L_{11} I_1 + L_{12} I_2, \\ \Phi_2 &= \mu \int_{S_2} (\mathbf{H}_1 + \mathbf{H}_2)_{n_2} d\sigma_2 = L_{21} I_1 + L_{22} I_2. \end{aligned} \quad (17)$$

The magnetic flux is thus now expressed by two elements (for n linear conductors by n elements) in terms of the two (or n) currents.

C. Application to the Selfinductance of a Two-Wire Line

As in §15E we regard conductor and return conductor as *one* closed circuit and designate, just as there, the separation of the wires with a , the radius with b . Since we cannot proceed to the limit $b \rightarrow 0$ our method of the scalar potential, which is restricted to linear conductors, is strictly inapplicable. However, we may regard Eq. (14), quite apart from its origin in this method, as the definition of the selfinductance L , or, more exactly, of that part of L which has its origin in the exterior of the wires (see Fig. 22).

The portion of the branch cut S which is of importance to us has here been shaded. It extends in the xy -plane (the plane containing the two axes of the wires) from the periphery of one wire to that of the other and is to have the length 1 in the y -direction. The magnetic field \mathbf{H} results, in gen-

eral, from the *vectorial* superposition of the magnetic fields H_1 and H_2 of the two wires. On the plane area S , H_1 and H_2 have, however, the same direction, since $I_1 = -I_2$, and are perpendicular to S . With x and $a - x$,

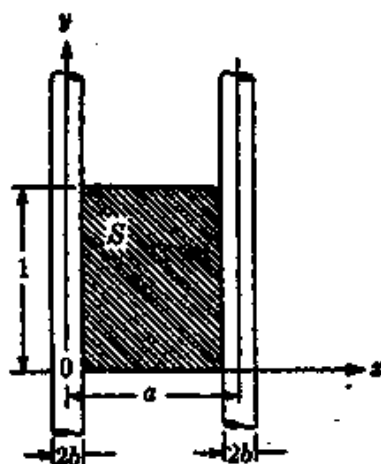


FIG. 22. Selfinductance of a two-wire line, computed from the magnetic flux through the branch cut S .

respectively, indicating the distance of the reference point on S from the conductor and the return conductor we obtain

$$H_x = \frac{I}{2\pi} \left(\frac{1}{x} + \frac{1}{a-x} \right);$$

hence we find from (14) for unity length of the two-wire line

$$L_a = \frac{\mu}{2\pi} \int_b^{a-b} \left(\frac{1}{x} + \frac{1}{a-x} \right) dx. \quad (18)$$

The subscript a of L indicates "external selfinductance". The evaluation of (18) yields:

$$L_a = \frac{\mu}{2\pi} \left(\log \frac{a-b}{b} - \log \frac{b}{a-b} \right) = \frac{\mu}{\pi} \log \frac{a-b}{b} \sim \frac{\mu}{\pi} \log \frac{a}{b}, \quad (19)$$

the last since $b \ll a$. If we compare this with (15.35), we see that the first part of the earlier formula corresponds to our "external" selfinductance and conclude from this that the second part will signify the "inner" selfinductance L_i .

We confirm this in the following manner: Employing the energetic definition of selfinductance in (15.17) we put

$$W_i = \frac{1}{2} L_i I^2. \quad (20)$$

W_i is the magnetic energy within a unit segment of our two-wire line. Within the individual wire the magnetic field is given, as in (15.33a), by

$$H = \frac{r}{b} \frac{I}{2\pi b},$$

if we neglect the magnetic field of the other single wire, which is weak in comparison. Hence the energy content of unit length of the individual wire becomes as in (15.33a)

$$\frac{\mu}{2} \int_0^b \int_0^{2\pi} H^2 r \, dr \, d\varphi = \mu\pi \int_0^b \left(\frac{r}{b} \frac{I}{2\pi b} \right)^2 r \, dr = \frac{\mu I^2}{4\pi b^4} \int_0^b r^3 \, dr = \frac{\mu I^2}{16\pi}.$$

This is one half of the energy W_i in (20). It hence follows from (20) that

$$L_i = \frac{\mu}{4\pi}. \quad (20a)$$

Together with (19) this confirms our earlier result (15.35).

D. Application to the Electromagnetic Current Measurement of Wilhelm Weber

We proceed from Eq. (8) and assume to begin with that the surface S bounded by the current I is *plane*; the reference point for which Ψ is to be computed is supposed to be at a *great* distance from S . Then the direction of dn and the magnitude of $1/r$ become the same for all elements $d\sigma$ of S . The integration over $d\sigma$ may be carried out directly and yields

$$4\pi\Psi = IS \frac{\partial}{\partial n} \frac{1}{r}. \quad (21)$$

At the same time we recall Eq. (7.10b) for the potential of a single magnetic pole p . From it we obtain for the potential of a dipole of the very small separation l between poles and the direction n of the axis:

$$4\pi\Psi = M \frac{\partial}{\partial n} \frac{1}{r}, \quad M = pl = \text{moment of the dipole}. \quad (22)$$

Comparison of (21) and (22) indicates: *The magnetic field of a current I bounding a surface S is, at large distance, equal to that of a dipole, we might also say a short bar magnet, which is placed perpendicular to the surface S and has the moment $M = IS$.*

A non-plane current path may be projected on three mutually perpendicular planes and the equivalent bar magnets may be arranged perpendicularly to the resulting plane current paths. Vectorial addition of their moments yields an obliquely oriented dipole, which at large distance again produces the same magnetic field as the original current path.

This *equivalence of current and magnetism* is the basis of the famous "electrodynamic determination of units" of Wilhelm Weber. Furthermore the electromagnetic system of units, which hails back to Weber, is based on it, i.e. on putting equal

$$I \cdot S = M. \quad (23)$$

The *electric* quantity I thus becomes a *magnetic* quantity. Or rather: For the electric quantity I a quantity M/S , which appears different in character, is substituted. This is possible only if a definite relation is established between the dimension of the magnetic pole, which as before we shall designate as P , and the dimension Q of charge. By (23) this relation is

$$Q \cdot \frac{\text{area}}{\text{time}} = P \cdot \text{length}$$

or

$$P = Q \cdot \frac{\text{length}}{\text{time}} = Q \cdot \text{velocity}.$$

We thus arrive at *Ampère's hypothesis* according to which magnetism is merely electricity in motion. However, we mentioned already on p. 47 that this hypothesis is today, after the discovery of the neutron as a basic element of all nuclear matter, no longer as binding as a hundred years ago; we also saw in §8B that Cohn's system of units is independent of this hypothesis and is recommended particularly by that fact.

Our system of the four units MKSQ bears a peculiar relation to the electromagnetic cgs-system introduced by Weber. As we know, our unit $Q = 1$ coulomb = 1 ampere-second is defined as 1/10 of the electromagnetic unit of charge. The fundamental constants of vacuum, with due regard for the experimental fact $\epsilon_0 \mu_0 = 1/c^2$, were therefore found to be

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{\text{joule S}^2}{Q^2 M} = 4\pi \cdot 10^{-7} \frac{\text{henry}}{M}, \text{ Eq. (15.18b)} \quad (24)$$

$$\epsilon_0 = \frac{1}{\mu_0 c^2} = \frac{10^9}{4\pi c^2} \frac{Q^2 M}{\text{joule S}^2} \simeq \frac{10^{-9}}{36\pi} \frac{\text{farad}}{M}, \text{ Eqs. (7.17) and (10.3b)} \quad (25)$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = 4\pi c 10^{-7} \frac{\text{joule S}^2}{Q^2 M} \simeq 120\pi \Omega, \text{ Eqs. (7.19) and (4.5c).} \quad (26)$$

However, we know that, apart from the electromagnetic (more briefly *magnetic*) system, also an electrostatic (more briefly *electric*) system is in use. Here the arbitrary, and only historically justifiable, convention (7.8) is made:

$$f = 4\pi e_0 = 1 \quad (27)$$

and from this an *electrostatic unit* of charge e_{st} is defined. We ask how Q is to be expressed in this unit, i.e. what value Q/e_{st} may have, having fixed the value of Q measured in terms of the *electromagnetic unit* of charge e_{mag} by

$$\frac{Q}{e_{mag}} = \frac{1}{10}. \quad (28)$$

From Eqs. (25) and (27):

$$4\pi\epsilon_0 = 1 = \frac{10^9}{c^2} \frac{(Q/e_{el})^2 \cdot 10^3 \text{ cm}}{10^7 \text{ erg sec}^2}.$$

From this and (28) it follows that

$$1 = \frac{1}{c^2} (e_{\text{magn}}/e_{el})^2, \quad \frac{e_{\text{magn}}}{e_{el}} = c \simeq 3 \cdot 10^{10} \frac{\text{cm}}{\text{sec}}. \quad (29)$$

The unit of charge in the magnetic system is $3 \cdot 10^{10}$ times larger than the unit of charge in the electric system. This corresponds to our metaphor of river and waterfall on p. 53. The numerical values of a given physical charge behave of course in inverse fashion. Thus the coulomb has, in the electric cgs-system, the numerical value

$$Q = \frac{1}{10} \cdot c = 3 \cdot 10^9;$$

see also our data on the charge of the electron in §8D:

$$4.80 \cdot 10^{-10} \text{ electric and } 1.60 \cdot 10^{-20} \text{ magnetic units.}$$

The conversion rule for the numerical value e of an arbitrary charge in and MKSQ system to the corresponding numerical values e_{magn} and e_{el} is

$$10^9 \left(\frac{e^2}{4\pi\epsilon_0} \right)_{\text{MKSQ}} = c^2 e_{\text{magn}}^2 = e_{el}^2. \quad (30)$$

Since the quotient D/e on the one hand and the product Ee on the other are independent of our fourth unit, the conversion of D and E may readily be derived from Eq. (30); for the conversion of E (30) yields

$$10 (4\pi\epsilon_0 E^2)_{\text{MKSQ}} = \frac{1}{c^2} E_{\text{magn}}^2 = E_{el}^2. \quad (31)$$

Rules (30) and (31) replace the rule given by H. A. Lorentz on p. 87 of Vol. V₂ of the *Mathematische Enzyklopädie*.

With this the unpleasant business of the electrical units may be regarded as definitely disposed of.

§17. Detailed Treatment of the Field of a Straight Wire and a Coil

We consider the apparently trivial case of an infinitely long straight wire carrying a stationary current with a return path in a coaxial hollow cylinder surrounding it. Let the radius of the wire be a , the inner radius of the hollow cylinder b , and the outer radius $c \rightarrow \infty$. For reasons of symmetry

the magnetic field is known directly, and similarly the current density. As in §4, Eqs. (10) to (13), we have

$$\begin{aligned} r < a, \quad H &= \frac{r}{a} \frac{I}{2\pi a}, \quad J = \frac{I}{\pi a^2}, \\ a < r < b, \quad H &= \frac{I}{2\pi r}, \quad J = 0, \\ b < r < \infty, \quad H &= \frac{I}{2\pi r}, \quad J = 0. \end{aligned} \quad (1)$$

The direction of H is everywhere azimuthal: $H = H_\varphi$.

In the interior the electric field is everywhere axial in direction and has by Ohm's law the value

$$E = E_z = \frac{J}{\sigma} = \frac{I}{\pi a^2 \sigma}, \quad r \leq a. \quad (2)$$

Similarly by Ohm's law we have within the hollow cylinder (conductivity σ_1)

$$E = \frac{J}{\sigma_1} = 0, \quad b \leq r < \infty. \quad (3)$$

In the region between wire and hollow cylinder the field is yet to be determined, from the differential equations

$$\mathbf{E} = -\text{grad } \Psi, \quad \Delta \Psi = 0 \quad (4)$$

and the boundary conditions

$$\frac{\partial \Psi}{\partial z} = -E_z = \begin{cases} -\frac{I}{\pi a^2 \sigma} & \text{for } r = a, \\ 0 & \text{for } r = b. \end{cases} \quad (5)$$

Since these conditions are independent of z and φ we can write for the solution of $\Delta \Psi = 0$

$$\Psi = \Psi_1(r)z. \quad (6)$$

We can omit an additive term $\Psi_2(r)$ independent of z since we can satisfy all the conditions of the problem with formula (6). We then obtain from (5)

$$\Psi_1(a) = -\frac{I}{\pi a^2 \sigma}; \quad \Psi_1(b) = 0. \quad (7)$$

The differential Eq. (4) demands

$$\frac{d}{dr} r \frac{d\Psi_1}{dr} = 0, \quad \Psi_1 = A \log r + B,$$

which yields, with (7)

$$\Psi_1(r) = -\frac{I}{\pi a^2 \sigma} \log \frac{r}{b} / \log \frac{a}{b}. \quad (8)$$

Now the field \mathbf{E} is known also in the intermediate space $a < r < b$. According to (6) and (8) it is represented by

$$E_z = -\Psi_1 = \frac{I}{\pi a^2 \sigma} \log \frac{r}{b} / \log \frac{a}{b}, \quad (9)$$

$$E_r = -\frac{\partial \Psi_1}{\partial r} z = \frac{Iz}{\pi a^2 \sigma r} / \log \frac{a}{b}. \quad (9a)$$

Thus the field is here by no means axial in direction, as in the interior of the wire; rather, its radial component is of the same order of magnitude as its axial component.

In passing from the interior to the exterior of the wire there occurs a jump in E_r and hence also in the corresponding component of the excitation D , which indicates the existence of a *surface charge*:

$$\omega = D_r = eE_r = \frac{eIz}{\pi a^2 \sigma} / \log \frac{a}{b}. \quad (10)$$

This surface charge decreases linearly along the wire, from positive to negative values, in formal language from $+\infty$ to $-\infty$. It depends only slightly, i.e. logarithmically, on the radius b of the outer return conductor. The zero point of the charge remains undetermined since the point $z = 0$ can be fixed arbitrarily. We may eventually identify it with the "center" of the wire, which, for infinite length, also remains indefinite.

We can obtain an idea as to the magnitude of this charge in the following manner: The dielectric constant in (10), which refers to the exterior of the wire, does not differ materially from that in the interior of the wire (though for metals it is rather hypothetical). Hence the quotient e/σ does not differ materially from the relaxation time T_r , defined in (4.9a), for the material of the wire; this is of the order of *microseconds*. The product eI/σ occurring in (10), which represents a charge, is thus not of the order of an ampere-second = Q , but of the order of a microampere-second = $10^{-6} Q$. The peripheral charge of the wire and the radial field strength corresponding to it are hence very small. This is the reason why they generally remain unnoticed both in theory and in experiment although, as we shall see, they are essential for an understanding of the current transport.

Implicit in the existence of a radial electric field is the appearance of a potential difference between wire and return conductor. According to (9a) it is given by

$$V = \int_a^b E_r dr = -\frac{Iz}{\pi a^2 \sigma}. \quad (11)$$

We compare it with the charge on unit length of the wire, $e = 2\pi a\omega$, or the equal and opposite charge on unit length of the return conductor. We find

$$\frac{e}{V} = \frac{2\pi e}{\log \frac{b}{a}}. \quad (11a)$$

This is the capacity per unit length of the cylindrical condenser formed by the wire and the return conductor (see Problem II.5). We speak here and in similar cases of a "distributed capacity."

Fig. 23 shows the shape of the *equipotential lines* $\Psi = \text{const.}$ in the z, r -plane, given according to Eqs. (6) and (8) by

$$z \log \frac{b}{r} = C. \quad (12)$$

For $C = 0$, $z = 0$ and $r = b$, corresponding to the broken lines ABC and ABD of the figure. The equipotential lines for $C > 0$ accommodate themselves within the area bounded by them. The angles under which they meet the surface of the wire deviate increasingly from a right angle with increasing distance from A . The orthogonal trajectories to the equipotential lines represent the *lines of force*; the arrows on them indicate the direction from positive to negative surface charge ω . Two bounding curves passing through the "equilibrium point" B (see footnote 2 after Eq. (9.14b)) belong to this family of curves. According to (12) the equipotential lines satisfy the differential equation

$$\log \frac{b}{r} dz - z \frac{dr}{r} = 0;$$

on the other hand, the lines of force, which are orthogonal thereto, (replacement of dz/dr by $-dr/dz$) are given by

$$z dz + r \log \frac{b}{r} dr = 0.$$

In the neighborhood of B we find, with $\rho = b - r$,

$$z dz - \rho d\rho = 0.$$

Thus *two* lines of force, with tangent directions $z = \pm\rho$, pass through the point B forming a right angle with each other. At a greater distance from these bounding curves, above and below them, the lines of force pass more or less radially from the wire surface to the outer conductor.

The equipotential lines represent at the same time the paths of the *energy flux* \mathbf{S} ; the arrows marked on them indicate the direction of \mathbf{S} . From

the formula $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, \mathbf{S} is perpendicular to \mathbf{H} and hence lies in the plane of the drawing, since \mathbf{H} is everywhere perpendicular thereto; in addition \mathbf{S} is perpendicular to \mathbf{E} and hence has everywhere the direction of the family of curves $\Psi = \text{const.}$ An application of the right-screw rule for the three vectors \mathbf{E} , \mathbf{H} , \mathbf{S} shows that the arrows are properly oriented.

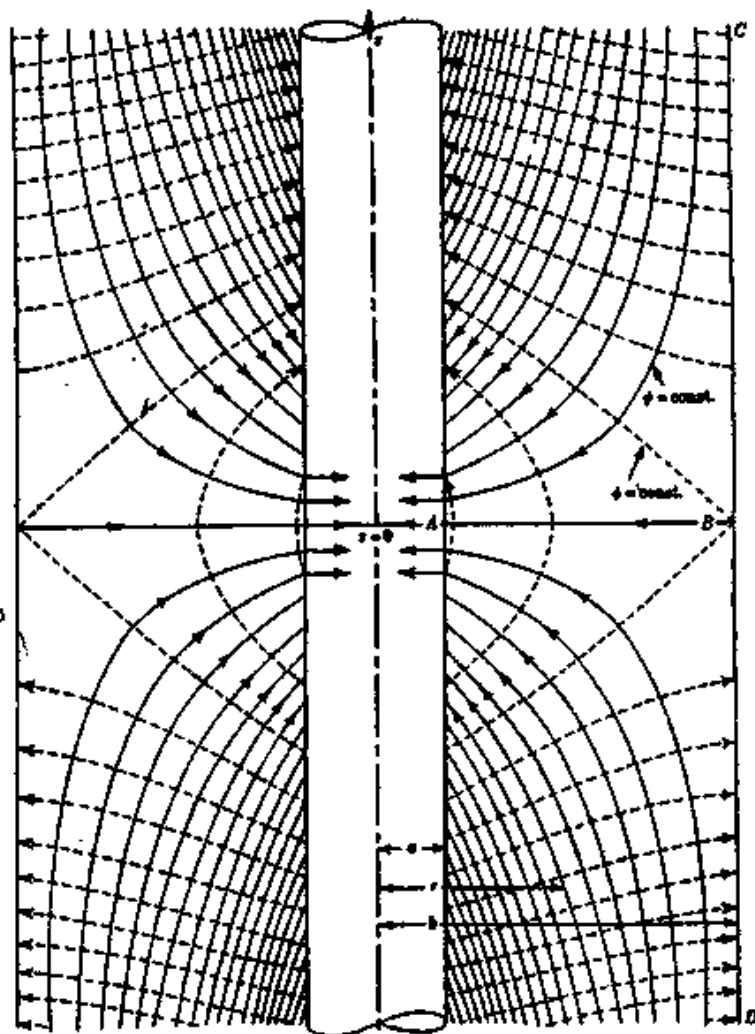


FIG. 23. Energy flux about a straight wire carrying stationary current with coaxial return conductor. Equipotential lines = stream lines of the energy drawn out full, electrical lines of force = excitation lines drawn dotted. They form tubes of constant charge starting and ending on the surface of the wire and the return conductor, respectively. Their number per unit length of the wire indicates the surface charge on the latter and its linear increase with distance from A , positive for $z < 0$, negative for $z > 0$.

Since $\mathbf{E} = \mathbf{J}/\sigma$ the lines of force within the wire (not drawn in the figure) are axial in direction; hence the vector \mathbf{S} is directed inward, perpendicular to the surface of the wire; here also it lies in the equipotential surfaces $\Psi = \text{const.}$ The energy flux is dissipated in the interior of the wire since it becomes zero for $r = 0$, in view of $\mathbf{H} = 0$. This is indicated in the figure by the terminated arrows at A . Such arrows should be imagined along the

entire surface $r = a$. The energy entering from the surface is converted into Joule heat in the interior of the wire.

According to (9) and (1) the magnitude of the energy flux for $r = a$ is

$$|S| = E \cdot H_r = \frac{I}{\pi a^2 \sigma} \frac{I}{2\pi a};$$

the energy supply to unit length of the wire from all sides hence becomes

$$2\pi a |S| = I^2 R_1, \quad R_1 = \frac{1}{q\sigma}, \quad q = \pi a^2. \quad (13)$$

R_1 is the ohmic resistance of unit length of the wire; (13) thus yields, in fact, the Joule heat claimed by this unit length.

Accordingly we obtain the following total picture of the behavior of the energy: *Outside of the wire the energy flows from the electrodes $z = \pm\infty$ from all sides toward the surface of the wire. After entering it it flows radially toward the axis of the wire, being converted at the same time into heat. There is no energy flux parallel to the wire axis within the wire.*

This picture is materially different from the popular concept of the energy transfer in a wire carrying current. From the Maxwellian standpoint there is no doubt, however, about the inner consistency and unique validity of our picture. It indicates the fundamental change which Maxwell's theory has brought in the concepts conductor and nonconductor: *The conductors are nonconductors of energy. Electromagnetic energy is transported without loss only in nonconductors; in conductors it is destroyed, or rather transformed.* The notation "conductor" and "nonconductor" refers only to the behavior with respect to *charge*; it is misleading if applied to behavior with regard to *energy*.

We thus come to the conclusion that our simple example is after all not as trivial as it appeared.

The next-simplest case of the circular conductor is already beyond elementary treatment. Even in the limiting case of the linear conductor the geometrical formula for the solid angle involves an elliptic integral. In fact, if we use the polar coordinates r, φ, z for the reference point and $\rho, \alpha + \varphi - \pi, 0$ for the point of integration we find from Eq. (16.8), denoting the distance between point of integration and reference point by R ,

$$\begin{aligned} \int \frac{\partial}{\partial n} \frac{1}{R} d\sigma &= \int_0^\pi \rho d\rho \int_0^{2\pi} d\alpha \frac{\partial}{\partial z} (r^2 + \rho^2 + 2r\rho \cos \alpha + z^2)^{-1/2} \\ &= 4 \frac{\partial}{\partial z} \int_0^\pi \frac{\rho d\rho}{\sqrt{(r+\rho)^2 + z^2}} \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}} \\ \beta &= \alpha/2 \\ k^2 &= \frac{4r\rho}{(r+\rho)^2 + z^2} \end{aligned}$$

The preceding integral with respect to β is a "complete elliptical integral of the first kind in the Legendre standard form," and k is the "modulus" thereof. We cannot of course here delve further into the treatment of this formula.

The exact treatment of a coil of wire of finite thickness with finite pitch would be even more complex. We hence pass directly to the limiting case of very small thickness and pitch, i.e. Ampère's solenoid, whose magnetic field was already discussed in Eq. (4.14), though only superficially. We now wish to compare it with the field of a permanent uniformly magnetized bar magnet of the same dimensions as the coil. We shall here assume the interior of the coil and the coil wire to be unmagnetic ($\mu = \mu_0$).

We shall show that the \mathbf{H} -field of the coil corresponds to the \mathbf{B} -field of such a bar magnet. As proof we write down the boundary conditions and differential equations for the two cases side by side:

<i>Coil</i>	<i>Bar Magnet</i>
$(\mathbf{H}_n - \mathbf{H}_t)_s = N_1 I,$	$(\mathbf{B}_n - \mathbf{B}_t)_s = \mu_0(\mathbf{H}_n - \mathbf{H}_t - \mathbf{M})_s = -\mu_0 \mathbf{M}_s,$
$(\mathbf{H}_n - \mathbf{H}_t)_n = 0,$	$(\mathbf{B}_n - \mathbf{B}_t)_n = 0,$
$\Delta\psi = 0,$	$\text{div } \mathbf{B} = \mu_0 \text{div } \mathbf{H} = -\mu_0 \Delta\psi = 0.$

The first line relates to the *mantel surface*. Its left half states simply, in the terminology introduced at Eq. (4.4e), that the surface curl on the mantel surface of the coil is equal to $N_1 I$, where N_1 is the number of turns per unit length. The right half of the first line follows from our equation $\mathbf{B} = \mu_0(\mathbf{H}_t + \mathbf{M})$ for the interior of the bar magnet, which for the exterior becomes $\mathbf{B} = \mu_0 \mathbf{H}_n$. It states that the surface curl of \mathbf{B} on the mantel surface of the bar is equal to $-\mu_0 \mathbf{M}$. A comparison of the two halves of the first line shows that, in the bar magnet, the quantity $-\mu_0 \mathbf{M}/N_1$ corresponds to the current I in the coil.

We apply the second line in particular to the two *end surfaces*. Here both \mathbf{B} and \mathbf{H} are of course continuous for the solenoid, for the bar magnet only \mathbf{B} . Since we have assumed $\mu = \mu_0$ the same equations apply also for the mantel surface.

The third line applies in the two cases both for the interior and for the exterior; here we must recall our assumption regarding the bar magnet that its magnetization was supposed to be uniform, since otherwise the term $\mu_0 \text{div } \mathbf{M}$ would have to be added to the term $\mu_0 \text{div } \mathbf{H}$ and this would spoil the comparison of the two cases.

We see therefore that our earlier Fig. 17, which represented the \mathbf{B} -field of the bar magnet, reproduces simultaneously the \mathbf{H} -field of the solenoid. Accordingly the earlier Eq. (4.14), which applies only for an infinitely long coil, is rounded out graphically by Fig. 17. This provides us now with a complete picture of the spreading of the lines of excitation at the ends of the coil and of their exit through the convex surface.

With respect to the representation of the coil field by the potential Ψ we wish to point out expressly its familiar multivalued character; the branch cut S , which makes it single valued, is a helical surface of infinitely small pitch which follows the turns of the wire. It follows that the equation

$$\oint \mathbf{H} \cdot d\mathbf{s} = 0$$

which is universally valid for the bar magnet loses its validity for the coil if the path of integration links one or several turns. Hence also the conclusion with regard to the demagnetizing character of the \mathbf{H} -field of the bar magnet is not applicable to the coil. For to reach this conclusion, we employed (see p. 85) a path which inside was along the axis of the bar and on the outside led back to the bar axis. This same path, for the coil, intersects the above-mentioned helical surface and therefore is not a permissible path of integration.

We can also make the following statement: The \mathbf{H} -field of the *bar magnet* is *lamellar* throughout, that of the *coil* is *not*; instead it has the *curl I concentrated* at the turns of wire. The \mathbf{H} -field of the coil is *solenoidal* throughout, since everywhere $\mathbf{B} = \mu_0 \mathbf{H}$, that of the bar magnet is *not*. For the uniformly magnetized bar the bar ends have a *surface distribution of divergence*. In view of the proportionality of \mathbf{B} and \mathbf{H} for the coil, Fig. 17 evidently also represents the \mathbf{B} -field of the coil. The \mathbf{H} -field of the bar magnet, Fig. 18, is materially different from Fig. 17.

The near-uniformity of the internal field evident in Fig. 17 suggests the computation of the magnetic energy W of the coil by the formula

$$W = \frac{\mu}{2} \mathbf{H}^2 V, \quad (14)$$

where $V = \pi a^2 l$ is the volume of the interior of the coil. In view of the relation $Hl = NI$ (Eq. (4.14)), where N is the number of turns along the full length of the coil, (14) takes the form

$$W = \frac{\pi a^2}{l} \mu N^2 \frac{I^2}{2}. \quad (14a)$$

A comparison with our energetic definition of selfinductance in (15.17) leads to the following value of the latter:

$$L = \frac{\pi a^2}{l} \mu N^2. \quad (15)$$

This formula applies, of course, as may also be seen from Fig. 17, only for a *very long* coil and hence can scarcely be used for forms encountered in practice. On the other hand, it retains its validity if the straight coil is bent into a *ring electromagnet*, because of the close approach to field uniformity

within the latter. Here the factor μ in (15) also becomes significant if the ring is provided with a soft-iron core. Eq. (15) shows that this arrangement, first suggested by Ampère, has a much greater selfinductance and hence realizes a much greater concentration of energy than the electromagnet without iron *for equal coil current*.

This increase in energy concentration applies qualitatively also to the straight coil with soft-iron core. The quantitative computation of field and energy would, however, be much more complex, since a boundary-value problem, corresponding to the passage of the magnetic lines of force from iron into air, would be added to the summation problem which we have treated.

§18. Quasi-Stationary Currents

Most of the problems of electrical engineering and of laboratory physics lie within the domain of the *slowly variable fields*. It is true that there is no one-word answer to the question "slow compared with what?". For vibrations which are periodic in time or exhibit a damped periodicity the answer may be identified with the demand that the light path corresponding to the period τ of the vibration be large compared to the dimensions l of the apparatus:

$$c\tau \gg l. \quad (1)$$

It is then permissible to neglect the "retardation of the fields" to be introduced in §19. At the end of this paragraph we shall, however, treat successfully also very long lines, which do not obey this condition, on a quasi-stationary basis, by subdivision into differential segments.

In general terms the *quasi-stationary* approximation consists in calculating all fields as for *stationary* processes. In this manner it becomes possible to establish a *linear* relation between the expressions occurring in the integral form of the Maxwell equations. We refer to the magnetic flux Φ through a closed curve, the current I through a cross section, and the electromotive forces V_a along segments of the path of integration, which add up to the loop e.m.f. V for this path:

$$\Phi = \int B_n d\sigma, \quad I = \int J_n d\sigma, \quad V_{12} = \int_1^2 E_s ds,$$

$$V = \sum V_a = \oint E_s ds.$$

We know that, under stationary conditions, the following relations exist between them:

$$\Phi_1 = L_{11}I_1 + L_{12}I_2 + L_{13}I_3 + \dots \text{ (magn. flux through circuit 1)}$$

$$I = \frac{V}{R} \text{ (Ohm's law), } I = \dot{V}K \text{ (condenser charge).}$$

The inductances L , resistances R , and capacities K depend, apart from the material constants, only on the geometry of the field and are the result of an integration over the *space* coordinates. Hence only an integration with respect to the *time* remains to be carried out. The mathematical simplification achieved in this manner is considerable: While the exact treatment of rapidly variable fields demands the integration of the Maxwell *partial differential equations*, the integration of *ordinary differential equations* with constant coefficients suffices for slowly varying fields; for periodic processes these reduce even to *algebraic equations*.

This method was developed by Gustav Kirchhoff (see beginning of §15) and was applied to metal wire loops and networks composed of them. It is in general appropriate to choose the closed path of integration along the metal wires; an interruption of the metallic path by nonconducting gaps (condensers) does not interfere with the method, however. The case of *thin* wires is particularly convenient, since here the field within the wire is practically uniform.

The *first* Maxwell equation in integral form then yields for the chosen closed path of integration

$$-\dot{\Phi} = V = \text{loop e.m.f.} \quad (2)$$

The *second* Maxwell equation finds expression in the computation of the coefficients of induction insofar as it represents the production of magnetic fields by currents and in the computation of the resistances and capacities insofar as it represents the origin of these currents from the electric field.

For a simple current loop without branching the two sides of (2) may be expressed in terms of the current I which is the *same* for all cross sections:

$$-I \sum L = I \sum R + \int I \, dt \sum \frac{1}{K} + E. \quad (3)$$

It is seen that here the selfinductance coefficients of all the magnetic fields can be combined in a single expression, which may be represented by an imaginary coil at an arbitrary point of the circuit. The same applies for the resistances and the capacitances. E is the presumably known e.m.f. between the terminals by which the current enters and leaves, i.e. the so-called *terminal e.m.f.* The introduction of this e.m.f. E conveniently avoids carrying the path of integration through apparatus whose action is not covered by the Maxwell theory (galvanic or thermoelements, photoelectric cells, electron tubes) or through "machines" which, though they function fully within the framework of the theory, would unduly complicate the problem.

If *several* loops are joined in a *network*, the latter may be regarded as made up of elements connecting the "junction points". Let the n th element carry a current I_n from one junction point to another. Since we do not as yet know its sign, we place a marker arrow on the element which is to

indicate in what direction we shall reckon the current as positive. In view of the absence of current sources we have for the junction points

$$\sum I_n = 0 \quad (4)$$

and for every circuit made up of arbitrary elements Eq. (2) applies again in the form

$$\sum I_n R_n + \sum V_{em} = -\dot{\Phi} \quad (= \text{e.m.f.}). \quad (5)$$

(4) and (5) are known as the "first" and "second" Kirchhoff equations. They date from the time preceding Maxwell's theory. Hence Kirchhoff places on the right side of (5) not $-\dot{\Phi}$, but the older concept of the *electromotive force* (e.m.f.) of all "current sources" which are inserted in the closed circuit. If we are dealing with currents which result from Faraday induction, e.g. in coils of machines, this e.m.f. becomes exactly identical with $-\dot{\Phi}$ and its concept is superfluous. It has, however, the advantage of covering also the effect of other current sources (batteries etc.) without requiring an examination of the physical processes taking place therein.

A. Energetic Interpretation of the Wave Equation

We consider in particular an unbranched circuit with selfinductance, capacitance, and ohmic resistance, which we imagine connected in series and, as in Eq. (3), concentrated at certain points of the circuit. Thus we insert a resistance box in place of the resistance which is distributed over the wire; a coil in place of the distributed selfinductance with which we became familiar in connection with the two-wire line in §15E; and we shall not consider distributed capacities in the circuit but assume instead the presence of an electric plate condenser.

The law of conservation of energy offers a particularly convenient approach to the treatment of such a system, just as to the treatment of material vibrations in mechanics. We write it in the form of Eq. (5.7a):

$$\dot{W}_m + \dot{W}_e + Q = \oint S_n d\sigma. \quad (6)$$

W_m is the magnetic energy concentrated in the coil, given according to Eq. (17.14a) by

$$W_m = \frac{L}{2} I^2. \quad (6a)$$

W_e is the electric energy within the plate condenser, according to Eq. (10.11) given by

$$W_e = \frac{1}{2K} e^2. \quad (6b)$$

Here $\pm e$ are the varying charges on the two condenser plates and

$$I = \frac{de}{dt}. \quad (6c)$$

Q is the Joule heat generated within the resistance box:

$$Q = RI^2. \quad (6d)$$

The energy flux S refers to the current source. By Eq. (15.40) we write for the energy supplied by it in unit time

$$\oint S_n d\sigma = EI(\text{volt-ampere} = \text{watt}). \quad (6e)$$

E is the terminal e.m.f. of the current source mentioned above.

From (6a, b) follows

$$\dot{W}_m = LI\dot{I}, \quad \dot{W}_e = \frac{1}{K} e\dot{e} = \frac{1}{K} eI. \quad (6f)$$

Substitution of (6d, e, f) in (6) leads to, after canceling of a factor I :

$$L\dot{I} + RI + \frac{1}{K} e = E, \quad (7)$$

and, after a second differentiation with respect to t , to the wave equation

$$L\ddot{I} + R\dot{I} + \frac{1}{K} I = \dot{E}. \quad (7a)$$

In the terminology of point mechanics there corresponds thus

\dot{E} to the exciting force,

L to the inertia, more particularly the mass of the vibrating particle,

R to the damping, and

K to the coefficient of the restoring force.

As in mechanics, we distinguish between free and forced vibrations.

a. Free Vibrations

We set $E = 0$ and ask for the solution of the homogeneous equation

$$L\ddot{I} + R\dot{I} + \frac{1}{K} I = 0. \quad (8)$$

We can assume a trigonometric form for I , but know from point mechanics that it is definitely preferable to use instead the exponential form at the start and to pass to the real part of I only after the integration. We therefore set¹

$$I = I_0 e^{i\omega t} \quad (8a)$$

¹ Temporarily we adhere to custom in employing the usual positive sign of i in the exponent, although we generally prefer the negative sign. See e.g. §6, Eq. (11).

and obtain for the circular frequency $\omega_0 = 2\pi/\tau_0$ of the free vibration the quadratic equation

$$-L\omega_0^2 + iR\omega_0 + \frac{1}{K} = 0. \quad (8b)$$

For no damping we find

$$\omega_0^2 = \frac{1}{KL}, \quad \tau = 2\pi\sqrt{KL}. \quad (9)$$

This is the Kirchhoff-Thomson formula. If it is necessary to take account of damping the solution of (8b) yields

$$\omega_0 = \frac{iR}{2L} \pm \sqrt{\frac{1}{KL} - \frac{R^2}{4L^2}}. \quad (9a)$$

The process is aperiodic or periodic, depending on whether

$$\frac{R}{2L} > \frac{1}{\sqrt{KL}} \quad \text{or} \quad \frac{R}{2L} < \frac{1}{\sqrt{KL}}. \quad (9d)$$

In the aperiodic case ω_0 is purely imaginary and the current (8a) decreases *monotonically*. In the periodic case usually found with condenser discharges the angular frequency is

$$\sqrt{\frac{1}{KL} - \frac{R^2}{4L^2}}. \quad (9c)$$

Since the ohmic damping occurs here only as a correction of the second order as compared with the first term on the right of (9a), (9c) can usually be replaced by (9)—in analogy with the mathematical pendulum, where the formula $\tau = 2\pi\sqrt{l/g}$ is not affected appreciably by air resistance etc; hence it is generally permissible to write in place of (8a):

$$I = I_0 e^{-Rt/(2L)} \cdot e^{\pm 2\pi i t/\tau_0}, \quad (9d)$$

where τ_0 now represents the value (9) with no damping. The double sign in (9d) evidently becomes unimportant when finally passing to the real part, but permits taking account of the phase of the oscillation which, like the amplitude, may be prescribed arbitrarily, through superposition of the two solutions.

b. Forced Vibrations

Here we proceed preferably from Eq. (7) and set

$$E = E_0 e^{i\omega t};$$

ω is the angular frequency of the alternating current source. The circuit oscillates with the same rhythm as soon as its characteristic vibrations,

which are determined by an arbitrary initial state, have decayed. We have therefore

$$I = I_0 e^{i\omega t}, \quad \dot{I} = i\omega I, \quad e = \frac{I}{i\omega} \quad (\text{see (6c)}).$$

We thus obtain from (7)

$$\left(i\omega L + R + \frac{1}{i\omega K} \right) I = E,$$

for which we shall write more briefly

$$\mathbf{R}I = E. \quad (10)$$

(10) is *Ohm's law for alternating currents*; the real ohmic resistance R is here replaced by the complex impedance

$$\mathbf{R} = R + i \left(\omega L - \frac{1}{\omega K} \right). \quad (10a)$$

If we put

$$\mathbf{R} = |\mathbf{R}| e^{i\alpha}, \quad (10b)$$

we evidently have

$$|\mathbf{R}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega K} \right)^2}, \quad \tan \alpha = \frac{\omega L - 1/(\omega K)}{R}. \quad (10c)$$

We introduce the following designations:

$$\begin{aligned} R &= \text{resistance} \\ \omega L - 1/(\omega K) &= \text{reactance} \\ \omega L &= \text{inductive reactance} \\ 1/(\omega K) &= \text{capacitive reactance} \\ |\mathbf{R}| &= \text{impedance.}^1 \end{aligned}$$

We give an interpretation of Eq. (10) in the complex Gaussian plane of Fig. 24. The two-dimensional "vector" I lags by the constant angle α behind the two-dimensional "vector" E . Of course only the real parts of E and I have physical meaning.

The engineer calls this representation a rotating vector diagram; The figure should in fact be thought of as rotating with the angular velocity ω as time progresses. The projections of the two-dimensional vectors E and I on the real axis give the instantaneous values of these quantities.

We shall establish furthermore how the energy flowing into the system

¹ In electrical engineering this term, incidentally, is used not only for $|\mathbf{R}|$, but also for our impedance operator \mathbf{R} itself.

is used up. To this end we multiply Eq. (7) with I , where now I and E are to represent their real parts. We find

$$\frac{L}{2} \frac{d}{dt} I^2 + RI^2 + \frac{1}{2K} \frac{d}{dt} e^2 = IE. \quad (11)$$

If we average over a period $\tau = 2\pi/\omega$ of the vibration, we obtain:

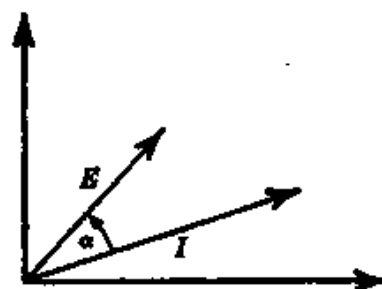
$$\frac{1}{\tau} \int_0^\tau RI^2 dt = \frac{1}{\tau} \int_0^\tau IE dt,$$

so that

$$R\bar{I}^2 = \bar{IE}. \quad (11a)$$

The contributions derived from L and K vanish since they are given by differentials. The average power \bar{IE} introduced into the system is thus dis-

FIG. 24. Representation of the complex E and I in the Gaussian plane. Lag of I with respect to E by the angle α .



sipated entirely in the ohmic resistance R . The imaginary part of our impedance operator R has a purely wattless effect on the e.m.f., i.e. it does not consume energy on a time average. We can calculate directly

$$\bar{I}^2 = \frac{1}{\tau} \int_0^\tau I_0^2 \cos^2(\omega t - \alpha) dt = \frac{1}{2} I_0^2; \quad (11b)$$

$$I_{eff} = \frac{1}{\sqrt{2}} I_0 \text{ and correspondingly } E_{eff} = \frac{1}{\sqrt{2}} E_0; \quad (11c)$$

I_{eff} is called the "effective current", E_{eff} the "effective voltage". For the average power we find

$$\begin{aligned} \bar{EI} &= \frac{1}{\tau} \int_0^\tau E_0 I_0 \cos \omega t \cos(\omega t - \alpha) dt \\ &= \frac{1}{\tau} E_0 I_0 \left\{ \int_0^\tau \cos^2 \omega t \cos \alpha dt + \int_0^\tau \cos \omega t \sin \omega t \sin \alpha dt \right\} \quad (12) \\ &= \frac{1}{2} E_0 I_0 \cos \alpha = I_{eff} E_{eff} \cos \alpha. \end{aligned}$$

Analog: Work = Path · Projection of the force on the path, where the projection must be carried out in the complex plane in the present example. This analogy applies however only for the time average, in which the second

integral of the middle line of (12) vanishes. The latter signifies an oscillation of the energy between storage (K, L) and the current source (E).

B. The Wheatstone Bridge

We distinguish between the four bridge arms a, b, c, d and the arms e and f , containing the current source and the galvanometer, respectively. The disappearance of the current in the galvanometer arm is attained by adjustment of the slide-wire contact S (or eventually the two sliding contacts S and T). Thus a *null method*, viewed with such favor in physical

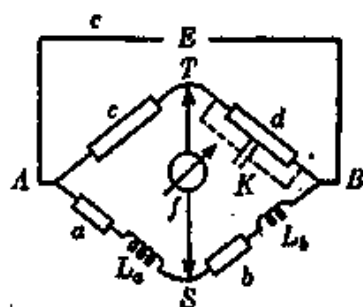
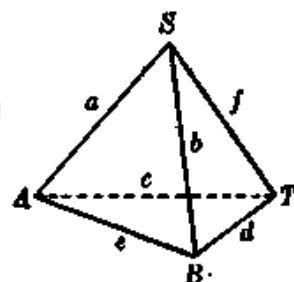


FIG. 25. Wheatstone bridge: a, b, c, d branches of bridge with resistance boxes and induction coils connected in series and, eventually, capacitances connected in parallel; f and e , branches containing galvanometer and voltage source; S and T slide contacts.

measurements, is realized. The geometrical structure of the bridge is best represented by a tetrahedron (Fig. 26), the six arms being transformed into the six sides without altering their relationship, as in analysis situs. The two arms e and f then become opposite sides, similarly the arms a, d and b, c , while the arms a, b etc. become "adjoining sides".

FIG. 26. Space representation of Wheatstone bridge as tetrahedron; adjoining and opposite branches.



From the great range of applications of the Wheatstone bridge we select two characteristic special cases; the trivial case of the comparison of two ohmic d.c. resistances will be a by-product:

a. Comparison of two selfinductances

b. Comparison of a selfinductance and a capacitance

a. Let the two selfinductances L_a and L_b be inserted in the adjoining arms a and b , in series with the ohmic resistances a and b , as shown in Fig. 25. In order that there may be no current in the arm f , we adjust the sliding contacts S and T so that there is no difference of potential between them. Starting from A we have, from (10) and (10a),

$$(a + i\omega L_a)I_1 = cI_2 \quad (13)$$

and starting from B :

$$(b + i\omega L_0)I_1 = dI_2, \quad (13a)$$

so that

$$(a + i\omega L_0)d = (b + i\omega L_0)c. \quad (14)$$

If this equation is to be satisfied, the equality must exist individually for the real and for the imaginary parts. Thus:

$$\begin{aligned} ad &= bc, & L_0 d &= L_0 c. \\ \frac{a}{b} &= \frac{c}{d} = \frac{L_0}{L_0}. \end{aligned} \quad (14a)$$

The first half of this double equation applies also for the equilibrium of an inductance-free bridge, irrespective of whether it is traversed by direct or by alternating current.

b. Let the selfinductance and the capacitance lie in two opposite arms of the bridge, e.g. a and d , in such fashion that the selfinductance L and the ohmic resistance a are connected in series, the capacitance K and the ohmic resistance d , in *parallel*. According to Kirchhoff (see p. 101) the potential drop across two parallel ohmic resistances R' and R'' with the currents I' and I'' is given by

$$RI \quad \text{with} \quad I = I' + I'', \quad \frac{1}{R} = \frac{1}{R'} + \frac{1}{R''}.$$

The corresponding voltage drop for alternating current is evidently

$$RI \quad \text{with} \quad I = I' + I'', \quad \frac{1}{R} = \frac{1}{R'} + \frac{1}{R''}. \quad (15)$$

For our bridge arm d the total current at its end points is I_2 (see Fig. 25); we therefore set

$$I = I_2, \quad R' = d, \quad R'' = \frac{1}{i\omega K}, \quad \frac{1}{R} = \frac{1}{d} + i\omega K$$

and obtain in place of Eq. (13a)

$$bI_1 = \frac{I_2}{\frac{1}{d} + i\omega K} \quad (15a)$$

In combination with the unchanged Eq. (13) this leads to:

$$\frac{I_2}{I_1} = \frac{a}{c} + i\omega \frac{L}{c} = \frac{b}{d} + i\omega bK,$$

which yields, separating real and imaginary parts,

$$ad = bc = \frac{L}{K}. \quad (16)$$

In the notation of p. 138 the last term of this double equation is the *product* of inductive and capacitive reactance. Eq. (16) shows that the determination of a capacity from bridge equilibrium rests, apart from ohmic resistances, on a known selfinductance (or vice versa).

In our examples we have restricted ourselves to cases in which the equilibrium conditions, Eqs. (14a) and (16), do not depend on ω . In such cases the bridge equilibrium exists not only for the purely periodic alternating current here assumed, but for an arbitrary time variation, e.g. excitation by an interrupter. There are other cases in which the equilibrium conditions depend on ω ; bridge equilibrium then exists only for sinusoidal alternating current.

C. Coupled Circuits

The law of conservation of energy sufficed to set up the oscillatory equation for one degree of freedom. For a system of two circuits (two degrees of freedom I_1 and I_2) the energy theorem no longer is adequate for setting up the differential equations, just as in mechanics. On the other hand, Kirchhoff's law (5) yields directly

$$\begin{aligned} L_{11}I_1 + L_{12}I_2 + R_1I_1 + \frac{e_1}{K_1} &= E_1, \\ L_{22}I_2 + L_{21}I_1 + R_2I_2 + \frac{e_2}{K_2} &= E_2. \end{aligned} \quad (17)$$

The generalization for more than two circuits is obvious.

From (17) we can calculate, on the one hand, the *free* vibrations of our coupled system, on the other, the forced vibrations produced by an e.m.f. In the first case we set $E_1 = E_2 = 0$ and reduce the resulting homogeneous system of equations, with the assumption

$$I_1 = A_1 e^{i\omega t}, \quad I_2 = A_2 e^{i\omega t}$$

to a biquadratic equation for ω , after elimination of the ratio A_2/A_1 . We can be brief in discussing the conclusions derived therefrom since the same problem has been treated in detail in Vol. I, §20 and the results are represented there in Figs. 34 and 36. The characteristic *beat phenomena* of the coupled pendulums in the case of *resonance* occur, in terms of our present notation, when the periods of the uncoupled circuits are equal, which leads to $K_1L_{11} = K_2L_{22}$, and when in addition the two "coupling coefficients" L_{12}/L_{11} and L_{21}/L_{22} are equal, which leads, in view of the universal equality

of L_{12} and L_{21} , to $L_{11} = L_{22}$. These beat phenomena become particularly impressive if we assume, as for the coupled pendulums, that damping is slight, in our case $R_1 \cong 0$, $R_2 \cong 0$.

Also for *forced* vibrations the treatment in Vol. I, §19 may serve as a model. If the free and forced frequencies are identical, $\omega = \omega_0$, damping becomes essential and the amplitude maximum and phase lag represented in Fig. 33 of Vol. I occur.

At the beginning of the §20 mentioned above it was pointed out that in the early stages of wireless telegraphy coupled mechanical oscillations commonly served as model for the coupled electrical oscillations which occurred in the open primary antenna circuit and the tuned secondary circuit introduced by Ferdinand Braun. It is true that for these rapid oscillations the quasistationary treatment is only a crude approximation; only the complete integration of the Maxwell equations in §19 can yield a satisfactory representation.

D. The Telegraph Equation

Quasistationary calculations may also be applied to short sections of a long two-wire line, for which condition (1) is not fulfilled. If the length of these sections is permitted to approach zero the total differential equations of the system become a *partial differential equation*. This was set up by W. Thomson even *before* Maxwell in the treatment of the propagation of telegraph signals in marine cables. Between two oppositely located and oppositely charged points of the double line there is a charging current with a change in voltage; Hence, in addition to the voltage $V(x)$, *also the current $I(x)$ varies continuously* along the length of the line. According to Kirchhoff's second law these two variables are related by

$$L \frac{\partial I}{\partial t} + RI + \frac{\partial V}{\partial x} = 0; \quad (18)$$

L and R relate to unit length of the double line. Furthermore, it follows from the absence of current sources, i.e. Kirchhoff's first law (4), that

$$\frac{\partial I}{\partial x} + K \frac{\partial V}{\partial t} + GV = 0. \quad (18a)$$

For the sake of completeness a conduction current GV through the eventually semiconducting dielectric has been added to the charging current $K\partial V/\partial t$. This term may also account for hysteresis losses in the dielectric. G is known as the "leakage" per unit length of the double line; K also refers to this unit length.

Elimination of V from (18) and (18a) leads to the partial differential equation

$$\left\{ LK \frac{\partial^2}{\partial t^2} + (RK + LG) \frac{\partial}{\partial t} + RG - \frac{\partial^2}{\partial x^2} \right\} I = 0, \quad (19)$$

which applies also for V . If, in particular, the dissipative coefficients R and G are set equal to zero, it assumes the simple form of the differential equation of the vibrating string:

$$\left\{ LK \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right\} I = 0$$

and is integrated, for a phenomenon advancing in the positive x -direction, by

$$I = af(x - ct), \quad c = \sqrt{\frac{1}{LK}}. \quad (20)$$

According to (18) and (18a) the corresponding value of V is in this special case

$$V = Lcaf(x - ct) = \sqrt{\frac{L}{K}} I. \quad (21)$$

The ratio of voltage V and current I is thus the quantity $(L/K)^{\frac{1}{2}}$, which is independent of x and t and represents a resistance. It is known as the *wave resistance*.

The following practical conclusion may be drawn from (21): If a finite double line is terminated by an ohmic resistance of the magnitude $(L/K)^{\frac{1}{2}}$, no discontinuity occurs in the current and voltage variation at the end, hence also *no reflection*.

Eq. (20) states that the current and hence also the voltage propagate themselves along the line *without distortion* and *without damping*. We may also ask for the condition for *undistorted damped* propagation along the line, i.e. that

$$I = e^{-\alpha x} f(x - ct) \quad (22)$$

represents a solution of the differential equation (19). By substituting (22) in (19) and setting the resulting factors of f'' , f' , and f equal to zero we find

$$c = \sqrt{\frac{1}{LK}}, \quad \alpha = \frac{1}{2} \frac{RK + LG}{\sqrt{LK}} = \sqrt{RG}. \quad (23)$$

The velocity of propagation c is the same as in (20). The double equation for α leads to

$$RK + LG = 2\sqrt{RKL G}, \quad \text{i.e. } \sqrt{\frac{RK}{LG}} + \sqrt{\frac{LG}{RK}} = 2,$$

and hence

$$\sqrt{\frac{RK}{LG}} = 1 \quad \text{or also} \quad \frac{K}{G} = \frac{L}{R}. \quad (23a)$$

This signifies equal decay time for the pure displacement current $K\dot{V} + GV = 0$ (Eq. (18a) with $\partial I/\partial x = 0$) and of the pure conduction current $LI + RI = 0$ (Eq. (18) with $\partial V/\partial x = 0$). By (23) and (23a) our damping coefficient α then becomes equal to $R\sqrt{K/L}$.

In the general case the current variation changes with progress along the line. It is then proper to analyze the process into component waves of the form $\exp i(kx - \omega t)$ with complex, frequency-dependent k which are periodic in time and damped spatially. The total phenomenon is now no longer *distortion-free*.

The ideal case of undamped *plane* waves is approached if the two wires are imagined flattened into wide bands, the intermediate space being vacuum and the band material a perfect conductor. Then, apart from the marginal portions, the electric field in the intermediate space is uniform, similarly the magnetic field. \mathbf{E} is perpendicular to the bands in direction, \mathbf{H} , parallel thereto. The current becomes $I = |\mathbf{H}|b$ (b = band width), the voltage $V = |\mathbf{E}|d$ (d = separation of the bands). The charge per unit length is $e = \epsilon_0 |\mathbf{E}|b$ and the capacity $K = e/V = \epsilon_0 b/d$. The magnetic flux per unit length, i.e. through a rectangle with the sides 1 and d , becomes $\Phi = \mu_0 |\mathbf{H}|d$, so that the selfinductance $L = \Phi/I = \mu_0 d/b$. From K and L we compute, by (20) and (21),

$$\text{wave velocity } c = \frac{1}{\sqrt{KL}} = \frac{1}{\sqrt{\epsilon_0 \mu_0}},$$

$$\text{wave resistance } \frac{V}{I} = \sqrt{\frac{L}{K}} = \sqrt{\frac{\mu_0 d}{\epsilon_0 b}}.$$

Herewith we have again come upon the quantity $(\mu_0/\epsilon_0)^{1/2}$, which in §6 we had described as the wave resistance of vacuum for the propagation of a plane wave. (To obtain agreement we must refer it now to a quadratic section of the wave surface, i.e. set $b = d$.)

We have inserted this sketchy note on the telegraph equation partly to refer the concept of the wave resistance (more generally wave impedance or surge impedance) to its historical origin, partly to prepare the way for the transition to rapidly variable fields in the next section.

§19. *Rapidly Variable Fields. The Electrodynamic Potentials*

Only in this paragraph do we make full use of the unabbreviated Maxwell equations. We indicate a general method of integration, which however is limited to the case of a uniform medium, e.g. vacuum. Hence throughout space we put $\epsilon = \epsilon_0$, $\mu = \mu_0$ and in addition imagine the charge density ρ and the current density \mathbf{J} to be given in all of space and for all times $t < t_0$ (t_0 = instant of observation). In this formulation of the problem we already take cognizance of the electron theory, which, however, we

shall take up only in the third part. We start from the Maxwell equations in the form (4.4) with the auxiliary conditions (4.4a, b, c). In view of our assumptions $\epsilon = \epsilon_0 = \text{const}$, $\mu = \mu_0 = \text{const}$ we can write instead

$$\dot{\mathbf{B}} = -\text{curl } \mathbf{E}, \quad (1)$$

$$\frac{1}{c^2} \dot{\mathbf{E}} + \mu_0 \mathbf{J} = \text{curl } \mathbf{B}, \quad (2)$$

$$\text{div } \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (3)$$

$$\text{div } \mathbf{B} = 0, \quad (4)$$

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \quad (5)$$

We satisfy Eq. (4) by our earlier formulation (15.3)

$$\mathbf{B} = \text{curl } \mathbf{A}. \quad (6)$$

Substituting this in (1) we obtain

$$\text{curl } (\mathbf{E} + \dot{\mathbf{A}}) = 0.$$

This has the necessary consequence that the vector after the curl sign is a gradient. Thus

$$\mathbf{E} = -\text{grad } \Psi - \dot{\mathbf{A}}. \quad (7)$$

We call Ψ *scalar potential*, \mathbf{A} *vector potential*.

We substitute expressions (6) and (7) in (2) and obtain

$$-\frac{1}{c^2} (\dot{\mathbf{A}} + \text{grad } \dot{\Psi}) + \mu_0 \mathbf{J} = \text{curl curl } \mathbf{A} = -\Delta \mathbf{A} + \text{grad div } \mathbf{A}. \quad (8)$$

We have here utilized the transformation of curl curl which has been repeatedly employed before (e.g. in Eq. (6.2)), but applies only for Cartesian components of the vector \mathbf{A} . We shall simplify Eq. (8) by splitting it up into two vector equations, namely into

$$\Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \quad (9)$$

and

$$\text{grad} \left(\text{div } \mathbf{A} + \frac{1}{c^2} \dot{\Psi} \right) = 0. \quad (9a)$$

If we refrain from the inappropriate addition of a function depending on t only, i.e. of a kind of "integration constant", the second of these becomes

$$\text{div } \mathbf{A} + \frac{1}{c^2} \dot{\Psi} = 0. \quad (10)$$

Our initial equations (1), (2), and (4) are thus satisfied; there remains, apart from Eq. (5), Eq. (3). In view of (7) this takes the form

$$\Delta\Psi + \operatorname{div} \dot{\mathbf{A}} = -\frac{\rho}{\epsilon_0}$$

or, taking account of (10),

$$\Delta\Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = -\frac{\rho}{\epsilon_0}. \quad (11)$$

Our two potentials \mathbf{A} and Ψ thus satisfy two differential equations of the same form. We call them "wave equations". As noted above, their right sides are given functions of x, y, z and of "past time," $t < t_0$. The desired solutions are related by condition (10).

We recognize that this condition is appropriate from the following: If we call its left side X and if we form

$$\operatorname{div} (9) + \epsilon_0 \mu_0 \frac{\partial}{\partial t} (11),$$

we obtain

$$\Delta X - \frac{1}{c^2} \frac{\partial^2 X}{\partial t^2} = -\mu_0 \left(\operatorname{div} \mathbf{J} + \frac{\partial \rho}{\partial t} \right). \quad (12)$$

The right side of this equation vanishes however because of our Eq. (5), which here at last is drawn into the consideration. Thus X also satisfies the *homogeneous* wave equation, which represents a wave process without external excitation, i.e. not a forced, but a free vibration. It can be foreseen from this that a suitable integration of the differential equations for \mathbf{A} and Ψ , which excludes the appearance of free vibrations, not only leads to Eq. (12) for X being satisfied, but also to $X = 0$, i.e. the satisfying of Eq. (10). Nevertheless this equation is neither superfluous nor obvious, since the splitting up of Eq. (8), i.e. the transition from (8) to (9) and (11), rests expressly on condition (10).

A. The Retarded Potentials

With respect to the integration of our wave equations (9) and (11) we shall be brief, since the next section will indicate the rational procedure. We write down directly the result of the integration:

$$4\pi\epsilon_0\Psi = \int \frac{[\rho]}{r} d\tau, \quad (13a)$$

$$4\pi \frac{1}{\mu_0} \mathbf{A} = \int \frac{[\mathbf{J}]}{r} d\tau. \quad (13b)$$

Ψ and A refer to the reference point x, y, z and the "reference time" t , for which we wish to calculate the values of Ψ and A . ξ, η, ζ is the point of integration and $d\tau$ is equivalent to $d\xi d\eta d\zeta$. The integration is carried out over all of infinite space and we have

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2.$$

$[\rho]$ and $[J]$ are, however, *not* the values of charge and current density at the time of observation t , but at the *earlier* time

$$t' = t - r/c. \quad (13c)$$

r/c is the time required by the "light" to travel from the point of integration to the reference point. Hence the expressions (13) are called *retarded potentials*. They are calculated from charge and current density at a time which is *set back* by r/c relative to the time of observation.

The method of integration (13) is mathematically unique if, for physical reasons, the addition of *advanced potentials*, which correspond to the later time

$$t'' = t + r/c, \quad (13d)$$

is excluded. It should be noted, however, that such advanced potentials have tentatively been introduced by Dirac into the theory of the electron and play an important role in more recent investigations (see §37).

If we apply the same method of calculation (13) to our quantity X we find directly

$$X = 0$$

since the right side of (12) was equal to zero. This result is also mathematically unique with the exclusion of "advanced" solutions, which alone, in combination with the retarded solutions, could give rise to free vibrations. This may be taken as confirmation of our earlier statement that condition (10) is satisfied automatically in the integration of the differential equations for A and Ψ . We note finally that a full understanding of the structure of the above formalism including the significance of our retarded and advanced potentials can be obtained only on the basis of the theory of relativity. What up to now may have appeared arbitrary and asymmetric will there assume an astonishingly unique and symmetrical form.

B. The Hertzian Dipole

We will explain our method of integration (13) for a particular case, the case of the *Hertzian dipole*. This is obtained if we combine a moving charge $+e$ with a neighboring stationary charge $-e$ to form a moment $p(t) = el$ varying with time, where l signifies the separation of the two charges.

We substitute $\mathbf{J} = \rho \mathbf{v}$ in (13b), denoting the space density of the moving charge by ρ and its velocity by \mathbf{v} , and obtain on carrying out the integration, where r and \mathbf{v} may be regarded as constant in space,

$$\int \frac{[\mathbf{J}]}{r} d\tau = \frac{[\mathbf{v}]}{r} \int \rho d\tau = \frac{e[\mathbf{v}]}{r} = \frac{e}{r} \left[\frac{\partial \mathbf{l}}{\partial t} \right] = \frac{1}{r} \left[\frac{\partial \mathbf{p}}{\partial t} \right].$$

We hence obtain from (13b), if we take due account of the meaning of the bracket symbol as given by (13c)

$$4\pi \mathbf{A} = \frac{\mu_0}{r} \frac{\partial}{\partial t} \mathbf{p} \left(t - \frac{r}{c} \right). \quad (14)$$

It is historically customary and convenient to introduce, in place of the vector potential \mathbf{A} , the *Hertzian vector* Π by writing

$$\mathbf{A} = \mu_0 \frac{\partial \Pi}{\partial t}, \quad 4\pi \Pi = \frac{1}{r} \mathbf{p} \left(t - \frac{r}{c} \right). \quad (15)$$

With this notation we follow the great paper of Hertz, already discussed in §1, p. 5:¹ "The Forces of Electrical Oscillations, Treated by Maxwell's Theory."

In all of space except at the origin of the coordinate system Π satisfies, by (9), the differential equation

$$\Delta \Pi - \frac{1}{c^2} \frac{\partial^2 \Pi}{\partial t^2} = 0, \quad (16)$$

which can also readily be verified from the explicit representation of Π given in (15). By (10) the corresponding value of Ψ becomes

$$e_0 \Psi = - \operatorname{div} \Pi. \quad (16a)$$

From (6) and (7) we obtain then as the representation of the electromagnetic field

$$\mathbf{H} = \operatorname{curl} \Pi, \quad e_0 \mathbf{E} = \operatorname{grad} \operatorname{div} \Pi - \frac{1}{c^2} \frac{\partial^2 \Pi}{\partial t^2}. \quad (17)$$

As an example we assume that the path of the mobile charge e is rectilinear and make its direction, which is also that of the vector Π , the axis of a spherical coordinate system r, ϑ, φ . We then have²

$$\Pi_r = \cos \vartheta \cdot \Pi, \quad \Pi_\vartheta = - \sin \vartheta \cdot \Pi, \quad \Pi_\varphi = 0,$$

¹ Ann. d. Physik 56, p. 1, 1888; Gesammelte Werke, Vol. II, p. 147.

² The positive r -direction forms the angle ϑ with the direction of Π , the positive ϑ -direction, the angle $\vartheta + \pi/2$; hence the factors $\cos \vartheta$ at Π_r and $\cos (\vartheta + \pi/2) = - \sin \vartheta$ at Π_ϑ .

where, according to (15), Π depends only on t and r , i.e. is independent of ϑ and φ . In these coordinates we obtain, by Problem I.3 of Vol. II,

$$\text{curl}_\vartheta \Pi = \frac{-\sin \vartheta}{r} \left(\frac{\partial(r\Pi)}{\partial r} - \Pi \right) = -\sin \vartheta \frac{\partial \Pi}{\partial r}, \quad \text{curl}_r \Pi = \text{curl}_\varphi \Pi = 0,$$

$$\text{div} \Pi = \frac{\cos \vartheta}{r^2} \frac{\partial}{\partial r} (r^2 \Pi) - \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin^2 \vartheta \Pi) = \cos \vartheta \frac{\partial \Pi}{\partial r},$$

$$\text{grad}_r \text{div} \Pi = \cos \vartheta \frac{\partial^2 \Pi}{\partial r^2}, \quad \text{grad}_\vartheta \text{div} \Pi = \frac{-\sin \vartheta}{r} \frac{\partial \Pi}{\partial r}, \quad \text{grad}_\varphi \text{div} \Pi = 0.$$

Hence, by (17):

$$H_r = H_\vartheta = E_\varphi = 0 \quad (18)$$

and by (15)

$$\begin{aligned} 4\pi H_\varphi &= -\frac{\sin \vartheta}{r} \left(\frac{\partial}{\partial r} \dot{p} - \frac{1}{r} \dot{p} \right), \\ 4\pi e_0 E_r &= \frac{\cos \vartheta}{r} \left(\frac{\partial^2}{\partial r^2} p - \frac{2}{r} \frac{\partial}{\partial r} p + \frac{2}{r^2} p - \frac{1}{c^2} \ddot{p} \right), \\ 4\pi e_0 E_\vartheta &= -\frac{\sin \vartheta}{r} \left(\frac{1}{r} \frac{\partial}{\partial r} p - \frac{1}{r^2} p - \frac{1}{c^2} \ddot{p} \right). \end{aligned} \quad (19)$$

We conclude from (18): The magnetic lines of force are circles about the direction of \mathbf{p} , while the electric lines of force lie in the meridional planes through this direction.

Because of the argument $t - r/c$ of p it is possible, in Eqs. (19), to transform the differentiation with respect to r into one with respect to t . We have

$$\frac{\partial p}{\partial r} = -\frac{1}{c} \dot{p}, \quad \frac{\partial^2 p}{\partial r^2} = \frac{1}{c^2} \ddot{p}. \quad (19a)$$

Then the term with $\frac{\partial^2 p}{\partial r^2}$ cancels that with \ddot{p} in the equation for E_r in (19).

At the same time we will limit ourselves to the "distant zone" (large distances from the origin, i.e. set $r \rightarrow \infty$). We will indicate the more precise meaning of this in a moment, in discussing the periodically oscillating dipole. Accordingly we neglect all terms in (19) which contain higher powers of $1/r$ than the first. We then obtain

$$\begin{aligned} 4\pi H_\varphi &= \frac{\sin \vartheta}{cr} \dot{p} \left(t - \frac{r}{c} \right), \\ 4\pi e_0 E_r &= 0, \\ 4\pi e_0 E_\vartheta &= \frac{\sin \vartheta}{c^2 r} \dot{p} \left(t - \frac{r}{c} \right). \end{aligned} \quad (20)$$

The vectors \mathbf{H} and \mathbf{E} are perpendicular to each other and to the radius vector r from the origin. Both \mathbf{H} and \mathbf{E} vanish on the axis $\vartheta = 0$ and $\vartheta = \pi$; the \mathbf{H} - and \mathbf{E} -fields have their maxima in the equatorial plane $\vartheta = \pi/2$.

From (20) we calculate

$$\frac{E_{\vartheta}}{H_{\varphi}} = \frac{1}{e_0 c} = \sqrt{\frac{\mu_0}{\epsilon_0}}. \quad (21)$$

This is the same ratio as that which was obtained from Eqs. (6.11) and (6.13) for the ratio E_y/H_z . The structure of the radiated electromagnetic field is thus that of a plane light wave. It is customary to say instead, in both cases, that \mathbf{E} and \mathbf{H} are equal, which however is dimensionally meaningless.

The amount of energy radiated per unit area and per unit time becomes

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = E_{\vartheta} H_{\varphi} = \frac{1}{16\pi^2 \epsilon_0 c^2} \frac{\sin^2 \vartheta}{r^2} \dot{p}^2. \quad (22)$$

The total energy radiated in unit time is obtained by integration over the spherical surface of radius r :

$$S = \int S d\sigma = 2\pi r^2 \int S \sin \vartheta d\vartheta = \frac{\dot{p}^2}{6\pi \epsilon_0 c^2}. \quad (23)$$

Since $\mathbf{p} = e\mathbf{l}$ (l = separation of the mobile and the stationary charge) $\dot{p} = e\dot{v}$, $\dot{p} = e\dot{v}$, where, of course, in accord with the meaning of \dot{p} , \dot{v} denotes the value of the acceleration at the earlier time $t - r/c$. We thus obtain from (23)

$$S = \frac{e^2 \dot{v}^2}{6\pi \epsilon_0 c^2}. \quad (24)$$

In atomic physics it is customary to write, in electric or magnetic cgs-units,

$$S = \frac{2}{3} \frac{e^2 \dot{v}^2}{c^2} \quad (24a)$$

or

$$S = \frac{2}{3} \frac{e^2 \dot{v}^2}{c} \quad (24b)$$

which, according to (16.30), corresponds to (24). J. J. Larmor¹ first gave this fundamental law of radiation in the form (24b).

Fig. 27 shows the radiation density S as function of ϑ . It is simply the

¹ Phil. Mag. 1897, p. 512. Larmor points out the relationship to Hertz's paper of 1888 in his book *Aether and Matter*, Cambridge, 1900, p. 225.

polar diagram of $\sin^2 \vartheta$. In the theory of wireless telegraphy (see Vol. VI, Chapter VI) it finds extensive application in the treatment of a linear antenna radiating freely into space. In fact such an antenna does not radiate energy in its own direction; the maximum of the radiation is directed *transversally*.

In place of a single dipole p we may, of course, also consider a discrete or continuous sequence of dipoles. In the latter case we write in place of (15):

$$4\pi\Pi = \int_C \frac{dp(t - r/c)}{r}. \quad (25)$$

Here the integration is to be extended over a given curve C and the difference in direction of the vectors dp must be considered.

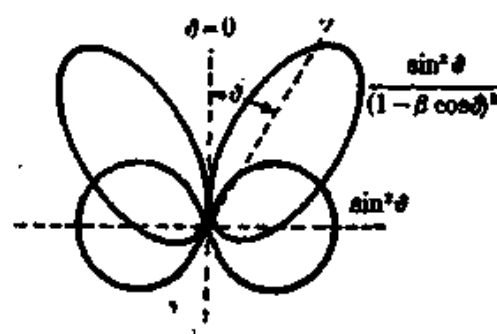


FIG. 27. Radiation of an electron accelerated longitudinally in the direction $\vartheta = 0$. Lower pair of curves: Hertz's formula (22), $v \ll c$. Upper pair of curves: corrected relativistically, v comparable with c .

A comparison of the preceding with Hertz's calculation in Cartesian coordinates, which is found in most textbooks, demonstrates the superiority of our vector representation of the problem or eventually, of our spherical polar coordinates, which fit the symmetry of the problem. It seems even more important that our presentation clearly indicates the dimensions of all field quantities, while the Gaussian system of units employed by Hertz obscures them.

C. Specialization for Periodic Processes

We obtain the simplest model for a *light source* by assuming that the electric moment p oscillates monochromatically with a certain circular frequency ω . For example we set

$$\begin{aligned} p(t) &= A \cos \omega t = A \operatorname{Re} e^{-i\omega t}, \\ p(t - r/c) &= A \operatorname{Re} \exp \{-i\omega(t - r/c)\}. \end{aligned} \quad (27)$$

If, as in (6.10a, b), we introduce the wave number $k = \omega/c$ and omit the sign indicating the real part, which is permissible for all field quantities except the quadratic ones S, S , we find

$$\frac{1}{r} p(t - r/c) = A \frac{e^{ikr}}{r} e^{-i\omega t}. \quad (27a)$$

We have thus arrived at the representation of the *spherical wave* in Vol. II, Eq. (13.18) if we also suppress the time factor in (27a). This may and will be done in the following. We then obtain from Eqs. (20) the following representation of the electromagnetic field:

$$E = E_{\vartheta} = -\frac{Ak^2}{4\pi\epsilon_0} \sin \vartheta \frac{e^{ikr}}{r}, \quad H = H_{\varphi} = -\frac{Ak\omega}{4\pi} \sin \vartheta \frac{e^{ikr}}{r}. \quad (28)$$

This applies, as already noted above (20), for the "distant zone"; we are now however in a position to define this term exactly. For, if we let the wave-length

$$\lambda = \frac{2\pi}{\omega} c$$

correspond to the angular frequency ω , the distant zone includes all distances for which

$$r \gg \lambda, \quad (29)$$

i.e. excludes only the immediate neighborhood of the light source. For aperiodic processes (29) is replaced by the inequalities

$$\frac{1}{c} |\dot{p}| \gg \frac{|\dot{p}|}{r}, \quad \frac{1}{c^2} |\dot{p}| \gg \frac{|\dot{p}|}{cr} \gg \frac{|p|}{r^2}. \quad (29a)$$

These statements justify precisely the approximations made in passing from (19) to (20).

As compared with a natural light source our model is specialized both with regard to its monochromatism and its intensity distribution. It radiates no energy in the directions $\vartheta = 0$ and $\vartheta = \pi$; for also now Fig. 27 and Eq. (22) apply to the radiation vector S . In (22) both the time factor and the phase factor e^{ikr} drop out in the time average. In fact, by (27),

$$\frac{\dot{p}}{r} = -\frac{A\omega^2}{r} \cos(kr - \omega t)$$

and the time average of the square thereof is

$$A^2 \omega^4 / (2r^2) = A^2 c^4 k^4 / (2r^2). \quad (29b)$$

Since $k = 2\pi/\lambda$ this is *inversely proportional to the fourth power of the wave-length*.

If (29b) is substituted in (22) or (23) we obtain the famous law of Lord Rayleigh, explaining the *blue sky*. The sun rays falling on the particles of the air generate in them electric moments which vibrate in harmony and radiate light in turn. Their radiation is much stronger at the blue end of the spectrum than at the red end. Since $\lambda_{\text{red}} \cong 2\lambda_{\text{blue}}$ their ratio is about 2^4 . The same law explains also the red color of the sun and moon when rising

and setting. In its path through the atmosphere, which is here particularly long, the blue light is scattered much more strongly out of its direct path than the red light; primarily red sun- or moonlight reaches our eyes. We shall not inquire whether a certain selectivity of water vapor in the atmosphere plays an added role.

D. The Characteristic Vibrations of a Metallic Spherical Oscillator

The problem of *electromagnetic* characteristic vibrations became significant as a result of Hertz's experiments. A metallic body consisting of two oppositely charged halves (Hertzian oscillator) discharges with the formation of a spark and radiates exponentially damped vibrations toward infinity. How do we calculate their frequency and damping? In the case of the sphere, which we imagine as subdivided into two closely adjoining oppositely charged halves, the question is answered directly by formulas which are already familiar to us. It is true that we must not start here from Eq. (20) for the distant zone, but must employ the more general formulas (19), since the wave length of the characteristic vibration generated will understandably be of the order of magnitude of the radius of the sphere; the surface of the sphere thus belongs to the near zone. If the sphere is assumed to be perfectly conducting we have on the surface, i.e. for $r = a =$ radius of sphere, $E_r = 0$ for all θ . Hence, by (19), we have as boundary condition

$$\left(\frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p = 0. \quad (30)$$

We let p take the same form as in (27), treating k not as a real number as up to now, but as an unknown complex number. The same applies then also for $\omega = ck$. To determine k (30) yields

$$\frac{ik}{a} - \frac{1}{a^2} + k^2 = 0. \quad (31)$$

The solution of this equation, which is quadratic in ka , is

$$ka = \frac{-i \pm \sqrt{3}}{2}. \quad (31a)$$

The imaginary part is negative, as we must demand, since we are dealing with a vibration which decreases with time. In the real part the positive sign is to be chosen in order that wave-length and frequency be positive. We thus obtain

$$\frac{2\pi a}{\lambda} = \frac{\sqrt{3}}{2}, \quad \lambda = \frac{4\pi}{\sqrt{3}} a,$$

i.e. in fact λ of the order of magnitude of the radius of the sphere. Damp-

ing is very great; as follows from (31a), the amplitude decreases by a factor

$$e^{-\lambda/(2a)} = e^{-2\pi/\sqrt{3}}$$

in the course of a single vibration.

Herewith the nature of the *fundamental* vibration of our spherical oscillator is described. There is however also an infinite number of harmonics for which the sphere is not divided into two oppositely charged halves, but into 4, 6, \dots alternately charged zones. While the fundamental vibration corresponds to the Hertzian dipole, these harmonics cannot be derived from the Hertzian vector Π . For them we must refer to Vol. VI and more particularly to Appendix II of Chapter V of that volume.

The case of the prolate spheroid, which comes closer to the Hertz oscillator than the spherical shape, was treated by Max Abraham,¹ after the problem of the spherical oscillator had been solved generally by J. J. Thomson as early as 1884.

E. Application to the Theory of X-Rays

The primary x-rays are produced by the incidence of cathode rays on the anticathode. Classically the initial velocity v of the incident electrons is reduced to a low value; the electrons experience a retardation $-v$. From Fig. 27 we expect that no radiation occurs in the direction of the cathode rays, insofar as this coincides with the direction of v . The proof of this is possible with extremely thin anticathodes (films a few microns in thickness), if the transmitted x-rays are observed; this has been shown by Kulenkampff and his students. For solid metal anticathodes the retardation takes place along a zigzag path; hence the variation with direction is smoothed out. We will show relativistically in §30, at Eq. (11), that the maximum of the radiation does not lie, as indicated by the pair of curves in Fig. 27, at $\vartheta = \pi/2$, but that it advances, instead, with increasing hardness of the cathode rays (increasing magnitude of v) more and more toward $\vartheta = 0$. The fact that the continuous or "brems"-spectrum discussed here has a short-wave limit is a consequence of the quantum theory, with which we shall not deal here. The same applies for the regular increase in hardness and intensity of the x-rays with the hardness of the cathode rays.

Here we shall only discuss the proof of the *transversal* nature of x-rays, which was given by Barkla in 1905, ten years after Röntgen's discovery. In planning his experiment Barkla assumed this transversality, drew the consequences of this assumption, and confirmed them by the experiment.

We consider, in Fig. 28, a broken line consisting of three mutually perpendicular segments, the "primary", "secondary", and "tertiary" segment. The *primary x-rays*, regarding whose polarization we shall make no assump-

¹ See Enzykl. d. mathem. Wiss., Vol. V, section 18, p. 498.

tion, travel along the first segment (for a very thin anticathode even these would be partly polarized). We imagine their electric field strength to be analyzed into its components along the directions 2 and 3 of the two other segments. They fall on a first scatterer Z_1 , whose electrons they set into vibrations. Those parallel to 2 have no effect along the secondary segment, while those parallel to 3 produce on it *secondary x-rays*, which vibrate parallel to 3 and are *totally polarized*. They fall on a scatterer Z_2 and set its electrons into vibrations in direction 3. In this fashion *tertiary x-rays* are produced which, however, have the *intensity zero* along the tertiary segment. They have maximum intensity in the direction of the primary segment. This behavior of the tertiary x-rays proves both the transversal nature of the primary and the total polarization of the secondary x-rays.

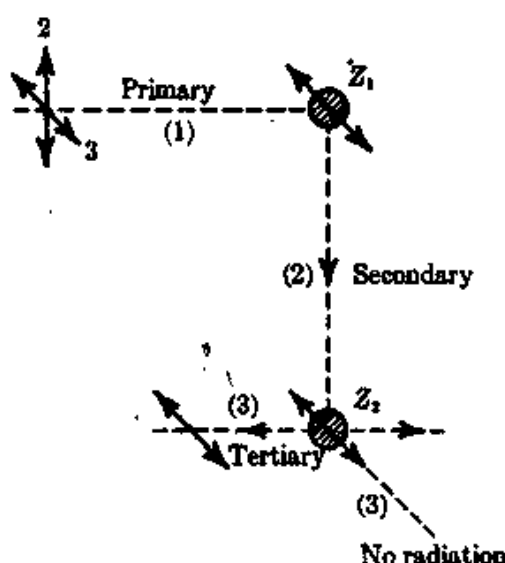


FIG. 28. Barkla's arrangement for demonstrating the transversal nature of x-rays. Z_1 , Z_2 scatterers (spheres of paraffin).

The scatterers Z_1 and Z_2 were spheres of paraffine; for heavier materials the "characteristic radiation" might have falsified the result.

§20. General Considerations on the Structure of Wave Fields of Cylindrical Symmetry. Applications to Alternating Current Impedance and Skin Effect

In the following section we will concern ourselves almost exclusively with *surface waves* which are *guided* along bodies of *cylindrical shape*. Let the excitation be such that the process is periodic in time with the circular frequency ω . The calculation of the *propagation and damping* of the waves as they progress in the direction of the cylinder axis, which we shall choose as the direction of the x -coordinate axis, then becomes of primary interest. We leave the cross section of the cylindrical conductor (or also nonconductor) temporarily indeterminate. We express propagation and damping by a single *complex wave number* h , which differs from the real wave number $k = \omega/c$ in vacuum. We consider thus a wave type with the dependence on x and t

$$\exp \{i(hx - \omega t)\};$$

for the assumed cylindrical structure of the wave field, h has necessarily the same value outside of and inside of the guiding surfaces; the same applies of course to ω .

We define, in the plane perpendicular to the x -axis, an orthogonal coordinate system u, v ; dx, du, dv , in this order, are to form a right-handed coordinate system. For the line element in space we write, in accord with (2.22) of Vol. II:

$$ds^2 = dx^2 + g_u^2 du^2 + g_v^2 dv^2. \quad (1)$$

g_u and g_v are here given functions of u and v . The cross section (which is constant, i.e. independent of x , for every conductor) may differ for different conductors. The coordinates u, v are to be fitted, in each particular case, to the shape of the cross section: polar coordinates for the single wire, bipolar coordinates for the two-wire line, Cartesian coordinates for semi-infinite space (limiting case of the single wire of infinitely great radius).

We set ourselves the problem of computing the transversal components E_u, E_v, H_u, H_v from the longitudinal components E_x, H_x . This is possible without making any special assumptions regarding the cross sections of the cylindrical conductors guiding the wave, without discussing the corresponding boundary conditions, and also without assuming that the wave equation is separable in the coordinates u, v . The general structure of the wave field so obtained applies not only for the exterior of the conductors, but also, with altered choice of the material constants, to their interior.

A. Longitudinal and Transversal Components

The longitudinal components E_x and H_x , which we shall designate by the single symbol X , satisfy, as Cartesian components, the simple wave equation

$$\left(\Delta - \epsilon\mu \frac{\partial^2}{\partial t^2} - \sigma\mu \frac{\partial}{\partial t} \right) X = 0. \quad (2)$$

It has been written in this form for the interior of the conductors, but applies also to the exterior, where $\sigma = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$. We put

$$\Delta = \frac{\partial^2}{\partial x^2} + \Delta_{uv},$$

where Δ_{uv} is the *two-dimensional* Laplace operator transformed to the curvilinear coordinates u, v . In view of the dependence of the phase factor on x and t we can write in place of (2)

$$(\Delta_{uv} + k^2 - h^2)X = 0, \quad k^2 = \epsilon\mu\omega^2 + i\mu\sigma\omega. \quad (3)$$

In the exterior of the conductors k is real ($= \sqrt{\epsilon_0\mu_0}\omega = \omega/c$), in the interior

it is complex. However, by introducing the complex dielectric constant ϵ' from Eq. (6.18) we can employ the same formula

$$k = \sqrt{\epsilon\mu}\omega \quad (3a)$$

in both cases, if we set

$$\begin{aligned} \epsilon &= \epsilon_0, & \mu &= \mu_0 & \text{in the exterior,} \\ \epsilon &= \epsilon', & \mu &= \mu & \text{in the interior.} \end{aligned} \quad (3b)$$

Actually to integrate Eq. (2) it would of course be necessary to pass to specific coordinates u, v , which are adapted to the shape of the conductor; we shall avoid this for the present, however. Nevertheless we may regard, for what follows, the longitudinal components E_z and H_z as known functions of space.

To calculate from them the transversal components we utilize the definition of the curl of an arbitrary vector \mathbf{A} in any curvilinear coordinates p_1, p_2, p_3 contained in Eq. (2.26) of Vol. II:

$$\text{curl}_1 \mathbf{A} = \frac{1}{g_2 g_3} \left(\frac{\partial(g_3 A_3)}{\partial p_2} - \frac{\partial(g_2 A_2)}{\partial p_3} \right) \quad (4)$$

and set in accord with (1),

$$p_1 = x, \quad p_2 = u, \quad p_3 = v; \quad g_1 = 1, \quad g_2 = g_u, \quad g_3 = g_v.$$

By cyclic interchange of the indices 1, 2, 3 and the coordinates x, u, v we obtain from (4) all the components of the curl occurring in the Maxwell equations.

We thus calculate from the first and second group of three Maxwell equations, taking account at the same time of the x, t dependence of the phase factor,

$$\begin{aligned} i\omega\mu H_z &= \text{curl}_z \mathbf{E} = i\hbar E_z - \frac{1}{g_u} \frac{\partial E_z}{\partial u} \\ -i\omega\epsilon E_z &= \text{curl}_z \mathbf{H} = -i\hbar H_z + \frac{1}{g_v} \frac{\partial H_z}{\partial v}. \end{aligned}$$

On the basis of our convention (3a, b) we substitute on the left $\omega = k/\sqrt{\epsilon\mu}$ and obtain

$$\begin{aligned} i \left(k \sqrt{\frac{\mu}{\epsilon}} H_z - \hbar E_z \right) &= -\frac{1}{g_u} \frac{\partial E_z}{\partial u} \\ i \left(\hbar \sqrt{\frac{\mu}{\epsilon}} H_z - k E_z \right) &= \frac{1}{g_v} \sqrt{\frac{\mu}{\epsilon}} \frac{\partial H_z}{\partial v}. \end{aligned}$$

From these equations E_u and H_v are obtained by simple elimination:

$$\begin{aligned} i(k^2 - h^2)E_u &= -\frac{h}{g_u} \frac{\partial E_z}{\partial u} - \frac{k}{g_v} \sqrt{\frac{\mu}{\epsilon}} \frac{\partial H_z}{\partial v} \\ i(k^2 - h^2) \sqrt{\frac{\mu}{\epsilon}} H_v &= -\frac{k}{g_u} \frac{\partial E_z}{\partial u} - \frac{h}{g_v} \sqrt{\frac{\mu}{\epsilon}} \frac{\partial H_z}{\partial v}. \end{aligned} \quad (5)$$

Thus our objective regarding the two transversal components E_u and H_v has been attained, since on the right there occur only the longitudinal components which are assumed to be known.

The calculation for E_v and H_u is carried out similarly. Here the Maxwell equations

$$\begin{aligned} i\omega\mu H_u &= -ihE_v + \frac{1}{g_v} \frac{\partial E_z}{\partial v} \\ -i\omega\epsilon E_v &= ihH_u - \frac{1}{g_u} \frac{\partial H_z}{\partial u} \end{aligned}$$

are employed, and ω substituted in them once more in accord with the convention (3a, b). After eliminating one of the two unknowns H_u or E_v , we obtain

$$\begin{aligned} i(k^2 - h^2)E_v &= -\frac{h}{g_v} \frac{\partial E_z}{\partial v} + \frac{k}{g_u} \sqrt{\frac{\mu}{\epsilon}} \frac{\partial H_z}{\partial u}, \\ i(k^2 - h^2) \sqrt{\frac{\mu}{\epsilon}} H_u &= \frac{k}{g_v} \frac{\partial E_z}{\partial v} - \frac{h}{g_u} \sqrt{\frac{\mu}{\epsilon}} \frac{\partial H_z}{\partial u}. \end{aligned} \quad (6)$$

We will be able to make good use of these rather brief and abstract considerations in the following §21–25, where we shall replace our general coordinates u, v partly by polar, partly by bipolar coordinates. Thus, they show directly, e.g., that for polar coordinates r, φ and non-dependence of the field on φ , the pairs of equations (5) and (6), which in general are coupled, separate into one pair which contains only E_z, E_r, H_φ and another which contains only H_z, H_r, E_φ . This simplification corresponds to the symmetry of the single wire. In the present paragraph we shall deal with the still simpler case of rectangular coordinates $u = y, v = z$ for non-dependence of the field on z . This corresponds to the transition to the limit: radius of wire $\rightarrow \infty$ in Fig. 30. In all these special cases the preceding general relations take on a readily understood form and may, as we shall see, be verified directly.

Independently of the choice of u and v a further conclusion may be drawn which applies for all cylindrical perfect conductors: *The velocity of propagation on them is always equal to the velocity of light.* We note first

that the electric lines of force must be perpendicular to the surface of the conductor and conclude hence $E_z = 0$. Since, furthermore, the energy flux can have no component in a direction toward the conductor, we must also have $H_z = 0$. To begin with this applies only to the surface of the conductors. We shall assume however that both longitudinal components vanish *everywhere* in the nonconductor without coming into conflict with the Maxwell equations. It then follows from (5) and (6) (excluding the trivial solution $E_u = E_v = H_u = H_v = 0$), that we must have $h = k$. The wave equation (3) which applies for every Cartesian component of the electric and magnetic field then passes over into the potential equation. We shall make use of this conclusion in a specific example (§25A).

B. The Wave Field of Semiinfinite Space and its Skin Effect

Let the metallic semiinfinite space be bounded by the plane $y = 0$. Let the positive y -axis point upwards into the empty space (eventually filled with air) $y > 0$ and the wave progress toward the right (positive x -axis). Of the two possible solutions (5) and (6) we choose the first, since it corresponds to the wire traversed by alternating current. We hence make the reasonable assumption

$$\left. \begin{array}{l} E_x = E(y) \\ E_y = F(y) \\ H_x = G(y) \end{array} \right\} \exp \{i(hx - \omega t)\}; \quad \left. \begin{array}{l} E_z \\ H_z \end{array} \right\} = 0 \quad (7)$$

The differential equation (3) for E_x then takes the form

$$\frac{d^2 E}{dy^2} + (k^2 - h^2)E = 0, \quad \text{with } k^2 = \epsilon\mu\omega^2 + i\mu\omega. \quad (8)$$

Its solution is

$$E(y) = Ae^{i\sqrt{k^2 - h^2}y} + Be^{-i\sqrt{k^2 - h^2}y}. \quad (8a)$$

For $y < 0$ k^2 is complex, for $y > 0$ it is real. However, we shall, for the present, assume a very small $\sigma > 0$ even for $y > 0$ and pass over to the limit $\sigma \rightarrow 0$ at a later stage.¹

We shall choose the sign of $\sqrt{k^2 - h^2}$ once and for all so that it has a positive imaginary part.

Since the state must remain finite for $y \rightarrow \pm \infty$ we must set in (8a)

$$\text{for } y > 0 : B = 0,$$

$$\text{for } y < 0 : A = 0.$$

¹ In this manner we circumvent some basic questions regarding the exterior of the wire which will be deferred until §22.

Furthermore, since E_z must be continuous at $y = 0$, B must be set equal to A in the two resulting expressions.

In the following we shall reserve the letter k for the real wave number for $y > 0$ in the limiting case $\sigma \rightarrow 0$. For the sake of distinction the value of k within the conductor will be denoted by k_L . Then (8a) assumes the final form:

$$E(y) = \begin{cases} A e^{i\sqrt{k^2 - h^2}y} & y > 0, \\ A e^{-i\sqrt{k_L^2 - h^2}y} & y < 0. \end{cases} \quad (9)$$

With $u = y$, $v = z$, $g_u = g_v = 1$, and $\frac{\partial}{\partial z} = 0$ we obtain from (5)

$$F(y) = \begin{cases} -\frac{hA}{\sqrt{k^2 - h^2}} e^{i\sqrt{k^2 - h^2}y} & y > 0, \\ \frac{hA}{\sqrt{k_L^2 - h^2}} e^{-i\sqrt{k_L^2 - h^2}y} & y < 0. \end{cases} \quad (9a)$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} \left\{ G(y) = \begin{cases} -\frac{kA}{\sqrt{k^2 - h^2}} e^{i\sqrt{k^2 - h^2}y} & y > 0, \\ \frac{k_L A}{\sqrt{k_L^2 - h^2}} e^{-i\sqrt{k_L^2 - h^2}y} & y < 0. \end{cases} \right. \quad (9b)$$

The factors of G , on the left, denote "reciprocal wave impedances", the upper one the real reciprocal impedance of vacuum as on p. 36, the lower one the correspondingly defined complex quantity for our conductor. There we learned already that the dimensions of \mathbf{H} and \mathbf{E} must differ by a factor with this dimension.

We now take account of the continuity of H_z at $y = 0$. In view of (3a, b) this demands

$$\frac{\mu_0 \sqrt{k^2 - h^2}}{k^2} = -\frac{\mu \sqrt{k_L^2 - h^2}}{k_L^2}. \quad (10)$$

From this we derive

$$\frac{1}{h^2} = \frac{\mu_0^2 \frac{k_L^2}{k^2} - \mu^2 \frac{k^2}{k_L^2}}{\mu_0^2 k_L^2 - \mu^2 k^2}. \quad (10a)$$

This value depends in a *symmetrical fashion* on the constants μ_0 , k and μ , k_L of the two media air and metal. We find, in particular, for $\mu = \mu_0$

$$\frac{1}{h^2} = \frac{1}{k^2} + \frac{1}{k_L^2}. \quad (10b)$$

For a well conducting metal the conduction current (the σ -term in k_L) exceeds the displacement current (the ϵ -term). We may thus assume

$$k_L \cong \sqrt{i\mu\sigma\omega} = \sqrt{2i} \sqrt{\frac{\mu\sigma\omega}{2}} = (1+i)\kappa, \quad \kappa^2 = \frac{\mu\sigma\omega}{2}. \quad (11)$$

We have then simultaneously

$$\kappa \gg k, \quad \text{i.e. by (10b)} \quad h \cong k. \quad (11a)$$

In this case one obtains as phase velocity simply:

$$\frac{\omega}{h} \cong \frac{\omega}{k} = c,$$

as has already been pointed out at the end of section A.

From (9) and (9a) there follows with assumption (11a)

in the exterior of the conductor: $|E| \ll |F|$,

in the interior of the conductor: $|E| \gg |F|$.

The electric lines of force thus have essentially the direction of the y -axis outside of the conductor, that of the x -axis inside of the conductor.

We will now determine that value of the y -coordinate within the metal for which E_x and hence also J_x have decreased to a fraction $1/e$ of the value at the surface. We call this value of y , $-d$. According to our assumptions (11) and (11a):

$$\kappa d = 1, \quad d = \frac{1}{\kappa} \ll \frac{1}{k}. \quad (12)$$

Since $k = 2\pi/\lambda$ this is small compared to the wave-length λ corresponding to the frequency ω . *The current is confined to a thin skin at the periphery of the conductor, while the whole interior is practically free of current.* We speak of a *skin effect* which is known to play an important role in alternating current practice. The thickness of the skin is smaller in the degree that the frequency is higher (by (11) κ increases in proportion to $\sqrt{\omega}$). The following table gives some values of d for Cu ($\sigma = 57.5 \cdot 10^6 \Omega^{-1}\text{M}^{-1}$, $\mu = \mu_0 = 4\pi \cdot 10^{-7} \Omega\text{M}^{-1}\text{S}$).

	Alternating Current Line	Telephone	Wireless Telegraphy	Hertzian Oscillations
$\frac{\omega}{2\pi} =$	60/sec	1000/sec	$3 \cdot 10^8/\text{sec}$	$10^9/\text{sec}$
$\lambda =$	$5 \cdot 10^3 \text{ km}$	300 km	1 km	30 cm
$\kappa =$	116 m^{-1}	$4.7 \cdot 10^3 \text{ m}^{-1}$	$8.15 \cdot 10^3 \text{ m}^{-1}$	$4.7 \cdot 10^4 \text{ m}^{-1}$
$d =$	8.6 mm	2.1 mm	0.13 mm	$2.1 \cdot 10^{-6} \text{ mm}$

We illustrate this by Fig. 29, which, however, does not indicate the circumstances of a semiinfinite space, but the practically more interesting ones for a wire of circular cross section. The straight line 00 for direct current or commercial alternating current (60/sec) passes over into the slightly concave curve 11 for telephone frequencies (1000/sec); curve 22 applies for high frequencies proper (e.g. 1 km wave-length) and shows a

FIG. 29. Variation of alternating-current amplitude in cross section of wire: 00, direct current; 11, telephone current; 22 high-frequency current.

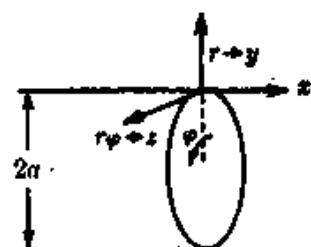


FIG. 30. Transition to the limit from the circular cross section of the wire (coordinates x, r, φ) to semi-infinite space (coordinates x, y, z).

pronounced skin effect. The three curves 00, 11, 22 have been drawn for the same total current I .

We also illustrate, by Fig. 30, the transition from the coordinates x, r, φ of the wire to the coordinates x, y, z of the semiinfinite space.

C. The Alternating-Current Impedance of a Semiinfinite Space

We cut out of the metallic semiinfinite space a rectangular parallelepiped which is infinitely long in the y -direction and whose upper end surface lies in the plane $y = 0$. Let the length of the side parallel to the z -direction be unity, that parallel to the x -direction be equal to the wave-length λ of the wave propagating itself in this direction. If, in the following, we neglect the imaginary part of h (the slight damping of the wave in the x -direction) we have $\lambda = 2\pi/h$. The total current flowing through the parallelepiped is:

$$I = \int_0^1 \int_0^\infty J_z dz dy. \quad (13)$$

We define the resistance of the parallelepiped energetically with the aid of the heat generated within it, which is given by the Joule heat Q integrated over the parallelepiped and averaged over the time. By Poynting's theorem we have for the parallelepiped:

$$\dot{W}_e + \dot{W}_m + Q = - \int S_n d\sigma. \quad (13a)$$

Here the first two terms on the left drop out on taking the time average because of the periodicity of the process. The Poynting vector on the right

is only that across the xz -surface (shaded in Fig. 31); the contributions of the xy -surfaces are zero since $H_z = H_y = 0$ and the contributions of the yz -surfaces cancel because of the periodicity with respect to x . Hence

$$\int S_n d\sigma = - \int_0^1 dz \int_0^\lambda S_x dx = \int_0^\lambda E_x H_z dx.$$

If, for E_x and H_z , we use the values within the metal (which, as we know, agree with those in air at $y = 0$ and may be simplified because $|h| \ll |k_z|$), we find from (9) and (9b), utilizing the representation (7)

$$\int S_n d\sigma = A^2 \int_0^\lambda \operatorname{Re}[e^{-i\omega t + ihx}] \cdot \operatorname{Re}\left\{\sqrt{\frac{e'}{\mu}} e^{-i\omega t + ihx}\right\} dx.$$

If, as a sufficient approximation, we substitute here $i\sigma/\omega$ for e' and write for \sqrt{i} its value $(1+i)/\sqrt{2}$, we can place the factor $(\sigma/(2\mu\omega))^{1/2}$ ahead of

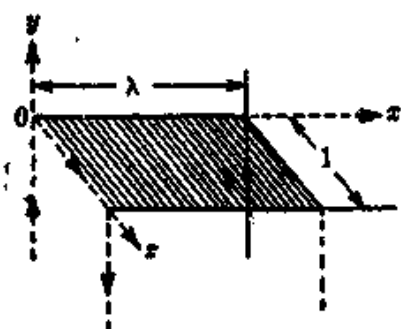


FIG. 31. A block is cut out of the metallic semi-infinite space $y < 0$ by the two pairs of planes $x = 0$, $x = \lambda$; $z = 0$, $z = 1$. Computation of its resistance for a surface wave progressing in the x -direction.

the integral sign, while the factor $1+i$ remains in the argument of Re . If we make the further substitution

$$u = hx - \omega t, \quad dx = \frac{du}{h} = \frac{\lambda du}{2\pi}, \quad u_0 = -\omega t,$$

the preceding equation passes over into

$$\int S_n d\sigma = \frac{\lambda A^2}{2\pi} \sqrt{\frac{\sigma}{2\mu\omega}} \int_{u_0}^{u_0+2\pi} \cos u (\cos u - \sin u) du = \frac{\lambda A^2}{2} \sqrt{\frac{\sigma}{2\mu\omega}}. \quad (13b)$$

Since this value has become independent of t , it represents at the same time the time average of Q , which we require for the definition of the resistance. Taking a time average of the energetic definition of R in (18.6d) and thus extending it for alternating current we write

$$R\bar{I}^2 = \bar{Q} = \frac{\lambda A^2}{2} \sqrt{\frac{\sigma}{2\mu\omega}}. \quad (14)$$

In order to compute the value of \bar{I}^2 which occurs here we make use of the loop integral of \mathbf{H} about the parallelepiped, e.g. in the plane $x = 0$

(indicated in Fig. 31 by heavy arrows). In view of the direction of \mathbf{H} only the edge $y = 0, 0 < z < 1$ of the parallelepiped yields a contribution. We thus obtain

$$\begin{aligned} I &= \int_0^1 H_z dz = H_z = G(0)e^{-i\omega t} \\ &= \sqrt{\frac{\epsilon'}{\mu}} A e^{-i\omega t} = \sqrt{\frac{\sigma}{2\mu\omega}} A(1 + i)e^{-i\omega t} \end{aligned} \quad (14a)$$

Before taking the mean square of this we must pass to the real part:

$$I = \sqrt{\frac{\sigma}{2\mu\omega}} A (\cos \omega t + \sin \omega t).$$

We then obtain

$$\bar{I}^2 = \frac{\sigma}{2\mu\omega} A^2. \quad (14b)$$

Substitution in (14) yields

$$R = \lambda \sqrt{\frac{\mu\omega}{2\sigma}} = \frac{\lambda\kappa}{\sigma}, \quad \text{referring to (11).} \quad (15)$$

The meaning of this formula becomes clear if we substitute the skin thickness d from Eq. (12). It then becomes

$$R = \frac{\lambda}{d\sigma}. \quad (15a)$$

λ is the "length" of our conductor segment measured in the direction of propagation of the waves. If we compare (15a) with the elementary formula for direct current

$$R_0 = \frac{l}{q\sigma}, \quad (15b)$$

we see that the cross section q becomes, in our alternating-current case, the rectangle

$$d \cdot 1 \text{ (} d \text{ in the } y\text{-direction, } 1 \text{ in the } z\text{-direction).} \quad (15c)$$

In place of the infinite cross section (the yz -surface of our parallelepiped) available to it the alternating current utilizes, in a sense, only the rectangle (15c); expressed differently: the alternating current, which drops off exponentially within the conductor, behaves, with respect to its resistance, just as a direct current which is distributed uniformly over the skin thickness d .

D. The Rayleigh Resistance of a Wire

We now pass from the resistance formula (15) for a conductor with plane boundary to that for a *circularly cylindrical wire* of radius a . We assume here

$$a \gg d,$$

so that the skin effect may develop freely at the surface of the wire and its interior remains free of current. We must now, however, consider a width $2\pi a$ of the parallelepiped, rather than the width 1, and imagine its current-carrying layer to be bent into the current-carrying surface layer

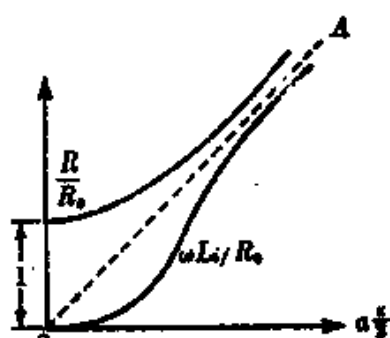


FIG. 32. Resistance and inner inductive reactance for alternating current as function of frequency. The abscissa is proportional to the square root of the frequency. OA = bisector of the angle between the two axes. The inductive reactance curve approaches it asymptotically, whereas the resistance curve runs parallel to it in the limit.

of the wire. Herewith we have changed the cross section $q = d \cdot 1$ defined in (15c) to $q = d \cdot 2\pi a$ and the resistance R found in (15a) to R_0 :

$$R_0 = \frac{R}{2\pi a} = \frac{\lambda}{2\pi a \sigma d}. \quad (16)$$

If, furthermore, we introduce the direct current resistance of the wire

$$R_0 = \frac{\lambda}{\pi a^2 \sigma} \quad (16a)$$

which, like R_0 , we refer to the same length λ and the same conductivity σ as our R , we obtain simply:

$$\frac{R_0}{R_0} = \frac{1}{2} \frac{a}{d} = \frac{a}{2} \kappa. \quad (17)$$

This is *Rayleigh's resistance formula* for high-frequency alternating current. Fig. 32 indicates its limits of validity. For small ω (stationary and quasistationary currents) $R_0 = R_0$, in contradiction with (17): Our plotted curve at $\omega = 0$ is tangent to the horizontal at a distance 1 from the axis of abscissas (to a higher order of tangency). The approximate representation (17) applies only for sufficiently large ω . Because of our choice of the scale of abscissas the direction of the curve for increasing ω follows the asymptote OA , which is inclined by 45° to the axes. The intermediate

region between small and large ω requires a more detailed analytical treatment (see end of this section).

E. The Alternating Current Inductance

The method given so far could yield only the *resistance*. To obtain in similar manner the *reactance* we would have to compute the *magnetic energy* W as a function of the current I :

$$W_m = \frac{L}{2} I^2$$

and this not only for the conductor (inner selfinductance L_i , see p. 122), but also for the surrounding air space (outer selfinductance L_a). On the other hand our analysis of the external field, as carried out up to this point, would not be adequate for application to the wire, so that in the following we shall confine ourselves to the interior field and the inner selfinductance L_i .

For this however, we have a more general and also simpler method available, namely that of the impedance operator of Eq. (18.10):

$$RI = E, \quad R = R - i\omega L,^1$$

We put E equal to the field strength E_s (voltage per unit length of our conductor) at its surface $y = 0$, so that, suppressing the phase factor, we have by Eq. (9) $E = A$, where now both R and L have to be referred to unit length. We obtain I from (14a), where we again suppress the time factor:

$$I = \sqrt{\frac{\sigma}{2\mu\omega}} A(1 + i) = \frac{\sigma}{2\kappa} A(1 + i).$$

We find as the ratio of the two

$$\frac{E}{I} = (1 - i) \frac{\kappa}{\sigma},$$

so that, by (18),

$$R = \frac{E}{I} = (1 - i) \frac{\kappa}{\sigma}, \quad R = \omega L_i = \frac{\kappa}{\sigma}. \quad (19)$$

With respect to R this agrees with (15) if λ is replaced by our present unit of length, and shows at the same time that the inner *inductive reactance*

¹ We have changed the sign of the imaginary unit as compared with §18 in order to be able to employ the preceding formulas for I and E directly. In them the time factor was written in the form $\exp(-i\omega t)$, while in §18 $\exp(+i\omega t)$ occurred in the corresponding formulas.

ωL , is equal to the resistance R . This applies generally for a conductor bounded by a plane, but applies also to the circularly cylindrical wire in Rayleigh's limiting case of sufficiently high frequency.

*F. Further Treatment of the Alternating Current Field of a
Circularly Cylindrical Wire*

In order to close the subject of this section it is necessary to utilize some formulas which will not be derived systematically until §22. We are concerned first of all with the longitudinal alternating current field in a wire of radius a :

$$E_z = C J_0(kr). \quad (20)$$

J_0 denotes the Bessel function of order 0 which is continuous at $r = 0$.¹ (Just as previously, the phase factor should be imagined to be added.) In the following k is to signify the complex wave number in the interior of the wire, not, as before, the wave number in air. The coefficient C in (20) is determined from the current density $J = \sigma E_z$ in the wire and the total current I by the later Eq. (22.3d):

$$C = - \frac{kI}{2\pi a \sigma} \frac{1}{J'_0(ka)}; \quad (20a)$$

From (20) and (20a) we obtain for the current density J

$$\frac{J}{J_0} = - \frac{ka}{2} \frac{J_0(kr)}{J'_0(ka)}; \quad (21)$$

$J_0 = I/(\pi a^2)$ is the direct-current value of J

Eq. (21) may serve to check our Fig. 29. For low frequencies the arguments kr and ka are small in absolute value. Then the expansion (22.3c) and its derivative may be employed:

$$\begin{aligned} J_0(\rho) &= 1 - \left(\frac{\rho}{2}\right)^2 + \frac{1}{4}\left(\frac{\rho}{2}\right)^4 - \frac{1}{36}\left(\frac{\rho}{2}\right)^6 \pm \dots \\ J'_0(\rho) &= -\frac{\rho}{2} \left(1 - \frac{1}{2}\left(\frac{\rho}{2}\right)^2 + \frac{1}{12}\left(\frac{\rho}{2}\right)^4 \pm \dots\right) \end{aligned} \quad (21a)$$

substituted in (21) this leads, with due regard of (11), to

$$\frac{J}{J_0} = \frac{1 - \left(\frac{kr}{2}\right)^2 + \frac{1}{4}\left(\frac{kr}{2}\right)^4 - \dots}{1 - \frac{1}{2}\left(\frac{ka}{2}\right)^2 + \frac{1}{12}\left(\frac{ka}{2}\right)^4 - \dots} = \frac{1 - \frac{i}{2}(kr)^2 - \frac{1}{16}(kr)^4}{1 - \frac{i}{4}(ka)^2 - \frac{1}{48}(ka)^4}. \quad (21b)$$

¹ We employ here and in the following (unlike Vol. VI) the symbol J , in accord with the practice followed in Vol. II of these lectures and in American physics and engineering literature generally.

From this follows

$$\left| \frac{J}{J_0} \right|^2 = \frac{1 + \frac{1}{8} (\kappa r)^4}{1 + \frac{1}{48} (\kappa a)^4} \quad (21c)$$

As compared to the direct-current straight line 00 in Fig. 29 there occurs thus a dip, which on the axis of the wire ($r = 0$) has the depth $(\kappa a)^4/48$, and a rise at the periphery of the wire ($r = a$) to a height which is five times as great.

At high frequencies we obtain asymptotically, according to (22.7) and (22.6a):

$$J_0(\kappa r) = (\pi \kappa r/2)^{-1/2} \cos(\kappa r - \pi/4) \quad (22)$$

$$\frac{J}{J_0} = \frac{\kappa a}{2} \left(\frac{a}{r} \right)^{1/2} \frac{\cos(\kappa r - \pi/4)}{\sin(\kappa a - \pi/4)} \quad \left| \frac{J}{J_0} \right| = \frac{\kappa a}{\sqrt{2}} \left(\frac{a}{r} \right)^{1/2} e^{-\kappa(a-r)} \quad (22a)$$

In view of the sharp decrease for $r < a$ the whole interior of the wire is practically free of current; the magnitude at the edge is $\kappa a/\sqrt{2}$ times as great as in the direct-current case.

The same formulas also yield a closed expression for the operator R in Eq. (18). By (20) and (20a) we have

$$R = \frac{E_z}{I} = - \frac{k}{2\pi a \sigma} \frac{J_0(\kappa a)}{J'_0(\kappa a)} = - \frac{\kappa a}{2} \frac{J_0(\kappa a)}{J'_0(\kappa a)} R_0, \quad (23)$$

where R_0 is once more the direct-current resistance per unit length, i.e. $1/(\pi a^2 \sigma)$. Hence we have for *low* frequencies (by (21a), expanding in powers of κa):

$$\frac{R}{R_0} - \frac{i\omega L}{R_0} = 1 - \frac{i}{4} (\kappa a)^2 + \frac{1}{48} (\kappa a)^4.$$

The separation of the real and imaginary parts yields:

$$\frac{R}{R_0} = 1 + \frac{1}{48} (\kappa a)^4, \quad \frac{\omega L}{R_0} = \frac{1}{4} (\kappa a)^2. \quad (23a)$$

On the other hand, we obtain, from (22) and (22a), for very *high* frequencies simply:

$$\begin{aligned} \frac{R}{R_0} - \frac{i\omega L}{R_0} &= -i \frac{\kappa a}{2} = (1 - i) \frac{\kappa a}{2} \\ \frac{R}{R_0} &= \frac{\omega L}{R_0} = \frac{\kappa a}{2}. \end{aligned} \quad (23b)$$

This agrees with our earlier results (17) and (19), obtained for the conductor with the plane boundary.

Our approximations (23a, b) permit us to check also Fig. 32 and to interpolate for the intermediate region between low and high frequencies. At low frequencies, by (23a), the resistance curve approaches the straight line $R = R_0$ as a parabola of the fourth order and at high frequencies it approaches the straight line OA in Fig. 32 from above. The curve for the inner inductive reactance starts at low frequencies as a parabola of the second order¹ and approaches² at high frequencies the same straight line from below.

§21. The Coil Carrying Alternating Current

Inasmuch as in §17 we had to defer the treatment of the direct-current field of the circular wire as mathematically too complicated, the rigorous treatment of the alternating-current field of a long coil appears to be out of the question. We hence make the same approximation as on p. 25, i.e. replace the coil, which we assume to consist of a single closely-wound layer, by an infinitely long hollow cylinder of uniform metal. Let its axis (x -axis) be vertical and let it be traversed by horizontal circular currents whose intensity distribution we shall determine. Let the inner radius of the hollow cylinder be a , the outer radius $a + d$.

A. The Field of the Coil

As in the direct-current case we assume that the magnetic excitation is zero outside of the coil, uniform³ within it and parallel to the coil axis, so that we may write

$$H = 0 \text{ for } r \geq a + d, \quad H = H_z = H_0 e^{-i\omega t} \text{ for } r \leq a. \quad (1)$$

We must then assume the H -field to be parallel to the cylinder axis also in the metallic conductor, i.e.

$$H = H_z = H(r) e^{-i\omega t} \text{ for } a < r < a + d.$$

¹ This statement applies to the product ωL plotted in the figure; L itself has for $\omega = 0$, in accord with the meaning of $\epsilon^2 = \mu\sigma\omega/2$, the non-vanishing value

$$L = \frac{R_0 \mu \sigma a^2}{4 \cdot 2} = \frac{\mu}{8\pi},$$

in agreement with §16C.

² This approach is a true tangency for $\omega \rightarrow \infty$; on the other hand, the curve for the resistance remains even for $\omega \rightarrow \infty$ a finite amount $R_0/4$ above the straight line OA , as would be shown by a more precise formulation of the approximation (23b).

³ This customary and practically unavoidable assumption for infinite length of the coil is, strictly speaking, not permissible in Maxwell's theory. It contradicts the equation, applying for the nonconducting interior space, $\mathbf{D} = \text{curl } \mathbf{H}$ and is, in view of $\text{curl } \mathbf{H} = 0$ equivalent to the neglect of the displacement current $\dot{\mathbf{D}}$.

H_z must satisfy the general wave equation (20.2), which yields, for $H(r)$, the differential equation

$$\Delta H(r) + k^2 H(r) = 0, \quad k^2 = \epsilon\mu\omega^2 + i\mu\sigma\omega. \quad (2)$$

Transformed to polar coordinates x, r, φ it is integrated in terms of Bessel functions. We do not require here, however, the particular solution $J_0(kr)$ as in (20.20), but the general solution containing two constants, which we write preferably in the form

$$C_1 H_0^1(kr) + C_2 H_0^2(kr). \quad (2a)$$

H^1, H^2 are the two Hankel cylinder functions, about which we shall give some information in the next section. In particular we shall familiarize ourselves there with their asymptotic behavior for large values of the argument $\rho = kr \rightarrow \infty$:

$$H_0^1(\rho) \rightarrow \sqrt{\frac{2}{\pi\rho}} e^{i(\rho - \pi/4)}, \quad H_0^2(\rho) \rightarrow \sqrt{\frac{2}{\pi\rho}} e^{-i(\rho - \pi/4)}. \quad (2b)$$

Since for high-frequency alternating current invariably $|k|a \gg 1$, we can limit ourselves in the integration of (2) to these asymptotic values and can write

$$H(r) = \sqrt{\frac{2}{\pi kr}} (C_1 e^{i(kr - \pi/4)} + C_2 e^{-i(kr - \pi/4)}). \quad (3)$$

From Eqs. (1) we have the boundary conditions

$$H(a + d) = 0 \quad \text{and} \quad H(a) = H_s.$$

They are satisfied if, by special choice of C_1, C_2 , expression (3) is transformed into

$$H(r) = \sqrt{\frac{a}{r}} H_s \frac{\sin[k(a + d - r)]}{\sin(kd)}. \quad (4)$$

Having found in this manner H_z as function of r (we may also say as a function of the polar coordinates x, r, φ) we can now utilize §20 A. It is true that now we are not dealing, as there, with a wave advancing along the x -axis, but with ordinary (stationary) alternating current, for which the wave number h given there vanishes. In fact our present Eq. (2) passes over into the earlier (20.3) for $X = H_z, h = 0$. Furthermore we must not employ, as in the preceding paragraph, Eq. (20.5) (electric type), but Eq. (20.6) (magnetic type). This yields the transversal components H_r, E_φ expressed in terms of the longitudinal component H_z . We are particularly interested in E_φ . With $g_u = 1, g_r = r, H = 0$, (20.6) yields:

$$E_\varphi = -\frac{i}{k} \sqrt{\frac{\mu}{\epsilon'}} \frac{dH(r)}{dr} e^{-i\omega t} = \frac{J}{\sigma}. \quad (5)$$

If we carry out the differentiation with respect to r only in the factor $\sin\{k(a + d - r)\}$ (the square-root factor is "slowly variable") and if we denote the value of J on the inner surface of the coil by J_a , we obtain

$$J = J_a \sqrt{\frac{a}{r}} \frac{\cos\{k(a + d - r)\}}{\cos(kd)}, \quad J_a = i\sigma \sqrt{\frac{\mu}{\epsilon'}} H_a \frac{\cos(kd)}{\sin(kd)} e^{-i\omega t}. \quad (6)$$

Passing to the discussion of the field, we distinguish two cases:

$$a. |k|d \gg 1 \quad \text{and} \quad b. |k|d \ll 1;$$

we of course continue to adhere to the original assumption $|k|a \gg 1$.

a. High-Frequency Alternating Current with not too Small Coil Thickness. Since we take the imaginary part of k to be positive we have then

$$|e^{-ikd}| \gg |e^{ikd}|$$

and, more particularly near the inner coil boundary also:

$$|e^{-ik(a+d-r)}| \gg |e^{ik(a+d-r)}|.$$

Hence (4) yields, for $r \cong a$,

$$H(r) \cong H_a e^{ik(r-a)}, \quad (7)$$

which signifies a steep exponential falling off at the inner boundary of the coil. By (6) the current density shows a similar steep exponential decline:

$$J \cong J_a e^{ik(r-a)}, \quad (7a)$$

$$J_a = i\sigma \sqrt{\frac{\mu}{\epsilon'}} H_a e^{-i\omega t}. \quad (7b)$$

We thus have a *pronounced skin effect* at the inner surface of the coil.

b. Small Coil Thickness and Relatively Low-Frequency Alternating Current. We may then expand Eqs. (4) and (6) in powers of kd and, particularly in the neighborhood of $r = a$, also in powers of $k(a + d - r)$. We indicate only the first term of these expansions:

$$H(r) \cong H_a \frac{a + d - r}{d}, \quad J \cong J_a. \quad (8)$$

Fig. 33 illustrates this graphically: At the left is shown the behavior of $H(r)$, at the right that of J . The curves 0 correspond to the limiting case b , the curves 2 to case a , and the curves 1 to an intermediate case. All three pairs of curves refer to the same H within the coil and hence also to the same total current I in the coil.

B. Resistance and Inner Inductive Reactance of the Coil

We wish to compute these quantities per unit length of the coil and must first know the total current passing through this unit length. It is obtained by integration of (5) with respect to r :

$$\int_a^{a+d} J dr = C \{H(a+d) - H(a)\} e^{-\omega t} = -CH_a e^{-\omega t};$$

in view of the meaning of k and ϵ' we have

$$-C = \frac{i\sigma}{k} \sqrt{\frac{\mu}{\epsilon'}} \cong 1. \quad (9)$$

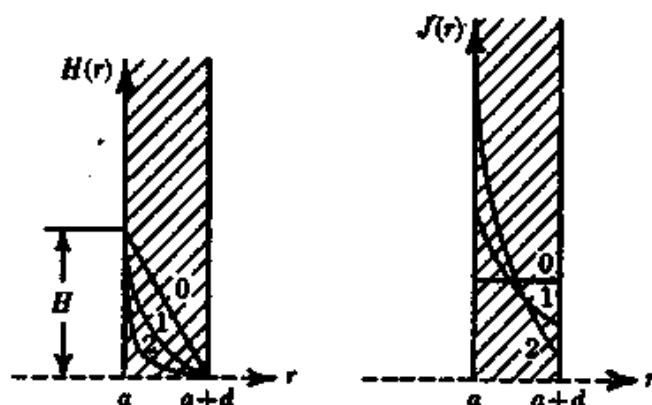


FIG. 33. Magnetic field and current distribution in a single-layer coil: At the left, magnetic field $H(r) = H_a$, at the right current density $J(r) = J_0$. The curves 0 and 2 correspond to the limiting cases of direct current and high-frequency alternating current, curves 1 to intermediate frequencies with the same total current as in the two limiting cases.

To refer this total current to the current I flowing in a *single wire* of the coil we set it equal to NI , where N is the number of turns per unit length:

$$I = \frac{1}{N} H_a e^{-\omega t}. \quad (9a)$$

With this value of I we write the equation

$$(R - i\omega L)I = E,$$

where we must set the voltage E equal to the field strength on the inner surface of the coil, i.e. equal to J_a/σ . We thus obtain from (6) and (9)

$$R - i\omega L = \frac{Nk \cos(kd)}{\sigma \sin(kd)} = i \frac{Nk}{\sigma} \frac{e^{ikd} + e^{-ikd}}{e^{ikd} - e^{-ikd}}. \quad (10)$$

For a discussion of this expression we write, as in (20.11), $k = (1 + i)\kappa$. The denominator of (10) then becomes

$$e^{(-1+i)\kappa d} - e^{(1-i)\kappa d}.$$

We multiply numerator and denominator with the complex conjugate of this quantity, whereupon they can be expressed in terms of the trigonometric and hyperbolic functions of $2\kappa d$. We obtain

$$R - i\omega L = (1 - i) \frac{N\kappa \sinh(2\kappa d) + i \sin(2\kappa d)}{\sigma \cosh(2\kappa d) - \cos(2\kappa d)} \quad (11)$$

so that, separating real and imaginary parts,

$$R = \frac{N\kappa \sinh(2\kappa d) + \sin(2\kappa d)}{\sigma \cosh(2\kappa d) - \cos(2\kappa d)}, \quad (11a)$$

$$\omega L = \frac{N\kappa \sinh(2\kappa d) - \sin(2\kappa d)}{\sigma \cosh(2\kappa d) - \cos(2\kappa d)}. \quad (11b)$$

For direct current ($\kappa d \rightarrow 0$) (11a) yields

$$R_0 = \frac{N}{\sigma d}, \quad (12)$$

corresponding to the direct-current resistance per unit length of a wire of rectangular cross section with the width d of the coil (not to be confused with the layer thickness d of p. 162) and the height $1/N$. We have therefore

$$\frac{R}{R_0} = \kappa d \frac{\sinh(2\kappa d) + \sin(2\kappa d)}{\cosh(2\kappa d) - \cos(2\kappa d)}, \quad (12a)$$

$$\frac{\omega L}{R_0} = \kappa d \frac{\sinh(2\kappa d) - \sin(2\kappa d)}{\cosh(2\kappa d) - \cos(2\kappa d)}. \quad (12b)$$

More particularly we consider the two limiting cases *a* and *b* of p. 172:

a. $\kappa d \gg 1$. Then $\sinh(2\kappa d) = \cosh(2\kappa d) \rightarrow \infty$, so that

$$\frac{R}{R_0} = \frac{\omega L}{R_0} = \kappa d, \quad (13a)$$

b. $\kappa d \ll 1$. By expansion in powers of $2\kappa d$, and retaining only the first nonvanishing term we find from (12a, b)

$$\frac{R}{R_0} = 1 + \frac{4}{45} (\kappa d)^4, \quad \frac{\omega L}{R_0} = \frac{2}{3} (\kappa d)^2. \quad (13b)$$

These results (13a, b) are represented qualitatively once more by Fig. 32. Also now the direct-current straight line $R/R_0 = 1$ is approximated at low frequencies to the fourth order and the axis of abscissas is parabolically tangent to the curve $\omega L/R_0$ at the origin. At high frequencies both curves again approach the 45° line asymptotically. Only the scale of abscissas, now given by κd , differs from that given before since the skin effect occurs here unilaterally, on the inner surface of the coil.

However much more important than the inner selfinductance, to which we have limited ourselves, is of course the *external selfinductance of the coil*. In the discussion of the coil traversed by direct current (§17, Eq. (15)) we were exclusively interested in the latter. With the limitation mentioned in footnote 3, p. 170 (neglect of the displacement current) we can transfer the value found there directly to the alternating-current case.

With our idealization of the problem (closed ring currents in horizontal planes) we could circumvent the determination of the electric field within the coil and leave the boundary conditions for this field out of consideration. Actually the currents are, in view of the finite pitch of the coil, not exactly horizontal and there exists, from turn to turn, an external electric field of complex character which is predominantly axial in direction. Our elementary treatment evidently does not suffice to determine this field.

C. The Multilayer Coil

We imagine the several layers, n in number, to be placed one over the other without intervening space, idealized as hollow cylinders of thickness d/n , and traversed in series by alternating current I of uniform magnitude. Let the insulation of the successive layers be perfect, though by assumption infinitely thin; the same is to apply for the insulation of successive turns in any single layer.

In view of the magnitude of the loop integral of \mathbf{H} about a single layer, i.e. over a rectangle of height 1 and width d/n , the magnetic field decreases by NI for each layer (N = number of turns per unit length along the axis of the coil). Hence we find at the boundaries of successive layers, from the inside outward:

$$H_0 = H, \quad H_1 = H - NI, \quad H_2 = H - 2NI, \dots, H_n = H - nNI = 0. \quad (14)$$

The factor $\exp(-i\omega t)$ must be thought of as added here, and in the following. In the ν th layer the magnetic field satisfies the differential equation and the boundary conditions:

$$\Delta H + k^2 H = 0, \quad H = \begin{cases} H_{\nu-1} & r = a + (\nu - 1) \frac{d}{n}, \\ H_\nu & r = a + \nu \frac{d}{n}. \end{cases} \quad (15)$$

The differential equation is again integrated by the superposition of the two Hankel functions, for which we can substitute their asymptotic values from Eq. (2b) since kr is very large; furthermore we can treat the denominator \sqrt{kr} as slowly variable in comparison with the exponential functions and

include it along with the remaining constants ($\exp\{-i\pi/4\}$ and $\sqrt{2/\pi}$) in the amplitudes C_1, C_2 . Eq. (3) is then simplified to

$$H = C_1 e^{ikr} + C_2 e^{-ikr}.$$

The following expression, which at the same time satisfies the boundary conditions (15) and hence represents the magnetic field in the ν th layer, is of this form:

$$H = H_{\nu-1} \frac{\cos \left\{ k \left(r - a - (\nu - \frac{1}{2}) \frac{d}{n} \right) \right\}}{\cos \left(k \frac{d}{2n} \right)} - NI \frac{\sin \left\{ k \left(r - a - (\nu - 1) \frac{d}{n} \right) \right\}}{\sin \left(k \frac{d}{n} \right)} \quad (16)$$

In particular we obtain for the first layer, $a < r < a + d/n$, $\nu = 1$

$$H = H_a \frac{\cos \left\{ k \left(r - a - \frac{d}{2n} \right) \right\}}{\cos \left(k \frac{d}{2n} \right)} - NI \frac{\sin \{ k (r - a) \}}{\sin \left(k \frac{d}{n} \right)}. \quad (16a)$$

We need consider only this formula if now we wish to determine the electric field E_r at the inner surface of the coil and from this the impedance operator $R = R - i\omega L$ of the multilayer coil.

To begin with we obtain for the current density J_r , for which we shall consider right away the value for $r = a$, from (5) (see also (9)):

$$J_a = c \left(\frac{dH}{dr} \right)_a = -Hk \frac{\sin \left(k \frac{d}{2n} \right)}{\cos \left(k \frac{d}{2n} \right)} + NI k \frac{1}{\sin \left(k \frac{d}{n} \right)}.$$

We here set $H = nNI$ (see last of Eqs. (14)) and obtain after simple trigonometric transformation,

$$J_a = NI k \frac{n \cos \left(k \frac{d}{n} \right) - (n - 1)}{\sin \left(k \frac{d}{n} \right)}.$$

Hence we obtain

$$E = E_p = \frac{1}{\sigma} J_n \quad \text{and} \quad R = \frac{E}{I} = \frac{Nk}{\sigma} \frac{n \cos\left(k \frac{d}{n}\right) - (n-1)}{\sin\left(k \frac{d}{n}\right)},$$

which, for $n = 1$, is identical with Eq. (10).

We divide R by the direct-current value R_0 of R , obtained by passing to the limit $\omega \rightarrow 0$, i.e. $k \rightarrow 0$:

$$R_0 = \frac{Nn}{\sigma d},$$

and find

$$\frac{R}{R_0} = kd \frac{\cos\left(k \frac{d}{n}\right) - \left(1 - \frac{1}{n}\right)}{\sin\left(k \frac{d}{n}\right)}. \quad (17)$$

The separation of (17) into real and complex parts is rather complicated. If as before we set $k = (1 + i)\kappa$, it yields

$$\frac{R/R_0}{\omega L/R_0} = kd \frac{\sinh\left(2\kappa \frac{d}{n}\right) \pm \sin\left(2\kappa \frac{d}{n}\right) - 2\left(1 - \frac{1}{n}\right) \cdot \left(\sinh\left(\kappa \frac{d}{n}\right) \cdot \cos\left(\kappa \frac{d}{n}\right) \pm \cosh\left(\kappa \frac{d}{n}\right) \cdot \sin\left(\kappa \frac{d}{n}\right)\right)}{\cosh\left(2\kappa \frac{d}{n}\right) - \cos\left(2\kappa \frac{d}{n}\right)}$$

Thus, thanks to our extensive (possibly excessive) idealization of the problem, we have obtained a quite simple final formula. The frequency band for which our formula is valid has an upper limit determined by the characteristic frequency of the coil; as we approach the latter our notion of equal current in all turns obviously becomes invalid. It should also be emphasized that our formula presumes the regular superposition of the layers and does not cover the spiral interweaving of the turns (Dolezalek, litz wire) which is preferred for practical reasons (suppression of the skin effect).

§22. The Problem of Waves on Wires

As is well known, the experiments of Heinrich Hertz dealt with "surface waves" progressing along wires as well as with "space waves" propagated freely through the air. Hertz expected their velocity also to be equal to c ,

but could confirm this result neither experimentally nor theoretically. The reason for his experimental failure was the influence of the walls of the laboratory; the reason for his theoretical failure, an excessive idealization of the problem. He treated the wire as infinitely thin and hence could not set up electromagnetic boundary conditions. This was first accomplished in a paper by the author¹ which yielded for the velocity of propagation a value nearly equal to—more precisely, slightly less than— c . It was here essential that a phase velocity exceeding c could be excluded by a condition at infinity. The experimental difficulties were overcome by E. Lecher (see §25) by using a *two-wire line*. In the present section we shall confine ourselves to Hertz's problem of the *single wire*.

A. The Field within and outside of the Wire

While in the preceding section the Maxwell equations were utilized only in part, inasmuch as not only was the displacement current within the conductor neglected, but to some extent also that in free space, we must now adhere strictly to these equations. The problem is symmetrical about the axis of the wire. We make it the x -axis of a cylindrical coordinate system x, r, φ . Then for all components $\partial/\partial\varphi = 0$ and only the components

$$E_x, E_r, H_\varphi \quad (1)$$

differ from zero. As in §20 we set them equal to products of the common factor

$$e^{-i\omega t + ikx} \quad (1a)$$

with a function of r only. Let the time variation be purely periodic, i.e. ω be real, and the phase propagation take place along the positive x -axis; k must then have a positive real part.

We deal first with the Cartesian longitudinal component E_x . It satisfies the wave equation (20.3), in which we put $u = r, v = \varphi$. If, in place of r , we introduce the dimensionless variable

$$\rho = \sqrt{k^2 - k^2} r \quad (1b)$$

and set

$$E_x = F(\rho)e^{-i\omega t + ikx}, \quad (2)$$

we obtain for F the differential equation

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dF}{d\rho} \right) + F = 0 \quad (3)$$

¹ Ann. d. Physik, Vol. 67, pp. 233-290, 1899.

or, with the differentiation carried out,

$$\frac{d^2 F}{d\rho^2} + \frac{1}{\rho} \frac{dF}{d\rho} + F = 0. \quad (3a)$$

This, as well as the more general equation

$$\frac{d^2 F}{d\rho^2} + \frac{1}{\rho} \frac{dF}{d\rho} + \left(1 - \frac{n^2}{\rho^2}\right) F = 0 \quad (3b)$$

is known as Bessel's differential equation. We have dealt with it already in Vol. II, §27. The solution which is continuous for $\rho = 0$ was represented there, in (27.7), by the series

$$J_n(\rho) = \frac{1}{n!} \left(\frac{\rho}{2}\right)^n - \frac{1}{1!(n+1)!} \left(\frac{\rho}{2}\right)^{n+2} + \frac{1}{2!(n+2)!} \left(\frac{\rho}{2}\right)^{n+4} - \dots \quad (3c)$$

For all that follows n may be assumed to be an integer. For $n = 0$ we obtain the representation (21a) for $J_0(\rho)$ employed in §20, from which we see directly

$$J_1(\rho) = -\frac{d}{d\rho} J_0(\rho), \quad \int_0^\rho \rho J_0(\rho) d\rho = \rho J_1(\rho). \quad (3d)$$

Within the wire, $r < a$ (a = radius of wire), where E_z must nowhere become infinite, our function F is thus determined but for a constant by its differential equation (3):

$$F = C J_0(\rho) \quad \text{for} \quad 0 < r < a. \quad (4)$$

Outside of the wire (in air) $\sigma = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$, so that $k = \omega/c$ is real. For the sake of differentiation we shall denote the complex value of k , which applies within the wire, by k_L as in §20. Since the condition of continuity at $r = 0$ plays no role outside of the wire, Eq. (3) has to be integrated generally. This is done by the two "Hankel functions of order 0", already mentioned on p. 171:

$$H_0^1(\rho) \quad \text{and} \quad H_0^2(\rho).$$

We shall deal in detail with these and the general Hankel functions of order n in Vol. VI, §19. It must here suffice to enumerate some of their principal properties:

a. The functions H_0^1 and H_0^2 become *logarithmically infinite* for $\rho = 0$, since we have, for small ρ :

$$H_0^{1,2}(\rho) = 1 \pm \frac{2i}{\pi} \log \frac{\gamma\rho}{2} + \dots = \pm \frac{2i}{\pi} \log \frac{\gamma\rho}{\pm 2i} \dots \quad (5)$$

γ is related, to the Euler-Mascheroni constant

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) = 0.5772 \dots;$$

For

$$\log \gamma = 0.5772 \dots, \gamma = 1.781 \dots \quad (5a)$$

(log here denotes, as always, the natural logarithm.)

b. The Hankel functions are branched in the complex ρ -plane, just as the logarithm. To make them single-valued we must provide a branch cut in the ρ -plane, e.g. along the negative imaginary axis. If we set $\rho = |\rho| e^{i\vartheta}$, we thus limit the angle ϑ to the values $-\pi/2 < \vartheta < 3\pi/2$. In this sense we speak of the principal branch of the Hankel functions, just as we speak of the principal branch of the logarithm.

c. The Hankel functions $H_n^1(\rho)$ and $H_n^2(\rho)$ of order n are defined as solutions of Eq. (3b) in such fashion that also their representation contains a logarithmic term for $\rho \rightarrow 0$, i.e. that the branching mentioned in b applies also to them. However, the determining factor for their singularity is not this logarithmic term, but the term which becomes most strongly infinite:

$$H_n^1(\rho) = \frac{(n-1)!}{i\pi} \left(\frac{2}{\rho}\right)^n, \quad H_n^2(\rho) = \frac{(n-1)!}{-i\pi} \left(\frac{2}{\rho}\right)^n. \quad (6)$$

The logarithmic singularity and the branching disappear for the sum of the two functions H_n . The regular solution of the differential equation (3b) is obtained in the form

$$J_n(\rho) = \frac{1}{2}(H_n^1(\rho) + H_n^2(\rho)). \quad (6a)$$

d. For $\rho \rightarrow \infty$ we have as asymptotic representation of the two principal branches:

$$H_n^1(\rho) = \sqrt{\frac{2}{\pi\rho}} e^{i(\rho - (n+1)\pi/2)}, \quad H_n^2(\rho) = \sqrt{\frac{2}{\pi\rho}} e^{-i(\rho - (n+1)\pi/2)}. \quad (7)$$

Thus H_n^1 vanishes for large ρ in the positively-imaginary ρ -halfplane, H_n^2 , in the negatively imaginary ρ -halfplane. The two together vanish for large ρ only on the real axis. J_n becomes, by (6a), infinite everywhere at infinity except on the real axis. In view of (6a) and (7) we have at infinity in the positively imaginary ρ -halfplane

$$\frac{J_n(\rho)}{J_n'(\rho)} = +i. \quad (7a)$$

After these insertions, which unfortunately were necessary for what follows, we return to our actual problem. We make the convention that the sign of the square root in (1b) is always to be chosen so that its imaginary

part is *positive*. This covers the case that the root is itself *complex*. We must however also take into account the possibility that the root is *real* and that $h < k$.

In the first case the only possible formula for the exterior of the wire is

$$F(\rho) = AH_0^1(\rho), \quad a < r < \infty, \quad A = \text{const}, \quad (8)$$

since, by (7), $H_0^2(\rho)$ becomes infinitely great for $r \rightarrow \infty$.

In the second case

$$F(\rho) = AH_0^1(\rho) + BH_0^2(\rho), \quad a < r < \infty. \quad (9)$$

is a possible formula since now, for real ρ , both H vanish, by (7), as ρ^{-1} . A and B are for the present arbitrarily disposable constants.

What is the meaning of this case? Since according to (2) the phase velocity of the wave is equal to ω/h and since ω/k is equal to c , it implies

$$\text{phase velocity} > \text{velocity of light}.$$

We supplement our representation of E_z by that of the transversal components E_r and H_ϕ , which is obtained most readily with the aid of the general rule (20.5) (where however the terms with H_z are of course omitted). If for the present we designate the functions of ρ appearing in E_r and H_ϕ with $G(\rho)$ and $V(\rho)$ this yields

$$G(\rho) = \frac{ih}{k^2 - h^2} \frac{\partial F(\rho)}{\partial r} = \frac{ih}{\sqrt{k^2 - h^2}} F'(\rho), \quad (10)$$

$$\sqrt{\frac{\mu}{\epsilon}} E(\rho) = \frac{ik}{k^2 - h^2} \frac{\partial F(\rho)}{\partial r} = \frac{ik}{\sqrt{k^2 - h^2}} F'(\rho). \quad (11)$$

B. The Boundary Condition at Infinity

As for the surface wave in §20 we are also now dealing with a process which draws its energy from the end of the wire at $x = -\infty$. We hence shall demand that the total energy flux through a cylindrical surface $r = R$ coaxial with the wire vanishes:

$$S = 2\pi \{r S_r\}_{r=R} = 0. \quad (12)$$

If as an abbreviation we designate the phase $hx - \omega t$ by Φ we obtain

$$S_r = E_z H_\phi = \text{Re}\{F(\rho)e^{i\Phi}\} \text{Re}\left\{\sqrt{\frac{\epsilon_0}{\mu_0}} \frac{-ik}{\sqrt{k^2 - h^2}} \frac{dF(\rho)}{d\rho} e^{i\Phi}\right\}.$$

We consider the *second case*, in which h , $\sqrt{k^2 - h^2}$, and ρ were real. Eq. (9) then applies for $F(\rho)$ and we obtain, indicating all immaterial constant factors by ... and utilizing the asymptotic expressions (7) with $n = 0$:

$$S = \dots \frac{r}{\rho} \{A^2 \cos^2(\Phi + \rho - \pi/4) - B^2 \cos^2(\Phi - \rho + \pi/4)\}. \quad (12a)$$

It is significant here that the factor r/ρ remains finite for arbitrarily large $r = R$ and that the phase Φ may take on any real values. Our requirement (12) can then be satisfied only by

$$A = B = 0.$$

This signifies: *In the second case wire waves cannot occur; they would have to be fed by an artificial arrangement of energy sources at infinity, which contradicts the physical meaning of the process.*

Conditions are different in the first case, where ρ has a positive imaginary part. According to Eqs. (8) and (7) the field outside of the wire decreases here exponentially as $r \rightarrow \infty$; the energy flux S vanishes in similar manner. Only this case is relevant for us. By Eqs. (2), (4), (8), (10), and (11) we calculate the corresponding field within and outside of the wire, and distinguish the complex k_L within from the real k outside, as well as the complex ϵ'/μ within from the real ϵ_0/μ_0 outside as in (20.9a, b). Furthermore it is convenient to redefine the constants C and A so that all components are multiplied with $\frac{\sqrt{k_L^2 - h^2}}{ih}$ inside and with $\frac{\sqrt{k^2 - h^2}}{ih}$ outside of the wire. We thus obtain the following tabulation:

$0 < r < a, \quad \rho = \sqrt{k_L^2 - h^2} r$ $E_z = \frac{\sqrt{k_L^2 - h^2}}{ih} C J_0(\rho)$ $E_r = C J'_0(\rho)$ $\sqrt{\frac{\mu}{\epsilon'}} H_\phi = \frac{k_L}{h} C J'_0(\rho)$	$a < r < \infty, \quad \rho = \sqrt{k^2 - h^2} r$ $E_z = \frac{\sqrt{k^2 - h^2}}{ih} A H_0(\rho)$ $E_r = A H'_0(\rho)$ $\sqrt{\frac{\mu_0}{\epsilon_0}} H_\phi = \frac{k}{h} A H'_0(\rho)$
------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------	--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

(13)

H_0 is identical with H_0^1 ; the prime at J_0 and H_0 indicates differentiation with respect to the argument ρ .

C. The Boundary Condition at the Surface of the Wire

E_z and H_ϕ must be continuous for $r = a$. Hence we must require

$$\rho_L C J_0(\rho_L) = \rho A H_0(\rho), \quad \sqrt{\frac{\epsilon'}{\mu}} k_L C J'_0(\rho_L) = \sqrt{\frac{\epsilon_0}{\mu_0}} k A H'_0(\rho) \quad (14)$$

$$\rho_L = \sqrt{k_L^2 - h^2} a, \quad \rho = \sqrt{k^2 - h^2} a.$$

By eliminating the amplitude factors A and C we obtain the *transcendental equation*

$$\frac{\rho H_0(\rho)}{H'_0(\rho)} = \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} \frac{k}{k_L} \frac{\rho_L J_0(\rho_L)}{J'_0(\rho_L)}. \quad (14a)$$

We regard this as the determining equation for the as yet unknown wave number h . However, Eq. (14a) can be greatly simplified by taking account

of the fact that if the material of the wire is a good conductor ρ_L is a large complex number with positive imaginary part so that Eq. (7a) is applicable. With its aid the right side of (14a) may be transformed into

$$i \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} \frac{k}{k_L} \rho_L \cong i \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} ka. \quad (14b)$$

Since $|\epsilon'| \gg \epsilon_0$ its absolute value is small compared with 1. Furthermore the left side may also be simplified. Since it must be small we may use Eq. (5). We then obtain for the left side of (14a)

$$\rho^2 \log \frac{\gamma \rho}{2i} = -\frac{2}{\gamma^2} u \log u \quad \text{with} \quad u = \left(\frac{\gamma \rho}{2i} \right)^2. \quad (14c)$$

Comparison with (14b) then yields as the final form of our transcendental equation

$$u \log u = v \quad \text{with} \quad v = -\frac{i\gamma^2}{2} \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} ka. \quad (15)$$

For its solution it is possible to employ a peculiar method reminiscent of the continued fraction. This rests on the fact that $\log u$ varies slowly in comparison with u . Hence if an n th approximation u_n has been found, an $n + 1$ st approximation may be obtained from

$$u_{n+1} \log u_n = v. \quad (15a)$$

We may begin, for example with $u_0 = v$ and put, in accord with (15a),

$$u_1 = \frac{v}{\log v}; \quad (15b)$$

the exact initial value is of little importance since it is corrected step by step in the subsequent approximations. Furthermore, by (15a):

$$u_2 = \frac{v}{\log u_1} = \frac{v}{\log \frac{v}{\log v}}, \quad u_3 = \frac{v}{\log \frac{v}{\log \frac{v}{\log v}}} \quad \text{etc.} \quad (15c)$$

Consider, for example, a copper wire with radius $a = 1$ mm and the frequency given farthest to the right on the table on p. 162, which corresponds to a wave-length of 30 cm and a value $ka = 2.1 \cdot 10^{-2}$. For the corresponding value of κ taken from the same table, (15) yields

$$v = -(1 + i) \cdot 7.2 \cdot 10^{-7}.$$

We begin with

$$u_1 = (1 + i) \cdot 3.6 \cdot 10^{-8}, \quad (16)$$

where $\log v$, occurring in (15b), has been approximated by -20 . We then find from (15a)

$$u_2 = (4.1 + 4.5i) \cdot 10^{-8}, \quad u_1 = (4.2 + 4.6i) \cdot 10^{-8}; \quad (16a)$$

We thus have already arrived at the limit of convergence of our "continued fraction."

From this value of u we find by (14c)

$$\rho^2 = -\frac{4}{\gamma^2} u = -(5.3 + 5.8i) \cdot 10^{-8}; \quad (16b)$$

and by (1b)

$$h^2 = k^2 + \frac{1}{a^2} (5.3 + 5.8i) \cdot 10^{-8}, \quad h = k \{1 + (6.0 + 6.6i) \cdot 10^{-4}\}. \quad (17)$$

We form

$$\frac{\omega}{h} = \frac{\omega}{k} \{1 - (6.0 + 6.6i) \cdot 10^{-4}\} \quad (17a)$$

and conclude from this, since $\omega/k = c$, that the *phase propagation lags behind c by only $6 \cdot 10^{-4} c$* . On the other hand we see from our field factor $\exp(ikh)$ that the amplitude is reduced by a factor $1/e$ only after traversal of a distance x given by

$$k \cdot 6.6 \cdot 10^{-4} x = 1, \quad x = 720 \text{ m.} \quad (17b)$$

This corresponds completely to what is expected: *Nearly undamped propagation with c as phase velocity.*

There are however also conditions for which this expectation proves to be erroneous. Consider, for example, a Wollaston wire of platinum with a radius $a = 2 \cdot 10^{-4}$ cm; the conductivity of platinum is 8 times less than that of copper. Let the wave-length in air be 1 meter. Then

$$\kappa = 9.2 \cdot 10^2 \text{ cm}^{-1}, \quad \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} = (1 - i) \cdot 0.34 \cdot 10^{-4}, \quad \rho_L = (1 + i) \cdot 0.2.$$

The argument of J_0 in (14) is then no longer large, so that we must use Eq. (3c) in place of (7a). It yields

$$\frac{J_0(\rho_L)}{J'_0(\rho_L)} = -(1 - i) \cdot 5.0$$

and for the right side of (15), with the meaning of u unchanged, the value $v = -i \cdot 7.0 \cdot 10^{-9}$. Our transcendental equation thus becomes

$$u \log u = v, \quad v = -i \cdot 7.0 \cdot 10^{-9}$$

If, once again, we put $u_1 = -v/20$, we obtain by (15c),

$$u_2 = (-0.29 + 3.5i) \cdot 10^{-10} \cong u_3$$

and

$$\rho^2 = -\frac{4}{\gamma^2} u = (0.36 - 4.4i) \cdot 10^{-10},$$

a value of the same order of magnitude as that found before in (16b). However the further calculation becomes quite different because of the smallness of $a = 2 \cdot 10^{-4}$ cm. In place of (17) we find in cm^{-2}

$$h^2 = k^2 - 0.0009 + 0.011 i.$$

It is now no longer adequate to retain a first term in a binomial expansion, since here $k^2 = (2\pi/\lambda)^2 = 0.0039$. Instead we obtain

$$h = 0.085 + 0.065i.$$

From this follows for the length of wire along which the wave amplitude has been reduced by a factor $1/e$:

$$\frac{1}{0.065} = 15 \text{ cm}$$

and for the phase velocity $\omega/0.085$. Division by the velocity of light ω/k yields

$$\frac{k}{0.085} = 0.74$$

as the ratio of the velocity of wave propagation and the velocity of light. *The former lags behind the latter by 26 per cent.*

The reason for this abnormal behavior evidently lies in the extreme thinness of the wire, which increases the alternating current impedance and prevents the development of a normal skin effect. The interior of the wire is then no longer free of current; the current distribution no longer has the character of curve 22 of Fig. 29, but that of curve 11. The field is then no longer "immunized" against Joule heat loss. Damping and propagation become anomalous.

§23. General Solution of the Wire-Wave Problem

In the preceding section we have derived that particular solution which is related to Hertz's original problem of wire waves. The question arises as to whether there is a more general solution. This question was proposed to D. Hondros as subject for his Munich thesis.¹ At the time it seemed of

¹ Ann. d. Phys. 50, p. 905, 1909.

purely theoretical interest, remote from any practical application. It has been found since that it possesses close analogies with the theory of cavity conductors (§24), at present a favored field of communications engineering. Furthermore, it may be utilized advantageously for the theory of the Lecher two-wire line which first converted Hertz's single wire into a functioning system. Also the "wire waves in nonconductors" which fit into Hondros's formulation of the problem have found practical application

A. Primary Wave and Electrical Secondary Waves

We call the solution in §22 the *principal wave*; for it $|\rho|$ was small and hence $|\rho_L|$ large. In the converse case that $|\rho|$ is large we speak of a *secondary wave*. Then, by (22.7), the left side of (22.14a) becomes equal to $-i\rho$, i.e. also large in absolute value. The denominator on the right side of (22.14a) must then become very small. ρ_L must hence approximate one of the infinitely many and frequently tabulated roots of

$$J'_0(\rho) = 0, \quad \rho = w'_1, \quad w'_2, \dots, \quad (1)$$

generally w'_r , where we shall note in particular

$$w'_1 = 3.83. \quad (1a)$$

By Eq. (22.3d) these w'_r are identical with the roots of $J_1(\rho) = 0$. (We want to reserve the symbol w_r for the roots of $J_0(\rho) = 0$.) In view of the meaning of ρ_L given by Eq. (22.14) we have then approximately

$$h_r^2 \cong k_L^2 - \left(\frac{w'_r}{a}\right)^2. \quad (2)$$

For a well-conducting wire h_r is thus approximately equal to k_L and we can write, for all moderate values of r (see (20.11)):

$$h_r \cong (1 + i)\kappa. \quad (2a)$$

From the formula for phase and damping $\exp(-i\omega t + i\kappa x)$, which is to be interpreted as before, we find: *All secondary waves are exceedingly strongly damped in their progress along the wire*; their amplitude decreases by the factor $1/e$ in the short distance $1/\kappa$. *Their phase velocity ω/κ is small compared to the velocity of light $c = \omega/k$, the ratio of the two being equal to k/κ .*

The secondary waves also behave oppositely to the principal wave with respect to the character of the field. By (2a) we have *outside of the wire*

$$\rho = \sqrt{k^2 - h_r^2} r \cong (-1 + i)\kappa r$$

and by (22.7)

$$H_0^1(\rho) \sim e^{-\alpha + i\beta r} \quad \text{for} \quad a < r < \infty.$$

By (22.8) this signifies a *skin effect outside of the wire*. In its interior, on

the other hand, the argument of $J_0(\rho)$ is real since

$$\rho = \sqrt{k_L^2 - h^2} r = w' \frac{r}{a} \quad (w', \text{ real})$$

and hence $J_0(\rho)$ is of the order of magnitude 1 for $0 \leq r \leq a$.

The entire interior is filled by current. Considerable Joule heat is generated here, which explains the rapid damping of the wave in its progress along the wire and makes any observation of the secondary waves *illusory*.

B. Magnetic Waves

While we derived the electrical principal and secondary waves from the general Eq. (20.5), we obtain the magnetic waves from (20.6). We here set $E_z = 0$, H_z equal but for a constant to the Bessel function J_0 inside, and equal to the first Hankel function H_0 outside of the wire. If we take the corresponding transversal components from (20.6) we obtain, with $u = r$, $v = \varphi$, $g_u = 1$, $g_v = r$:

$$\begin{aligned} 0 < r < a, \quad \rho = \sqrt{k_L^2 - h^2} r & \quad a < r < \infty, \quad \rho = \sqrt{k^2 - h^2} r \\ \sqrt{\frac{\mu}{\epsilon'}} H_z = \frac{\sqrt{k_L^2 - h^2}}{ih} DJ_0(\rho) & \quad \sqrt{\frac{\mu_0}{\epsilon_0}} H_z = \frac{\sqrt{k^2 - h^2}}{ih} BH_0(\rho) \\ \sqrt{\frac{\mu}{\epsilon'}} H_r = DJ'_0(\rho) & \quad \sqrt{\frac{\mu_0}{\epsilon_0}} H_r = BH'_0(\rho) \\ -E_\varphi = \frac{k_L}{h} DJ'_0(\rho) & \quad -E_\varphi = \frac{k}{h} BH'_0(\rho). \end{aligned} \quad (3)$$

From the requirement of continuity of H_z and E_φ we now obtain the boundary conditions

$$\rho_L DJ_0(\rho_L) = \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} \rho BH_0(\rho), \quad k_L DJ'_0(\rho_L) = k BH'_0(\rho). \quad (4)$$

Here we have put $\rho = \sqrt{k^2 - h^2} a$, $\rho_L = \sqrt{k_L^2 - h^2} a$, as in (22.14). The reader may prove to his own satisfaction that the second of these conditions assures at the same time the continuity of B_r , taking account of the relation $k^2 = \epsilon_0 \mu_0 \omega^2$, which we shall also use in the following.

Elimination of B and D in (4) leads to the transcendental equation

$$\rho \frac{H_0(\rho)}{H'_0(\rho)} = \sqrt{\frac{\epsilon' \mu_0}{\epsilon_0 \mu}} \frac{k}{k_L} \frac{\rho_L J_0(\rho_L)}{J'_0(\rho_L)}, \quad (5)$$

whose right side differs materially, even in order of magnitude, from that of (22.14a). We ask whether (5) has a solution of the type of the *principal wave* $h \cong k$, i.e. $\rho \ll 1$. Then the left side would, by (22.5), be of the order of $\rho^2 \log \rho$, i.e. in absolute value $\ll 1$, while the right side, by (22.7a) and since $\rho_L \cong k_L a$, would be approximately equal to $\{\epsilon' \mu_0 / (\epsilon_0 \mu)\}^{1/2} k a$, i.e. in ab-

solute value $\gg 1$ since $|\epsilon'| \gg \epsilon_0$. This applies even for soft iron where μ is much larger than μ_0 . The assumption $h \cong k$ thus leads to a contradiction. *There is no magnetic principal wave*; the magnetic waves all have the character of *secondary waves*:

$$|\rho| \gg 1, \quad J'_0(\rho_L) \cong 0, \quad h_r^2 \cong k_L^2 - \left(\frac{w_r}{a}\right)^2,$$

as in (1). The earlier comments on the *electric* secondary waves may be transferred without change to the *magnetic secondary waves*. Their field also shows a skin effect outside of the wire and is rapidly damped in the interior by Joule heat.

C. Asymmetric Waves of the Electromagnetic Type

We now consider processes without rotational symmetry about the axis of the wire and employ for the representation of E_z the more general solutions of the Bessel differential equation

$$J_n(\rho) \cos(n\varphi) \quad \text{or} \quad H_n(\rho) \cos(n\varphi) \quad (6)$$

in place of the functions $J_0(\rho)$ and $H_0(\rho)$. We readily convince ourselves then that the former three-component solutions E_z, E_r, H_φ and H_z, H_r, E_φ are no longer sufficient, but that all six components of \mathbf{E} and \mathbf{H} must occur in the solution. We must now combine with the formula (6) for E_z the formula

$$\left. \begin{matrix} J_n(\rho) \\ H_n(\rho) \end{matrix} \right\} \sin(n\varphi) \quad (6a)$$

for H_z , so as to give all terms in (20.5) the common factor $\cos(n\varphi)$, all terms in (20.6) the common factor $\sin(n\varphi)$. We thus obtain from (20.5, 6) for the interior of the wire, with $\rho = \sqrt{k_L^2 - h^2} r$

$$\begin{aligned} E_z &= \frac{\sqrt{k_L^2 - h^2}}{ih} C J_n(\rho) \cos(n\varphi) \\ E_r &= \left\{ C J'_n(\rho) + \frac{k_L n}{h \rho} D J_n(\rho) \right\} \cos(n\varphi) \\ -E_\varphi &= \left\{ \frac{n}{\rho} C J_n(\rho) + \frac{k_L}{h} D J'_n(\rho) \right\} \sin(n\varphi) \\ \sqrt{\frac{\mu}{\epsilon'}} H_z &= \frac{\sqrt{k_L^2 - h^2}}{ih} D J_n(\rho) \sin(n\varphi) \\ \sqrt{\frac{\mu}{\epsilon'}} H_r &= \left\{ \frac{n}{\rho} \frac{k_L}{h} C J_n(\rho) + D J'_n(\rho) \right\} \sin(n\varphi) \\ \sqrt{\frac{\mu}{\epsilon'}} H_\varphi &= \left\{ \frac{k_L}{h} C J'_n(\rho) + \frac{n}{\rho} D J_n(\rho) \right\} \cos(n\varphi). \end{aligned} \quad (7)$$

The phase factor $\exp(-i\omega t + i h x)$ is again to be thought as included. The constant coefficients in E_z and H_z have been so chosen that (7) passes over into (22.13) for $D = 0$ and $n = 0$ and (after the permissible interchange of \cos and \sin) into (3) for $C = 0$ and $n = 0$.

Proceeding likewise for the exterior of the wire we obtain, with $\rho = \sqrt{k^2 - h^2} r$

$$\begin{aligned}
 E_z &= \frac{\sqrt{k^2 - h^2}}{ih} A H_n(\rho) \cos(n\varphi) \\
 E_r &= \left\{ A H'_n(\rho) + \frac{kn}{h\rho} B H_n(\rho) \right\} \cos(n\varphi) \\
 -E_\varphi &= \left\{ \frac{n}{\rho} A H_n(\rho) + \frac{k}{h} B H'_n(\rho) \right\} \sin(n\varphi) \\
 \sqrt{\frac{\mu_0}{\epsilon_0}} H_z &= \frac{\sqrt{k^2 - h^2}}{ih} B H_n(\rho) \sin(n\varphi) \\
 \sqrt{\frac{\mu_0}{\epsilon_0}} H_r &= \left\{ \frac{n}{\rho} \frac{k}{h} A H_n(\rho) + B H'_n(\rho) \right\} \sin(n\varphi) \\
 \sqrt{\frac{\mu_0}{\epsilon_0}} H_\varphi &= \left\{ \frac{k}{h} A H'_n(\rho) + \frac{n}{\rho} B H_n(\rho) \right\} \cos(n\varphi).
 \end{aligned} \tag{8}$$

We now turn to the boundary conditions between interior and exterior at $r = a$. With $\rho = \sqrt{k^2 - h^2} a$ and $\rho_L = \sqrt{k_L^2 - h^2} a$ we obtain from the continuity of E_z and H_z

$$\rho_L C J_n(\rho_L) = \rho A H_n(\rho), \tag{9}$$

$$\rho_L D J_n(\rho_L) = q \rho B H_n(\rho), \quad q = \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}}. \tag{9a}$$

and from the continuity of E_φ and H_φ the two conditions

$$C \frac{n}{\rho_L} J_n(\rho_L) + D \frac{k_L}{h} J'_n(\rho_L) = A \frac{n}{\rho} H_n(\rho) + B \frac{k}{h} H'_n(\rho), \tag{10}$$

$$C \frac{k_L}{h} J'_n(\rho_L) + D \frac{n}{\rho_L} J_n(\rho_L) = A \frac{qk}{h} H'_n(\rho) + B \frac{qn}{\rho} H_n(\rho). \tag{10a}$$

The constants A , B , C , and D are to be eliminated from these four equations (9), (9a), (10), and (10a), most simply in the form of a four-row determinant. We divide their columns immediately by H_n and J_n and find:

$$\begin{vmatrix}
 \rho & 0 & \rho_L & 0 \\
 0 & q\rho & 0 & \rho_L \\
 \frac{n}{\rho} & \frac{k}{h} \frac{H'_n}{H_n} & \frac{n}{\rho_L} & \frac{k_L}{h} \frac{J'_n}{J_n} \\
 q \frac{k}{h} \frac{H'_n}{H_n} & \frac{qn}{\rho} & \frac{k_L}{h} \frac{J'_n}{J_n} & \frac{n}{\rho_L}
 \end{vmatrix} = 0 \tag{11}$$

This transcendental equation is to be regarded as the *equation determining the wave number h* , which occurs not only explicitly, but also implicitly in ρ , ρ_L , H'/H , and J'/J . The electric and magnetic components of the wave are *coupled* by it. An *uncoupling* occurs only in the symmetrical case $n = 0$, for which (11) may be separated:

$$\left\{ \rho k_L \frac{J'_0}{J_0} - \rho_L q k \frac{H'_0}{H_0} \right\} \left\{ \rho_L k \frac{H'_0}{H_0} - \rho q k_L \frac{J'_0}{J_0} \right\} = 0. \quad (11a)$$

When set individually equal to zero the two parentheses correspond exactly with the transcendental equations for the symmetric magnetic and the symmetric electrical case, i.e. with the earlier Eqs. (5) and (22.14a).

We utilize (11) below only to answer the question whether in the unsymmetrical case a state of the character of the *principal wave* is possible. We thus assume

$$h \cong k, \quad \rho \cong 0, \quad |\rho_L| \gg 1, \quad \frac{J'_n}{J_n} \rightarrow -i, \quad \frac{H'_n}{H_n} \rightarrow -\frac{n}{\rho};$$

the last two statements follow from Eqs. (22.6) and (22.7a). We then can neglect, in the first two rows of (11), not only $q\rho$, but also ρ in comparison with ρ_L . Then the determinant (11) breaks up into the product of the two subdeterminants

$$\begin{vmatrix} \rho_L & 0 \\ 0 & \rho_L \end{vmatrix} = \rho_L^2 \quad \text{and} \quad q \left(\frac{n}{\rho} \right)^2 \begin{vmatrix} 1 & -\frac{k}{h} \\ -\frac{k}{h} & 1 \end{vmatrix} = -\frac{qn^2}{a^2 h^2}. \quad (12)$$

The product of the two yields

$$(h^2 - k_L^2) \frac{qn^2}{h^2}$$

Set equal to zero this leads, for $n = 0$, to $h = \pm k_L$, which contradicts our requirement $h \cong k$. *There is hence no asymmetric principal wave.* We must refer to Hondros's thesis for the rather complicated solution of the transcendental equation for the secondary waves.

D. Wire Waves on a Nonconductor

The dissipation of the secondary waves on the metallic wire by Joule heat raises the question as to whether secondary waves on a *dielectric* wire might be observable. According to Hondros and Debye this question is to be answered in the affirmative.¹

¹ D. Hondros and P. Debye, *Ann. d. Phys.* 32, p. 465, 1910.

We shall consider a "water wire" (which may be imagined surrounded by an infinitely thin-walled glass cylinder). In view of the absence of absorption h is real, so that $\sqrt{k^2 - h^2}$ is either real or purely imaginary. The first possibility ($h < k$, propagation with a velocity exceeding that of light) is excluded by the prohibition of radiation, in accord with §22B. Hence $\sqrt{k^2 - h^2}$ and our former $\rho = \sqrt{k^2 - h^2} a$ become purely imaginary. On the other hand, $\rho_L = \sqrt{k_L^2 - h^2} a$ is real. For we have now, since $\sigma = 0$

$$k_L^2 = \epsilon \mu \omega^2 = \frac{\epsilon \mu}{\epsilon_0 \mu_0} \cdot \epsilon_0 \mu_0 \omega^2 = n^2 k^2,$$

where n is now to denote the *refractive index*, in accord with Maxwell's law in Eq. (6.7) (not, as up to now, the order of the Bessel functions!). For water we have in the high-frequency range (decimeter waves) $n \cong 9$.

We introduce the two *real* quantities

$$\xi = \sqrt{h^2 - k^2} a, \quad \eta = \sqrt{n^2 k^2 - h^2} a, \quad (13)$$

which will serve as rectangular coordinates for a graphical representation.

All our earlier formulas, in particular those for symmetrical waves, remain valid for our present case of real ξ , η insofar as they do not contain approximations. Eq. (22.14a) now takes the form

$$i\xi \frac{H_0(i\xi)}{H'_0(i\xi)} = \frac{\eta}{n^2} \frac{J_0(\eta)}{J'_0(\eta)}. \quad (14)$$

For $\xi \rightarrow 0$ and $\xi \rightarrow \infty$ its left side varies, according to (22.5) and (22.7) as

$$\frac{i\xi \log \frac{\gamma \xi}{2}}{1/i\xi} = \xi^2 \log \frac{2}{\gamma \xi} \quad \text{and as} \quad \frac{i\xi}{i} = \xi, \quad \text{respectively.}$$

It thus becomes equal to zero and infinity along with ξ . Hence the right side of (14) must also vanish for $\xi = 0$; this is not the case when $\eta = 0$ (since $J'_0(0) = 0$), but only when

$$J_0(\eta) = 0. \quad (15)$$

On the other hand, the right side of (14) becomes infinite for

$$J'_0(\eta) = -J_1(\eta) = 0. \quad (15a)$$

We shall represent the variation given by (14) graphically in the $\xi\eta$ -plane. On the ordinate axis we mark the roots of (15) and (15a), which alternate with each other. As on p. 186 we call the sequence of points

$$w_1, w_2, w_3, \dots \quad \text{and} \quad w'_1, w'_2, w'_3, \dots$$

and note in particular

$$w_1 = 2.40. \quad (15b)$$

We draw lines parallel to the axis of abscissas through the points $\eta = w'_i$, $\xi = 0$. The points $\eta = w_i$, $\xi = 0$ are initial points of curve branches of the desired representation, which must approach the horizontal straight lines $\eta = w'_i$ asymptotically for $\xi = \infty$.

There exists however, according to (13), also the relation

$$\xi^2 + \eta^2 = (n^2 - 1)k^2 a^2 = (n^2 - 1) \left(\frac{2\pi a}{\lambda} \right)^2 \quad (16)$$

between ξ and η . This means that the desired solutions of (14) must also lie on circles about the origin of the $\xi\eta$ -plane with radius

$$r_\lambda = 2\pi \sqrt{n^2 - 1} \frac{a}{\lambda}, \quad (16a)$$

where λ is the "wave-length in air" corresponding to our state of vibration. Hence we must let the curve branches intersect with the circles (16).

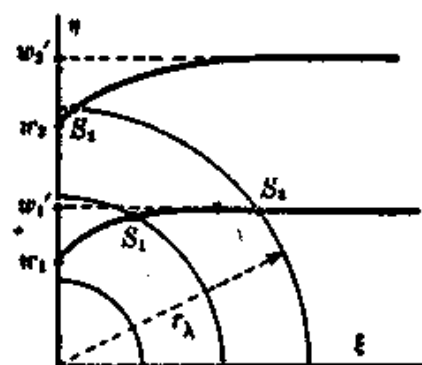


FIG. 34. Wire waves on a nonconductor. Plane of the real coordinates $\xi = \sqrt{h^2 - k^2} a$, $\eta = \sqrt{n^2 k^2 - h^2} a$. Construction of the roots of Eq. (14) with the aid of the roots w_1, w_2, \dots of Eq. (15), $J_0(\eta) = 0$, and the roots w'_1, w'_2, \dots of Eq. (15a), $J_1(\eta) = 0$.

Depending on the magnitude of λ there are zero, one, two, or even more intersections. Our figure shows directly:

For $r_\lambda < w_1$ there is no intersection.

For $w_1 < r_\lambda < w_2$ there is one intersection.

For $w_2 < r_\lambda < w_3$ there are two intersections etc.

According to (15b) the first root of $J_0(\eta) = 0$ is $w_1 = 2.40$. The *maximum* permissible value of λ which corresponds to this least value of r_λ for which a wire wave is just still possible is, according to (16a):

$$\lambda_{\max} = \frac{2\pi}{2.40} \sqrt{n^2 - 1} a$$

For our "water wire" with radius $a = 1$ cm this is

$$\lambda_{\max} = \frac{2\pi\sqrt{80}}{2.40} = 23.4 \text{ cm.}$$

Longer wave-lengths than this cannot be propagated along it. Thus we find

ourselves in the range of the "decimeter waves" which is of such great interest at present.

As λ is reduced there is one possibility of propagation, represented by the first intersection S_1 in the figure; as λ is reduced still further, corresponding to $r_\lambda > 5.52$ being increased, there are two possibilities, given by the two intersections S_2, S_3 in the figure. The first, at small ξ , yields $h \cong k$ (velocity along the wire nearly equal to c), the second at larger ξ (h materially larger than k , velocity of propagation appreciably less than c) corresponds to an η which is nearly equal to the first root of $J_1(\eta) = 0$, which according to Eq. (23.1a) is $w'_1 = 3.83$. In the first of these two cases the electric lines of force are nearly *perpendicular* to the surface of the wire and the decrease of the field outward is slow (the asymptotic decrease e^{-kr} is attained only for large r). In the second case, where ξ is quite large, we have a *skin effect* outwards, as for our *auxiliary waves* for the metallic wire. The first case has the character of the *principal wave* on the outside, with the difference that the interior of the dielectric is filled by current (η real and of moderate magnitude). Similarly, for still smaller λ , the large number of vibration states then possible are arranged between the limiting cases of principal and auxiliary waves. At the same time the phase velocity of the wave varies between the velocity c in vacuum and that "in water."

The above results predicted by Hondros and Debye were verified most successfully by G. Southworth in the Bell Laboratories. Also in Germany such dielectric wire waves have been profitably applied in communications.

§24. On the Theory of Wave Guides

In the preceding section we have seen that electromagnetic fields may be held together and guided by the surface of a non-conducting rod and that they protect themselves against outward radiation by a skin effect. This protection will be complete if we embed the non-conductor in a metallic tube, whereupon the condition of a sufficiently high dielectric constant may be omitted and the dielectric within the tube may also be air. We thus arrive at the configuration of the *wave guides*, which have become important in high frequency practice.

We consider in particular the cylindrical wave guide, since its treatment may be deduced directly from the preceding formulas. Let a be the radius of the metallic envelope, which for the present will be assumed to be a *perfect* conductor, and h the wave number of the propagation. It is real since the wave is damped neither by Joule heat nor by radiation. There are electric and magnetic waves of symmetric type and also of asymmetric type.

We write for the *symmetric electric waves*, as in (22.13), omitting the amplitude coefficient C and the phase factor $\exp(-i\omega t + ihx)$:

$$E_z = \frac{\sqrt{k^2 - h^2}}{ih} J_0(\rho), \quad E_r = J'_0(\rho), \quad \sqrt{\frac{\mu_0}{\epsilon_0}} H_\phi = \frac{k}{h} J'_0(\rho). \quad (1)$$

For known E_z these formulas also follow directly from the relationship of transversal and longitudinal components in (20.5). The boundary conditions reduce to the single equation $E_z = 0$ for $r = a$, since for H_z the required condition of continuity is satisfied by a surface current induced in the envelope. Thus, with $\rho = \sqrt{k^2 - h^2} a$ we have

$$J_0(\rho) = 0, \quad \rho = w_1, w_2, \dots, w, \dots \text{ with } w_1 = 2.40. \quad (2)$$

From the definition of ρ it follows that

$$h_r^2 = k^2 - \left(\frac{w_r}{a}\right)^2, \quad h_r < k, \quad \frac{\omega}{h_r} > \frac{\omega}{k} = c. \quad (2a)$$

The phase velocity ω/h_r along the tube thus exceeds the velocity of light. As for our dielectric wire there is a lower limit for the wave number k , i.e. an upper limit for the corresponding "primary" wave-length $\lambda = 2\pi/k$. It corresponds to $h = h_1 = 0$ (phase velocity infinite) and yields by (2a)

$$k_{\min} = \frac{w_1}{a}, \quad \lambda_{\max} = \frac{2\pi}{k_{\min}} = \frac{2\pi}{w_1} a = \frac{2\pi}{2.40} a. \quad (3)$$

Since a is of the order of magnitude of centimeters all the following considerations apply to the *centimeter-wave* region. The number of possible states (or "modes") of type (1) depends on the frequency ω or, what is the same, on the primary wave number $k = \omega/c$. According to (2a) this number is equal to the number of roots w , which are less than ka .

The *magnetic symmetrical waves* are represented, according to (23.3),

$$\sqrt{\frac{\mu_0}{\epsilon_0}} H_z = \frac{\sqrt{k^2 - h^2}}{ih} J_0(\rho), \quad \sqrt{\frac{\mu_0}{\epsilon_0}} H_r = J'_0(\rho), \quad -E_\varphi = \frac{k}{h} J'_0(\rho), \quad (4)$$

corresponding to the general scheme of Eq. (20.6). The single boundary condition which must here be fulfilled is $E_\varphi = 0$ for $\rho = \sqrt{k^2 - h^2} a$. It demands

$$J'_0(\rho) = 0, \quad \rho = w'_1, w'_2, \dots, w', \dots \text{ with } w'_1 = 3.83 \quad (5)$$

and yields, as in (2a), values of h_r which are $< k$. The upper limit for the primary wave-length lies somewhat lower than for the electric type. It is

$$\lambda_{\max} = \frac{2\pi}{w'_1} a = \frac{2\pi}{3.83} a. \quad (5a)$$

To pass over to the asymmetric types we start from Eqs. (23.7). In view of the reduced number of boundary conditions we may now however set

¹ By the "primary" wave-length we understand that of the *exciting* oscillation, which of course has the same frequency ω as the wave guide oscillation excited by it. This primary wave-length is actually simply a measure of the frequency ω which is familiar to the engineer and convenient in dimension. The wave-length in the wave guide can be determined uniquely only in the axial direction and is $\lambda = 2\pi/h$, whereas the primary wave-length is $2\pi/k = 2\pi c/\omega$. For $\lambda_{\text{prim}} = \lambda_{\max}$, $\lambda_{\max} = \infty$ since $h = 0$.

one of the two amplitudes C and D equal to zero, the other equal to 1. This simplifies the formulas considerably and leads to an *asymmetric electric* ($D = 0$) and an *asymmetric magnetic* case ($C = 0$).

For the asymmetric electric type we obtain:

$$\begin{aligned} E_z &= \frac{\sqrt{k^2 - h^2}}{ih} J_n(\rho) \cos(n\varphi), & H_z &= 0 \\ E_r &= J'_n(\rho) \cos(n\varphi), & \sqrt{\frac{\mu_0}{\epsilon_0}} H_r &= \frac{k}{h} \frac{n}{\rho} J_n(\rho) \sin(n\varphi) \\ E_\varphi &= -\frac{n}{\rho} J_n(\rho) \sin(n\varphi), & \sqrt{\frac{\mu_0}{\epsilon_0}} H_\varphi &= \frac{k}{h} J'_n(\rho) \cos(n\varphi) \end{aligned} \quad (6)$$

and for the asymmetric magnetic case, if, for convenience, $n\varphi$ is exchanged for $n\varphi + \pi/2$,

$$\begin{aligned} E_z &= 0, & \sqrt{\frac{\mu_0}{\epsilon_0}} H_z &= \frac{\sqrt{k^2 - h^2}}{ih} J_n(\rho) \cos(n\varphi) \\ E_r &= -\frac{k}{h} \frac{n}{\rho} J_n(\rho) \sin(n\varphi), & \sqrt{\frac{\mu_0}{\epsilon_0}} H_r &= J'_n(\rho) \cos(n\varphi) \\ E_\varphi &= -\frac{k}{h} J'_n(\rho) \cos(n\varphi), & \sqrt{\frac{\mu_0}{\epsilon_0}} H_\varphi &= -\frac{n}{\rho} J_n(\rho) \sin(n\varphi) \end{aligned} \quad (7)$$

For $n = 0$ (6) and (7) pass over into (1) and (4).

The boundary conditions for $r = a$, $\rho = \sqrt{k^2 - h^2} a$ require

for (6): $E_z = E_\varphi = 0$, i.e. $J_n(\rho) = 0$,

for (7) $E_\varphi = 0$, i.e. $J'_n(\rho) = 0$.

As in (2) and (5) we call the roots of these two equations again w_ν and w'_ν and distinguish them when necessary from the former by the addition of the argument n , writing thus $w_\nu(n)$ in place of $w_\nu(0)$, $w'_\nu(n)$ in place of $w'_\nu(0)$.

The following table indicates the relative position of the smallest roots in the doubly-indexed twofold system w, w' :

$J'_1 = 0$	$J_0 = 0$	$J'_0 = J_1 = 0$
$w'_1(1) = 1.84$	$w_1(0) = 2.40$	$w'_1(0) = 3.83 = w_1(1)$,
$w'_2(1) = 5.33$	$w_2(0) = 5.52$	$w'_2(0) = 7.02 = w_2(1)$.

It shows, contrary to expectation, that the *magnetic asymmetric wave* with $n = 1$, and not the *electric symmetric wave* with $n = 0$, possesses the smallest root. Also for the second root $\nu = 2$, in the second row of the table, the sequence of these two waves is the same as for the root $\nu = 1$. The third column shows finally that the electric symmetric wave is succeeded by the

magnetic symmetric wave with $n = 0$, which yields the same root as the electric asymmetric wave for $n = 1$ since $J'_0 = -J_1$.

In view of this the wave-length λ_{\max} given by (3) is not the absolute upper limit for all waves that can be propagated in the guide, but rather the wave-length

$$\lambda_{\max} = \frac{2\pi}{1.84} a. \quad (8)$$

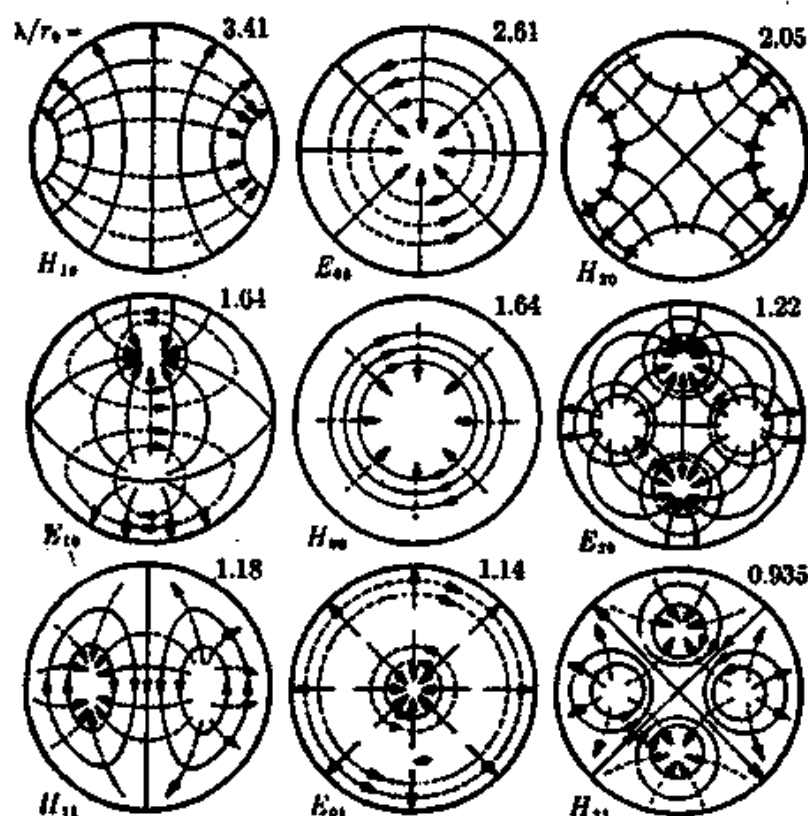


FIG. 35. Transversal fields of cylindrical-guide waves, ordered according to the limiting wave-length λ_{\max}/a . Full lines, electric; broken lines, magnetic lines of force, both represented in the cross section of their antinodes. Open beginnings or ends of the lines of force indicate their being bent out of or into the direction of the axis. Customary engineering notation: TM magnetic, TE electric type; first subscript, azimuthal number of nodal lines; second subscript, radial number of nodal lines within the guide.

Thus, if the frequency is increased continuously (continuous reduction of the primary wave-length) and if the exciting elements are suitably disposed in space, the electric symmetric wave is not the first to appear, but instead the magnetic asymmetric wave with $n = 1$. The electric symmetric wave follows and, after it, the magnetic symmetric wave simultaneously with the first asymmetric electric wave $n = 1$, as illustrated in the series of pictures in Fig. 35.

So far we have discussed only exactly circularly cylindrical tubes. Every deviation from circular symmetry occurring along the tube is equivalent

to a disturbance in the symmetry of the excitation and hence occasions the appearance of new secondary waves of different symmetry, which leads to difficulties in the practical application of the desired modes of oscillation.

Also a tube of elliptical crosssection may be treated directly by the general method of §20A, since the wave equation is separable in the elliptical coordinates u, v (see Vol. II, Problem IV 3). It is only necessary to put E_z, H_z equal to a *Mathieu function* (function of the elliptical cylinder) $F(iu)$ or equal to the product of two such functions $F(iu) \cdot F(iv)$; the transversal components $E_x, H_x; H_y, E_y$ may then be written down immediately with the aid of Eqs. (20.5) and (20.6).

We shall still glance briefly at tubes with *rectangular cross section*. Since there can be no question of a symmetrical wave in view of the shape of the rectangle (sides b and c in the y - and z -directions) we give right away the general formulas (n and m arbitrary integers) corresponding to Eqs. (6) and (7); the amplitudes of E_z in (9) and H_z in (10) have, just as in (6) and (7), been chosen in a manner convenient for what follows. As before, the phase factor $\exp i(hx - \omega t)$ must be imagined as included.

$$E_z = \pi^2 \left(\frac{n^2}{b^2} + \frac{m^2}{c^2} \right) \sin \left(n\pi \frac{y}{b} \right) \sin \left(m\pi \frac{z}{c} \right), \quad H_z = 0,$$

$$E_y = ih \frac{n\pi}{b} \cos \left(n\pi \frac{y}{b} \right) \sin \left(m\pi \frac{z}{c} \right), \quad \sqrt{\frac{\mu_0}{\epsilon_0}} H_x = -\frac{k}{h} E_z, \quad (9)$$

$$E_z = ih \frac{m\pi}{c} \sin \left(n\pi \frac{y}{b} \right) \cos \left(m\pi \frac{z}{c} \right), \quad \sqrt{\frac{\mu_0}{\epsilon_0}} H_x = +\frac{k}{h} E_y.$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} H_z = \pi^2 \left\{ \frac{n^2}{b^2} + \frac{m^2}{c^2} \right\} \cos \left(n\pi \frac{y}{b} \right) \cos \left(m\pi \frac{z}{c} \right), \quad E_z = 0,$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} H_y = -ih \frac{n\pi}{b} \sin \left(n\pi \frac{y}{b} \right) \cos \left(m\pi \frac{z}{c} \right), \quad E_y = \frac{k}{h} \sqrt{\frac{\mu_0}{\epsilon_0}} H_z, \quad (10)$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} H_x = -ih \frac{m\pi}{c} \cos \left(n\pi \frac{y}{b} \right) \sin \left(m\pi \frac{z}{c} \right), \quad E_z = -\frac{k}{h} \sqrt{\frac{\mu_0}{\epsilon_0}} H_y.$$

The wave number h is determined in both cases by the differential equation $\Delta X + k^2 X = 0$, which must be satisfied for every one of the Cartesian components of \mathbf{E} and \mathbf{H} . Substitution of either (9) or (10) readily leads to

$$h^2 + \pi^2 \left(\frac{n^2}{b^2} + \frac{m^2}{c^2} \right) = k^2 = \left(\frac{2\pi}{\lambda} \right)^2, \quad \lambda = \text{primary wave-length.}$$

The maximum value of λ for given n and m , below which the tube is capable of oscillations, occurs for $h = 0$ and is

$$\lambda_{\max} = \frac{2}{\sqrt{\frac{n^2}{b^2} + \frac{m^2}{c^2}}}.$$

If, as we may assume, $b > c$ the absolute maximum is attained for $n = 1$, $m = 0$ and is

$$\lambda_{\max} = 2b.$$

As Leon Brillouin has noted, these and similar oscillations in guides can be constructed elegantly and instructively by the superposition of ordinary plane space waves which interfere at the tube walls.

The same idea leads also directly from the progressive waves derived above to characteristic standing waves in e.g. a rectangular parallelepiped or a circular cylinder of finite length. Essentially, the wave number h must simply be replaced by an integer multiple of π/a where a is the length of the third side of the parallelepiped or the length of the cylinder, respectively. We will discuss this in greater detail in Problems II 7 and II 8 and treat, in Problem II 9, the radially symmetric characteristic vibrations of the sphere as well. The characteristic vibrations of the rectangular parallelepiped, in particular, find a useful application in microwave practice for the determination of the frequency of the primary excitation by the resonance principle.

More difficult questions arise in the practical application of wave guides, where, instead of perfectly conducting walls, the finite conductivity of real metals and the heat loss in them must be considered; the latter has, up to the present, prevented the propagation of waves in guides over great distances. Also the shaping of the ends of the guides into conical or horn-shaped openings raises questions upon which we shall not enter here.¹

§25. The Lecher Two-Wire Line

Mathematically this is the generalization for *high-frequency alternating currents* of the *quasistationary* two-wire line treated in §18. Its advantage, compared with the *single wire* traversed by alternating current, rests in the fact that the field outside of the wires decreases more rapidly than for the single wire, so that the disturbances by the surroundings discussed on p. 178 are avoided.

The *phase* of the alternating current in the two wires advances in the *same direction*, say in the positive x -direction; the direction of the current itself, on the other hand, as in §15E, is *opposite* in the two wires. We might say: For the same x positive charge flows in one wire through a given cross section, negative charge in the other. Also the charge accumulated on the surface has at any moment, for equal x , the opposite sign in the two wires.

¹ We refer to the comprehensive textbook of S. A. Schelkunoff, "Electromagnetic Waves," Van Nostrand, New York, 1943, which was published as a Bell Monograph and is widely employed in the United States, as well as to the lectures of L. de Broglie, "Problèmes de propagation guidée des ondes électromagnétiques." Paris. Gauthier-Villars, 1941.

We call this a *push-pull excitation*. However, the mode in which charge of equal sign flows in both wires (and is accumulated at their surface) may also be realized. We then speak of *parallel excitation*. The conditions of excitation determine which of the two states occurs. Any asymmetry of excitation results in the appearance of both wave types. However, we only call the push-pull arrangement a "Lecher system." For parallel excitation the situation is quite similar to that for the single wire (§22) and is fraught with the same experimental drawbacks.

G. Mie¹ succeeded in giving a complete theoretical treatment of the Lecher problem as early as 1900. The following representation,² which is both simplified and rounded out to some extent, deviates from that of Mie more in form than in substance. Like Mie, we introduce a system of *bipolar coordinates*, to which the circumferences of the two cross sections, assumed circular, belong. These coordinates would be the ideal mathematical medium if the wave equation were separable in them. Unfortunately this is not the case (see Vol. II, Problem IV.1). We hence must employ methods of approximation which rest on the replacement of the wave equation by the potential equation in the yz -plane. However, this approximation is valid only for sufficiently good conductivity of the material of the wire and in the exterior of the wires. Inside we must calculate with ordinary cylindrical polar coordinates. The comparison of the two formulas at the surface of the wires leads to a clear-cut equation for the determination of h , the wave number, which in the push-pull case becomes even simpler than for the single wire, being algebraic in place of transcendental. In the parallel case it is practically identical with that for the single wire.

G. Gentile Jr. has proposed a procedure which differs from ours and from Mie's.³ In accord with the general methods of perturbation theory he superposes on the symmetric wave propagated along the first wire the totality of asymmetric Hondros waves from §23, each multiplied by a disposable coefficient. He seeks to fit these coefficients to the boundary conditions on the first and second wires, in which process he has to utilize the generalized addition theorems of the Bessel and Hankel functions. This leads him to an infinite system of simultaneous linear equations for the coefficients. However he and his collaborator T. Magri failed to obtain an approximate solution of it. On the other hand, our procedure leads to a direct and explicit determination of the infinite number of coefficients which must be introduced.

¹ Ann. d. Phys. 2, 201, 1900.

² It rests on a detailed study of the problem by Mr. J. Jaumann; he also has made available to me the elegant treatment of the limiting case $\sigma \rightarrow \infty$ given in the succeeding section A, which we owe his late father, the wellknown physicist G. Jaumann of Brünn.

³ Nuovo Cimento, Vol. I, pp. 161 and 190, 1943.

A. The Limiting Case of Infinite Conductivity

For $\sigma \rightarrow \infty$ the waves propagate themselves with the velocity of light c , so that $h = k$, as has already been pointed out at the end of §20A and follows directly for the single wire from Eq. (22.15). Then the three-dimensional wave equation for each Cartesian field component becomes the two-dimensional potential equation, in accord with Eq. (20.3). This may be solved by the method of conformal mapping for arbitrary cross-section peripheries (which need not be circular, nor even the same for the two wires). The method can also be applied when the excitation is not purely periodic and monochromatic, i.e., when our phase factor $\exp[i(hx - \omega t)]$ is replaced by an arbitrary function $f(x - ct)$.

It is true that the longitudinal components E_z, H_z , which were considered first in Eq. (20.3), vanish in the limit $h \rightarrow k$, since for infinite conductivity the electric lines of force are perpendicular to the surface of the two conductors, and the magnetic lines of force also lie in the planes $z = \text{const.}$ Hence we have for the longitudinal components, in a first approximation

$$E_z = 0, \quad H_z = 0. \quad (2)$$

On the other hand, the Cartesian transversal components E_x, E_y, H_x, H_y may be determined almost directly from the fact that as solutions of the two-dimensional potential equation they form an electrostatic and a corresponding magnetostatic field. They are most simply combined in the vector formula¹

$$\mathbf{E} + i\sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{H} = \text{grad } w, \quad w = u + iv. \quad (2a)$$

$w = w(\zeta)$ is a function of the complex variable $\zeta = y + iz$, which may be constructed by conformal mapping; the transversal \mathbf{E} - and \mathbf{H} -components are obtained as gradients of the real and imaginary parts, u and v , of this complex function.

The conformal mapping for our two identical circular cross sections is known to us from §19 of Vol. II. Fig. 26 given there is reproduced in the following figure with the notation to be employed here. Both systems of lines of force are circles. The electrical lines of force $v = \text{const.}$ proceed from the fixed points Q_1, Q_2 of the family of circles; the magnetic lines $u = \text{const.}$ have their centers on the real axis of the ζ -plane, on which Q_1 and Q_2 also lie. Let the center of the system M be the origin $\zeta = 0$. Our function w is given by Eq. (19.10) of Vol. II which in the present notation (u, v, ζ, ζ_0 in place of ρ, φ, z, c) and with a convenient choice of the constant A takes the form

$$w = \log \frac{\zeta - \zeta_0}{\zeta + \zeta_0}. \quad (3)$$

¹ As before, the factor $(\mu_0/\epsilon_0)^{1/2}$ must be applied to \mathbf{H} for dimensional reasons.

$\pm \xi_0$ are the (real) values of ξ corresponding to Q_2 and Q_1 , respectively. u and v , as real and imaginary parts of w , have the same meaning as the parameters ρ and φ of the *bipolar coordinate system* defined in Eq.(19.10b) of Vol. II. Of the magnetic lines of force, those have been drawn heavy in Fig. 36 which are supposed to correspond to the cross sections of the wires (radius a). Their centers O_1, O_2 do not coincide with the points Q_1, Q_2 . We call the latter, as sources of the electric lines of force, *source points* (three-dimensionally they are *source lines* parallel to the axes of the wires).

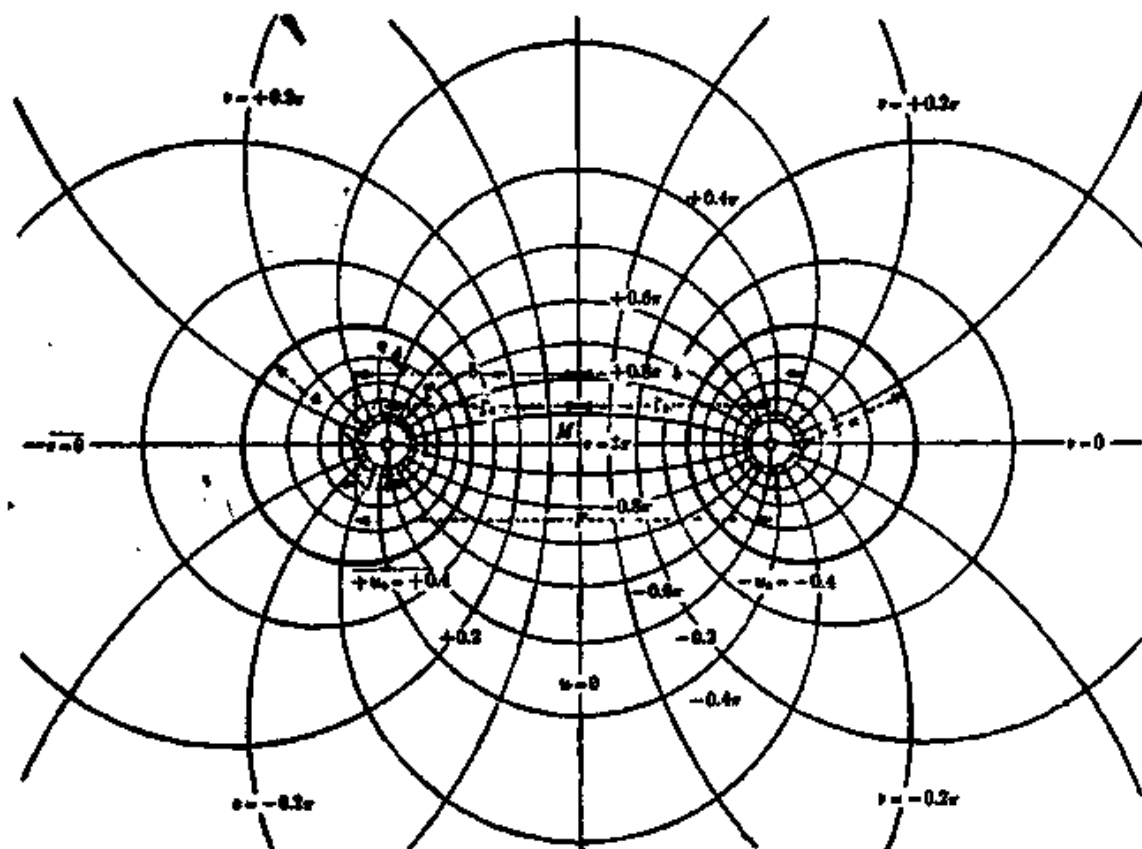


FIG. 36. The families of circles of the bipolar coordinates $u = \text{const}$, $v = \text{const}$ with the fixed points (source points) Q_1, Q_2 . The peripheries of the two wire cross sections $u = \pm u_0$ are indicated by heavier lines; O_1, O_2 are their centers, a their radius. The center M of the figure is the origin of the complex variable $\xi = x + iy$.

In the sense of Eq. (9.8) Q_1 and Q_2 are their *mutual electrical images* with reference to the *two* circular cross sections, i.e. are transformed into each other by the "transformation of reciprocal radii." With the designations

$$O_1Q_1 = O_2Q_2 = f, \quad O_1Q_2 = O_2Q_1 = F, \quad O_1M = O_2M = b$$

we have hence

$$fF = a^2, \quad f + F = 2b, \quad F - f = 2\xi_0. \quad (3a)$$

Solution of a simple quadratic equation then leads to

$$F = b + \sqrt{b^2 - a^2}, \quad f = b - \sqrt{b^2 - a^2}, \quad \xi_0 = \sqrt{b^2 - a^2}. \quad (3b)$$

B. The Exterior of the Wires

According to Problem IV.1 of Vol. II the line element of the bipolar coordinates may be written

$$ds^2 = g^2(du^2 + dv^2), \quad \frac{1}{g} = \frac{\cosh u - \cos v}{(F - f)/2}. \quad (4)$$

As compared with the general orthogonal line element in Eq. (20.1) we thus have here the special (isometric) case $g_u = g_v = g$.

The electric lines of force, represented in Fig. 36 by the family of circles $v = \text{const}$, have the direction of increasing u , the magnetic lines of force, represented by the orthogonal family of circles $u = \text{const}$, have the direction of increasing v . For the particular line elements ds_u , ds_v of the two systems of lines of force we obtain from (4)

$$\frac{ds_u}{du} = \frac{ds_v}{dv} = g. \quad (4a)$$

We write Eq. (2a) separately for the u - and v -directions:

$$E_u + i\sqrt{\frac{\mu_0}{\epsilon_0}} H_v = \frac{\partial w}{\partial u} \frac{du}{ds_u} = \frac{1}{g},$$

$$E_v + i\sqrt{\frac{\mu_0}{\epsilon_0}} H_u = \frac{\partial w}{\partial v} \frac{dv}{ds_v} = \frac{i}{g}.$$

Separation of real and imaginary parts leads to

$$E_u = \sqrt{\frac{\mu_0}{\epsilon_0}} H_v = \frac{1}{g} = \frac{\cosh u - \cos v}{(F - f)/2}, \quad E_v = \sqrt{\frac{\mu_0}{\epsilon_0}} H_u = 0. \quad (5)$$

The last is obvious since the electric lines of force have the u -direction, the magnetic, the v -direction. Furthermore, in spite of the vanishing of the longitudinal components E_z and H_z noted in Eq. (2), we wish to obtain a somewhat closer approximation for them as solutions of the two-dimensional potential equation. This is, in terms of u and v ,

$$\Delta_{uv} = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = 0.$$

It is integrated by particular solutions of the form

$$\left. \begin{matrix} \sinh(nu) \\ \cosh(nu) \end{matrix} \right\} \cos(nv), \quad \left. \begin{matrix} \cosh(nu) \\ \sinh(nu) \end{matrix} \right\} \sin(nv). \quad (5a)$$

Here n is a positive integer. For $n = 0$ these functions are replaced by the linear function

$$au. \quad (5b)$$

The addition of a term bv is not permissible since E_x and H_x must be unique functions of space, whereas the coordinate v changes by $\pm 2\pi$ after revolving once about one of the wires, i.e. is multivalent. Furthermore, the addition of a constant c to (5b) is excluded since E_x and H_x must vanish at infinity ($u = v = 0$). In view of the symmetry of our problem we write E_x as an odd function of u and as an even function of v , H_x , vice versa, as an even function of u and an odd function of v :

$$\begin{aligned} E_x &= E_0 u + E_1 \sinh u \cos v + E_2 \sinh (2u) \cos (2v) + \dots \\ \sqrt{\frac{\mu_0}{\epsilon_0}} H_x &= H_1 \cosh u \sin v + H_2 \cosh (2u) \sin (2v) + \dots \end{aligned} \quad (6)$$

In justification we consider two symmetrically placed points u, v and $-u, v$ to the right and to the left in the figure. In the push-pull case, which alone interests us to begin with, the currents flow in opposite directions in the two wires; the same applies to the x -components of the displacement currents outside of the wires. Hence E_x is, in our two points, equal and opposite. On the other hand H_x has the same sign in the two points in view of their position to the left and to the right of the two wires. Consider now two points u, v and $u, -v$ in the figure, above and below the straight line $v = \frac{\pi}{2}$. In them E_x has the same sign and H_x opposite signs. The formulation (6) is hence justified.

We already know of the coefficients $E_0, E_1, \dots, H_1, \dots$, from (2), that they vanish in the first order for $h \rightarrow k$. To obtain more detailed information we best turn back to the general relations (20.5) and (20.6) which before served for the calculation of the transversal components from the longitudinal ones, and which we shall now employ to determine the longitudinal components to the first order from the transversal components known to the zero order of approximation. Since $g_u = g_v = g$, substitution of (5) and (6) in (20.5) and (20.6) leads to

$$\begin{aligned} i(k^2 - h^2) &= -hE_0 - (hE_1 + kH_1) \cosh u \cos v \dots && \text{from (20.5)} \\ i(k^2 - h^2) &= -kE_0 - (kE_1 + hH_1) \cosh u \cos v \dots \\ 0 &= (hE_1 + kH_1) \sinh u \sin v \dots && \text{from (20.6)} \\ 0 &= - (kE_1 + hH_1) \sinh u \sin v \dots \end{aligned}$$

We conclude therefore:

$$E_0 \cong \frac{i(h^2 - k^2)}{h} \cong \frac{i(h^2 - k^2)}{k} \cong 2i(h - k), \quad (7)$$

$$H_1 \cong -\frac{h}{k} E_1 \cong -\frac{k}{h} E_1 \cong -E_1. \quad (8)$$

The sign \cong signifies here "equal but for higher terms in $h - k$ ". It is readily seen that the same applies for H_2, H_3, \dots as for H_1 . We have

thus determined E_0 and $H_1, H_2, H_3, \dots, H_n$. The E_1, E_2, \dots, E_n remain indefinite from this point on and are disposable for what follows.

To conclude the consideration of the exterior of the wires we write down the expression for H_z on the periphery of the first wire as we approach the latter from the outside: Since this periphery is a magnetic line of force, we have here $u = \text{const}$, say $= +u_0$. We utilize the abbreviation

$$p = e^{-u_0} \quad (9)$$

and obtain from (5)

$$\sqrt{\frac{\mu_0}{\epsilon_0}} H_z = \frac{1}{F-f} \left(p + \frac{1}{p} - 2 \cos v \right) \quad \text{for} \quad u = +u_0. \quad (9a)$$

C. The Interior of the Wires

Since, as in the preceding sections, we have within the wire

$$k_L = \sqrt{\epsilon \mu \omega^2 + i \mu \sigma \omega} = \sqrt{\epsilon' \mu} \omega$$

in place of k and since $|k_L| \gg h$, we do not attain our goal with solutions of the potential equation in bipolar coordinates, but must employ actual solutions of the wave equation in ordinary cylindrical coordinates. Hence we introduce at the center e.g. of the first wire a new complex variable $\eta = re^{i\varphi}$ with the origin O_1 and we must deal with the mutual transformation of our two systems, polar and bipolar coordinates, particularly at the periphery of the wires. This is furnished by Eq. (3) if there we express ζ in terms of the new variable η . Referring to Fig. 36 and Eqs. (3a, b) we set

$$\zeta + b = \eta, \quad \zeta \mp \zeta_0 = \eta - b \mp \sqrt{b^2 - a^2} = \begin{cases} \eta - F \\ \eta - f \end{cases}$$

and obtain from (3)

$$e^v = e^{u+iv} = \frac{\zeta - \zeta_0}{\zeta + \zeta_0} = \frac{\eta - F}{\eta - f}; \quad (10)$$

inversion leads to

$$\eta = re^{i\varphi} = \frac{fe^{u+iv} - F}{e^{u+iv} - 1}. \quad (10a)$$

The coordinates r, φ are thus expressed in terms of the coordinates u, v and vice versa.

Thus by forming the absolute value of (10) and squaring it we find for the periphery of the first wire, where we should have $u = +u_0$ and $r = a$,

$$e^{-2u_0} = p^2 = \frac{ae^{i\varphi} - f}{ae^{i\varphi} - F} \cdot \frac{ae^{-i\varphi} - f}{ae^{-i\varphi} - F} = \frac{a^2 + f^2 - 2af \cos \varphi}{a^2 + F^2 - 2aF \cos \varphi}.$$

Since p is independent of φ this equation is satisfied only if, after multiplying through with the denominator, the factors of $\cos \varphi$ on the two sides are equal, i.e.

$$-2ap^2F = -2af, \quad p = \sqrt{\frac{f}{F}}.$$

or, in greater detail, in view of the reciprocity relation in Eq. (3a),

$$p = \sqrt{\frac{f}{F}} = \frac{a}{F} = \frac{f}{a}. \quad (11)$$

From the same Eq. (10) we find for $r = a$ since $p < 1$:

$$\begin{aligned} e^{iv} &= p \frac{e^{iv} - F/a}{e^{iv} - f/a} = \frac{pe^{iv} - 1}{e^{iv} - p} = (p - e^{-iv})(1 + pe^{-iv} + p^2e^{-2iv} + \dots) \\ &= p + (p^2 - 1)e^{-iv} + (p^3 - p)e^{-2iv} + \dots \end{aligned} \quad (11a)$$

and hence

$$\begin{aligned} \cos v &= p + (p^2 - 1) \cos \varphi + p(p^2 - 1) \cos (2\varphi) \\ &\quad + p^2(p^2 - 1) \cos (3\varphi) + \dots \end{aligned} \quad (12)$$

On the other hand we find from (10a) for $r = a$, making use of (11):

$$e^{iv} = \frac{p - e^{-iv}}{1 - pe^{-iv}}, \quad e^{in\varphi} = (p - e^{-iv})^n (1 - pe^{-iv})^{-n}. \quad (13)$$

The real part of this equation is, for $n = 1, 2, 3$,

$$\begin{aligned} \cos \varphi &= p - (1 - p^2) \cos v - (p - p^3) \cos (2v) \\ &\quad - (p^2 - p^4) \cos (3v) + \dots \\ \cos (2\varphi) &= p^2 - 2(p - p^3) \cos v + (1 - 4p^2 + 3p^4) \cos (2v) + \dots \\ \cos (3\varphi) &= p^3 - 3(p^2 - p^4) \cos v + (3p - 9p^3 + 6p^5) \cos (2v) + \dots \end{aligned} \quad (14)$$

As the general expression for the field inside of the first wire we use the superposition of the system of partial waves in (23.7), where however we need write down only the expressions for E_z and H_φ :

$$\begin{aligned} E_z &= \frac{\sqrt{k_L^2 - h^2}}{ih} \sum_{n=0}^{\infty} C_n J_n(\rho) \cos(n\varphi), \\ \sqrt{\frac{\mu}{\epsilon}} H_\varphi &= \sum_{n=0}^{\infty} \left\{ \frac{k_L}{h} C_n J'_n(\rho) + \frac{n}{\rho} D_n J_n(\rho) \right\} \cos(n\varphi). \end{aligned} \quad (15)$$

By superposing here all possible asymmetric Hondros waves we implicitly and in the most general fashion take account of the unilateral effect of the "second wire" on the interior of the "first".

D. The Boundary Condition $H_z = H_z$

Previously the amplitudes C , D could be chosen arbitrarily for the individual partial wave. Here we do not consider the individual partial wave, but the superposition of all of them, and must fix the amplitudes C_n , D_n of each partial wave by the requirement of continuity in passing over to the exterior field.

To this end we compare the expression (9a) for H_z , after having replaced in it $\cos v$ with the series (12), with the representation of H_z in (15). Taking account also of (11) we obtain as factor of $\cos(n\varphi)$ in (9a)

$$\sqrt{\frac{\epsilon_0}{\mu_0}} \frac{2p^{n-1}}{F-f} (1-p^2) = \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{2p^n}{a}. \quad (16)$$

We write for the corresponding factor in (15):

$$\sqrt{\frac{\epsilon'}{\mu}} C_n \frac{k_L}{h} J'_n(\rho) \left\{ 1 + \frac{h}{k_L} \frac{n}{\sqrt{k_L^2 - h^2 a}} \frac{D_n}{C_n} \frac{J_n}{J'_n} \right\}. \quad (16a)$$

Here J_n/J'_n has the order of magnitude unity, as was noted at (22.7a). Of the multiplying factors the first two are very small quantities; the same applies for D_n/C_n , as we shall confirm later on. A comparison of (16) and (16a) thus yields directly

$$C_n = - \sqrt{\frac{\epsilon_0}{\epsilon'} \frac{\mu}{\mu_0}} \frac{h}{k_L} \frac{2p^n}{a J'_n(\rho)}. \quad (17)$$

The case $n = 0$ requires special consideration because of the constant term appearing in the expression (12) for $\cos v$. We here obtain in place of (16) and (16a)

$$\sqrt{\frac{\epsilon_0}{\mu_0}} \frac{1}{F-f} \left(-p + \frac{1}{p} \right) = \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{1}{a} \text{ and } - \sqrt{\frac{\epsilon'}{\mu}} \frac{k_L}{h} C_0 J'_0(\rho), \text{ respectively}$$

and hence

$$C_0 = - \sqrt{\frac{\epsilon_0}{\epsilon'} \frac{\mu}{\mu_0}} \frac{h}{k_L} \frac{1}{a J'_0(\rho)}. \quad (17a)$$

With these expressions for C_n and C_0 the continuity condition $H_z = H_z$ is satisfied.

E. The Boundary Condition for E_z and the Law of Phase Propagation

Substitution of (17) and (17a) in (15) leads to the expression for E_z

$$- \sqrt{\frac{\epsilon_0}{\epsilon'} \frac{\mu}{\mu_0}} \frac{\sqrt{k_L^2 - h^2}}{i k_L a} \left(\frac{J_0(\rho)}{J'_0(\rho)} + 2 \sum_{n=1}^{\infty} \frac{J_n(\rho)}{J'_n(\rho)} p^n \cos(n\varphi) \right), \quad (18)$$

which, in view of $|k_L| \gg |h|$ and $J_n/J'_n \cong i$, may be simplified to

$$-\sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} \frac{1}{a} \left(1 + 2 \sum_{n=1}^{\infty} p^n \cos(n\varphi) \right). \quad (18a)$$

Here we imagine the Fourier series (14) substituted for $\cos(n\varphi)$ and compare then (18a) with the representation (6) for the exterior of the wires, where we put $u = +u_0$:

$$E_0 u_0 + E_1 \sinh u_0 \cos v + E_2 \sinh(2u_0) \cos(2v) + \dots \quad (18b)$$

Equating of the factors of $\cos v$, $\cos(2v)$, $\cos(3v)$ on the two sides yields

$$E_1 \sinh u_0 = \frac{2p}{a} \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} \{1 - p^2 + 2p^2(1 - p^2) + 3p^4(1 - p^2) + \dots\}, \quad (19)$$

$$E_2 \sinh(2u_0) = \frac{2p^2}{a} \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} \{1 - p^2 - (1 - 4p^2 + 3p^4) + \dots\},$$

$$E_3 \sinh(3u_0) = \frac{2p^3}{a} \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} \{1 - p^2 + \dots\}.$$

Thus the coefficients E_n of the exterior field, left indefinite up to this point, are determined for $n > 0$; their values may be further simplified by utilizing

$$\sinh u_0 = \frac{1}{2p} (1 - p^2), \quad \sinh(2u_0) = \frac{1}{2p^2} (1 - p^4), \dots$$

What, however, is the status of the coefficient E_0 , which was fixed already by Eq. (7) and hence is not disposable for satisfying the boundary condition? Fortunately it contains the as yet undetermined quantity h . The remaining condition, containing E_0 , thus serves for the *final determination of the propagation constant h* , which here as for the single wire is of primary interest. The equation in question is obtained by comparison of the terms in (18a, b) which are independent of v and is

$$-E_0 u_0 = \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} \frac{1}{a} \left(1 + 2 \sum_{n=1}^{\infty} p^{2n} \right).$$

If here we substitute values from (7) and (9) for E_0 and u_0 and carry out the summation on the right, we obtain:

$$h = k + \frac{i}{2} \sqrt{\frac{\epsilon_0 \mu}{\epsilon' \mu_0}} \frac{1}{a} \frac{1 + p^2}{1 - p^2} / \log \frac{1}{p}. \quad (20)$$

We see that h is given directly and by an elementary expression, not, as for the single wire, by a transcendental equation. Furthermore for a perfectly

conducting material, $\epsilon_0/\epsilon' \rightarrow 0$, $h = k$, as should be the case. For finite conductivity ϵ' is essentially positive imaginary, $\sqrt{\epsilon'}$ hence of the type $\exp(i\pi/4)$ and $(\epsilon_0/\epsilon')^{1/2}$ of the type

$$e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}.$$

Hence the correction term in (20) is of the type $1+i$. The real part of h thus becomes *greater* than k ; this signifies a *propagation velocity less than c* . At the same time the real part of ih becomes negative. For our form $\exp\{i(hx - \omega t)\}$ this means *damping for propagation along the positive x -axis*. The correction term thus indicates a reasonable physical behavior in both its real and its imaginary parts.

Finally, to test the dependence of the correction term on the geometrical data of the Lecher system, i.e. the wire radius a and the wire separation $2b$, we may, for $2b \gg a$, replace F by $2b$. We then find from formula (11)

$$p = \frac{a}{2b} \ll 1, \quad \frac{1+p^2}{1-p^2} \log \frac{1}{p} \cong 1/\log \frac{2b}{a}. \quad (20a)$$

The correction term then has only a logarithmic dependence on the separation of the wires and is inversely proportional to the wire radius.

* If, conversely, b is only slightly greater than a , i.e. $b = a(1+\alpha)$ with $\alpha \ll 1$, we obtain from (3b) and (11)

$$F = a(1 + \sqrt{2\alpha}), \quad p = 1 - \sqrt{2\alpha}, \quad \log \frac{1}{p} = \sqrt{2\alpha},$$

and hence

$$\frac{1}{a} \frac{1+p^2}{1-p^2} \log \frac{1}{p} = \frac{1}{a} \left(\frac{1}{\sqrt{2\alpha}} - 1 \right) / \sqrt{2\alpha} = \frac{1}{2a\alpha} = \frac{1}{2(b-a)}. \quad (20b)$$

We have carried out this short calculation to show that our final formula (20) also covers the case of slightly separated wires, where the mutual influence of the wires is very great and their skin effect must be enhanced unilaterally. We thus make it clear that our treatment is quite general with respect to the *geometric* circumstances. On the other hand, the limitation to wires of *high conductivity* already introduced in section A applies throughout.

F. Supplement Regarding the Remaining Boundary Conditions

To demonstrate the completeness of our solution and its freedom from contradiction we shall survey briefly the remaining boundary conditions. These are the continuity conditions for H_z on the one hand, for $E_z = \bar{E}_z$ on the other.

The representation of the interior field to be obtained from (23.7) by

summation over n contained, in H_z , the coefficients D_n which till now we have suppressed. The boundary condition relates them to the constants H_n of the external field which occur in (6) and which, according to (8), are equal and opposite to the already known constants E_n . The boundary condition for H_z thus serves the determination of the constants D_n of the internal field. Without entering into the calculation, we find

$$|D_n| \ll |C_n|,$$

which we have already used in (17). Finally, no disposable coefficients are left over for the fulfilment of the boundary condition

$$E_r(\text{inside}) = E_r(\text{outside}).$$

On the other hand we know from our approximate calculation that everywhere outside, and in particular at the surfaces of the wires, $E_r = 0$. It is thus necessary to show that according to our representation (23.7) also the sum of all the E_r (inside) is much smaller than that of the other field components. This is in fact so, by several orders of magnitude. The proof must however be omitted here.

G. Parallel and Push-Pull Operation

The approach already described is entirely adapted to push-pull excitation. In the parallel case the transversal components must be derived not from the potential (3), but from

$$w = \log(\zeta - \zeta_0) + \log(\zeta + \zeta_0), \quad (21)$$

and this only for sufficiently thin wires ($a \ll b$; otherwise the equipotential lines of (21) are not approximated by circles!). The bipolar coordinates then lose their usefulness, since they no longer coincide with the system of equipotentials of (21). In particular the electric lines of force no longer pass, as in Fig. 36, from Q_1 to Q_2 , but repel each other and pass from Q_1 and Q_2 separately to infinity. We hence are now obliged to use ordinary cylindrical coordinates at the centers of the first and second wires, r, φ and $\bar{r}, \bar{\varphi}$, respectively, for the exterior of the wires as well. The field is now to be constructed by the superposition of the contributions of the two wires with equal sign. Furthermore, it is now necessary¹ to formulate these representations directly as solutions of the wave equation, which formerly, because of the limited range of the field, could be avoided. We thus return also for the exterior to Hondros' formulation (23.8) where, restricting ourselves to thin wires, we can limit our attention to the zero-order terms. In

¹ For the radiation condition for $r \rightarrow \infty$, which now becomes essential, see the discussion for the single wire in §22.

the representation of E_z we then obtain for the superposition of the two wires the sum:

$$\frac{\sqrt{k^2 - h^2}}{h} \left\{ H_0(\sqrt{k^2 - h^2}r) + H_0(\sqrt{k^2 - h^2}\bar{r}) \right\}. \quad (22)$$

On the periphery of the first wire $r = a$, $\bar{r} \cong 2b$ we obtain by (22.5), because of the smallness of $\sqrt{k^2 - h^2}$ (we of course continue to assume high conductivity),

$$\frac{\sqrt{k^2 - h^2}}{h} \frac{2i}{\pi} \left(\log \left\{ \frac{\gamma \sqrt{k^2 - h^2}}{2i} a \right\} + \log \left\{ \gamma \frac{\sqrt{k^2 - h^2}}{2i} 2b \right\} \right). \quad (22a)$$

This much about the exterior of the wires. Inside it is necessary to use the earlier perfectly general formulation (15) for E_z . The coefficient C_0 appearing here is determined with the aid of the continuity condition for H_z . Herewith the longitudinal field E_z inside of the wire, more particularly its zero-order partial wave, is also determined. This must agree with the term (22) of the external field for $r = a$. We thus obtain an equation of the form

$$(k^2 - h^2) \left(\log \left\{ \frac{\gamma \sqrt{k^2 - h^2}}{2i} a \right\} + \log \left\{ \frac{\gamma \sqrt{k^2 - h^2}}{2i} 2b \right\} \right) = \text{const},$$

whose right side is known. It must be fulfilled by proper choice of h and is *transcendental* in character, as with the single wire. (It evidently can be made to correspond to the single-wire equation (22.15) by combining the two logarithms.) We now see the reason why the corresponding equation for h becomes elementary instead of transcendental in the Lecher case: The two logarithms are here superposed with the $-$ sign in place of the $+$ sign; in combining them the factor $\gamma \sqrt{k^2 - h^2}/(2i)$ under the log-sign cancels and only $\log \{a/(2b)\}$ remains, as in Eq. (20a). We also see that the calculation here outlined, without bipolar coordinates, would have been successful also in the Lecher case, but that it would have been much more involved than the earlier method, particularly without restriction to extremely thin wires.

Not only in mathematical formulation but also in physical structure the parallel wave resembles the single-wire wave. It decreases much more slowly outwards than the push-pull wave and is hence much more disturbed by the surroundings. It is obvious that at large distance the two similarly directed currents of the parallel wave must produce the same field as the alternating current of the single wire.

Experimentally a pure excitation of the push-pull wave is always desirable to avoid disturbances from the surroundings. However if the arrangement is not quite symmetrical parallel waves are also occasionally excited,

which make the position of the nodal points, on which the wave-length determination rests, unsharp.

Hence even from a purely experimental standpoint it is important to keep in mind the possibility of the parallel processes and to take account of the theory of the single wire, which we have treated before the rest in §22, although as compared with the theory of the Lecher system it is of secondary practical importance.

PART III

THEORY OF RELATIVITY AND ELECTRON THEORY

§26. *The Invariance of the Maxwell Equations in the Four-Dimensional World*

The path taken by Einstein in 1905 in the discovery of the special theory of relativity was steep and difficult. It led through the analysis of the concepts of time and space and some ingenious imaginary experiments. The path which we shall take is wide and effortless. *It proceeds from the universal validity of the Maxwell equations* and the tremendous accumulation of experimental material on which they are based. It ends almost inadvertently at the *Lorentz transformation* and all its relativistic consequences.

A. *The Four-Potential*

We refer to the electrodynamic potentials in §19, which at that point still remained in the fog of an unsatisfactory formalism. I wish to create the impression in my readers that the true mathematical structure of these entities will appear only now, as in a mountain landscape when the fog lifts.

In the two differential equations (19.9) and (19.11), satisfied by \mathbf{A} and Ψ , we had on the left the operator

$$\Delta = \frac{1}{c^2} \frac{\partial^2}{\partial t^2}. \quad (1)$$

We now introduce in place of x, y, z, t the new coordinates

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict, \quad (2)$$

where the proper remarks regarding the imaginary unit in x_4 will be made later. We call these x_i world coordinates since all events in the world are determined in space-time. The operator (1) then becomes the four-dimensional generalization of the Laplace operator and may be designated by¹

$$\square = \sum_{i=1}^4 \frac{\partial^2}{\partial x_i^2}. \quad (3)$$

¹ Certain more advanced theories of Einstein and Kaluza employ also the five-dimensional symbol $\square = \sum_{i=1}^5 \frac{\partial^2}{\partial x_i^2}$.

Like the independent coordinates x_i we combine the potentials A, Ψ in a four-dimensional entity, the four-potential Ω . Let its four components be

$$\Omega_1 = A_x, \quad \Omega_2 = A_y, \quad \Omega_3 = A_z, \quad \Omega_4 = \frac{i}{c} \Psi. \quad (4)$$

The factor i in Ω_4 is reasonable in view of the definition of x_4 ; the factor $1/c$ gives the four components of Ω the same dimension and will be justified below (7). The differential equations (19.9) and (19.11) then take the form

$$\square \Omega = -\mu_0 \Gamma. \quad (5)$$

The quantity Γ here introduced may be called the *four-current density*. It follows from (4), (19.9), and (19.11) that its four components are

$$\Gamma_1 = J_x, \quad \Gamma_2 = J_y, \quad \Gamma_3 = J_z, \quad \Gamma_4 = ic\rho; \quad (6)$$

they all have the dimension Q/M²S of a current density.

We now turn to the Eq. (19.10) relating the potentials A and Ψ . The second term on its left is, in view of (4),

$$\frac{1}{c^2} \ddot{\Psi} = \frac{1}{ic} \frac{\partial \Omega_4}{\partial t} = \frac{\partial \Omega_4}{\partial x_4}.$$

Eq. (19.10) thus becomes

$$\frac{\partial \Omega_1}{\partial x_1} + \frac{\partial \Omega_2}{\partial x_2} + \frac{\partial \Omega_3}{\partial x_3} + \frac{\partial \Omega_4}{\partial x_4} = 0.$$

We write this more briefly

$$\text{Div } \Omega = 0, \quad \text{Div} = \sum_{i=1}^4 \frac{\partial}{\partial x_i}. \quad (7)$$

We shall call the operator Div the *four-dimensional divergence*. Its four-dimensional symmetry indicates *isotropy* in space and time of world events. Our operator \square shows the same isotropy.

Thus we see the reason that we may regard our four-potential Ω and our four-current density Γ as *vectors* in four-dimensional space, more briefly, as *four-vectors*. This notation contains a statement regarding the behavior of the quantities Ω and Γ when the coordinates x_i are changed. As was shown in Vol. II, §2, an ordinary "three-vector" is a quantity which, for orthogonal transformation of the x, y, z , behaves just as the radius vector $r = (x, y, z)$. Thus the four-vector attains a meaning in the four-dimensional world which is independent of the choice of the coordinate system. At the same point in Vol. II we defined a scalar as a quantity which is *invariant* with respect to orthogonal transformations; in particular, we showed by a simple calculation that the divergence of a three-vector possesses this property. This calculation may be transferred directly to

four dimensions and shows that the divergence defined in (7), applied to any four-vector, yields a *scalar*, i.e. a four-dimensional invariant.

By reducing the Maxwell equations to the four-vector Ω and the invariant operators \square , Div we have demonstrated at the same time their *general validity*, independent of the coordinate system. The isotropy of three-dimensional space found adequate expression in the vector calculus of parts I and II. It is now replaced, in view of the world isotropy, by the *four-dimensional vector calculus*. This states that, for a transition to a "primed" coordinate system x'_i , the Maxwell equations remain invariant, i.e. have the same form in the primed field components and coordinates as in the original "unprimed" ones. *This invariance is simply the principle of relativity in its electrodynamic formulation.* The Maxwell equations satisfy the relativity postulate from the very beginning. They need not be subsequently adapted to it, like the equations of mechanics (see §32).

B. The Six-Vectors of Field and Excitation

We now turn to the representation of the field component E . By Eq. (19.7) we have e.g.

$$E_x = -\frac{\partial \Psi}{\partial x} - \frac{\partial A_x}{\partial t}.$$

According to Eqs. (2) and (4) this is equivalent to

$$E_x = ic \left(\frac{\partial \Omega_4}{\partial x_1} - \frac{\partial \Omega_1}{\partial x_4} \right). \quad (8)$$

This relation suggests the introduction of the four-dimensional curl

$$\text{Curl}_{mn} \Omega = \frac{\partial \Omega_m}{\partial x_n} - \frac{\partial \Omega_n}{\partial x_m}. \quad (9)$$

As a two-indices quantity it has six components (according to Vol. II, Eq. (2.17) it was preferable to give the three-dimensional curl as well two indices instead of one index). Evidently

$$\text{Curl}_{nn} = 0, \quad \text{Curl}_{mn} = -\text{Curl}_{nm}. \quad (9a)$$

$\text{Curl} \Omega$ is called a *six-vector* or, preferably, in view of the symmetry properties indicated by (9a), an *antisymmetric six-tensor*. The term "surface tensor" is also applied to it. The six components which differ from zero and from each other may be divided into three space-time and three space-space forms, corresponding to the arrangement of the indices

$$14, 24, 34 \quad \text{and} \quad 23, 31, 12. \quad (9b)$$

According to (8) the first three belong to the electric vector, the last three to the magnetic vector. However, we must not couple the entity of quan-

tity \mathbf{H} with the entity of intensity \mathbf{E} to form a four-dimensional unit, but must employ for this the entity of intensity \mathbf{B} or rather the quantity $c\mathbf{B}$, which has the same dimension as \mathbf{E} . We therefore write for example,

$$cB_z = c \operatorname{curl}_{yz} A = c \left(\frac{\partial \Omega_1}{\partial x_2} - \frac{\partial \Omega_2}{\partial x_1} \right). \quad (10)$$

Combining (8) and (10) and extending them cyclically to the remaining field components we obtain the following representation of the *six-component field vector*, where () is to indicate merely the combination of the two three-dimensional vectors into one four-dimensional quantity:

$$F = (c\mathbf{B}, -i\mathbf{E}) = c \operatorname{Curl} \Omega. \quad (11)$$

This *field vector* F is to be given two indices in accord with the sequence (9b), like the six-vector $\operatorname{Curl} \Omega$.

We next ask about the *excitation vector*, which we shall denote by f . We form it from the two equally dimensioned quantities \mathbf{H} and $c\mathbf{D}$. To pass from \mathbf{H} and \mathbf{D} to \mathbf{B} and \mathbf{E} we employ the constants for vacuum μ_0 and ϵ_0 , since our part III will be limited throughout to space free from matter (e.g. a vacuum tube). Since

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \sqrt{\frac{\epsilon_0}{\mu_0}} c\mathbf{B} \quad \text{and} \quad c\mathbf{D} = c\epsilon_0 \mathbf{E} = \sqrt{\frac{\epsilon_0}{\mu_0}} \mathbf{E}$$

we obtain from (11) simply

$$f = (\mathbf{H}, -ic\mathbf{D}) = \sqrt{\frac{\epsilon_0}{\mu_0}} c \operatorname{Curl} \Omega. \quad (12)$$

Hence

$$f = \sqrt{\frac{\epsilon_0}{\mu_0}} F. \quad (13)$$

To our satisfaction the geometric mean of the two three-dimensional constants of vacuum ϵ_0 and $\mu' = 1/\mu_0$ appears here. Already at the introduction of the permeability in §4, p. 21 we emphasized that the true magnetic analog of the dielectric constant ϵ is not μ , but its reciprocal μ' . In any case, Eq. (13) combines the three-dimensional relations between excitations and fields symmetrically, with a single constant of vacuum. At the same time our formulation (11) and (12) translates the earlier, highly heterogeneous representations (19.7) and (19.6) of \mathbf{E} and \mathbf{H} into an entirely symmetric and harmonious form.

To create a clear visual image of the structure of the antisymmetric tensor we write down the array of all the components of f in matrix form; the arrangement of the components of F is obtained herefrom by multiplication with $\sqrt{\mu_0/\epsilon_0}$ and simultaneous exchange of \mathbf{H} with $c\mathbf{B}$ and of $c\mathbf{D}$

with \mathbf{E} . We distinguish the components of \mathbf{H} and \mathbf{D} in the customary three-dimensional fashion by the indices x, y, z ; from our present standpoint it would be preferable to designate them by the double indices (9b), as in the matrix at the left. We point out specifically the order 12, 13, 14 in the first row, 21, 23, 24 in the second row, etc. as well as the change in sign for the converse order:

$$f = \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ f_{21} & 0 & f_{23} & f_{24} \\ f_{31} & f_{32} & 0 & f_{34} \\ f_{41} & f_{42} & f_{43} & 0 \end{pmatrix} = \begin{pmatrix} 0 & H_z & -H_y & -icD_x \\ -H_z & 0 & H_x & -icD_y \\ H_y & -H_x & 0 & -icD_z \\ icD_x & icD_y & icD_z & 0 \end{pmatrix} \quad (14)$$

C. The Maxwell Equations in Four-Dimensional Form

We also want to write the original Maxwell equations with double indices. Proceeding from the equation

$$\dot{D}_x - \frac{\partial H_z}{\partial y} + \frac{\partial H_y}{\partial z} = -J_x$$

we obtain from the first row of the array (14), making use of the definition of Γ in (6),

$$-\frac{\partial f_{12}}{\partial x_1} - \frac{\partial f_{13}}{\partial x_2} - \frac{\partial f_{14}}{\partial x_3} = -\Gamma_1$$

and corresponding equations for the second and third of this triplet of Maxwell equations. We hence have in general form for $m = 1, 2, 3$

$$\sum_{n=1}^3 \frac{\partial f_{mn}}{\partial x_n} = \Gamma_m. \quad (15)$$

If we extend this form to $m = 4$, we obtain by (14) and (6)

$$ic \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = ic\rho = \Gamma_4.$$

Our original definition of the charge density ρ in (4.4b) thus proves to be the four-dimensional completion of the second Maxwell triplet (4.4).

The operation $\sum \frac{\partial}{\partial x_n}$ carried out in (15) bears the name "reduction" or "divergence" in general tensor analysis (see Vol. II, p. 60); it reduces a four-dimensional tensor to a four-vector. We symbolize it by Div , placing it parallel to the operation Div defined in (7), which reduces a four-vector to a scalar. We thus write in place of (15)

$$\text{Div}_m f = \sum_{n=1}^4 \frac{\partial f_{mn}}{\partial x_n} = \Gamma_m. \quad (16)$$

Generally for any antisymmetric tensor T

$$\text{Div Div } T_{mn} = 0, \quad (16a)$$

which follows directly from $T_{mn} = -T_{nm}$. We conclude therefore that the *divergence of the four-current vanishes*:

$$\text{Div } \Gamma = 0; \quad (16b)$$

this is simply the *continuity equation* (4.4c) in a refined, four-dimensional form.

What of the first triplet of the Maxwell equations (4.4)? Its x -component is

$$\dot{B}_x + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0.$$

In view of the meaning of our field tensor F , which is analogous to (14):

$$F = \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ F_{21} & 0 & F_{23} & F_{24} \\ F_{31} & F_{32} & 0 & F_{34} \\ F_{41} & F_{42} & F_{43} & 0 \end{pmatrix} = \begin{pmatrix} 0 & cB_x & -cB_y & -iE_x \\ -cB_x & 0 & cB_z & -iE_y \\ cB_y & -cB_z & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix} \quad (17)$$

it may be rewritten in the form

$$i \left(\frac{\partial F_{23}}{\partial x_4} + \frac{\partial F_{34}}{\partial x_2} + \frac{\partial F_{42}}{\partial x_3} \right) = 0. \quad (17a)$$

This somewhat confusing distribution of subscripts becomes quite plain if we introduce the "dual" six-vector of F ,

$$F^* = (-i\mathbf{E}, c\mathbf{B}), \quad (17b)$$

which is obtained from F by an exchange of the real and the imaginary constituents. The determination of the individual components of F^* is fixed by the rule

$$F^*_{mn} = F_{kl} \quad (17c)$$

with the prescription that the sequence of subscripts

$$k l m n \text{ arises from } 1 2 3 4 \quad (17d)$$

by an *even number of exchanges*. By this requirement we have uniquely

$$F_{23} = F^*_{14}, \quad F_{34} = F^*_{12}, \quad F_{42} = F^*_{13}, \quad (17e)$$

so that Eq. (17a) becomes

$$\frac{\partial F^*_{12}}{\partial x_3} + \frac{\partial F^*_{13}}{\partial x_2} + \frac{\partial F^*_{14}}{\partial x_4} = 0. \quad (17f)$$

It has thus become the first component of

$$\text{Div } F^* = 0. \quad (18)$$

The other two components of the Maxwell triplet in question take on similar forms.

But what is the meaning of the fourth component of (18)? It is

$$\frac{\partial F_{41}^*}{\partial x_1} + \frac{\partial F_{42}^*}{\partial x_2} + \frac{\partial F_{43}^*}{\partial x_3} = 0$$

and, by (17b) and (17) may be transformed into

$$-c \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) = 0.$$

It is thus identical with the familiar absence of *sources of the magnetic field intensity* \mathbf{B} . This appears now, from the four-dimensional relativistic standpoint, as a *formally necessary completion* of our first Maxwell triplet, while originally, in Eq. (4.4a), it had to be *postulated* separately as an empirical fact.

We have thus a complete representation of Maxwell's theory for vacuum in the statements

$$\text{Div } f = \Gamma, \quad \text{Div } F^* = 0, \quad F = \sqrt{\frac{\mu_0}{\epsilon_0}} f, \quad (19)$$

which parallel and are equivalent to the potential relations

$$\square \Omega = -\mu_0 \Gamma, \quad \text{Div } \Omega = 0, \quad F = c \text{Curl } \Omega, \quad \mu_0 f = \text{Curl } \Omega. \quad (20)$$

All quantities and operations appearing in both formulations have proper citizenship in the four-dimensional world and hence satisfy the principle of relativity.

It should be emphasized on this occasion that the theory of relativity leaves no doubt that the vectors \mathbf{E} and \mathbf{B} on the one hand, and \mathbf{D} and \mathbf{H} on the other, belong together, as parts of the higher entities F and f . This seemed clear to us from the beginning, partly for dimensional reasons, partly because of their different significance as entities of intensity and quantity. In particular, the theory of relativity leaves no doubt that this distinction is as necessary in vacuum as in any ponderable medium, i.e. that here also both six-vectors F and f (the four three-vectors \mathbf{E} , \mathbf{B} , \mathbf{D} , \mathbf{H}) have to be employed side by side.

D. On the Geometric Character of the Six-Vector and its Invariants

The *four-vector* is represented, as a matter of course, by a *straight line segment* in four-dimensional space (by an R_1 with sense of direction). It might appear appropriate to represent the *six-vector* by a two-dimensional

segment of a plane, i.e. by its magnitude and position in four-dimensional space (an R_2 , one of whose sides is designated as positive). However, this representation is too specialized. Such a segment of area has only 5 independent parameters, not 6, as a six-vector, i.e. *one* parameter indicating its size (shape is to be indifferent) and *four*¹ indicating its orientation (parallel displacements do not count). To obtain a geometrical interpretation of the general six-vector we note that every R_2 in four-dimensional space has uniquely correlated with it a second R_2 perpendicular² to it. If a segment of area is also prescribed in the second, a single further parameter is obtained (since the orientation in space is already determined by the orientation of the first segment of area), leading to the desired total number of six independent parameters. The geometric picture of the six-vector is thus not *one* segment of area, but *two mutually perpendicular segments of area of arbitrary size*. The components of the six-vector are equal to the sums of the projections of the *two* segments of area on the six coordinate planes (x_n, x_m). If the sizes of the two segments are interchanged, the original six-vector F passes over into the *dual* six-vector F^* ; this confirms the relation (17c) between the components of F and F^* .

The four-vector has only *one* invariant, the square of its length, equal to the sum of the squares of its four components (the fourth of them taken, of course, with negative sign, in view of its imaginary character). On the other hand, every six-vector F has *two invariants*

$$F \cdot F \quad \text{and} \quad F \cdot F^*$$

both given, in accord with the rule for the scalar product, by summation over the six components with equal indices. We carry this out for the electrodynamic case. By (17) and (17a)

$$\begin{aligned} F \cdot F &= F_{12}^2 + F_{21}^2 + F_{31}^2 + F_{13}^2 + F_{23}^2 + F_{32}^2 = c^2 \mathbf{B}^2 - \mathbf{E}^2 \\ F \cdot F^* &= F_{12}F_{34} + F_{23}F_{14} + F_{31}F_{24} + \dots = -2ic\mathbf{B} \cdot \mathbf{E} \end{aligned} \quad (21)$$

Here \dots signifies repetition of the three preceding products with reversal of the sequence of the factors, i.e. simply the doubling of the sum. For the *vacuum light wave* both invariants (21) are zero. In fact, by §6, $\mathbf{B} \perp \mathbf{E}$ and $c|\mathbf{B}| = |\mathbf{E}|$. In view of our statement regarding invariance the

¹ If its orientation is thought of as defined by two four-vectors proceeding from the same point and lying in R_2 , these are given by 3 quantities, e.g. the ratios of their four components. Both may however be rotated arbitrarily within R_2 , so that the number $2 \cdot 3$ is reduced to $2 \cdot 3 - 2 = 4$.

² One R_2 may be designated as the "axis" of the other, since one can be rotated about the other arbitrarily within itself. The relation of the two is of course mutual or "dual". The axis of an R_2 in R_4 is thus a two-dimensional, not as in R_3 a one-dimensional manifold.

light wave retains this property in all reference systems of the four-dimensional world.

The excitation vector has of course the corresponding invariants

$$\begin{aligned} f \cdot f &= \mathbf{H}^2 - c^2 \mathbf{D}^2, \\ f \cdot f^* &= -2ic\mathbf{H} \cdot \mathbf{D}. \end{aligned} \quad (21a)$$

The following mixed invariants differ from (21) and (21a) only by a factor:

$$\begin{aligned} f \cdot F &= f_{12}F_{12} + f_{23}F_{23} + f_{31}F_{31} + f_{14}F_{14} + f_{24}F_{24} + f_{34}F_{34} \\ &= c\mathbf{H} \cdot \mathbf{B} - c\mathbf{D} \cdot \mathbf{E}, \end{aligned} \quad (22)$$

$$\begin{aligned} f \cdot F^* &= f_{12}F^*_{12} + f_{23}F^*_{23} + f_{31}F^*_{31} + f_{14}F^*_{14} + f_{24}F^*_{24} + f_{34}F^*_{34} \\ &= -i\mathbf{H} \cdot \mathbf{E} - ic^2\mathbf{D} \cdot \mathbf{B} = -2i\mathbf{E} \cdot \mathbf{H}. \end{aligned} \quad (23)$$

We define

$$\Lambda = \frac{1}{2c} f \cdot F = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} - \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \quad (24)$$

as the *Lagrange density* (Lagrange function of the moving electron per unit volume of the field). On the other hand the energy density W , given by $\frac{1}{2}\mathbf{H} \cdot \mathbf{B} + \frac{1}{2}\mathbf{D} \cdot \mathbf{E}$, will prove to be a *component* of a world tensor; by itself it has no meaning independent of the frame of reference.

Our invariants may be expressed as follows in terms of the areas a and b which are correlated in the six-vector:

$$F \cdot F = a^2 + b^2, \quad F \cdot F^* = 2ab, \quad \Lambda = \frac{e_0}{2} (a^2 + b^2). \quad (25)$$

It follows from this that a particular six-vector ($b = 0$) is distinguished from the general one by the condition $F \cdot F^* = C$.

E. Relativistically Invariant Three-Vectors

We now ask what properties a three-vector must have in order that it may exist legitimately in the four-dimensional world. For this purpose we consider a *six-vector which is dual to itself*. We may write it in the form

$$\bar{F} = F_{mn} + F^*_{mn} = F_{mn} + F_{kl} \quad (26)$$

This six-vector has in fact only *three* independent components. We can define them by

$$\begin{aligned} a_x &= \bar{F}_{14} = \bar{F}_{23} = F_{14} + F_{23}, \\ a_y &= \bar{F}_{24} = \bar{F}_{31} = F_{24} + F_{31}, \\ a_z &= \bar{F}_{34} = \bar{F}_{12} = F_{34} + F_{12}, \end{aligned} \quad (26a)$$

and obtain as specific tensor arrangement for the three-vector a by (17):

$$\bar{F} = \begin{pmatrix} 0 & a_x & -a_y & a_z \\ -a_x & 0 & a_z & a_y \\ a_y & -a_z & 0 & a_x \\ -a_z & -a_y & -a_x & 0 \end{pmatrix} \quad (26b)$$

In the special case of the electrodynamic tensor F we obtain as corresponding three-vector

$$a_x = -i(E_x + icB_x), \quad a_y = -i(E_y + icB_y), \quad a_z = -i(E_z + icB_z).$$

Thus the complex three-vector

$$\mathbf{E} + ic\mathbf{B} = i\mathbf{a} \quad (27)$$

may be regarded as a four-dimensional tensor of the form (26), which is dual to itself.

On the other hand, we may also write, in place of (26),

$$\bar{F} = F_{mn} - F^*_{mn} = F_{mn} - F_{kl}.$$

This six-vector is *oppositely dual* to itself and leads to the three-vector

$$\mathbf{E} - ic\mathbf{B} = -i\mathbf{b}. \quad (27a)$$

Even before the theory of relativity it was often noted that the complex combinations $\mathbf{E} \pm ic\mathbf{B}$ and the analogous $\mathbf{H} \pm ic\mathbf{D}$ have certain characteristic advantages for the integration of the Maxwell equations.

We can now also give the Hertzian vector Π its proper place in the four-dimensional world. It appeared in §19 as a *three-vector*, but it is in fact a *disguised six-vector*, with the structure of an electrostatic field vector F_{stat} . For it $\mathbf{B}_{stat} = 0$, \mathbf{E}_{stat} equals a three-vector which, temporarily, we will denote by P_x, P_y, P_z . By (17) we then have

$$F_{stat} = \begin{pmatrix} 0 & 0 & 0 & -iP_x \\ 0 & 0 & 0 & -iP_y \\ 0 & 0 & 0 & -iP_z \\ iP_x & iP_y & iP_z & 0 \end{pmatrix}. \quad (28)$$

Reduction of this tensor leads to a four-vector which for the present will be denoted by Ω , like our four-potential:

$$\Omega = \text{Div } F_{stat}. \quad (28a)$$

It has the components

$$\Omega_{1,2,3} = -i \frac{\partial P_{x,y,z}}{\partial x_4} = -\frac{1}{c} \dot{\mathbf{P}}, \quad \Omega_4 = i \text{div } \mathbf{P}.$$

If, by Eq. (4), we pass from this $\Omega_{1,2,3,4}$ to the electrodynamic potential A and Ψ , we find

$$A = -\frac{1}{c} \dot{P}, \quad \Psi = c \operatorname{div} P. \quad (28b)$$

If, finally, we set $P = -\mu_0 c \Pi$, our Eqs. (28b) pass over exactly into Eqs. (19.15) and (19.16a), by which we had originally defined the Hertzian vector. Thus, like P , the three-vector Π has been reduced to the six-vector (28). This six-vector is here referred to a coordinate system in which the Hertzian dipole *rests*. The transition from this "system at rest" to an arbitrary reference system can be carried out by the rules of the next section.

§27. *The Group of the Lorentz Transformations and the Kinematics of the Theory of Relativity*

In his "Erlangen Program" Felix Klein¹ has classified the several geometric disciplines on the basis of the group of transformations permitted in them. *Projective geometry* regards all figures as the same, which pass over into each other by *central projections* in three-dimensional space. *Affine geometry* keeps the infinitely distant plane fixed, and hence permits only *parallel projections*. For *elementary geometry* also the shapes of figures, their angles and ratios of linear dimensions, are of importance. Its group is that of the *orthogonal transformations in three-dimensional space*, extended by the similarity transformations. Here the imaginary sphere circle contained in the infinitely distant plane is kept fixed in addition to this plane itself. The geometry of the general *point transformations* can transform any surface into any other, but subjects any small region of space only to linear (projective) changes. The group of the *contact transformations* dissolves even the content of surfaces in space and leaves only the combined position of surface point and tangential plane untouched.

The fact that we have spoken here not of individual transformations, but only of transformation *groups* evidently derives from the necessity of regarding transformations resulting from a sequence or combination of transformations as equally valid. If attention is focused on the unaltered properties, rather than the changes of geometric structures, we speak of the *theory of invariants* belonging to the transformation group in question.

The transformation group of *classical mechanics* is that of the *Galilei transformations* (see Vol. I, p. 10). It may be divided into the group of the orthogonal transformations of space and the displacements of the time scale, corresponding to Newton's idea of absolute space and absolute time. Its invariants are the square of the separation in space and the time difference. The group of *Maxwell's electrodynamics* is, as we saw in the last

¹ Comparative Study of More Recent Researches in Geometry, Erlangen, 1872.

section, that of the *orthogonal transformations in space-time*. In honor of the great Dutch physicist Hendrick Antoon Lorentz, Poincaré has called them the *Lorentz transformations*. Just as the nature of the several geometries is characterized by their particular group, the essence of Maxwell's theory rests in its *invariance within the Lorentz group*. Its fundamental invariant is the *separation of two world points*, in particular the *four-dimensional line element*, i.e. the separation of two neighboring points in space-time.

A. The General and the Special Lorentz Transformation

We have recorded the general pattern of the Lorentz transformation already in (2.10) of Vol. I. It is contained in the formulas

$$x'_i = \sum \alpha_{ik} x_k, \quad x_k = \sum \alpha_{ik} x'_i, \quad \left. \begin{matrix} i \\ k \end{matrix} \right\} = 1, 2, 3, 4, \quad (1)$$

$$\sum_{j=1}^4 \alpha_{ij} \alpha_{kj} = \sum_{j=1}^4 \alpha_{ji} \alpha_{jk} = \delta_{ik} = \begin{cases} 0 & k \neq i \\ 1 & k = i \end{cases}. \quad (2)$$

The orthogonal transformations of three-dimensional space, or its rotations within itself, form a subgroup. Correspondingly the general Lorentz transformation may be designated as a *rotation in space-time*. We have also already derived, in Vol. I, Eq. (2.14), the *special Lorentz transformation* in which two coordinates remain unchanged. If we choose for the latter the *y*- and *z*-coordinates, the transformation matrix reduces to

	x_1	x_2	x_3	x_4	
x'_1	α_{11}	0	0	α_{14}	
x'_2	0	1	0	0	
x'_3	0	0	1	0	
x'_4	α_{41}	0	0	α_{44}	(2a)

According to conditions (2) we must have

$$\alpha_{11}^2 + \alpha_{14}^2 = \alpha_{41}^2 + \alpha_{44}^2 = \alpha_{11}^2 + \alpha_{41}^2 = \alpha_{14}^2 + \alpha_{44}^2 = 1, \quad (3)$$

so that

$$\alpha_{11}^2 = \alpha_{44}^2, \quad \alpha_{14}^2 = \alpha_{41}^2. \quad (4)$$

we put

$$\alpha_{11} = \alpha_{44} = \alpha \quad (5)$$

and find from (2)

$$\alpha_{11}\alpha_{44} + \alpha_{41}\alpha_{14} = \alpha(\alpha_{14} + \alpha_{41}) = 0. \quad (6)$$

We thus may write, introducing a new constant β :

$$\alpha_{14} = -\alpha_{41} = i\alpha\beta. \quad (7)$$

The added factor i is necessary because of the imaginary character of x_4 and x'_4 if real values are to be obtained for x'_1 and x_1 in the equations of our system

$$x'_1 = \alpha x_1 + \alpha_{14}x_4 \quad \text{and} \quad x_1 = \alpha x'_1 + \alpha_{41}x'_4.$$

Substitution of (5) and (7) in (3) finally leads to

$$\alpha^2(1 - \beta^2) = 1, \quad \alpha = \frac{1}{\sqrt{1 - \beta^2}}. \quad (8)$$

Thus our system (2a) becomes

$$\begin{aligned} x'_1 &= \frac{1}{\sqrt{1 - \beta^2}} (x_1 + i\beta x_4), & x'_2 &= x_2, & x'_3 &= x_3, \\ x'_4 &= \frac{1}{\sqrt{1 - \beta^2}} (-i\beta x_1 + x_4) \end{aligned} \quad (9)$$

or, in real terms,

$$\begin{aligned} x' &= \frac{1}{\sqrt{1 - \beta^2}} (x - \beta ct), & y' &= y, & z' &= z, \\ t' &= \frac{1}{\sqrt{1 - \beta^2}} \left(t - \frac{\beta}{c} x \right). \end{aligned} \quad (10)$$

If, in (10), we carry out the transition to the limit

$$c \rightarrow \infty, \quad \beta \rightarrow 0, \quad \text{but} \quad \beta c = v = \text{finite},$$

we obtain

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t. \quad (10a)$$

We follow Ph. Frank in calling these equations the *Galilei transformation*. From the standpoint of this transformation group time and space have become "absolute". It takes the place of the Lorentz group only when

$$v \ll c, \quad \text{i.e.} \quad \beta \ll 1. \quad (10b)$$

The universally accepted notation $\beta = v/c$ may recall the β -rays, which have a velocity comparable with c , so that the Galilei group is not applicable to them.

Eqs. (10) signify that the two systems (x, t) and (x', t') move with respect to each other with the velocity $v = \beta c$. If we consider a particular point $x' = \text{const}$, we find for it from (10)

$$x - \beta ct = x - vt = \text{const.}$$

The primed system thus progresses along the positive x -axis, which coincides with the x' -axis, with the velocity v . The two other axes y' and z' displace themselves, in space, with the same velocity v , remaining parallel to the axes y and z . For $t = 0$ the "moving system" and the "system at rest" coincide.¹

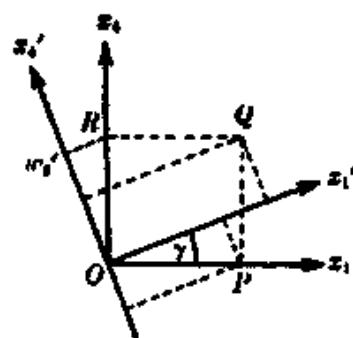
Problem III.1 will treat the somewhat more general case that the relative motion of the two systems will not be along the x -axis, but for example in some other direction lying in the xy -plane.

B. The Relative Nature of Time

From Eqs. (10) and (10a) we see that the course of time is absolute only in the limit $c \rightarrow \infty$, whereas for finite c it depends on the frame of reference of the observer: The "primed" observer measures a different time than the "unprimed" observer.

This becomes obvious if we return from the real representation (10) to the complex representation (9), but nevertheless plot the entities occurring

Fig. 37. The system x_1, x_4 is transformed into the system x'_1, x'_4 by the imaginary angle of rotation γ . The two events R, Q which are simultaneous in t then receive different coordinates x'_1 , just as the two events P, Q with the same x -coordinate have different coordinates x'_1 .



there as real quantities (Fig. 37). We may then write, as in plane analytic geometry,

$$x'_1 = x_1 \cos \gamma + x_4 \sin \gamma, \quad x'_4 = -x_1 \sin \gamma + x_4 \cos \gamma \quad (11)$$

with

$$\cos \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \sin \gamma = \frac{i\beta}{\sqrt{1 - \beta^2}}, \quad \tan \gamma = i\beta. \quad (11a)$$

The first Eq. (11), as is well known, signifies the projection of the broken line OPQ on the x'_1 -axis, the second that on the x'_4 -axis. However, the relative angle of rotation of the systems is here *imaginary* and similarly also $\sin \gamma$ and $\tan \gamma$; $\cos \gamma$ is actually a hyperbolic cosine and hence > 1 , since $\beta < 1$.

Fig. 37 also shows directly that two "events" (points in space-time) \bar{Q}

¹ It is of course equally permissible to regard the primed system as "system at rest," relative to which the unprimed system moves with the velocity v in the direction of the negative x -axis. This will occur occasionally in §§28 and 33.

and R , which are simultaneous in the unprimed system, are no longer simultaneous in the primed system. This *removal of the sameness in time* (simultaneity) now surprises us no more than the *removal of the sameness of the x -value* of the events Q and P .

Our pseudo-real representation in Fig. 37 will also prove useful in the future and can scarcely lead to misunderstandings. It is true that we deviate herein from the great example of Hermann Minkowski. In his classic lecture "Space and Time" before the Kölner Naturforscher-Gesellschaft in 1908 he calculates throughout with real quantities. What we would call the unit circle $x_1^2 + x_2^2 = 1$ is for him the hyperbola $x^2 - c^2t^2 = 1$; two straight lines which, to us, are perpendicular to each other, then become conjugate diameters of this hyperbola. It need scarcely be emphasized that, in spite of this (only superficial) difference, we have stood on Minkowski's shoulders even in the preceding paragraph and will continue to follow his conception of the theory of relativity.

C. The Lorentz Contraction

This was proposed by H. A. Lorentz even before the theory of relativity as an *ad hoc* hypothesis to explain the negative result of the Michelson experiment. Deferring discussion of this experiment to Vol. IV, we state Lorentz's hypothesis in the following manner: *To an observer at rest a rod of "intrinsic length" l_0 appears, if moving with uniform velocity v in the direction of its length, shortened to $l = l_0 \sqrt{1 - \beta^2}$.*

Let the rod rest in the moving system x', t' and let its endpoints in this system have the coordinates x'_a, x'_b ; their difference is the intrinsic length $l_0 = x'_b - x'_a$. We are here not concerned with the times t'_a, t'_b of the measurements. The situation is different for the observer at rest. He must arrange the measurements of the two ends of the rod so that, from his standpoint, they occur *simultaneously*, i.e. at the same instant $t_a = t_b$. He thus finds the points x_a and x_b and regards their difference, $x_b - x_a$, as the length of the rod l . From the first equation (10) it follows that

$$x'_a = \frac{1}{\sqrt{1 - \beta^2}} (x_a - \beta c t_a), \quad x'_b = \frac{1}{\sqrt{1 - \beta^2}} (x_b - \beta c t_b),$$

and hence, since $t_a = t_b$, $x_b - x_a = l$, $x'_b - x'_a = l_0$,

$$l_0 = \frac{l}{\sqrt{1 - \beta^2}}, \quad l = l_0 \sqrt{1 - \beta^2}. \quad (12)$$

The hypothetical Lorentz contraction is thus a direct consequence of the Lorentz transformation.

It is not superfluous to interpret this result graphically. The location of the rod in the moving system is represented in Fig. 38 by the strip which

is shaded parallel to the x_1' -axis. The observer at rest makes a cut of this strip parallel to the x_1 -axis. From the figure its length is

$$l = \frac{l_0}{\cos \gamma}.$$

In our pseudo-real representation it appears longer than the intrinsic length l_0 , although in fact it is shorter since $\cos \gamma = (1 - \beta^2)^{-1/2}$.

As a result of this contraction a moving sphere of radius a is flattened into an oblate spheroid with the small axis $b = a\sqrt{1 - \beta^2}$ and with the large axis a . Lorentz based on this his hypothesis of the *deformable electron*, first stated in 1903 as final result of his great paper on electron theory.¹

The frequently raised question whether the Lorentz contraction is "real" or "apparent" is of course just as idle as the question whether a body "actually" moves. The distinction between a moving system and a system

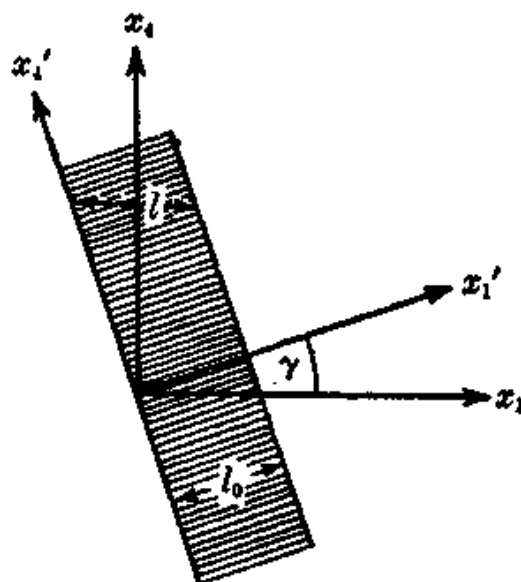


FIG. 38. A rod at rest in the primed system, moving with respect to the unprimed system, is represented by the shaded strip. Its length l in the unprimed system appears longer than its intrinsic length l_0 in the figure, but is in fact shorter, in view of the fact that γ is imaginary: Lorentz contraction.

at rest, which we permitted in the preceding for the sake of simplicity of expression, is equally meaningless and arbitrary.

D. The Einstein Dilatation of Time

Let a clock² rest in the primed system and mark, by its pendulum swings, the times and successive time differences

$$t'_1, t'_2, t'_3, \dots, t'_2 - t'_1 = t'_3 - t'_2 = \dots = \tau'.$$

They project themselves in the unprimed system into

$$t_1, t_2, t_3, \dots, t_2 - t_1 = t_3 - t_2 = \dots = \tau.$$

In Fig. 39 τ seems shortened as compared with τ' , but is actually expanded in view of

¹ Enzyklopaedie der Math. Wiss., Vol. V₂. Pages 277-279 of this article are fore-runners of the theory of relativity.

² Instead of speaking, with Einstein, of a clock we may adhere to electromagnetic patterns by thinking of a tuned circuit and its natural period.

$$\tau = \tau' \cos \gamma = \frac{\tau'}{\sqrt{1 - \beta^2}}. \quad (13)$$

This is realized most simply analytically if Eq. (10) is inverted, in which v simply changes sign, as may be verified by calculation. Thus

$$x = \frac{1}{\sqrt{1 - \beta^2}} (x' + \beta ct'), \quad t = \frac{1}{\sqrt{1 - \beta^2}} \left(t' + \frac{\beta x'}{c} \right). \quad (14)$$

Furthermore, since $x' = \text{const}$, the second of these equations yields for the successive differences $t_2 - t_1, t_3 - t_2, \dots; t'_2 - t'_1, \dots$

$$\tau = \frac{\tau'}{\sqrt{1 - \beta^2}}.$$

Such a clock is realized in a rapidly moving atom which emits a monochromatic spectral line, e.g. a hydrogen canal ray. Einstein regarded the expected red shift of the spectral line as the crucial experiment of the

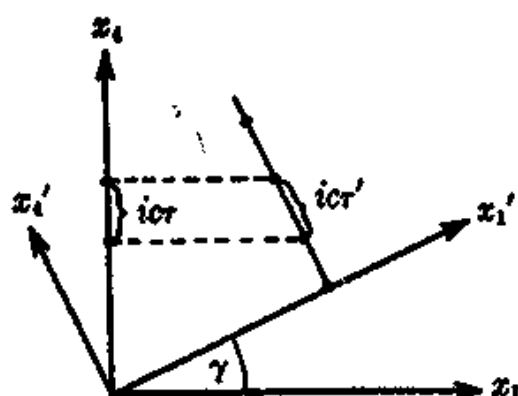


FIG. 39. The period of oscillation of the moving clock τ' appears shortened in the figure for an observer at rest in the x_1, x_2 -system to τ , is however actually lengthened in view of the imaginary character of γ : *Einstein's time dilatation.*

theory of relativity and spoke of a "transversal Doppler effect," considering observation at 90° relative to the canal rays. Ives has shown, however, that the observation may as well be carried out at an arbitrary angle, preferably a small angle, where the primary light of the canal rays can be compared with light reflected by a mirror from the (oppositely directed) canal rays. We then observe, e.g. for the H_α line of the hydrogen canal rays, in addition to the primary light, which is shifted toward the blue, the reflected light, which is shifted toward the red. The arithmetic mean of the two wave-lengths does not, however, coincide with the spectral line H_α of the atom at rest, but is displaced from it by the relativistic red shift independently of the direction of viewing. The experiment¹ fully confirms Einstein's expectation.

¹ H. E. Ives and G. R. Stillwell, J. Optical Soc. Am. 28, 215, 1938; H. E. Ives, J. Optical Soc. Am. 29, 183 and 294, 1939. G. Otting, Munich thesis, Phys. Z. 40, 681, 1939. There is a difference in the theoretical interpretation of the American papers and the simultaneous German thesis which is notable in view of the times (1939):

A radioactive sample also has an intrinsic time τ' in the form of its mean life. Hence, observed in the canal ray, it should have a longer life than at rest. This experiment is realized under the most favorable circumstances (β nearly equal to 1) in the meson disintegration of cosmic rays. Rasetti found for the life of mesons which had become trapped in an absorber and hence were practically at rest the value $\tau' \cong 1.5 \cdot 10^{-6}$ sec, determining the time difference between the incidence of the meson and the appearance of the secondary electron produced in the disintegration. On the other hand, absorption measurements on the mesons of cosmic rays lead to a most probable range of the order of 20 km. In the (unprimed) time measure of the terrestrial observer this corresponds to a mean life $\tau \cong 20 \text{ km}/c \cong 7 \cdot 10^{-6}$ sec. The time expansion thus has here the enormous value¹

$$\frac{\tau}{\tau'} \cong \frac{7 \cdot 10^{-6}}{1.5 \cdot 10^{-6}} \cong 50.$$

We hence find for the velocity of the mesons, by (13),

$$\frac{1}{\sqrt{1 - \beta^2}} = 50, \quad v = c \left(1 - \frac{1}{5000} \right).$$

This consideration is confirmed by the experimental determination of the energy of the mesons; for the most commonly occurring mesons approximately 50 times the rest energy is found, which fully agrees with the dependence of the kinetic energy on the velocity (see §32).

E. The Addition Theorem for the Velocity

Two velocities v_1 and v_2 having the same direction do not combine relativistically according to the rule

$$v = v_1 + v_2.$$

We have instead

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}. \quad (15)$$

Here v_2 is the velocity with which a point 2 moves relative to a body 1, which itself moves in the same direction with the velocity v_1 . When Einstein in 1905 proposed this formula, it naturally aroused surprise. It becomes entirely reasonable, however, when we note that we are here dealing with the composition of two Lorentz transformations, and that each of them, according to Fig. 37, denotes a rotation.² Let γ_2 be the angle of rota-

the American papers seek to retain the concept of the absolute ether, while the German paper assumes the relativistic standpoint from the very beginning.

¹ W. Heisenberg, *Vorträge über kosmische Strahlen*, Springer, 1943, pp. 78 ff.

² The two rotations γ_1 and γ_2 as well as their resultant γ take place about the same "axis", i.e. the R_z perpendicular to the x_1x_2 -plane (see p. 219, footnote 2).

tion which, by Eq. (11a), pertains to v_2 , and γ_1 that pertaining to v_1 . The result of the composition of the two rotations is a rotation through the angle

$$\gamma = \gamma_1 + \gamma_2. \quad (15a)$$

Thus the *angles* of rotation are added, not their *tangents*. For the latter we have instead

$$\tan \gamma = \frac{\tan \gamma_1 + \tan \gamma_2}{1 - \tan \gamma_1 \tan \gamma_2}.$$

By (11a) this leads to

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}, \quad (15b)$$

which agrees with (15). The addition theorem for the velocities is hence in essence merely the addition formula for the tangent function.

The same formula may be obtained quite readily, though more indirectly, by superposing the two Lorentz transformations. They may be written, e.g., in the form of Eqs. (14):

$$\begin{aligned} \sqrt{1 - \beta_1^2} x &= x_1 + \beta_1 ct_1, & \sqrt{1 - \beta_2^2} x_1 &= x_2 + \beta_2 ct_2, \\ \sqrt{1 - \beta_1^2} t &= t_1 + \frac{\beta_1}{c} x_1, & \sqrt{1 - \beta_2^2} t_1 &= t_2 + \frac{\beta_2}{c} x_2. \end{aligned} \quad (15b)$$

Elimination of x_1, t_1 then yields for the direct transition from x, t to x_2, t_2 :

$$\begin{aligned} \frac{\sqrt{1 - \beta_1^2} \sqrt{1 - \beta_2^2}}{1 + \beta_1 \beta_2} x &= x_2 + \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} ct_2, \\ \frac{\sqrt{1 - \beta_1^2} \sqrt{1 - \beta_2^2}}{1 + \beta_1 \beta_2} t &= t_2 + \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \frac{x_2}{c}. \end{aligned} \quad (15c)$$

This is again a Lorentz transformation of the form (14) if we put

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}, \quad \sqrt{1 - \beta^2} = \frac{\sqrt{1 - \beta_1^2} \sqrt{1 - \beta_2^2}}{1 + \beta_1 \beta_2}. \quad (15d)$$

The first of these formulas agrees with (15a); the second follows from it, as may be verified by a simple calculation.

For small velocities ($v_1 \ll c$ and $v_2 \ll c$) (15) of course passes over into the elementary superposition formula.

F. c as Upper Limit for All Velocities

If by repeated superposition of velocities the resultant approaches the velocity of light, the further addition of any arbitrary velocity is without effect. In fact we have, by (15b), for $\beta_1 \cong 1$:

$$\beta \cong \frac{1 + \beta_2}{1 + \beta_2} \cong 1.$$

The velocity of light c can only be approached, never exceeded. Even a cyclotron or betatron, which operates with continuous increases in velocity, cannot yield velocities greater than that of light.

We shall define our statement more precisely. To begin with, we obviously mean by "velocity" "relative velocity". But that does not suffice. Consider a sample of radium. It emits electrons with almost the velocity of light. Two electrons which fly off simultaneously in opposite directions have very nearly the relative velocity $2c$, viewed from the laboratory in which the sample of radium is at rest. However, in order to properly define relative velocity as used in our statement we must view one electron from the other. Then and then only the seemingly paradoxical equation $c + c = c$ applies. We are thus concerned, in our statement, with the *relative velocity of a moving point with respect to a reference system which is transformed to a state of rest*.

The moving point need not be a material point; it may also be a *process resulting in material changes*. Such a process is called a *signal* and we then speak of the *signal velocity*. If this should ever exceed c , the whole time sequence would be disturbed (see below). In wireless telegraphy and radar a bundle of electromagnetic waves serves as signal; a *monochromatic wave*, on the other hand, constitutes no signal, since a purely periodic wave has neither beginning nor end. Its velocity of propagation is hence not governed by our statement. In fact, we found for wave guides, in §24, phase velocities ω/h which were greater than c . Similarly, we will see in Vol. IV that phase velocities greater than c may occur in the anomalous dispersion of light waves. There are also quite trivial processes with velocities exceeding that of light, which, then, obviously cannot serve as signals. An example is the intersection of the edge of a ruler with a straight line with a very acute angle. If we displace the ruler at right angles to itself even with only moderate velocity, the intersection will move along the straight line with a velocity exceeding that of light, provided only that the angle has been chosen small enough.

A formal indication of the prohibition of $v > c$ is evidently given already by the Lorentz transformation in the form (10), since here $\sqrt{1 - \beta^2}$ and consequently also x' and t' would become imaginary.

G. Light Cone; Space-Like Vectors and Time-Like Vectors; Intrinsic Time

The four-dimensional form

$$\sum_{i=1}^4 x_i^2 = 0, \quad \text{in real terms } r^2 - c^2 t^2 = 0, \quad (16)$$

is characteristic for the metric of the Lorentz transformations. It represents, in three dimensions, a sphere expanding with the velocity of light, in four dimensions, a conic R_1 with rotational symmetry about the t -axis.

We call it, with Minkowski, the *light cone*. The interior is called the *forecone* and the *aftercone*, depending on whether $t < 0$ or $t > 0$.

All four-vectors leaving the origin which lie outside of the light cone are called *space-like*, those which lie inside of it, *time-like*. Thus the vector \mathbf{r} in the equatorial plane of the light cone is space-like, whereas all permitted velocities leaving the origin are time-like.

The sequence of all four-dimensional positions assumed by a moving material point is called its *world line*. The world line of a point at rest is parallel to the t -axis. All world lines passing through the origin lie in the aftercone for $t > 0$, in the forecone for $t < 0$.

We consider the *element of a world line*

$$ds = \sqrt{\sum_{i=1}^4 dx_i^2}.$$

As the distance between two neighboring world points, it is Lorentz-invariant (see p. 213). The same applies to *Minkowski's intrinsic time*

$$\begin{aligned} d\tau = \frac{ds}{ic} &= \sqrt{dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)} \\ &= dt \sqrt{1 - \frac{v^2}{c^2}} = dt \sqrt{1 - \beta^2}. \end{aligned} \quad (17)$$

We will now define the *four-vector of the velocity* along a world line. The form

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, ic = (\mathbf{v}, ic)$$

would not be a permissible definition, since dt has no invariant meaning. This does not apply, however, to

$$\mathbf{V} = \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, ic \frac{dt}{d\tau} = \frac{dt}{d\tau} (\mathbf{v}, ic). \quad (18)$$

The square of its length is

$$\mathbf{V} \cdot \mathbf{V} = \frac{dx^2 + dy^2 + dz^2 - c^2 dt^2}{d\tau^2} = -c^2, \quad (18a)$$

i.e. in fact an invariant which, furthermore, has the same value for all velocity-vectors \mathbf{V} . The *four-vector of the acceleration* should be defined correspondingly by

$$\mathbf{W} = \frac{d\mathbf{V}}{d\tau} = \frac{d^2x}{d\tau^2}, \frac{d^2y}{d\tau^2}, \frac{d^2z}{d\tau^2}, ic \frac{d^2t}{d\tau^2}. \quad (18b)$$

It is, in the four-dimensional sense, perpendicular to the four-vector of the velocity; differentiation of (18a) with respect to τ leads to:

$$\mathbf{V} \cdot \mathbf{W} = 0. \quad (18c)$$

H. The Addition Theorem for Velocities of Different Directions

Velocities of the *same* direction were to be combined in such fashion that their *angles* were added. Since in their elementary meaning angles denote arcs on the unit circle, their addition is equivalent to the joining of arcs on a circle which in our case, it is true, has the radius i instead of 1. The formulas of *plane* trigonometry could be applied to the joining of these arcs.

In order to combine velocities with *different* directions it is necessary to pass from the circle to the sphere, i.e. from the formulas of plane to those of *spherical* trigonometry, and for a sphere of radius i , not of radius 1. Combination of the velocities v_1 and v_2 to form the resultant v is therefore equivalent to the combination of the angles γ_1 and γ_2 to the resultant angle γ , i.e. the construction of a spherical triangle with the sides γ_1 , γ_2 , and γ . If α is the angle of inclination of v_2 relative to v_1 , then α appears as the *external* angle between the sides γ_1 and γ_2 in the spherical triangle. We then have by the cosine law (see Problem I.4):

$$\cos \gamma = \cos \gamma_1 \cos \gamma_2 - \sin \gamma_1 \sin \gamma_2 \cos \alpha. \quad (19)$$

This is the desired *generalized addition theorem*. For $\alpha = 0$ we obtain $\cos \gamma = \cos(\gamma_1 + \gamma_2)$; $\gamma = \gamma_1 + \gamma_2$, i.e. the earlier Eq. (15a). In view of the relation $\cos \gamma = (1 - \beta^2)^{-1/2}$ etc. (19) is equivalent with the rather untransparent formula

$$\beta^2 = \frac{\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 \cos \alpha - \beta_1^2\beta_2^2 \sin^2 \alpha}{(1 + \beta_1\beta_2 \cos \alpha)^2}. \quad (19a)$$

which was given already by Einstein. In Problem III.2 it will be proved analytically by application of the Lorentz transformation.

The introduction of our sphere of radius i may seem an arbitrary trick; actually, it is merely an expression of the fact that the arcs γ_1 , γ_2 which we must combine are imaginary, according to (11a).

We shall mention one more interesting result, which may be read off directly on Fig. 40: In the theory of relativity the sequence of differently oriented velocities is *not exchangeable*; the result of the combination of v_1 and v_2 differs from that of the combination of v_2 and v_1 . Though the *magnitude* of the resultant is the same, the *direction* differs. The difference in the two directions increases as the velocities increase; in fact, as we shall know, it is equal to the spherical excess of the spherical triangle formed in our construction.

In Fig. 40 the angle α between v_1 and v_2 has been chosen equal to $\pi/2$

to simplify the drawing and the arc γ_1 corresponding to v_1 has been placed on the equator of the sphere. The extension of the arc γ_2 then passes through the northpole N . If, on the other hand, starting from the same point A , we first record γ_2 (denoted by γ'_2 on the figure), perpendicular to the equator and with its extension also passing through N , we must draw through the endpoint B' of γ'_2 a great circle perpendicular to the meridian AB' and measure off on it $\gamma'_1 = \gamma_1 = AB$. The point A' located in this manner does not coincide with C ; instead, the connecting arcs AC and $C'A'$ enclose a certain angle ε . In view of the equality of the two triangles ABC and $A'B'C'$ we have here $\angle BAC = \angle B'A'C'$ and $\angle ACB = \angle A'C'B'$. If we call

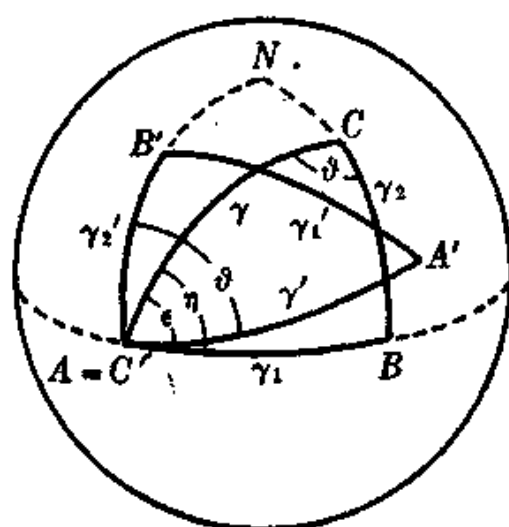


FIG. 40. Combination of two differently directed velocities v_1 and v_2 to form the resultant v , corresponding to the circular arcs γ_1 , γ_2 , and γ on a sphere of radius i . For convenience in representation the angle between v_1 and v_2 has been set equal to $\pi/2$. The figure shows the non-commutative character of the components: $v_1, v_2 = ABC \neq v_2, v_1 = C'B'A$; the angle ε between AC and $C'A'$ is equal to the spherical excess of the triangle ABC (and that of the triangle $A'B'C'$ which is congruent to it).

these two angles η and ϑ , we see that the right angle at A is formed by η , ϑ , and ε in the following manner:

$$\frac{\pi}{2} = \eta + \vartheta - \varepsilon.$$

Hence

$$\varepsilon = \eta + \vartheta - \pi/2 = \eta + \vartheta + \pi/2 - \pi. \quad (20)$$

ε thus is in fact the spherical excess of our right spherical triangle ABC and the congruent triangle $A'B'C'$. (The same applies for a general spherical triangle.)

The limiting case $\gamma_1 = \gamma_2 = \pi/2$, where the two triangles ABC and $A'B'C'$ become equal to the same spherical octant, is particularly simple. Here the resultants are evidently perpendicular to each other and, in view of $\eta = \vartheta = \pi/2$, the spherical excess is also $\pi/2$.

J. The Principles of the Constancy of the Velocity of Light and of Charge

Einstein in 1905 expressly added the first of these principles to the principle of relativity as an empirical postulate. It states that the velocity

of propagation of light is independent of the state of rest or motion of the emitting body. This principle is already included in the original formulation of our world geometry insofar as we have demanded the universal validity of the Maxwell equations. Like the velocity, the spherical propagation of the light is invariant in the transition from $x_1 \cdots x_4$ to $x'_1 \cdots x'_4$. The Lorentz transformation does not change the light sphere into a light *ellipsoid*, but leaves it a light *sphere*. (This does not apply to the *wave-length* of the light, which is not invariant but is known to depend on the frame of reference of the observer: Doppler effect.)

In the earlier but long since discarded theory of the universal *ether*, the independence of the light wave from the state of motion of the emitting body was readily understood: once transferred to the ether, it propagates itself in accord with the (elastic or electromagnetic) properties of this medium. Constancy of the velocity of light was here equivalent with field action. The same does not apply for a mechanical emission theory such as that surmised by Newton. Here a transfer of the velocity of the emitting body to the emitted light particles seems almost unavoidable.¹ We may say: The constancy of the velocity of light is today the only valid *remnant of the ether concept*. If at present we should speak of an ether, we would have to assign a separate ether to every frame of reference, i.e. speak e.g. of a primed and an unprimed ether. We now regard Lenard's "absolute ether (Uräther)" merely as a freak and the Aristotelian and scholastic "quintessence" (the fifth element, added to fire, water, air, and earth) as an historical curiosity. Thus in parts I and II, we have almost never spoken of the "ether", but used instead the not readily misinterpreted word "vacuum".

The principle of *constant charge* is as important as that of the constancy of the velocity of light. The charge is the same for every frame of reference. This is not obvious, but follows from the Maxwell equations if we can claim their universal validity for all frames of reference. On the other hand the principle of the constancy of mass with change of the system of reference, formerly regarded as obvious, cannot be upheld, as we shall see presently. *The charge is an absolute invariant with respect to Lorentz transformations; mass and, as we shall also see, energy, are not.*

Summarizing the content of this and the preceding section we may say: From the standpoint of the Maxwell equations the theory of relativity is obvious. A mathematician whose eyes had been trained by Klein's Erlangen program could have read from the form of the Maxwell equations its transformation group along with all its kinematic and optical consequences.

¹ The fact that Newton's emission theory could in a sense, experience a resurrection in the present theory of the light quanta rests solely on the addition theorem of the theory of relativity according to which effectively $c + v = c$ (c = velocity of light quanta, v = velocity of the emitting body).

§28. Preparation for the Electron Theory

Maxwell had directed attention away from the charges and toward the lines of force. Since the discovery of the electron and Helmholtz's earlier remarks¹ on the atomism of electricity, interest has once more returned to the sources of the lines of force, the electrons and ions. H. A. Lorentz² has created the secure mathematical basis for this new electrodynamics (which might be called *electron dynamics*). The judgment exhibited by him here is remarkable; he introduced only concepts which retained their substance in the later theory of relativity. We will abbreviate our treatment by inverting the historical development and basing the electron theory on the theory of relativity.

Unlike Maxwell, Lorentz does not recognize a host of media differing electrically and magnetically; all events take place in a single uniform medium, the vacuum. The different properties of matter arise simply from the varied binding and state of motion of the electrons and ions. In dielectrics the electrons are bound to ions, in conductors they are more or less freely mobile, and in magnetic materials we are dealing with electrons which, as the result of their spin, are aligned in the magnetic field.

In this explanation of the electromagnetic properties of matter we have the simultaneous action of great numbers of electrons, i.e. a *statistics of electrons*. We shall treat this subject in greater detail in Vol. V. In the present volume we must limit ourselves to the *theory of the individual electron*. It is true that the basic question regarding the *nature* of the electron will remain unclarified. *The electron is a stranger in electrodynamics*, as Einstein has said on occasion. We cannot comprehend, from the electrodynamic standpoint, how the finite electron charge e , concentrated in a point or in a very small volume, can cohere stably in spite of the Coulomb forces between parts of the charge. For a solution of this problem we must look to a *general theory of the elementary particles*, the electron, proton, neutron, neutrino, positron, meson (and other elementary particles which are yet to be discovered). It is clear, however, that such a theory is at the moment still remote.

¹ In his Faraday Lecture in 1881: "If we accept atoms for the chemical elements we cannot avoid concluding that also both positive and negative electricity is subdivided into certain elementary quanta which behave like atoms of electricity."

² In his book "Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern," Leyden, 1895; unaltered reprinting, Teubner, 1906. See also the later "Theory of Electrons," Teubner, 1909. Emil Wiechert reached the same conclusions and formulas independently of Lorentz at almost the same time in "The Theory of Electrodynamics and Röntgen's Discovery," Abh. der Physikalisch-ökonomischen Gesellschaft zu Königsberg.

A. The Transformation of the Electric Field. Introduction to The Lorentz Force

In order to calculate in the most elementary fashion, i.e. only with four-vectors, we return to the four-potential Ω , which transforms itself like the coordinate vector. We employ Eqs. (27.11) which, applied to Ω , take the form

$$\begin{aligned}\Omega'_1 &= \cos \gamma \Omega_1 + \sin \gamma \Omega_4, & \Omega'_2 &= \Omega_2, & \Omega'_3 &= \Omega_3, \\ \Omega'_4 &= -\sin \gamma \Omega_1 + \cos \gamma \Omega_4.\end{aligned}\quad (1)$$

The same equations (27.11), solved for x , yield

$$\begin{aligned}x_1 &= \cos \gamma x'_1 - \sin \gamma x'_4, & x_2 &= x'_2, & x_3 &= x'_3, \\ x_4 &= \sin \gamma x'_1 + \cos \gamma x'_4.\end{aligned}\quad (2)$$

We concern ourselves first with E'_y and corresponding to (26.9), form

$$\text{Curl}'_{24} \Omega' = \frac{\partial \Omega'_4}{\partial x'_2} - \frac{\partial \Omega'_2}{\partial x'_4}. \quad (3)$$

From (1) and (2) we obtain

$$\frac{\partial \Omega'_4}{\partial x'_2} = \frac{\partial \Omega_4}{\partial x_2} = -\sin \gamma \frac{\partial \Omega_1}{\partial x_2} + \cos \gamma \frac{\partial \Omega_4}{\partial x_2}, \quad (3a)$$

$$\frac{\partial \Omega'_2}{\partial x'_4} = \frac{\partial \Omega_2}{\partial x_1} \frac{\partial x_1}{\partial x'_4} + \frac{\partial \Omega_2}{\partial x_4} \frac{\partial x_4}{\partial x'_4} = -\sin \gamma \frac{\partial \Omega_2}{\partial x_1} + \cos \gamma \frac{\partial \Omega_2}{\partial x_4} \quad (3b)$$

and as the difference of the right sides of (3a, b)

$$\begin{aligned}\cos \gamma \left(\frac{\partial \Omega_4}{\partial x_2} - \frac{\partial \Omega_2}{\partial x_4} \right) + \sin \gamma \left(\frac{\partial \Omega_2}{\partial x_1} - \frac{\partial \Omega_1}{\partial x_2} \right) \\ = \cos \gamma \text{Curl}_{24} \Omega + \sin \gamma \text{Curl}_{12} \Omega.\end{aligned}\quad (4)$$

This is at the same time the right side of (3). We hence have

$$\text{Curl}'_{24} \Omega' = \cos \gamma \text{Curl}_{24} \Omega + \sin \gamma \text{Curl}_{12} \Omega. \quad (5)$$

By Eq. (26.11) we conclude therefore

$$-iE'_y = \cos \gamma (-iE_y) + \sin \gamma (cB_x). \quad (6)$$

In view of (27.11a) we may write instead

$$E'_y = \frac{E_y - \beta c B_x}{\sqrt{1 - \beta^2}}. \quad (6a)$$

A corresponding calculation yields

$$E'_x = \frac{E_x + \beta c B_y}{\sqrt{1 - \beta^2}}. \quad (6b)$$

The calculation for the x -component is somewhat more complicated insofar as it leads first to 8 terms, of which 4 are multiplied with $\sin \gamma \cos \gamma$, and two each with $\sin^2 \gamma$ and $\cos^2 \gamma$, respectively. The first cancel each other, whereas the remaining ones may be reduced to

$$\frac{\partial \Omega_1}{\partial x_4} - \frac{\partial \Omega_4}{\partial x_1}$$

and yield simply

$$E'_x = E_x. \quad (7)$$

So as to remove the distinctive treatment of the x -axis we indicate by the subscripts \parallel and \perp the direction parallel and perpendicular to the relative motion of the two systems. The Eqs. (7) and (6a, b) then become

$$E'_{\parallel} = E_{\parallel}, \quad E'_{\perp} = \left(\frac{\mathbf{E} + \mathbf{v} \times \mathbf{B}}{\sqrt{1 - \beta^2}} \right)_{\perp}. \quad (8)$$

Since $(\mathbf{v} \times \mathbf{B})_{\parallel} = 0$ we may write instead

$$E'_{\parallel} = (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\parallel}, \quad E'_{\perp} = \left(\frac{\mathbf{E} + \mathbf{v} \times \mathbf{B}}{\sqrt{1 - \beta^2}} \right)_{\perp}. \quad (8a)$$

The quantity $\mathbf{E} + \mathbf{v} \times \mathbf{B}$, which appears here automatically, when multiplied with e , has the dimension "newton" and is called the *Lorentz force*

$$\mathbf{K} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (9)$$

Through its formulation (more precisely, the formulation of the force density \mathbf{k} to be introduced presently) Lorentz put an end to the fruitless discussions of the older theory with regard to the ponderomotive forces on moving charges. In spite of its amazing simplicity Eq. (9) represents the sum total of the forces acting in arbitrary electromagnetic fields. An experiment of W. Wien on hydrogen canal rays confirms this directly.¹ After J. Stark had demonstrated the splitting of the Balmer lines in an electric field, Wien could produce qualitatively the same effect by letting a magnetic field corresponding to the electric field act on the rays. He thus replaced \mathbf{E} by the $\mathbf{v} \times \mathbf{B}$ which is equivalent to it.

It may incidentally be noted that $\mathbf{v} \times \mathbf{H}$ is commonly written in place of $\mathbf{v} \times \mathbf{B}$ in (9); from our dimensional standpoint this is an absurdity.

B. The Magnetic Analog to the Lorentz Force

We must now calculate \mathbf{B}' , i.e. the space-space components of the curl of $\mathbf{\Omega}'$, instead of the preceding space-time components. This becomes very

¹ Preuss. Akad., January 1914.

simple for the component in the direction of motion. In view of (1) and (2) it becomes

$$\text{Curl}'_{23} \Omega' = \frac{\partial \Omega'_3}{\partial x'_2} - \frac{\partial \Omega'_2}{\partial x'_3} = \frac{\partial \Omega_3}{\partial x_2} - \frac{\partial \Omega_2}{\partial x_3} = \text{Curl}_{23} \Omega.$$

By Eq. (26.11) this leads directly to

$$B'_z = B_z. \quad (10)$$

For B'_y we proceed as before for E'_y , noting that

$$\text{Curl}'_{31} \Omega' = \frac{\partial \Omega'_1}{\partial x'_3} - \frac{\partial \Omega'_3}{\partial x'_1} = \frac{\partial}{\partial x_3} \{ \cos \gamma \Omega_1 + \sin \gamma \Omega_4 \} - \frac{\partial \Omega_3}{\partial x'_1},$$

$$\frac{\partial \Omega_3}{\partial x_1} = \left\{ \cos \gamma \frac{\partial}{\partial x_1} + \sin \gamma \frac{\partial}{\partial x_4} \right\} \Omega_4, \quad \text{so that}$$

$$\begin{aligned} \text{Curl}'_{31} \Omega &= \cos \gamma \left(\frac{\partial \Omega_1}{\partial x_3} - \frac{\partial \Omega_3}{\partial x_1} \right) + \sin \gamma \left(\frac{\partial \Omega_4}{\partial x_3} - \frac{\partial \Omega_3}{\partial x_4} \right) \\ &= \cos \gamma \text{Curl}_{31} \Omega + \sin \gamma \text{Curl}_{34} \Omega. \end{aligned}$$

From this we obtain by Eqs. (26.11) and (27.11a):

$$B'_y = \frac{B_y + \beta E_z/c}{\sqrt{1 - \beta^2}}. \quad (10a)$$

Similarly we find

$$B'_z = \frac{B_z - \beta E_y/c}{\sqrt{1 - \beta^2}}. \quad (10b)$$

These formulas (10) may be generalized vectorially to

$$B'_{||} = (\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2)_{||}, \quad B'_{\perp} = \left(\frac{\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2}{\sqrt{1 - \beta^2}} \right)_{\perp}. \quad (11)$$

The quantity $\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2$ appearing in (11) is at the same time the ponderomotive force on a magnetic pole of strength 1, i.e. the *magnetic analog* of the Lorentz force exerted by the field on the charge 1.

C. The Intrinsic Field of an Electron in Uniform Motion

In a frame of reference x, y, z which moves with the electron the intrinsic field of the electron is electrostatic in character. Thus, for $r = \sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{E} = -\frac{e}{4\pi\epsilon_0} \text{grad} \frac{1}{r} \quad \text{and} \quad \mathbf{B} = 0. \quad (12)$$

For an observer at rest, with respect to whom the electron moves in the direction of the negative x -axis (see footnote at the end of §27A), we then have in view of (6a, b), (7), (10), and (10a, b)

$$\begin{aligned} E'_x &= E_x & E'_y &= \frac{1}{\sqrt{1-\beta^2}} E_y, & E'_z &= \frac{1}{\sqrt{1-\beta^2}} E_z. \\ B'_x &= 0, & B'_y &= \frac{v}{c^2 \sqrt{1-\beta^2}} E_x, & B'_z &= -\frac{v}{c^2 \sqrt{1-\beta^2}} E_y. \end{aligned} \quad (12a)$$

We express these primed fields in terms of the primed coordinates x', y', z' of the point of the field considered, which, like the x, y, z , we shall measure from the momentary position of the electron and consider the Lorentz contraction along the x -coordinate:

$$x' = \sqrt{1-\beta^2} x, \quad y' = y, \quad z' = z. \quad (13)$$

Simultaneously we set $s(x', y', z') \equiv r(x, y, z)$ or

$$s = \sqrt{\frac{x'^2}{1-\beta^2} + y'^2 + z'^2}. \quad (13a)$$

We then obtain from (12) and (12a)

$$E'_x, \quad E'_y, \quad E'_z = \frac{e}{4\pi\epsilon_0 \sqrt{1-\beta^2}} \frac{x', y', z'}{s^3}, \quad (14)$$

$$B'_x, \quad B'_y, \quad B'_z = \frac{ev}{4\pi\epsilon_0 c^2 \sqrt{1-\beta^2}} \frac{0, z', -y'}{s^3}. \quad (14a)$$

Thus our primed observer, unlike one moving with the electron, is aware of a *magnetic* field in addition to the electric field. By (14a) its lines of force are circles about the direction of motion; its intensity is, if we replace $\epsilon_0 c^2$ in the denominator of (14a) by μ_0 in the numerator and pass over from the field strength \mathbf{B} to the excitation $\mathbf{H} = \mathbf{B}/\mu_0$,

$$|\mathbf{H}| = \frac{ev}{4\pi\sqrt{1-\beta^2}} \frac{\sin \vartheta}{s^2}, \quad \sin \vartheta = \frac{\sqrt{y'^2 + z'^2}}{s}. \quad (14b)$$

This expression should be compared with the expression (15.12) for the Biot-Savart force, from which (14b) differs only by relativistic corrections of the second order in β . Thus in a sense a moving electron in a cathode ray realizes the commonly mentioned, but unreal, current element of the earlier theory; ev here takes the place of $I ds$.

The electric lines of force, on the other hand, are according to (14) straight lines diverging from the instantaneous position of the electron in the primed system (in view of the proportionality of the components of \mathbf{E}' in (14) with x', y', z') as well as in the unprimed system; however, they do

not have the same density in all directions in the former as in the latter case. Rather, they are squeezed together in the equatorial plane $x' = 0$. Because of the meaning of s in (13a), $s \rightarrow \infty$ and $E' \rightarrow 0$ for $\beta \rightarrow 1$ unless $x' = 0$. In this limiting case the electric field would be concentrated entirely in the equatorial plane. Thus the electron is *flattened* in the limit $v \rightarrow c$ not only in respect to its shape (with which we are not concerned here), but also in respect to its field.

In the preceding we have convinced ourselves that the determination of the field for uniform motion is merely a matter of *algebraic transformation*, while in the older electrodynamics it involved at least some integration.¹ §30 will deal with the field of accelerated motion.

We emphasize in general: The electric and magnetic fields form a single *unit* and can be distinguished only with reference to the particular reference system employed. Together they form a six-vector. In changing the frame of reference its electric components contribute to the magnetic ones and vice-versa. We are here dealing with an *effect of perspective in four dimensions*. The aspect of a cube furnishes the three-dimensional analog: For a particular choice of the viewing direction we see only the ("electric") front face, for other, oblique, directions the ("magnetic") lateral faces as well.

D. An Invariant Approach to the Lorentz Force; the Four-Vector of the Force Density

From the four-vector $x_1 \cdots x_4$ we obtain as the difference in position of two neighboring world points the *four-vector*

$$dx_1, dx_2, dx_3, ic dt. \quad (15)$$

Furthermore the four-dimensional volume element

$$dx_1 dx_2 dx_3 \cdot ic dt \quad (15a)$$

is also independent of the choice of coordinates, just like the volume element $dx_1 dx_2 dx_3$ in three dimensions. Since the corresponding charge Δe (the number of electrons contained in the element of volume), just like e itself, is also *invariant*, division of Δe by (15a) leads to another invariant scalar and multiplication of this scalar by (15) to another four-vector. As in (26.6) we call it the four-current density Γ :

$$\Gamma = \frac{\Delta e}{dx_1 dx_2 dx_3} \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, ic \right) = \rho(\mathbf{v}, ic). \quad (16)$$

ρ is the usual three-dimensional charge density. A comparison of the preceding definition of Γ with that in (26.6) shows that the current density \mathbf{J}

¹ See Oliver Heaviside, Phil. Mag. 1889. The surface $s = \text{const}$ (Eq. (13a)) is known as the Heaviside ellipsoid; see Problem III.3.

of electrodynamics passes over into the *convection current* density $\rho \mathbf{v}$ in the electron theory and, furthermore, that the four-vector Γ has the same direction as the velocity four-vector V defined in (27.18), in view of the relation

$$\Gamma = \rho \frac{d\tau}{dt} V = \rho \sqrt{1 - \beta^2} V. \quad (16a)$$

We now multiply the four-vector Γ with the six-vector F of the field. This results, by the process of "reduction", again in a four-vector, just as for the operation Div in (26.16). After having divided it by c , for dimensional reasons, we call it *force density* and denote it by \mathbf{k} , its n th component by k_n :

$$\mathbf{k} = \frac{1}{c} \Gamma \cdot F, \quad ck_n = \sum_{m=1}^4 \Gamma_m F_{nm}, \quad n = 1, 2, 3, 4. \quad (17)$$

Written term by term this becomes

$$\begin{aligned} ck_1 &= \Gamma_2 F_{12} + \Gamma_3 F_{13} + \Gamma_4 F_{14}, \\ ck_2 &= \Gamma_1 F_{21} + \Gamma_3 F_{23} + \Gamma_4 F_{24}, \\ ck_3 &= \Gamma_1 F_{31} + \Gamma_2 F_{32} + \Gamma_4 F_{34}, \\ ck_4 &= \Gamma_1 F_{41} + \Gamma_2 F_{42} + \Gamma_3 F_{43} \end{aligned} \quad (17a)$$

or, in three-dimensional coordinates,

$$\left. \begin{aligned} k_1 = k_x &= \rho(v_y B_z - v_z B_y + E_x) \\ k_2 = k_y &= \rho(v_z B_x - v_x B_z + E_y) \\ k_3 = k_z &= \rho(v_x B_y - v_y B_x + E_z) \end{aligned} \right\} = \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (17b)$$

In his original theory Lorentz operates primarily with the three-dimensional vector on the right.

We are of course also interested in the fourth component. It is, by (17a),

$$ck_4 = i\rho(v_x E_x + v_y E_y + v_z E_z) = i\rho \mathbf{v} \cdot \mathbf{E} = i\rho L. \quad (17c)$$

L is here the *power* expended by the electric field strength on a unit charge moving with velocity \mathbf{v} .

It may be shown readily from the representation (17a) that the four-vector \mathbf{k} is perpendicular to the world line of the charge. In view of the proportionality of Γ and V in (16a) and the antisymmetric character of F we obtain for the scalar product of \mathbf{k} and V :

$$\mathbf{V} \cdot \mathbf{k} = 0. \quad (17d)$$

We pass from the *force density* to the *force* itself. It is not permissible here, however, to simply change ρ into the electronic charge e by integration over space, since the three-dimensional volume element is not a relativistic invariant, but an arbitrary section through the "world tube" described by the electron (perpendicular to the also arbitrarily chosen time axis). It is much more appropriate to place the section perpendicular to the *world line* of the electron, which is independent of the orientation of the t -axis, or, what is the same thing, perpendicular to the generatrices of the mantle surface of the world tube. If we denote the angle between the world line and t -axis by γ , the three-dimensional $dx dy dz$ projects itself into the "world tube cross section"

$$dx dy dz \cos \gamma = \frac{dx dy dz}{\sqrt{1 - \beta^2}} \quad (18)$$

with the general meaning of γ given by Eq. (27.11a). By integration over this cross section we obtain

$$\int \rho dx dy dz \cos \gamma = \frac{e}{\sqrt{1 - \beta^2}}. \quad (18a)$$

From the representation (17b) for the force density we thus find directly

$$\int \frac{\mathbf{k} dx dy dz}{\sqrt{1 - \beta^2}} = \frac{e}{\sqrt{1 - \beta^2}} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \mathbf{K} / \sqrt{1 - \beta^2}. \quad (19)$$

\mathbf{K} is the Lorentz force of Eq. (9). It is not directly a part of a four-vector, but becomes one after division by $\sqrt{1 - \beta^2}$.

The corresponding fourth *energetic component* of this four-vector is according to (17c)

$$\int \frac{k_4 dx dy dz}{\sqrt{1 - \beta^2}} = \frac{ei}{c\sqrt{1 - \beta^2}} \mathbf{v} \cdot \mathbf{E} = \frac{eiL}{c\sqrt{1 - \beta^2}}. \quad (19a)$$

We call the *four-vector of the force*, completed in this manner, \mathbf{F} ; Its four components may be expressed collectively by

$$\mathbf{F} = \left\{ \frac{\mathbf{K}}{\sqrt{1 - \beta^2}}, \quad \frac{eiL}{\sqrt{1 - \beta^2}} \right\}. \quad (19b)$$

In view of its derivation from the Lorentz force density \mathbf{k} and of the relation (17d) it is *perpendicular* to the world line of the electron:

$$\mathbf{V} \cdot \mathbf{F} = 0. \quad (19c)$$

E. The General Orthogonal Transformation of a Tensor of the Second Rank

As generalization of the antisymmetric field tensor F we now consider an arbitrary (symmetric or asymmetric) tensor of the second rank T_{nm} ,

whose components T_{nn} need not vanish and for which we do not necessarily have $T_{nm} = T_{mn}$. Here we define as tensor a quantity whose components T_{nn} and T_{nm} behave like the squares and products x_n^2 and $x_n x_m$ of the four-dimensional coordinates in the orthogonal transformation (27.1). The formula which, by (27.1), applies for the product $x_n x_m$:

$$x'_n x'_m = \sum_{i=1}^4 \sum_{k=1}^4 \alpha_{ni} \alpha_{mk} x_i x_k$$

may be transferred to T in the following manner:

$$T'_{nm} = \sum_{i=1}^4 \sum_{k=1}^4 \alpha_{ni} \alpha_{mk} T_{ik}. \quad (20)$$

We shall encounter a symmetric tensor in §31. It remains symmetric in the transformation.

For an *antisymmetric* tensor (20) may be written

$$T'_{nm} = \sum_{i>k} (\alpha_{ni} \alpha_{mk} - \alpha_{mi} \alpha_{nk}) T_{ik} = \sum_{i>k} \begin{vmatrix} \alpha_{ni} & \alpha_{nk} \\ \alpha_{mi} & \alpha_{mk} \end{vmatrix} T_{ik}. \quad (20a)$$

Here the components $T_{ki} = -T_{ik}$ are already accounted for with the components T_{ik} . Hence the double sum in (20a) must be carried out in such fashion that, whereas i traverses all values from 1 to 4, k only assumes the values $k < i$. The value $k = i$ evidently need not be considered since $T_{ii} = 0$ and, in addition, since all the determinants in (20a) vanish in this case. An antisymmetric tensor remains antisymmetric in the transformation since the interchange of n and m reverses the sign of all the determinants.

After these general considerations we return once more to the behavior of the six-vector in the special Lorentz transformation (27.2a). We convince ourselves that all the subdeterminants of this matrix vanish with the exception of

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{31} & \alpha_{32} \end{vmatrix} = \begin{vmatrix} \alpha_{22} & \alpha_{24} \\ \alpha_{42} & \alpha_{44} \end{vmatrix} = \begin{vmatrix} \alpha_{33} & \alpha_{34} \\ \alpha_{43} & \alpha_{44} \end{vmatrix} = \frac{1}{\sqrt{1-\beta^2}},$$

$$\begin{vmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{vmatrix} = \begin{vmatrix} \alpha_{11} & \alpha_{14} \\ \alpha_{41} & \alpha_{44} \end{vmatrix} = 1,$$

$$\begin{vmatrix} \alpha_{12} & \alpha_{14} \\ \alpha_{22} & \alpha_{24} \end{vmatrix} = \begin{vmatrix} \alpha_{12} & \alpha_{14} \\ \alpha_{32} & \alpha_{34} \end{vmatrix} = - \begin{vmatrix} \alpha_{21} & \alpha_{22} \\ \alpha_{41} & \alpha_{42} \end{vmatrix} = - \begin{vmatrix} \alpha_{31} & \alpha_{32} \\ \alpha_{41} & \alpha_{42} \end{vmatrix} = \frac{-i\beta}{\sqrt{1-\beta^2}},$$

in view of the values of the α_{ik} given by Eqs. (27.5, 7, 8). On this basis the sums of six terms in (20a) are reduced to one or two terms. We find specifically

$$\begin{aligned}
T'_{12} &= \frac{1}{\sqrt{1-\beta^2}} (T_{12} - i\beta T_{23}), & T'_{21} &= T_{21}, \\
T'_{31} &= \frac{1}{\sqrt{1-\beta^2}} (T_{31} + i\beta T_{23}), \\
T'_{14} &= T_{14}, & T'_{24} &= \frac{1}{\sqrt{1-\beta^2}} (T_{24} + i\beta T_{12}), \\
T'_{34} &= \frac{1}{\sqrt{1-\beta^2}} (T_{34} - i\beta T_{12}).
\end{aligned}$$

If here we substitute for the $T_{\alpha\beta}$ the electromagnetic equivalent of the $F_{\alpha\beta}$ given by the matrix arrangement (26.17) we see readily that the preceding transformation formulas correspond to Eqs. (8) and (11). The present procedure for the derivation of these equations may be slower than the earlier one, but it is fundamentally more elementary and certainly more general, since it covers any tensors of the second rank.

§29. Integration of the Differential Equation of the Four-Potential

We now turn to the differential equation (26.5),

$$\square \Omega = -\mu_0 \Gamma, \quad (1)$$

where we are dealing with a problem of *four-dimensional potential theory*. The three-dimensional potential theory, Eqs. (7.4a) and (7.5), may serve us as example. To begin with, we require the four-dimensional analog to Newton's potential $1/r$. It is given by

$$U = \frac{1}{R^2}, \quad R^2 = (\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + (\xi_3 - x_3)^2 + (\xi_4 - x_4)^2. \quad (2)$$

As proof we calculate

$$\frac{\partial}{\partial \xi_i} \frac{1}{R^2} = -2 \frac{\xi_i - x_i}{R^4}, \quad \frac{\partial^2}{\partial \xi_i^2} \frac{1}{R^2} = -\frac{2}{R^4} + \frac{8}{R^6} (\xi_i - x_i)^2;$$

From this we deduce

$$\square U = \sum_{i=1}^4 \frac{\partial^2}{\partial \xi_i^2} \frac{1}{R^2} = -\frac{8}{R^4} + \frac{8R^2}{R^6} = 0, \quad (3)$$

which is valid for all points except the "source point" $\xi_i = x_i, i = 1, \dots, 4$. It may be proved similarly that in a space of $p+2$ dimensions the centrally symmetric potential is represented by $U = R^{-p}$ if the meaning of R^2 is generalized correspondingly.

Furthermore, as a preliminary, we shall determine the "surface" of the sphere $R = \text{const}$, i.e. a three-dimensional structure in four-dimensional space. If ω signifies the surface area of the unit sphere, it is

$$\omega R^3 \quad \text{with} \quad \omega = 2\pi^2. \quad (4)$$

In proof we consider the integral

$$J = \iiint\limits_{-\infty}^{\infty} \exp \left\{ - \sum_{i=1}^4 \xi_i^2 \right\} d\xi_1 d\xi_2 d\xi_3 d\xi_4. \quad (4a)$$

By carrying out the integral for each coordinate separately and utilizing the familiar value of the Laplace integral we obtain the fourth power of $\sqrt{\pi}$ or π^2 . On the other hand, if we introduce polar coordinates, with $r^2 = \sum \xi_i^2$, we find

$$J = \omega \int_0^{\infty} e^{-r^2} r^3 dr = \frac{\omega}{2}. \quad (4b)$$

A comparison of (4a) and (4b) proves Eq. (4). Similarly, for space with $p + 2$ dimensions

$$\omega = 2\pi^{p/2+1} / \Gamma(p/2 + 1). \quad (4c)$$

(The reader may check the validity of this formula in the three-dimensional and two dimensional cases, $p = 1$ and $p = 0$).

A. Four-Dimensional Form of the Potential Ω

We now apply Green's theorem to our two potentials Ω and U :

$$\int (\Omega \square U - U \square \Omega) d\xi_1 \cdots d\xi_4 = \int \left(\Omega \frac{\partial U}{\partial n} - U \frac{\partial \Omega}{\partial n} \right) d\sigma. \quad (5)$$

The integration at the left is to be extended over infinite four-dimensional space, with the exclusion of the source point $x_i = \xi_i$ by means of a sphere K of radius $R \rightarrow 0$. The integration on the right is to be carried out over this sphere K and a sphere $R \rightarrow \infty$, which however, as in the three-dimensional case (see p. 39), does not contribute to the integral. Substitution from the differential equations (1) and (3) then yields for (5) (Γ now denotes the four-current, not the Γ -function):

$$\mu_0 \int \Gamma \frac{d\xi_1 \cdots d\xi_4}{R^2} = \int_K \Omega \frac{\partial}{\partial n} \frac{1}{R^2} d\sigma - \int_K \frac{\partial \Omega}{\partial n} \frac{d\sigma}{R^2}. \quad (5a)$$

Since by Eq. (4) $\int d\sigma = 2\pi^2 R^3$, the second integral on the right vanishes for $R \rightarrow 0$. Since furthermore $dn = -dR$ (the normal is to be taken positive if it points outward from the space of integration, i.e. inward into the sphere K),

$$\int_K \Omega \frac{2}{R^3} d\sigma = 4\pi^2 \Omega.$$

Ω here denotes the value of our potential at $R = 0$, i.e. for $\xi_i = x_i$. Hence we find from (5a)

$$4\pi^2 \Omega(x_1, x_2, x_3, x_4)/\mu_0 = \int \Gamma \frac{d\xi_1 \cdots d\xi_4}{R^2}, \quad (6)$$

in perfect analogy to (7.5). (6) represents the four-potential in an arbitrary world point $x_1 \cdots x_4$ by a four-dimensional integration over the four-current density Γ , which is assumed to be known.

However, Γ is known to us only for the real times $\tau < t$, which precede the time of observation t ; we might also say, for the times $\tau < 0$, if, without loss of generality, we put the time of observation t temporarily equal to 0.

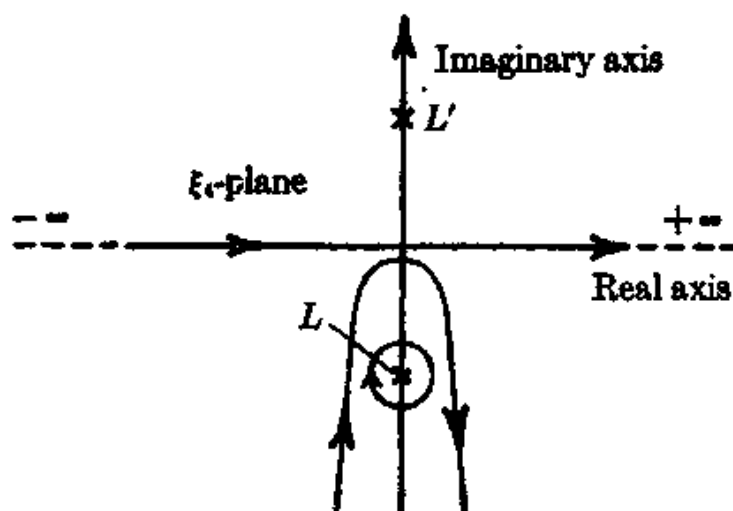


FIG. 41. Integration of the four-dimensional potential equation $\square\Omega = \mu_0\Gamma$. Deformation of the original path of integration along the real ξ_4 -axis into a loop about the "light point" L on the negative imaginary axis.

Accordingly Γ is not known to us along the real ξ_4 axis, as we have implicitly assumed till now, but only for the negative imaginary values

$$\xi_4 = ic\tau = -ic|\tau|. \quad (6a)$$

Accordingly we shall distort the path of integration for ξ_4 along the real axis, $-\infty < \xi_4 < \infty$ into a loop about the negative imaginary half-axis, which leads from $-i\infty$ by way of the neighborhood of the origin of the complex ξ_4 -plane back to $-i\infty$, as shown in Fig. 41. This does not alter our representation (6) or the fact that (6) satisfies our differential equation (1).

We still want to convince ourselves that (6) satisfies also the auxiliary condition (26.7), $\text{Div } \Omega = 0$. This follows from the fact that Γ satisfies the continuity equation (26.16b) $\text{Div } \Gamma = 0$. In fact, if we indicate the differentiation with respect to x_i and ξ_i by subscripts, and carry it out under the fourfold integral sign we find

$$\begin{aligned}
 4\pi^2 \operatorname{Div} \Omega / \mu_0 &= \int \left(\Gamma \operatorname{Grad}_x \frac{1}{R^2} \right) d\xi_1 \cdots d\xi_4 \\
 &= - \int \left(\Gamma \operatorname{Grad}_\xi \frac{1}{R^2} \right) d\xi_1 \cdots d\xi_4,
 \end{aligned} \tag{7}$$

i.e., after carrying out an integration by parts,

$$4\pi^2 \operatorname{Div} \Omega / \mu_0 = \int \operatorname{Div}_\xi \Gamma \frac{d\xi_1 \cdots d\xi_4}{R^2} = 0, \tag{7a}$$

which was to be proved.

For the further treatment of the representation (6) we can carry out first either the integration with respect to ξ_4 or that with respect to ξ_1 , ξ_2 , and ξ_3 . For the present we shall follow the first course.

B. Retarded Potentials

With reference to Fig. 41, we look for those points of the complex ξ_4 -plane at which the denominator R^2 vanishes. We write

$$R^2 = r^2 + (x_4 - \xi_4)^2, \tag{8}$$

where r signifies the three-dimensional distance between the point of integration ξ_1, ξ_2, ξ_3 and the reference point x_1, x_2, x_3 , and where we have dropped our temporary convention $x_4 = 0$, which merely served the more convenient description of Fig. 41.

There are two points at which $R^2 = 0$, i.e.

$$x_4 - \xi_4 = +ir \tag{8a}$$

and

$$x_4 - \xi_4 = -ir. \tag{8b}$$

We call the first, with Minkowski, the "light point" L ; the second¹ is designated with L' in the figure.

In the neighborhood of L we have according to (8) and (8a)

$$R^2 = (x_4 - \xi_4 - ir)(x_4 - \xi_4 + ir) \cong 2ir(x_4 - \xi_4 - ir). \tag{8c}$$

By Cauchy's theorem we can now distort the path of integration in Fig. 41 into a circuit about L , yielding by the method of residues

$$\oint \Gamma \frac{d\xi_4}{R^2} = \frac{\Gamma_L}{2ir} \oint \frac{d\xi_4}{x_4 - \xi_4 - ir} = \frac{\Gamma_L}{2ir} (+2i\pi) = \frac{\pi\Gamma_L}{r}. \tag{9}$$

Γ_L is the value of Γ at the light point. The factor $(+2\pi i)$ results from the fact that, on the one hand, $-\xi_4$ occurs in the denominator, on the other,

¹ The distortion of the original real path of integration into a loop about L' would lead to the "advanced" instead of the retarded potentials (see p. 148).

the integration path is traversed clockwise about L , i.e. in the negative direction from a function-theoretical standpoint.

Substituting (9) in (6) we find

$$4\pi\Omega/\mu_0 = \int \frac{\Gamma_L}{r} d\xi_1 d\xi_2 d\xi_3. \quad (10)$$

Resolved into components this yields, by (26.4) and (26.6)

$$4\pi\mathbf{A}/\mu_0 = \int \frac{\mathbf{J}_L}{r} d\xi_1 d\xi_2 d\xi_3, \quad 4\pi\epsilon_0\Psi = \int \frac{\rho_L}{r} d\xi_1 d\xi_2 d\xi_3. \quad (10a)$$

These are, however, exactly the representations of the *retarded potentials* in Eq. (19.13). In fact our present \mathbf{J}_L and ρ_L have the same meaning as our earlier $[\mathbf{J}]$ and $[\rho]$ in §19. For if we designate the time of the light point, which precedes that of the observation, by τ , as in (6a), we find from (8a)

$$ict = ic\tau + i\tau, \quad \tau = t - r/c. \quad (10b)$$

This is however exactly the time defined in (19.13c), for which $[\mathbf{J}]$ and $[\rho]$ were to be calculated. In this manner the formerly suppressed proof of (19.13) has been brought in a mathematically particularly appropriate fashion. It should be noted that G. Herglotz had devised the method given here even before the theory of relativity, just on the basis of mathematical symmetry and elegance.¹

C. The Lienard-Wiechert Approximation

We now take the second course mentioned above and carry out the integration over ξ_1, ξ_2, ξ_3 . We here imagine current and charge to be concentrated in a single point, the *electron*, rather than spacially distributed as up to now. We use for Γ its electron-theory value (28.16), by which the conduction current \mathbf{J} was interpreted as convection current, and with e as electron charge, obtain from it

$$\int \Gamma d\xi_1 d\xi_2 d\xi_3 = e(\mathbf{v}, ic) = -e\dot{\mathbf{R}}. \quad (11)$$

\mathbf{R} is the radius vector from the electron to the reference point, $\dot{\mathbf{R}}$ its derivative with respect to t for fixed reference point:

$$\dot{\mathbf{R}} = -\left(\frac{d\xi_1}{dt}, \frac{d\xi_2}{dt}, \frac{d\xi_3}{dt}, ic\right) = -(\mathbf{v}, ic). \quad (11a)$$

We then obtain from (6), carrying out the first three integrations,

$$4\pi^2\Omega/\mu_0 = -e \oint \frac{\dot{\mathbf{R}}}{R^2} d\xi_4. \quad (12)$$

¹ See Göttinger Nachr., 1904.

As in (9), the integration is to be carried out about the light point in Fig. 41. However, the locus of the electron ξ_1, ξ_2, ξ_3 is not an independent point of integration, as up to now, but itself depends on the integration variable ξ_4 . We must therefore consider the *world line of the electron in the neighborhood of the light point L* and expand R^2 as follows:

$$R^2 = R_L^2 + (\xi_4 - \xi_{4L}) \frac{dR^2}{d\xi_4} + \dots, \quad (13)$$

so as to be able to apply the method of residues.

Here we have

$$R_L^2 = 0 \quad \text{and} \quad \frac{dR^2}{d\xi_4} = \frac{1}{ic} \frac{d(\mathbf{R} \cdot \mathbf{R})}{dt} = \frac{2}{ic} \dot{\mathbf{R}} \cdot \mathbf{R}.$$

The expansion of R^2 becomes hence

$$R^2 = (\xi_4 - \xi_{4L}) \frac{2}{ic} \mathbf{R} \cdot \dot{\mathbf{R}} + \dots \quad (13a)$$

and Eq. (12) passes over into

$$4\pi^2 \Omega / \mu_0 = -\frac{e}{2} \frac{ic \dot{\mathbf{R}}}{\mathbf{R} \cdot \dot{\mathbf{R}}} \oint \frac{d\xi_4}{\xi_4 - \xi_{4L}}. \quad (13b)$$

In view of the sign of ξ_4 in the denominator, which is opposite to that in (9), the integral is now equal to $-2\pi i$. We thus obtain

$$4\pi \Omega / \mu_0 = -\frac{ec \dot{\mathbf{R}}}{\mathbf{R} \cdot \dot{\mathbf{R}}}. \quad (14)$$

According to (8a) the vector \mathbf{R} has the time component ir , whereas its space component (light point to reference point) is \mathbf{r} ; by (11a) the time and space components of $\dot{\mathbf{R}}$ are $-ic$ and $-\mathbf{v}$. Hence

$$\mathbf{R} \cdot \dot{\mathbf{R}} = rc - \mathbf{v} \cdot \mathbf{r} = rc \left(1 - \frac{\mathbf{v} \cdot \mathbf{r}}{rc} \right) = rc \left(1 - \frac{v_r}{c} \right), \quad (14a)$$

where v_r denotes the projection of \mathbf{v} on the direction of \mathbf{r} . Substitution in (14) and separation into real and imaginary parts yields the remarkably simple formulas of Lienard (1898) and Wiechert (1900):

$$4\pi \mathbf{A} / \mu_0 = \frac{e}{r} \frac{\mathbf{v}}{1 - v_r/c}, \quad 4\pi \epsilon_0 \Psi = \frac{e}{r} \frac{1}{1 - v_r/c}. \quad (15)$$

Our derivation shows that, like r , \mathbf{v} and v_r must be taken for the earlier time of the light point. It is interesting to note that the denominator $1 - v_r/c$ will recur in Vol. IV in connection with the Doppler effect.

Actually the original integral form (6) of the four-potential will prove

more useful for what follows than the formulas (15) or (10a), where the integration has been carried out.

§30. *The Field of the Accelerated Electron*

The advantage of Eq. (29.6) rests in the fact that the variables $x_1 \cdots x_4$ of the reference point occur here only in the denominator R^2 . We have to differentiate only the latter if we wish to calculate the field of an *electron in any state of motion*. In this manner we obtain from (29.6) first:

$$4\pi^2 \text{Curl}_{nm} \Omega = \mu_0 \int \left(\Gamma_m \frac{\partial}{\partial x_n} \frac{1}{R^2} - \Gamma_n \frac{\partial}{\partial x_m} \frac{1}{R^2} \right) d\xi_1 \cdots d\xi_4. \quad (1)$$

Since

$$\frac{\partial}{\partial x_n} \frac{1}{R^2} = -2 \frac{x_n - \xi_n}{R^4} = -\frac{2R_n}{R^4},$$

the parenthesis in (1) becomes

$$-\frac{2}{R^4} (\Gamma_m R_n - \Gamma_n R_m) = +\frac{2}{R^4} (\Gamma \times R)_{nm}.$$

Here we have transferred the usual symbol (\times) of the three-dimensional vector product to the product of our two four-vectors, which evidently is a quantity with six components. The same applies for the left side of Eq. (1), where by (26.12) $\text{Curl}_{nm} \Omega$ is the nm -component of the six-vector $\mu_0 f$. We thus obtain from (1)

$$2\pi^2 f_{nm} = \int \frac{1}{R^4} (\Gamma \times R)_{nm} d\xi_1 \cdots d\xi_4. \quad (1a)$$

We now carry out the integration with respect to ξ_1, ξ_2 , and ξ_3 , in which process, by Eq. (29.11), Γ transforms itself into $-e\dot{\mathbf{R}}$, and ξ_4 refers, from this point on, to the point electron. We find

$$2\pi^2 f = -e \oint \frac{(\dot{\mathbf{R}} \times \mathbf{R})}{R^4} d\xi_4. \quad (2)$$

The integration is here to be carried out over a circuit about the light point, as in Fig. 41. The difference from the previous calculations consists only in the fact that the denominator now vanishes to the second order, so that we have to carry the expansion in denominator and numerator one term further. If we abbreviate

$$\frac{\xi_4 - \xi_{42}}{ic} = u,$$

we write in place of (29.13a)

$$R^2 = 2u \dot{R} \cdot R + u^2 \{ \ddot{R} \cdot R + \dot{R} \cdot \dot{R} \} + \dots$$

$$R^4 = 4u^2 (\dot{R} \cdot R)^2 \left(1 + u \frac{\ddot{R} \cdot R + \dot{R} \cdot \dot{R}}{\dot{R} \cdot R} + \dots \right).$$

and, since $\dot{R} \times \dot{R} = 0$,

$$R \times \dot{R} = (R \times \dot{R})_L + u(R \times \ddot{R})_L + \dots$$

Thus, if at this point we transfer the denominator in part to the numerator and suppress the subscript L (2) becomes:

$$2\pi^2 f = \frac{eic}{4(\dot{R} \cdot R)^2} \oint \frac{du}{u^2} (R \times \dot{R} + u(R \times \ddot{R})) \left(1 - u \frac{\ddot{R} \cdot R + \dot{R} \cdot \dot{R}}{R \cdot R} \right).$$

Here we need write out only the term multiplied with u^{-1} , since only this is involved in determining the residue, and can omit the terms with u^{-2} , u^0 , u^1 . . . We thus obtain

$$2\pi^2 f = \frac{eic}{4(\dot{R} \cdot R)^2} \oint \frac{du}{u} \left(R \times \ddot{R} - R \times \dot{R} \frac{\ddot{R} \cdot R + \dot{R} \cdot \dot{R}}{\dot{R} \cdot R} \right).$$

Since the integration indicated in Fig. 41 amounts simply to the addition of the factor $-2\pi i$, we find finally

$$\frac{4\pi f}{ec} = \frac{R \times \ddot{R}}{(\dot{R} \cdot R)^2} - R \times \dot{R} \frac{\ddot{R} \cdot R + \dot{R} \cdot \dot{R}}{(\dot{R} \cdot R)^2}. \quad (3)$$

According to (29.8a) and (29.11a) the expressions on the right must be formulated specifically for

$$R = (r, ir), \quad \dot{R} = -(\mathbf{v}, ic), \quad \ddot{R} = (-\dot{\mathbf{v}}, 0). \quad (3a)$$

We examine this general representation first for the special case of the

A. Electron in Uniform Motion

Here, since $\ddot{R} = 0$,

$$\frac{4\pi f}{ec} = -R \times \dot{R} \frac{\dot{R} \cdot \dot{R}}{(\dot{R} \cdot R)^2} = R \times \dot{R} \frac{c^2 - v^2}{c^2 r^3 (1 - v_r/c)^3}. \quad (4)$$

According to (3a) the six-vector $R \times \dot{R}$ is given, in matrix notation, by

$$R \times \dot{R} = - \begin{bmatrix} r_x & r_y & r_z & ir \\ v_x & v_y & v_z & ic \end{bmatrix}. \quad (5)$$

We calculate its space-space and space-time components as subdeterminants of the matrix. In the notation of ordinary three-dimensional vector calculus we obtain

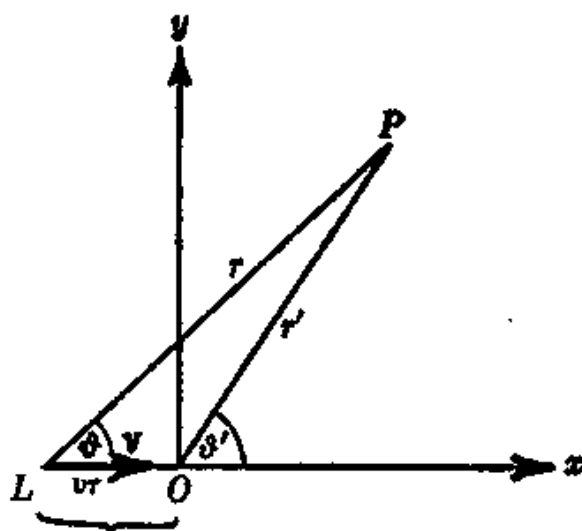
$$\mathbf{R} \times \dot{\mathbf{R}} = \begin{cases} \mathbf{v} \times \mathbf{r} & \text{for the space-space components,} \\ i(r\mathbf{v} - c\mathbf{r}) & \text{for the space-time components.} \end{cases} \quad (5a)$$

If this is substituted on the right side of (4) and f is separated into its space-space portion \mathbf{H} and its space-time portion $-icD$ on the left as well, we find

$$\begin{aligned} \frac{4\pi\mathbf{H}}{e} &= \mathbf{v} \times \mathbf{r} \frac{1 - v^2/c^2}{r^3(1 - v_r/c)^3}, \\ -\frac{4\pi D}{e} &= \left(r\frac{\mathbf{v}}{c} - \mathbf{r}\right) \frac{1 - v^2/c^2}{r^3(1 - v_r/c)^3}. \end{aligned} \quad (6)$$

These expressions appear basically different from the expressions (28.14) and (28.14a), with which we represented previously the field of the electron in uniform motion (there designated by \mathbf{H}' , \mathbf{E}'), but can actually be

FIG. 42. The field of an electron in uniform motion. The electron moves along the x -axis with the velocity v ; O is the location of the electron which is simultaneous with the observation at P , L the light point, so that $LO = vr$, where r is the retarded time of the light signal emitted from L to P .



transformed into each other by elementary geometrical considerations. We will show this in Problem III.3. We will then make use of Fig. 42, which pictures at the same time the different viewpoint of the present and the earlier formulas: In the present formulas r and r' refer to the light point L , in which the electron was at the time $t - r/c$, t being the time coordinate of the reference point P . The earlier formulas, on the other hand, concerned the position of the electron simultaneous with t , which is designated in the figure by O ; the coordinates of the reference point with respect to O are given by x' , y' , z' as in (28.14).

B. The Accelerated Electron

If in (3) we omit the part (4) we obtain the pure "acceleration field"

$$\frac{4\pi f}{ec} = \frac{\mathbf{R} \times \ddot{\mathbf{R}}}{(\dot{\mathbf{R}} \cdot \mathbf{R})^2} - \frac{(\mathbf{R} \times \dot{\mathbf{R}})(\mathbf{R} \cdot \ddot{\mathbf{R}})}{(\dot{\mathbf{R}} \cdot \mathbf{R})^3}. \quad (7)$$

To analyze this, we calculate from (3a)

$$\mathbf{R} \times \ddot{\mathbf{R}} = \begin{cases} -\mathbf{r} \times \dot{\mathbf{v}}, & \text{the space-space portion,} \\ i r \dot{v}, & \text{the space-time portion} \end{cases} \quad (7a)$$

as well as

$$\mathbf{R} \cdot \ddot{\mathbf{R}} = -\mathbf{r} \cdot \dot{\mathbf{v}}. \quad (7b)$$

We then obtain from (7) with due regard to (5a) and (29.14a)

$$\begin{aligned} \frac{4\pi\mathbf{H}}{ec} &= -\frac{\mathbf{r} \times \dot{\mathbf{v}}}{c^2 r^2 (1 - v_r/c)^2} - \frac{(\mathbf{r} \times \mathbf{v})(\mathbf{r} \cdot \dot{\mathbf{v}})}{c^2 r^2 (1 - v_r/c)^2}, \\ \frac{4\pi\mathbf{D}}{e} &= -\frac{r \dot{v}}{c^2 r^2 (1 - v_r/c)^2} + \frac{(c\mathbf{r} - r\mathbf{v})(\mathbf{r} \cdot \dot{\mathbf{v}})}{c^2 r^2 (1 - v_r/c)^2}. \end{aligned} \quad (8)$$

From this we conclude directly

$$r\mathbf{H}/c = \mathbf{r} \times \mathbf{D} \quad \text{and} \quad \mathbf{r} \cdot \mathbf{D} = 0. \quad (8a)$$

\mathbf{H} , \mathbf{D} , and \mathbf{r} or, as we might also say, \mathbf{H} , \mathbf{E} , and \mathbf{r} are mutually perpendicular to each other. Furthermore, taking the absolute value in the first Eq. (8a) in view of the second Eq. (8a), leads to

$$\frac{1}{c} |\mathbf{H}| = |\mathbf{D}| \quad (8b)$$

or, expressed in other terms, to

$$|\mathbf{H}| = \sqrt{\frac{\epsilon_0}{\mu_0}} |\mathbf{E}|.$$

We thus have a typical transversal field, as for the plane light wave in Eqs. (6.11) and (6.13). Its strength decreases with increasing r as $1/r$; for the denominators in (8a) have each *one* factor r more than the numerators, not *two* as for the electron in uniform motion (Eq. (6)). Hence at great distances (6) may be neglected as compared to (7), and (7) represents the entire field of the accelerated electron.

C. The Longitudinally Accelerated Electron

Let us assume specifically that \mathbf{v} and $\dot{\mathbf{v}}$ have the same direction (rectilinear motion, longitudinally accelerated electron); we then see readily that

$$\dot{v}(\mathbf{r} \cdot \mathbf{v}) = \mathbf{v}(\mathbf{r} \cdot \dot{\mathbf{v}})$$

and hence also

$$(\mathbf{r} \times \dot{\mathbf{v}})(\mathbf{r} \cdot \mathbf{v}) = (\mathbf{r} \times \mathbf{v})(\mathbf{r} \cdot \dot{\mathbf{v}}).$$

If, now, the denominators of the right sides of Eqs. (8) are made the same, two terms cancel each other in each case. These Eqs. (8) then reduce to

$$\frac{4\pi\mathbf{H}}{ec} = -\frac{\mathbf{r} \times \dot{\mathbf{v}}}{c^2 r^2 (1 - v_r/c)^3}, \quad \frac{4\pi\mathbf{D}}{e} = \frac{-r\dot{\mathbf{v}} + r\dot{v}_r}{c^2 r^2 (1 - v_r/c)^3}. \quad (9)$$

If we make the common direction of \mathbf{v} and $\dot{\mathbf{v}}$ the axis $\vartheta = 0$ of a spherical polar coordinate system r, ϑ, φ , we have

$$\begin{aligned} v_r &= v \cos \vartheta, & v_\vartheta &= -v \sin \vartheta, & v_\varphi &= 0, \\ \dot{v}_r &= \dot{v} \cos \vartheta, & \dot{v}_\vartheta &= -\dot{v} \sin \vartheta, & \dot{v}_\varphi &= 0, \\ \mathbf{r} \times \dot{\mathbf{v}} &= (\mathbf{r} \times \dot{\mathbf{v}})_\varphi, & H &= H_\varphi, & D &= D_\varphi \end{aligned}$$

and we obtain from (9)

$$4\pi H_\varphi = \frac{e\dot{v}}{cr} \frac{\sin \vartheta}{(1 - \beta \cos \vartheta)^3}, \quad 4\pi c D_\varphi = \frac{e\dot{v}}{cr} \frac{\sin \vartheta}{(1 - \beta \cos \vartheta)^3} \quad (10)$$

These are the same expressions as (19.20), with the addition of the *relativistic denominator* $(1 - \beta \cos \vartheta)^3$, which of course was lacking in the nonrelativistic calculation ($\beta \rightarrow 0$). In fact our earlier factor $\ddot{\mathbf{p}}(t - r/c)$ is the same as our present factor $e\dot{\mathbf{v}}$, computed for the light point. Correspondingly we find in place of the radiation S in (19.22)

$$S = \frac{e^2 \dot{v}^2}{16\pi^2 \epsilon_0 c^4 r^3} \frac{\sin^2 \vartheta}{(1 - \beta \cos \vartheta)^6}. \quad (11)$$

Accordingly the maximum of the radiation no longer lies at $\vartheta = \pi/2$, but advances, as β approaches 1, from $\vartheta = \pi/2$ toward $\vartheta = 0$.¹ We already referred on p. 155 to this phenomenon, which is characteristic for x-ray theory.

§31. The Maxwell Stresses and the Stress-Energy Tensor

So far we have only dealt with the kinematics of the electron, prescribing its motion and inquiring regarding the accompanying field. We now turn to the *statics* and then to the *dynamics* of the electron. With respect to the

¹ By differentiation of (11) with respect to ϑ we obtain as condition for S_{\max} a quadratic equation for $\cos \vartheta$, which, for small ϑ , yields

$$\cos \vartheta = 3\beta, \quad \vartheta = \frac{\pi}{2} - 3\beta,$$

and for β nearly equal to 1,

$$\cos \vartheta = 1 - \frac{1 - \beta^2}{15}, \quad \vartheta = \sqrt{\frac{1 - \beta^2}{5}}.$$

statics of the electron we have familiarized ourselves till now only with the Lorentz force, acting at the locus of the electron. A field concept cannot be satisfied herewith, however, but must follow up the transfer of force actions in vacuum, where there are no charges. This was Faraday's intimation when he spoke of lines of force as of elastic bands which transmit tension and compression. Maxwell was also here able to place Faraday's notions into clear mathematical focus. This was the origin of Maxwell's *stress tensor*, which may be expanded relativistically into a *stress-energy tensor*.

We proceed from the Lorentz force density in Eq. (28.17),

$$\mathbf{k} = \frac{1}{c} \Gamma \cdot \mathbf{F}, \quad ck_n = \sum_{r=1}^4 \Gamma_r F_{nr}, \quad (1)$$

and replace Γ , in accord with Maxwell's equations (26.16), by the six-vector of the excitation f . We then obtain from (1)

$$ck_n = \sum_{r=1}^4 \text{Div}_r f \cdot F_{nr} = \sum_{r=1}^4 \sum_{m=1}^4 \frac{\partial f_{rm}}{\partial x_m} F_{nr}. \quad (2)$$

We will show that \mathbf{k} may be expressed as the four-dimensional divergence of a tensor T , i.e. that

$$k_n = \sum_{m=1}^4 \frac{\partial}{\partial x_m} T_{nm} \quad (3)$$

and

$$T_{nm} = -\frac{1}{c} \sum_{r=1}^4 F_{nr} f_{mr} + \delta_{nm} \Lambda, \quad (4)$$

where Λ denotes the Lagrange density in (26.24).

Since we here enter the domain of tensor quantities the following rather abstract computations with double indices cannot be avoided.

To transform the right side of (2) we utilize the identity

$$\frac{\partial f_{rm}}{\partial x_m} F_{nr} = \frac{\partial}{\partial x_m} (f_{rm} F_{nr}) - f_{rm} \frac{\partial F_{nr}}{\partial x_m}. \quad (5)$$

Change of the order of summation yields for the first term on the right of (5), summed as indicated in (2),

$$\sum_m \frac{\partial}{\partial x_m} \sum_r f_{rm} F_{nr} = - \sum_m \frac{\partial}{\partial x_m} \sum_r f_{mr} F_{nr}. \quad (6)$$

The second term on the right side of (5) becomes, after carrying out the summation over r and m (including the negative sign)

$$\sum_r \sum_m f_{mr} \frac{\partial F_{nr}}{\partial x_m}.$$

We write this expression once more, reversing both the symbols for the summation subscripts r, m and the sequence of the subscripts of f and F :

$$\sum_r \sum_m f_{mr} \frac{\partial F_{mn}}{\partial x_r}$$

and form half the sum of these equal expressions:

$$\frac{1}{2} \sum_r \sum_m f_{mr} \left(\frac{\partial F_{mn}}{\partial x_r} + \frac{\partial F_{nr}}{\partial x_m} \right). \quad (7)$$

Now we make use of Maxwell's Eq. (26.18). According to it (the three terms are formed by the cyclic interchange of the subscripts m, n, r)

$$\frac{\partial F_{mn}}{\partial x_r} + \frac{\partial F_{nr}}{\partial x_m} + \frac{\partial F_{rm}}{\partial x_n} = 0.$$

Taking care of the negative sign by changing the sequence of subscripts of F , we then can write instead of (7)

$$\frac{1}{2} \sum_r \sum_m f_{mr} \frac{\partial F_{mr}}{\partial x_n}. \quad (7a)$$

This is the result of the summation of the second term on the right side of (5), whereas that for the first term was given by (6). Hence we obtain, finally, from (2), (6), and (7a),

$$ck_n = - \sum_m \frac{\partial}{\partial x_m} \sum_r f_{mr} F_{nr} + \frac{1}{2} \sum_r \sum_m f_{mr} \frac{\partial F_{mr}}{\partial x_n}. \quad (8)$$

Here the first term is already identical with the first half of the representation of the tensor T in (3) and (4). To prove fully the correctness of the representation we must still demonstrate that the second term on the right of (8) is equal to

$$c \sum_{m=1}^4 \delta_{nm} \frac{\partial \Lambda}{\partial x_m} = c \frac{\partial \Lambda}{\partial x_n}.$$

By (26.24) this is actually the case. The statement (3) is thus proved.

Our expression (4) for T represents a *symmetric tensor of the second rank*. Its symmetry follows directly from the proportionality of f and F and from the meaning of δ_{nm} ; the tensor character in the sense of p. 244 follows from the behavior of the six-vectors f and F in a Lorentz transformation. This calculation, overloaded with indices and formal as it may seem, leads to far-reaching physical consequences.

From (4) we compute the components of T individually, beginning with the diagonal terms of the matrix, all of which have the term with Λ in common. We find

$$\begin{aligned}
T_{11} &= -\frac{1}{c} \{f_{12}F_{12} + f_{13}F_{13} + f_{14}F_{14}\} + \Lambda \\
&= -H_x B_x - H_y B_y + D_z E_z + \frac{1}{2} \mathbf{H} \cdot \mathbf{B} - \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \\
&= -\mathbf{H} \cdot \mathbf{B} + H_x B_x + D_z E_z + \frac{1}{2} \mathbf{H} \cdot \mathbf{B} - \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \\
&= H_x B_x + D_z E_z - W,
\end{aligned}$$

where W denotes the energy density. Similarly,

$$\begin{aligned}
T_{22} &= H_y B_y + D_y E_y - W, \\
T_{33} &= H_z B_z + D_z E_z - W.
\end{aligned}$$

On the other hand

$$\begin{aligned}
T_{44} &= -\frac{1}{c} \{f_{41}F_{41} + f_{42}F_{42} + f_{43}F_{43}\} + \Lambda \\
&= D_x E_x + D_y E_y + D_z E_z + \frac{1}{2} \mathbf{H} \cdot \mathbf{B} - \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \\
&= \frac{1}{2} \mathbf{D} \cdot \mathbf{E} + \frac{1}{2} \mathbf{H} \cdot \mathbf{B} = W.
\end{aligned}$$

We now turn to the nondiagonal elements, beginning with those having the subscript 4, such as

$$\begin{aligned}
T_{14} = T_{41} &= -\frac{1}{c} (f_{42}F_{12} + f_{43}F_{13}) \\
&= ic(D_y B_x - D_x B_y) = -ic\epsilon_0\mu_0(\mathbf{E} \times \mathbf{H})_z = -\frac{i}{c} S_z.
\end{aligned}$$

Similarly:

$$T_{24} = T_{42} = -\frac{i}{c} S_y, \quad T_{34} = T_{43} = -\frac{i}{c} S_x.$$

The remaining nondiagonal elements are

$$T_{12} = T_{21} = -\frac{1}{c} (f_{13}F_{23} + f_{14}F_{24}) = \begin{cases} H_y B_x + D_x E_y = H_y B_x + D_y E_z, \\ H_x B_y + D_y E_x = H_x B_y + D_z E_z, \end{cases}$$

where the equality of the last four expressions again follows from the proportionality of \mathbf{D} and \mathbf{E} , and similarly

$$T_{13} = T_{31} = \begin{cases} H_x B_z + D_z E_x, \\ H_z B_x + D_x E_z, \end{cases} \quad T_{23} = T_{32} = \begin{cases} H_y B_z + D_z E_y, \\ H_z B_y + D_y E_z. \end{cases}$$

Thus the complete T matrix becomes, in abbreviated notation

$$T = \left(\begin{array}{c|c} \sigma & -\frac{i}{c} \mathbf{S} \\ \hline -\frac{i}{c} \mathbf{S} & W \end{array} \right) \quad (9)$$

Here σ is the *three-dimensional* matrix of the so-called Maxwell stresses:

$$\begin{aligned} H_x B_x + D_x E_x - W, & \quad H_y B_x + D_y E_x, & \quad H_z B_x + D_z E_x, \\ H_x B_y + D_x E_y, & \quad H_y B_y + D_y E_y - W, & \quad H_z B_y + D_z E_y, \\ H_x B_z + D_x E_z, & \quad H_y B_z + D_y E_z, & \quad H_z B_z + D_z E_z - W. \end{aligned} \quad (10)$$

The electrical portion of σ represents a tension of the magnitude W in the direction of the lines of force and a compression of the same magnitude in the directions perpendicular thereto. This is seen immediately if the x -axis is placed in the direction of \mathbf{E} and \mathbf{B} is put equal to 0. Then

$$\sigma_{xx} = \frac{1}{2} \mathbf{D} \cdot \mathbf{E}, \quad \sigma_{yy} = \sigma_{zz} = -\frac{1}{2} \mathbf{D} \cdot \mathbf{E}, \quad \sigma_{ik} = 0.$$

The same may be shown for the magnetic lines of force if the x -axis is placed in their direction and \mathbf{E} is put equal to 0. We have thus returned to the model which Faraday had constructed purely on the basis of intuition.

This stress tensor σ , by itself, is however not a legitimate physical quantity in the sense of the theory of relativity. It becomes one only by its extension with the energy quantities \mathbf{S} and W , forming the "stress-energy tensor" T . This has the characteristic property that its "trace" (sum of all the four terms on the principal diagonal) vanishes. We have in fact:

$$T_{11} + T_{22} + T_{33} + T_{44} = \mathbf{H} \cdot \mathbf{B} + \mathbf{D} \cdot \mathbf{E} - 3W + W = 0.$$

We now return to the relationship between T and the Lorentz force density \mathbf{k} as given by (3), and consider first the fourth line of (3). In view of (9) it is

$$k_4 = -\frac{i}{c} \operatorname{div} \mathbf{S} + \frac{\partial W}{\partial x_4}.$$

If we replace k_4 by its value $i\rho L/c$ from (28.17c), we find, since $x_4 = ict$,

$$\frac{\partial W}{\partial t} + \operatorname{div} \mathbf{S} + \rho L = 0, \quad L = \mathbf{v} \cdot \mathbf{E}. \quad (11)$$

This is Poynting's theorem, Eq. (5.7), where the former energy loss by Joule heat is replaced by the work done on the moving charge ρ .

Consider now one of the space components of Eq. (3), e.g. the first line:

$$k_1 = k_x = \operatorname{div}_x \sigma - \frac{i}{c} \frac{\partial S_x}{\partial x_4}$$

which, written out in detail, becomes

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} - \frac{1}{c^2} \frac{\partial S_x}{\partial t} = k_x. \quad (12)$$

If we omit the last term on the left, i.e. confine ourselves to a stationary state, we obtain the characteristic equation (8.11) of elastic equilibrium in Vol. II. Just as there the volume force F_x is absorbed and balanced by the stresses σ_{ik} , insofar as they point in the x -direction. The Lorentz force density may be completely replaced by these stresses in our case. They are defined throughout the field by the tensor array (10), even where, in view of the absence of charge density, the Lorentz force is nonexistent. We have thus attained the goal set at the beginning of this section, of following up the transmission of the force through vacuum (without the use of a test body).

However, what do we know of the nonstationary state and the term with $\partial S/\partial t$ which is then added in (12)? The answer is given by Eq. 14.1 in Vol. II, where the corresponding term, there designated by $-\rho \partial^2 s/\partial t^2$, represented the *inertial resistance* of unit volume of the elastic body or, with positive sign, its *change in momentum*. We learn from this that there exists a momentum per unit volume G also in the electromagnetic field, and that it is to be defined, in direction and magnitude, by

$$G = \frac{1}{c^2} S. \quad (13)$$

We already know that the electromagnetic field possesses *energy* and how this is to be localized in space. We now see that we must also attribute to the field *momentum*, continuously distributed through space wherever there is an energy flux S and of the same direction with the latter.

Correspondingly a *light wave* carries momentum and exerts a pressure on a nonreflecting (black) body on which it is incident—the *light pressure* discovered by Maxwell. Similarly, if a light wave is emitted by a body, it imparts to it a recoil which is equal and opposite to the momentum carried by it. We call the latter body the “transmitter,” the former the “receiver,” and assume that both were at rest for $t < 0$. At $t = 0$, when the wave is emitted by the transmitter, the latter receives a recoil. The center of gravity of the transmitter and receiver is then set into motion and remains in motion for the duration $0 < t < T$. At the time $t = T$ the light wave is absorbed by the receiver (without reflection, as we shall assume for the sake of brevity). The wave then imparts to the receiver an equal impulse forward. From this point on the center of gravity of the two bodies is once more at rest, though it has been displaced a certain distance during the interim T , corresponding to the backward motion of the transmitter. This contradiction with the law governing the center of gravity vanishes only if we assign a momentum to the light wave itself during its lifetime T . Then momentum is neither created nor destroyed, both in emission and in absorption, and the center of gravity remains *permanently* at rest.

As is well known, it is difficult to demonstrate the pressure of light in

the laboratory. The radiometers constructed for this purpose indicate generally convection currents of residual gases, caused by the thermal effect of the radiation. The proof of the pressure of light in the heavens is much grander. The tails of the comets, pointed away from the sun, show it (Lebedew), also the solar corona, where luminous particles are balanced by the pressure of light or radiation at a height equal to as much as the radius of the sun. The inner constitution of the sun and the bright fixed stars generally is also controlled by the common action of the pressure of radiation and the thermodynamic gas pressure (Karl Schwarzschild, †1916, for the surface of the sun; quite generally, A. S. Eddington, †1944). We must here content ourselves with pointing out some general relationships between momentum, energy, and light pressure.

From the definition of \mathbf{S} we obtain for a transversal plane wave ($\mathbf{H} \perp \mathbf{E}$, $|\mathbf{H}| = \sqrt{\frac{\epsilon_0}{\mu_0}} |\mathbf{E}|$):

$$|\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = |\mathbf{E}| |\mathbf{H}| = \begin{cases} \sqrt{\frac{\epsilon_0}{\mu_0}} E^2 = \epsilon_0 c E^2 \\ \sqrt{\frac{\mu_0}{\epsilon_0}} H^2 = \mu_0 c H^2 \end{cases} = cW.$$

From this follows, in view of the definition (13) of \mathbf{G}

$$|\mathbf{G}| = \frac{W}{c}. \quad (14)$$

The momentum incident on a screen is hence equal in absolute magnitude, but for the factor $1/c$, to the *energy density in front of the screen* (this applies not only for vacuum, but for any non-absorbing medium).

We consider a bundle of parallel rays, a "wave packet", of length l and cross section q . Let W be the energy contained in it, G the momentum contained in it:

$$W = qlW, \quad G = qlG = ql \frac{W}{c} = \frac{W}{c}; \quad (15)$$

the last follows from (14). We speak of a "photon" or a "light quantum" if the energy W of the bundle is equal to $h\nu$ (h = Planck's constant, ν = number of vibrations per second). By (15) the momentum of this bundle is

$$G = \frac{h\nu}{c}. \quad (15a)$$

In the theory of light quanta the light pressure is thus identified with a "hail of photons", to which every photon contributes the quantity $h\nu/c$.

We test this statement once again with the aid of the representation (10) of the stress tensor σ . Let the light wave, assumed plane, be incident perpendicularly, in the positive x -direction, on a plate. Because of the transversal nature of light E_x , D_x , B_x , and H_x are zero and the first row of (10) reduces to

$$\sigma_{xx} = -W, \quad \sigma_{xy} = \sigma_{xz} = 0.$$

If we are dealing with the light bundle described by (15), the force

$$-\sigma_{xx}q = Wq$$

acts on the plate during the time $T = l/c$; its time integral yields the impulse imparted to the plate. We calculate:

$$G = \int_0^T Wq dt = \frac{1}{c} \int_0^l Wq dl = \frac{Wql}{c} = \frac{W}{c},$$

which agrees with (15).

If the light wave is not incident perpendicularly on the plate, but at an angle α with respect to the normal, the quadratic character of the coefficients in the tensor transformation formula (28.20) leads to

$$\sigma_{xx} = W \cos^2 \alpha.$$

This dependence on angle is reasonable since the area bombarded by the light pencil is now $q/\cos \alpha$ and only the component in the x -direction of the momentum of the light rays is effective as light pressure.

§32. Relativistic Mechanics

Unlike electrodynamics, which fits the requirements of the theory of relativity from the very start so that we could actually base this theory on it, classical mechanics must undergo fundamental revision to harmonize it with the theory of relativity. This revision even affects, for the individual particle, the definition of its *momentum* (its "quantitas motus") as a *four-vector*. In agreement with Vol. I, §2 we assume it to be proportional to the four-vector V of the velocity in Eq. (27.18) and call it again G :

$$G = m_0 V, \quad V = \frac{dx_1}{d\tau}, \quad \frac{dx_2}{d\tau}, \quad \frac{dx_3}{d\tau}, \quad \frac{dx_4}{d\tau}. \quad (1)$$

The coefficient m_0 is the *rest mass* of the particle, $d\tau = \sqrt{1 - \beta^2} dt$ is the differential of the intrinsic time. We can also write in place of (1)

$$G = \frac{m_0}{\sqrt{1 - \beta^2}} (\mathbf{v}, ic), \quad \mathbf{v} = \frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt}. \quad (1a)$$

The quantity

$$m = \frac{m_0}{\sqrt{1 - \beta^2}} \quad (2)$$

is called the *mass in motion*. It is not constant, as in classical mechanics, but increases for $\beta \rightarrow 1$, $v \rightarrow c$ toward infinity; accordingly it depends on the frame of reference and is hence not a legitimate world entity. This applies not only for the electron, but for every mass—though a large mass cannot be accelerated to velocities close to c in the same manner as an electron. (However, in cosmic rays with their tremendous energies the variation of mass finds expression also for the heavy and semi-heavy particles, the protons and mesons.)

The law of inertia, Newton's *first law*, now becomes in relativistic formulation

$$G = \text{const.} \quad (3)$$

Correspondingly the *second law* may be written as four-dimensional vector equation as follows:

$$\frac{dG}{d\tau} = F. \quad (4)$$

Here F is the external force, extended to a four-vector. We know that for the individual electron the Lorentz force density \mathbf{k} is such a four-vector, but not the Lorentz force \mathbf{K} itself. The latter becomes one only after it has been divided by $\sqrt{1 - \beta^2}$ and has thus been placed into the invariant relationship with \mathbf{k} which is expressed by Eqs. (28.19) and (28.19a). This leads to the following definition of the four-vector F for the individual electron in terms of \mathbf{K} :

$$F_{1,2,3} = \frac{\mathbf{K}}{\sqrt{1 - \beta^2}} = \frac{e(\mathbf{E} + \mathbf{v} \times \mathbf{B})}{\sqrt{1 - \beta^2}}, \quad F_4 = \frac{ei}{c\sqrt{1 - \beta^2}} \mathbf{v} \cdot \mathbf{E}. \quad (4a)$$

When this is substituted in (4) the factor $\sqrt{1 - \beta^2}$ on the right cancels the factor $\sqrt{1 - \beta^2}$ contained in $d\tau$. We thus obtain instead of the first three components of (4)

$$\frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt} \frac{m_0 \mathbf{v}}{\sqrt{1 - \beta^2}} = \mathbf{K}. \quad (5)$$

This is the equation of motion (4.6) in Vol. I. As was first pointed out by Planck,¹ the *Lorentz force* \mathbf{K} here takes the place of the classical *Newtonian force*. For the sake of distinction the four-force F in (4) is designated as *Minkowski force*.

As was already shown in Vol. I, §4, (5) becomes, for *longitudinal* and *transversal* direction of the force ($\mathbf{K} \parallel \mathbf{v}$ and $\mathbf{K} \perp \mathbf{v}$),

$$\frac{m_0}{(1 - \beta^2)^{3/2}} \frac{d\mathbf{v}}{dt} = \mathbf{K} \quad \text{and} \quad \frac{m_0}{(1 - \beta^2)^{1/2}} \frac{d\mathbf{v}}{dt} = \mathbf{K}, \quad \text{respectively.}$$

¹ Verhandl. d. deutsch. phys. Ges. 4, p. 136, 1906.

The designations *longitudinal* and *transversal* mass for $m_0(1 - \beta^2)^{-3/2}$ and $m_0(1 - \beta^2)^{-1/2}$ were discussed and criticized at the same place in Vol. I.

We supplement Eq. (5) by the fourth energetic component, which follows from (4) and (4a):

$$\frac{d}{dt} \frac{m_0 c^2}{\sqrt{1 - \beta^2}} = e\mathbf{v} \cdot \mathbf{E} = \mathbf{v} \cdot \mathbf{K}. \quad (6)$$

The equality of $e\mathbf{v} \cdot \mathbf{E}$ and $\mathbf{v} \cdot \mathbf{K}$ postulated here follows in the electrodynamic case simply from the fact that $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$. Applied to an arbitrary force law it signifies that the four-force \mathbf{F} must be *perpendicular* to the world line of the particle (see p. 243).

$\mathbf{v} \cdot \mathbf{K}$ is the work done on the moving particle by the force \mathbf{K} in unit time, i.e. it is equal to dA/dt . Accordingly the left side of Eq. (6) is simply the change in the kinetic energy T effected by the force \mathbf{K} . We hence have

$$T = \frac{m_0 c^2}{\sqrt{1 - \beta^2}} + \text{const.} \quad (6a)$$

In Exercise III.4 we shall convince ourselves of the fact that Eq. (6) may be derived from the equation of motion (5) also by the formalism customary in the derivation of the energy theorem in elementary mechanics, namely scalar multiplication with \mathbf{v} . Since, by definition, T must vanish for $v \rightarrow 0$, the constant in (6a) must be put equal to $-m_0 c^2$. Hence, in view of (2),

$$T = m_0 c^2 \left(\frac{1}{\sqrt{1 - \beta^2}} - 1 \right) = (m - m_0) c^2. \quad (7)$$

The classical expression $T = mv^2/2$ follows from this by passing to the limit $c \rightarrow \infty$, as was already noted at the end of §4 in Vol. I.

A. The Equivalence of Energy and Mass

Just as we considered the rest mass m_0 apart from the mass in motion m , we introduce apart from the *energy in motion* E the *rest energy* E_0 , where then $T = E - E_0$. We can hence write in place of (7)

$$E - E_0 = (m - m_0) c^2. \quad (7a)$$

We render this equation more specific by the statement

$$E = mc^2 \quad (8)$$

and the consequent relation

$$E_0 = m_0 c^2. \quad (8a)$$

This is the theorem of the *inertia of energy*, which according to Einstein is the most important result of the (special) theory of relativity. We

quote Einstein literally: "The mass of a body is a measure of its energy content; if the energy changes by ΔE , the mass changes in the same direction by $\Delta E/c^2$. It is not out of question that for bodies whose energy content is variable in a high degree (e.g. for radium salts) a test of the theory may be successful."¹

This test has since been carried out on a huge scale: The atomic transformations which have been discovered in the meantime and been studied in detail for most of the light elements have led, by the use of the equivalence theorem, to an undreamed of increase in the precision of the chemical atomic weights² and the fission of the heaviest element, uranium—more precisely, the uranium isotope of atomic weight 235—which was discovered by Otto Hahn only toward the end of 1938 has, in accord with the loss of mass occurring in it, had a terrifying effect in the *destruction caused by the uranium bomb*. We shall concern ourselves here only with the second example and this only briefly and superficially.

The uranium 235 atom, after capture of a neutron (atomic weight 1) has assumed the atomic weight $M = 236$, but retained the atomic number $Z = 92$ of the original uranium atom. It may, e.g., split into krypton, $Z = 36$, and barium, $Z = 56$, or into xenon, $Z = 54$, and strontium, $Z = 38$. Both fission possibilities are observed. The conservation of the nuclear charge eZ is here assured, since

$$92 = 36 + 56 = 54 + 38.$$

However, the mass is not conserved. Instead, the mass excess of the atomic weight over the integer 235 (the so-called "packing fraction") is set free, i.e. transformed into energy. If we assume that it amounts to one unit in the first decimal (for the heavier isotopes the atomic weights are not yet precisely known), we obtain for the energy available from one gram-atom

$$0.1 c^2 = 9 \cdot 10^{10} \text{ g} \cdot \text{cm}^2 \cdot \text{sec}^{-2} = 9 \cdot 10^{12} \text{ joules}.$$

Computed for a kilogram of fissioned uranium 235 it is 1000/235 times as much, or $38 \cdot 10^{12}$ joules. We transform this into heat units (one large calorie $\cong 4.2 \cdot 10^3$ joules) and obtain

$$\frac{38}{4.2 \cdot 10^3} \cdot 10^{12} \text{ cal} \cong 10^{10} \text{ cal}.$$

¹ A. Einstein, "Does the inertia of a body depend on its energy content?", Ann. Physik, Vol. 17, 1905. Einstein here explains the specialization of Eq. (7a) to the equivalence theorem (8) by an imaginary experiment: a moving body emits radiation and is observed from a system at rest.

² H. Betha, Phys. Rev. 47, 633, 1935; Oliphant, Kempton, and Rutherford, Proc. Roy. Soc. London 149, 406, 1935. The almost simultaneous publication of these two papers on the two sides of the Atlantic shows once more the inevitable course of development of the understanding of physics as prescribed by the experimental material available at the time.

If we note that the energy transfers of ordinary molecular processes lie in the range from 100 to 1000 calories, we see that our uranium process supplies many million times as much energy. On this basis we may understand both the terrible effect of the *uranium bomb* and the beneficial effect of the *uranium engine*, i.e. a controllable, continuously operating uranium process, which could remove all economic ills of the times. The fact that the practical realization of the uranium process differs from that here considered, i.e. that it is carried out by way of a transuranium element (plutonium), does not require mention. The validity of the proof indicated by our simplified process is not affected thereby.

B. Relationship between Momentum and Energy

In classical mechanics the components of momentum are *derivatives of the kinetic energy* with respect to the velocity components, e.g. for an individual particle in Cartesian coordinates:

$$G_k = m\dot{x}_k = \frac{\partial T}{\partial \dot{x}_k}, \quad T = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \quad \text{with } m = \text{const.} \quad (9)$$

This no longer applies in relativistic mechanics. It may readily be verified however that the relativistic momentum components (1a) are *derivatives of the following quantity*:

$$K = -m_0 c^2 \sqrt{1 - \beta^2} + \text{const.} \quad (9a)$$

with respect to the \dot{x}_k . Following Helmholtz,¹ a function which accomplishes this is called a "kinetic potential". If, again, K is normalized so that it vanishes for $\beta = 0$, the constant must be chosen equal to $m_0 c^2$, yielding

$$K = m_0 c^2 (1 - \sqrt{1 - \beta^2}). \quad (9b)$$

Hence the definition of the momentum of an individual point mass replacing (9) becomes

$$G_k = \frac{\partial K}{\partial \dot{x}_k} = \frac{m_0 \dot{x}_k}{\sqrt{1 - \beta^2}} \quad (10)$$

in agreement with the definition (1a). For $c \rightarrow \infty$ K evidently passes over into T and (10) into (9).

C. The Principles of D'Alembert and Hamilton

What are the consequences of this changed meaning (10) of the momentum coordinates for the general principles of mechanics? We shall first discuss *D'Alembert's principle*. The inertial reaction forces introduced by D'Alembert (see Vol. I, Eq. (10.1)) are also now given by $-G_k$. (The

¹ In his general studies on the principle of least action.

usual definition as mass \times acceleration is of course now invalid). The statements of D'Alembert's principle in Vol. I, §10 then continue to apply literally: "The inertial reaction forces balance themselves against the physically impressed forces" (Vol. I, p. 57). "The sumtotal of the lost forces is in equilibrium on the system." (Vol. I, p. 58). The condition on p. 49 of Vol. I serves as definition of the word "mechanical system:" "The virtual work of the reactions within the system is equal to zero."

Hamilton's principle is derived from D'Alembert's principle in the manner of §33 of Vol. I. Here T is to be replaced by K and, for forces possessing a potential, (33.12) of Vol. I is replaced by

$$\delta \int_{t_0}^{t_1} (K - V) dt = 0. \quad (11)$$

Here the variation is to be carried out as in the earlier example: The space coordinates are varied, whereas the endpoints of the path and the time for its transversal remain fixed.

Application to a single point mass yields

$$\delta \int_{t_0}^{t_1} K dt = \delta \int_{t_0}^{t_1} m_0 c^2 (1 - \sqrt{1 - \beta^2}) dt = -\delta \int_{t_0}^{t_1} m_0 c^2 \sqrt{1 - \beta^2} dt. \quad (11a)$$

Here we have already taken account of the fact that the times t_0 and t_1 are not to be varied, i.e. that $\delta(t_1 - t_0) = 0$. Hence (11) becomes

$$\delta \int_{t_0}^{t_1} (m_0 c^2 \sqrt{1 - \beta^2} + V) dt = 0. \quad (12)$$

We can readily convince ourselves that this variational prescription agrees with our equation of motion (5), and this not only for the Lorentz force \mathbf{K} , to which (5) was limited, but for any given potential energy and an arbitrary force $\mathbf{K} = -\text{grad } V$ derived from it.

In the variation we must replace x, y, z by $x + \delta x, y + \delta y, z + \delta z$, obtaining

$$\delta \frac{dx}{dt} = \frac{d\delta x}{dt}, \dots, \quad \delta V = \frac{\partial V}{\partial x} \delta x + \dots = -K_x \delta x - \dots,$$

where the \dots represent corresponding expressions in y and z . Similarly we must form

$$\begin{aligned} \delta \sqrt{1 - \beta^2} &= \delta \sqrt{1 - \frac{1}{c^2} \left\{ \left[\frac{dx}{dt} \right]^2 + \left[\frac{dy}{dt} \right]^2 + \left[\frac{dz}{dt} \right]^2 \right\}} \\ &= \frac{1}{2 \sqrt{1 - \beta^2}} \left[-\frac{2}{c^2} \right] \left[\frac{dx}{dt} \frac{d\delta x}{dt} + \dots \right] \end{aligned}$$

and hence

$$\delta \int_{t_0}^{t_1} m_0 c^2 \sqrt{1 - \beta^2} dt = \int_{t_0}^{t_1} \left\{ -\frac{m_0}{\sqrt{1 - \beta^2}} \frac{dx}{dt} \frac{d\delta x}{dt} + \dots \right\} dt. \quad (12a)$$

An integration by parts, in which the terms without integral sign vanish (because $\delta x = 0$ for $t = t_0$ and $t = t_1$) according to our original assumption, transforms this into

$$\int_{t_0}^{t_1} \left\{ \frac{d}{dt} \left[\frac{m_0}{\sqrt{1-\beta^2}} \frac{dx}{dt} \right] \delta x + \dots \right\} dt. \quad (12b)$$

Thus (12) yields altogether

$$\int_{t_0}^{t_1} \left\{ \left[\frac{d}{dt} \left(\frac{m_0}{\sqrt{1-\beta^2}} \frac{dx}{dt} \right) - K_x \right] \delta x + \dots \right\} dt = 0. \quad (12c)$$

Since δx , δy , and δz are independent of each other the factor of δx must vanish, as well as those of δy and δz . We thus obtain in fact our earlier equation of motion (5) for arbitrary \mathbf{K} ; it is valid, incidentally, even if \mathbf{K} cannot be derived from a potential energy.

If there are no external forces ($V = \text{const}$) (12) may be abbreviated to

$$\delta \int_{t_0}^{t_1} \sqrt{1-\beta^2} dt = \delta \int_{\tau_0}^{\tau_1} d\tau = 0. \quad (13)$$

This is *Fermat's principle of least time*, which now however does not relate to the conventional time t , but to the Lorentz-invariant intrinsic time. Since $d\tau$ corresponds to the four-dimensional line element ds but for the factor ic , we can also write in place of (13)

$$\delta \int_A^E ds = 0. \quad (13a)$$

This is the *principle of the shortest path* for given starting point A and end point E , or, as we called it in Vol. I, Eq. (37.14), the *principle of the geodesic line*, extended to four dimensions and made Lorentz-invariant. We shall therefore call it more precisely the *principle of the shortest world line*.

D. Lagrange Function and Lagrange Equations

In our formulation (11) of Hamilton's principle the relativistic Lagrange function

$$L_{\text{rel}} = K - V \quad (14)$$

replaces the classical Lagrange function $L_{\text{cl}} = T - V$ in Vol. I, Eq. (33.13). The *general Lagrange equations* for arbitrary position and velocity coordinates are derived from L_{rel} by carrying out the variation prescribed in (11), just as they are derived from L_{cl} in §34 of Vol. I:

$$\frac{d}{dt} \frac{\partial L_{\text{rel}}}{\partial \dot{q}_k} - \frac{\partial L_{\text{rel}}}{\partial q_k} = 0. \quad (14a)$$

In spite of their similarity with those of classical mechanics, these equations, when applied to specific cases, yield results which differ decidedly from those of the latter. For example, in the Kepler problem of the hydrogen atom they lead to an ellipse with precessing perihelion instead of to a closed ellipse as a consequence of the relativistic variation of mass; see also §38 regarding the perihelion of mercury.

E. Schwarzschild's Principle of Least Action

In his fundamental papers "On Electrodynamics" Schwarzschild¹ introduced with the designation "electrokinetic potential" the quantity

$$L = \Psi - \mathbf{v} \cdot \mathbf{A}. \quad (15)$$

We shall show that when multiplied with the charge density ρ this is a relativistic invariant. To this end we form the scalar product of the four-vector Γ of current density (Eq. (28.16)) and the four-potential Ω (Eq. (26.4)). We obtain

$$\Gamma \cdot \Omega = \rho(\mathbf{v} \cdot \mathbf{A} - \Psi).$$

We call $-\Gamma \cdot \Omega$ the *Schwarzschild invariant*. In view of (15)

$$\Gamma \cdot \Omega = -\rho L. \quad (16)$$

Schwarzschild adds to this invariant the Lagrange density Λ from (26.24) which, we know, is also Lorentz-invariant, and forms, with T = kinetic energy,

$$T - \Lambda - \rho L. \quad (17)$$

We shall replace (17) by

$$K' = T' - 2\Lambda - \rho L = T' - 2\Lambda + \Gamma \cdot \Omega. \quad (17a)$$

Here T' represents the relativistic value of the kinetic energy from (7), where, however, the rest mass m_0 is to be replaced by the rest-mass density μ_0 . (Also the remaining terms in (17a) are densities, referring to unit volume.) Hence

$$T' = \mu_0 c^2 ((1 - \beta^2)^{-1/2} - 1). \quad (17b)$$

The factor 2 of Λ in (17a), on the other hand, derives from our basic distinction between the entities of quantity f and of intensity F ; Schwarzschild, who puts $\mathbf{D} = \mathbf{E}$ and $\mathbf{B} = \mathbf{H}$ and hence writes in our Eq. (26.24)

¹ K. Schwarzschild, Göttinger Nachr. 1903. See in particular the first of the three papers. The notation L is the same as Schwarzschild's; Schwarzschild uses ϕ in place of our Ψ . Note the date of publication 1903! Thus Schwarzschild arrived intuitively at the correct postulate of the theory of invariants six years ahead of Minkowski.

$H^2 - E^2$ instead of $H \cdot B - D \cdot E$, gains a factor 2 in the variation, which we must supply in (17a). In detail our formula (17a) becomes

$$K' = \mu_0 c^2 ((1 - \beta^2)^{-1} - 1) - 2\Lambda + \Gamma \cdot \Omega. \quad (18)$$

From this point on we follow Schwarzschild's procedure. He integrates (17) over an arbitrary region of space-time and constructs in this manner an action function W , which he subjects to the requirement $\delta W = 0$. We form correspondingly

$$W = \iiint K' dx dy dz dt \quad (19)$$

and also set

$$\delta W = 0. \quad (19a)$$

According to Schwarzschild this variation is to be carried out in following fashion:

a. The components $\Omega_1, \Omega_2, \Omega_3$, and Ω_4 of the potential and
 b. the coordinates x_1, x_2, x_3 , and x_4 of the electrons are subjected to arbitrary small variations; these variations are to vanish on the boundaries of the region. The variations a and b are independent of each other¹ and may be carried out individually, e.g. also for each component of Ω . If several electrons are present we can limit ourselves to one of them since the effects of the rest on it are contained in the potential Ω . The fact that we use the four-potential Ω , originally introduced for convenience of calculation, rather than the six-vector F as fundamental field quantity represents a new departure, to which we shall return in §37.

a. Since the first term on the right of Eq. (18) is independent of Ω we are only concerned with Λ and $\Gamma \cdot \Omega$. If the variation is limited to $\delta\Omega_1$ we find

$$\delta(\Gamma \cdot \Omega) = \Gamma_1 \delta\Omega_1. \quad (20)$$

In the expression (26.24) for Λ we must imagine F as expressed (by (26.11)) by $c \text{Curl } \Omega$, whereas f is to be regarded as an unknown. Hence, for the specific variation mentioned above, (-2Λ) reduces to the following three terms (the remaining terms of the Curl, to be formed with Ω_2, Ω_3 , and Ω_4 , drop out):

$$\delta(-2\Lambda) = f_{12} \frac{\partial \delta\Omega_1}{\partial x_2} + f_{13} \frac{\partial \delta\Omega_1}{\partial x_3} + f_{14} \frac{\partial \delta\Omega_1}{\partial x_4} = - \left[\frac{\partial f_{12}}{\partial x_2} + \frac{\partial f_{13}}{\partial x_3} + \frac{\partial f_{14}}{\partial x_4} \right] \delta\Omega_1 + \dots$$

¹ Schwarzschild does not take account of the auxiliary condition $\text{Div } \Omega = 0$ which was satisfied automatically in our method of integration in §29. We adhere to Schwarzschild's prescription also in this respect.

The dots refer to partial derivatives with respect to the coordinates which vanish in the later integration over our world region (since $\delta\Omega = 0$ on its boundary). Together with (20) we thus find for the factor of $\delta\Omega_1$ in the integrand of (19)

$$-\left[\frac{\partial f_{12}}{\partial x_2} + \frac{\partial f_{13}}{\partial x_3} + \frac{\partial f_{14}}{\partial x_4}\right] + \Gamma_1.$$

It must vanish since $\delta W = 0$. If we substitute for f_{ik} the values in the array (26.14) and for Γ its value from (28.16) we find

$$-\frac{\partial H_x}{\partial y} + \frac{\partial H_y}{\partial z} + \frac{\partial D_z}{\partial t} + \rho v_x = 0. \quad (21)$$

This is exactly the first component of the three Maxwell equations $\dot{\mathbf{D}} + \mathbf{J} = \text{curl } \mathbf{H}$, where here, for vacuum, the convection current density $\rho\mathbf{v}$ represents \mathbf{J} . The second and third component are evidently obtained similarly by the variation of Ω_2 and Ω_3 and the condition $\text{div } \mathbf{D} = \rho$ from that of Ω_4 . We cannot, of course, expect to derive the other set of Maxwell equations and the condition $\text{div } \mathbf{B} = 0$ in the same manner, since these are already implicit in the existence of the potential.

This clarifies also the basis for our earlier name "Lagrange density" for Λ (see p. 220): In our electrodynamical variation principle Λ takes the place of the earlier Lagrange function L or L_{rel} (Eq. (32.14)).

b. The potential Ω is not varied; hence $\delta\Lambda = 0$.

On the other hand, the world line of the electron is to be compared with neighboring world lines, so that the first term on the right of (18) and the Schwarzschild invariant $\Gamma \cdot \Omega$ are to be varied. We shall first deal with the term $\Gamma \cdot \Omega$. It is here convenient to replace the world volume element $dx dy dz dt = dV dt$ in (19) by $dV_\perp d\tau$, where dV_\perp represents its three-dimensional cross section perpendicular to the world line. Since the charge density ρ occurring in Γ is concentrated on the world line of the electron we obtain in the integration over dV_\perp (not in that over dV !) the electron charge e .

At the same time, in the expressions for Γ and Ω , we pass from the coordinates $x_1 \cdots x_4$ used so far to the coordinates $\xi_1 \cdots \xi_4$ of the world line element considered at the moment ($d\xi_4 = ic d\tau$, $d\tau$ = element of the intrinsic time). Then $d\xi_j/d\tau$ replaces dx_j/dt and we obtain

$$\iiint \Gamma \cdot \Omega dV_\perp d\tau = e \int \sum_j \frac{d\xi_j}{d\tau} \Omega_j d\tau. \quad (22)$$

We must note that in the variation not only ξ_j is changed by $\delta\xi_j$, but also Ω_j is changed by

$$\delta\Omega_j = \Omega_j(\xi_1 + \delta\xi_1, \cdots, \xi_4 + \delta\xi_4) - \Omega_j(\xi_1, \cdots, \xi_4) = \sum_i \frac{\partial \Omega_j}{\partial \xi_i} \delta\xi_i$$

(the charge e is of course conserved). Hence (22) leads to

$$\delta \iiint \Gamma \cdot \Omega dV_n d\tau = e \int \left(\sum_j \frac{d\delta\xi_j}{d\tau} \Omega_j + \sum_j \sum_i \frac{d\xi_j}{d\tau} \frac{\partial \Omega_j}{\partial \xi_i} \delta\xi_i \right) d\tau. \quad (22a)$$

The first of the two terms on the right is transformed by integration by parts and yields (since $\delta\xi_j = 0$ on the boundary of the world region)

$$- \sum_j \frac{d\Omega_j}{d\tau} \delta\xi_j = - \sum_j \sum_i \frac{\partial \Omega_j}{\partial \xi_i} \frac{d\xi_i}{d\tau} \delta\xi_j.$$

Thus (22a), after interchange of the subscripts i, j in the double sum, becomes

$$e \int \left(\sum_j \sum_i \frac{d\xi_i}{d\tau} \left[\frac{\partial \Omega_i}{\partial \xi_j} - \frac{\partial \Omega_j}{\partial \xi_i} \right] \delta\xi_j \right) d\tau. \quad (23)$$

Here we have in the parenthesis $\text{Curl}_j \Omega$, i.e. except for the factor e the component F_{ji} of the field (see (26.11)). On the other hand, we have, by (28.17)

$$\Gamma \cdot F = ck \quad (k = \text{force density}) \quad (24)$$

and by (28.19)

$$\int k dV_n = \frac{K}{\sqrt{1-\beta^2}} \quad (K = \text{Lorentz force}).$$

Hence our expression (23), which was obtained by carrying out the integration over V_n , signifies simply

$$\int \sum_j K_j \delta\xi_j \frac{d\tau}{\sqrt{1-\beta^2}} = \int \sum_j K_j d\xi_j dt. \quad (25)$$

We must add to this from the first term of the right side of (18), if we again put $dV dt = dV_n d\tau$ and integrate over the world line cross section dV_n :

$$m_0 c^2 \delta \int ((1-\beta^2)^{-1} - 1) d\tau = m_0 c^2 \delta \int (1 - \sqrt{1-\beta^2}) dt = \delta \int K dt.$$

We have already carried out the variation of this integral over K in Eq. (11a) and the succeeding equations. We found there, translated into the present notation ξ_j of the world line coordinates (see (12b)):

$$- \int \sum_j \frac{d}{dt} \left(\frac{m_0}{\sqrt{1-\beta^2}} \frac{d\xi_j}{dt} \right) \delta\xi_j dt. \quad (26)$$

Together with (25) we obtain as variation of the action integral

$$\delta W = - \int \left\{ \sum_j \frac{d}{dt} \left[\frac{m_0}{\sqrt{1-\beta^2}} \frac{d\xi_j}{dt} \right] - K_j \right\} \delta\xi_j dt. \quad (27)$$

We require that this integral should vanish for arbitrary displacements. This is only possible if the $\{ \}$ vanishes for $j = 1, 2, 3, 4$. In this manner we have derived our earlier Eq. (5) including the corresponding fourth component, and this in more explicit form: Our present derivation yields not only this equation of motion, but also the Lorentz force impressed on the electron by the field. *Schwarzschild's principle of least action thus combines Maxwell's electrodynamics and the Lorentz electron theory in a single four-dimensionally invariant formulation.*

From a historical point of view it may be noted that Schwarzschild, starting with the kinetic potential (17), also obtains the Maxwell equations and the equation of motion of the electron including the expression for the Lorentz force. Only the variation of mass of the electron escapes him, since, in (17), he employs the classical value of the kinetic energy. It is true that his derivation of the Maxwell equations is not quite correct from our point of view because of the missing factor 2 in Λ , which is compensated in Schwarzschild's treatment by putting f and F ($D = E$, $H = B$) equal. In our representation the proportionality of f and F is also contained in the Schwarzschild principle. It is only necessary to eliminate Γ from the Eqs. (21) and (26.5), which we can write

$$\text{Div } f = \Gamma \quad \text{and} \quad \text{Div } F = c\mu_0\Gamma.$$

Schwarzschild's action principle is very suggestive. It could be made the starting point of the theory and the Maxwell equations be regarded as its consequences. There would be at the same time the inviting possibility of refining the Maxwell equations by extending the kinetic potential (18) (addition of other field invariants, taking account of interactions between the electrons, their magnetic moment and spin). We will enter upon such questions in §37.

§33. *Electromagnetic Theory of the Electron*

At the turn of the century interest was focused on the variable mass of the electron. The assumption of the *rigid electron*, which appeared appropriate in the theory of the absolute ether, led to a different, much more complicated law of transformation (Max Abraham) than Lorentz's assumption of the *deformable electron*, which soon afterwards attained an assured basis in the theory of relativity. The experiments of Kaufmann, Bucherer, Neumann and many others were concerned with this law of transformation.

The theoretical treatment of the problem (also for the rigid electron) rested on the definition (31.13) of the electromagnetic momentum. Without detaining ourselves with the rigid electron we shall show that the same starting point, with the relativistic treatment of the momentum, leads to the same law (32.2) of the variation of the mass as the theory of relativity, which however extends it immediately to any arbitrary mass m .

We shall obtain as a by-product an interesting formula for the rest mass m_0 of the electron.

Below we understand by \mathbf{G} the *total momentum* of the field in infinite space; we shall call the momentum per unit volume, designated by \mathbf{g} in (31.13), \mathbf{g} . With dV as three-dimensional volume element we then have

$$\mathbf{G} = \int \mathbf{g} dV = \frac{1}{c^2} \int \mathbf{S} dV. \quad (1)$$

To be able to carry out the indicated integration we utilize the ideas and symbols of Eqs. (28.12). Let x, y, z be the frame of reference moving with the electron, x', y', z' a coordinate system at rest, with respect to which the electron has the instantaneous velocity v in the positive x' -direction. In the x, y, z system we then have of course $\mathbf{G} = 0$; the field is electrostatic so that $\mathbf{H} = 0$ and $\mathbf{S} = 0$. We are interested in the momentum \mathbf{G}' and more particularly in its x -component:

$$G'_x = \frac{1}{c^2} \int S'_x dV' = \frac{1}{c^2} \int (E'_x H'_y - E'_y H'_x) dV'. \quad (2)$$

We express the primed quantities in terms of the unprimed ones in accord with Eq. (28.12a), in which however, in view of the opposite direction of motion, the sign of v must be changed:

$$E'_x = E_x, \quad E'_y = \frac{1}{\sqrt{1-\beta^2}} E_y, \quad E'_z = \frac{1}{\sqrt{1-\beta^2}} E_z, \quad (2a)$$

$$H'_x = 0, \quad H'_y = \frac{-v}{\mu_0 c^2 \sqrt{1-\beta^2}} E_z, \quad H'_z = \frac{B'_x}{\mu_0} = \frac{v}{\mu_0 c^2 \sqrt{1-\beta^2}} E_y, \\ dV' = dV \sqrt{1-\beta^2} \quad (\text{Lorentz contraction}). \quad (2b)$$

Thus we obtain from (2)

$$G'_x = \frac{v}{\mu_0 c^4 \sqrt{1-\beta^2}} \int (E_y^2 + E_z^2) dV. \quad (3)$$

In the xyz -system the \mathbf{E} -field is spherically symmetrical, so that

$$\int E_x^2 dV = \int E_y^2 dV = \int E_z^2 dV = \frac{1}{3} \int E^2 dV. \quad (3a)$$

The same applies for the charge distribution. It seems most natural to spread the charge e uniformly over a sphere of radius a ("radius of the electron"). Then, as follows e.g. from (7.6a),

$$E = E_r = \begin{cases} 0 & \text{for } r < a \\ \frac{e}{4\pi\epsilon_0 r^2} & \text{for } r \geq a \end{cases} \quad (3b)$$

and hence

$$\int E^2 dV = 4\pi \int E_r^2 r^2 dr = \frac{e^2}{4\pi\epsilon_0^2} \int_0^\infty \frac{dr}{r^2} = \frac{e^2}{4\pi\epsilon_0^2 a}. \quad (3c)$$

(3) and (3a) then lead to

$$G'_x = \frac{v}{\mu_0 c^4 \sqrt{1-\beta^2}} \cdot \frac{2}{3} \frac{e^2}{4\pi\epsilon_0^2 a} = \frac{e^2}{6\pi\epsilon_0 c^2 a} \frac{v}{\sqrt{1-\beta^2}}. \quad (4)$$

We may also readily convince ourselves that,

$$G'_y = G'_z = 0 \quad (4a)$$

as must be expected for the spherical symmetry in the xyz -system. For if we form

$$G'_y = \frac{1}{c^2} \int S'_y dV' = \frac{1}{c^2} \int (E'_z H'_x - E'_x H'_z) dV'$$

in analogy to (2) and again make use of Eqs. (2a, b),

$$G'_y = \frac{v}{\mu_0 c^2} \int E_z E_y dV = 0$$

since for a spherically symmetrical field E_x and E_y are proportional to x and y , and xy integrated over the sphere vanishes.

Eqs. (4) and (4a) can be combined to

$$\mathbf{G}' = m\mathbf{v}, \quad m = \frac{m_0}{\sqrt{1-\beta^2}}. \quad (5)$$

The mass factor m here introduced has thus the dependence on velocity familiar to us from §32. For the rest mass m_0 we find from (4):

$$m_0 = \frac{e^2}{6\pi c^2 \epsilon_0 a}. \quad (6)$$

The reader may check the dimensional correctness of this formula, i.e. the independence of the choice of the unit of charge Q and the unit of length M . The factor ϵ_0 , which in the Gaussian system is set equal to 1, is from our point of view indispensable. If it is suppressed the formula becomes dimensionally meaningless.

With the value of the rest mass computed by (6) Eq. (5) states: *The mechanical momentum of the electron is equal to the momentum contained in the electromagnetic field as defined by Eq. (2):*

$$\mathbf{G}_{\text{electron}} = \mathbf{G}_{\text{field}}. \quad (7)$$

We read directly in Eq. (6) that the transition to the limit $a \rightarrow 0$ is unfortunately impossible; it would lead to $m_0 \rightarrow \infty$ and $e/m_0 \rightarrow 0$. To

determine the numerical value of a , we must know the experimental values of e and e/m_0 . In MKSQ units, with $Q = 1$ coulomb, these are:

$$e = 1.60 \cdot 10^{-19} Q, \quad e/m_0 = 1.76 \cdot 10^{11} Q/K. \quad (8)$$

From this we compute

$$m_0 = 0.9 \cdot 10^{-30} K. \quad (8a)$$

Eq. (6) then yields, with (7.18a),

$$a = \frac{2}{3} \frac{e^2}{4\pi\epsilon_0 c^2 m_0} = \frac{2}{3} \frac{e^2}{10^7 m_0} \frac{MK}{Q^2} = \frac{2}{3} \frac{(1.60 \cdot 10^{-19})^2}{0.9 \cdot 10^{-28}} M \cong 2 \cdot 10^{-13} \text{ cm.} \quad (9)$$

This is a subatomic dimension, of the same order of magnitude as nuclear dimensions.

It is of course quite arbitrary that we have here assumed a surface charge. We might equally well have distributed the electron charge e uniformly over the electron volume.

If we then call its radius once more a , we find¹ instead of (3b)

$$E = E_r = \begin{cases} \frac{er}{4\pi\epsilon_0 a^3} & \text{for } r \leq a \\ \frac{e}{4\pi\epsilon_0 r^2} & \text{for } r \geq a \end{cases} \quad (10)$$

and instead of (3c)

$$\int E^2 dV = \frac{e^2}{4\pi\epsilon_0^2} \left\{ \int_0^a \frac{r^4 dr}{a^3} + \int_a^\infty \frac{dr}{r^2} \right\} = \frac{e^2}{4\pi\epsilon_0^2 a} \left\{ \frac{1}{5} + 1 \right\}. \quad (10a)$$

The factor 6/5 is thus to be added to the formula for G' in (4) so that we obtain in place of (6)

$$m_0 = \frac{e^2}{5\pi c^2 \epsilon_0 a}. \quad (10b)$$

The order of magnitude of the value of a found in (9) is not affected.

The following remark is of greater importance: Who can guarantee that the Maxwell equations can be extrapolated right up to the surface or into the interior of the electron? May not their simplicity and linearity be a consequence of the fact that they are exactly valid only for weak fields and that they must be corrected, in the immediate neighborhood of concentrated charges, by higher terms, in some such manner as the theory of

¹ The first line of (10) evidently follows from the fact that elements of charge whose distance from the center is less than r may be thought of as concentrated at the center of the sphere, while those distant by more than r from the center do not contribute to the field strength.

dilute solutions in thermochemistry? We shall return to this question in §37. It will here merely be emphasized that the derivation of the law governing the variation of mass with velocity is not subject to this criticism, since, in §32, it could be derived from the general principles of relativistic mechanics, whereas our present computation of m_0 is affected; the latter is anyhow beyond experimental verification, in view of the hypothetical character of the electron radius. The derivation of the mass-velocity law in §32 is, like all considerations of the special theory of relativity, only tied to the condition that the occurring relative motions should be nearly uniform. We express this here by the demand that the electron motion be *quasistationary*. We mean hereby that its velocity change in the time taken by a light wave to sweep over the electron (i.e. the time $2a/c$) be small compared to v . We thus demand only;

$$\dagger \frac{2a}{c} \ll v. \quad (11)$$

All processes in vacuum tubes satisfy this requirement.

With respect to formula (6) for the rest mass we note furthermore that it may be derived in the following very elementary manner: We consider a *slowly* moving electron. Its mass is equal to the rest mass m_0 and its kinetic energy

$$T = \frac{m_0}{2} v^2. \quad (12)$$

If this is of electromagnetic origin we must set it equal to the *magnetic* energy of the field since the electric energy is constant for small fields, i.e. not proportional to v^2 . We hence put

$$T = \frac{\mu_0}{2} \int \mathbf{H}^2 dV. \quad (12a)$$

Here we can substitute for \mathbf{H} the value (15.12) from the law of Biot-Savart

$$\mathbf{H} = \frac{ev \sin \vartheta}{4\pi r^2}.$$

We then obtain for surface charge

$$T = \frac{\mu_0}{2} \left(\frac{ev}{4\pi} \right)^2 \int_0^\infty \frac{dr}{r^3} \int_0^\pi \sin^3 \vartheta d\vartheta \int_0^{2\pi} d\varphi.$$

The three integrals are, in sequence,

$$\frac{1}{a}, \frac{4}{3}, 2\pi.$$

Hence

$$T = \frac{\mu_0 e^2 v^2}{2} \frac{1}{6\pi a}. \quad (13)$$

Comparison with (12) yields

$$m_0 = \frac{\mu_0 e^2}{6\pi} \frac{1}{a}$$

which is identical with (6) since $\epsilon_0 \mu_0 c^2 = 1$.

Since the kinetic energy of the electron computed with (6) proved to be equal to the magnetic energy of the surrounding field we may suspect that its *rest energy* will correspond to the *electrostatic energy* of the Coulomb field. In the simple case of surface charge we find that this is equal to

$$E_{\text{stat}} = \frac{\epsilon_0}{2} \int \mathbf{E}^2 dV = \frac{\epsilon_0}{2} 4\pi \int_{r=a}^{\infty} E_r^2 r^2 dr$$

and obtain in view of (3c)

$$E_{\text{stat}} = \frac{e^2}{8\pi\epsilon_0 a}. \quad (14)$$

In contrast to this Eq. (6) yields for the rest energy of our electron by Einstein's law of equivalence of mass and energy:

$$E_0 = m_0 c^2 = \frac{e^2}{6\pi\epsilon_0 a}. \quad (15)$$

Thus only $\frac{2}{3}$ of this rest energy is explained electromagnetically by our preceding (admittedly primitive) considerations. The program indicated by the title of this section is as yet incapable of realization.

As was already said on p. 236, the electron is a stranger in electrodynamics. The forces which, opposing the Coulomb forces, prevent its explosion are unknown to us, just like the theory of the elementary particles in general. Poincaré introduced (as early as 1906, in the *Rendiconti di Palermo*) a cohesion pressure of unknown origin which was supposed to envelop the electron at rest like a membrane under uniform tension; the missing quarter of the rest energy was supposed to be hidden herein. The hypothesis of rigidity of the absolute theory could transfer this cohesion pressure to the electron in motion. It did not suffice, however, for a purely electromagnetic description of the electron. Even the assumption of rigidity contradicts the group-theoretical nature of Maxwell's electrodynamics which, as we know, demands the deformable electron of Lorentz.

Altogether, we should face the fact that our electrodynamic theory of the electron is as yet very incomplete. We have known for 20 years that the electron possesses in addition to its charge a quite definite spin and a

quite definite magnetic moment. Both can only be defined on the basis of the quantum theory and are inaccessible to Maxwell's electrodynamics. The secret of the spin was first discovered in the more precise analysis of the Zeeman effect; the secret of the magnetic moment was actually, as we know now, clearly and tangibly demonstrated in ferromagnetism. It is strange that practical electronics remained untouched by these fundamental facts and could get along with the notion of the charged point mass or the minute charged sphere.

Our Problems III.5 to III.10 deal with this application of electron theory. The varied electron trajectories which occur in vacuum tubes and which, in the e/m experiments, first served to clarify the nature of the electron are at the same time in a way the simplest and best defined examples of the mechanics of an isolated point mass.

PART IV

MAXWELL'S THEORY FOR MOVING BODIES AND OTHER ADDENDA

§34. *Minkowski's Equations for Moving Media*

The extension of Maxwell's theory from media at rest to those in motion was a favorite problem of the older electrodynamics. Heinrich Hertz had failed in this effort (see his paper cited in footnote 2 on p. 2) because he adhered consistently to classical theory (the "Galilei transformation"). His friend Emil Cohn¹ came closer to the goal, but was not yet (in 1902!) in possession of the necessary tools, the Lorentz transformation. Even H. A. Lorentz did not quite attain the final form in his papers in the *Enzyklopädie* (1903), particularly not for magnetizable bodies. Einstein called his paper of 1905 "On the electrodynamics of moving bodies" and indicated in this manner a principal goal of his theory of relativity; however he does not enter upon the general structure of the equations for ponderable bodies but confines himself instead to the questions arising for the isolated electron. Minkowski, in 1908, at long last in full possession of the principle of relativity, was the first to solve the problem completely.²

Minkowski's logic was simple: The Maxwell equations for a state of rest apply within the laboratory. Consider a point of space-time P of a body moving³ with respect to the laboratory at the laboratory time t ; let it have the velocity \mathbf{v} . Let P be transformed to rest by the introduction of the coordinates x', y', z', t' for the description of the processes in the neighborhood of P, t . In this system Maxwell's equations for a state of rest apply to the quantities $\mathbf{E}', \mathbf{B}', \mathbf{D}', \mathbf{H}', \mathbf{J}', \rho'$:

$$\frac{\partial \mathbf{B}'}{\partial t'} = -\text{curl } \mathbf{E}', \quad \frac{\partial \mathbf{D}'}{\partial t'} + \mathbf{J}' = \text{curl } \mathbf{H}', \quad (1)$$

$$\text{div } \mathbf{D}' = \rho', \quad \text{div } \mathbf{B}' = 0,$$

with material constants differing from those for vacuum:

$$\mathbf{D}' = \epsilon \mathbf{E}', \quad \mathbf{B}' = \mu \mathbf{H}', \quad \mathbf{J}' = \sigma \mathbf{E}'. \quad (2)$$

¹ Göttinger Nachr. 1901, p. 74; Ann. Physik 7, 29, 1902.

² Göttinger Nachr. 1908, p. 53; *Gesammelte Werke* II, p. 352.

³ The motion may be variable in space and time and must merely be capable of quasistationary treatment in the sense of Eq. (33.11). Thus \mathbf{v} need not be a pure translation and the body need not be rigid. Only the fixed value of \mathbf{v} in the space-time point P, t enters in the following Lorentz transformations.

These constants have the same values as if the body were at rest with respect to the laboratory, since it knows nothing of its motion. The operations curl and div in (1) refer of course, just like the time t' , to the primed system. Now the inverse Lorentz transformation is to be carried out, which transforms the primed system back into the original one of the laboratory. In the latter Eqs. (1) apply once more if all primes are omitted, in view of the basic property of covariance of the Maxwell equations with respect to the Lorentz transformations. However, Eqs. (2), transformed to the unprimed system, take on a new form.

We know the relationship of the \mathbf{E}' , \mathbf{B}' and the \mathbf{E} , \mathbf{B} from Eqs. (28.8a) and (28.11):

$$\begin{aligned} E'_{||} &= (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{||}, & E'_{\perp} &= \left[\frac{\mathbf{E} + \mathbf{v} \times \mathbf{B}}{\sqrt{1 - \beta^2}} \right]_{\perp} \\ B'_{||} &= \left(\mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E} \right)_{||}, & B'_{\perp} &= \left\{ \frac{\mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E}}{\sqrt{1 - \beta^2}} \right\}_{\perp} \end{aligned} \quad (3)$$

$||$ and \perp signify as before "parallel" and "perpendicular to the velocity \mathbf{v} ". We shall supplement this by the corresponding relations between \mathbf{D}' , \mathbf{H}' and \mathbf{D} , \mathbf{H} . In view of the definition of the six-vectors

$$\mathbf{f} = (\mathbf{H}, -ic\mathbf{D}), \quad \mathbf{F} = (c\mathbf{B}, -i\mathbf{E})$$

they are obtained from (3) by replacing \mathbf{E} by $c\mathbf{D}$ and \mathbf{B} by \mathbf{H}/c . We thus obtain

$$\begin{aligned} D'_{||} &= \left(\mathbf{D} + \frac{1}{c^2} \mathbf{v} \times \mathbf{H} \right)_{||}, & D'_{\perp} &= \left[\frac{\mathbf{D} + \frac{1}{c^2} \mathbf{v} \times \mathbf{H}}{\sqrt{1 - \beta^2}} \right]_{\perp}, \\ H'_{||} &= (\mathbf{H} - \mathbf{v} \times \mathbf{D})_{||}, & H'_{\perp} &= \left[\frac{\mathbf{H} - \mathbf{v} \times \mathbf{D}}{\sqrt{1 - \beta^2}} \right]_{\perp}, \end{aligned} \quad (4)$$

Substitution of (3) and (4) in (2) yields, for both the parallel and the perpendicular components for which the denominator cancels on the two sides,

$$\begin{aligned} \mathbf{D} + \frac{1}{c^2} \mathbf{v} \times \mathbf{H} &= \epsilon(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ \mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E} &= \mu(\mathbf{H} - \mathbf{v} \times \mathbf{D}). \end{aligned} \quad (5)$$

Here \mathbf{B} may, for example, be eliminated in the first equation by means of the second, so that \mathbf{D} is expressed only in terms of \mathbf{E} and \mathbf{H} ;¹ similarly

¹ Here we make use of the transformation $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ and of the relation $\epsilon\mu c^2 = 1$. It should be noted that according to (5a, b) the identity in direction of \mathbf{D} and \mathbf{E} as well as that of \mathbf{B} and \mathbf{H} has ceased to apply even for the isotropic medium.

elimination of \mathbf{D} leads to an expression of \mathbf{B} in terms of \mathbf{E} and \mathbf{H} . The resulting equations become simpler if they are written separately for the components \parallel and \perp . They then become

$$D_{\parallel} = \epsilon E_{\parallel}, \quad B_{\parallel} = \mu H_{\parallel} \quad (5a)$$

$$\left[1 - \frac{\epsilon\mu}{\epsilon_0\mu_0}\beta^2\right] \begin{cases} D_{\perp} = \epsilon(1 - \beta^2)E_{\perp} + (\epsilon\mu - \epsilon_0\mu_0)\mathbf{v} \times \mathbf{H}, \\ B_{\perp} = \mu(1 - \beta^2)H_{\perp} + (\epsilon_0\mu_0 - \epsilon\mu)\mathbf{v} \times \mathbf{E}. \end{cases} \quad (5b)$$

Having taken care of the first two Eqs. (2) we now turn to the third Eq. (2), "Ohm's law for moving conductors". What is the relationship of \mathbf{J}' and \mathbf{J} ? We know from Eq. (26.6) that \mathbf{J} is the space component of a four-vector Γ , whose time component is $ic\rho$. We also know that every four-vector transforms itself like the coordinate vector x_1, x_2, x_3, x_4 . We therefore have for the specialized Lorentz transformation ($\mathbf{v} \parallel x$):

$$J'_x = \frac{J_x - v\rho}{\sqrt{1 - \beta^2}}, \quad J'_y = J_y, \quad J'_z = J_z, \quad \rho' = \frac{\rho - \frac{v}{c^2}J_x}{\sqrt{1 - \beta^2}}.$$

For an arbitrary direction of \mathbf{v} this becomes

$$J'_{\parallel} = \left[\frac{\mathbf{J} - \rho\mathbf{v}}{\sqrt{1 - \beta^2}} \right]_{\parallel}, \quad J'_{\perp} = J_{\perp}, \quad \rho' = \frac{\rho - \frac{1}{c^2}\mathbf{v} \cdot \mathbf{J}}{\sqrt{1 - \beta^2}}, \quad (6)$$

where we may also write the middle equation in the form

$$J'_{\perp} = (\mathbf{J} - \rho\mathbf{v})_{\perp}, \quad (6a)$$

since by definition $v_{\perp} = 0$.

With this meaning of \mathbf{J}' and the meaning (3) of \mathbf{E}' our Ohm's law becomes

$$\begin{aligned} \left(\frac{\mathbf{J} - \rho\mathbf{v}}{\sqrt{1 - \beta^2}} \right)_{\parallel} &= \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\parallel}, \\ (\mathbf{J} - \rho\mathbf{v})_{\perp} &= \sigma \left(\frac{\mathbf{E} + \mathbf{v} \times \mathbf{B}}{\sqrt{1 - \beta^2}} \right)_{\perp}. \end{aligned} \quad (7)$$

These two equations can also be combined into a single one, though only in a somewhat artificial manner. We here make use of the following notation, which is customary also elsewhere in the literature and will be useful later on:¹

$$\mathbf{E}^* = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \quad \mathbf{H}^* = \mathbf{H} - \mathbf{v} \times \mathbf{D}. \quad (8)$$

¹ The * here employed of course bears no relation to the earlier * of the dual six-vector

We can then write in place of (7)

$$\mathbf{J} - \rho\mathbf{v} = \sigma \frac{\mathbf{E}^* - \frac{\mathbf{v}}{c} \left(\frac{\mathbf{v}}{c} \cdot \mathbf{E}^* \right)}{\sqrt{1 - \beta^2}}. \quad (9)$$

Since $v_{\perp} = 0$ the perpendicular component of this is identical with the second Eq. (7). Furthermore, since $\mathbf{v} \cdot \mathbf{E}^* = vE_{\parallel}$ the parallel component of (9) is

$$(\mathbf{J} - \rho\mathbf{v})_{\parallel} = \frac{\sigma}{\sqrt{1 - \beta^2}} \left[E_{\parallel}^* - \frac{v}{c} \cdot \frac{v}{c} E_{\parallel}^* \right] = \frac{\sigma}{\sqrt{1 - \beta^2}} E^*(1 - \beta^2),$$

which agrees with the first Eq. (7). We call

$$\mathbf{J}_1 = \mathbf{J} - \rho\mathbf{v} \quad (9a)$$

the "conduction current".

Eq. (9) expresses the fact that the *convection current* $\rho\mathbf{v}$ and the *conduction current* \mathbf{J}_1 are superposed and that their differentiation depends on the *frame of reference of the observer*. The reason for this evidently rests in the four-dimensional combination of \mathbf{J} and $\rho\mathbf{v}$ in the four-vector $\mathbf{\Gamma}$. Just as for the six-vector \mathbf{F} the distinction between its electric and magnetic aspect depended on the reference frame of the observer (see p. 241), a change in the reference frame now adds the time component $ic\rho$ of the $\mathbf{\Gamma}$ vector and the corresponding convection current $\rho\mathbf{v}$ to the conduction current \mathbf{J}_1 . The former, like the latter, produces a *magnetic field*.

This conclusion was contained already in the *Rowland effect* discovered in 1878. Since we are here dealing exclusively with charge in motion and since therefore the conduction term in (9) is lacking, the convection current $\rho\mathbf{v}$ alone is magnetically active and takes the place of \mathbf{J} in the appropriate Maxwell equation.

The question naturally arises whether also the so-called "free charge",¹ which occurs at the surface of a homogeneous dielectric in an electric field, is magnetically active when the dielectric is set into motion. This led Roentgen to his fundamental experiment:² A dielectric plate is placed in a

¹ We have avoided this notation elsewhere (like Röntgen, who expressly designated his dielectric plate as uncharged) since the "free charge" is not a charge dimensionally, but a divergence of field strength (see p. 40), in our case a surface divergence of the electric field strength.

² W. C. Roentgen, Ann. Physik Vol. 35, p. 264, 1888. In a supplement to this paper Roentgen reports the negative result of an experiment with a rotatably suspended condenser so oriented with respect to the motion of the earth that the "ether wind" passed through the condenser plates. Does this ether wind generate a magnetic field and, as a result, a deflection of the condenser? From our present relativistic point of view the negative result of the experiment is a foregone conclusion. A similar, refined, arrangement became famous at a later date in the Trouton-Noble experiment.

plate condenser, parallel to the plate electrodes, and is moved perpendicularly to the lines of force in the condenser (it was rotated about an axis normal to the condenser plates in the experiment). Does this motion produce a magnetic field? Roentgen could answer this affirmatively and Lorentz, as a result, named the current equivalent to the motion the *Roentgen current*. In agreement with later experiments and considerations of Eichenwald¹ the magnitude of this current for the experimental arrangement in question is:

$$\mathbf{R} = \mathbf{v}(\epsilon - \epsilon_0) |\mathbf{E}_0| = \mathbf{v} |\mathbf{P}_0|. \quad (10)$$

The plate is here assumed to be unmagnetic ($\mu = \mu_0$), its motion a parallel displacement \mathbf{v} ; \mathbf{E}_0 is the field strength in the charged condenser and \mathbf{P}_0 the corresponding polarization of the plate, both referring to the plate at rest, as indicated by the subscript 0. Since the "free charge" is concen-

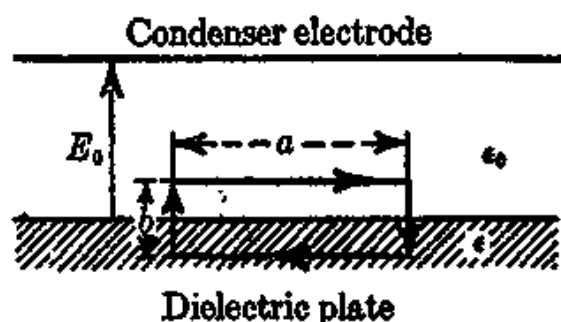


FIG. 43. Explanation of the Röntgen current. Section perpendicular to the direction of motion of the dielectric plate and the condenser electrode. The location of the Röntgen current is the surface of the plate. The portion linked by a rectangular loop a , b , a , b is indicated by a heavy line.

trated on the surface of the dielectric plate, the Roentgen current is also a pure surface current: It is absent both from the air gap and from the interior of the plate and occurs only at their interface; its direction is that of \mathbf{v} , just as for the Rowland current.

We shall show that (10) follows from (5a, b) if $\beta^2 = (|\mathbf{v}|/c)^2$ is neglected (which is of course fully justified under the conditions of the experiment) and if furthermore $\mu = \mu_0$, and on the right the values of \mathbf{E} and \mathbf{H} for the plate at rest are substituted, namely $E_{\perp} = E_0$, $E_{\parallel} = 0$, $H = H_0 = 0$. We then find

$$D_{\parallel} = B_{\parallel} = 0, \quad D = D_{\perp} = \epsilon E_0, \quad B = B_{\perp} = \mu_0(\epsilon_0 - \epsilon)\mathbf{v} \times \mathbf{E}_0.$$

Fig. 43 represents a section normal to \mathbf{v} (\mathbf{v} is directed into the plane of the paper) in which the shaded portion below indicates the dielectric plate, the upper portion, the air gap of the condenser. We compute the line integral of $\mathbf{H} = \mathbf{B}/\mu_0$ about the rectangular loop which has been drawn, the direction of the integration being related to the direction of \mathbf{v} by a right-handed screw motion. In view of the direction of $\mathbf{v} \times \mathbf{E}_0$, \mathbf{H} has the direction of the arrow on the upper side of the rectangle a , but

¹ A. Eichenwald, Ann. Physik, Vol. 11, pp. 1 and 241, 1903.

vanishes on it since $\epsilon = \epsilon_0$; the same applies for the sides b of the rectangle. Thus there remains only the lower side a of the rectangle, which is traversed in a direction opposite to a . It yields

$$\oint \mathbf{H} \cdot d\mathbf{s} = -a(\epsilon_0 - \epsilon) \mathbf{v} \times \mathbf{E}_0.$$

This magnetic circuit is equal to the surface current flowing through its interior, which in the figure is indicated by the heavy line through the middle of the rectangle. It is $a \cdot R$ if we call the surface current per unit length R . We thus obtain

$$\oint \mathbf{H} \cdot d\mathbf{s} = -a R, \quad R = (\epsilon - \epsilon_0) \mathbf{v} |\mathbf{E}_0| = \mathbf{v} |\mathbf{P}_0|, \quad (10a)$$

where by the vectorial symbol \mathbf{v} we also indicate the positive direction of R (pointing into the plane of the paper like \mathbf{v}). Thus Eq. (10) is verified.

The experiments of Eichenwald in which the dielectric plate and the two condenser plates were rotated about their common normal as a unit, so that the convection currents $\mathbf{v} |\mathbf{D}|$ of the condenser plates (surface density $\omega = |\mathbf{D}|$) are added to the Roentgen currents $\mathbf{v} |\mathbf{P}_0|$ on the dielectric plate, are of special interest. Since \mathbf{D} and \mathbf{P} differ, a residual magnetic field arises here also, contrary to Hertz's earlier theory and in spite of the opposite signs of the Rowland and Roentgen currents. (The sign of the condenser charge is opposite to that of the charge on the plate induced by it.) Eichenwald (on p. 331) states expressly regarding this residual field: "The magnetic effect is independent of the material of the dielectric." In fact, $\mathbf{D} - \mathbf{P} = \epsilon_0 \mathbf{E}$ is the vacuum component of \mathbf{D} , for which the term "dielectric displacement" is not particularly appropriate, but which is very characteristic for Maxwell's theory and its optical application.

We finally want to mention a kind of inversion of Roentgen's experiment, the experiment of H. A. Wilson¹: A hollow dielectric cylinder is placed between the electrodes of an uncharged cylindrical condenser in a uniform magnetic field parallel to the cylinder axis. *If the cylinder is rotated the condenser is charged.*

We have followed Minkowski closely so far and believe to have thus even improved on the clarity of the otherwise insurpassable representation in W. Pauli's article in the *Enzyklopädie*. We shall now establish contact with H. A. Lorentz's article in the *Enzyklopädie* which, in its mathematical formulation, follows the paper of H. Hertz (1891) and older papers of Helmholtz. To this end we introduce for the quantities referred to the

¹ Phil. Trans. Vol. 204, p. 121, 1904; see also H. A. Wilson and M. Wilson, Proc. Roy. Soc., Vol. 89, p. 99, 1913.

laboratory (i.e. the unprimed quantities) on the right of Eq. (1) in place of \mathbf{E} and \mathbf{H} the quantities \mathbf{E}^* and \mathbf{H}^* from (8):

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -\text{curl } \mathbf{E}^* + \text{curl } (\mathbf{v} \times \mathbf{B}), \\ \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} &= \text{curl } \mathbf{H}^* + \text{curl } (\mathbf{v} \times \mathbf{D}).\end{aligned}\quad (11)$$

We shift the last terms on the right over onto the left and take account of the auxiliary conditions in (1):

$$\text{div } \mathbf{B} = 0, \quad \text{div } \mathbf{D} = \rho. \quad (11a)$$

We then can write instead of (11)

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \text{div } \mathbf{B} - \text{curl } (\mathbf{v} \times \mathbf{B}) &= -\text{curl } \mathbf{E}^*, \\ \frac{\partial \mathbf{D}}{\partial t} + \mathbf{v} \text{div } \mathbf{D} - \text{curl } (\mathbf{v} \times \mathbf{D}) + \mathbf{J} - \rho \mathbf{v} &= \text{curl } \mathbf{H}^*.\end{aligned}\quad (11b)$$

We have encountered the aggregates on the left already in Vol. II, (18.7c). There we computed for an arbitrary vector \mathbf{A} and a surface element $d\sigma$ which moves with the velocity \mathbf{v} , varying from point to point, and in the process changes size and shape itself, the "A-flux through $d\sigma$ "

$$\frac{d}{dt} (A_n d\sigma) = \left[\frac{\partial A}{\partial t} + \mathbf{v} \text{div } \mathbf{A} - \text{curl } (\mathbf{v} \times \mathbf{A}) \right]_n d\sigma.$$

Here we employ the abbreviation introduced by Lorentz¹

$$\underline{\dot{A}} = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \text{div } \mathbf{A} - \text{curl } (\mathbf{v} \times \mathbf{A}), \quad (12)$$

The preceding equation then passes into

$$\frac{d}{dt} (A_n d\sigma) = \underline{\dot{A}}_n d\sigma \quad (12a)$$

or, for a finite surface σ ,

$$\frac{d}{dt} \int A_n d\sigma = \int \underline{\dot{A}}_n d\sigma. \quad (12b)$$

We then obtain in place of (11b) the basic form of the *Maxwell-Minkowski equations in moving bodies (viewed from our laboratory)* given by Lorentz and Pauli:

$$\begin{aligned}\underline{\dot{B}}_n &= -\text{curl } \mathbf{E}^*, \\ \underline{\dot{D}} + \mathbf{J} - \rho \mathbf{v} &= \text{curl } \mathbf{H}^*.\end{aligned}\quad (13)$$

¹ See Enzyklopaedie, Vol. V, part 2, p. 75, Eq. (5).

Their advantage rests in the fact that they lead directly to the *integral form*:

$$\begin{aligned} \int \underline{\dot{B}}_n d\sigma &= - \oint \mathbf{E}^* \cdot d\mathbf{s}, \\ \int \underline{C}_n d\sigma &= \oint \mathbf{H}^* \cdot d\mathbf{s}, \quad \underline{C} = \underline{\dot{D}} + \mathbf{J} - \rho \mathbf{v}. \end{aligned} \quad (14)$$

At the left we integrate over a surface moving with the velocity \mathbf{v} , on the right, over its boundary s , the direction of traversal of s and the normal n of σ being correlated by the right-screw rule. We here recall footnote 3 on p. 280, according to which \mathbf{v} can be arbitrary, i.e. σ and s be attached to an arbitrarily moving and deformed body. In this manner we have arrived at a formulation which is closely related to our original axioms, Eqs. (3) and (4) of §3, and generalizes them greatly. Only in the present generalized form do they do justice to the facts, already mentioned in §3, of the induction for moving conductors and moving magnets.

The *boundary conditions* for moving bodies also follow from Eqs. (14), in the same manner as those for media at rest in §3. They require the *continuity of the tangential components of \mathbf{E}^* and \mathbf{H}^* as well as of the normal component of \mathbf{B}* . Here we must note that the velocity \mathbf{v} occurring in \mathbf{E}^* and \mathbf{H}^* is to be regarded as a constant of the Lorentz transformation, by which the point P of the moving body is transformed to rest. Thus \mathbf{v} has the same value on the two sides of the boundary surface, namely that in the point P , or is at least continuous in passing through the boundary.

The situation is different if \mathbf{v} jumps discontinuously from the value 0 (laboratory) to the value \mathbf{v} (moving solid body). We consider in particular the case, important for unipolar induction, that the field is stationary ($\partial/\partial t = 0$) and the surface of the body is displaced with the velocity \mathbf{v} ($v = v_{\text{tang}}$). We shall show that then *not the tangential components of \mathbf{E}^* , \mathbf{H}^* , but the tangential components (as seen from the laboratory) of \mathbf{E} , \mathbf{H} must be continuous along the boundary layer*.

We note in preparation that the two conditions which are here compared signify the same for a tangential direction of \mathbf{E} or \mathbf{H} *parallel* to \mathbf{v} (because of the meaning of the vector products in (8)), but that they are actually contradictory for every other tangential direction, in particular that *at right angles* to \mathbf{v} .

Let us consider now, just as in Fig. 3, a rectangular loop $\Delta s \Delta h$, which is initially placed normal to the boundary surface; this is now distorted, since the side Δs parallel to the boundary within the body is displaced, whereas the opposite side, in vacuum, remains fixed. Then, since $\partial \mathbf{B} / \partial t = 0$ and $\text{div } \mathbf{B} = 0$, $\underline{\dot{B}}$ is, by (12), equal to $-\text{curl } (\mathbf{v} \times \mathbf{B})$; the integral on the

left of the first Eq. (14) becomes, making use of Stokes' theorem,

$$-\int \text{curl}_n (\mathbf{v} \times \mathbf{B}) d\sigma = -\oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s},$$

i.e. in general *not equal to 0* as for constant or continuously varying \mathbf{v} .

On the other hand, the integral on the right of the same equation, carried out over the same distorted loop, becomes in view of the meaning of \mathbf{E}^*

$$-\oint \mathbf{E} \cdot d\mathbf{s} - \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{s}.$$

Equating the two expressions leads to the requirement

$$\oint \mathbf{E} \cdot d\mathbf{s} = 0, \quad \text{i.e. continuity of } E_{\text{tang}},$$

in accord with our earlier conclusion (3.9).

The same consideration applied to the second Eq. (14) yields (since $\partial \mathbf{D} / \partial t = 0$ and $\text{div } \mathbf{D} = \rho$)

$$\mathbf{Q} = \mathbf{v}\rho - \text{curl} (\mathbf{v} \times \mathbf{D}) + \mathbf{J} - \rho\mathbf{v} = -\text{curl} (\mathbf{v} \times \mathbf{D}) + \mathbf{J}$$

and for the integrals on the left and right sides of the second Eq. (14)

$$-\oint (\mathbf{v} \times \mathbf{D}) \cdot d\mathbf{s} + \int J_n d\sigma \quad \text{and} \quad \oint \mathbf{H} \cdot d\mathbf{s} - \oint (\mathbf{v} \times \mathbf{D}) \cdot d\mathbf{s}, \text{ respectively.}$$

Equating of the two leads to

$$\int J_n d\sigma = \oint \mathbf{H} \cdot d\mathbf{s}.$$

If in the limit $\Delta h \rightarrow 0$ the surface integral over \mathbf{J} is put equal to zero as in Eq. (3.8) (see also footnote 1 at that point) we obtain

$$\oint \mathbf{H} \cdot d\mathbf{s} = 0, \quad \text{i.e. continuity of } H_{\text{tang}}. \quad (15a)$$

This closes our consideration of the boundary conditions in the special case of a moving interface between two different media.

The existence of the *Rowland* and *Roentgen currents* attests the fact that the preceding theory is not only of importance for the large velocities of the theory of relativity. The same follows from the problem of *unipolar induction* which has been famous since the days of Arago and Faraday. The literature on this subject is voluminous and by no means free of contradictions, since this problem is concerned with the exact laws of the electrodynamics of moving bodies. We shall discuss this problem only

qualitatively here and defer quantitative considerations to Problem IV.1. Furthermore, we are primarily interested in the fields which occur here; hence we pass over the phenomena of motion, which are realized in apparatus of many types and have claimed most attention in experimental work.

If, for example a bar magnet, suitably supported, is rotated about its axis, induction currents arise in a wire of which one end glides, for example, on the middle of the magnet while the other is connected to the bearing at one of the ends of the axis of the magnet. Since here only the magnet pole adjoining this end is effective, we speak of "unipolar induction". This arrangement has been employed not only in laboratory experiments, but at times also on a large scale in electric generators.

We simplify the statement of the problem if we separate the conductor from the body generating the magnetic field. Let us consider for example a copper disk between the pole pieces of an electromagnet. It is known that such a "Faraday disk" is raised to incandescence if maintained in rotation and that alternatively an initial rotational momentum of the disk is rapidly damped by the magnetic field.

This occurs, however, only for an inhomogeneous field, such as is normally realized experimentally, where the disk extends beyond the innermost homogeneous portion of the field of the pole pieces. In order to deal with a well-defined and easily solvable problem we assume that the magnetic field is uniform throughout and introduce into it a metal rod with its axis perpendicular to the magnetic field B , which we set into uniform translatory motion along its axis. Its surface is charged hereby. No Joule heat arises in the interior since the conduction current is everywhere zero; the total charge is of course also zero. Interior and exterior field join continuously, but with discontinuous normal gradient, corresponding to the presence of surface charge. The *interior field* is perpendicular both to the axis of the rod and to B and can be given immediately for any form of the cross section.

In contrast with the interior field, the *exterior field* cannot be given immediately but requires the solution of a *boundary-value problem*: the continuous fitting of the potential in the exterior to the surface values of the potential known from the interior field. In the special case of the circular cross section this boundary-value problem is readily solved; see Problem IV.1.

Along with the interior field the potential difference between two surface points is determined. If, by means of sliding contacts on a connecting wire, it is to be used for the generation of current, the interior is no longer free of current. The present description of the interior field then becomes invalid.

A static magnetic field, arising from the Rowland currents at the sur-

face of the rod, occurs along with the electric field. It is, however, evidently very small compared to the original inducing field and can hence be neglected.

Our description applies throughout for an observer at rest in the laboratory; for an observer moving with the rod the electric field within the rod is zero.

In the actual realization of the experiment the rod is of course replaced by a metallic body of revolution and the translation by a rotation about the axis of symmetry of the latter. A mathematical difficulty which arises here is also indicated in Problem IV.1.

§35. The Ponderomotive Forces and the Stress-Energy Tensor

We return to §31 and generalize the concepts introduced there from vacuum to a body of arbitrary ϵ , μ , which, however, we shall assume to be both *homogeneous* and *isotropic*, although the anisotropic body would be of special interest in connection with electro- and magnetostriction. We shall, furthermore, regard the body as at rest, since, for the questions at issue, we can place our frame of reference on this body.

Our earlier definitions of §26

$$F = (cB, -iE), \quad f = (H, -icD), \quad \Gamma = (J, ic\rho) \quad (1)$$

as well as the Maxwell equations in the differential form given there

$$\text{Div } F^* = 0, \quad \text{Div } f = \Gamma \quad (1a)$$

retain their validity; it can be readily demonstrated that the factors c arising in (1) are not derived from the vacuum constants ϵ_0 , μ_0 , but from the time measurement $x_4 = ict$, which, in the special theory of relativity, applies quite generally for all ponderable media. However, the relation

$$f = \sqrt{\frac{\epsilon_0}{\mu_0}} F \quad (2)$$

between excitation and field is to be changed to

$$f = \left\{ \frac{\mu_0/\mu}{\epsilon/\epsilon_0} \right\} \sqrt{\frac{\epsilon_0}{\mu_0}} F, \quad (2a)$$

where the upper line refers to the space-space, the lower to the space-time components of f and F .

A comparison of (1) and (2a) shows that this change leads in fact to the required relations between excitation and field:

$$\begin{aligned} H &= \frac{\mu_0}{\mu} \sqrt{\frac{\epsilon_0}{\mu_0}} cB, \quad \text{i.e. } H = B/\mu, \\ -icD &= \frac{\epsilon}{\epsilon_0} \sqrt{\frac{\epsilon_0}{\mu_0}} (-iE), \quad \text{i.e. } D = \epsilon E. \end{aligned}$$

We see from (2a) that the simple proportionality (2) between f and F which was characteristic for vacuum passes over, for the ponderable body, into a type of linear vector function with two different proportionality constants for the space-space and the space-time components. In the anisotropic body this is replaced by a much more general vector function (see p. 28) with in general 12 different material constants.

Starting from the universally valid representation of the Lorentz force density in (31.1) we convince ourselves, by a critical consideration of the individual steps, that the transformations up to Eq. (31.8) remain unaltered, and are not influenced by the different proportionality factors for the electric and magnetic quantities in (2a). The same statement applies also for the diagonal elements of the tensor T , so that the diagonal sum of the latter retains its earlier value

$$\sum_{n=1}^4 T_{nn} = 0. \quad (3)$$

The same holds for the nondiagonal elements T_{nm} , provided only that n and m differ from 4. On the other hand we compute from (31.4) e.g.

$$\begin{aligned} T_{14} &= -\frac{1}{c} (F_{12} f_{42} + F_{13} f_{43}) & T_{41} &= -\frac{1}{c} (F_{42} f_{12} + F_{43} f_{13}) \\ &= -ic(B_z D_y - B_y D_z) & &= -\frac{i}{c} (E_y H_z - E_z H_y) \\ &= -ice\mu(\mathbf{E} \times \mathbf{H})_x = -\frac{i}{c} \frac{e\mu}{\epsilon_0 \mu_0} S_x, & &= -\frac{i}{c} (\mathbf{E} \times \mathbf{H})_x = -\frac{i}{c} S_x. \end{aligned} \quad (4)$$

If the subscript 1 is here replaced by 2 or 3 the result remains the same except that the subscript x of \mathbf{S} is replaced by y and z , respectively.

This different behavior of the two groups of the T_{nm} (n and $m \neq 4$ as against n or $m = 4$) has the result that whereas the three-dimensional stress tensor can still be written in the form (31.10), the complete four-dimensional tensor T takes on the *asymmetric* form

$$T = \left(\begin{array}{c|c} \sigma & -\frac{i}{c} \frac{e\mu}{\epsilon_0 \mu_0} \mathbf{S} \\ \hline -\frac{i}{c} \mathbf{S} & W \end{array} \right). \quad (5)$$

This asymmetry has questionable consequences. We know from hydrodynamics and the theory of elasticity that an asymmetric stress tensor leads to torques which do not correspond to observation (see e.g. Vol. II, §10 and §8). Also in electrodynamics torques may be deduced from the

asymmetric character of our tensor T with respect to its principal diagonal; these torques are very small and scarcely observable, but are even so improbable. M. Abraham has hence proposed a *symmetric* form of the tensor T , differing from Minkowski's, and M. von Laue has followed Abraham's suggestion.¹ The two points of view are compared with respect to their physical consequences by W. Pauli in his oft-quoted article in the *Enzyklopädie*, p. 665.

Following once again Minkowski, we deduce from the matrix (5) that the fourth component of our earlier Eq. (31.3) remains unchanged and corresponds to *Poynting's theorem* also in a ponderable body. The first three components of the same equation, which are affected by the changed upper portion of our matrix (5) on the other hand, lead to a definition of the electromagnetic momentum density differing from (31.13). Whereas we found for vacuum

$$\mathbf{G} = \frac{1}{c^2} \mathbf{S} = \frac{1}{c^2} \mathbf{E} \times \mathbf{H} \quad (6)$$

we now obtain the different (though, of course, dimensionally equal) expression

$$\mathbf{G} = \frac{1}{c^2} \frac{\epsilon\mu}{\epsilon_0\mu_0} \mathbf{S} = \mathbf{D} \times \mathbf{B}. \quad (6a)$$

This conclusion of Minkowski's theory is also not universally accepted.

As noted initially, we have been able to confine ourselves to bodies at rest in this section. In view of the behavior of the world tensor T in a Lorentz transformation, known to us from (28.20), our formulas can be transferred directly to bodies in motion. The problem of the ponderomotive forces would be solved for them also as soon as the ultimate form of our tensor T for bodies at rest had been determined. The fact that this has not been accomplished in a unique fashion signifies physically really only an esthetic defect and is certainly no serious objection to the theory of relativity. In fact from our present electron-theoretical standpoint all processes take place in vacuum, for which the question of the stress-energy tensor has received a satisfying and generally recognized solution in §31. From this point of view the ponderable bodies with their continuous material constants ϵ, μ are simply convenient abstractions and are not physical realities.

¹ In his excellent textbook "Die Relativitätstheorie," Vol. I: "Das Relativitätsprinzip der Lorentztransformation," and Vol. II: "Die allgemeine Relativitätstheorie und Einsteins Lehre von der Schwerkraft," which have been published as Nos. 38 and 58 of the series "Wissenschaft" by Vieweg.

§36. *The Energy Loss of an Accelerated Electron by Radiation and Its Reaction on the Motion*

We know that, unlike the electron in uniform motion, the accelerated electron radiates. According to (19.24) the energy radiated per unit time is, for a velocity small compared with c ,

$$S = \frac{e^2 \dot{v}^2}{6\pi\epsilon_0 c^3}. \quad (1)$$

This energy loss must of course find expression in the equation of motion of the electron. To take account of it we will replace it by an equivalent force. Consider the effect of a short acceleration interval from t_1 to t_2 . Before and after the interval, as well as at its limits, the motion is to be regarded as uniform, i.e. \dot{v} as equal to zero. In view of the briefness of the interval the velocity is changed but little, so that we may put $\beta_1 \cong \beta_2 \cong \beta$. We call the desired force the "reaction force of the radiation" and denote it by R . (Please excuse the use of the same symbol R as for the Roentgen current and, before that, for the impedance operator!) It must satisfy the condition that the work done by it on the electron in the interval from t_1 to t_2 be equal to the negative radiation loss of the electron, i.e.

$$\int R \cdot ds = - \int_{t_1}^{t_2} S dt. \quad (2)$$

From the identity

$$\dot{v}^2 = \frac{d}{dt}(v\dot{v}) - v\ddot{v}$$

and our assumption $\dot{v}(t_1) = \dot{v}(t_2) = 0$ we find

$$\int \dot{v}^2 dt = v\dot{v} \Big|_{t_1}^{t_2} - \int \dot{v}v dt = - \int \ddot{v} ds,$$

so that, by (1) and (2),

$$\int R \cdot ds = \frac{e^2}{6\pi\epsilon_0 c^3} \int \ddot{v} ds. \quad (3)$$

We thus obtain as the simplest formulation

$$|R| = \frac{e^2 \ddot{v}}{6\pi\epsilon_0 c^3}; \quad (4)$$

furthermore, it may be shown that other expressions consistent with (3) deviate from (4) only by terms of a smaller order of magnitude (see the

discussion after Eq. (27)). In the magnetic cgs-system this becomes, by (16.30),

$$|R| = \frac{2}{3} \frac{e^2 \ddot{\theta}}{c}; \quad (4a)$$

this may be compared with Larmor's formula (19.24b).

We shall study the effect of the reaction force for the very simple case of an electron vibrating about its position of rest, which, as in §19c, may serve as an idealized model of a light source. Let the vibration be rectilinear; in view of the reaction force it is damped. We call the distance of the electron from its position of rest ξ and set

$$\xi = \xi_0 e^{-i\omega t}, \quad \omega = \omega_0(1 + \alpha). \quad (5)$$

$\omega_0 = 2\pi/\tau$ is the angular frequency in the absence of damping, τ the corresponding period, α a complex number; it is very small in absolute value and, for our expression for ξ , must have a negative imaginary part. We shall demonstrate both facts.

The equation of motion of the electron is

$$m_0 \ddot{\xi} + f\xi = R. \quad (6)$$

The restoring force, which may arise in some fashion from the atomic binding, has been set equal to $-f\xi$ and been transferred to the left. We divide (6) by m_0 and put

$$\frac{f}{m_0} = \omega_0^2, \quad (7)$$

$$\frac{R}{m_0} = \frac{a}{c} \ddot{\xi}. \quad (7a)$$

(7) follows from Eq. (5), according to which ω_0 is the characteristic frequency of the oscillation for $R = 0$; in (7a) a signifies, by (4), a length of the same order of magnitude as the electron radius in (33.6). Eq. (6) then becomes

$$\ddot{\xi} + \omega_0^2 \xi = \frac{a}{c} \ddot{\xi}. \quad (8)$$

Substitution from (5) yields, after cancellation of $\omega_0^2 \xi$,

$$-(1 + \alpha)^2 + 1 = i \frac{a}{c} \omega_0 (1 + \alpha)^3 = 2\pi i \frac{a}{\lambda} (1 + \alpha)^3. \quad (9)$$

λ is the wave-length of the emitted light; even in the x-ray region it is very large compared to the radius a of the electron. We may hence neglect α as

compared with 1 on the right side of the equation and α^2 as compared with 2α on the left side. We thus find from (9)

$$\alpha = -\pi i \frac{a}{\lambda}. \quad (10)$$

The sign of our result agrees with the expectation expressed at (5). The fact that, in our approximate calculation, α has become purely imaginary indicates that the period of the oscillation is not changed appreciably by the reaction force, just as in (18.9d), where the period of the quasistationary current oscillation did not depend materially on the resistance. Substitution of (10) in (5) yields

$$\left| \frac{\xi}{\xi_0} \right| = \exp \left(-\pi \frac{a}{\lambda} \omega_0 t \right) = \exp \left(-2\pi^2 \frac{a}{\lambda} \frac{t}{\tau} \right). \quad (11)$$

The amplitude is hence reduced by a factor $1/e$ in a time, measured in periods of the oscillation,

$$\frac{t}{\tau} = \frac{\lambda}{2\pi^2 a}. \quad (12)$$

The corresponding light path measured in wave-lengths, x/λ , has the same value. For $\lambda = 4 \cdot 10^{-6}$ cm, $a = 2 \cdot 10^{-13}$ cm (12) yields

$$\frac{x}{\lambda} = 10^7, \quad x = 400 \text{ cm} = 4\text{M}. \quad (12a)$$

The "distance of coherence" of light waves, measured for particularly sharp (i.e. particularly monochromatic) spectral lines, is of the same order of magnitude. There are no absolutely sharp spectral lines. Every broken or damped wave train, when subjected to Fourier analysis (see Vol. VI, Exercise I.4), yields a finite spectral width (more precisely, half-value width). The Doppler effect, which is the result of the thermal motion of the emitting particles and is hence temperature-dependent, has the same consequence. The reciprocal of the damping time t given by (12) is defined as the *natural classical line width*. With the notation $D = 1/t$ and with $\tau = \lambda/c$ we obtain

$$D = 2\pi^2 \frac{ac}{\lambda^2}$$

or, with the value of a given by (33.6),

$$D = \frac{\pi}{3} \frac{e^2}{\epsilon_0 m_0 c} \frac{1}{\lambda^2}. \quad (12b)$$

This classical line width is the lower limit of the observable line width, at the lowest possible temperatures (elimination of the Doppler effect) and

the lowest possible pressures (elimination of so-called collision damping). We specialize the expression for the cgs-units customary in spectroscopy ($e^2/\epsilon_0 = 4\pi c^2 e_{\text{magn}}^2$ by Eq. (16.30)) and find

$$D = \frac{4\pi^2}{3} \frac{ce_{\text{magn}}^2/m_0}{\lambda^2}, \quad \begin{aligned} e_{\text{magn}} &= 1.60 \cdot 10^{-20}, & c &= 3 \cdot 10^{10}, \\ e_{\text{magn}}/m_0 &= 1.76 \cdot 10^7. \end{aligned} \quad (12c)$$

If in particular we set $\lambda = 4 \cdot 10^{-5}$ cm as before, we obtain

$$D = 7 \cdot 10^7 \text{ sec}^{-1}. \quad (12d)$$

This is very small, even compared with the minute frequency separation of the hydrogen doublet of the first Balmer line:

$$\Delta\nu_B = \frac{Rc\alpha^2}{2^4} \cong 10^{10} \text{ sec}^{-1} \quad \begin{aligned} Rc &= \text{Rydberg frequency} \\ \alpha &= \text{fine-structure constant} \end{aligned}$$

The reaction force R plays an important role in Planck's theory of black-body radiation (see Vol. V). It determines the amplitude to which a linear oscillator is raised in equilibrium with thermal radiation, from which it then accomplishes its emission of quanta.

So far we have considered only the slowly moving electron insofar as we have determined the force of reaction only from the standpoint of an observer moving with the electron. The theory of relativity makes it possible, however, to change the frame of reference and to determine then the reaction force for an electron moving with arbitrary velocity. The fact that a result differing materially from (4) will be obtained follows from the fact that Eq. (1) then ceases to be valid and, instead, (30.11) yields for the energy radiated per unit area and per unit time at an angle ϑ with respect to the direction of motion

$$S = \frac{c^2 \dot{v}^2}{16\pi^2 \epsilon_0 c^2 r^2} \frac{\sin^2 \vartheta}{(1 - \beta \cos \vartheta)^3}. \quad (13)$$

The acceleration is here assumed to be longitudinal, i.e. in the direction of motion. It should furthermore be noted that (13) refers to the time scale of the observer at rest, whereas we must know for the determination of R the radiation per unit time in the system of the moving electron. The retardation relation (29.10b) exists between the two time scales (t , time of the observer, τ , time of the electron as well as "intrinsic time" of the latter):

$$\tau = t - \frac{r}{c}, \quad d\tau = dt + \frac{v_r}{c} dt, \quad \frac{dt}{d\tau} = 1 - \frac{v_r}{c} = 1 - \beta \cos \vartheta.$$

Thus to refer S to the time scale of the moving electron we must multiply (13) by

$$\frac{dt}{d\tau} = 1 - \beta \cos \vartheta. \quad (14)$$

Furthermore, to pass from S to the total radiation S of the electron we must integrate over the sphere of radius r and with the element of angle $d\omega = 2\pi \sin \vartheta d\vartheta$. With the abbreviation $u = 1 - \beta \cos \vartheta$ we thus obtain

$$2\pi \int_0^\pi \frac{\sin^2 \vartheta \sin \vartheta d\vartheta}{(1 - \beta \cos \vartheta)^3} = \frac{2\pi}{\beta} \int_{1-\beta}^{1+\beta} \left\{ 1 - \left(\frac{1-u}{\beta} \right)^2 \right\} \frac{du}{u^3}.$$

This integral may be evaluated in an elementary manner and yields

$$\frac{2\pi}{3} \frac{4}{(1 - \beta^2)^{3/2}}.$$

Hence

$$S = \frac{e^2 \dot{\vartheta}^2}{6\pi \epsilon_0 c^3} \frac{1}{(1 - \beta^2)^{3/2}}. \quad (15)$$

The value of R which is now sought—we shall call it R' —must (for longitudinal acceleration) be implicitly related to (15) by (2). However, it is not determined uniquely hereby; for this it would be necessary, following Abraham, to add conservation of momentum to conservation of energy. We hence prefer to proceed from the relativistic equation of motion of the electron, which will yield an explicit value for R' ; at the same time we need no longer limit ourselves to the longitudinal case.

In accord with Eq. (32.4) etc. and adding the desired reaction force R' we write the equation of motion in the form:

$$m_0 W = F + R'. \quad (16)$$

W is the four-vector of the acceleration introduced in (27.18b), F , the four-force acting at any moment, which in the electrodynamic case is related to the Lorentz force K by Eq. (28.19b):

$$F = \gamma K, \quad \gamma = (1 - \beta^2)^{-1/2}. \quad (17)$$

W and F are perpendicular to the world line of the electron in the four-dimensional meaning of the term, i.e. (see (27.18c) and (28.19c))

$$V \cdot W = 0 \quad \text{and} \quad V \cdot F = 0. \quad (18)$$

Accordingly (16) leads to the requirement that R' also be perpendicular to the world line:

$$V \cdot R' = 0. \quad (18a)$$

It would seem most reasonable, following (4), to define \mathbf{R}' by

$$\mathbf{R}' = \dot{\mathbf{W}}, \quad b = \frac{e^2}{6\pi\epsilon_0 c^4}, \quad \dot{\mathbf{W}} = \frac{d\mathbf{W}}{d\tau} \quad (19)$$

(dots will indicate differentiation with respect to the intrinsic time also in what follows). This would, in fact, be a Lorentz-invariant definition of \mathbf{R}' which, specialized to the frame of reference of the electron, would agree directly in its space components with (4). However, this definition would contradict the requirement (18a). We hence modify it to

$$\mathbf{R}' = b(\dot{\mathbf{W}} + \alpha\mathbf{V}) \quad (19a)$$

and determine the constant α here introduced from the condition

$$\mathbf{V} \cdot \dot{\mathbf{W}} + \alpha \mathbf{V} \cdot \mathbf{V} = 0, \quad \alpha = -\frac{\mathbf{V} \cdot \dot{\mathbf{W}}}{\mathbf{V} \cdot \mathbf{V}}. \quad (20)$$

Like (19) this definition of \mathbf{R}' satisfies the requirement of being identical with (4) in the frame of reference of the electron, since the first three components of \mathbf{V} vanish here; furthermore if, as in (4), correction terms of a smaller order of magnitude are neglected the definition is also unique.

The value (20) of α can be further simplified: First, by (27.18a), $\mathbf{V} \cdot \mathbf{V} = -c^2$; second, we may deduce from $\mathbf{V} \cdot \mathbf{W} = 0$ by differentiation with respect to τ :

$$\mathbf{V} \cdot \dot{\mathbf{W}} + \dot{\mathbf{V}} \cdot \mathbf{W} = 0, \quad \mathbf{V} \cdot \dot{\mathbf{W}} = -\dot{\mathbf{V}} \cdot \mathbf{W} = -\mathbf{W} \cdot \dot{\mathbf{W}}.$$

Hence we may write

$$\alpha = \frac{1}{c^2} \mathbf{W} \cdot \dot{\mathbf{W}} \quad (20a)$$

and

$$\mathbf{R}' = b \left(\dot{\mathbf{W}} - \frac{\mathbf{W} \cdot \dot{\mathbf{W}}}{c^2} \mathbf{V} \right). \quad (21)$$

This is the very concise formulation of the reaction force, valid in every frame of reference. The conciseness is lost when we pass over to three dimensions, to permit comparison with the formulations known from the literature. We proceed from Eqs. (27.18) and (27.18b):

$$\mathbf{V} = (\eta\mathbf{v}, ic\eta), \quad \mathbf{W} = \dot{\mathbf{V}} = (\eta\dot{\mathbf{v}} + \eta\dot{\mathbf{v}}, ic\dot{\eta}) \quad (22)$$

and compute with the aid of the definition (17) of η

$$\eta^2 = \frac{1}{1 - \beta^2}, \quad \eta\dot{\eta} = \frac{1}{(1 - \beta^2)^2} \mathbf{v} \cdot \dot{\mathbf{v}}, \quad \frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{c^2} = \frac{\eta^4}{c^2} \mathbf{v} \cdot \dot{\mathbf{v}}, \quad \frac{\dot{\eta}}{\eta} = \frac{\eta^3}{c^2} \mathbf{v} \cdot \dot{\mathbf{v}}. \quad (22a)$$

We then obtain from (22)

$$W = \eta \left(\frac{\eta^2}{c^2} (\mathbf{v} \cdot \dot{\mathbf{v}}) \mathbf{v} + \dot{\mathbf{v}}, \frac{i}{c} \eta^2 \mathbf{v} \cdot \dot{\mathbf{v}} \right), \quad (22b)$$

$$W \cdot W = \eta^2 \left\{ \frac{\eta^4}{c^4} (\mathbf{v} \cdot \dot{\mathbf{v}})^2 v^2 + \frac{2\eta^2}{c^2} (\mathbf{v} \cdot \dot{\mathbf{v}})^2 + \dot{\mathbf{v}}^2 - \frac{\eta^4}{c^2} (\mathbf{v} \cdot \dot{\mathbf{v}})^2 \right\}.$$

The first and last term of the { } can be combined to yield

$$\frac{\eta^4}{c^2} (\mathbf{v} \cdot \dot{\mathbf{v}})^2 \left(\frac{v^2}{c^2} - 1 \right) = -\frac{\eta^2}{c^2} (\mathbf{v} \cdot \dot{\mathbf{v}})^2,$$

which combines with the second term. We thus obtain

$$W \cdot W = \eta^2 \left\{ \frac{\eta^2}{c^2} (\mathbf{v} \cdot \dot{\mathbf{v}})^2 + \dot{\mathbf{v}}^2 \right\} \quad (23)$$

$$\frac{W \cdot W}{c^2} \mathbf{v} = \frac{\eta^2}{c^2} \left\{ \frac{\eta^2}{c^2} (\mathbf{v} \cdot \dot{\mathbf{v}})^2 + \dot{\mathbf{v}}^2 \right\} \mathbf{v}, \dots \quad (23a)$$

In the last equation we have written down only the three space components of the four-dimensional vector. We do the same in the computation of \dot{W} , i.e. the differentiation of (22b) with respect to τ :

$$\dot{W} = \frac{3\eta^2}{c^2} \dot{\eta} (\mathbf{v} \cdot \dot{\mathbf{v}}) \mathbf{v} + \frac{\eta^2}{c^2} \{ \dot{v}^2 \mathbf{v} + (\mathbf{v} \cdot \ddot{\mathbf{v}}) \mathbf{v} + (\mathbf{v} \cdot \dot{\mathbf{v}}) \dot{\mathbf{v}} \} + \eta \dot{\mathbf{v}} + \eta \ddot{\mathbf{v}}.$$

If we take account of (22a) we find

$$\dot{W} = \frac{3\eta^2}{c^4} (\mathbf{v} \cdot \dot{\mathbf{v}})^2 \mathbf{v} + \frac{\eta^2}{c^2} \{ \dot{v}^2 \mathbf{v} + (\mathbf{v} \cdot \ddot{\mathbf{v}}) \mathbf{v} + 2(\mathbf{v} \cdot \dot{\mathbf{v}}) \dot{\mathbf{v}} \} + \eta \ddot{\mathbf{v}}. \quad (24)$$

We obtain for the difference of (24) and (23a)

$$\frac{R'}{b} = \frac{2\eta^2}{c^4} (\mathbf{v} \cdot \dot{\mathbf{v}})^2 \mathbf{v} + \frac{\eta^2}{c^2} \{ (\mathbf{v} \cdot \ddot{\mathbf{v}}) \mathbf{v} + 2(\mathbf{v} \cdot \dot{\mathbf{v}}) \dot{\mathbf{v}} \} + \eta \ddot{\mathbf{v}}. \quad (25)$$

Finally we pass from the intrinsic time of the electron, to which $\dot{\mathbf{v}}$ and $\ddot{\mathbf{v}}$ are referred, to the time scale t of the observer, in which we shall denote the corresponding quantities by \mathbf{v}' and \mathbf{v}'' . We set

$$\dot{\mathbf{v}} = \frac{d\mathbf{v}}{d\tau} = \eta \frac{d\mathbf{v}}{dt} = \eta \mathbf{v}', \quad \eta = \frac{\eta^4}{c^2} \mathbf{v} \cdot \mathbf{v}'$$

$$\ddot{\mathbf{v}} = \frac{d}{d\tau} (\eta \mathbf{v}') = \dot{\eta} \mathbf{v}' + \eta^2 \mathbf{v}'' = \frac{\eta^4}{c^2} (\mathbf{v} \cdot \mathbf{v}') \mathbf{v}' + \eta^2 \mathbf{v}''.$$

Thus we obtain from (25)

$$\frac{R'}{b} = \frac{3\eta^2}{c^4} (\mathbf{v} \cdot \mathbf{v}')^2 \mathbf{v} + \frac{\eta^2}{c^2} (\mathbf{v} \cdot \mathbf{v}'') \mathbf{v} + \frac{3\eta^2}{c^2} (\mathbf{v} \cdot \mathbf{v}') \mathbf{v}' + \eta^2 \mathbf{v}''. \quad (25a)$$

This value of R' is to be employed in the equation of motion (16). If we transform the latter into the customary form of the momentum equation (32.5), i.e. replace $m_0 W$ by $dG/dt = m_0 W/\eta$ and F by $K = F/\eta$, R' must also be replaced by $R^* = R'/\eta$. We thus obtain for R^* from (25a) after substituting the values of b and η :

$$R^* = \frac{e^2}{6\pi\epsilon_0 c^3} \frac{1}{1 - \beta^2} \left(\ddot{v}'' + \dot{v}' \frac{3\dot{v} \cdot \dot{v}'}{c^2(1 - \beta^2)} + \frac{\dot{v}}{c^2(1 - \beta^2)} \left[\dot{v} \cdot \ddot{v}'' + \frac{3(\dot{v} \cdot \dot{v}')^2}{c^2(1 - \beta^2)} \right] \right). \quad (26)$$

For small velocities ($\beta \rightarrow 0$) we obtain, of course, $R^* = R$, i.e. the value from Eq. (4). Eq. (26) was derived first by Abraham from electrodynamics and by v. Laue from the theory of relativity. Our procedure follows the suggestions of Pauli.¹

We must still establish the limits of validity of the formulas derived here. By using the relativity transformation for uniform motion we have assumed the acceleration to be "small" without otherwise restricting the magnitude of the velocity. We are hence dealing with a process of approximation or a series expansion which is broken off. H. A. Lorentz² carries out this process, representing the retarded potentials as power series of the relaxation time $t - r/c$ and computing the mechanical force of the intrinsic field. The first term of this expansion is the *inertial reaction* of the electron, which, for small velocities,³ is given by

$$-m_0 \ddot{v} = -\frac{e^2 \ddot{v}}{6\pi\epsilon_0 c^2 a}. \quad (27)$$

The reaction force R of Eq. (4) appears as second term:

$$R = \frac{e^2 \ddot{v}}{6\pi\epsilon_0 c^3}. \quad (27a)$$

Lorentz emphasizes that R is the only term of the expansion which does not depend on a ("on the shape of the electron"). The higher terms, which are not computed, are of the form

$$m_0 \left(\frac{a}{c}\right)^2 \ddot{v}, \quad m_0 \left(\frac{a}{c}\right)^3 \ddot{v}, \dots \quad (27b)$$

Here a/c is the time required by the light to traverse the "radius of the electron". The term (27a) can be brought into the same form; in view of

¹ Enzykl. d. Math. Wiss. Vol. V₂, p. 654.

² "The Theory of Electrons," Teubner, 1909, Note 18, p. 251.

³ \dot{v}' , \ddot{v}' ... are then identical with, \dot{v} , \ddot{v} ...

the meaning of m_0 it can be written

$$m_0 \frac{a}{c} \ddot{\mathbf{v}}.$$

The terms of the series must decrease in order that the series may converge, i.e. be practically useful. Hence we must have

$$|\ddot{\mathbf{v}}| < \frac{c}{a} |\dot{\mathbf{v}}|, \quad |\dot{\mathbf{v}}| < \frac{c}{a} |\mathbf{v}|, \dots \quad (28)$$

It is clear however that in processes involving very large energies, such as the flight of an electron close to an atomic nucleus, not only very great accelerations, but also very great changes in acceleration can occur. The termination of the series with the term R would then be no longer permissible. The same applies for the acceleration process in the betatron (see Problem III.10) and synchrotron. The formulation of the radiation resistance in such extreme cases constitutes an as yet unsolved problem which has led to many discussions (Wessel, Dirac, Bopp, Stückelberg).¹

§37. *Approaches to the Generalization of Maxwell's Equations and to the Theory of the Elementary Particles*

Gustav Mie took the first step in this direction in 1912 in his famous papers² "Foundations of a Theory of Matter." Their goal is no less than the generalization of the Maxwell equations so that they include the *existence of the electron*. In order that the generalization may not lose itself in limitless possibilities it is subjected from the start to the principle of relativity and derived from a "world function" which may depend only on Lorentz-invariant quantities. Here a distinction is made—possibly for the first time in a consistent fashion—between entities of intensity and entities of quantity, i.e. written in our notation and units, between

$$F = (c\mathbf{B}, -i\mathbf{E}), \quad \Omega = \left(\mathbf{A}, \frac{i}{c} \Psi \right)$$

on the one hand and

$$f = (\mathbf{H}, -ic\mathbf{D}), \quad \Gamma = (\rho\mathbf{v}, i\rho c)$$

¹ The most recent contributions to this question are given by the papers of W. Heitler and H. W. Peng, Proc. Cambridge Phil. Soc. 38, 296 (1942) and, from the standpoint of Einstein's latest methods, N. Hu, Proc. R. Irish Academy, 57, 87 (1947). *

² Ann. d. Phys: First communication, Vol. 37, p. 511; second communication, Vol. 39, p. 1. The third communication (Vol. 40, p. 1, 1913) deals with the theory of gravitation and is of course outdated, having been originated before the *general* theory of relativity.

on the other. Mie tests the invariants which may be set up with the entities of intensity and the entities of quantity respectively. We record three¹ of the former (see (26.24), (26.21), and (26.4)), suppressing constant factors insofar as they are dimensionally superfluous:

$$\Lambda = \frac{\epsilon_0}{2} (c^2 \mathbf{B}^2 - \mathbf{E}^2), \quad (1)$$

$$\mathbf{M} = c\mathbf{E} \times \mathbf{B}, \quad (2)$$

$$|\Omega|^2 = \Lambda^2 - \Psi^2/c^2. \quad (3)$$

He finds that only the invariants (1) and (3) need be considered for the description of quasistationary processes and constructs a world function² W such that at large distance from the electron the ordinary Maxwell equations apply, whereas the equations are modified at the electron and in its immediate neighborhood. Like Schwarzschild's function, the world function is to be integrated over an arbitrary region of the four-dimensional world and to be varied in suitable manner.

What must be the form of the world function if it is to yield the ordinary Maxwell equations at an adequate distance from the electron? According to our experience with Schwarzschild's principle of action we must then have $W = \Lambda$. In fact, in pure vacuum ($\Gamma = 0$ and no kinetic energy of matter) the kinetic potential K' in Eq. (32.18) reduces to the middle term, proportional to Λ and when subjected to the variation, yields the Maxwell equations of vacuum. At the same time the change in the world function in the neighborhood of the electron is to be such that from it a definite value e of the electron charge (or, at least, of the specific charge e/m of the electron) may be computed. This is certainly not so if the two invariants Λ and $|\Omega|^2$ are superposed linearly, since then also the resulting differential equations would be linear in the field and potential components respectively and their integrals would consequently involve coefficients which could be chosen arbitrarily. On the other hand, both requirements might be satisfied by the formula

$$W = \Lambda + a |\Omega|^n \quad (4)$$

where n is a sufficiently large number. Actually, the second term may then be neglected at sufficiently large distance from the electron since it

¹ Weyl, in §28 of his book to be quoted on p. 321, points out a fourth field invariant, constructed from \mathbf{F} and Ω .

² Mie himself calls the function constructed with entities of intensity "Hamiltonian function H " and designates as "world function" that constructed with entities of quantity. We have taken the liberty of reversing the nomenclature so as to establish correspondence with Schwarzschild's action function. The entities of intensity are then obtained from Mie's world function by differentiation with respect to the entities of quantity.

vanishes as r^{-n} , whereas a singularity of high order occurs at the location of the electron. For mathematical reasons Mie puts specifically $n = 6$; in this manner he obtains a spatially highly concentrated charge distribution, which, however, is not stable in the field of another electron.

It would, after all, have been indeed surprising if the fundamental problem of the elementary particles could have been solved by clever guessing. Today we are convinced that much experimental preparation will be required instead. Nevertheless, blazing the path to the problem was an act of great merit, as is evident from the fact that all later workers in the field have followed in Mie's tracks.

Pauli, in Nr. 64 of his paper in the *Enzyklopädie*, had already emphasized that dependence on the absolute values of the electrodynamic potentials led to serious difficulties in Mie's theory. Hence we shall avoid use of invariant (3) in the formulations to be discussed below. Born and Infeld¹ in particular utilize in their theory a world function which depends only on invariants (1) and (2). The nonlinearity of the electromagnetic field, which is required here also, follows from the choice of W as nonlinear function of Λ and M . The conjugate four-current Γ drops out along with the potential Ω and, just as in Lorentz's electron theory, must be brought in as a foreign element.

W is chosen so that an infinity of the field at the location of the electron is avoided. In this manner a finite value for the self-energy of the electron is obtained and a difficulty of classical theory, which yields infinite energy for the point electron ($a = 0$), is circumvented. For quasistationary problems, in which the above invariant (2) does not enter, the formulation of Born and Infeld is

$$W = \epsilon_0 b^2 \left\{ \sqrt{1 + \frac{2\Lambda}{\epsilon_0 b^2}} - 1 \right\}. \quad (5)$$

The universal constant b here introduced has the dimension of an electric field strength since, by (1), Λ has the dimension $\epsilon_0 E^2$.

This formulation follows the pattern of the action function of classical and relativistic mechanics. In the classical mechanics of the point mass not acted upon by forces we have as the integrand of the Hamiltonian principle the kinetic energy

$$T = \frac{m}{2} v^2. \quad (6)$$

In relativistic mechanics this is replaced by the "kinetic potential" in Eq. (32:9b)

$$K = m_0 c^2 \left\{ 1 - \sqrt{1 - \frac{v^2}{c^2}} \right\}. \quad (6a)$$

¹ M. Born, *Proc. Roy. Soc. London (A)* 143, 410, 1933/34; M. Born and L. Infeld, *loc. cit.* 144, 425, 1934. M. Born, *Ann. de l'Inst. Henri Poincaré*, Tome VII,

which passes over into (6) for $v \ll c$ and then becomes independent of c . Similarly (5) passes for $\Lambda \ll \epsilon_0 b^2$ over into the value $W = \Lambda$, which corresponds to Maxwell's theory, and becomes independent of b . Whereas (6a) sets an upper limit c to v , (6) imposes no restriction on v . Similarly (5) sets an upper limit b to the field strength E in the electrostatic case ($B = 0$, $\Lambda = -\epsilon_0 E^2/2$), whereas the formula $W = \Lambda$ permits an unlimited increase in the field strength.

The Maxwell equations in vacuum for E , B and D , H are retained in the theory of Born and Infeld. In the electrostatic case, to which we shall limit ourselves in the following, we obtain for a centrally symmetric field and a point charge e at $r = 0$, just as for the conventional theory

$$D_r = \frac{e}{4\pi r^2}. \quad (7)$$

D_r may also be determined from the world function by the general rule of Mie's theory:

$$D_r = -\frac{\partial W}{\partial E_r}. \quad (7a)$$

According to (5) this yields, with $B = 0$ and $W = \epsilon_0 b^2 \{\sqrt{1 - E^2/b^2} - 1\}$

$$D_r = \frac{\epsilon_0 E_r}{\sqrt{1 - E_r^2/b^2}}. \quad (8)$$

It follows that

$$\epsilon_0 E_r = \frac{D_r}{\sqrt{1 + D_r^2/(\epsilon_0^2 b^2)}}. \quad (9)$$

If (7) is substituted in (9) we obtain

$$E_r = \frac{e}{4\pi\epsilon_0} \frac{1}{\sqrt{r_0^4 + r^4}}, \quad r_0 = \sqrt{\frac{e}{4\pi\epsilon_0 b}}. \quad (10)$$

The quantity r_0 may be regarded as the electron radius. E_r is now everywhere finite, since for $r = 0$ we have $E_r = b = e/(4\pi\epsilon_0 r_0^2)$. D_r , on the other hand, becomes infinitely large at the same place. For $r > r_0$, E_r differs little from the Coulomb field $e/(4\pi\epsilon_0 r^2)$. The electrostatic potential is

$$\Psi(r) = \int_r^\infty E_r dr = \frac{e}{4\pi\epsilon_0 r_0} f\left(\frac{r}{r_0}\right) \quad (11)$$

with

$$f(x) = \int_x^\infty \frac{dy}{\sqrt{1 + y^4}}.$$

At the origin we have

$$\Psi(0) = \frac{e}{4\pi\epsilon_0 r_0} f(0), \quad f(0) = 1.854.$$

The Hamiltonian function H is related to our world function W by the general formula

$$H = W + \mathbf{E} \cdot \mathbf{D} - \mathbf{B} \cdot \mathbf{H}.$$

Thus we find in the electrostatic case

$$H = e_0 b^2 (\sqrt{1 - E_r^2/b^2} - 1) + E_r D_r. \quad (12)$$

If E_r is eliminated with the aid of (9) this contracts to

$$H = e_0 b^2 (\sqrt{1 + D_r^2/(e_0^2 b^2)} - 1), \quad (13)$$

We obtain therefore for the total energy

$$W = 4\pi \int_0^\infty H r^2 dr.$$

Substitution of D_r from (7) yields finally

$$W = \frac{e^2}{4\pi\epsilon_0 r_0} \int_0^\infty (\sqrt{1 + y^4} - y^2) dy \quad (14)$$

The numerical value of the integral is 1.236. If (14) is put equal to the selfenergy of the electron $m_0 c^2$ we obtain

$$r_0 = 1.236 \frac{e^2}{4\pi\epsilon_0 m_0 c^2}, \quad (15)$$

i.e. very nearly the classical radius a of the electron from (33.9). Then (10) yields for b the value

$$b = \frac{e}{4\pi\epsilon_0 r_0^2} \cong \frac{e}{4\pi\epsilon_0 a^2},$$

very nearly equal to the classical field strength E at the "edge of the electron".

In this manner an electron radius of the proper order of magnitude and a very high critical field strength b are obtained. The field of several point charges, also, can be determined in unique fashion. Within these limits the theory of Born and Infeld thus leads to sensible results, although its fundamental formula (5) can claim only heuristic validity.

We shall finally discuss the problem of the "scattering of light by light". This problem arose from Dirac's theoretical discovery of the *positron* and *pair production*. (Pair production, i.e. the simultaneous generation of an electron and a positron from hard gamma radiation, was realized experi-

mentally soon afterwards by Irene and Frederic Joliot-Curie, whereas the positron was observed in cosmic radiation by Anderson and Blackett and Occhialini.) It is clear that this problem also involves a change in the Maxwell equations for vacuum which is equivalent to a "non-linear theory of the electromagnetic field".¹ The linear Maxwell equations could not account for such scattering, but would imply that the fields of two interpenetrating light waves are simply to be superposed.

The problem has been treated quantum-mechanically by Euler and Kockel under the direction of Heisenberg² and hence lies entirely outside of the framework of our presentation. We can merely indicate the procedure for its solution.

The world function is here chosen so that for weak fields it reduces, as before, to the Lagrange density Λ . In the second approximation it is written as a function of the second degree in Λ and M . Only even powers of M may occur here, however, since for a transition from a right-handed to a left-handed coordinate system B and hence M change their sign (in the terminology of Euler M is not "mirror-invariant"). The next term of the expansion must hence have the form

$$\alpha\Lambda^2 + \beta e_0^2 M^2. \quad (17)$$

In order that α and β may be pure numbers and this second term have the same dimension as the first term Λ , we divide (17) by $e_0 \times$ the square of a critical field strength, which, as before, we shall call b . It is here defined as the field strength "at the edge of the electron", i.e. at a distance $r = a$ from its "center" (a = classical electron radius, multiplied by the fine-structure constant $1/137$).

We obtain thus the formula

$$W = \Lambda + \frac{1}{e_0 b^2} (\alpha\Lambda^2 + \beta e_0^2 M^2) + \dots \quad (18)$$

As in Mie's theory the components of the entities of quantity D and H are obtained from this world function as partial derivatives with respect to the corresponding components of E and B :

$$D = - \frac{\partial W}{\partial E}, \quad H = \frac{\partial W}{\partial B}. \quad (19)$$

¹ This is the title of the paper of Born mentioned in the last footnote.

² H. Euler and B. Kockel, *Naturwiss.*, **23**, 1935; Euler, Leipzig thesis, *Ann. d. Phys.* **28**, 1936; Heisenberg and Euler, *Z. Physik* **98**, 1936. See also the simplified representation in the paper of M. Born cited at the end of footnote 1 on p. 303. The problem of pair production was approached simultaneously from a different angle by R. Serber and E. A. Uhling, *Phys. Rev.* **48**, 1933.

Since, by (1) and (2),

$$\frac{\partial \Lambda}{\partial \mathbf{E}} = -\epsilon_0 \mathbf{E}, \quad \frac{\partial M}{\partial \mathbf{E}} = c\mathbf{B}, \quad \frac{\partial \Lambda}{\partial \mathbf{B}} = \epsilon_0 c^2 \mathbf{B} = \mathbf{B}/\mu_0,$$

$$\frac{\partial M}{\partial \mathbf{B}} = c\mathbf{E},$$

we obtain from (19)

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \left\{ \mathbf{E} + \frac{2}{\epsilon_0 b^2} (\alpha \Lambda \mathbf{E} - \beta \epsilon_0 M c \mathbf{B}) + \dots \right\}, \\ \mathbf{H} &= \frac{1}{\mu_0} \left\{ \mathbf{B} + \frac{2}{\epsilon_0 b^2} (\alpha \Lambda \mathbf{B} + \beta \epsilon_0 M \mathbf{E}/c) + \dots \right\}. \end{aligned} \quad (20)$$

The most difficult part is the determination of the numerical coefficients α and β from Dirac's theory of pair production, for which we refer to the original papers.

The final result is the following: Maxwell's equations for vacuum, relating \mathbf{B} , \mathbf{E} and \mathbf{D} , \mathbf{H} retain their form. However, just as in the theory of Born and Infeld, \mathbf{D} is no longer proportional to \mathbf{E} ; a correction term occurs in the expressions for \mathbf{D} and \mathbf{H} which depends on \mathbf{E} , \mathbf{B} , Λ , and M and is negligible compared with the principal term for weak fields. No arbitrary assumptions of any kind are made here; it is merely presumed that an expansion in ascending powers of the field strength is possible, this being indicated in Eqs. (18) and (20) by \dots . In the paper of Heisenberg and Euler the expansion has been extended by an additional term and has even been expressed in closed form. In any case, this work demonstrates the necessity of modifying Maxwell's equations for extremely strong fields even in vacuum.

§33. *General Theory of Relativity; Unified Theory of Gravitation and Electrodynamics*

In this paragraph also we must limit ourselves to a mere outline. A full presentation of the subject would require a separate textbook; it would be premature to write such a one at this time since many pertinent questions are as yet undecided.

For the present we shall follow the original presentation of Einstein as recorded particularly effectively in his Princeton lectures.¹ He proceeds entirely in the spirit of Klein's Erlangen program: *Classical physics* belongs to the group of elementary geometry (isotropy of space within itself).

¹ The four lectures on the theory of relativity, held at Princeton in May 1921, have been reprinted in A. Einstein, *The Meaning of Relativity*, 3rd Ed., Princeton University Press, 1950.

extended by the displacement of the time axis along itself. The *special theory of relativity* is founded on the group of the linear orthogonal transformations of the four world coordinates x_1, x_2, x_3 , and $x_4 = ict$ (isotropy of the four-dimensional world, Lorentz transformations). We are led to the *general theory of relativity* if we start from the broader group of point transformations which, in Vol. I, p. 16 we have characterized by the formulas

$$x'_k = f_k(x_1, x_2, x_3, x_4), \quad k = 1, 2, 3, 4. \quad (1)$$

In this manner all possible frames of reference become legitimate, not only those which move with a constant velocity $v < c$ relative to each other. "Space and time lose the last vestige of their absolute character postulated by Newton and become merely means for the description of physical phenomena." Such a program had already been set up by Ernst Mach. However, he gave up with a negative point of view (which, strangely enough, he denoted as positivism) and remained an opponent of Einstein's theory of relativity to the end of his days. The latter, on the other hand, assumed a *positive* attitude by inquiring into those space time relations which are conserved in all point transformations. The general theory of relativity signifies the *invariant* or *covariant* theory of this group of transformations.

The basis for this had been created in part by Gauss¹ in his theory of surfaces and by Riemann² in his initiation lecture.

Gauss studied the *inner* properties of a surface, apart from its external shape and position in three-dimensional space. For this purpose he represents the line element ds , i.e. the separation of two neighboring points of the surface, by the formula

$$ds^2 = E dp^2 + 2F dp dq + G dq^2. \quad (2)$$

p and q are parameters of two families of (in general not orthogonal) curves on the surface, and E, F , and G particular functions of p and q . For a pure bending of the surface (without dilatation or shearing) the totality of line elements and hence also the system of the coefficients E, F , and G is conserved.

Gauss shows that the *measure of curvature*

$$K = \frac{1}{R_1 R_2} \quad (3)$$

¹ *Disquisitiones generales circa superficies curvas* 1827, Ges. Werke, Vol. IV, translated into German in Ostwalds Klassiker Nr. 5.

² *Über die Hypothesen, welche der Geometrie zugrunde liegen*, 1854, Ges. Werke 2nd Edition, p. 272.

introduced by him (R_1 and R_2 are the two "principal radii of curvature" of the surface) may be expressed by the E, F, G and their first and second derivatives with respect to p and q . In this manner he arrives at his "Theorema egregium": If a curved surface is bent into another shape (without dilatation!) the measure of curvature remains invariant in all points. The measure of curvature hence expresses an *inner property* of the surface, whereas the definition $1/(R_1 R_2)$ (just like that by the "spherical image") appears to depend on the external shape of the surface and does not indicate its invariance.

Gauss commends his method of fixing attention on the inner properties of surfaces as "most worthy of being diligently exploited by geometers". We shall see that this challenge was heeded by Riemann and Einstein.

The character of the *geodesic*, or *shortest*, *lines* is, of course, also conserved in the bending since it rests solely on the extremal property of the integrated line element. We also mention the approximation of the surface by one of its tangential planes, although it does not belong to the inner relations of the surface; locations will here be indicated not in the curvilinear coordinates p, q , but in ordinary Cartesian coordinates.

We now consider, with Riemann, an n -dimensional manifold of very general structure. As the generalization of (2) and already written in Einstein's notation, its line element is

$$ds^2 = \sum_{\mu} \sum_{\nu} g_{\mu\nu} dx_{\mu} dx_{\nu}, \quad g_{\mu\nu} = g_{\nu\mu} (\mu, \nu = 1, 2 \dots n). \quad (4)$$

The $g_{\mu\nu}$ are given functions of the quite arbitrarily chosen parameters $x_1, x_2 \dots x_n$. Riemann studies the inner, invariant (in more general terms, covariant or contravariant) properties of such a manifold.

To begin with, however, we shall answer the simple question: What must be the dimension N of an Euclidean space in order that the n -fold manifold may be embedded in it? We shall employ Cartesian coordinates X_1, \dots, X_N in this Euclidean space. On the n -fold manifold they may be represented as functions of the n parameters x_1, \dots, x_n :

$$X_1 = F_1(x_1, \dots, x_n); \quad \dots \quad X_N = F_N(x_1, \dots, x_n). \quad (5)$$

If we form the Euclidean line element

$$dX_1^2 + dX_2^2 + \dots + dX_N^2$$

this contains the first derivatives of the functions F_1, \dots, F_N . In order that it may assume the form (4) on the embedded n -fold manifold with arbitrarily prescribed $g_{\mu\nu}$, the number N of the arbitrarily prescribable F must suffice for the determination of the also arbitrarily prescribable $g_{\mu\nu}$.

and hence be equal to the number of the $g_{\mu\nu}$, which is $n(n+1)/2$. We thus have¹

$$N = \frac{n(n+1)}{2}. \quad (6)$$

In the case of Einstein's four-dimensional world we have

$$N = \frac{4 \cdot 5}{2} = 10; \quad (6a)$$

in the Gaussian case of the two-parameter surface we have of course

$$N = \frac{2 \cdot 3}{2} = 3. \quad (6b)$$

For every point of the n -fold manifold a "plane" (Euclidean) manifold may be constructed which, departing from the n -fold manifold, plays the same role as the tangential plane in three-dimensional Euclidean space.

One of the inner properties of the n -fold manifold is, in particular, the minimal property of the geodetic (shortest or straightest) lines. As on the two-parameter surface, they are at the same time the paths of a point mass not subjected to forces. We demonstrate this with the aid of an example: Let a plane table top be covered with a cloth under which there may lie a stone. The geodetic paths are in general straight, but curved close to the stone. The mass point, which is assumed not to be acted upon by forces, including gravity, will here be deflected out of its straight path, in accord with the prevailing surface curvature. This is the simplest example of Einstein's theory of gravitation (stone = sun, point mass = planet).

Riemann investigates the generalization of the Gaussian concept of curvature to the n -dimensional manifold. Following Riemann, Christoffel has defined his three-indices symbols and Einstein his $\Gamma_{\mu\nu}^{\sigma}$, which are identical with them (see Appendix I, Eq. (3) in Vol. II of these Lectures). They depend only on the components $g_{\mu\nu}$ of the "fundamental tensor" and its derivatives with respect to the coordinates and are hence inner properties of the manifold. The "Riemannian curvature tensor" is formed from the g , Γ , and their derivatives; its vanishing is the condition for the manifold being "plane" (Euclidean). The Riemannian "symmetric curvature tensor" $R_{\mu\nu}$ is derived from it by "reduction" (summation with respect to one of the pairs of indices). The "Riemannian scalar" R is derived from the curvature tensor $R_{\mu\nu}$ in similar fashion; it is the generalization of the Gaussian measure of curvature K .

¹ See e.g. H. Lang, Ann. Physik, Vol. 61, 1919 (Munich thesis). I am informed that theorem (6) was stated by Schläfli as early as 1871 (Ann. Mat. pura appl. 5, p. 190), and has been proved by E. Cartan and M. Janet (E. Cartan, La géométrie riemannienne et ses généralisations. Encycl. Française, t.1, 1937).

As we saw from our primitive example of the table top, the curvature properties find expression in the paths of mass points subjected to no other forces; they act on them like forces of physical origin. Einstein recognized herein the *origin of gravitation*, giving quantitative content to an idea of Mach. As the most general force action, superseding all other physical agencies, it is attributed by Einstein solely to the curvature conditions of the space-time continuum.

However, how are these curvature conditions determined? *They are determined by the energies distributed in space and time.* Space and time exist only by virtue of the physical processes which occur in them. Their structure is derived from the latter. We are inclined to recall Goethe's grand vision of the "Mothers" in Faust II (corresponding, in a sense, to the Platonic ideas which existed before the creation of the world):

Göttinen thronen behr in Einsamkeit,
Um sie kein Raum, noch wen'ger eine Zeit.
Von ihnen sprechen ist Verlegenheit.
Nichts wirst du sehn in ewig leerer Ferne,
Den Schritt nicht hören, den du tust,
Nichts Festes finden, wo du ruhst.

The approach to the curvature conditions of Einstein's world which will now be described may seem as disconcerting to the reader as the voyage to the Mothers seemed to Faust; we shall guide the reader along a less forbidding path presently.

The curvature tensor $R_{\mu\nu}$ is to be related to the stress-energy tensor $T_{\mu\nu}$ of material and electromagnetic phenomena by the system of 10 equations ($\mu, \nu = 1, 2, 3, 4$)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}, \quad (7)$$

as is shown by Einstein. The factor of proportionality κ here introduced is in essence the constant G of Newton's law of gravitation. Since the $R_{\mu\nu}$ and R may be expressed by the Γ 's and g 's, and the Γ 's, in turn, by the g 's and their derivatives, Eqs. (7) are in effect a system of differential equations for the $g_{\mu\nu}$. A general solution of this system would of course be extremely involved. Einstein could show however that they lead in a first approximation to the statements of Newton's theory of gravitation for weak fields or $g_{\mu\nu}$ which differ only little from the Euclidean ones of the special theory of relativity (or, more exactly, the pseudo-Euclidean ones, in view of the negative sign of dx_4^2).¹

¹ Einstein wrote the author November 28, 1915:

"Last month I passed through one of the most exciting, absorbing and, at the same time, most productive periods of my life. I could not think of writing.

I recognized that my former field equations of gravitation were quite without basis. This is indicated by the following factors . . .

Having lost all confidence in the earlier theory, I saw clearly that a satis-

Even the first approximation, i.e. the mere fact of Newtonian attraction, reveals the non-Euclidean structure of the scale determination. In the second approximation there occur deviations from the Newtonian law, which are of course greatest in the neighborhood of large concentrations of energy. Hence the anomaly in the path of Mercury, the planet closest to the sun, the deflections (observable only during solar eclipses) of light rays passing very close to the edge of the sun, and the red shift of the spectral lines of the white dwarfs resulting from their extraordinarily high densities.

Gravitational and Inertial Mass

These are the abstract mathematical foundations of Einstein's theory of gravitation. At a much earlier date,¹ almost immediately after the discovery of the special theory of relativity, Einstein recognized a concrete physical basis in the *equivalence of gravitation and acceleration*. The phenomena observed in an elevator which is imagined to be freed from the influence of gravitation and is moving upward with the constant acceleration g are exactly the same as in the same system at rest or in uniform motion when it is subject to the influence of gravity. In both cases a thrown body describes a parabola, a body at rest on the floor presses against it with the force mg , and a pendulum of equal length has the same period of oscillation. Conversely, an elevator falling freely in a gravitational field is not subject to the influence of gravitation: The pressure on the floor ceases, the period of oscillation of a pendulum becomes infinite, and a thrown body describes a straight line. Such a freely falling system realizes a world free from gravitation and curvature, in which pseudo-Euclidean measure is valid, and hence corresponds to the tangential plane to the Riemannian space which was discussed before.

The prerequisite for this is the *identical character of gravitational and*

factory solution could be attained only on the basis of general covariant theory, i.e. of Riemann's covariant $R_{\mu\nu}$. Unfortunately I have immortalized the last errors of this conflict in the Academy papers which I shall send you soon. The final result is the following: . . . The Christoffel symbols $\left(\begin{smallmatrix} \mu\nu \\ \sigma \end{smallmatrix}\right)$ are to be regarded as the natural representation of the "components" of the gravitational field . . .

The splendid thing which I experienced was not only that now Newton's theory was obtained as first approximation, but that in addition, the precession of the perihelion of Mercury (43" per century) followed as second approximation. The magnitude of the deflection of light at the sun became twice as large as before."

And on February 8 he remarks on a post card:

"You will be convinced by the general theory of relativity when you have studied it. Hence I do not lose a word to defend it to you."

¹ Jahrbuch f. Radioakt. und Elektronik, Vol. 4, 1907, further elaborated in Ann. d. Phys., Vol. 35, 1911.

inertial mass, which was expressed in Vol. I, §3 by the equation

$$m_g = m_i. \quad (8)$$

Only if this is satisfied is the "weight" $m_{\text{grav}} g$ equal to the "inertial reaction" $m_{\text{inert}} g$ and only then is the period of oscillation the same for all pendulums of equal length. In detail the formula for this period is

$$\tau = 2\pi \sqrt{\frac{m_{\text{inert}} l}{m_{\text{grav}} g}}. \quad (8a)$$

Already Newton saw that a profound physical problem was hidden herein and Bessel pursued the problem by making extremely careful measurements on pendulums of different materials.¹ R. Eötvös increased the precision of such measurements by powers of ten with his torsion balance. However, Einstein was the first to interpret Eq. (8) in the final form

$$\text{gravitation} = \text{inertia} (= \text{world curvature}).$$

We shall show that this equivalence principle suffices for the elementary calculation of the $g_{\mu\nu}$ in a specific case,² i.e. to solve a problem which was formulated generally in Eq. (7), but was postponed as being too difficult.

Consider a centrally symmetric gravitational field, e.g. that of the sun, of mass M , which may be regarded as at rest. Let a box K_∞ fall in a radial direction toward M . Since it falls freely, K_∞ is not aware of gravitation and therefore carries continuously with itself the Euclidean metric valid at infinity. Let the coordinates measured within it be x_∞ (longitudinal, i.e. in the direction of motion), y_∞ , z_∞ (transversal), and t_∞ . K_∞ arrives at the distance r from the sun with the velocity v . v and r are to be measured in the system K of the sun, which is subject to gravitation. In it we use as coordinates r , ϑ , φ , and t . Between K_∞ and K there exist the relations of the special Lorentz transformation, where K_∞ plays the role of the system "moving" with the velocity $v = \beta c$, K that of the system "at rest". The relations are

$$dx_\infty = dr / \sqrt{1 - \beta^2} \quad (\text{Lorentz contraction}),$$

$$dt_\infty = dt \cdot \sqrt{1 - \beta^2} \quad (\text{Einstein dilatation}),$$

$$dy_\infty = r d\vartheta \quad (\text{invariance of the transversal lengths})$$

$$dz_\infty = r \sin \vartheta d\vartheta d\varphi$$

¹ F. W. Bessel, "Experiments on the Force, with which the Earth Attracts Different Kinds of Bodies," *Abhandlgen d. Preuss. Akad.* 1830; "Studies on the Length of the Second Pendulum", loc. cit. 1826—reprinted in Ostwald's *Klassiker* Nr. 7.

² On the basis of an unpublished paper of W. Lenz, which he kindly communicated to the author in 1944. In the planned publication he will render the argument given in the text more rigorous. He also intends, following Schwarzschild (see below), to extend the consideration to the interior of a sphere filled with an incompressible fluid.

Hence the Euclidean world line element

$$ds^2 = dx_{\infty}^2 + dy_{\infty}^2 + dz_{\infty}^2 - c^2 dt_{\infty}^2 \quad (9)$$

passes over into

$$ds^2 = \frac{dr^2}{1 - \beta^2} + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) - c^2 dt^2 (1 - \beta^2). \quad (9a)$$

The factor $1 - \beta^2$, which occurs here twice, is meaningful so far only in connection with our specific box experiment. In order to determine its meaning in the system of the sun we write down the energy equation for K_{∞} , as interpreted by an observer on K . Let m be the mass of K_{∞} , m_0 its rest mass. The equation then is:

$$(m - m_0)c^2 - \frac{GmM}{r} = 0. \quad (10)$$

At the left we have the sum of the kinetic energy in accord with Eq. (32.7) and of the (negative) potential energy of gravitation. The energy constant on the right was to be put equal to zero since at infinity $m = m_0$ and $r = \infty$. We have computed the potential energy from the Newtonian law, which we shall consider as a first approximation. We divide (10) by mc^2 and obtain then, since $m = m_0/\sqrt{1 - \beta^2}$,

$$1 - \sqrt{1 - \beta^2} = \frac{\alpha}{r}, \quad \alpha = \frac{GM}{c^2} = \frac{\kappa M}{8\pi} \text{ (see Eq. (7a)).} \quad (10a)$$

With $M = 3.3 \cdot 10^6 M_{\text{earth}}$ and $g = G M_{\text{earth}}/R^2$, $R = \text{radius of earth} = (2/\pi) 10^7$ meter we obtain

$$\alpha = 3.3 \cdot 10^6 g \left(\frac{2}{\pi} \frac{10^7}{3 \cdot 10^8} \right)^2 = 14.6 \cdot 10^3 \text{ meter} \cong 1 \text{ km.}$$

It follows from (10a) that

$$\sqrt{1 - \beta^2} = 1 - \frac{\alpha}{r}, \quad 1 - \beta^2 \cong 1 - \frac{2\alpha}{r}, \quad (11)$$

and hence, from (9a),

$$ds^2 = \frac{dr^2}{1 - 2\alpha/r} + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) - c^2(1 - 2\alpha/r) dt^2. \quad (12)$$

This is the line element derived by K. Schwarzschild¹ from Einstein's Eqs. (7). In Eddington's presentation² the 40 components $\Gamma_{\mu\nu}^{\sigma}$ of the gravi-

¹ Preuss. Akad., Sitzungsber. 1916, p. 189.

² See his excellent book: "The Mathematical Theory of Relativity," Cambridge, 1923.

tational field are computed and (12) is shown to be the exact solution of the ten equations contained in (7). Our derivation claims only to yield an approximation, since it utilizes the Newtonian law as first approximation and neglects, in the second Eq. (11), the term $(\alpha/r)^2$; nevertheless, our result is, as shown by Schwarzschild and Eddington, exact in the sense of Einstein's theory.

It might be asked at this point: What is the relativistically exact formulation of the Newtonian law? The question is wrongly put if a vector law is meant hereby. The gravitational field is not a vector field, but has a much more complex tensor character. For the single point mass it is completely described by the four coefficients g of the line element (12) and the vanishing of the remaining $g_{\mu\nu}$.

B. Observable Deductions from the General Theory of Relativity

We shall first deduce the anomaly of the perihelion of Mercury from the line element (12); the general formalism of tensor calculus will not be required here.

The law of the geodesic paths demands

$$\delta \int ds = 0. \quad (13)$$

Of the four coordinates r, ϑ, φ, t in (12) we choose φ as "independent variable" and hence write in place of (13)

$$\delta \int v d\varphi = 0, \quad (13a)$$

$$v^2 = \frac{\dot{r}^2}{1 - 2\alpha/r} + r^2(\dot{\vartheta}^2 + \sin^2 \vartheta) - c^2 \dot{t}^2(1 - 2\alpha/r), \quad (14)$$

$$\dot{r} = \frac{dr}{d\varphi}, \quad \dot{\vartheta} = \frac{d\vartheta}{d\varphi}, \quad \dot{t} = \frac{dt}{d\varphi}. \quad (14a)$$

We designate the "dependent variables" r, ϑ, t collectively by q . The method of the calculus of variation, which we utilized in Vol. I, §34 for the proof of the Lagrange equations, leads to the "Euler equation" (see the first footnote in the section referred to)

$$\frac{d}{d\varphi} \frac{\partial v}{\partial \dot{q}} - \frac{\partial v}{\partial q} = 0. \quad (15)$$

For $q = \vartheta$ (14) yields

$$\frac{\partial v}{\partial \dot{\vartheta}} = \frac{r^2 \vartheta}{v}, \quad \frac{\partial v}{\partial \vartheta} = \frac{r^2 \sin \vartheta \cos \vartheta}{v},$$

and hence, by (15),

$$\frac{d}{d\varphi} \frac{r^2 \dot{\vartheta}}{v} = \frac{r^2 \sin \vartheta \cos \vartheta}{v}.$$

The last equation is fulfilled for $\vartheta = \text{const} = \pi/2$; the other possibility $\vartheta = \text{const} = 0$ represents no planetary orbit, but a meteor falling centrally straight into the sun. Our denoting the plane of the planetary orbit by $\vartheta = \pi/2$ is obviously simply a convenient choice of our system of polar coordinates.

For $q = \bar{t}$ (14) yields

$$\frac{\partial v}{\partial \bar{t}} = -\frac{c^2 \bar{t}(1 - 2\alpha/r)}{v}, \quad \frac{\partial v}{\partial t} = 0, \quad \text{so that } \frac{d}{d\varphi} \frac{\bar{t}(1 - 2\alpha/r)}{v} = 0.$$

From this we conclude

$$\frac{\bar{t}(1 - 2\alpha/r)}{v} = \text{const.} \quad (16)$$

Since \bar{t} is real and v (as well as ds) is purely imaginary, the constant also is purely imaginary. We put it equal to ik and find from (16)

$$\bar{t} = \frac{ikv}{1 - 2\alpha/r}. \quad (17)$$

k is a first integration constant of the path of the planet.

For $q = r$ (15) would yield a differential equation for \bar{r} , whose integration would provide a second integration constant of the problem. It is simpler, however, to employ a general theorem, which we have established in Vol. I for an arbitrary variation problem and which corresponds in mechanics to the law of conservation of energy, i.e. Eq. (41.18a). We replace L in this equation by v in our present problem and the constant on the right side by ih (numerical value of the Hamiltonian function H in Eq. (41.18) of Vol. I and, at the same time, second integration constant of our problem). We thus find

$$\sum_j \frac{\partial v}{\partial \dot{q}_j} \dot{q}_j - v = ih \quad (18)$$

or, after multiplication with v ,

$$\frac{1}{2} \sum_j \frac{\partial v^2}{\partial \dot{q}_j} \dot{q}_j - v^2 = ihv. \quad (18a)$$

We evaluate the left side with the aid of (14), whereby all terms with \dot{r} , $\dot{\vartheta}$, and \dot{t} cancel. It then reduces to

$$-r^2 \sin^2 \vartheta = -r^2 \text{ since } \vartheta = \pi/2.$$

Hence (18a) yields

$$v = -\frac{r^2}{ih}. \quad (19)$$

(17) thus becomes

$$\dot{t} = -\frac{k}{h} \frac{r^2}{1 - 2\alpha/r} \quad (20)$$

and (14) yields, with $\vartheta = \pi/2$,

$$-\frac{\dot{r}^2}{h^2} = \frac{\dot{r}^2}{1 - 2\alpha/r} + r^2 - \frac{c^2 k^2}{h^2} \frac{r^4}{1 - 2\alpha/r}. \quad (21)$$

This is a differential equation for r which takes the place of the (once integrated Euler) equation for the dependent variable $q = r$ in Eq. (15). It is simplified if $u = 1/r$ is introduced as a new variable and if it is multiplied with $1 - 2\alpha/r = 1 - 2\alpha u$:

$$\dot{u}^2 + u^2(1 - 2\alpha u) + \frac{1 - c^2 k^2 - 2\alpha u}{h^2} = 0,$$

for which we may also write

$$\dot{u}^2 + u^2 = \frac{2\alpha u}{h^2} + 2\alpha u^3 - \frac{1 - c^2 k^2}{h^2}. \quad (22)$$

We differentiate with respect to the independent variable and cancel out $\dot{u} = du/d\varphi$. We thus obtain

$$\ddot{u} + u = \frac{\alpha}{h^2} + 3\alpha u^2. \quad (23)$$

For comparison we treat the same problem by Newton's theory. We start with the energy equation (W = sum of kinetic and potential energy):

$$\frac{1}{2} \left\{ \left(\frac{\partial r}{\partial t} \right)^2 + r^2 \left(\frac{\partial \varphi}{\partial t} \right)^2 \right\} - \frac{GM}{r} = \frac{W}{m}. \quad (24)$$

According to the law of equal areas we have, with the area constant denoted by hc :

$$r^2 \frac{\partial \varphi}{\partial t} = hc. \quad (24a)$$

On the left side of (24) we factor out $(d\varphi/dt)^2 = h^2 c^2 / r^4$, put once more $u = 1/r$, and obtain after division by $h^2 c^2$

$$\frac{1}{2} (\dot{u}^2 + u^2) - \frac{GM}{h^2 c^2} u = \frac{W}{mh^2 c^2}.$$

Also here we differentiate again with respect to φ , whereupon \dot{u} cancels out, and find in view of (10a)

$$\ddot{u} + u = \frac{GM}{h^2 c^2} = \frac{\alpha}{h^2}. \quad (25)$$

The relativistic equation (23) differs from (25) only in the correction term $3\alpha u^2$. This has no appreciable influence on the size or shape of the orbit, but affects merely the position of the perihelion. To recognize this we place the direction $\varphi = 0$ at the perihelion, which may be defined by $u = u_{\max}$ and hence $\dot{u} = 0$. Then u becomes an even function of φ . We may then, beginning with the solution of (25), expand this in a Fourier cosine series:

$$u = A + B \cos \varphi + \dots; \quad (26)$$

it is then found that the higher terms, indicated by \dots , vanish. In (23) we substitute¹

$$u = A + B \cos(\gamma\varphi) + C \cos(2\gamma\varphi) \dots \quad (26a)$$

and find for the determination of the constant γ here introduced from (23) the equation

$$\begin{aligned} A + (1 - \gamma^2)B \cos(\gamma\varphi) + (1 - 4\gamma^2)C \cos(2\gamma\varphi) + \dots \\ = \frac{\alpha}{h^2} + 3\alpha(A + B \cos(\gamma\varphi))^2 \\ = \frac{\alpha}{h^2} + 3\alpha A^2 + 6\alpha AB \cos(\gamma\varphi) + \frac{3}{2}\alpha B^2(1 + \cos(2\gamma\varphi)). \end{aligned}$$

Here we have already dropped the higher terms of the series in the correction term. A comparison of the coefficients yields

$$\begin{aligned} A &= \frac{\alpha}{h^2} + 3\alpha A^2 + \frac{3\alpha}{2} B^2 \\ (1 - \gamma^2)B &= 6\alpha AB \\ (1 - 4\gamma^2)C &= \frac{3}{2}\alpha B^2. \end{aligned} \quad (27)$$

¹ In a manner similar as for the fine structure of the hydrogen atom; the following calculation may be clearer than the customary astronomical one (Eddington). It should be noted that every deviation from Newton's or Coulomb's law effects a motion of the perihelion of the Kepler ellipse. The motion brought about by the variation of mass is, however, much smaller (by a factor $\frac{1}{2}$) than that arising from the gravitational correction.

Since B drops out of the middle equation, it serves to determine γ :

$$1 - \gamma^2 = 6\alpha A, \quad \gamma \cong 1 - 3\alpha A, \quad 1 - \gamma = 3\alpha A. \quad (28)$$

A may be determined geometrically in terms of the perihelion and aphelion distances (a and ϵ denote the major axis and the numerical eccentricity of the ellipse):

$$u_{\max} = \frac{1}{r_{\min}} = \frac{1}{a(1 - \epsilon)} = A + B + \dots \quad \text{for } \gamma\varphi = 0,$$

$$u_{\min} = \frac{1}{r_{\max}} = \frac{1}{a(1 + \epsilon)} = A - B + \dots \quad \text{for } \gamma\varphi = \pi,$$

so that

$$A = \frac{1}{a(1 - \epsilon^2)}.$$

Hence we find from (28)

$$1 - \gamma = \frac{3\alpha}{a(1 - \epsilon^2)}.$$

The precession $\delta\tilde{\omega}$ of the perihelion in the course of one revolution is also determined geometrically, namely by the formula

$$\gamma(2\pi + \delta\tilde{\omega}) = 2\pi, \quad \delta\tilde{\omega} = 2\pi \frac{1 - \gamma}{\gamma} \cong \frac{6\pi\alpha}{a(1 - \epsilon^2)} \quad (29)$$

if a term with α^2 is neglected. For Mercury the secular displacement of the perihelion is hence found to be $43''$, in agreement with observation.

With the aid of the preceding calculations the second test of the general theory of relativity, the *light deflection at the edge of the sun*, can also be readily treated. Light paths are geodetic lines, for which $ds = 0$. In the special theory of relativity they were the generatrices of the light cone $\Sigma dx_i^2 = 0$; now they are given by $\Sigma g_{ik} dx_i dx_k = 0$, i.e. in our case, according to Eq. (14), by $v = 0$. Hence we must set $h = \infty$ in Eq. (19). Eq. (23) then becomes

$$\ddot{u} + u = 3\alpha u^2.$$

In the integration it is permissible, as an approximation, to let γ approach 1. Then (27) leads to $\alpha A \rightarrow 0$, $C \rightarrow -\frac{1}{2}\alpha B^2$, $A \rightarrow \frac{2}{3}\alpha B^2$ (the last in view of $h = \infty$). Hence, by (26a)

$$u = \frac{3\alpha}{2} B^2 + B \cos \varphi - \frac{\alpha}{2} B^3 \cos(2\varphi). \quad (30)$$

For $\varphi = 0$ the light is to be tangent to the edge of the sun ($r = R$). We must hence have

$$\frac{1}{R} = \frac{3\alpha}{2} B^2 + B - \frac{\alpha}{2} B^2 = B + \alpha B^2,$$

$$B = \frac{1}{R} \frac{1}{1 + \alpha B} \cong \frac{1}{R} \frac{1}{1 + \alpha/R} \cong \frac{1}{R},$$

since $\alpha \cong 1$ km is very small in comparison with R . With $x = r \cos \varphi$, $y = r \sin \varphi$ (30) then yields, after multiplication with rR :

$$R = \frac{3}{2} \frac{\alpha}{R} \sqrt{x^2 + y^2} + x - \frac{1}{2} \frac{\alpha}{R} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$$

The light path comes to resemble a hyperbola, just as the path of the planet resembled an ellipse. With the assumption $|y| > |x|$ we obtain

$$R = \pm \frac{3}{2} \frac{\alpha}{R} y + x \pm \frac{1}{2} \frac{\alpha}{R} y = x \pm \frac{2\alpha}{R} y \quad (31)$$

The angle between the two asymptotes, which is equal to the deflection of the light from its original path, is $4\alpha/R = 1.75''$ and agrees well with the results of the solar eclipse expeditions. It is twice as large as the value obtained by an earlier more primitive calculation (Soldner as well as Einstein before 1915; see footnote on p. 311).

We note furthermore in this connection that in addition to the direction, the velocity of the light is changed by the gravitational field. In a radial direction, e.g. along the radius $\vartheta = 0$, taking account of $ds = 0$, it is

$$\frac{dr}{dt} = (1 - 2\alpha/r)c \text{ by (12)}. \quad (31a)$$

Finally, the last of the enumerated tests of the theory, the *red shift of the spectral lines in the gravitational field*, can be understood without any calculation. Consider a point of the curved world and construct there the (gravity-free, Euclidean) tangential plane. Let the coordinate changes in the latter, $dX_1, \dots, dX_4 = icdT$ coincide in direction with the coordinate changes $dr, \dots, dx = icdt$ in the gravitational field. In view of the equality of the two line elements we then have for a particle at rest which is radiating light

$$-c^2 dT^2 = -c^2(1 - 2\alpha/r)dt^2. \quad (32)$$

The measures of time dt and dT are hence different; the same applies to the frequencies ν and ν_0 (in absence of gravity), which are inversely proportional to these times. According to (32) we have

$$\nu = \sqrt{1 - 2\alpha/r} \nu_0 \cong (1 - \alpha/r) \nu_0$$

$$\nu - \nu_0 \cong -\frac{\alpha}{r} \nu_0. \quad (33)$$

The frequency is reduced by the gravitational field. In view of the meaning of α , given by Eq. (10a), the magnitude of the red shift is

$$\frac{1}{c^2} \frac{GM}{r} = \frac{|V|}{c^2}, \quad (34)$$

where V (Eq. (10)) is the gravitational potential. The spectrum of Sirius B and of other white dwarfs provides the experimental confirmation.

C. Unified Theory of Gravitation and Electrodynamics

Following Gauss and Riemann, Einstein put metric first, i.e., required the invariance of ds^2 and the tensor character of the $g_{\mu\nu}$. "Balance, rod, and clock" were the basic elements which he manipulated in the general, just as in the special theory of relativity. With them he was able to geometrize gravitation.

However, Maxwell's electrodynamics of vacuum also constitutes a complex of phenomena overshadowing material processes. The amazingly simple form which it assumes in the special theory of relativity and which may be transferred without appreciable changes to the realm of the general theory of relativity, covering arbitrary frames of reference, suggests similarly geometrization. However, the metric proves too restricted for this. Einstein attempted to broaden it by demanding, instead of the invariance of ds^2 , merely that of $ds^2 = 0$ (i.e. that of the line elements of the light cone).¹ We are then concerned only with the ratios of the $g_{\mu\nu}$, rather than with the $g_{\mu\nu}$ themselves.

Hermann Weyl had recognized even a short time before this that it was simpler and more natural to drop the metric departure and to begin directly with Einstein's $\Gamma_{\mu\nu}^\sigma$. The resulting system is known as *affine world geometry*. It provides a rule for "parallelism at a distance", i.e. a prescription for proceeding along a world line without departing from the initial direction. Both gravitation and electrodynamics fit into this system quite naturally. The $\Gamma_{\mu\nu}^\sigma$ were here assumed symmetric in the μ and ν . The result of this theory is recorded in his classic book "Raum-Zeit-Materie", Springer, Berlin, 1918.²

However the system can be generalized even further: $\Gamma_{\mu\nu}^\sigma$ and $\Gamma_{\nu\mu}^\sigma$ can

¹ "On a reasonable extension of the basis of the general theory of relativity", Preuss. Akad. 1921, p. 261, as well as the following notes on the unified field theory: loc. cit. 1925, p. 414; 1928, p. 3; and 1929, p. 3.

² English edition: H. Weyl, "Space-Time-Matter," Methuen, London, 1922.

be chosen to be different. There results an *asymmetric affine* theory, which gives rise to a new antisymmetric tensor. It has been sketched by Erwin Schrödinger¹ and is being developed by him in friendly competition with Einstein.

The incentive was provided by the following: *Nuclear physics* has joined atomic physics as a younger sister science. Whereas atomic physics from the corpuscular standpoint rests on the electrodynamic interactions between electrons and protons, the forces of nuclear physics must be ascribed to the *mesons*, which, in the meantime, have come to be demanded by theory and have been discovered experimentally. (The name meson derives from the fact that these elementary particles have a mass intermediate between those of the electron and the proton). Nuclear physics hence is *meson theory*. It demands an antisymmetric tensor differing from that of electrodynamics. Such a tensor is furnished by the *asymmetric affine* world geometry, which thus would create a triple bond between gravitation, electrodynamics, and nuclear theory. Its detailed structure has not yet been determined, however. When at last it has been fully elaborated Maxwell's theory, too, will be revealed in its full beauty and symmetry.

¹ See his note in *Nature*, May 13, 1944, and, following it, several papers in the *Proceedings of the Irish Academy* for the years 1944-46, the last, with the title "The general affine field laws", in Vol. 51, p. 41.

SYMBOLS EMPLOYED THROUGHOUT THE TEXT AND THEIR DIMENSIONS

Note: As a matter of course all equations in this volume are written in a dimensionally consistent manner and hence not tied to any particular choice of units (e.g. M = meter, Q = coulomb). They are, to use a preferred current expression, "equations of quantities". The "numerical equations", which are correct only for a specific choice of units, are fortunately falling more and more into disuse even in engineering.

e, q charge	Q , practical unit: 1 coulomb
ρ charge density	QM^{-3}
ω surface charge density	QM^{-2}
p, P magnetic pole strength	QMS^{-1} (in §8 P serves as fifth independent unit)
ρ_m magnetic density	$QM^{-2}S^{-1}$
ω_m magnetic surface density	$QM^{-1}S^{-1}$
E electric field strength	newton/ Q = $MKS^{-2}Q^{-1}$, 1 newton = 10^5 dynes
D electric excitation (displacement)	QM^{-2} , $\text{div } D = \rho$, $D_n - D'_n = \omega$, Eq. (3.11)
J conduction current density	$QM^{-2}S^{-1} = AM^{-2}$, A = ampere
I total conduction current	$QS^{-1} = A$, $I = \int J_n d\sigma$
\dot{D} displacement current density	$QM^{-2}S^{-1} = AM^{-2}$
C total current density = $J + \dot{D}$	$QM^{-2}S^{-1} = AM^{-2}$
V electric potential difference $= \int E \cdot ds$	joule Q^{-1} = volt = V ;
B magnetic field strength (induction)	newton/ P = $KS^{-1}Q^{-1} = VSM^{-2}$, $\text{div } B = 0$; 1 gauss = 10^{-4} VSM^{-2}
H magnetic excitation (ampere-turns per meter)	$PM^{-2} = QM^{-1}S^{-1} = AM^{-1}$, $\text{div } H = \rho_m$, $H_n - H'_n = \omega_m$, 1 oersted = $10^3/(4\pi)$ AM^{-1}
U magnetic potential difference $= \int_1^2 H \cdot ds$	$QS^{-1} = A$
Φ magnetic flux = $\int B_n d\sigma$	$M^2KS^{-1}Q^{-1} = VS$

Ψ electric potential	
$\mathbf{E} = -\text{grad } \Psi$	joule/Q = V
Ψ_m magnetic potential	
$\mathbf{H} = -\text{grad } \Psi_m$	$QS^{-1} = A$
\mathbf{A} vector potential, $\mathbf{B} = \text{curl } \mathbf{A}$	$MKS^{-1}Q^{-1} = VSM^{-1}$
\mathbf{S} radiation vector = Poynting vector = energy flux density	joule $M^{-2}S^{-1} = \text{watt } M^{-2}$
W energy density,	joule M^{-3}
$W = W_e + W_m, \quad W_e = \frac{1}{2}\mathbf{D} \cdot \mathbf{E}, \quad W_m = \frac{1}{2}\mathbf{H} \cdot \mathbf{B}$	
W_J Joule heat per unit volume	joule $M^{-3}S^{-1}$
$= \mathbf{E} \cdot \mathbf{J}$	
W energy in given volume	joule
ϵ dielectric constant	$Q^2/(\text{joule } M) = SM^{-1}\Omega^{-1} = \text{farad} \cdot M^{-1}$
μ permeability	$MKQ^{-2} = \Omega SM^{-1} = \text{henry} \cdot M^{-1}$
ϵ_0, μ_0 vacuum constants	$(\epsilon_0\mu_0)^{-1/2} = \text{velocity of light, } MS^{-1}$
σ conductivity	$(\mu_0/\epsilon_0)^{1/2} = \text{wave resistance, } \Omega$
$\epsilon' = \epsilon + i\sigma/\omega = \text{complex dielectric constant}$	$M^{-2}K^{-1}SQ^2 = M^{-1}\Omega^{-1}$
R resistance of a wire	$M^{-2}K^{-1}S^2Q^2 = SM^{-1}\Omega^{-1}$
L selfinductance	volt/ampere = $M^2KS^{-1}Q^{-2} = \Omega$
R impedance = $R + i\omega L$	$\Omega S = \text{henry}$
K capacitance	Ω
P dielectric polarization	$Q^2/\text{joule} = \Omega^{-1}S = \text{farad}$
M magnetization	$QM^{-2}, \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, see p. 74
η electric susceptibility	$QM^{-1}S^{-1}, \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, see p. 91
	pure number, $\mathbf{P} = \eta\epsilon_0\mathbf{E}$, $\epsilon = \epsilon_0(1 + \eta)$
κ magnetic susceptibility	pure number, $\mathbf{M} = \kappa\mathbf{H}$, $\mu = \mu_0(1 + \kappa)$
ω angular frequency	S^{-1} , $\omega = 2\pi/\tau$, $\tau = \text{period of vibration}$
k wave number in vacuum	$M^{-1}, k = 2\pi/\lambda, \lambda = \text{wave-length}$
k_L wave number in conductor	$M^{-1}, k_L^2 = \epsilon\mu\omega^2 + i\mu\sigma\omega = \epsilon'\mu\omega^2$
h wave number of surface waves on cylindrical guide	M^{-1}
κ in §20 to 25 = $\sqrt{\mu\sigma\omega/2}$ for alternating currents	$M^{-1}, 1/\kappa = d = \text{layer thickness in skin effect}$

Additional Symbols in Parts III and IV

x_4 imaginary time coordinate	$x_4 = ict$
ds world line element	$ds^2 = \sum_{j=1}^4 dx_j^2$

$d\tau$ element of intrinsic time = $ds/(ic)$	$d\tau = \sqrt{1 - \beta^2} dt, \beta = v/c$
γ imaginary angle of rotation in the Lorentz transformation	$\tan \gamma = i\beta$
R four-dimensional radius vector	$R = x_1, x_2, x_3, x_4$
V, W four-vectors of velocity and acceleration	$V = dR/d\tau, W = dV/d\tau$
Ω four-potential = $A, i\Psi/c$	VSM^{-1}
Γ four-current = $J, ic\rho$	$QM^{-2}S^{-1}$, in vacuum = $\rho(v, ic)$
F six-vector of the field	$VM^{-1}, F = cB, -iE = c \text{Curl } \Omega$
f six-vector of the excitation	$AM^{-1}, f = H, -icD = \frac{1}{\mu_0} \text{Curl } \Omega$
F^*, f^* dual six-vectors	$F^* = -iE, cB, f^* = -icD, H$
Λ Lagrange density	$\Lambda = \frac{1}{2c} f \cdot F = \frac{1}{2} H \cdot B - \frac{1}{2} D \cdot E$
M second invariant of the field	$M = \frac{i}{2} F \cdot F^* = c B \cdot E$
k force density = $\frac{1}{c} \Gamma \cdot F$	$k_{1,2,3} = \rho(E + v \times B), k_4 = \frac{i\rho}{c} v \cdot E$
K Lorentz force, three-dimensional	$K = e(E + v \times B)$
F four-force, $F \cdot V = 0$	$F_{1,2,3} = \frac{K}{\sqrt{1 - \beta^2}}, F_4 = \frac{ie v \cdot E}{c \sqrt{1 - \beta^2}}$
T stress-energy tensor	$T_{nm} = -\frac{1}{c} \sum_{r=1}^4 F_{nr} f_{mr} + \delta_{nm} \Lambda$
G momentum of point mass	$G = m(v, ic), m = m_0/\sqrt{1 - \beta^2}$
E_0 rest energy	$E_0 = m_0 c^2$
K kinetic potential	$K = m_0 c^2 (1 - \sqrt{1 - \beta^2})$
$-\Gamma \cdot \Omega$ Schwarzschild invariant	$\Gamma \cdot \Omega = \rho(v \cdot A - \Psi)$
E^* } Abbreviations in the equations	$E^* = E + v \times B$
H^* } for moving media	$H^* = H - v \times D$
\underline{A} Lorentz symbol	$\frac{\partial A}{\partial t} + v \text{div } A - \text{curl } (v \times A)$
J_l conduction current density	$J_l = J - \rho v$
R in §36: reaction force of radiation	depending on the frame of reference also denoted by R' or R^*
$\Gamma \cdot \Omega = \sum_{n=1}^4 \Gamma_n \Omega_n$	scalar product of two four-vectors yields a scalar
$\Gamma \cdot F_n = \sum_{m=1}^4 \Gamma_m F_{nm}$	scalar product of a four- and a six- vector yields a four-vector
$f \cdot F = \frac{1}{2} \sum_n \sum_m f_{nm} F_{nm}$	scalar product of two six-vectors yields a scalar
	$= \sum_{n < m} \sum_{m=1}^4 f_{nm} F_{nm}$

$(\mathbf{R} \times \mathbf{V})_{nm} = R_n V_m - R_m V_n$	vector product of two four-vectors yields a six-vector
$\text{Div } \Omega = \sum_{n=1}^4 \frac{\partial \Omega_n}{\partial x_n}$	divergence of a four-vector yields a scalar
$\text{Curl}_{nm} \Omega = \frac{\partial \Omega_n}{\partial x_m} - \frac{\partial \Omega_m}{\partial x_n}$	curl of a four-vector yields a six- vector
$\text{Div}_m T = \sum_{n=1}^4 \frac{\partial T_{mn}}{\partial x_n}$	vector divergence of a (general or also antisymmetric) tensor
$\text{Div}_m^* F = \text{Div}_m F^* = \sum_{n=1}^4 \frac{\partial F_{mn}^*}{\partial x_n}$	yields a four-vector
$\text{Div}^* F = 0$	dual vector divergence of a six- vector
$\text{Div Div } F = 0$	results in $F = \text{Curl } \Omega$
$\text{Div Curl } \Omega = \text{Grad Div } \Omega - \square \Omega$	applies for every six-vector
$\square \Omega_m = \sum_{n=1}^4 \frac{\partial^2 \Omega_m}{\partial x_n^2}$	

Numerical Values, Results of Measurements, and Definitions

- c = velocity of light in vacuum = $3.00 \cdot 10^8 \text{ MS}^{-1}$ (measurement)
 μ_0 = permeability of vacuum = $4\pi \cdot 10^{-7} \Omega \text{ SM}^{-1}$ (definition)
 ϵ_0 = dielectric constant of vacuum = $10^7 / (4\pi c^2) \text{ M}^{-1} \text{ S} \Omega^{-1}$
 $\quad \quad \quad = 10^{-2} / (36.00\pi) \text{ M}^{-1} \text{ S} \Omega^{-1}$ (consequence)
 $(\mu_0 / \epsilon_0)^{1/2}$ = wave resistance of vacuum = $120.0\pi \Omega$ (consequence)
 e = electronic charge = $1.60 \cdot 10^{-19} \text{ Q}$ (measurement)
 e/m_0 = specific charge of the electron = $1.76 \cdot 10^{11} \text{ Q/K}$ (measurement)
 m_0 = rest mass of the electron = $0.90 \cdot 10^{-30} \text{ K}$ (consequence)
 eV = electron volt = $1.60 \cdot 10^{-19} \text{ joule}$ (consequence)
 $m_0 c^2$ = rest energy of the electron = $\frac{1}{2}$ million eV = $0.81 \cdot 10^{-13} \text{ joule}$ (consequence)

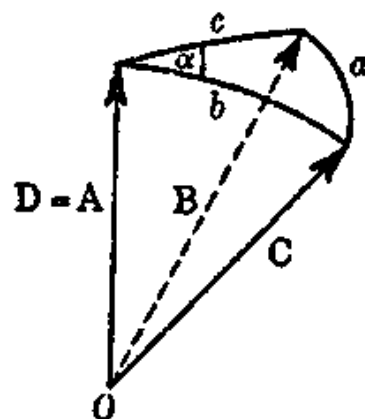
PROBLEMS FOR PART I

I.1. The Boundary Conditions of Maxwell's Theory. Derive Eqs. (3.7a) to (3.12) for \mathbf{E} , \mathbf{B} , \mathbf{H} , and \mathbf{D} by the differential method. The transition from medium 1 to 2 must then be assumed to be continuous ("boundary layer" instead of "boundary surface"). Use a rectangular coordinate system x, y, z and let z be perpendicular to the boundary surface which, in the limit of infinite smallness, may be regarded as plane. In the boundary layer ($h \rightarrow 0$) the derivatives with respect to z occurring in the differential equations (4.8) must be continuous in order that these equations may be meaningful.

I.2. The Magnetic Excitation Inside and Outside of an Infinitely Long Wire. Proof of Eqs. (4.10) to (4.13) from the differential equations.

I.3. The Magnetic Excitation within an Infinitely Long Solenoid. Proof of Eq. (4.14) from Maxwell's equations.

FIG. 44.



I.4. The Cosine Law of Spherical Trigonometry as Special Case of a General Vector Formula. Prove the vector formula:

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

and deduce from it, for the special case $\mathbf{D} = \mathbf{A}$ and with reference to Fig. 44, the cosine law

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

PROBLEMS FOR PART II

II.1. The Charging Potential of a Conducting Ellipsoid of Revolution. Let a be the major axis, b the minor axis, and $c = \sqrt{a^2 - b^2}$ the linear eccentricity. For fixed c and variable a

$$\frac{x^2 + y^2}{a^2 - c^2} + \frac{z^2}{a^2} = 1$$

represents the family of confocal ellipsoids with the separation of focal points $2c$. Show that on each of them the expression for Ψ given in Eq. (9.4) is constant (independent of x, y, z).

II.2. The Infinitely Long Rubbed Glass Rod and its Comparison with the Conducting Paraboloid of Revolution. Calculate the potential of an infinitely long uniformly charged straight line terminated at one end, and show that its equipotential surfaces are the same as those for the conducting paraboloid of revolution which is obtained from (9.4) by transition to the limit $c \rightarrow \infty, a \rightarrow \infty$.

II.3. Comparison of the Dielectric and the Conducting Sphere. For a dielectric sphere $r = a$ placed in an originally uniform electric field there always exists a concentric conducting sphere $r = b < a$, whose exterior field agrees, for $r > a$, with the exterior field of the dielectric sphere. Fig. 45 shows, for $r > a$, the field of the dielectric sphere, for $a > r > b$, not the (uniform) field within this sphere, but the analytical continuation of the exterior field, which is identical with the field of the conducting sphere of radius b . Prove that

$$b = a \sqrt{\frac{\epsilon - 1}{\epsilon + 2}}.$$

The figure shows how the singularity of the equilibrium point of the conductor (see footnote concerning Fig. 9a) develops continuously from the regular behavior of the force lines for the nonconductor.

II.4. Edge Correction for the Plate Condenser According to Kirchhoff. Convince yourself that the relation

$$z = \frac{a}{2\pi} f(\zeta), \quad f(\zeta) = 1 + \frac{2\pi i \zeta}{V} - \exp\left(\frac{2\pi i \zeta}{V}\right) \quad \begin{cases} z = x + iy \\ \zeta = \Psi + i\Phi \end{cases}$$

represents the fringe field of the unilaterally terminated condenser in Fig. 46. $\Psi = \text{const}$ are the equipotential lines in the x, y -plane, $\Phi = \text{const}$, the lines of force. Show that the two families of curves correspond qualitatively to the dotted lines in the figure, and that the line of force $\Phi = 0$ (drawn as full line in the figure) is an arc of a cycloid which joins the two edge points $x = 0, y = 0$ and $x = 0, y = a$.

II.5. The Capacitance of a Leyden Flask (Cylindrical Condenser). Let the dimensions be: height $h = 20$ cm, inner radius $r_1 = 5$ cm, wall thickness $d = 1$ mm. Let the dielectric constant of the glass be $6 \epsilon_0$. Boundary corrections are to be neglected. The capacity is to be expressed in microfarads.

II.6. On the Definition of the Capacitance of Two Conductors with Equal and Opposite Charges. If in (10.15) we put $E_1 = -E_2 = E$ and $v = \Psi_1 - \Psi_2$ we obtain

$$\begin{aligned} 2W = VE &= (H_{11} + H_{22} - 2H_{12})E^2 \\ &= K_{11}\Psi_1^2 + K_{22}\Psi_2^2 + 2K_{12}\Psi_1\Psi_2. \end{aligned} \quad (1)$$

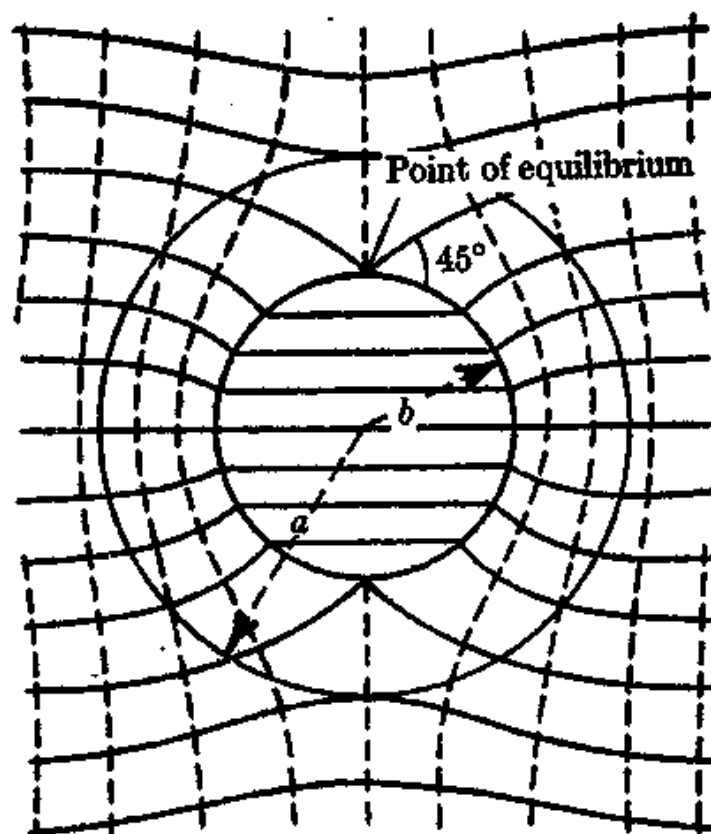


FIG. 45. The field of the *dielectric* sphere of radius $r = a$, which is produced by a uniform field on the outside, yields, when continued analytically into the interior, at the same time the field in the exterior of a *conducting* sphere of radius $r = b < a$ which is produced by the same uniform field.

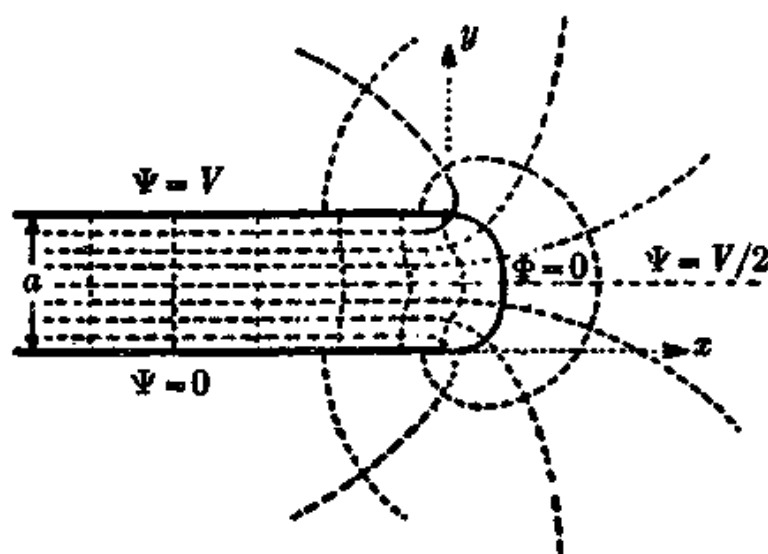


FIG. 46. Shape of the equipotentials $\Psi = \text{const}$ and lines of force $\Phi = \text{const}$ at the edge of a plate condenser.

Show by comparison with (10.11) that the following relation exists between the elementary definition of capacitance K and the coefficients H_{ij} and K_{ij} :

$$K = \frac{1}{H_{11} + H_{22} - 2H_{12}} = \frac{\Delta'}{K_{11} + K_{22} + 2K_{12}}, \quad \Delta' = \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} \quad (2)$$

II.7. Characteristic Oscillations and Characteristic Frequencies of a Completely Conducting Cavity Bounded by a Rectangular Parallelepiped. Using Eqs. (24.9) and (24.10), represent the completely continuous field within a rectangular parallelepiped with the sides a, b, c , with the condition $E_{\text{tang}} = 0$ on the three pairs of bounding surfaces

$$x = \begin{cases} 0 \\ a \end{cases}, \quad y = \begin{cases} 0 \\ b \end{cases}, \quad z = \begin{cases} 0 \\ c \end{cases}.$$

II.8. Characteristic Oscillations and Characteristic Frequencies of the Interior of a Perfectly Conducting Circular Cylinder of Finite Length. Using Eqs. (24.6) and (24.7), represent the continuous field within a circular cylinder of radius a and length l , with the condition $E_{\text{tang}} = 0$ both on the mantle surface $r = a$ and on the two end surfaces $x = \begin{cases} 0 \\ l \end{cases}$.

II.9. Characteristic Oscillations within a Cavity Bounded by a Metal Sphere. As in §19, start with a Hertzian vector π which is directed along a diameter of the sphere ($\vartheta = \begin{cases} 0 \\ \pi \end{cases}$), is periodic in t , and otherwise depends only on r . In contrast with §19 π must now be continuous also at the center of the sphere. The state then corresponds not to a spherical wave emitted from this point, but to a superposition of a spherical wave radiated outward and a spherical wave (reflected by the spherical surface) radiated inward. Determine the characteristic wave numbers k and the corresponding characteristic frequencies $\omega = kc$ from the boundary condition $E_{\text{tang}} = 0$ at the surface of the sphere $r = a$.

II.10. Determination of the Propagation Constants of Wire Waves from Kelvin's Telegraph Equation and from Rayleigh's Alternating Current Resistance a. for a Lecher two-wire line, b. for return conduction through the ground (let conduction in the forward direction be perfect).

PROBLEMS FOR PARTS III AND IV

III.1. The Lorentz Transformation for a Relative Motion Deviating from the x -Axis.

Let α be the angle between the relative motion \mathbf{v} and the x -axis of the "system at rest". Let the xy -plane of the latter coincide with the plane through x and \mathbf{v} . We consider an "intermediate system" x_1, y_1, z_1, t_1 whos x_1 axis is to coincide with the \mathbf{v} -direction and whose x_1y_1 -plane coincides with the xv -plane. The transformation

$$x, y, z, t \rightarrow x_1, y_1, z_1, t_1, \quad z_1 = z, \quad t_1 = t \quad (1)$$

is then an ordinary rotation through α in the xy -plane. Let a moving system x'_1, y'_1, z'_1, t'_1 be so placed that its x'_1 - and y'_1 -axes agree with the x_1 - and y_1 -axes of the intermediate system for $t = 0, t' = 0$. The transformation

$$x_1, y_1, z_1, t_1 \rightarrow x'_1, y'_1, z'_1, t'_1, \quad y'_1 = y_1, \quad z'_1 = z_1 \quad (2)$$

is then a special Lorentz transformation and is hence represented by Eq. (27.10).

If, finally, x'_1, y'_1, z'_1, t'_1 is rotated again through the angle $-\alpha$ in the $x'_1y'_1$ -plane, corresponding to the transformation

$$x'_1, y'_1, z'_1, t'_1 \rightarrow x', y', z', t', \quad z'_1 = z', t'_1 = t' \quad (3)$$

the system of coefficients of the resulting total transition

$$x, y, z, t \rightarrow x', y', z', t' \quad (4)$$

is simplified. Convince yourself of the obvious fact that this transformation is orthogonal in four dimensions and of the not obvious fact that it may be represented by a three-dimensional vector formula.

III.2. On the Addition Theorem for Two Differently Directed Velocities. Prove Einstein's formula (27.19a) by the method of the Lorentz transformation.

III.3. The Field of an Electron in Uniform Motion. Transform the representation (30.6) by means of considerations of elementary geometry applied to Fig. 42 into the representation (28.14), (28.14a).

III.4. On the Relativistic Energy Theorem for the Electron. Derive the expression (32.7) for the kinetic energy and the energy theorem (32.6) from the equation of motion (32.5) of the electron by the usual method (scalar multiplication with the velocity).

III.5. The Electron in the Uniform Electrostatic Field. An electron enters a (vacuum) condenser with a transparent upper plate with the velocity v at an angle α . Let the plate separation be d , the potential difference of the upper with respect to the lower plate, V volts.

What curve does the electron describe in a non-relativistic treatment? How closely does it approach the lower plate?

For what velocity does it reach the lower plate?

(Example: $v = 5 \cdot 10^6$ meter/sec; $d = 10^{-2}$ meter; $V = 110$ volts.)

What potential field must an electron which is initially at rest traverse to attain the velocity $v = 5 \cdot 10^6$ M/S?

How do the conditions change for a relativistic treatment?

III.6. The Electron in a Uniform Magnetostatic Field. If the initial velocity of the electron is perpendicular to the lines of force a circular path is described. Determine its radius. If a component parallel to the lines of force is present, the path becomes a helix with circular projection. This applies in the relativistic just as in the non-relativistic case.

III.7. The Electron in a Uniform Electric Field and a Uniform Magnetic Field which is Parallel thereto. In Kaufmann's arrangement for the measurement of e/m the β -rays emitted by a radium sample pass first through a

narrow aperture D and then cross a uniform electric field $\pm E$ and magnetic field B parallel thereto ($\pm E$ signifies reversal of polarity of the condenser). Assume that both fields begin at the aperture D and reach to the photographic plate, mounted perpendicular to the beam direction at a distance a from D . What curve is recorded on the plate if the β -rays are emitted with all possible velocities v ? Neglect the change in the total velocity as compared with the large v , and represent the coordinates of the points of incidence as function of the parameter v .

III 8. The Electron in a Uniform Electric Field and a Uniform Magnetic field Perpendicular thereto. The path is a trochoid. Under the influence of

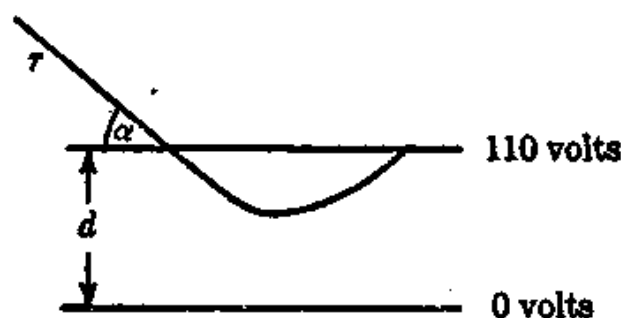


FIG. 47. The electron describes a ballistic parabola in the uniform condenser field.

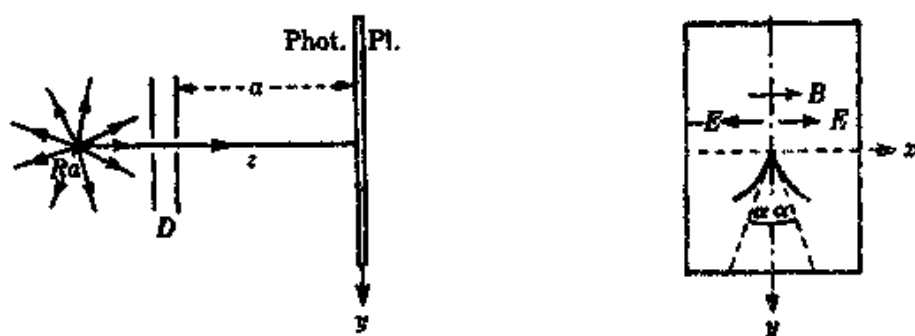


FIG. 48. Kaufmann's arrangement for the measurement of e/m . At the left: Lateral view. On the right: The pattern observed on the plate for β -rays with a continuous velocity spectrum emitted by the radium sample.

the electric field the circular motion produced by the magnetic field is converted into the motion of a point on a rolling circular disk. For what initial conditions is a *simple cycloid* obtained?

Motions of this type occur in the "magnetron" electron tube.

III 9. The Characteristic of the Thermionic Diode According to Langmuir and Schottky. In practice a *cylindrical* configuration is generally employed: The cathode is a wire along the axis of cylinder whose mantel surface coincides with the anode. The *plane* configuration is mathematically simpler: Here the cathode at $x = 0$ and the anode at $x = l$ are plane circular disks separated at the edge by an insulating cylindrical tube. Let $V(x)$ be the

potential at the point x between the cathode $V(0) = 0$ and the anode $V(l) = V$. Let the number of the electrons leaving the cathode per second be so large that it is permissible to assume a continuous space charge $-\rho$ of the electrons. Between cathode and anode the Poisson equation $\Delta V(x) = d^2V(x)/dx^2 = \rho/\epsilon_0$ is valid. The current density $J = \rho v$ transported through the tube is independent of x . v is determined from $mv^2/2 = eV(x)$. Integrate the Poisson equation by assuming a power law and deduce herefrom the so-called characteristic of the tube (I as function of the applied voltage V).

Convince yourself that the same method of assuming a power law is applicable also to the cylindrical configuration.

III.10. The Acceleration of an Electron in the Betatron. In the betatron¹ electrons are injected in the plane of symmetry between the axially symmetric pole pieces of an *alternating-current* electromagnet. The magnetic field forces them into a circular orbit. The pulsing of the magnetic field is accompanied by a vortex-like electric field which accelerates the electrons in their orbit. There is a radius $r = r_0$ of the path which remains unaltered with the pulsation of the magnetic field and with increasing electron velocity. After countless revolutions the electrons reach a velocity approaching that of light; they then resemble the beta-rays of radioactive materials, whence the name "betatron".

Let the axially symmetric magnetic field distribution $B(r, t)$ (its axial component) between the pole pieces, which decreases monotonically outwards, and the tangential initial velocity v_A of the electrons be given. We require

1. the attainable momentum mv of the electrons, their velocity, mass, and energy in eV (electron volts),
2. the radius r_0 of the equilibrium orbit,
3. the frequency of revolution at the end of the acceleration period and the total number of revolutions, and
4. the reaction force of the radiation at this point.

IV.1. The Field of Unipolar Induction. Let a bar be inserted in a uniform magnetic field B perpendicular to the lines of force and be displaced with uniform velocity along its axis. Compute

- a. the electric field in the interior and its potential,
- b. the voltage between its two sides,
- c. the external field, specifically for a circular cross section, and
- d. the surface charge.
- e. What change results if we pass from uniform translation of the bar

¹ This has also been called a rheotron, beam transformer, or electron centrifuge. The original idea of the device was given in the Aachen thesis of R. Wideröe in the year 1928.

along its axis to uniform rotation of a body of revolution about its axis of symmetry?

ANSWERS AND COMMENTS

I.1. The derivatives

$$\frac{\partial E_y}{\partial z} \quad \text{and} \quad \frac{\partial H_y}{\partial z}$$

occur in the x -components of Eqs. (4.8). These must remain finite in the transition to the limit $h \rightarrow 0$ so that the left sides of the equations in question do not become infinite, since this would make B_z and D_z infinite. In view of the y -components of Eqs. (4.8) this applies also to the derivatives

$$\frac{\partial E_z}{\partial z} \quad \text{and} \quad \frac{\partial H_z}{\partial z}.$$

The continuity of the *tangential components* E_x, E_y, H_x, H_y (see (3.9) and (3.8a)) follows herefrom.

In $\text{div } \mathbf{B} = 0$ (Eq. (4.4a)) there occurs the derivative $\partial B_z / \partial z$, which must also be continuous in the boundary layer: hence the condition for the continuity of the *normal component* B_z , Eq. (3.7a). The z -component of the Maxwell equation $\dot{\mathbf{B}} = -\text{curl } \mathbf{E}$ does not suffice for this conclusion. It is true that on the right there occur only the tangential components E_x and E_y and differentiations with respect to x and y , so that the right side is continuous. However, the continuity of the left side, which may be deduced herefrom, would be consistent with a time-independent discontinuity of B_z . Hence the auxiliary condition $\text{div } \mathbf{B} = 0$ becomes necessary.

The same considerations, applied to Eq. (4.4b) for nonconductors, $\text{div } \mathbf{D} = \rho$, lead in the limit $h \rightarrow 0$ to $\rho \rightarrow \infty$. This conclusion is by no means to be rejected, but indicates that at the boundary of two nonconductors there may exist a surface charge ω , which corresponds to an infinitely great charge density and is equal to the jump in the normal component of \mathbf{D} on the two sides. By the z -component of the Maxwell equation $\dot{\mathbf{D}} = \text{curl } \mathbf{H}$, this jump must be time-independent. In the general case of a conductor and a nonconductor surface charge may also occur, in accord with Eq. (4.4c) or the z -component of the Maxwell equation $\dot{\mathbf{D}} + \mathbf{J} = \text{curl } \mathbf{H}$; however, this surface charge need not be constant, but may decay as indicated by the current density J_z in the conductor.

I.2. We concern ourselves only with the magnetic field. Of the electric field, which we shall investigate in greater detail in §17, we need only know that it gives rise to a uniform current field J_z within the wire, $0 < r < a$, and to an also uniform current field J_{-z} , corresponding to an equal but oppositely directed total current I , in the return conductor:

$$I = \pi a^2 J_z = -\pi(c^2 - b^2) J_{-z}. \quad (1)$$

We employ polar coordinates r, φ, z with $r = 0$ as the axis of the wire. In view of the symmetry of the problem

$$\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial z} = 0$$

for all three components H_r, H_φ, H_z . The φ - and z -components of the Maxwell equation $\text{curl } \mathbf{H} = \mathbf{J}$ then, with reference to the table in Problem I.3 of Vol. II, reduce to

$$\frac{dH_z}{dr} = 0 \quad (2) \quad \text{and} \quad \frac{1}{r} \frac{d}{dr} (rH_\varphi) = \begin{cases} J_z, & 0 < r < a, \\ 0, & a < r < b, \\ J_{-z}, & b < r < c, \\ 0, & c < r < \infty, \end{cases} \quad (3)$$

whereas the r -component takes on the form $0 = 0$.

From (2) we obtain

$$H_z = \text{const} = 0,$$

the latter since H_z certainly must vanish for $r = \infty$. The fact that H_r must also vanish follows from the condition $\text{div } \mathbf{H} = 0$. The magnetic lines of excitation $\mathbf{H} = H_\varphi$ are hence coaxial circles about $r = 0$. We write $H_\varphi = H$ and tabulate the integration of (3) below:

Differential Equation	Solution	Determination of Constants
$0 < r < a \quad \frac{d}{dr} (rH) = J_z r$	$H = J_z \frac{r}{2} + \frac{A}{r}$	$A = 0$ since $H(0)$ is finite
$a < r < b \quad \frac{d}{dr} (rH) = 0$	$H = \frac{B}{r}$	$B = J_z a^2/2 = I/(2\pi)$ by continuity of H at $r = a$ and by Eq. (1)
$b < r < c \quad \frac{d}{dr} (rH) = J_{-z} r$	$H = J_{-z} \frac{r}{2} + \frac{C}{r}$	$C = \frac{I}{2\pi} - J_{-z} \frac{b^2}{2}$ $= \frac{I}{2\pi} \frac{c^2}{a^2} \frac{a^2}{b^2}$ by continuity of H at $r = b$ and by Eq. (1)
$c < r < \infty \quad \frac{d}{dr} (rH) = 0$	$H = \frac{D}{r}$	$D = 0$ by continuity of H at $r = c$ and by Eq. (1)

This solution agrees with Eqs. (4.10) to (4.13).

I.3. Coordinates r, φ, z and field symmetry $\partial/\partial\varphi = \partial/\partial z = 0$ as in I.2; a and b inner and outer radius of the solenoid. Now

$$J_r = J_z = 0, \quad \mathbf{J} = \begin{cases} J_\varphi & \text{for } a < r < b, \\ 0 & \text{for } r < a \text{ and } r > b. \end{cases}$$

The differential equation $\text{curl } \mathbf{H} = 0$ is fulfilled throughout by symmetry, whereas the differential equation $\text{curl } \mathbf{H} = 0$ demands $d(rH_\phi)/dr = 0$, $H_\phi = A/r$, $A = 0$ because of the continuity of \mathbf{H} at $r = 0$. Similarly $H_r = 0$ since $\text{div } \mathbf{H} = 0$. The table shows how H_z is to be evaluated:

Differential Equation	Solution	Determination of Constants
$0 < r < a \quad \frac{\partial H_z}{\partial r} = 0$	$H_z = \text{const}$	$\text{const} = H = \text{interior field}$
$a < r < b \quad \frac{\partial H_z}{\partial r} = -J_\phi$	$H_z = -\int_a^r J_\phi dr + A$	$A = H$ because of continuous joining with interior field
$b < r < \infty \quad \frac{\partial H_z}{\partial r} = 0$	$H_z = B$	$B = H - \int_a^b J_\phi dr$ because of continuity at $r = b$

Since $H_z = 0$ for $r = \infty$ we must have $B = 0$ and therefore $H = \int_a^b J_\phi dr$,

which is equivalent with $H = N_1 I$ in Eq. (4.14): $\int_a^b J_\phi dr$ is the total current which passes through the cross section of the solenoid of width $b - a$ and length 1 in the z -direction. This current is $N_1 I$, where N_1 is the number of turns per unit length.

1.4. The proof of the vector formula is obtained directly if the abbreviation $\mathbf{P} = \mathbf{A} \times \mathbf{B}$ is introduced, and the cyclic permutation rule $\mathbf{P} \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{C} \cdot (\mathbf{D} \times \mathbf{P})$, and the formula (6.2a), are employed.

To prove the cosine law set $\mathbf{D} = \mathbf{A}$. With $\mathbf{A}, \mathbf{B}, \mathbf{C}$ as radii of the unit sphere we have

$$\mathbf{A} \cdot \mathbf{B} = \cos c, \dots, |\mathbf{A} \times \mathbf{B}| = \sin C, \dots$$

The angle between the directions of $\mathbf{A} \times \mathbf{B}$ and $\mathbf{A} \times \mathbf{C}$ is equal to the angle α in the spherical triangle mapped out by \mathbf{A}, \mathbf{B} , and \mathbf{C} .

II.1. From the equation of the ellipsoid given in the Problem we calculate

$$x^2 + y^2 = (a^2 - c^2) \left(1 - \frac{z^2}{a^2}\right),$$

$$x^2 + y^2 + (z + c)^2 = a^2 + \frac{c^2}{a^2} z^2 + 2cz = \left(a + \frac{cz}{a}\right)^2.$$

In order to obtain the numerator and denominator occurring in the logarithm in (9.4) the root must be extracted so that it is positive for all $|z| \leq a$. This leads to

$$z + c + \sqrt{x^2 + y^2 + (z + c)^2} = z + c + a + \frac{cz}{a} = (a + c)\left(1 + \frac{z}{a}\right),$$

$$z - c + \sqrt{x^2 + y^2 + (z - c)^2} = z - c + a - \frac{cz}{a} = (a - c)\left(1 + \frac{z}{a}\right).$$

The quotient of the two is equal to $(a + c)/(a - c)$, i.e. a constant for each one of the confocal ellipsoids. Thus Eq. (9.4) is proved.

It is true that this constant changes if we proceed from an ellipsoid with principal axes $a_1 > a$, $b_1 > b$, c instead of from that with axes a , b , c . This does not, however, affect the identity of the fields in the exterior of the former ellipsoid, since no physical significance attaches to the numbering of the equipotential surfaces.

II.2. Let the rubbed glass rod have the constant charge λ per unit length and be infinitely thin. According to (7.5b) its potential is given by

$$\begin{aligned} 4\pi e\Psi &= \lambda \int_0^\infty \frac{d\xi}{\sqrt{x^2 + y^2 + (z - \xi)^2}} \\ &= \lambda \log(z - \xi + \sqrt{x^2 + y^2 + (z - \xi)^2}) \Big|_0^\infty \\ &= \lambda \log \left\{ \frac{z + \sqrt{x^2 + y^2 + z^2}}{x^2 + y^2} \right\} + \text{const.} \end{aligned}$$

The constant here becomes infinite, namely in a sense equal to $-\lambda \log 0$.

Eq. (9.4), with $z' = z + c$, $E = 2c\lambda$ leads to the same expression (with z' replacing z) in the limit $c \rightarrow \infty$. The equipotential surfaces of the paraboloidal conductor agree with those of the glass rod of course only outside of the former, since within it $\Psi = \text{const.}$

II.3. According to Eqs. (9.13) and (9.14) the potential of the dielectric sphere, for $r > a$ ($\epsilon =$ relative dielectric constant of the sphere with reference to its surroundings), is:

$$\Psi_1 = -F \left(r - \frac{\epsilon - 1}{\epsilon + 2} \frac{a^3}{r^2} \right) \cos \Theta.$$

This is at the same time the analytical continuation of the potential into the interior of the sphere. For $r = b < a$ it yields

$$\Psi_1 = -Fb \left(1 - \frac{\epsilon - 1}{\epsilon + 2} \frac{a^3}{b^3} \right) \cos \Theta.$$

If here we set

$$b = a \sqrt[3]{\frac{\epsilon - 1}{\epsilon + 2}},$$

Ψ_1 becomes independent of Θ , namely, as we must demand for the grounded conducting sphere, $\Psi_1 = 0$.

II.4. This problem is mathematically related to the conformal mapping problems in Vol. II, §29, 30, 31, and makes use of the identity of two-dimensional potential theory and the theory of functions of a complex variable $z = x + iy$ elucidated in Vol. II, §19. The combination of the potential Ψ and the stream function Φ to form the complex variable ζ of the present problem was discussed at that point. We give z as function of ζ , rather than ζ as function of z , because z is a single-valued function of ζ .

The proof of the mapping function $f(\zeta)$ rests on the following:

$$\text{If we set } \Psi = \begin{cases} 0 \\ V \end{cases}, \text{ we find } f(\zeta) = \begin{cases} 0 \\ 2\pi i \end{cases} + 1 - \varphi - e^{-\varphi}, \varphi = \frac{2\pi\Phi}{V};$$

$$\text{Hence } y = \begin{cases} 0 \\ a \end{cases}, x = \frac{a}{2\pi} (1 - \varphi - e^{-\varphi}) \leq 0 \text{ for } -\infty < \Phi < +\infty.$$

The $\begin{cases} \text{lower} \\ \text{upper} \end{cases}$ condenser plate $y = \begin{cases} 0 \\ a \end{cases}, x \leq 0$ is thus at the potential

$$\Psi = \begin{cases} 0 \\ V \end{cases}.$$

Also the bisecting plane of the condenser is an equipotential surface. If we set $\Psi = V/2$, we find $f(\zeta) = \pi i + 1 - \varphi + e^{-\varphi}$; hence $y = a/2, x = \frac{a}{2\pi} (1 - \varphi + e^{-\varphi})$. However, now $x \geq 0$ as φ varies between $-\infty$ and $+\infty$. We are therefore dealing not with a semi-infinite straight line or plane, but with a bilaterally infinite line or plane.

The boundary points $x = 0, y = 0$ and $x = 0, y = a$, corresponding to $\Psi = 0, \Phi = 0$ and $\Psi = V, \Phi = 0$, respectively, are branching points of the conformal mapping. The line of force $\Phi = 0$, which joins the two boundary points, is given in parametric form (with $\psi = 2\pi\Psi/V$) by

$$x = \frac{a}{2\pi} (1 - \cos \psi), \quad y = \frac{a}{2\pi} (\psi - \sin \psi).$$

This is the equation of the simple cycloid (see e.g. the quite similar representation in Vol. I, Eq. (17.2)).

The lines of force in the interior of the condenser and at a large distance from the boundary points belong to the parameter values

$$\varphi \gg 1, \quad 0 < \psi < 2\pi.$$

Since here $\exp(i\psi - \varphi)$ vanishes with increasing φ , we obtain simply

$$x + iy = \frac{a}{2\pi} (1 + i\psi - \varphi), \quad \text{i.e.} \quad x = \frac{a}{2\pi} (1 - \varphi), \quad y = \frac{a}{2\pi} \psi.$$

Since $\Phi = \text{const}$ on these lines of force, we have also $\varphi = \text{const}$ and hence $x = \text{const}$. On the other hand, Ψ varies on them between 0 and V , and hence

ψ between 0 and 2π and y between 0 and a . The lines of force hence approximate closer and closer to straight lines perpendicular to the condenser plates, as is to be expected.

II.5. The differential equation of the potential in the cylindrical coordinates r, φ, z is, when independent of φ and z ,

$$\frac{1}{r} \frac{d}{dr} r \frac{d\Psi}{dr} = 0.$$

It yields

$$\frac{d\Psi}{dr} = -E_r = \frac{A}{r}, \quad D_r = \frac{-\epsilon A}{r}.$$

Hence the surface density on the inner and outer electrode is

$$\omega_1 = \frac{-\epsilon A}{r_1}, \quad \omega_2 = \frac{\epsilon A}{r_2}$$

and the charge per unit length of the z -coordinate

$$e_1 = 2\pi r_1 \omega_1 = -2\pi \epsilon A, \quad e_2 = 2\pi r_2 \omega_2 = +2\pi \epsilon A = -e_1.$$

A second integration leads to

$$\Psi = A \log r + B, \quad \Psi_1 - \Psi_2 = V = -A \log \frac{r_2}{r_1} = \frac{e_1}{2\pi \epsilon} \log \frac{r_2}{r_1},$$

so that

$$K_1 = \frac{e_1}{V} = 2\pi \epsilon / \log \frac{r_2}{r_1} = \text{capacity per unit length.}$$

For $d \ll r_1$ we find

$$\log \frac{r_2}{r_1} = \log \left(1 + \frac{d}{r_1} \right) \cong \frac{d}{r_1}$$

and, neglecting end corrections,

$$K = K_1 h = \frac{2\pi r_1 h \epsilon}{d} = \frac{\text{surface}}{\text{separation}} \cdot \text{dielectric constant.}$$

In MKSQ-units:

$$r_1 h = 100 \text{ cm}^2 = 10^{-2} \text{ M}^2, \quad d = 1 \text{ mm} = 10^{-3} \text{ M},$$

$$\epsilon = 6\epsilon_0 = \frac{1}{6\pi} \cdot 10^{-9} \frac{\text{Q}^2}{\text{joule M}}, \quad K = \frac{1}{3} \cdot 10^{-6} \text{ farad} = \frac{1}{3} \cdot 10^{-2} \text{ microfarad.}$$

II.6. The first of the relations given in Eq. (2) of the Problem is obvious in view of Eq. (1). The second is obtained as follows:

By (10.14):

$$\begin{aligned} E &= K_{11}\Psi_1 + K_{12}\Psi_2, \\ -E &= K_{21}\Psi_1 + K_{22}\Psi_2. \end{aligned} \quad (1)$$

Hence

$$0 = (K_{11} + K_{21})\Psi_1 + (K_{12} + K_{22})\Psi_2. \quad (2)$$

In addition, the following linear relation exists between Ψ_1 and Ψ_2 :

$$V = \Psi_1 - \Psi_2. \quad (3)$$

From (2) and (3) we compute

$$\Psi_1 = \frac{K_{12} + K_{22}}{K_{11} + 2K_{12} + K_{22}} V, \quad \Psi_2 = -\frac{K_{11} + K_{12}}{K_{11} + 2K_{12} + K_{22}} V.$$

Substitution in (1) yields

$$E = \frac{K_{11}K_{22} - K_{12}^2}{K_{11} + 2K_{12} + K_{22}} V.$$

The factor of V is the capacity in the elementary sense. Hence also the second relation (2) of the Problem has been proved.

II.7. If the phase factor which must be thought of as added to (24.9) namely $\exp(ikx)$, is replaced by $\frac{\cos}{\sin} \pi l \frac{x}{a}$, where l is an integer (standing instead of travelling wave) and if $\frac{\cos}{\sin}$ is chosen with due regard of the boundary conditions prescribed for the individual \mathbf{E} -components, a particular characteristic electromagnetic oscillation of the interior of the parallelepiped is obtained for which $H_z = 0$; similarly, proceeding from (24.10), one for which $E_z = 0$ is obtained.

However these are not yet the general characteristic oscillations of the parallelepiped, as is evident from the specific values $H_z = 0$ and $E_z = 0$, respectively. The general system, which has complete symmetry with respect to the three axes, is

$$\begin{aligned} E_x &= A \cos\left(\pi l \frac{x}{a}\right) \sin\left(\pi n \frac{y}{b}\right) \sin\left(\pi m \frac{z}{c}\right), \\ E_y &= B \sin\left(\pi l \frac{x}{a}\right) \cos\left(\pi n \frac{y}{b}\right) \sin\left(\pi m \frac{z}{c}\right), \\ E_z &= C \sin\left(\pi l \frac{x}{a}\right) \sin\left(\pi n \frac{y}{b}\right) \cos\left(\pi m \frac{z}{c}\right), \end{aligned} \quad (1)$$

$$\begin{aligned}
\sqrt{\frac{\mu_0}{\epsilon_0}} H_z &= A' \sin\left(\pi l \frac{x}{a}\right) \cos\left(\pi n \frac{y}{b}\right) \cos\left(\pi m \frac{z}{c}\right), \\
\sqrt{\frac{\mu_0}{\epsilon_0}} H_y &= B' \cos\left(\pi l \frac{x}{a}\right) \sin\left(\pi n \frac{y}{b}\right) \cos\left(\pi m \frac{z}{c}\right), \\
\sqrt{\frac{\mu_0}{\epsilon_0}} H_x &= C' \cos\left(\pi l \frac{x}{a}\right) \cos\left(\pi n \frac{y}{b}\right) \sin\left(\pi m \frac{z}{c}\right)
\end{aligned} \tag{2}$$

Since $\text{div } \mathbf{E} = 0$ the A, B, C must satisfy the condition

$$A \frac{l}{a} + B \frac{n}{b} + C \frac{m}{c} = 0, \tag{3}$$

whereas the A', B', C' are determined from the A, B, C by the equations

$$\begin{aligned}
ikA' &= \frac{n\pi}{b} C - \frac{m\pi}{c} B, \\
ikB' &= \frac{m\pi}{c} A - \frac{l\pi}{a} C, \\
ikC' &= \frac{l\pi}{a} B - \frac{n\pi}{b} A.
\end{aligned} \tag{4}$$

This system will play an important role in the problem of black-body radiation in Vol. V, just as the general system of elastic characteristic vibrations in §44 of Vol. II was of importance for the problem of specific heats.

II.8. We find from (24.6) with $h = \pi m/l$ ($m = \text{integer}$, $l = \text{length of the cylinder}$), if, again, $\exp(ihx)$ is replaced by $\frac{\cos}{\sin} hx$ and care is taken, by the proper choice of the cosine or sine, to fulfill the boundary conditions $E_{\text{tang}} = 0$,

$$\begin{aligned}
E_z &= \frac{\sqrt{k^2 - h^2}}{h} J_n(\rho) \cos(n\varphi) \cos(hx), & \sqrt{\frac{\mu_0}{\epsilon_0}} H_z &= 0 \\
E_r &= -J'_n(\rho) \cos(n\varphi) \sin(hx), \\
\sqrt{\frac{\mu_0}{\epsilon_0}} H_r &= \frac{k}{ih} \frac{n}{\rho} J_n(\rho) \sin(n\varphi) \cos(hx),
\end{aligned} \tag{1}$$

$$E_\varphi = \frac{n}{\rho} J_n(\rho) \sin(n\varphi) \sin(hx),$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} H_\varphi = \frac{k}{ih} J'_n(\rho) \cos(n\varphi) \cos(hx).$$

Here, as in (24.6), $\rho = \sqrt{k^2 - h^2} r$ and $\sqrt{k^2 - h^2} a = w$, is one of the infinitely many roots of $J_n(w) = 0$. The characteristic wave number k and the characteristic frequency ω are accordingly given by

$$k^2 = h^2 + \frac{w^2}{a^2}, \quad \omega = kc (c = \text{velocity of light}). \quad (2)$$

The system of characteristic functions represented by (1) is triply infinite and is ordered by the numbers n , ν , and the integer m contained in h . Convince yourself that (1) satisfies not only the boundary conditions, but also the relations between \mathbf{E} and \mathbf{H} demanded by the Maxwell equations.

Similarly we obtain from (24.7), with the same meaning of h ,

$$\begin{aligned} \sqrt{\frac{\mu_0}{\epsilon_0}} H_z &= \frac{\sqrt{k^2 - h^2}}{h} J_n(\rho) \cos(n\varphi) \sin(hx), & E_z &= 0, \\ \sqrt{\frac{\mu_0}{\epsilon_0}} H_r &= J'_n(\rho) \cos(n\varphi) \cos(hx), \\ E_r &= -\frac{k}{i h} \frac{n}{\rho} J_n(\rho) \sin(n\varphi) \sin(hx), \end{aligned} \quad (3)$$

$$\begin{aligned} \sqrt{\frac{\mu_0}{\epsilon_0}} H_\varphi &= -\frac{n}{\rho} J_n(\rho) \sin(n\varphi) \cos(hx), \\ E_\varphi &= -\frac{k}{i h} J'_n(\rho) \cos(n\varphi) \sin(hx). \end{aligned}$$

The characteristic wave number and the characteristic frequency are now given by

$$k^2 = h^2 + \frac{w'^2}{a^2}, \quad \omega = kc \quad (4)$$

where w' is one of the infinitely many roots of $J'_n(w') = 0$. The series of characteristic oscillations (3) is again triply infinite.

Do (1) and (3) supply the complete system of characteristic vibrations of the interior of the cylinder?

II.9. Except for a multiplying constant the appropriate solution, continuous at $r = 0$, of the differential equation (19.16) is

$$\Pi = \frac{\sin(kr)}{r} e^{-i\omega t}.$$

The separation into two parts

$$\Pi = \frac{1}{2ir} (e^{ik(r-ct)} - e^{-ik(r+ct)})$$

indicates the superposition, mentioned in the Problem, of a spherical wave radiated outwards and one radiated inwards. Eqs. (19.17) and the following equations yield, without the time factor

$$\epsilon_0 E_r = \cos \vartheta \left(\frac{d^2}{dr^2} + k^2 \right) \frac{\sin(kr)}{r}, \quad \epsilon_0 E_\vartheta = -\sin \vartheta \left(\frac{1}{r} \frac{d}{dr} + k^2 \right) \frac{\sin(kr)}{r},$$

$$E_\varphi = 0, \quad H_r = H_\vartheta = 0, \quad H_\varphi = \frac{i\omega}{r} \sin \vartheta \left(\frac{d}{dr} \sin(kr) - \frac{\sin(kr)}{r} \right).$$

The boundary condition $E_\vartheta = 0$ demands for $r = a$

$$\cos(ka) - \frac{\sin(ka)}{ka} [1 - (ka)^2] = 0.$$

Therefore the transcendental equation

$$\tan x = \frac{x}{1 - x^2}, \quad x = ka.$$

Its graphical solution yields a first root x_1 which is somewhat smaller than π and an infinite series of additional roots which asymptotically approach the value $x_n = n\pi$.

In addition to this singly infinite system of characteristic functions, for which the electric lines of force lie in the meridional plane $\varphi = \text{const}$ and the magnetic lines are perpendicular thereto, there are ∞^2 less symmetric characteristic functions, with a Legendre function dependence in ϑ and φ .

II.10. The telegraph equation (18.19), with $G = 0$ and the assumption $I = I_0 \exp i(hx - \omega t)$, yields

$$h^2 = \omega K(\omega L + iR). \quad (1)$$

For large ω , more exactly, for $\omega L \gg |R|$, this leads to

$$h - k = \frac{i}{2} \frac{R}{Z}; \quad (2)$$

Here

$$k = \omega \sqrt{KL} \quad (2a)$$

is the wave number of the perfectly conducting line and

$$Z = \sqrt{\frac{L}{K}} \quad (2b)$$

is its wave impedance. R is the impedance operator from Eq. (20.19), composed of the real resistance R and the inner inductive reactance ωL_i (numerically equal to the resistance in the presence of the skin effect) to form a complex quantity:

$$R = R - i\omega L_i = (1 - i)R. \quad (2c)$$

It may be noted, incidentally, that R may be replaced directly by R in (1) if we interpret L as the sum $L_s + L_i$ of the outer and inner selfinductance and not, as was done in the telegraph equation, as the outer selfinductance L_s alone. We assume that ω is large enough that a fully developed skin effect occurs. According to (20.12) the depth of penetration is then given by

$$d = 1/\sqrt{\mu\sigma\omega/2} \quad (3)$$

We shall furthermore assume that the wave amplitude may vary slowly along the circumference of the wire so that the validity of the solution originally obtained for the plane problem in §20 B is not impaired. The alternating current resistance of a surface strip of length 1 and width 1, measured in the direction of the circumference, is then by (20.15a)

$$R_1 = \frac{1}{\sigma d}. \quad (4)$$

The current through this metal strip (lying directly underneath) is equal to the line integral of \mathbf{H} about this strip, which in our case reduces to the value of \mathbf{H} at the surface. Hence we obtain for the Joule heat developed in the strip, utilizing (4),

$$R_1 \mathbf{H}^2 = \frac{1}{\sigma d} \mathbf{H}^2$$

and for the Joule heat developed in unit length of the conductor as a whole (ds = line element of the circumference, \oint = integration over the circumference):

$$\frac{1}{\sigma d} \oint \mathbf{H}^2 ds = RI^2 \quad \text{with} \quad I = \oint \mathbf{H} \cdot ds. \quad (5)$$

R and I are resistance and total current of this unit length. From (5) we compute

$$R = \frac{1}{\sigma d} \oint \mathbf{H}^2 ds / \left(\oint \mathbf{H} \cdot ds \right)^2. \quad (6)$$

\mathbf{H} is to be obtained from the *quasistationary* field in the dielectric. This yields also the external selfinductance L_s of unit length of the conductor and its capacity K , as well as its wave impedance z in Eq. (2b).

a. *The Lecher Two-wire Line.* Wire radius a , separation of wire axes $2b$, separation of the two source lines $2\zeta_0$ (see Fig. 36); u, v bipolar coordinates for representing the field in the dielectric, $u = \text{const}$ magnetic, $v = \text{const}$ electric lines of force;

$$ds = g dv, \quad g = \frac{\zeta_0}{\cosh u - \cos v}, \quad \zeta_0 = \sqrt{b^2 - a^2}. \quad (7)$$

On the circumference of the wire $u = u_0$, $\cosh u_0 = b/a$.

Since the bipolar coordinate v signifies the magnetic potential directly, the desired magnetic field component at the periphery of the wire is

$$H = \frac{dv}{ds} = \frac{1}{g} = \frac{1}{\xi_0} (\cosh u_0 - \cos v). \quad (8)$$

From this the integrals occurring in (6) may be calculated:

$$\oint \mathbf{H} \cdot d\mathbf{s} = \int_0^{2\pi} \frac{1}{g} g dv = 2\pi,$$

$$\oint H^2 ds = \int_0^{2\pi} \frac{1}{g^2} g dv = \frac{1}{\xi_0} \int_0^{2\pi} (\cosh u_0 - \cos v) dv = \frac{2\pi}{\xi_0} \cosh u_0 = \frac{2\pi}{\xi_0} \frac{b}{a}$$

and hence (6) leads to (the factor 2 to be added to (6) arises from the two-wire line):

$$R = \frac{2}{\sigma d} \frac{2\pi}{\xi_0} \frac{b}{a} / (2\pi)^2 = \frac{1}{\pi \sigma \xi_0 d} \frac{b}{a} \Omega M^{-1}. \quad (9)$$

Since furthermore the bipolar coordinate u determines the electric potential, the difference in potential between the two wires is

$$2u_0 = 2 \operatorname{arccosh} \frac{b}{a} = 2 \log \frac{b + \xi_0}{a}.$$

From this we find for the capacity and selfinductance per unit length

$$\frac{\epsilon_0}{K} = \frac{L}{\mu_0} = \frac{2u_0}{2\pi} = \frac{1}{\pi} \log \frac{b + \xi_0}{a} \quad (10)$$

and for the wave impedance

$$Z = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{\pi} \log \frac{b + \xi_0}{a} \Omega. \quad (11)$$

Taking account of (2c), (9) and (11) yields for the propagation constant in Eq. (2)

$$h - k = \frac{1+i}{2} \sqrt{\frac{\epsilon}{\mu_0}} \frac{1}{\sigma \xi_0 d} \frac{b}{a} / \log \frac{b + \xi_0}{a}. \quad (12)$$

This expression agrees with the value calculated in (25.20). To realize this it is merely necessary to substitute the value of d from Eq. (3) in (12), and to express the conductivity σ by the complex dielectric constant $\epsilon' = \epsilon + i\sigma/\omega \cong i\sigma/\omega$, and to note the meaning of the abbreviation $p(1/p = (b + \xi_0)/a)$ in (25.20).

b. *Return through ground.* The earth's surface now takes the place of the plane of symmetry $u = 0$ of the bipolar coordinates. In formula (6) for R the factor 2 is now to be omitted, since the forward conduction through

the wire is to be assumed to be resistance-free, so that R refers only to the return conduction through ground. On the other hand K is to be doubled, L and z to be halved. b now signifies the height of the wire above ground, b_0 the height of the source line above ground, which does not differ appreciably from b . We then obtain from (9), (11), and (12)

$$R = \frac{1}{2\pi\sigma bd}, \quad z = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{2\pi} \log \frac{2b}{a}, \quad (13)$$

$$h - k = \frac{1+i}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{1}{\sigma bd} \log \frac{2b}{a}. \quad (14)$$

Numerical example: frequency in the range of radiotelephony $10^4 S^{-1}$, σ (earth) = $10^{-2} - 10^{-4} \Omega^{-1} M^{-1}$, $b = 10 M$, $a = 1 mm$. We compute from (3)

$$d = 50.4 M \text{ to } 5.04 M$$

and obtain from (13) and (14)

$$R = 3.2 \text{ to } 0.32 \Omega/M$$

$$|h - k| = 38 \text{ to } 3.8 \cdot 10^{-4} M^{-1}.$$

At low frequencies the penetration is so great compared with the usual height of the wire b that (in contrast with that in §25) our present method of calculation fails.

For the single wire without return conductor the external field can no longer be calculated in a quasistationary manner, so that the above approximate method no longer constitutes a simplification as compared with §22. Our numerical example indicates, by the way, how greatly the field of the single wire is disturbed even by a non-metallic return conductor; see in this connection the note at the beginning of §22 regarding the failure of Hertz's original experiments with wire waves and the influence of the laboratory walls.

III.1. The transformation (4) of the Problem is to be built up out of the transformations (1), (2), and (3) in the following manner:

$$L' = DLD^{-1} \quad (a)$$

(L' and L = Lorentz transformations, D = rotation, D^{-1} = inverse rotation). Here

$$D: \quad \begin{aligned} x_1 &= x \cos \alpha + y \sin \alpha, & z_1 &= z, \\ y_1 &= -x \sin \alpha + y \cos \alpha, & t_1 &= t; \end{aligned}$$

$$L: \quad \begin{aligned} x'_1 &= \frac{x_1 - t_1 \beta c}{\sqrt{1 - \beta^2}}, & y'_1 &= y_1, \\ t'_1 &= \frac{t_1 - x_1 \beta/c}{\sqrt{1 - \beta^2}}, & z'_1 &= z_1; \end{aligned}$$

$$D^{-1}: \quad \begin{aligned} x' &= x_1' \cos \alpha - y_1' \sin \alpha, & z' &= z_1', \\ y' &= x_1' \sin \alpha + y_1' \cos \alpha, & t' &= t_1'. \end{aligned}$$

By their successive combination we find, with the abbreviation

$$\eta = (1 - \beta^2)^{-1/2}$$

the following system of coefficients for L' :

	x	y	z	ict
x'	$1 + (\eta - 1) \cos^2 \alpha$	$(\eta - 1) \cos \alpha \sin \alpha$	0	$i\beta\eta \cos \alpha$
y'	$(\eta - 1) \cos \alpha \sin \alpha$	$1 + (\eta - 1) \sin^2 \alpha$	0	$i\beta\eta \sin \alpha$
z'	0	0	1	0
ict'	$-i\beta\eta \cos \alpha$	$-i\beta\eta \sin \alpha$	0	η

It is four-dimensionally orthogonal (sum of the squares equal to 1, sum of the products equal to 0, both in the horizontal rows and the vertical columns) and hence can be read just as well from top to bottom as from left to right.

The velocity \mathbf{v} of transformation (2) forming an angle α with the x -axis has, in the x, y, z -system, the components $v \cos \alpha, v \sin \alpha, 0$. With $\mathbf{r} = x, y, z$; $\mathbf{r}' = x', y', z'$ we therefore obtain

$$\frac{\mathbf{v}}{v} \cdot \mathbf{r} = x \cos \alpha + y \sin \alpha, \quad \frac{\mathbf{v}}{v} \cdot \mathbf{r}' = x' \cos \alpha + y' \sin \alpha.$$

With these abbreviations our system, read from left to right, yields

$$\mathbf{r}' = \mathbf{r} + \frac{\mathbf{v}}{v} \left\{ (\eta - 1) \frac{\mathbf{v}}{v} \cdot \mathbf{r} - \beta \eta c t \right\} \quad (\text{b})$$

$$t' = \eta \left\{ t - \frac{\beta}{c} \frac{\mathbf{v}}{v} \cdot \mathbf{r} \right\}$$

and, read from top to bottom,

$$\mathbf{r} = \mathbf{r}' + \frac{\mathbf{v}}{v} \left\{ (\eta - 1) \frac{\mathbf{v}}{v} \cdot \mathbf{r}' + \beta \eta c t' \right\} \quad (\text{c})$$

$$t = \eta \left\{ t' + \frac{\beta}{c} \frac{\mathbf{v}}{v} \cdot \mathbf{r}' \right\}.$$

Whereas (b) follows from symbol (a), (c) corresponds to the inversion of (a):

$$L = D^{-1} L' D. \quad (\text{d})$$

By taking the derivative of (b) with respect to t' or of (c) with respect to t , respectively, we obtain for the three-dimensional velocity vectors $\mathbf{q}' = d\mathbf{r}'/dt'$ and $\mathbf{q} = d\mathbf{r}/dt$

$$\mathbf{q}' = \frac{\mathbf{q} - \mathbf{v}\{\eta - (\eta - 1)\mathbf{v} \cdot \mathbf{q}/v^2\}}{\eta(1 - \mathbf{v} \cdot \mathbf{q}/c^2)}, \quad (\text{e})$$

$$\mathbf{q} = \frac{\mathbf{q}' + \mathbf{v}\{\eta + (\eta - 1)\mathbf{v} \cdot \mathbf{q}'/v^2\}}{\eta(1 + \mathbf{v} \cdot \mathbf{q}'/c^2)}. \quad (\text{f})$$

III.2. The Problem can be formulated in the following manner: Let a system x_1, t_1 move with respect to a system x, t with the velocity $\beta_1 c$ along the x -axis. In system x_1, t_1 let a point P move with the velocity $\beta_2 c$ at an angle α with respect to the x_1 -axis (and at the same time, the x -axis). What is the resultant velocity of point P as observed from system x, t ?

Apart from additive constants the motion of P in the x_1, t_1 -system is described by

$$x_1 = \beta_2 c t_1 \cos \alpha, \quad y_1 = \beta_2 c t_1 \sin \alpha. \quad (1)$$

At the same time the Lorentz transformation

$$x_1 = \frac{x - \beta_1 c t}{\sqrt{1 - \beta_1^2}}, \quad y_1 = y, \quad t_1 = \frac{t - \beta_1 x/c}{\sqrt{1 - \beta_1^2}} \quad (2)$$

applies for every point of the x_1, t_1 -system.

Substitution of x_1 from (1) and t_1 from (2) leads to

$$x - \beta_1 c t = \beta_2 c \cos \alpha (t - \beta_1 x/c)$$

or, after collecting terms with x and t ,

$$x(1 + \beta_1 \beta_2 \cos \alpha) = (\beta_1 + \beta_2 \cos \alpha) c t.$$

Hence

$$\frac{dx}{dt} = \frac{\beta_1 + \beta_2 \cos \alpha}{1 + \beta_1 \beta_2 \cos \alpha} c; \quad (3)$$

in addition, (2) and (1) lead to

$$y = \frac{\beta_2 c \sin \alpha}{\sqrt{1 - \beta_1^2}} (t - \beta_1 x/c),$$

$$\frac{dy}{dt} = \frac{\beta_2 c \sin \alpha}{\sqrt{1 - \beta_1^2}} \left(1 - \frac{\beta_1}{c} \frac{dx}{dt}\right),$$

and, in view of (3),

$$\frac{dy}{dt} = \frac{\beta_2 c \sin \alpha}{\sqrt{1 - \beta_1^2}} \left(1 - \frac{\beta_1^2 + \beta_1 \beta_2 \cos \alpha}{1 + \beta_1 \beta_2 \cos \alpha}\right) = \frac{\beta_2 c \sin \alpha \sqrt{1 - \beta_1^2}}{1 + \beta_1 \beta_2 \cos \alpha}. \quad (4)$$

The resultant in the x, t -system is

$$q = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

If we put $q/c = \beta$, we obtain

$$\begin{aligned}\beta^2 &= \frac{(\beta_1 + \beta_2 \cos \alpha)^2 + \beta_2^2(1 - \beta_1^2) \sin^2 \alpha}{(1 + \beta_1 \beta_2 \cos \alpha)^2} \\ &= \frac{\beta_1^2 + 2\beta_1 \beta_2 \cos \alpha + \beta_2^2 - \beta_1^2 \beta_2^2 \sin^2 \alpha}{(1 + \beta_1 \beta_2 \cos \alpha)^2} \quad \text{Q.E.D.}\end{aligned}\quad (5)$$

III.3. In (30.6) r denotes the vector $L \rightarrow P$, and r , hence, one side of the triangle LOP in Fig. 42. Let O be the position of the electron at the time of the field observation at P . Let the velocity \mathbf{v} of the electron be directed, unlike §28, along the positive x -axis, leading to a specialization of the formulas in §30. The length LO then becomes equal to $v\tau$ with $\tau = r/c$, so that

$$LO = v\tau = \beta r.$$

The length OP is the separation between electron and point of reference at the time of the observation and will be designated by r' ,

$$r' = \sqrt{x'^2 + y'^2 + z'^2},$$

where x', y', z' are the coordinates of P relative to O . ϑ and ϑ' are the angles at L and O shown in the figure and we have

$$r' \cos \vartheta' = x', \quad r \cos \vartheta = x' + \beta r. \quad (1)$$

Furthermore, by the Pythagorean theorem,

$$r^2 = r'^2 + (\beta r)^2 + 2r'\beta r \cos \vartheta', \quad (2)$$

so that

$$r^2(1 - \beta^2) - 2\beta r x' = r'^2.$$

The solution of this quadratic equation for r yields

$$r = \frac{\beta x'}{1 - \beta^2} + \frac{1}{\sqrt{1 - \beta^2}} \sqrt{x'^2 + y'^2 + z'^2 + \frac{\beta^2}{1 - \beta^2} x'^2}. \quad (3)$$

As in (28.13a) we denote the square root on the right by s :

$$s = \sqrt{x'^2 \left(1 + \frac{\beta^2}{1 - \beta^2}\right) + y'^2 + z'^2} = \sqrt{\frac{x'^2}{1 - \beta^2} + y'^2 + z'^2}$$

and obtain by (3)

$$r = \frac{s}{\sqrt{1 - \beta^2}} + \frac{\beta}{1 - \beta^2} x'. \quad (4)$$

Now, $v_r = v \cos \vartheta$ and, in view of (1),

$$r \frac{v_r}{c} = \beta r \cos \vartheta = \beta x' + \beta^2 r;$$

Accordingly, by (4),

$$r \left(1 - \frac{v_r}{c} \right) = r(1 - \beta^2) - \beta x' = \sqrt{1 - \beta^2} s. \quad (5)$$

Thus the quotient occurring in (30.6) becomes

$$\frac{1 - v^2/c^2}{r^2(1 - v_r/c)^2} = \frac{1 - \beta^2}{(1 - \beta^2)^{3/2}} \frac{1}{s^2}. \quad (6)$$

At the same time the vectorial factors multiplied herewith in view of the assumed direction of \mathbf{v} , are resolved into their components:

$$\mathbf{r} \times \mathbf{v} = 0, \quad z'v, -y'v. \quad (7)$$

$$r \frac{v_x}{c} - r_x = r\beta, -r \cos \vartheta = r\beta - x' - r\beta = -x', \quad (7a)$$

$$r \frac{v_y}{c} - r_y = -y', \quad r \frac{v_z}{c} - r_z = -z'.$$

If all of this is substituted in Eq. (30.6) taking due account of the changed sign of v , Eqs. (28.14) and (28.14a) are obtained.

The equation $s = \text{const}$ defines the family of mutually similar "Heaviside ellipsoids", flattened in the direction of motion (see §28C); the electric lines of force in x', y', z' space are the orthogonal trajectories of the family.

III.4. Scalar multiplication with \mathbf{v} of the left side of (32.5) results in

$$\mathbf{v} \cdot \frac{d}{dt} \frac{m_0 \mathbf{v}}{\sqrt{1 - \beta^2}} = m_0 \frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{\sqrt{1 - \beta^2}} + m_0 v^2 \frac{d}{dt} \frac{1}{\sqrt{1 - \beta^2}}.$$

The second term on the right is equal to

$$m_0 v^2 \frac{\beta \cdot \dot{\beta}}{(1 - \beta^2)^{3/2}} = m_0 \beta^2 \frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{(1 - \beta^2)^{3/2}}.$$

Together with the first term this yields

$$m_0 \frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{\sqrt{1 - \beta^2}} \left(1 + \frac{\beta^2}{1 - \beta^2} \right) = m_0 \frac{\mathbf{v} \cdot \dot{\mathbf{v}}}{(1 - \beta^2)^{3/2}} = \frac{d}{dt} \frac{m_0 c^2}{\sqrt{1 - \beta^2}}. \quad (1)$$

On the right side of (32.5) scalar multiplication with \mathbf{v} yields

$$\mathbf{v} \cdot \mathbf{K} = \mathbf{v} \cdot \mathbf{E} + \mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = \mathbf{v} \cdot \mathbf{E} + \mathbf{B} \cdot (\mathbf{v} \times \mathbf{v}) = \mathbf{v} \cdot \mathbf{E}. \quad (2)$$

Hence, setting (1) and (2) equal to each other, we obtain

$$\frac{d}{dt} \frac{m_0 c^2}{\sqrt{1 - \beta^2}} = \mathbf{v} \cdot \mathbf{E}, \quad (3)$$

as in Eq. (32.6).

III.5. The path is of course a ballistic parabola with the acceleration

$$g = \frac{e}{m} \frac{V}{d}.$$

The distance of its vertex from the upper and lower plate, respectively, is

$$h = \frac{(v \sin \alpha)^2}{2g} = \frac{m}{e} \frac{d}{V} \frac{v^2 \sin^2 \alpha}{2}, \quad d - h = d \left(1 - \frac{m}{e} \frac{v^2 \sin^2 \alpha}{2V} \right).$$

The lower plate is reached with

$$v \sin \alpha = \sqrt{\frac{e}{m} 2V}.$$

For a velocity $v = 5 \cdot 10^8$ M/S this is not satisfied even for $\alpha = \pi/2$. With the value e/m_0 from Eq. (33.8) we have then instead

$$\frac{d - h}{d} = 1 - \frac{25 \cdot 10^{12}}{2 \cdot 110 \cdot 1.76 \cdot 10^{11}} = 0.35.$$

The voltage required to produce v is found to be, from $eV = mv^2/2$,

$$V = \frac{1}{2} \frac{25 \cdot 10^{12}}{1.76 \cdot 10^{11}} = 70 \text{ volts.}$$

The "electron volt" ev is a unit of energy much used in atomic physics, particularly in the form "million electron volts", Mev. Since $e = 1.60 \cdot 10^{-19}$ Q we have in our units

$$1 \text{ Mev} = 10^6 \cdot 1.60 \cdot 10^{-19} \text{ joule} = 1.60 \cdot 10^{-13} \text{ joule.}$$

One half of this is almost exactly equal to the rest energy of the electron, i.e.

$$m_0 c^2 = 0.80 \cdot 10^{-13} \text{ joule, } m_0 = 0.90 \cdot 10^{-30} \text{ K.}$$

If, in the condenser field, the velocity suffers changes which are comparable with c , the constancy of the x -momentum (x parallel to the plates) results in the fact that v_x cannot be constant, and hence x cannot be proportional to t . Correspondingly, the equation of motion for the y -direction shows that y is not proportional to t^2 . Hence the path is not a parabola, but a transcendental curve (catenary). Similarly in §32d the Kepler orbit, which in the limiting case of an infinitely distant center of attraction becomes the ballistic parabola, was not an ellipse, but a transcendental curve (ellipse with precessing perihelion).

To compute extremely high velocities from the number z of the corresponding Mev we may use the energy equation

$$\frac{1}{\sqrt{1 - \beta^2}} - 1 = z \frac{\text{Mev}}{m_0 c^2}.$$

For an energy of 200 Mev, such as occurs in cosmic radiation, we have

$$\frac{1}{\sqrt{1-\beta^2}} = 1 + 400 \cong 4 \cdot 10^2, \quad \beta = 1 - \frac{1}{32} \cdot 10^{-4}.$$

III.6. we consider immediately the case of high velocities (the familiar case $v \ll c$ is contained therein).

Let the direction of \mathbf{B} be the z -direction. In the plane perpendicular thereto let s be the projection of the direction of motion, n the direction perpendicular to s , s , n , and z forming a right-handed system. We then have always

$$v_n = 0, \quad (\mathbf{v} \times \mathbf{B})_s = 0, \quad (\mathbf{v} \times \mathbf{B})_n = v_s B = 0, \quad (\mathbf{v} \times \mathbf{B})_z = -v_s B.$$

Hence the momenta in the s - and z -directions are constant:

$$\frac{v_s}{\sqrt{1-\beta^2}} = C, \quad \frac{v_z}{\sqrt{1-\beta^2}} = C'.$$

Since $v_n = 0$, squaring and adding leads to $\frac{\beta^2}{1-\beta^2} = \text{const}$, so that also β , v_s , and v_z are constant.

The equation of motion for the n -direction is (the charge of the electron is negative!)

$$\frac{d}{dt} \frac{\dot{v}_n}{\sqrt{1-\beta^2}} = \frac{\dot{v}_n}{\sqrt{1-\beta^2}} = \frac{-e}{m_0} (\mathbf{v} \times \mathbf{B})_n = \frac{e}{m_0} v_s B;$$

Hence

$$\dot{v}_n = e \frac{\sqrt{1-\beta^2}}{m_0} v_s B = \frac{e}{m} v_s B.$$

\dot{v}_n is the centrifugal acceleration, and as such equal to v_s^2/ρ , where ρ is the radius of curvature of the path projected on the s , n -plane. Therefore

$$\frac{1}{\rho} = \frac{e}{m} \frac{B}{v_s}.$$

The same formula applies also for the non-relativistic calculation, where however $m = m_0 = \text{const}$. The curvature $1/\rho$ vanishes nonrelativistically only for $v_s = \infty$, whereas relativistically it becomes zero for $\beta = 1$, i.e. $v_s^2 + v_z^2 = c^2$. The product ρB (commonly written ρH) is the experimental measure of the "stiffness" of the cathode ray.

III.7. If \mathbf{x} is the common direction of the electric and the magnetic field and z is the direction of the β -ray leaving D , the equations of motion of the β -particle are, with Lorentz's expression $\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ for the force (negative sign because of the negative charge of the electron)

$$\frac{d}{dt} \frac{v_x}{\sqrt{1-\beta^2}} = \mp \frac{eE}{m_0},$$

$$\frac{d}{dt} \frac{v_y}{\sqrt{1-\beta^2}} = -\frac{e(\mathbf{v} \times \mathbf{B})_y}{m_0} = -\frac{ev_x B}{m_0}, \quad \frac{d}{dt} \frac{v_z}{\sqrt{1-\beta^2}} = \frac{ev_y B}{m_0}.$$

Since v_x and v_y can be neglected in comparison with $v_z \cong v$, we have $\beta^2 \cong v^2/c^2$, and from the third equation of motion to the same approximation $v \cong \text{const.}$ Hence the first two equations of motion can be integrated directly and we obtain, if the time t is measured from the moment of passage through D so that the instant of incidence on the photographic plate may be set equal to $t = a/v$,

$$x = \mp \frac{eE}{m_0} \sqrt{1 - \frac{v^2}{c^2}} \frac{a^2}{2v},$$

$$y = -\frac{eB}{m_0} \sqrt{1 - \frac{v^2}{c^2}} \frac{a^2}{2v}.$$

This is the parametric representation of the two branches of the curve which result when the polarity of the electric field is reversed.

If v^2 is neglected in comparison with c^2 elimination of the parameter v leads to the two branches of a parabola

$$y^2 = \mp Cx, \quad C = \frac{e}{m_0} \frac{B^2}{E} \frac{a^2}{2}.$$

They touch at the point $x = y = 0$ with a vertical tangent. This point corresponds to the value of the parameter $v = \infty$.

If the relativity factor $\sqrt{1 - v^2/c^2}$ is retained, elimination of v leads to the curve of the fourth order

$$y^4 + D^2 y^2 = C^2 x^2, \quad D = \frac{eB}{m_0 c} \frac{a^2}{2} \quad (C \text{ as above}),$$

which takes the place of both branches of the above parabola. At the point $x = y = 0$, which now corresponds to the parameter $v = c$, it has a cusp; the two tangents at this point have the two distinct directions

$$\frac{dy}{dx} = \pm \frac{C}{D} = \pm \frac{Bc}{E},$$

and form accordingly a finite angle 2α (see Fig. 48, on the right) with each other. This is clearly evident from Kaufmann's photographic records. However it was not possible to arrive at a definite decision between Lorentz's and Abraham's variation of mass (see the beginning of §33), as intended by Kaufmann, although this should be possible in principle from

the complete shape of the curve; the fields employed were not uniform and their distribution had to be established by laborious probe measurements.

III.8. The two mutually perpendicular fields $E_z = E$ and $B_z = B$ give rise to the Lorentz force

$$F_x = -e \frac{dy}{dt} B, \quad F_y = -e \left(E - \frac{dx}{dt} B \right), \quad F_z = 0.$$

For not too great velocities ($m = m_0$) the equations of motion are

$$\begin{aligned} m \frac{d^2 x}{dt^2} + e \frac{dy}{dt} B &= 0, \\ m \frac{d^2 y}{dt^2} - e \frac{dx}{dt} B &= -eE. \end{aligned}$$

If the second equation is multiplied by i , and if we set $\zeta = x + iy$, we obtain by addition

$$\ddot{\zeta} - i\alpha \dot{\zeta} = -i \frac{e}{m} E, \quad \alpha = \frac{e}{m} B.$$

The general integral is

$$\zeta = A e^{i\alpha t} + \frac{E}{B} t + C.$$

The time $t = 0$ can be so chosen that for it $dy/dt = 0$, so that $\zeta_0 = x_0$ becomes real. We then have $x_0 = i\alpha A + E/B$, $\zeta_0 = A + C$ and hence

$$\zeta - \zeta_0 = \frac{i}{\alpha} \left(x_0 - \frac{E}{B} \right) (1 - e^{i\alpha t}) + \frac{E}{B} t$$

Separating real and imaginary parts we obtain with $\varphi = \alpha t$, $a = E/(\alpha B)$, $b = (x_0 - E/B)/\alpha$:

$$x - x_0 = a\varphi + b \sin \varphi, \quad y - y_0 = b(1 - \cos \varphi).$$

This is the equation of the general cycloid or trochoid (overlapping or stretched, depending on $a \lesseqgtr b$). For $x_0 = 0$ (i.e. $b = -a$) we obtain a representation of the ordinary cycloid, such as occurred, with the same notation, in Vol. I, Eq. (17.1) in connection with the cycloidal pendulum (where however we had put $x_0 = y_0 = 0$).

III.9. Since we are dealing with a stationary state, J is constant in both time and space as

$$\operatorname{div} J = \frac{\partial J}{\partial x} = 0. \quad (1)$$

At the same time v and ρ are constant in time but not constant in space since

$$v = \sqrt{2 \frac{e}{m} V(x)}, \quad \rho = J/v. \quad (2)$$

Poisson's equation becomes

$$\sqrt{V(x)} \frac{d^2 V(x)}{dx^2} = C, \quad C = J/(e_0 \sqrt{2e/m}). \quad (3)$$

It may be integrated by putting

$$V(x) = Ax^\alpha. \quad (4)$$

(3) then leads to

$$A^{3/2} \alpha(\alpha - 1) x^{\alpha/2 + \alpha - 2} = C,$$

i.e.

$$\frac{\alpha}{2} + \alpha - 2 = 0, \quad \alpha = \frac{4}{3}; \quad A^{3/2} \frac{4}{9} = C, \quad A = \left(\frac{9}{4} C \right)^{2/3}. \quad (5)$$

For $x = l$ we obtain from formula (4) and the meaning of C in (3)

$$V = Al^{\frac{4}{3}} = \left(\frac{9}{4} \frac{l^2}{e_0 \sqrt{2e/m}} \right)^{2/3} J^{2/3}, \quad (6)$$

$$J = \frac{4}{9} \frac{e_0 \sqrt{2e/m}}{l^2} V^{3/2}. \quad (7)$$

The total current $I = \pi a^2 J$ (a = radius of cathode and anode) becomes

$$I = \frac{4\pi}{9} e_0 \sqrt{2e/m} \frac{a^2}{l^2} V^{3/2}. \quad (8)$$

This is the desired equation of the characteristic. The reader may convince himself that owing to our factor e_0 it is correct dimensionally, namely has the dimension Q/S also on the right side. We note expressly that, according to (3) and (4), V does not increase linearly with x , and that $\partial V/\partial x$ is equal to zero at the cathode. Here, according to (1), $\rho = \infty$ and $v = 0$. The last corresponds to the fact that, in the statement of the problem, we have neglected the (small) velocity of emission of the thermionic electrons as compared with the velocity impressed on them by the field.

For the cylindrical arrangement (radius of the hot-filament cathode: $r = 0$, radius of the cylinder-mantel anode: $r = a$, length of the cylinder mantel: l) the preceding equations change as follows:

$$\operatorname{div} \mathbf{J} = \frac{1}{r} \frac{\partial(r\mathbf{J})}{\partial r} = 0, \quad r\mathbf{J} = \text{const} = \frac{I}{2\pi l} \quad (1')$$

$$\rho = J/v = I / \left(2\pi l r \sqrt{2 \frac{e}{m} V(r)} \right) \quad (2')$$

$$\sqrt{V(r)} \frac{d}{dr} \left(r \frac{dV(r)}{dr} \right) = C, \quad C = I / (2\pi \epsilon_0 l \sqrt{2e/m}). \quad (3')$$

$$V(r) = Ar^\alpha \quad (4')$$

$$\alpha = \frac{2}{3}, \quad A = \left(\frac{9}{4} C \right)^{2/3} \quad (5')$$

$$\text{for } r = a, \quad V = Aa^\alpha = \left(\frac{9}{8\pi \epsilon_0 l \sqrt{2e/m}} \right)^{2/3} I^{2/3} \quad (6')$$

$$I = \frac{8\pi}{9} \epsilon_0 \sqrt{2e/m} \frac{l}{a} V^{3/2}. \quad (8')$$

At the cathode dV/dr now becomes infinitely large according to (4') because of $r = 0$, in contrast with dV/dr for the plane configuration. Nevertheless the total charge on the filament approaches zero with vanishing r : this is the reason for the absence of the logarithmic singularity of the potential occurring otherwise for a charged wire, whereas by (4') V vanishes for $r = 0$. For this reason (8') applies not only for $r = 0$, but also for wires of small finite thickness with sufficient accuracy.

III.10. The magnetic flux through the electron path of radius r

$$\Phi = 2\pi \int_0^r B(r, t) r dr$$

yields

$$\frac{\partial \Phi}{\partial r} = 2\pi r B(r, t), \quad \frac{\partial \Phi}{\partial t} = 2\pi \int_0^r \dot{B}(r, t) r dr. \quad (1)$$

According to the law of induction we have

$$2\pi \int_0^{r_0} \dot{B}(r, t) r dr = -2\pi r_0 E(r_0, t). \quad (2)$$

By multiplication with the absolute value e of the charge of the electron we obtain herefrom as accelerating force in the orbit $r = r_0$:

$$-eE(r_0, t) = \frac{e}{r_0} \int_0^{r_0} \dot{B}(r, t) r dr = \frac{e}{2\pi r_0} \frac{\partial \Phi}{\partial t}. \quad (3)$$

The equation for the change in momentum of the electron can then be integrated with respect to t and yields

$$mv - (mv)_A = \frac{e}{2\pi r_0} (\Phi - \Phi_A), \quad (4)$$

where Φ_A denotes the magnetic flux for the initial state $v = v_A$, $m = m_A$. Accordingly, for given initial momentum, mv is to be regarded as known. We hence calculate for the answer to question (1):

$$\beta = \frac{v}{c} = \frac{1}{\sqrt{1 + m_0^2 c^2 / (m^2 v^2)}}, \quad m = m_0 \sqrt{1 + \left(\frac{mv}{m_0 c}\right)^2}, \quad (5)$$

$$eV = (m - m_0)c^2.$$

Numerical example: For a path diameter $2r_0 = 10^{-1}M$ a flux amplitude $\Phi_{\max} = 10^{-2}$ VS is readily attainable in practice. Let the initial flux be very small, $\Phi_A \cong 0$. We can also assume $v_A \cong 0$ since the initial velocity is insignificant in comparison with the great final velocity. We then find from (4)

$$\frac{(mv)_{\max}}{m_0 c} = \frac{e}{2\pi r_0} \frac{\Phi_{\max}}{m_0 c} = \frac{1.6 \cdot 10^{-18}}{\pi} \frac{10^{-2}}{0.9 \cdot 10^{-20} \cdot 3 \cdot 10^8} = 18.8,$$

$$\beta_{\max} = \left(1 + \frac{1}{18.8^2}\right)^{-1/2} = 1 - \frac{1}{710}, \quad m_{\max} = 18.8 m_0, \quad (6)$$

$$m_{\max} - m_0 = 17.8 m_0, \quad eV_{\max} = 17.8 m_0 c^2.$$

Since, from the discussion of Problem III.5, $m_0 c^2$ is equal to $\frac{1}{2} \cdot 10^6$ electron volts, we have

$$eV_{\max} \cong 9 \cdot 10^6 \text{ ev}. \quad (7)$$

On question 2.: The orbit $r = r_0$ was assumed to be known till now; it will now be computed. On every circular orbit there must be equilibrium between the centrifugal force and the force of Biot-Savart:

$$\frac{mv^2}{r} = evB(r). \quad (8)$$

This signifies according to Eq. (1)

$$mv = \frac{e}{2\pi} \frac{\partial \Phi}{\partial r}. \quad (9)$$

Substitution from Eq. (4) with $\Phi_A = 0$, $v_A = 0$ yields

$$\frac{\Phi}{r} = \frac{\partial \Phi}{\partial r}. \quad (10)$$

Plot ordinate B , as a function of the abscissa r , as a monotonically decreasing curve which eventually may have to be determined experimentally. Multiplication with $2\pi r$ then yields, by Eq. (1), the curve for $\partial \Phi / \partial r$, and integration with respect to r that for Φ . Its ordinates are to be divided by

r and the resulting curve must be pursued to its intersection with the curve for $\partial\Phi/\partial r$. The abscissa of the point of intersection is the desired value $r = r_0$.

In order that this orbit may be *stable* the field distribution must satisfy certain conditions, which, for example, have been clearly set forth by Gans.¹

On question 3.: From β_{\max} we obtain for the frequency of revolution

$$\nu_{\max} = \frac{\beta_{\max} c}{2\pi r_0} \cong 10^9 \text{ S}^{-1}. \quad (11)$$

In order to be able to compute the number of revolutions with ease, we assume that the flux Φ does not increase sinusoidally, but linearly from the initial state $\Phi_A = 0$ to the final state Φ_{\max} . For 500-cycle alternating current in the windings of the electromagnet the time of rise (= a quarter period) is then $1/2000 \text{ S}$. We hence obtain

$$\frac{\partial\Phi}{\partial t} = 2000 \Phi_{\max} \text{ S}^{-1} = 20 \text{ V}. \quad (12)$$

Since by (3) the accelerating force is then also constant in time, this is at the same time the gain in energy in one revolution, measured in electron volts. Since by (7) the maximum kinetic energy measured in this manner was $9 \cdot 10^5$, the number of revolutions becomes

$$\frac{9 \cdot 10^5}{20} = 450000. \quad (13)$$

On question 4.: The state of the maximum number of revolutions is reached for $\Phi = \Phi_{\max}$, i.e. $\partial\Phi/\partial t = 0$. According to Eq. (3) we then have $E(r_0) = 0$, i.e. $\dot{v}_{\text{tang}} = 0$. From the equation of the circle

$$\mathbf{r} = r_0 e^{i\omega t}$$

we find for the magnitude and direction of the derivatives of \mathbf{r} :

$$\mathbf{v} = i\omega r_0 e^{i\omega t}, \quad \dot{\mathbf{v}} = -\omega^2 r_0 e^{i\omega t}, \quad \ddot{\mathbf{v}} = -i\omega^3 r_0 e^{i\omega t}. \quad (14)$$

We conclude herefrom: $\ddot{\mathbf{v}}$ is opposite in direction to \mathbf{v} ; $\ddot{\mathbf{v}} = -\omega^2 \mathbf{v}$, furthermore, $\mathbf{v} \cdot \dot{\mathbf{v}} = 0$, $\mathbf{v} \cdot \ddot{\mathbf{v}} = -\omega^2 v^2$.

For the reaction force of radiation we find hence by (36.26) (the symbols \mathbf{v}' , \mathbf{v}'' given there have the same meaning as our $\dot{\mathbf{v}}$, $\ddot{\mathbf{v}}$):

$$\begin{aligned} \mathbf{R}^* &= \frac{e^2}{6\pi\epsilon_0 c^3} \frac{v}{1-\beta^2} \left(-\omega^2 - \frac{\omega^2 v^2}{c^2(1-\beta^2)} \right) \\ &= -\frac{e^2}{6\pi\epsilon_0 c^3} \frac{v\omega^2}{1-\beta^2} \left(1 + \frac{\beta^2}{1-\beta^2} \right) = -\frac{e^2}{6\pi\epsilon_0 c^3} \frac{v\omega^2}{(1-\beta^2)^2} \quad (15) \\ |\mathbf{R}^*| &= \frac{e^2 v^3}{6\pi\epsilon_0 r_0^3 c^3} (1-\beta^2)^{-2}. \end{aligned}$$

¹ R. Gans. Zeits. f. Naturforschung, Vol. 1, p. 485, 1946.

In our numerical example we find with $v \cong c$, $\beta = 1 - 1/710$:

$$|R^*| = \frac{e^2(710)^2}{24\pi\epsilon_0 r_0^2}.$$

With $2r_0 = 10^{-1}M$ and $36\pi\epsilon_0 = 10^{-9} \frac{Q^2}{\text{joule } M}$ (Eq. (7.18)):

$$|R^*| = \frac{6(710)^2 e^2 V}{10^{-11} M Q} = 6 \cdot 7.1^2 \cdot 1.6 \cdot 10^{-4} \frac{eV}{M} \cong 0.048 \frac{eV}{M}. \quad (16)$$

This may be compared with the force of the electric circulating field, which by (3) and (12) is

$$\frac{20}{2\pi r_0} \frac{eV}{M} = \frac{200}{\pi} \frac{eV}{M}.$$

The reaction force increases with the fourth power of the particle energy. As the reaction force becomes comparable with the accelerating force of the circulating field the balance between increase in particle mass and centripetal force is upset. Thus the reaction force sets a limit to the maximum energy which the betatron can impart to an electron.

The principal purpose of the betatron is the production of x-rays of very great hardness. Since their limiting energy $h\nu$ is given by the maximum energy of the betatron electrons it depends, in accord with Eq. (5), on the momentum mv which can be attained. By Eq. (4) this is determined by the ratio Φ/r_0 . For a proportional increase of all of the dimensions of the magnet (B_{\max} and hence also B_{av} are fixed by the saturation of the magnet) Φ increases quadratically, Φ/r_0 hence linearly. We found for our path diameter $2r_0 = 10^{-1}M$ in (7)

$$eV_{\max} = 9 \cdot 10^6 \text{ eV} = 9 \cdot 1.602 \cdot 10^{-6} \text{ erg} = 1.45 \cdot 10^{-5} \text{ erg}.$$

Hence, for x-rays

$$1.45 \cdot 10^{-6} \text{ erg} = h\nu = \frac{hc}{\lambda} = 2 \cdot 10^{-18} \frac{\text{erg} \cdot \text{cm}}{\lambda}, \quad \lambda = 1.4 \cdot 10^{-11} \text{ cm} = 1.4 \text{ X}.$$

The X-unit = 10^{-11} cm here introduced is the unit of length customary in x-ray spectroscopy; thus the K-radiations of the heaviest elements have wave-lengths of about 100 X-units. We thus find ourselves with our betatron of relatively modest dimensions in a domain far beyond the short-wave-length limit of ordinary x-ray spectra and even beyond that of the natural γ -rays which is reached, for ThC, at $\lambda = 4.7 \text{ X}$. By increasing the betatron dimensions the limit can be lowered still further and the energy of 9 Mev, found above, be increased.

IV.1. We proceed from the fact that for an observer moving with it, the field within the rod is both free of current and free of charge: $J' = 0$

and $\rho' = 0$. It then follows from Eqs. (34.6) with the definition of the conduction current in (34.9a), except for correction terms of the order β^2

$$\mathbf{J}_i = \mathbf{J} - \rho \mathbf{v} = 0, \quad \rho = 0. \quad (1)$$

The rod hence has neither conduction current nor volume charge for an observer at rest in the laboratory as well. However, it possesses surface charge ω and a Rowland current (which, in fact, is demanded by $\mathbf{J}_i = 0$) $\mathbf{J} = \omega \mathbf{v}$.

Viewed from the laboratory, the field thus consists of the superposition of a stationary electric and magnetic field, the latter added to the original uniform field and derived from the Rowland current $\mathbf{J} = \omega \mathbf{v}$. We therefore have

$$\text{curl } \mathbf{E} = 0, \quad \mathbf{E} = -\text{grad } \Psi, \quad \text{curl } \mathbf{H} = \mathbf{J}. \quad (2)$$

Show that this agrees with the general Eqs. (34.13), if in them the meaning of \mathbf{E}^* , \mathbf{H}^* is substituted from (34.8) and that of $\dot{\mathbf{B}}$, $\dot{\mathbf{D}}$ from (34.12). For $\partial/\partial t = 0$ and the auxiliary conditions from (34.11a) they then become

$$-\text{curl}(\mathbf{v} \times \mathbf{B}) = -\text{curl}(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \text{ so that } \text{curl } \mathbf{E} = 0,$$

$$\rho \mathbf{v} - \text{curl}(\mathbf{v} \times \mathbf{D}) + \mathbf{J} - \rho \mathbf{v} = \text{curl}(\mathbf{H} - \mathbf{v} \times \mathbf{D}), \text{ so that } \text{curl } \mathbf{H} = \mathbf{J}.$$

By (34.7) we infer from $\mathbf{J}_i = 0$, $\mathbf{E}^* = 0$, i.e. to sufficient accuracy ($\mathbf{B} = \mathbf{B}_0 = \text{original field}$):

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}_0, \quad E = -vB_0, \quad \Psi = vB_0x + C. \quad (3)$$

We have here assumed that \mathbf{B}_0 has the z -direction, \mathbf{v} , the y -direction, and that x, y, z constitute a right-handed system. C is a constant of integration which is independent of x and, in view of the symmetry of the problem, also of y and z . Question a. is thus answered.

On question b.: Consider two points x_1, x_2 on the periphery of the rod, e.g. $x_1 = \text{point of entrance}$, $x_2 = \text{point of exit of the } x\text{-axis}$. The difference of potential is then by (3):

$$V = \Psi_1 - \Psi_2 = vB_0(x_1 - x_2). \quad (4)$$

As a difference of potential this is independent of the path (connecting wire of infinitely high resistance) by which we imagine points 1 and 2 to be joined; this path may be imagined either in the exterior or the interior of the body.

c. We consider the external field. Here the boundary conditions (34.15) take effect. They demand continuity of the tangential component of \mathbf{E} (not of \mathbf{E}^*) and are equivalent to continuity of the potential at the surface of the rod, whereas nothing is stated regarding the normal derivative. Since Ψ is known in the interior by (3), the surface values $\bar{\Psi}$ of Ψ are also

known. We hence must solve a *boundary-value problem* for the exterior, with the normalization condition $\Psi = 0$ at infinity. A solution can be attained for any shape of the rod. It becomes elementary for the circular cross section, to which, by conformal mapping, every other cross section may be reduced, transferring the boundary values prescribed for the latter. It hence suffices to deal with the circular cross section.

If r, φ are ordinary polar coordinates, $r = 0$ is the center of the circle, $r = a$ its periphery, and φ is measured from the x -axis, (3) yields for the interior and the periphery of the circle

$$\begin{aligned} E = E_z = -vB_0, \quad E_r = E_z \cos \varphi = -vB \cos \varphi, \\ E_\varphi = -E_z \sin \varphi = vB_0 \sin \varphi. \end{aligned} \quad (5)$$

The potential in the exterior can generally be expressed as Fourier series

$$\Psi = \sum \left(\frac{a}{r}\right)^n (A_n \cos(n\varphi) + B_n \sin(n\varphi));$$

In view of the boundary condition only the term with A_1 differs from zero. Hence

$$\Psi = A_1 \frac{a}{r} \cos \varphi, \quad E_\varphi = -\frac{1}{r} \frac{\partial \Psi}{\partial \varphi} = A_1 \frac{a}{r^2} \sin \varphi, \quad (6)$$

and, in view of the continuity of E_φ for $r = a$

$$A_1 = avB_0, \quad \Psi = \frac{a^2}{r} vB_0 \cos \varphi, \quad \frac{\partial \Psi}{\partial r} = -\left(\frac{a}{r}\right)^2 vB_0 \cos \varphi. \quad (7)$$

d. To determine the surface charge we must pass from \mathbf{E} to \mathbf{D} . We may utilize for this purpose in the interior of the rod Eq. (34.5), whose right side vanishes in view of $\mathbf{E}^* = 0$. Thus within the rod we do not have $\mathbf{D} = \epsilon \mathbf{E}$, but

$$\mathbf{D} = -\frac{1}{c^2} \mathbf{v} \times \mathbf{H} = -\epsilon_0 \mu_0 \mathbf{v} \times \mathbf{H} \cong -\epsilon_0 \mathbf{v} \times \mathbf{B}_0 = \epsilon_0 \mathbf{E}. \quad (8)$$

Since this ϵ_0 is derived from the general relation $\epsilon_0 \mu_0 = 1/c^2$, ϵ_0 represents the dielectric constant of vacuum and is not, in general, identical with the dielectric constant of the surroundings. It is characteristic and satisfying that in an exact application of Minkowski's theory in (8) there appears the well-defined dielectric constant of vacuum rather than the somewhat problematical and scarcely measurable dielectric constant of the metal.

We conclude from (8) to begin with for the whole interior of the rod (since $\mathbf{E} = \text{const}$):

$$\text{div } \mathbf{D} = \rho = 0,$$

which agrees with the initial equation (1). The interior of the rod is free from space charge, even as observed from the laboratory.

At the surface of the rod, as judged from its interior, we have by (5) and (8),

$$D_n = \epsilon_0 E_n = -\epsilon_0 E_r = \epsilon_0 v B_0 \cos \varphi$$

if n denotes the normal directed toward the interior.

For the sake of simplicity we set the dielectric constant of the exterior (air) also equal to ϵ_0 . Then, by (7), we have for $r = a$, as seen from the outside (n denoting the normal directed outwards),

$$D_n = -\epsilon_0 \frac{\partial \Psi}{\partial r} = \epsilon_0 v B_0 \cos \varphi.$$

The sum of these two D_n yields the surface divergence of \mathbf{D} at the surface, i.e. the surface charge

$$\omega = 2 \epsilon_0 v B_0 \cos \varphi. \quad (9)$$

It varies from place to place and has its maximum values for $\varphi = 0$ and $\varphi = \pi$, i.e. $\pm 2\epsilon_0 v B_0$.

The electric lines of force, which in the interior are straight lines and perpendicular to the axis of the rod, are bent in the exterior from the points of positive surface charge to those of negative surface charge along the shortest possible paths, particularly in the neighborhood of the two points $\varphi = \pi/2$ and $\varphi = 3\pi/2$. Only for $\varphi = 0$ and π are the lines of force perpendicular to the surface and flow off to infinity.

e. If the straight rod is bent into a circular ring, and this is rotated about the axis of symmetry perpendicular to its midplane, every section of the ring is subject to approximately the same conditions as the corresponding section of the straight rod, provided only that the radius of curvature of the ring is large compared to the radius of its cross section. The same applies for a circular disk ring, provided that the radius of its inner bounding cylinder is not too small. Since however the velocity is small in the excluded section of the disk, and the phenomenon of unipolar induction becomes insignificant at small velocities, this restriction may be overlooked and our results be extended to the whole disk and eventually also to an arbitrary body of revolution. We can then apply our Eqs. (3) and (8) also to the field in its interior:

$$\mathbf{E} = -\nabla \times \mathbf{B}_0, \quad \mathbf{D} = \epsilon_0 \mathbf{E} \quad (10)$$

and deduce therefrom the corresponding values of the voltage V and the interior potential Ψ , whereas the potential on the outside must be obtained by the solution of a complex three-dimensional boundary-value problem.

However, the following interesting difficulty arises: If the general (or, rather, too specialized) rule $\rho = \text{div } \mathbf{D}$ is employed to compute the space charge within the rotor, we obtain by (10), since now \mathbf{v} and hence also \mathbf{D} vary in space,

$$\rho = -\epsilon_0 \text{div} (\mathbf{v} \times \mathbf{B}_0) = -\epsilon_0 \mathbf{B}_0 \cdot \text{curl } \mathbf{v} = -2\epsilon_0 \phi B_0,$$

(ϕ = angular velocity of the rotation). This is not zero, as was the case for the translation and as we might have expected from the standpoint of the observer rotating with the body. This contradiction is, however, no objection to *Minkowski's theory of moving media*, which (see footnote 3 at the beginning of §34) is based on the Lorentz transformation of uniform translation, but merely an indication that it is not directly applicable to problems involving rotation.

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OPTICS

Lectures on Theoretical Physics, Vol. IV

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PREFACE

This volume is closely connected with "Electrodynamics," Vol. III of my lectures. Not only the formalism of Maxwell's equations but also their intrinsic character, the invariance with respect to the group of Lorentz transformations, is adopted from Vol. III and is assumed to be known.

Chapter I is entitled "Reflection and Refraction of Light." Only the (never realizable) ideal case of the monochromatic plane wave which is necessarily completely (and, in general, elliptically) polarized is treated in this chapter. Reflection and refraction are regarded throughout as boundary value problems associated with a single boundary surface or (in the case of the plate) with two boundary surfaces. It is surprising how much material falls in this category: It extends from the classical Fresnel formulae to the very timely problem of the tunnel effect and covers non-reflecting lenses, the Perot-Fabry etalon, and the (no longer timely) problem of the "black submarine." The fundamental question of the "coherence or non-coherence of light" is touched upon briefly only in fig. 2 of this chapter. Not until the last chapter, Sec. 49, will we return to the problem of characterizing white light.

Chapter II deals at once with the optics of moving media. Indeed, these questions seem to me to be basically simpler and more fundamental than the contents of the later chapters because one is dealing here with the universal character of the velocity of light and with its physical and astronomical consequences. The first doubts about the classical wave nature of light appear at the end of this chapter in connection with the Doppler and photo-electric effects, and the equivalent corpuscular nature of light makes its first appearance.

Chapter III deals with the theory of dispersion from Drude's semi-phenomenological point of view, which is based on the classically formulated resonance oscillations of electrons bound to atoms. However, it seemed to me unavoidable to add to this chapter a section in which the theory of dispersion is treated *wave-mechanically*, that is, where the characteristic oscillations are replaced by transitions between two different energy levels.

Chapter IV is dedicated to crystal optics, the favorite subject of physics in the last century. Here again the treatment is phenomenological even in the problem of the rotation of the plane of polarization in acentric crystals, which turns out to be particularly simple thanks to our use of the complex notation.

Chapter V and most of Chapter VI are devoted to the problem of diffraction. Diffraction by gratings (including three-dimensional ones) is treated first. Then follows Huygens' principle for scalar diffraction problems, which is applied to the question of "light and shadow" with its manifold paradoxical contradictions of geometrical optics. Chapter V closes with a presentation of the rigorously solvable boundary value problem of the perfectly reflecting straight edge.

Chapter VI begins with the problem of the narrow slit, which Lord Rayleigh solved in the first approximation more than fifty years ago. The problem leads to an integral equation from which higher approximations can be derived if proper use is made of the insight gained in the problem of the straight edge by regarding the behavior of the branched solutions at the edge of the screen. In the succeeding paragraphs a more or less new comprehensive point of view is applied to the question of the resolving powers of spectral apparatus (including Michelson's mirrors for the measurement of the diameters of fixed stars). Thomas Young's theory of diffraction in the formulation given to it by Rubinowicz, and Debye's formulation of focal point diffraction are presented next. Finally, the difference between the scalar and the vector diffraction problems is emphasized and the vectorial generalization of Huygens' principle is discussed. This latter discussion follows the most recent and particularly lucid treatment of the problem by W. Franz.

The presentation of the Cerenkov electron in Sec. 47 reaches beyond the limits of the conventional conception of optics and enters, so to speak, the realm of velocities greater than that of light. Section 48 deals with the (so far almost entirely neglected) *geometrical optics*. The introduction of the eikonal (and the unit vector associated with it) enables us to give a very brief presentation of several of the fundamental problems of geometrical optics. The very large field of *physiological optics*, on the other hand, could only be touched on in the introduction even though it is of primary importance with regard to our actual experience.

The last section is concerned with the nature of *white light* which possesses not a trace of periodicity and attains its wave character only upon passing through a spectral apparatus. The wave representation which appears here only as a secondary attribute of light is missing entirely in geometrical optics and is replaced by a corpuscular conception in Fermat's principle. The corpuscular concept points the way to the modern *theory of photons* and the *complementarity* of wave and corpuscle which was already stated at the end of the second chapter. Finally, it is impressed upon the reader that our presentation, which is essentially based upon the classical wave concept, forms only a part of the entire field of optics; in particular it does not en-

compass the primary processes in the retina because these are photoelectric in nature and therefore their discussion must be based on the theory of photons and not on the wave theory.

The text of this volume is based upon a careful record of my lectures on optics made by L. Waldmann in 1934. However, the last few subjects discussed go considerably beyond the contents of my lectures at that time.

As in the case of Vol. III, I enjoyed the invaluable cooperation of Mr. J. Jaumann in the preparation of this volume. In our many discussions he not only communicated to me his rich experience in experimental optics, but in many instances he prepared the first drafts for the manuscript. I mention, in particular, Sections 3 C, 6 C, 7 C, 30 C, 41, and 42. His part in the writing of this book should not be underestimated. My colleague, Dr. Q. Buhl, subjected the entire manuscript to his critical inspection and has helped me with many useful remarks. Dr. P. Mann has kindly checked the exercises.

Munich, end of 1949.

Arnold Sommerfeld

TRANSLATORS' NOTE

The translators of this volume have endeavored to adhere to the spirit of the original as much as possible and to keep changes to a minimum. In addition to certain changes in notation, some modifications of the text have proved to be inevitable. These are especially contained in Sections 27 and 28 which were kindly contributed by Professor P. P. Ewald. Furthermore, Sec. 47 should be read in the light of a recent paper by H. Motz and L. I. Schiff, *Am. J. Phys.* 21, 258, 1953. A completely new author and subject index was prepared.

O. L.

P. A. M.

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Lectures on Theoretical Physics

VOLUME I: Mechanics. 1952. Translated by Martin O. Stern.

VOLUME II: Mechanics of Deformable Bodies. 1950. Translated by G. Kuerti.

VOLUME III: Electrodynamics. 1952. Translated by Edward G. Ramberg.

VOLUME V: Thermodynamics and Statistical Mechanics. Translation in preparation.

VOLUME VI: Partial Differential Equations in Physics. 1949. Translated by Ernst G. Straus.

INTRODUCTION

1. Geometrical, Physical, and Physiological Optics. Historical Chart

The eye is our noblest sense organ. It is therefore not surprising that even the natural philosophers of antiquity were concerned with the science of light. Leonardo da Vinci called optics "the paradise of mathematicians". Of course, by optics he meant only *geometrical* or *ray optics*, the theory of perspective and the distribution of light and shadow. How much more justified would his assertion have been had he known the *wave optics* with its marvelous color phenomena arising from diffracted light or the polarized light of crystals. It is in particular these latter phenomena which one has in mind when one speaks of *physical optics*. Physical optics is related to ray optics in the same way in which wave mechanics is related to classical mechanics. This fact was recognized by Schrödinger on the basis of the profound work of Hamilton.

There is, however, still a third branch of optics which is called *physiological optics* after the title of Helmholtz's principal work. Also in this field fundamental laws hold which, however, are based on the operation of the sense organs and the mind. But these laws are not encompassed by our physical theory. It was the tragedy in the life of Goethe that he would not recognize the distinction between physical and physiological optics; this was the reason for his fruitless fight against Newton. Today we understand without difficulty that the sensation yellow which is caused by the *D*-lines of sodium is a phenomenon which is entirely different from the wavelengths $\lambda = 5890 \text{ \AA}$ and $\lambda = 5896 \text{ \AA}$ by which we must describe these lines physically. For, we know that the psychological response to an event is something entirely different from the physical event itself; the two are different in nature and incommensurable.

In this volume we shall be able to deal only briefly with ray optics and unfortunately not at all with physiological optics. Wave optics, which we shall develop directly from the results of Vol. III and which, through spectroscopy, opens the way to modern atomic physics, will give us enough to do. We shall not, for instance, enter upon the interesting field of the theory of color which was formulated in a classical manner by Thomas Young and Helmholtz, was further developed particularly by Grassmann, Maxwell and

Schrödinger, and is even today not a closed subject. We shall here only demonstrate very briefly that, quite aside from the quality of colors and their contrast effects, there exists a profound difference between subjective perception and objective fact even in regard to the quantitative determination of intensity. The phenomenon in question is that of the so-called "half-shadow".

This phenomenon played a role in the earliest attempts to determine the wavelength of X-rays. On X-ray plates there appear half-shadow regions

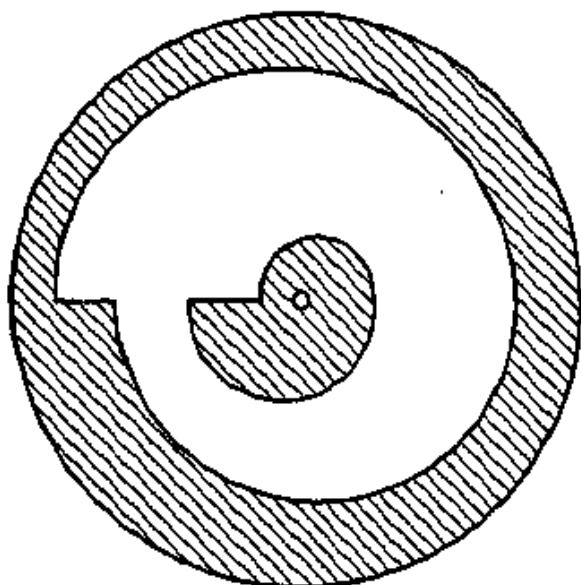


Fig. 1.

Rotating disc for the demonstration of a physiological optical illusion.

between the complete shadow and the region of full illumination. These are due to secondary X-rays which originate, for instance, at the edges of a slit. To the eye these half-shadow regions appear as bright and dark fringes which were at first interpreted as interference lines. However, Haga and Wind were able to show that these fringes were subjective in origin and they called attention to a phenomenon which had been investigated by E. Mach¹ and had also been recognized by H. Seeliger in his studies of eclipses of the moon. We shall describe it here as our sole example of physiological-optical phenomena.

Consider a white circular cardboard disc which is partially blackened as shown in fig. 1. The boundary between the black and white fields consists of two spirals of Archimedes and portions of a radius of the disc. Let us consider the *average* brightness (or blackness) along each circle concentric with the edge of the disc; this quantity determines, in accordance with a law due to Talbot, the perception of brightness when the disc is rotated sufficiently fast. The center of the disc is then perfectly black and so is its edge. Between the center and the edge there is a zone of maximum brightness. The transition between darkness and brightness consists of two half-shadow regions. Since the radius vectors of the spirals of Archimedes increase (or decrease) linearly with the central angle, the intensity in the half-shadow region also increases (or decreases) linearly with the distance from the center of the disc. If the disc is set into rapid rotation on the axis of a motor, then

¹ See, for instance, his book *Prinzipien der Physikalischen Optik*, p. 158, J. A. Barth, publ. 1921.

the intensity distribution presented to the eye is that represented by the dotted line in fig. 1a. But what does the eye see? Instead of the linearly varying half-shadows the eye perceives a uniform *average* brightness; where the half-shadow borders on the completely black regions it perceives *dark* fringes which are considerably blacker than the regions of complete blackness; at the limits of full brightness it sees *bright* fringes which appear much brighter than the region of full brightness. The eye (or the mind?) is, as it were, startled by the transition from the half-shadow to full illumination; it exaggerates the contrast. The same exaggeration takes place at the transition from the half-shadow to complete blackness. The eye (or the mind) judges only contrasts and not objective intensity values; it is affected more by the derivatives of the intensity curve than by the absolute values of its ordinate. The bright and dark fringes (which on the rotating disc are, of course, circles about the center) are so definitely pronounced that a naive observer would swear to their genuineness.

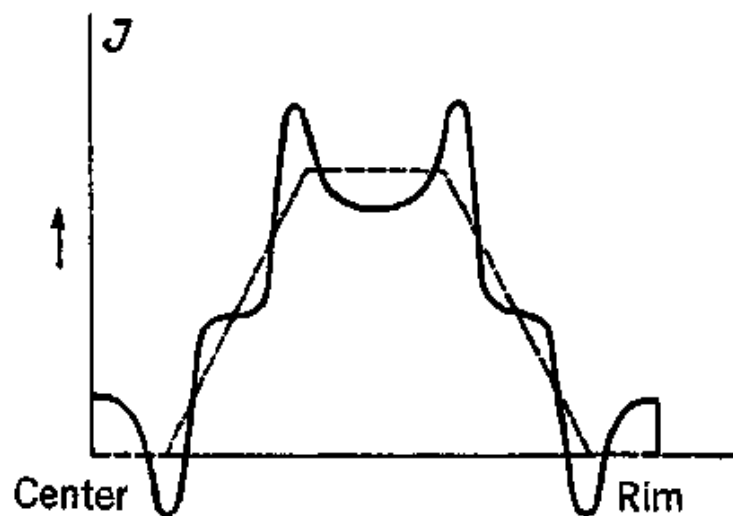


Fig. 1a.

The subjective intensity distribution perceived by the eye (full line) and the objective distribution of intensity (dotted line) when the disc is rotated.

Similar fringes are seen wherever extended light sources produce half-shadows according to geometrical optics, as for instance, behind a pencil which is illuminated by a Welsbach mantle. Also the bright border which one sees about one's own shadow on the road when the sun is behind one's back and which has the effect of a sort of halo about the head and limbs is at least partly due to this optical illusion. Such fringes also played a part in certain occasional arguments between the author and a group of Munich painters which revolved around the old controversy "Goethe vs. Newton". The opponents in these discussions understandably enough considered these subjective phenomena as objective and offered them as proof of the falseness of the physical theories.

It might be thought that this illusion could not be photographed and would thereby betray its subjective character. This is not so. Even though the number of blackened grains on the photographic plate corresponds to the correct intensity, the eye interprets the photographic image in the same way

as the original object and is deceived by its subjective contrast perceptions. This is illustrated by the following experiment¹: a micrometer slit illuminated from the rear with parallel light is photographed. At the beginning of the exposure the slit may have a width $2b$. It is then slowly and uniformly opened to a width $2a$, whereupon the exposure is terminated. Thus the center portion $2b$ of the photographic plate is continuously illuminated during the exposure; the adjoining portions $a - b$ are illuminated a shorter time which decreases linearly to zero. On the photograph one sees again bright and dark fringes at the limits of the half-shadows (if the slit is opened non-uniformly, there appear also secondary fringes inside the half-shadow regions $a - b$ which correspond to discontinuities in the derivative of the curve depicting illumination vs. time).

So much (or rather so little) for physiological optics. In order to provide a general summary of the wealth of material to be covered in this volume we continue with a historical list of the most important optical discoveries.

Snell's law of refraction (which became known only through Huygens) and Descartes, *Dioptrics*, 1637. The first theory of the rainbow is also due to Descartes.

Grimaldi, *Physico-mathesis de lumine, coloribus et iride*, Bononiae (Bologna) 1655; first textbook on optics; deviations from rectilinear ray paths; diffraction.

Olaf Roemer, 1675; determination of the velocity of light from the eclipses of the satellites of Jupiter.

Christian Huygens, *Traité de la Lumière*, Leiden 1690; wave theory without closer investigation of the nature of the oscillations (whether longitudinal or transverse). Huygens' principle; wave surfaces. Double refraction in calcite.

Newton, *Opticks* 1706, English 1675. Colors of thin plates. Spectral colors and their composition into white light. Theory of emission with lateral "fits".

Bradley, 1728; aberration of light.

Thomas Young, *Lectures on Natural Philosophy*, 1807. Interference of light; diffraction; theory of color; the color triangle; Young also deciphered hieroglyphics.

Malus, *Sur une propriété des forces répulsives qui agissent sur la lumière*, 1809. Polarization by reflection.

Biot, Brewster, Arago, crystal physics, Arago, 1811, rotatory power of quartz.

¹ J. Drecker, *Physikal. ZS.* 2, 145, 1900.

Fraunhofer, 1787 – 1826, Fresnel, 1788 – 1827, the two classics of wave optics, both of whom died young after lives filled with work, success and fame. Fraunhofer was the greatest glass technician and telescope constructor of his time; he made the first diffraction gratings and was the father of spectroscopy and astrophysics because of his discovery of the Fraunhofer lines in the spectrum of the sun and the planets. Fresnel developed the wave theory; his *dragging coefficient* was a forerunner of the theory of relativity; he was an untiring experimenter in crystal optics. Fraunhofer and Fresnel diffraction.

Bessel, in 1838, measured the first fixed star parallaxes in the constellation Cygnus by means of a Fraunhofer telescope.

Christian Doppler, 1842, "Über das farbige Licht der Doppelsterne und einiger anderer Gestirne des Himmels". (On the Colored Light of Double Stars and Several Other Stars.)

Terrestrial determination of the velocity of light, Fizeau 1849 by means of a toothed wheel; Foucault 1850 by means of a rotating mirror; also Michelson, beginning in 1926.

Faraday, 1845, "On the Magnetization of Light, and the Illumination of Magnetic Lines of Force".

Maxwell, 1861, discovery of the electro-magnetic theory of light; Treatise, 1873.

The experiment planned by Maxwell to determine by interferometry a possible dependence of the velocity of light on the azimuth of the earth in its orbit about the sun was carried out by Michelson in 1881, improved by Michelson and Morley in 1887, and repeated with greatest accuracy by Joos in 1930 at the Zeiss works, Jena.

The theory of dispersion based on an elastic theory was developed by Ketteler and Sellmeier. The electromagnetic theory of dispersion was started by Helmholtz and was completed by Drude on the basis of the theory of electrons; Drude, *Lehrbuch der Optik*, 1900. 1926, wave-mechanical theory of dispersion by Schrödinger.

Abbe, 1840 – 1905, diffraction theory of optical images; also simultaneous work by Helmholtz and Lord Rayleigh.

Standing light waves, O. Wiener, 1890, on their basis Lippmann's color photography.

Lord Rayleigh, 1842 – 1919, explanation of the blue color of the sky; introduction of group velocity into optics; resolving power of the prism. Conception of natural white light as a completely random, non-periodic process.

Zeeman effect, 1896; explanation of the normal Zeeman effect by H. A. Lorentz.

Einstein in 1905 deduced from the quantum theory the notion of photons (light quanta).

CHAPTER I

REFLECTION AND REFRACTION OF LIGHT

2. Review of Electrodynamics. Basic Principles of Ideal and Natural Light

In Vol. II, Sec. 45 we showed that at the interface between two optically different media the elastic theory of light provides more boundary conditions than are consistent with the known facts of polarization, i. e. the transverse character, of light. Hence we turn our attention to the electromagnetic theory of light, which, in contrast to the elastic theory, specifies only two boundary conditions for the electric field strength E and two for the associated magnetic "disturbance" H , namely, the equality of the components *tangential* to the boundary surface.

We shall assume the light to be monochromatic and will hence perform our calculations using a *single frequency* ω . This assumption involves a far-reaching idealization of true conditions and has the following meaning: we consider the light to be spectrally decomposed and use only an infinitesimally small portion of the spectrum as a light source. The spectroscopic apparatus used for this purpose is called a *monochromator*. Furthermore, we do not consider any arbitrary bundle of light rays but, again by a far-reaching idealization of true conditions, we consider the mathematically much simpler case of a plane wave with a well-defined direction of propagation. This means that we use a *collimator* (a tube with a convex lens in whose focal plane is a slit) by means of which we can obtain a system of parallel rays of a certain width from an originally diverging bundle of light rays.

Natural light does not possess either of these two properties. This holds even for sunlight which is completely irregular as regards frequency, and also lacks sufficient parallelism because of the finite size of the sun's disc.

We will first consider ideal light which has passed through an ideal monochromator and collimator. Later, we will discuss the characteristics of natural light.

We choose the x -axis in the direction of propagation of our plane wave, denote its electromagnetic field by the two vectors E and H , and represent

these by the real parts of the following expressions in which we suppress¹ the symbol Re which is to be thought of as attached to the right-hand sides:

$$(1) \quad \mathbf{E} = \mathbf{A} e^{i(kx - \omega t)}, \quad \mathbf{H} = \mathbf{A}' e^{i(kx - \omega t)}$$

k is called the wave number of the light. \mathbf{A} is a constant which is independent of x and t but which has different and generally complex values for the different components of \mathbf{E} . \mathbf{A}' is determined by \mathbf{A} . Instead of \mathbf{H} we could, of course, have used \mathbf{B} to represent the wave. We prefer \mathbf{H} mainly because the conditions at the boundary between two optically different media require the continuity of \mathbf{H} as well as of \mathbf{E} ; and partly also because the radiation vector \mathbf{S} which is fundamental in optics is given by

$$(1a) \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}$$

(this contains no additional factors, thanks to our M K S Q system of units which was introduced in Vol. III and upon which also this volume will be based). Furthermore, we are thus in better agreement with the usual literature in which \mathbf{H} (given the designation "field strength", however) is almost always used in conjunction with \mathbf{E} .

In an isotropic medium which is free of charges electromagnetism requires the transverse character of light because of the condition $\text{div } \mathbf{E} = 0$ which, see Vol. III, Sec. 6, leads to $E_x = 0$ for the case of our plane wave given by (1). Hence, there exist only the two components E_y and E_z . The same is true of \mathbf{H} . k is connected with ω by the equation (see Vol. III, Sec. 6)

$$(2) \quad k = \sqrt{\epsilon \mu} \omega \quad \begin{cases} \epsilon = \text{dielectric constant} \\ \mu = \text{permeability.} \end{cases}$$

Dimensionally, $\sqrt{\epsilon \mu}$ is an inverse velocity. We shall call it $1/u$ where u is the phase velocity of propagation in the particular medium. This follows from the exponents in expression (1), which, on differentiation with respect to t and setting equal to zero, give:

$$k \frac{dx}{dt} - \omega = 0.$$

In a vacuum with $\epsilon = \epsilon_0$, $\mu = \mu_0$ we have

$$(3) \quad u = c, \quad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \sim 3 \cdot 10^8 \text{ meters/second.}$$

¹ In exercise I. 1. we will illustrate by means of a very simple example the advantage of computing wave problems using complex exponential functions rather than using their real parts.

The connection between H and E or, what is the same, between the constants A' and A occurring in (1) follows from Maxwell's equations for nonconducting media

$$(4) \quad \mu \frac{\partial H}{\partial t} = - \text{curl } E, \quad \varepsilon \frac{\partial E}{\partial t} = \text{curl } H.$$

When specialized to the case of our plane wave, the first of these expressions gives, since $E_x = 0$, $H_x = 0$,

$$-i\omega\mu A_y' = ikA_z, \quad -i\omega\mu A_z' = -ikA_y$$

and hence

$$(5) \quad A_y' = -\frac{k}{\mu\omega} A_z = -\sqrt{\frac{\varepsilon}{\mu}} A_z, \quad A_z' = \frac{k}{\mu\omega} A_y = \sqrt{\frac{\varepsilon}{\mu}} A_y.$$

These expressions also follow, of course, from the second eq. (4).

Depending on the choice of the constants A_y , A_z , there results as a *necessary consequence of Maxwell's equations* a uniquely determined state of oscillation or, as we can also say, a uniquely determined *polarization* of our monochromatic plane wave.

In exercise I.2. we shall observe that in general eq. (1) represents *elliptic polarization*. This means that if the electric vector E is drawn so as to originate from the point $y = 0, z = 0$, its tip describes during the time $\tau = \frac{2\pi}{\omega}$ an ellipse with its principal axes in the yz -plane. The same is true of the vector H . It need hardly be said that, according to our conception of the Maxwell theory, nothing material oscillates in this process, no motion of the "light ether" takes place. We will see later how this ideal case of elliptic polarization can be realized practically to a good degree of approximation. (Total reflection, reflection on metals, crystal optics.)

An important special case of elliptic polarization is *circular polarization* which results when

$$(6) \quad |A_y| = |A_z|, \quad \frac{A_z}{A_y} = \pm i.$$

Linear polarization is characterized by

$$(6a) \quad \frac{A_z}{A_y} \text{ real (positive or negative),}$$

in particular, of course, by the special cases $A_z = 0$ or $A_y = 0$.

In the following discussion we shall have occasion to speak not only of a monochromator and collimator but also of a polarizer (Nicol prism, quarter wave plate, etc.). According to the above discussion, such a polarizer does

not actually produce polarized light but rather serves to *transform* the light from one type of polarization to another. The presence of general elliptic polarization is already guaranteed by the monochromator and collimator and by Maxwell's equations, paradoxical as this may sound to the experimentalist in practical optics. Theoretically it is correct to say that even after using an ideal monochromator and collimator, the four parameters contained in the arbitrary constants A_y and A_z (two amplitudes $|A_y|$ and $|A_z|$ and two phase constants α_y and α_z) remain undetermined; only through the use of a polarizer are the values of these parameters restricted.

Of course, an actual monochromator or collimator never functions ideally. Consequently, we are, in reality, never faced with ideal light but rather with a continuous superposition of ideal cases in which, at best, a certain region of frequency or a certain direction of propagation is strongly favored. Even light which has passed through the monochromator is, thus, not strictly monochromatic but has a certain spectral width even when it is reduced to a single spectral line. By the same token, the light emerging from the collimator is actually characterized by an integral over a region of spatial directions, though one of these directions contributes most strongly. Likewise, the various polarization devices also are endowed with a certain indefiniteness.

For *natural* light the region of integration is in every respect unlimited; it comprises all frequencies $0 < \omega < \infty$ and in the diffuse case all possible directions of incidence. Moreover, for natural light no one direction of polarization takes precedence over all others. Planck had to analyze the consequences of this situation in order to set up his thermodynamic law of radiation and this law, in turn, led to the discovery of the quantum theory. He represented the radiation of a black body not by eq. (1), but for every ever-so-small frequency range $\Delta\omega$ by an infinite sum of terms of the form (1) in which the constants A_i , even for neighboring ω 's change their absolute values and phases arbitrarily. This formulation is due to the fact that the elementary processes of black-body radiation originate in single atoms which emit independently of one another. Only the quantity $\sqrt{A_y^2 + A_z^2}$ has a given value which is determined by the mean energy of all elementary processes. The A_y , A_z taken singly, however, remain completely undetermined especially as regards their phases. All this is also true for what we call "natural light". To be sure, even the naked eye exercises a certain amount of selection among the infinitely numerous components of the natural light. By fixating a certain point the eye acts as a collimator since, like the latter, it is equipped with a lens. Furthermore, by its spectrally selective sensitivity and its perception of color, the eye limits the frequency region.

The more the preference for a certain frequency and direction is emphasized in any single case, the larger is the *region of space and time in which the plane wave represents a sufficiently good approximation of the natural light field*. Outside of this region the phases are subject to statistically distributed variations in space and time. Fig. 2. attempts to illustrate this by means of an example.

This figure represents the superposition of six plane monochromatic waves; that is, the superposition of the six real parts of exponential functions of the form (1); their frequencies ω are in the ratios

$$95 : 97 : 99 : 101 : 103 : 105;$$

their directions of propagation k deviate in pairs by $\pm 1/20$ radian from the direction of the middle pair; they are denoted successively by $+1/20, +1/20, 0, 0, -1/20, -1/20$.

The system of solid lines which are mostly straight shows the instantaneous nodes of the resulting state of oscillation. Between these nodes lie alternately wave crests and troughs whose amplitudes are denoted by the broken lines in a manner similar to the contour lines on a map. Successive contour lines have a difference in amplitude of 1. Except for the numbers 0, only the height of the tallest amplitude crest is given in each case. One sees that the regular sequence of waves is interrupted only at points where the amplitude vanishes; at such points a wave of new period infiltrates, so to speak. From this it follows that behind such a point the wavefronts overtake, or fall behind, the progress of the undisturbed waves corresponding to a locally closer succession of waves, or, effectively, to an increase in the frequency by 1. One must imagine that this whole wave picture propagates with the velocity of light in the direction of the arrows while its shape changes gradually.

Visual inspection shows that there are finite regions, several wavelengths in extent, which have, to a sufficient degree of approximation, the character of a homogeneous plane wave and also retain this character as the wave propagates. Hence, these regions indeed satisfy the above postulated conditions for "regions of good approximation". Only the zero points seem to be exceptions. However, just because the amplitude vanishes there, they do not produce any stronger effect than other points of varying intensity.

In order to verify experimentally any results calculated by means of plane monochromatic waves, one must make certain that the entire object under observation is contained in such a region of good approximation during the entire time of observation. If this condition is satisfied, one speaks of the production of *coherent light*. To be more precise, one should speak of the

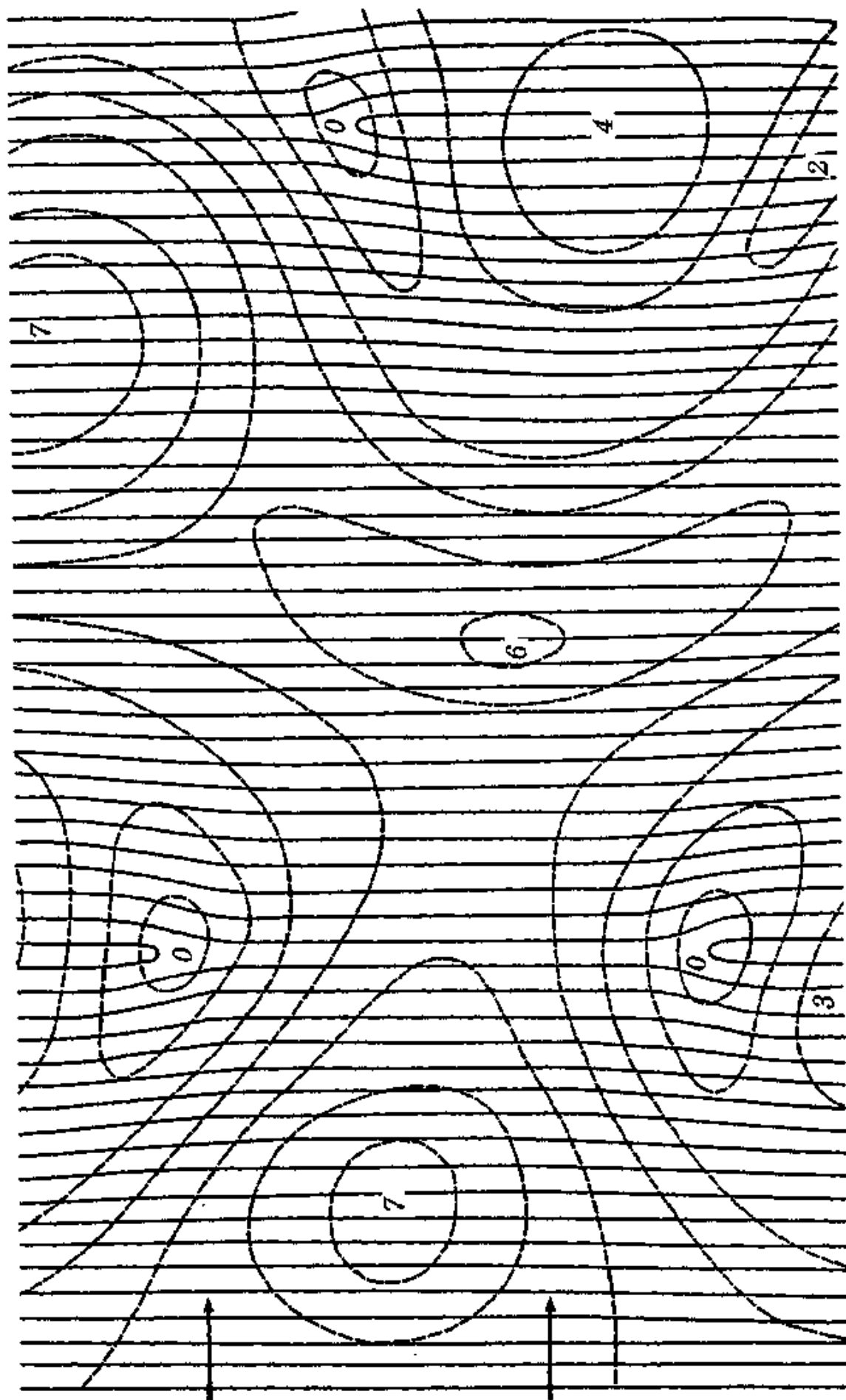


Fig. 2. Instantaneous picture of a "wave-packet" composed of 6 simple waves: — lines of constant phase, --- lines of constant amplitude.

production of a sufficiently large *region of coherence* in space-time because even a natural light field always has small regions of coherence.

The size of the region that is sufficient in each particular case, and hence the demands which are to be made upon the monochromatic quality and the directional uniformity of the rays, depend upon the relative sizes of the object and the wavelength. Anticipating a later discussion, we shall give a few examples.

No optical instruments are required in order to see the interplay of colors resulting from colloidal particles or in a lunar corona:

For the observation of the colors of *thin plates* a parallel direction of the incident light is of almost no importance. Also, in this case, the spectral selectivity of the eye suffices to limit the frequency range.

From *thick plates*, however, one obtains interference fringes only if sharp spectral lines and well-defined rays are used; a telescope is necessary to separate the fringes.

One cannot observe diffraction at a slit in sunlight unless the slit is extremely narrow. A collimator or auxiliary slit is necessary. Only rays originating from one and the same region of coherence can interfere with one another. For such rays the *light vectors* add. Radiation fields from more distant regions combine energetically in a time average sense. For these the intensities add.

Before we turn to the actual subject of this chapter, namely reflection and refraction, we shall review briefly the units used above as well as some of the quantities from Vol. III which we will use later on.

Using the four units M (meter), K (kilogram of mass), S (second), Q (electric charge)¹, one obtains

Unit of force: $1 \text{ newton} = \text{MK S}^{-2} = 10^5 \text{ dyne}$

Unit of energy: $1 \text{ joule} = 10^7 \text{ erg}$

Unit of power: $1 \text{ joule S}^{-1} = 1 \text{ watt}$

Electric field strength:

$$\frac{\text{Force}}{\text{Charge}} = \frac{\text{newton}}{Q} = \frac{\text{volt}}{M}, \quad 1V = 1 \frac{\text{joule}}{Q}$$

Current strength: $QS^{-1} = \text{amp.}$

Current density:

$$J = \frac{\text{amp.}}{M^2} = QM^{-2} S^{-1}$$

¹ For numerical calculations one uses the coulomb as a unit for Q, i. e. the charge one ampere second.

The "displacement current" D has the same dimension. Hence the dimension of Maxwell's "electric displacement" D which is our "electric excitation" becomes QM^{-2} . Because $D = \epsilon E$, the dimension of the dielectric constant becomes $\frac{Q^2 M^{-2}}{\text{newton}} = M^{-1} S \Omega^{-1}$, $1 \Omega = 1 \text{ joule } S / Q^2 = 1 \text{ ohm}$.

Magnetic moment = Pole strength \times Lever arm = Pl. Following Wilhelm Weber this is defined as Current \times Area. Hence,

Pole strength $P = QMS^{-1}$ = "moving charge" in the sense of Ampere.

Magnetic Induction (really "field strength") $B = \text{newton} / P = \text{newton } SQ^{-1} M^{-1} = \text{volt } S / M^2$

Magnetic Excitation $H = P / M^2 = QM^{-1} S^{-1} = \text{amp} / M$

Permeability $\mu = \frac{B}{H} = \text{newton } S^2 Q^{-2} = M^{-1} S \Omega$

$\epsilon \mu = M^{-2} S^2 = (\text{velocity})^{-2}$

$\mu / \epsilon = \Omega^2$, $\sqrt{\mu_0 / \epsilon_0} = \text{wave impedance of the vacuum}$

Convention for the purpose of eliminating 4π from the formulae: $\mu_0 = 4\pi \cdot 10^{-7} M^{-1} S \Omega$. From this it follows that: $4\pi c^2 \epsilon_0 = 10^7 MS^{-1} \Omega^{-1}$

Radiation vector $S = E \times H = \text{joule } M^{-2} S^{-1}$

We shall not introduce here the magnetic pole strength P as a fifth unit (see Vol. III 8 B).

3. Fresnel's Formulae. Transitions from Rarer to Denser Media

We can consider the boundary between two optically different media to be plane if we limit our consideration to a small portion of that boundary (for example, a piece several hundred wavelengths in size). We call this boundary the plane $y = 0$ of a right-handed cartesian coordinate system. Let a linearly polarized plane wave coming from the half-space $y > 0$ impinge upon this boundary. The plane of incidence shall be the plane of the paper in figures 3 a, b, that is, the yx -plane. Let the direction of propagation of the wave form the angle α with the negative y -axis. For the solution of the "boundary value problem" under consideration we will introduce, besides the refracted wave in the second medium (angle β with the negative y -axis), a reflected plane wave whose angle with respect to the positive y -axis shall, for the present, be termed α' . We denote the electric amplitudes of the three waves by A, B, C , where A belongs to the incident ray S_i , B to the refracted or "transmitted" ray S_d , C to the reflected ray S_r . We first consider the case:

A. ELECTRIC VECTOR PERPENDICULAR TO PLANE OF INCIDENCE.

Besides omitting the symbol for the real part in the relations (2.1) we shall also omit the time factor $\exp(-i\omega t)$, which is the same for all three waves. E is everywhere in the direction of the z -axis. We now introduce, temporarily, polar coordinates r, φ in the xy -plane. For the direction of propagation occurring in (2.1) and there denoted by α , we substitute $x = r \cos \varphi$ or, more generally, we take account of the differences between the three directions of propagation by writing $x = r \cos(\varphi - \gamma)$. According to fig. 3 a we must then use the following values for γ :

$$\text{for the incident wave } -\frac{1}{2}\pi + \alpha, \quad r \cos(\varphi - \gamma) = x \sin \alpha - y \cos \alpha,$$

$$\text{for the refracted wave } -\frac{1}{2}\pi + \beta, \quad r \cos(\varphi - \gamma) = x \sin \beta - y \cos \beta,$$

$$\text{for the reflected wave } +\frac{1}{2}\pi - \alpha', \quad r \cos(\varphi - \gamma) = x \sin \alpha' + y \cos \alpha',$$

where the x, y in the last column indicate, in contrast to the relation (2.1), the coordinates defined in the figure. The resulting superposition of the incident and reflected wave in medium 1 and the resulting refracted wave in medium 2 are given by

$$(1) \quad E_z = A e^{ik_1(x \sin \alpha - y \cos \alpha)} + C e^{ik_1(x \sin \alpha' + y \cos \alpha')}$$

and

$$(1 a) \quad E_z = B e^{ik_2(x \sin \beta - y \cos \beta)}$$

respectively. The boundary condition at $y = 0$ demands

$$(2) \quad A e^{ik_1 x \sin \alpha} + C e^{ik_1 x \sin \alpha'} = B e^{ik_2 x \sin \beta}.$$

Because of its dependence on x , this condition can be fulfilled by a choice of the constants $A : B : C$ only if the exponential factors cancel; hence if

$$(3) \quad \alpha = \alpha', \quad k_1 \sin \alpha = k_2 \sin \beta.$$

The first of these equations is the *law of reflection*; the second is the *law of refraction* which, recalling (2.2), becomes

$$(3 a) \quad \frac{\sin \alpha}{\sin \beta} = \frac{k_2}{k_1} = \sqrt{\frac{\epsilon_2 \mu_2}{\epsilon_1 \mu_1}}.$$

The right-hand side of this double equation defines the (relative) index of refraction of the media 1 and 2

$$(3 b) \quad n_{12} = \sqrt{\frac{\epsilon_2 \mu_2}{\epsilon_1 \mu_1}}.$$

If we assume medium 1 to be air, whose constants ϵ_1 and μ_1 are almost the same as those of vacuum, and if we denote the constants of medium 2 merely by ϵ and μ , then we obtain the definition of the *index of refraction with respect to air*:

$$(4) \quad n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} = \frac{c}{u}.$$

Here, as in (2.3), u denotes the phase velocity in medium 2. After putting $\mu = \mu_0$ and $\epsilon/\epsilon_0 = \epsilon_{rel}$ (dielectric constant relative to vacuum), this is usually written:

$$(4a) \quad n = \sqrt{\epsilon_{rel}},$$

which is Maxwell's relation. As Boltzmann has shown, this relation is fairly well satisfied for homopolar gases and vapors but not at all for solid and liquid media, especially not for those with infrared resonance vibrations. For water, for instance, one gets $\sqrt{\epsilon_{rel}} \sim 9$ as against $n \sim 4/3$. Maxwell's relation does not explain dispersion (dependence of n on frequency) at all.

Because of (3), (2) reduces simply to

$$(5) \quad A + C = B.$$

We obtain a second equation for the ratio of the constants $A : B : C$ by writing the boundary condition for the tangential component H_x . According to (2.5) the amplitude factor of H follows from that of E by multiplication by $\pm \sqrt{\epsilon/\mu}$ where the sign is to be decided according to the right-handed screw rule: E , H and S form for all three waves a right-handed coordinate system. At an instant when E is positive, that is, directed out of the paper in fig. 3 a,

H is generated by a clockwise rotation from the S direction. Hence, according to the ray directions as drawn, H must point to the *left* for the incident and refracted wave and to the *right* for the reflected wave. Accordingly, H_x is found by multiplying the corresponding amplitudes of the electric vector by $-\cos \alpha$ and $-\cos \beta$ in the case of the incident and refracted wave, respectively, and by $+\cos \alpha$ in the case of the reflected wave. One obtains then instead of (1)

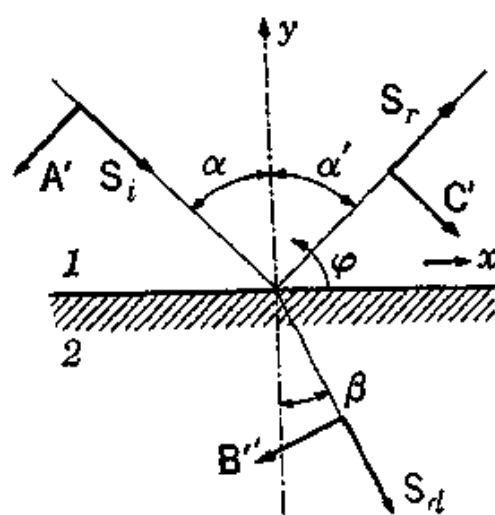


Fig. 3 a.

Illustration for the derivation of Fresnel's formulae for transition from a rarer to a denser medium. Electric vector perpendicular to the plane of the drawing.

$$(6) \quad H_x = \sqrt{\frac{\epsilon_1}{\mu_1}} \cos \alpha e^{i k_1 x \sin \alpha} \{-A e^{-i k_1 y \cos \alpha} + C e^{i k_1 y \cos \alpha}\}$$

and instead of (1 a)

$$(6a) \quad H_x = - \sqrt{\frac{\epsilon_2}{\mu_2}} \cos \beta e^{i k_2 x \sin \beta} B e^{-i k_2 y \cos \beta}.$$

Upon simplification by means of the law of refraction, the boundary condition takes the form

$$(7) \quad \sqrt{\frac{\epsilon_1}{\mu_1}} \cos \alpha \{-A + C\} = - \sqrt{\frac{\epsilon_2}{\mu_2}} \cos \beta B,$$

which we shall write in the form

$$(8) \quad A - C = m_{12} \frac{\cos \beta}{\cos \alpha} B, \quad m_{12} = \sqrt{\frac{\epsilon_2}{\epsilon_1} \frac{\mu_1}{\mu_2}}.$$

m is generally (for $\mu_2 \neq \mu_1$) different from n . While n indicates the *ratio of two wave velocities*, we have to look upon m , according to our table of dimensions, as the *ratio of two wave impedances*.

By addition and subtraction of the two equations (5) and (8), one obtains

$$(9) \quad \left. \begin{matrix} 2A \\ 2C \end{matrix} \right\} = \left(1 \pm m_{12} \frac{\cos \beta}{\cos \alpha} \right) B.$$

In order to conform to the usual way of writing Fresnel's formulae, we put, using (8) and (3 b),

$$(10) \quad m_{12} = n_{12} \frac{\mu_1}{\mu_2} = \frac{\mu_1}{\mu_2} \frac{\sin \alpha}{\sin \beta}$$

and obtain instead of (9)

$$(11) \quad \left. \begin{matrix} 2A \\ 2C \end{matrix} \right\} = \left(1 \pm \frac{\mu_1}{\mu_2} \frac{\sin \alpha \cos \beta}{\sin \beta \cos \alpha} \right) B.$$

In the case $\mu_2 \sim \mu_1 = \mu_0$ which usually obtains, we can write the *first Fresnel formula* more simply as:

$$(12) \quad A : B : C = \sin(\beta + \alpha) : [\sin(\beta + \alpha) + \sin(\beta - \alpha)] : \sin(\beta - \alpha).$$

B. MAGNETIC VECTOR PERPENDICULAR TO PLANE OF INCIDENCE.

We start now with the magnetic vector H in the direction of the z -axis. As in (2.1), we denote the amplitude factor of the incident wave by A' and the amplitudes of the refracted and reflected waves by B' and C' , respectively. Because of the continuity of H_z at $y=0$, we have instead of (2) the boundary condition:

$$(13) \quad A' e^{i k_1 x \sin \alpha} + C' e^{i k_1 x \sin \alpha'} = B' e^{i k_2 x \sin \beta}$$

The laws of reflection and refraction follow from this just as in (3), (3 a) and (13) reduces to

$$(14) \quad A' + C' = B'.$$

For the E-waves (factor $\sqrt{\epsilon_1/\mu_1}$ with A' and C' , factor $\sqrt{\epsilon_2/\mu_2}$ with B') this gives

$$(14 a) \quad A + C = m_{12} B.$$

where the meaning of m_{12} is given by (8).

A second condition follows from the continuity of E_x at $y = 0$. In order to determine the signs we look at fig. 3 b. and consider again an instant when H , represented as an arrow perpendicular to the plane of the paper, is directed toward the reader. From the fact that E , H and S , in that order, must form a right-handed system, the E arrows follow as drawn in fig. 3 b. Projecting these onto the x -axis one obtains, instead of (8), as a second boundary condition

$$\cos \alpha (A - C) = \cos \beta B,$$

and combining this with (14 a)

$$(15) \quad \begin{cases} 2A \\ 2C \end{cases} = \left(m_{12} \pm \frac{\cos \beta}{\cos \alpha} \right) B = \left(\frac{\mu_1 \sin \alpha}{\mu_2 \sin \beta} \pm \frac{\cos \beta}{\cos \alpha} \right) B.$$

In the usual case $\mu_1 \sim \mu_2$ one obtains the simplification

$$(15 a) \quad \frac{4A}{B} = \frac{\sin 2\alpha + \sin 2\beta}{\sin \beta \cos \alpha} = \frac{2 \sin (\alpha + \beta) \cos (\alpha - \beta)}{\sin \beta \cos \alpha},$$

$$(15 b) \quad \frac{4C}{B} = \frac{\sin 2\alpha - \sin 2\beta}{\sin \beta \cos \alpha} = \frac{2 \cos (\alpha + \beta) \sin (\alpha - \beta)}{\sin \beta \cos \alpha}.$$

From this it follows immediately that

$$A : C = \tan (\alpha + \beta) : \tan (\alpha - \beta).$$

On the other hand, from (15 a) one can easily calculate

$$\begin{aligned} A : B &= \tan (\alpha + \beta) : \frac{2 \sin \beta \cos \alpha}{\cos (\alpha + \beta) \cos (\alpha - \beta)} = \tan (\alpha + \beta) : \\ &: \frac{\sin (\alpha + \beta) - \sin (\alpha - \beta)}{\cos (\alpha + \beta) \cos (\alpha - \beta)} = \tan (\alpha + \beta) : \left(\frac{\tan (\alpha + \beta)}{\cos (\alpha - \beta)} - \frac{\tan (\alpha - \beta)}{\cos (\alpha + \beta)} \right). \end{aligned}$$

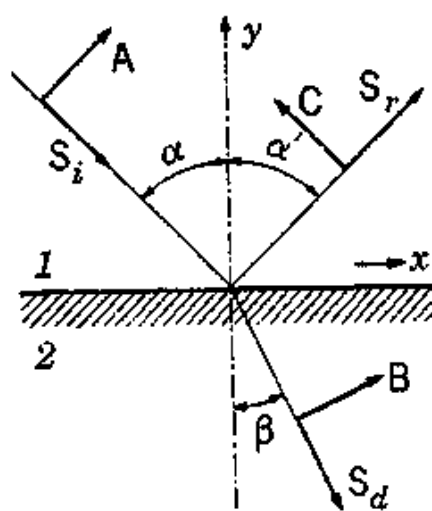


FIG. 3 b.

Illustration for the derivation of Fresnel's formulae for transitions from a rarer to a denser medium. Electric vector in the plane of the drawing.

Collecting these two ratios, one obtains the *second Fresnel formula*

$$(16) \quad A : B : C = \tan(\alpha + \beta) : \left(\frac{\tan(\alpha + \beta)}{\cos(\alpha - \beta)} - \frac{\tan(\alpha - \beta)}{\cos(\alpha + \beta)} \right) : \tan(\alpha - \beta).$$

In order to complete the above calculation, we will convince ourselves in exercise I.3 of the fact that no electric charge is induced on the surface $y = 0$ and that, therefore, no discontinuity in the electric excitation $D_y = \epsilon E_y$ occurs. (In case A this is self-evident because of $E_y = 0$.)

C. ARTIFICIAL SUPPRESSION OF REFLECTION FOR PERPENDICULAR INCIDENCE.

The more complete solutions (9) and (15) of our problem in which we have $\mu_1 \neq \mu_2$ are of some historic interest. During the war the problem arose to find, as a counter measure against allied radar, a largely non-reflecting ("black") surface layer of small thickness. This layer was to be particularly non-reflecting for perpendicular or almost perpendicular incidence of the radar wave. In this case α , and because of the law of refraction also β , are almost equal to zero. The problem is solved according to (9) and (15) by making

$$(17) \quad m_{12} = 1.$$

The criterion is, thus, not the index of refraction n but the ratio of wave resistances m . In order to "camouflage" an object against radar waves, one must cover it with a layer for which this ratio of wave resistances has the value 1 in the region of centimeter waves. According to (8) this means that if we call the constants of the desired material ϵ and μ and those of air ϵ_0 and μ_0 , then

$$(18) \quad \frac{\epsilon}{\epsilon_0} = \frac{\mu}{\mu_0}.$$

Hence, the problem concerns not only the dielectric constant but also the relationship between the dielectric constant and the permeability. A substance must be formed whose *relative permeability* μ/μ_0 is of the same magnitude as its relative dielectric constant ϵ/ϵ_0 .

But thereby the problem is not yet solved. For at its back surface the layer borders on the object (metal) which is to be camouflaged, and this second surface still reflects strongly. Hence, the further condition must be

imposed that the layer should absorb sufficiently strongly. This requires a *complex* rather than a real dielectric constant and because of the requirement (18) a corresponding *complex permeability*. The material must, therefore, be *ferromagnetic* and must possess a *strong hysteresis* or a structural relaxation that acts correspondingly. Thus, a difficult technological problem was posed which, though not unsolvable, required extensive preparatory work.

Because of the urgent war situation the solution which had to be used resulted from the following considerations. According to our presentation reflection appears as a consequence of the discontinuity of the material constants between the two media 1 and 2. The question of whether a completely *continuous* transition also causes reflection is an old one and was disputed for a long time. It has been answered only lately and in a completely affirmative sense, owing to the special interest which wave mechanics¹ as well as ionosphere research has recently brought to this question.

It turns out, however, that reflection becomes *extremely small* when the change of the material constants is spread over a distance equal to or larger than one wavelength, while an increase within a distance which is less than $1/4$ wavelength acts appreciably like a discontinuous increase. By material constants we mean the complex dielectric constant (it must be complex because of the necessary absorption; the permeability can be left out of this consideration).

In practice it was necessary to approximate the required continuous rise of ϵ by a series of steps, that is, by the application of several layers whose dielectric constants (especially their imaginary parts, which are the more important) increase with depth from one layer to the next. *In this manner the reflected intensity could be reduced to 1% of the value given by Fresnel's formula* for all wavelengths less than an upper limit whose value depends on the thickness of the layer. This could be accomplished without exceeding the admissible additional weight of the layers.

We will treat another method for diminishing reflection (extinction by interference) in Sec. 7.

¹ In wave mechanics one is concerned with the penetration of an electron into a region of increasing repulsive potential which according to the energy theorem of classical mechanics, would be inaccessible to the electron with its given kinetic energy. See also the later remarks on the tunnel effect, Sec. 5 c. In particular S. Epstein found a special profile for the increase in refractive index for which the reflection can be calculated rigorously (by means of hypergeometric functions). For further details see, for instance, "Atombau und Spektrallinien" Vol. II p. 29. A general discussion of the various methods of computation is given in Kofink and Menzer, Ann d. Phys. (Lpz) 89, 388, 1941; Kofink, *ibid.* 1, 119, 1947.

4. Graphical Discussion of Fresnel's Formulae. Brewster's Law.

Let medium 1 be the optically *rarer* medium, for instance air, and let medium 2 be the optically *denser* medium, for instance water or glass. Since these media are non-magnetic ($\mu = \mu_0$), m has the same value as n . The designations "denser" and "rarer" have their origins in the elastic (or rather quasielastic) theory of Fresnel.

In our fig. 4 we use for the abscissa the angle of incidence α , where $0 < \alpha < \pi/2$. In the direction of the ordinate we plot the amplitude ratios for the transmitted and reflected rays

$$(1) \quad D = \frac{B}{A}, \quad R = -\frac{C}{A}.$$

The negative sign of R is desirable because throughout the greater part of our figure the sign of the reflected amplitude C is opposite from that of the incident amplitude A . A change in sign during reflection obviously means a phase difference of π , that is, the addition of a phase factor

$$e^{i\pi} = -1.$$

We shall now distinguish the two cases A and B of the preceding paragraph by means of the indices p and s . Their meaning is: "plane of polarization parallel or perpendicular to the plane of incidence", and they contain a definition of the otherwise arbitrary term "plane of polarization". The significance of this definition is merely historical; see the beginning of Sec. 8.

A. PLANE OF POLARIZATION PARALLEL TO PLANE OF INCIDENCE.

We begin with R_p which is given according to (1) and (3.12) by

$$(2) \quad R_p = \frac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)}.$$

For small α we have, according to the law of refraction,

$$(3) \quad \beta = \frac{\alpha}{n}, \quad \text{hence} \quad R_p = \frac{n-1}{n+1};$$

For $n = 4/3$ (water) and $n = 3/2$ (mean value in the spectrum of light crown glass) this gives the results:

$$R = 1/7 \text{ and } 1/5, \text{ respectively.}$$

Correspondingly, the ratios of reflected to incident intensities are

$$R_p^2 = 2\% \text{ and } 4\%, \text{ respectively.}$$

Neither water nor glass can serve as *mirrors* for *perpendicular* incidence. If we look perpendicularly at water, we see our own mirror image less clearly than the bottom or the water's own color in the case of deep water. Ordinary mirrors are not glass mirrors but metal mirrors. The glass serves only for the protection of the silver on the reverse side¹.

We wish to improve the approximation for small α by one order. Hence, we set

$$\sin(\alpha \mp \beta) = (\alpha \mp \beta) \left\{ 1 - \frac{1}{6} (\alpha \mp \beta)^2 \right\}$$

and obtain instead of (3), see exercise I.4,

$$(4) \quad R_p = \frac{n-1}{n+1} \left(1 + \frac{\alpha^2}{n} \right).$$

Hence our representation of R_p in fig. 4 begins at a distance $\frac{n-1}{n+1}$ from the abscissa with a *horizontal tangent* and increases parabolically.

We now move from perpendicular to grazing incidence, $\alpha = \pi/2$. Here we get, according to the law of refraction,

$$(5) \quad \sin \beta = \frac{1}{n},$$

$$\sin(\alpha \mp \beta) = \cos \beta = \frac{\sqrt{n^2 - 1}}{n},$$

hence $R_p = 1$. For grazing incidence reflection is *complete*. This is the reason for the beautiful mirror image of the opposite shore in the waters of a mountain lake, as well as for the mirror image of the setting sun in a smooth sea; this image approaches the sun itself in intensity.

We also wish to determine at what angle our R_p curve approaches its end point $R_p = 1$ at $\alpha = \pi/2$. For this purpose we compute $dR_p/d\alpha$ at that point. We note that (again because of the law of refraction)

$$(6) \quad \cos \alpha d\alpha = n \cos \beta d\beta, \text{ hence } \frac{d\beta}{d\alpha} = 0 \text{ as we approach } \cos \alpha = 0.$$

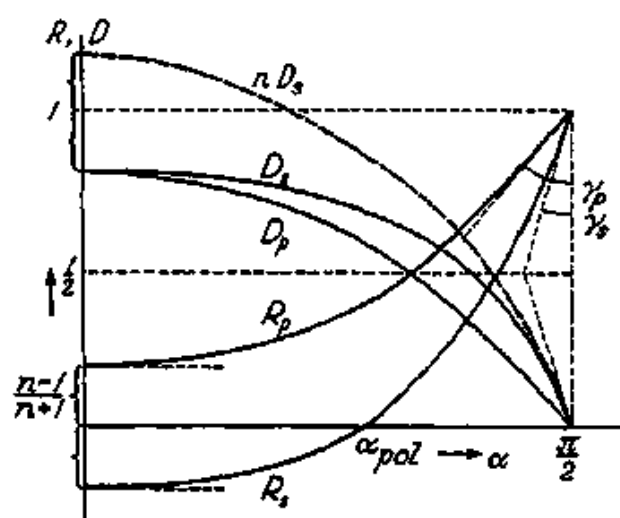


FIG. 4.

The amplitude ratios R , for the reflected ray, and D for the transmitted ray, as functions of the angle of incidence α .

¹ To be sure, the reflection from the front surface of the glass, weak as it is, makes such back-silvered mirrors unsuitable for optical purposes. For these the front surface of the glass must be covered with metal, preferably with rhodium.

Hence we need to differentiate (2) only partially with respect to α and by so doing obtain for $\alpha = \pi/2$:

$$(6a) \quad \frac{dR_p}{d\alpha} = 2 \tan \beta = \frac{2}{\sqrt{n^2 - 1}} \quad \text{because} \quad \sin \beta = \frac{1}{n}.$$

It follows that the angle denoted in the figure by γ_p is given by:

$$(7) \quad \tan \gamma_p = \frac{\sqrt{n^2 - 1}}{2}.$$

It is now very simple to draw also the curve for D_p . In our present¹ notation (1) we have, according to (3.5),

$$(7a) \quad D_p = 1 - R_p.$$

The ordinates of the two curves add up to 1. We obtain D_p by reflecting R_p on the center line of the figure (ordinate 1/2). Hence the curve D_p starts with a horizontal tangent at a distance $\frac{n-1}{n+1}$ below the ordinate 1 and proceeds as a parabola which terminates at the point $\alpha = \pi/2$, $D_p = 0$. In the case of grazing incidence no light passes from the rarer into the denser medium. D_p falls off at the endpoint at the same angle γ_p as determined by (7).

B. PLANE OF POLARIZATION PERPENDICULAR TO PLANE OF INCIDENCE.

According to (3.16) C/A is *positive* for angles of incidence that are not too large; hence, following our definition (1), R_s becomes *negative*:

$$(8) \quad R_s = -\frac{\tan(\alpha - \beta)}{\tan(\alpha + \beta)}.$$

If we let $\alpha \rightarrow 0$ then, except for the sign, equation (3) holds true also here and its next degree of approximation becomes, see exercise I.4, instead of eq. (4)

$$(9) \quad R_s = -\frac{n-1}{n+1} \left(1 - \frac{\alpha^2}{n} \right).$$

Hence the curve for R_s is a quadratic parabola which starts with a horizontal tangent at a distance $\frac{n-1}{n+1}$ below the abscissa. It ends according to (8) at $\alpha = \pi/2$ with the *positive* ordinate

¹ Our quantities R and D are not to be confused with the quantities r and d which will be defined on the basis of energy in section E.

$$(10) \quad R_s = - \frac{\tan\left(\frac{\pi}{2} - \beta\right)}{\tan\left(\frac{\pi}{2} + \beta\right)} = +1.$$

Its slope is steeper than that of R_p ; the angle γ_s , computed in a manner similar to (7), is given by

$$(11) \quad \tan \gamma_s = \frac{\sqrt{n^2 - 1}}{2n^2} < \tan \gamma_p.$$

Between its negative beginning (9) and its positive end (10) the R_s curve crosses the abscissa. We call this point

$$(12) \quad \alpha = \alpha_{pol} = \text{angle of polarization.}$$

From eq. (8) we conclude that at this point the denominator must suddenly change its value from $+\infty$ to $-\infty$ and that hence

$$(13) \quad \alpha_{pol} + \beta = \frac{\pi}{2}, \quad \beta = \frac{\pi}{2} - \alpha_{pol}, \quad \sin \beta = \cos \alpha_{pol}.$$

On the other hand, according to the law of refraction,

$$(13a) \quad \sin \beta = \frac{1}{n} \sin \alpha_{pol}.$$

By comparing (13) and (13a) it follows that

$$(14) \quad \tan \alpha_{pol} = n.$$

For glass $\left(n = \frac{3}{2}\right)$ and water $\left(n = \frac{4}{3}\right)$

$\alpha_{pol} = 57^\circ$ and 53° , respectively.

Since R_s vanishes at this angle, *the reflected light is polarized completely parallel to the plane of incidence*. This was the discovery of Malus. Our eq. (13) also contains "Brewster's law"; see the later fig. 5: *the reflected ray S_r is perpendicular to the refracted ray S_d* .

Next we complete fig. 4 by drawing the curve for D_s . This can be derived from the relation (3.14a) which, in our present notation, reads

$$(15) \quad 1 - R_s = n D_s.$$

If we now make the same construction as in the case of D_p , that is, if we mirror the curve R_s about the center line of the figure, then we are carried beyond the ordinate 1, as indicated by the broken curve denoted by $n D_s$ in the figure. The starting point of this curve has the ordinate

$$1 + \frac{n-1}{n+1} = \frac{2n}{n+1}.$$

To obtain D_s itself we must reduce this curve by a factor of $1/n$. Then we obtain the same starting point ordinate as in the plot for D_p , namely

$$1 - \frac{n-1}{n+1} = \frac{2}{n+1}.$$

C. PRACTICAL PRODUCTION OF POLARIZED LIGHT.

While the *reflected* component provides *complete* polarization at the angle of polarization, it provides only *low* intensity. Though the refracted component is only incompletely polarized, its intensity is *greater*. Indeed, according to (13) and (14), for $\alpha = \alpha_{pol}$

$$\begin{aligned}\sin(\alpha + \beta) &= 1 \\ \sin(\alpha - \beta) &= \sin^2 \alpha - \cos^2 \alpha = \frac{n^2 - 1}{n^2 + 1}\end{aligned}$$

and hence, according to (2),

$$R_p = \frac{n^2 - 1}{n^2 + 1} = \frac{5}{13} \quad \text{for } n = 3/2 \text{ (glass).}$$

The efficiency of this "polarizer" (ratio of reflected p -intensity to the entire incident ($p + s$) - intensity) amounts, therefore, to only

$$\frac{1}{2} R_p^2 = 7.4\%.$$

For other angles of incidence $\alpha \neq \alpha_{pol}$ the reflected light is also polarized parallel to the plane of incidence but only partially so.

On the other hand, a glance at fig. 4 shows that $D_s > D_p$ for every α . The refracted light is always *partially* polarized *perpendicularly* to the plane of incidence. For instance, for the special case $\alpha = \alpha_{pol}$ we get according to (15) and (7 a)

$$(16) \quad D_s = \frac{1}{n}, \quad D_p = 1 - \frac{n^2 - 1}{n^2 + 1} = \frac{2}{n^2 + 1}, \quad \frac{D_s}{D_p} = \frac{n^2 + 1}{2n} > 1.$$

If we consider the passage of light through a plate at the back surface of which a second transition with index of refraction $1/n$ takes place, then at this point the amplitude ratios are again the same¹ because

$$\frac{\frac{1}{n^2} + 1}{2/n} = \frac{1 + n^2}{2n}.$$

¹ For the generality of this relationship, see exercise I.2.

Thus for a glass plate the resulting ratio of refracted amplitudes $D_s : D_p$ is

$$\left(\frac{13}{12}\right)^2 = 1.17$$

and the ratio of intensities is

$$(1.17)^2 = 1.37.$$

Hence, by means of a *stack of glass plates* the polarization can be successively increased without *decreasing the intensity* (if the material is completely transparent and the surfaces are clean). In this way half of the intensity of the incident natural light (namely the s-polarized half) is completely utilized. The efficiency of an ideal stack of glass plates would thus be 50%. To be sure, complete polarization is approached only asymptotically as the number of glass plates is increased to infinity.

While the production of polarization by means of crystal structure is readily understandable, its origin in an isotropic material is, because of the complete lack of structural elements, somewhat paradoxical. This situation will be clarified in the next section.

D. BREWSTER'S LAW FROM THE POINT OF VIEW OF ELECTRON THEORY.

We now leave, temporarily, the phenomenological viewpoint of Maxwell's theory and interpret the process of refraction as scattering of the light by the atoms of the second medium (the first medium can be thought of as a vacuum). From this physically more profound viewpoint refraction takes place only because the electric field acting in the second medium sets the atomic electrons into oscillations, these oscillations being in the direction of the field. Thus, we are concerned with real material oscillations and not merely with alternating fields as before.

Fig. 5 represents the case of "plane of polarization *perpendicular* to the *plane of incidence*" in which the electric vector oscillates *in the plane of incidence*. Its direction of oscillation in the second medium is, of course, perpendicular to the direction of the refracted ray. The electrons oscillate in this same direction. They act like Hertzian oscillators and like them radiate *no* light in the direction of their oscillations (the same is well-known to be true for the antennas of radio transmitters). Regular reflection in the first medium can occur only if the electrons of the second medium deliver radiant energy in the direction of reflection (as determined by the law of reflection). This is not the case when this direction is parallel to the oscillations of the electrons,

hence perpendicular to the refracted ray, *in agreement with Brewster's law*. In other directions of reflection the electrons yield part of their radiation, which explains the variation of the strength of reflection with varying angles of incidence.

We also see immediately that this consideration does not affect the other case: "plane of polarization *parallel* to plane of incidence". In this case the electric vector and hence also the electron oscillations are *perpendicular* to the plane of incidence, hence also perpendicular to every position of the reflected

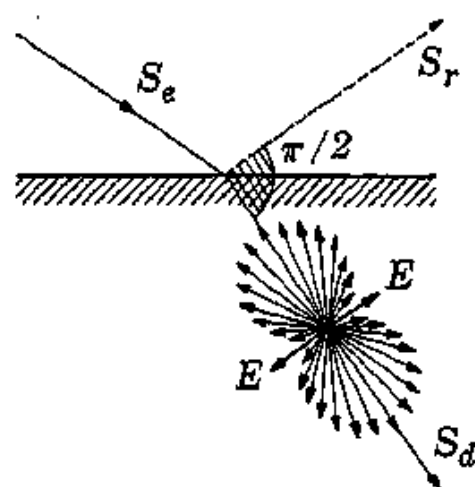


Fig. 5.

Brewster's law from the point of view of electron theory. At the angle of polarization the reflected and refracted rays are mutually perpendicular.

ray. Every one of these directions is a direction of maximum radiation from the electrons. Thus there is no reason for a forbidden direction of reflection such as Brewster's law demands.

We do not claim that the reflecting power can be calculated in this simple manner; for that purpose our method is still too primitive. Furthermore, it must be observed that only the layer next to the surface can be taken into consideration because at greater depths the contributions of the single atoms cancel by interference. But nevertheless, the null effect of Brewster's law is incontestably illustrated by our method.

These considerations also show that polarization depends, even in the case of isotropic substances, on a structural property of the material. This structure is, however, not prescribed crystallographically but is brought about by the electromagnetic field itself in creating a directed dipole structure in the otherwise unordered atoms.

E. ENERGY CONSIDERATIONS. REFLECTING POWER r AND TRANSMISSIVITY d .

Clearly these phenomena conserve energy in every case. In order to see this one may consider the flow of energy through an arbitrary cross section q of the incident wave. The corresponding cross section in the reflected wave is again q . But on the refracting plane q subtends the larger area $\frac{q}{\cos \alpha}$ and for the transmitted light the corresponding cross section is given by

$$(17) \quad q' = q \frac{\cos \beta}{\cos \alpha}.$$

We write for the time average of the energy flow through these three cross-sections

$$S_i = q \bar{S}_i, \quad S_r = q \bar{S}_r, \quad S_d = q' \bar{S}_d, \quad \bar{S} = \overline{E \times H} = \sqrt{\frac{\epsilon}{\mu}} \bar{E}^2$$

and define the reflecting power and the transmissivity by

$$(18) \quad r = \frac{S_r}{S_i} = \left| \frac{C}{A} \right|^2, \quad d = \frac{S_d}{S_i} = \left| \sqrt{\frac{\epsilon_2 \mu_1}{\mu_2 \epsilon_1}} \frac{q'}{q} \frac{B}{A} \right|^2 = m \frac{\cos \beta}{\cos \alpha} \left| \frac{B}{A} \right|^2$$

where the meaning of m is the same as in (3.8). We will convince ourselves in exercise I.5 that, in compliance with the energy law, for every case

$$(19) \quad r + d = 1,$$

and also that r and d are the same for both transitions: rarer \rightleftharpoons denser medium.

Our energy equation (19) is to be well distinguished from the amplitude equations (7 a) and (15)

$$R_p + D_p = 1 \quad \text{and} \quad R_s + n D_s = 1.$$

5. Total Reflection

In principle our formulae of Secs. 3 and 4 remain unchanged if we transfer the incident wave into the denser medium and investigate its reflection into the same medium and its refraction in the rarer medium. In particular, no changes need be made in the derivation of the law of refraction in (3.3). But because we want to preserve the former meaning of $n > 1$, we will substitute $1/n$ for n and must hence write

$$(1) \quad \frac{\sin \alpha}{\sin \beta} = \frac{1}{n}.$$

From this it follows that $\beta > \alpha$ for small α but that β is *imaginary* for $n \sin \alpha > 1$. In this latter case the coefficients $A : B : C$ in Fresnel's formulae also become complex.

In the older literature this situation was rejected as being unphysical, but it is entirely consistent with our viewpoint for we consider the problem of

reflection and refraction as a *boundary value problem*. Any procedure which leads to a self-consistent solution of this problem is justified. Calculations with complex magnitudes are just as admissible and advisable in optics as they are, for instance, in two-dimensional potential theory.

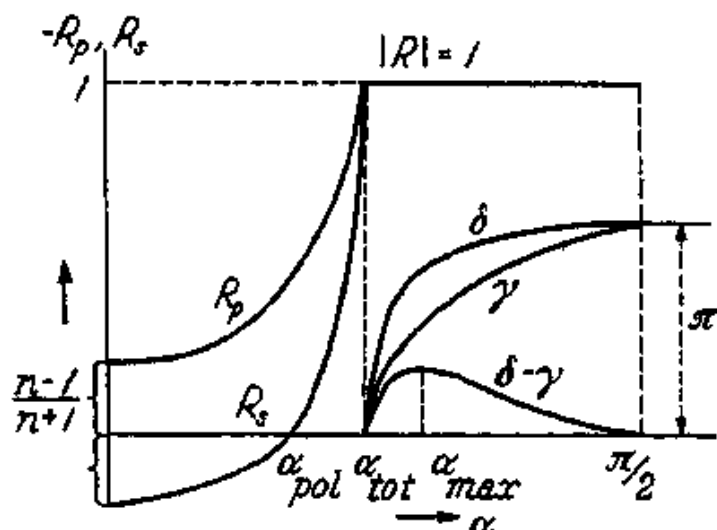


FIG. 6.
Reflection ratios R_p and R_s before and after total reflection occurs.

A. DISCUSSION OF FRESNEL'S FORMULAE.

We shall adopt immediately the graphical method of Sec. 4 and again neglect the distinction which, in principle, exists between n and m . The abscissa of fig. 6 again represents the angle of incidence $0 < \alpha < \pi/2$. On it we mark the point $n \sin \alpha = 1$ at which the angle of refraction β attains its largest real value $\beta = \pi/2$. We call this point

$$(2) \quad \alpha_{tot} = \text{Limiting angle of total reflection.}$$

For glass against air

$$\sin \alpha_{tot} \sim \frac{2}{3}, \quad \alpha_{tot} \sim 42^\circ.$$

Limiting ourselves for the time being to reflected light, we shall plot the following quantities along the ordinate:

$$(3) \quad R = \frac{C}{A}, \quad R_p = \frac{\sin(\beta - \alpha)}{\sin(\beta + \alpha)}, \quad R_s = -\frac{\tan(\beta - \alpha)}{\tan(\beta + \alpha)}.$$

Consequently, we now choose the sign of R oppositely from that in (4.1). Thereby we attain the advantage that in spite of the opposite state of affairs ($\beta < \alpha$ in fig. 4., $\beta > \alpha$ in fig. 6) the curves for R_p and R_s are similar to those in fig. 4 for small α . They both start with horizontal tangents at the same

ordinates $\pm \frac{n-1}{n+1}$ as before. But they reach the ordinate 1 not at $\alpha = \pi/2$ but already at $\alpha = \alpha_{\text{tot}}$. Indeed, since at that point $\beta = \pi/2$,

$$(4) \quad R_p = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\sin\left(\frac{\pi}{2} + \alpha\right)} = 1, \quad R_s = -\frac{\tan\left(\frac{\pi}{2} - \alpha\right)}{\tan\left(\frac{\pi}{2} + \alpha\right)} = 1.$$

Before reaching that point the curve for R_s will have crossed the abscissa at the angle of polarization

$$\alpha_{\text{pol}} + \beta = \frac{\pi}{2}, \quad \tan \alpha_{\text{pol}} = \frac{1}{n}$$

which corresponds to equations (4.12) and (4.14).

We are interested in the slopes at which the R_p and R_s curves approach the ordinate 1 at α_{tot} . To this end we note that in contrast to (4.6) we now have

$$\frac{d\alpha}{d\beta} = 0 \quad \text{because} \quad \sin \beta = 1, \quad \cos \beta = 0,$$

and, hence, that we need to differentiate the expressions (3) only with respect to β in order to be able to compute the angle in question. Thus by setting $\beta = \pi/2$ after differentiating, we first obtain the result,

$$\frac{dR_p}{d\beta} = 2 \frac{\sin \alpha}{\cos \alpha}.$$

Then recalling (4.6) with n replaced by $1/n$, we obtain in the neighborhood of the critical point

$$(5) \quad \frac{dR_p}{d\alpha} = 2 \frac{\sin \alpha}{\cos \alpha} \frac{d\beta}{d\alpha} = \frac{2n \sin \alpha}{\cos \beta} = \frac{2}{\cos \beta}.$$

The limit as $\beta \rightarrow \pi/2$, becomes ∞ ; the R_p curve has a *vertical tangent* at the point in question.

Correspondingly, one obtains

$$(5a) \quad \frac{dR_s}{d\alpha} = \frac{2}{\sin \alpha \cos \alpha} \frac{d\beta}{d\alpha} = \frac{2n}{\sin \alpha \cos \beta} = \frac{2n^2}{\cos \beta}.$$

The R_s curve runs even steeper than the R_p curve in the vicinity of the critical point. It also has a *vertical tangent* at that point.

This circumstance is of special experimental importance; it led Abbe and F. Kohlrausch to the construction of total reflection meters (total refraction meters, respectively). Because of the abrupt increase of the reflected light (the abrupt disappearance of the refracted light, respectively), the limit of total reflection can be determined very closely and thereby the index of refraction can be computed, according to the formula $n = \frac{1}{\sin \alpha_{tot}}$, with great accuracy.

We complete fig. 6 by drawing through the critical point $R = 1$ a line parallel to the abscissa and denoting it by $|R| = 1$. This means the following for both components R_p and R_s : *reflected intensity equals incident intensity*, hence indeed *total reflection*. To justify this assertion we follow the course of the β -point as given by the law of refraction in a complex β -plane. Its path follows the real axis from 0 to $\pi/2$, as α goes from 0 to α_{tot} . At this point the path of β splits into two, mathematically equally justifiable, branches $\beta = \pi/2 \pm i\beta'$ which run parallel to the imaginary axis. For both branches

$$(6) \quad \sin \beta = \sin \left(\frac{\pi}{2} \pm i\beta' \right) = \cos (\pm i\beta') = \cosh \beta' > 1$$

as is required by the law of refraction $\sin \beta = n \cdot \sin \alpha > 1$. Using (6) one obtains from (3)

$$(7) \quad R_p = \frac{\sin \left(\frac{\pi}{2} \pm i\beta' - \alpha \right)}{\sin \left(\frac{\pi}{2} \pm i\beta' + \alpha \right)} = \frac{\cos (\alpha \mp i\beta')}{\cos (\alpha \pm i\beta')} = 1 \cdot e^{i\gamma}$$

$$(7 a) \quad R_s = - \frac{\tan \left(\frac{\pi}{2} \pm i\beta' - \alpha \right)}{\tan \left(\frac{\pi}{2} \pm i\beta' + \alpha \right)} = \frac{\cot (\alpha \mp i\beta')}{\cot (\alpha \pm i\beta')} = 1 \cdot e^{i\delta}$$

Since the numerators and denominators of R_p and R_s contain mutually conjugate quantities, the absolute values of these quotients are equal to 1. γ and δ are real phase angles whose experimental utilization we will consider in section D. In preparation we have already sketched their curves in the right-hand part of fig. 6. $|R| = 1$ immediately explains the excellent action of prismatic field glasses whose operation is based on the principle of total reflection.

B. LIGHT PENETRATING INTO THE RARER MEDIUM.

The general formulae for the refracted wave in Sec. 3 give a field in the rarer medium not only for $\alpha < \alpha_{\text{tot}}$ but also for $\alpha > \alpha_{\text{tot}}$.

In the case of p -polarization we start with eq. (3.1 a) by substituting

$$\beta = \frac{\pi}{2} \pm i\beta', \quad \sin \beta = \cosh \beta', \quad \cos \beta = \mp i \sinh \beta'$$

and obtain for $k_2 = k$ (vacuum)

$$E_z = B e^{ik(x \cosh \beta' \pm iy \sinh \beta')}$$

We see that only the lower sign in front of i is physically admissible (E_z must remain finite as $y \rightarrow -\infty$) so that we must set

$$(8) \quad E_z = B e^{ky \sinh \beta'} e^{ikx \cosh \beta'}.$$

This wave has an entirely different structure from that of the usual "homogeneous" plane wave. It is called "inhomogeneous". Though this wave propagates without attenuation along the boundary surface, *its strength decreases perpendicularly to it*. Since $k = 2\pi/\lambda$, the wave is noticeable only within a distance of a few wavelengths from the boundary surface.

We compute, next, by means of Maxwell's equation $\mu_0 \dot{\mathbf{H}} = -\text{curl } \mathbf{E}$, the magnetic excitation \mathbf{H} belonging to (8). We consider the right-hand side of (8) to be provided with the time factor $\exp(-i\omega t)$ and take into account the relationship $\omega/k = c = (\epsilon_0 \mu_0)^{-1/2}$. We thus obtain besides $H_z = 0$:

$$(9) \quad \left. \begin{aligned} H_x &= -i \sqrt{\frac{\epsilon_0}{\mu_0}} \sinh \beta' \\ H_y &= - \sqrt{\frac{\epsilon_0}{\mu_0}} \cosh \beta' \end{aligned} \right\} \cdot B e^{ky \sinh \beta'} e^{ikx \cosh \beta'}.$$

Both components of \mathbf{H} have the same inhomogeneous structure as (8).

We now turn to the radiation vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. We must not, of course, multiply the complex expressions (8) and (9) but only their real parts taking account thereby of both the time factor and the complex nature of (8) and (9). We write

$$(10) \quad \left. \begin{aligned} S_x &= -E_z H_y \\ S_y &= +E_z H_x \end{aligned} \right\} = \sqrt{\frac{\epsilon_0}{\mu_0}} |B|^2 e^{2ky \sinh \beta'} \begin{cases} \cosh \beta' \cos^2 \tau \\ \sinh \beta' \sin \tau \cos \tau \end{cases}$$

where $\tau = \omega t - kx \cosh \beta'$. We see that the x -component of \mathbf{S} , which is the component parallel to the boundary surface, is always positive. On the other hand, the flow of energy in the direction perpendicular to the boundary

surface changes its sign periodically. Its time average vanishes while an actual energy flow takes place parallel to the boundary surface.

This seems to contradict both the name "total reflection" and our oft-repeated statement that no energy is lost in this process. We must, however, consider the fact that we have always performed our calculations for the ideal case of an infinitely wide wave front. For the actual, laterally restricted waves energy can very well pass from the denser into the rarer medium or, respectively, flow back from the rarer into the denser medium at the lateral boundaries¹ of the wave. This is the energy which is transported parallel to the boundary surface or, so to say, meanders about it.

If one may be permitted to use a military analogy, we can describe the situation in the following manner: an army marching in closed ranks comes in its advance upon difficult territory which forces it to change its direction of march. The wing of the army detaches a weak patrol with orders to penetrate the difficult region and to secure the flank. This patrol needs to be only a few men strong in depth. After carrying out its orders the patrol returns to the army.

Just as the lack of such a precaution would violate all rules of military caution, so would a sudden discontinuity of our totally reflected wave violate all rules of electrodynamics.

C. THE TUNNEL EFFECT OF WAVE MECHANICS.

The experimental proof of the existence of the inhomogeneous wave in the rarer medium has posed a difficult problem. Quincke tried his hand at it for many decades. He placed two precisely cut glass plates side by side at a distance of a few wavelengths, and allowed light to be reflected totally in the first plate. He believed he was then able to observe traces of transmitted light in the second plate. He considered this as an indication that the air gap between the plates was bridged by the light field. Woldemar Voigt repeated similar experiments with an improved set-up.

The experiment becomes very simple with Hertz waves. In the *Bose-Institute*² in Calcutta the following set-up is demonstrated: two asphalt

¹ These boundaries also have to do with the "lateral displacement" of the totally reflected ray as studied recently by F. Goos and H. Hänchen experimentally, and by K. Artmann theoretically, *Ann. d. Phys. (Lpz)* 1, 333, 1948 respectively 2, 87, 1948. See also C. v. Fragstein *ibid.* 4, 271, 1948.

² The botanist Sir Jagadis Bose in his younger years imitated experiments of classical optics with short Hertz waves, e.g. $\lambda = 20$ cm. See *Collected Physical Papers*, especially No. VI of the year 1897, Longmans, Green and Co., 1927.

prisms 1 and 2, fig. 7, are placed opposite each other at a distance of several centimeters. The waves are incident perpendicularly to 1 and are "totally reflected" on the back face of 1. Still, one obtains distinct signals in a receiver placed behind 2 and these increase in strength as the distance between the prism faces is decreased.

In wave mechanics quite analogous situations occur under the name of "tunnel effects" (Condon and Gurney, 1928). By assigning a wave to a particle (electron, ion), according to L. de Broglie, and making the former obey the Schrödinger equation, one shows that the particle can, as a wave, pass through a potential barrier which, considering its kinetic energy, the particle could not surmount according to classical mechanics. This happens with a specific probability which depends on the thickness of the wall and the original energy of the particle. In the wave mechanical for-

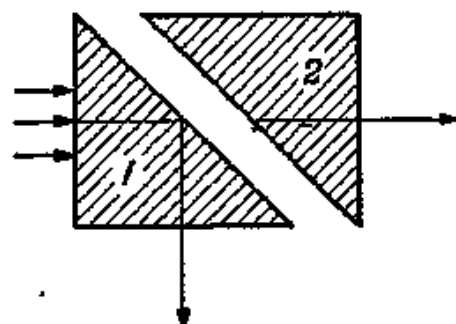


Fig. 7.

Experiment to prove that Hertz waves enter the rarer medium. The distance between the two prisms is a fraction of the wavelength.

mulation the potential barrier plays exactly the same role as the air space in the experiments on total reflection. The parallelism between classical and quantum mechanics on the one hand and ray optics and wave optics on the other is hereby well illustrated. The wave mechanical tunnel effects proved to be fundamental in the theories of chemical binding, of "cold" electron emission of metals, of radioactivity, and also of the process of uranium fission.

D. PRODUCTION OF ELLIPTICALLY AND CIRCULARLY POLARIZED LIGHT

Starting from equations (7), (7 a) we assume that the incident light is linearly polarized at an angle of 45° to the plane of incidence, a situation which is attainable by means of a Nicol prism. Then the amplitude factors A of the incident p - and s -waves will be equal and, according to the above-mentioned equations, the amplitudes of both totally reflected components will be the same but their phases γ and δ will differ; for by dividing (7) by (7 a) one obtains

$$(11) \quad e^{i(\gamma-\delta)} = \frac{\sin(\alpha \mp i\beta')}{\sin(\alpha \pm i\beta')}.$$

For $\alpha = \alpha_{\text{tot}}$ the right-hand side of this equation becomes 1, hence the phase difference becomes zero because there $\beta' = 0$. The same is also true for $\alpha = \pi/2$ because there $\sin(\pi/2 - i\beta') = \sin(\pi/2 + i\beta') = \cosh \beta'$. Hence between these two limits lies a point of maximum phase difference. The magnitude of this difference and the associated angle of incidence α_{max} are given by

$$(12) \quad \tan \frac{\delta - \gamma}{2} = \frac{n^2 - 1}{2n}, \quad \sin^2 \alpha_{\text{max}} = \frac{2}{n^2 + 1}.$$

We shall derive these expressions in exercise I.6. Here we shall apply the result to glass of refractive index $n = 1.51$ and find

$$\tan \frac{\delta - \gamma}{2} = 0.424, \quad \delta - \gamma = 45^\circ 36', \quad \sin \alpha_{\text{max}} = 0.781, \quad \alpha_{\text{max}} = 51^\circ 20'.$$

We cannot attain the special case of *circular* polarization by a *single* total reflection since $\gamma - \delta \leq 45^\circ 36'$ for all angles of incidence; but by two such reflections this case can be produced. For this purpose Fresnel constructed a glass prism with a parallelogram for a base. If linearly polarized light is incident perpendicularly upon the shorter prism face then, after being totally reflected twice at the longer prism faces, it emerges at the opposite short face as circularly polarized light.

6. Metallic Reflection

In Maxwell's theory metals are characterized by the conductivity σ . However, actual conduction of electricity has to be thought of as a phenomenon consisting of the interactions between free electrons and metallic ions at fixed positions and as being brought about by an averaging of many elementary processes. Only in stationary or slowly varying fields does this averaging lead to a constant which is independent of frequency. One cannot expect that the phenomenological Maxwell theory will still suffice in the visible region of the spectrum. We have already encountered such a failure of the theory in the optics of transparent media (the failure of water, for instance, to satisfy the Maxwell relation $n^2 = \epsilon_{\text{rel}}$. See p. 15). Again, the phenomenological description of metallic reflection proves to be insufficient in the visible domain though it agrees well with experiment in the infrared spectral range (see section B). Hence, the Maxwell theory of metallic reflection has general significance only in the sense of being a limiting theory.

As is well known, Maxwell's equations for conductors differ from the equations (2.4) for non-conductors only in that the Ohmic current $\sigma \mathbf{E}$ is

added to the displacement current $\varepsilon \dot{\mathbf{E}}$. In the periodic case this means that $-\varepsilon i\omega$ is to be replaced by $-\varepsilon i\omega + \sigma$; hence ε is to be replaced by the complex dielectric constant

$$(1) \quad \varepsilon' = \varepsilon + i \frac{\sigma}{\omega}.$$

We accept this formulation in optics because it embodies, for a given ω , the most general linear relationship between \mathbf{D} and \mathbf{E} . But according to the above discussion we must not expect that ε and σ will retain their electrodynamic, ω -independent meanings.

Together with ε the index of refraction of eq. (3.4) also becomes complex:

$$(2) \quad n' = \sqrt{\frac{\varepsilon' \mu}{\varepsilon_0 \mu_0}} = n (1 + i \kappa).$$

The significance of the real quantities n and κ which are here introduced follows by squaring (2) and equating real and imaginary parts on both sides of the equation:

$$(2 a) \quad \frac{\varepsilon}{\varepsilon_0} \frac{\mu}{\mu_0} = n^2 (1 - \kappa^2), \quad \frac{\sigma}{\varepsilon_0 \omega} \frac{\mu}{\mu_0} = 2 n^2 \kappa.$$

The metallic index of refraction n and the absorption coefficient κ thus defined are the optical constants of the metal. In connection with this usual designation "absorption coefficient" it should be remarked that the "ideal" conductor $\sigma \rightarrow \infty$ of electrodynamics is characterized, not by $\kappa \rightarrow \infty$, but by

$$(2 b) \quad \kappa \rightarrow 1, \quad n \rightarrow \infty.$$

Indeed by dividing the two equations (2 a) one obtains

$$\frac{\varepsilon}{\sigma} = \frac{1 - \kappa^2}{2 \omega \kappa} \quad \text{and hence} \quad \kappa^2 \rightarrow 1 \text{ as } \sigma \rightarrow \infty$$

from which it also follows, by referring back to (2 a), that $n \rightarrow \infty$.

Together with n the resistance ratio m and the wave number k become complex. Corresponding to (3.8) and (2.2) we let

$$(3) \quad m' = n \frac{\mu_0}{\mu} (1 + i \kappa), \quad k' = k n (1 + i \kappa), \quad k = \frac{\omega}{c}.$$

We shall first concern ourselves with the structure of a monochromatic, linearly polarized, plane wave which propagates, for instance along the x -axis, in the metal. This wave is no longer homogeneous as in non-conductors. Its inhomogeneity is, however, of an entirely different nature from that which we

encountered in the case of total reflection. We write, as in (2.1), omitting the time factors,

$$(4) \quad \begin{aligned} E_y &= A e^{ik'x} = A e^{-\kappa k n x} e^{ik n x} \\ H_z &= A' e^{ik'x} = A' e^{-\kappa k n x} e^{ik n x}, \quad A' = m' A = n \frac{\mu_0}{\mu} (1 + i\kappa) A. \end{aligned}$$

It follows from this that the phase velocity is

$$\frac{dx}{dt} = \frac{\omega}{k n} = \frac{c}{n} \quad \text{and the wavelength} \quad \lambda = \frac{2\pi}{k n}.$$

Furthermore, we see from (4) that the wave is damped *longitudinally* in its propagation in the x -direction, not *transversely* as in total reflection. The decrease in amplitude per wavelength amounts to $\exp(-2\pi\kappa)$. In addition, the complex nature of A' shows that there exists a constant phase difference between the magnetic and electric components. The nodes and the maxima of the two wave components no longer coincide as in the non-conductor but are displaced from each other by an amount depending on κ .

A. FRESNEL'S FORMULAE

Formally, we can take over Fresnel's formulae without change from Secs. 3 and 4, as well as the laws of reflection and refraction from (3.3). The former says again that angle of reflection = angle of incidence = real magnitude. The latter becomes because of (2)

$$(5) \quad \frac{\sin \alpha}{\sin \beta} = n (1 + i\kappa)$$

which shows that the angle of refraction β is complex for all α , not merely for $\alpha > \alpha_{tot}$ as in total reflection.

Since the wave refracted into the interior of the metal is well-nigh unobservable because of its strong absorption, we must deduce the optical properties of a metal exclusively from the reflected light. Hence, we shall only have to discuss formulae (4.2) and (4.8) for R_p and R_s :

$$(6) \quad \begin{aligned} R_p &= \frac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)} = |R_p| e^{i\gamma}, \\ R_s &= -\frac{\tan(\alpha - \beta)}{\tan(\alpha + \beta)} = |R_s| e^{i\delta}. \end{aligned}$$

Because of the complex value of β , γ and δ differ from zero and from each other.

We consider first reflection in the case of almost *perpendicular* incidence. Then α and $|\beta|$ are small and from (5)

$$\frac{\alpha}{\beta} = n(1 + i\kappa).$$

Hence, according to (6)

$$R_p = \frac{n-1+i n \kappa}{n+1+i n \kappa} = -R_s.$$

Therefore, we obtain for both components the *reflecting power*

$$(7) \quad r = |R|^2 = \frac{(n-1)^2 + n^2 \kappa^2}{(n+1)^2 + n^2 \kappa^2} = 1 - \frac{4n}{(n+1)^2 + n^2 \kappa^2}.$$

Assuming that we have a good conductor ($n \rightarrow \infty$ according to eq. (2 b)), we get $r \sim 1$. In contrast to the glass or water mirror (see p. 21) the metallic mirror is a *complete reflector*.

Turning now to *oblique* angles of incidence, we assume, as in eq. (5.11), that both components of the incident light, p and s , are equal in amplitude and phase. Then the quantities given by (6), namely the amplitude ratio $|R_p/R_s|$ and the phase difference $\gamma - \delta$, determine directly the nature of the reflected light. Generally, it is *elliptically* polarized. It will be *circularly* polarized only when $\gamma - \delta = \pi/2$ which we assume to be true for the special angle of incidence $\alpha = \alpha_k$, the so-called "principal angle of incidence". One observes this angle and converts the circular polarization, for instance by means of a $\lambda/4$ plate, to *linear* polarization. The azimuth α_p of the plane of polarization associated with the latter is called the "azimuth of the restored polarization". From α_p and α_k the metal constants n and κ can then be computed. The latter are to be looked upon as phenomenological substitutes for the real metal properties in the spectral range of visible light.

B. EXPERIMENTS BY HAGEN AND RUBENS

We come now to the experiments which demonstrate for infrared rays the validity of eq. (7) which was derived from Maxwell's theory.

Hagen and Rubens, Ann. d. Phys. (Lpz) 1903, used for their experiments so-called residual rays which were left over from a larger spectral range after repeated reflections from crystals of alkaline earth halides (CaF_2 , CaCl_2). These crystals possess pronounced resonances in the region from $\lambda = 10$ to 25.5μ and hence have a highly selective reflecting power for such wavelengths. Then, according to (2 b), $\kappa \sim 1$ and, according to (7),

$$(8) \quad 1-r = \frac{4n}{2n^2 + \dots} = \frac{2}{n} \text{ is proportional to } \lambda^{-1/2}.$$

This proportionality to $\lambda^{-\frac{1}{2}}$ follows from the second eq. (2 a) (n^2 proportional to ω^{-2}). $1 - r$ is the loss in reflection and $100(1 - r)$ is the loss in reflection in %. Hagen and Rubens observed r and computed from it the values of $100(1 - r)$ given in the following table:

	Ag	Au	Cu	Pt
$\lambda_1 = 12\mu$	9.05	13.8	12.1	10.6
$\lambda_2 = 25\mu$	7.07	8.10	6.67	6.88
Ratio	1.2	1.7	1.8	1.5

The last line gives the ratio of the two numbers above. According to (8) this should be constant and equal to

$$\sqrt{\frac{\lambda_2}{\lambda_1}} = \sqrt{\frac{25}{12}} = 1.46.$$

This is almost exactly the arithmetic mean of the four numbers in the last line of our table. Hagen and Rubens were also able to confirm the dependence of their observations on temperature which is to be expected according to (2 a) because σ is proportional to $1/T$ (T = absolute temperature) as well as the agreement of the conductivity σ thus obtained with the electromagnetic value. At very low temperatures these simple laws no longer hold even for infrared light¹ because then the mean time between two collisions of an electron with metallic ions becomes comparable with the period of oscillation of the light and, hence, the averaging procedure mentioned at the beginning of this paragraph fails.

C. SOME REMARKS ON THE COLOR OF METALS, GLASSES AND PIGMENTS.

The dependence on wavelength contained in equations (7) and (8) would in itself result in a kind of coloring of the reflected light. The real color of metals, however, is caused by the characteristic oscillations of the electrons or ions which will be treated in Chapter III for the case of transparent bodies. Gold appears yellow when viewed directly. Very thin layers of gold allow green light to shine through. Except for metals real surface colors appear only where the optical constants n and κ are of a different *order of magnitude* from those of the bordering air and depend strongly upon wavelength. Dried *red* ink (solution of fuchsin) shines *yellow-green* in the incident light. Its red color on white paper is due to the light passing through the ink.

¹K. Weiss, Ann. d. Phys. (Lpz) 2, p. 1., 1948 or E. Vogt, ibid. 3, p. 82., 1948 (Planck volume).

All other materials reflect practically unselectively; after reflection the incident white light remains almost *white*. This is the origin of the "glare" well-known to painters. The latter can also be observed, for instance, on the beautiful blue copper sulfate crystals or on ruby glass. One must however be careful to prevent any internally reflected light which has passed through the material and is therefore colored from entering the eye in addition to the glare. For otherwise the glare light itself will appear complementarily colored owing to the purely physiological "simultaneous contrast".

Colored glass obtains its color only from the transmitted light. Since the lengths of the light paths through the glass always amount to many hundreds of wavelengths even a *very weak selective absorptivity* suffices to color the glass intensively. That the glass shows the same color when looked at as when looked through is due to the fact that the light seen by looking at it has actually passed through the glass from the other side. If the back surface of the glass is painted with black lacquer, the color of the glass becomes invisible and only the colorless glare of the front surface remains. If, however, the glass is laid on a white surface, it looks colored because the light reflected by the white surface and leaving through the front surface has passed through the glass twice.

If a white textile is soaked in a dye solution, the crystal-clear fiber substance becomes selectively absorbing. The light reflected by the rear surface of the fibers, or by the fiber surfaces lying further back, has passed through the fibers several times. If the dyed fabric is soaked in water — or better, in a mixture of alcohol and benzene, then it appears dark and colorless because the reflection at the inner fiber surfaces has been stopped as a result of the equalization of the index of refraction.

Many inorganic "pigments" are pulverized melts. In their compact state they are dark; after pulverization and mixture with a binding substance reflecting surfaces are formed in the interior; the color becomes visible.

The green of leaves consists of transparent green grains. In order for foliage to appear light green, it must have in its interior sufficiently many inhomogeneities at which the light is reflected. If these are absent (coniferae, box-wood trees), a black-green color appears. Nevertheless, the chlorophyll of the coniferae is the same as that of other trees and plants.

When mixed, pigments act *subtractively* like color filters placed behind each other. Every component of the mixture extinguishes by absorption its own region of the spectrum. The colors of pigments placed *alongside each other*, on the other hand, add when mixed, as for instance, in the sectorially colored disc of a top. Also the illumination coming through colored church

windows and the picture of a Lumière plate is put together *additively* from its single colors.

Very beautiful colors are produced by diffraction of light by larger aggregates of atoms, so-called *colloidal particles*. Lapis lazuli, for instance, owes its deep blue color to colloidal sulfur particles. The blue of the sky is, according to Einstein, brought about by the density variations of the air molecules which are to be expected statistically; Lord Rayleigh originally explained this phenomenon in a somewhat more special way by the diffraction on the (irregularly distributed) air molecules themselves.

Nature attains her most beautiful color ornament through interference colors, see Secs. 7 and 8 c, as witnessed in the wings of butterflies, the plumage of the tropical humming bird, the opal and mother-of-pearl. What opportunities would present themselves to painting if it were possible to develop a convenient interference color technique!

7. Colors of Thin Membranes and Thick Plates

In this section we shall discuss on one hand the age-old observations of Newton which motivated him to assume a kind of spatial structure of light and which might almost have led him to interference and the wave theory.

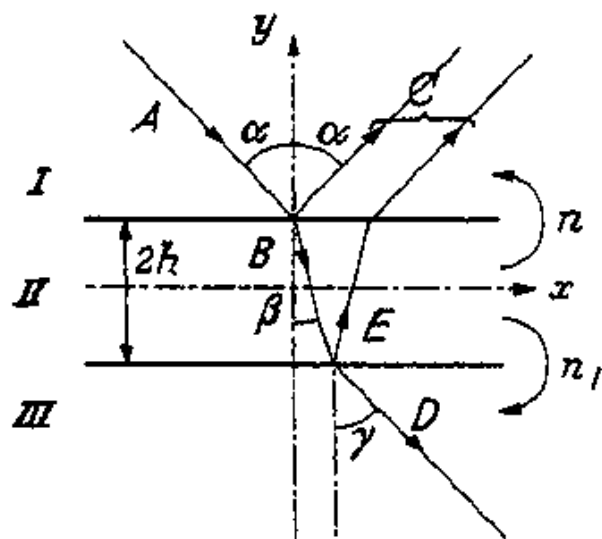


Fig. 8.

Reflection and refraction in a plate with plane parallel surfaces, considered as a boundary value problem.

On the other hand we shall describe the most modern experimental arrangements which serve in the most exact analysis of spectra. The mathematical problem which is the basis of both is that of the transparent plate with parallel surfaces, that is to say, the *problem of reflection and refraction at two boundary surfaces*. Until now we have actually solved this problem for only *one* boundary surface. Ordinarily, the two-surface problem is reduced to that of one boundary surface by dealing with repeated reflections and refractions. In con-

trast to this we shall treat the problem of the plate directly as a *boundary value problem*¹. We seek, therefore, the extension of Fresnel's

¹Of course this method has occasionally been used before; however only for special cases. See, e. g., M. Born, *Optik*, Berlin, Springer 1933, p. 125.

formulae corresponding to the generalized problem as described above; we thereby avoid the trouble of performing the summations over an infinite number of single processes which are necessitated by the other method. It is obvious, and will be shown in section E, that both methods must lead to the same result. However, we emphasize here already that in the case of the plate the above-mentioned single process is no longer a fundamental process and disappears in our extended boundary value problem.

A. THE GENERAL CASE

While in the case of *one* boundary condition *two* amplitude ratios $A:B:C$ were sufficient, we now require *four* of these ratios:

$$(1) \quad A:B:C:D:E$$

The meaning of the five amplitude factors $A \dots E$ is evident from fig. 8. The angle of incidence α is also the angle of reflection and would be encountered again as the angle of emergence of the transmitted light D if we allowed the bottom surface of the plate to border on the same material (air) as the top surface. We prefer, however, to let the medium behind the plate be arbitrary and to denote the angle of emergence by γ . The index of refraction of the plate with respect to that medium shall be n_1 and that with respect to air n . The thickness of the plate shall be $2h$; at the top and bottom surfaces we let $y = \pm h$. The z -axis is to be thought of as pointing out of the paper. The fact that the upward-reflected wave is represented by two arrows, bracketed together, is due only to the nature of the drawing. In reality all arrows in the figure represent, as before, not rays but unbounded plane waves.

We consider, for instance, the case of p -polarization, that is E parallel to the z -axis, and obtain above the plate

$$(I) \quad E_z = A e^{i k_1 (x \sin \alpha - y \cos \alpha)} + C e^{i k_1 (x' \sin \alpha + y \cos \alpha)}$$

as in (3.1). Inside the plate eq. (3.1 a) holds true but is to be completed by adding to the right-hand side the second particular solution in this region ($+i y$ instead of $-i y$) multiplied by the arbitrary factor E :

$$(II) \quad E_z = B e^{i k_1 (x \sin \beta - y \cos \beta)} + E e^{i k_1 (x \sin \beta + y \cos \beta)}.$$

Only below the plate does the field consist of a single wave because we must include among the conditions of the problem the fact that the plate is not irradiated from below:

$$(III) \quad E_z = D e^{i k_1 (x \sin \gamma - y \cos \gamma)}.$$

The two laws of refraction at $y = \pm h$

$$(2) \quad \frac{\sin \alpha}{\sin \beta} = \frac{k_2}{k_1} = n, \quad \frac{\sin \gamma}{\sin \beta} = \frac{k_2}{k_3} = n_1$$

are necessary conditions to enable us to cancel the x -dependent factors from all terms in the following expressions. If, furthermore, we write k for k_1 , we obtain for $y = +h$, instead of (3.5)

$$(3) \quad A e^{-ikh \cos \alpha} + C e^{ikh \cos \alpha} = B e^{-ikn h \cos \beta} + E e^{ikn h \cos \beta}$$

and for $y = -h$

$$(4) \quad B e^{ikn h \cos \beta} + E e^{-ikn h \cos \beta} = D e^{ik \left(\frac{n}{n_1}\right) h \cos \gamma}$$

Next we must write the expressions for H_x in I, II, and III in analogy to (3.5) and must require their continuity at $y = \pm h$. If we assume the plate to be non-magnetic, i. e. we set $m = n$, we obtain for $y = +h$ instead of (3.8):

$$(5) \quad A e^{-ikh \cos \alpha} - C e^{ikh \cos \alpha} = n \frac{\cos \beta}{\cos \alpha} (B e^{-ikn h \cos \beta} - E e^{ikn h \cos \beta})$$

and for $y = -h$

$$(6) \quad B e^{ikn h \cos \beta} - E e^{-ikn h \cos \beta} = \frac{\cos \gamma}{n_1 \cos \beta} D e^{ik \left(\frac{n}{n_1}\right) h \cos \gamma}.$$

We have, then, four linear homogeneous equations for the 5 unknowns $A \dots E$, which we can collect in the form

$$(7) \quad a_i A + b_i B + c_i C + d_i D + e_i E = 0, \quad i = 1, 2, 3, 4.$$

From this, the values of $A:B \dots E$ are computed as the ratios between the five corresponding four-rowed determinants of the scheme of coefficients $a b c d e$. Hence the Fresnel formulae of our plate problem assume the same form as before for the case of the single boundary surface, except that in place of the three-fold proportion, a five-fold proportion appears.

Since, however, the computation of this scheme of determinants becomes too cumbersome for the general case, we will in the following special examples do better to use the unsolved equations (3) to (6) as a starting point.

B. THE OIL SPOT ON WET ASPHALT.

Everyone has seen on the pavement the beautiful interference colors on a thin layer of oil. Medium I is air, medium II a layer of oil which is to be assumed as bounded by parallel planes. If the oil were to lie on dry asphalt, its lower boundary would not be plane but would be optically rough.

Hence, as medium III we must add a layer of water wetting the asphalt. The asphalt as a black material serves to absorb the transmitted wave D and thereby prevents further reflection processes.

We look at the oil spot perpendicularly from above and assume for convenience that the (actually diffuse) illumination also comes perpendicularly from above. Then $\alpha = \beta = \gamma = 0$. Furthermore, to abbreviate, we let

$$(8) \quad \eta = e^{i k h}.$$

From (4) and (6) we obtain by eliminating D

$$B \eta^n + E \eta^{-n} = n_1 (B \eta^n - E \eta^{-n}), \quad \text{hence} \quad E = \frac{n_1 - 1}{n_1 + 1} B \eta^{2n}.$$

Then (3) and (5) become

$$\begin{aligned} A \eta^{-1} + C \eta &= B \eta^{-n} \left(1 + \frac{n_1 - 1}{n_1 + 1} \eta^{4n} \right), \\ A \eta^{-1} - C \eta &= n B \eta^{-n} \left(1 - \frac{n_1 - 1}{n_1 + 1} \eta^{4n} \right). \end{aligned}$$

By eliminating B , a relation between A and C results which we can write

$$(9) \quad \frac{C}{A} = -\eta^{-2} \frac{n-1}{n+1} \frac{1-\nu_1 \eta^{4n}}{1-\nu_2 \eta^{4n}}, \quad \begin{cases} \nu_1 = \frac{n+1}{n-1} \frac{n_1-1}{n_1+1}, \\ \nu_2 = \frac{n-1}{n+1} \frac{n_1-1}{n_1+1}. \end{cases}$$

Since we are only interested in the reflected *intensity* (or rather its ratio with the incident intensity), we can simplify (9) to

$$(10) \quad \left| \frac{C}{A} \right|^2 = \left(\frac{n-1}{n+1} \right)^2 \left| \frac{1-\nu_1 \eta^{4n}}{1-\nu_2 \eta^{4n}} \right|^2.$$

In order to discuss this formula we calculate

$$(11) \quad (1-\nu \eta^{4n})(1-\nu \eta^{-4n}) = 1-\nu(\eta^{4n} + \eta^{-4n}) + \nu^2 = 1-2\nu \cos \varphi + \nu^2$$

($\nu = \nu_1, \nu_2$ is real, η has the absolute value 1). The angle φ introduced here is, according to (8), defined by

$$\eta^{4n} = e^{i\varphi}, \quad \varphi = 4nkh. \quad (12)$$

In section E we shall discuss the physical meaning of this "phase difference" φ in terms of optical path length.

Equation (10) becomes, because of (11)

$$(13) \quad \left| \frac{C}{A} \right|^2 = \left(\frac{n-1}{n+1} \right)^2 \frac{1 + \nu_1^2 - 2\nu_1 \cos \varphi}{1 + \nu_2^2 - 2\nu_2 \cos \varphi}.$$

The phase differences between intensity extremals are obtained by differentiating (13) with respect to φ and hence are found from the equation

$$0 = \{2\nu_1(1 + \nu_2^2 - 2\nu_2 \cos \varphi) - 2\nu_2(1 + \nu_1^2 - 2\nu_1 \cos \varphi)\} \sin \varphi = \\ = 2(\nu_1 - \nu_2)(1 - \nu_1\nu_2) \sin \varphi$$

to be

$$(14) \quad \varphi = z\pi, \quad z = \text{integer}$$

Substituting this in (13), one finds by elementary computation

$$(15) \quad \begin{aligned} \varphi = \pi, 3\pi, 5\pi, \dots; \left| \frac{C}{A} \right|^2 &= \left(\frac{n n_1 - 1}{n n_1 + 1} \right)^2, \quad \text{maxima,} \\ \varphi = 2\pi, 4\pi, 6\pi, \dots; \left| \frac{C}{A} \right|^2 &= \left(\frac{n - n_1}{n + n_1} \right)^2, \quad \text{minima.} \end{aligned}$$

The denotations "maxima" and "minima" refer to the case $n_1 < n$ which prevails in the case of oil on water. In the opposite case the denotations are reversed.

The layer of oil is very thin. Though not monomolecular, it has, however, only a thickness of perhaps a wavelength λ_v at the violet end of the spectrum. If we assume this, i. e. let $2h = \lambda_v$, and estimate the index of refraction of oil to be 1.5, then we obtain, according to the definition (12) of φ

$$\varphi = 6 \cdot 2\pi \frac{h}{\lambda_v} = 6\pi.$$

This means, according to (14) and (15), that the reflected *violet* light has a *minimum* for $z = 6$. On the other hand, since $\lambda_r \sim 2\lambda_v$, one obtains for the red end of the spectrum

$$\varphi = 6 \cdot 2\pi \frac{h}{\lambda_r} = 3\pi.$$

Hence, according to (14) and (15), one obtains a *maximum* of reflected *red* light for $z = 3$. The middle portion of the spectrum gives rise to a further minimum and maximum corresponding to $z = 4$ and 5. Hence, the light reflected by the oil spot has a mixed color, namely a predominantly blue-green tint under our assumptions. If the thickness of the layer varies locally, the color also varies.

C. COATED (NON-REFLECTING) LENSES

The light passing through a system of lenses is weakened by reflection. Even though for central rays (perpendicularly incident light) the attenuation arising from a single reflection is small (4% according to p. 20), it becomes

considerable for a system of lenses. Many optical firms strive to eliminate this reflection, which is especially disadvantageous in the design of photographic apparatus. The problem is solved by applying thin layers to all surfaces at which the lenses of the system border on air. Such a layer was originally produced by a *structural modification* of the glass surface (etching or solution of components of the glass flux). Nowadays one prefers to evaporate onto the glass a layer of suitable material whose index of refraction is lower than that of glass, taking care to make the layer as uniform as possible.

If we consider only one boundary layer and because of its thinness neglect the curvature, we are again faced with our problem of three media:

I	air,	index of refraction 1
II	surface layer,	index of refraction n
III	lens,	index of refraction $n_1 = n/n_g$

where n_g is the index of refraction of the lens glass relative to air. Since our n_1 should correspond to the transition $II \rightarrow III$ in the formulae (15), the transition $I \rightarrow III$ is characterized by $n_g = n/n_1$ in conformity with eq. (3.3 b). When, in view of this, we let $n_1 = n/n_g$ in the first eq. (15), we obtain *zero reflection*, for

$$(16) \quad \frac{n^2}{n_g} = 1, \quad n = \sqrt{1 \cdot n_g}.$$

Hence, the index of refraction n of the evaporated layer II should be the *geometric mean* of the indices of refraction 1 and n_g of media I and III (both relative to air). The optical industry tries to fulfill this requirement together with the first condition (15), namely $\varphi = \pi$, by a choice of a suitable material, e. g. lithium fluoride, and by a suitable thickness of the layer to be precipitated. Since however, n , n_g and therefore also φ depend on wavelength, the condition $\varphi = \pi$, in particular, cannot be satisfied for all wavelengths. One favors the brightest spot of the spectrum ($\lambda = 0.55 \mu$ in the yellow-green) and suppresses reflection as completely as possible for this wavelength. Then the reflection of the complementary purple is, to be sure, not zero but it is nevertheless small. Indeed a lens prepared in this way has a weak purplish tint.

We have here assumed the layer to be homogeneous, hence n to be a constant for a given wavelength. For literature on reflection in inhomogeneous layers we refer to footnote¹ on p. 19. However, we must still note that reflection is to be avoided not only at the front surface but also at the back surface of the lens as well as at all other surfaces of the lens system which border on air. Because of the interchangeability of media I and III in our condition $n = \sqrt{1 \cdot n_g}$, this is accomplished by the same compensation procedure, that is,

by evaporating the same thickness of the same material on the back of the lens as on the front. Our result is true not only for the central ray ($\alpha = \beta = \gamma = 0$) but it is true sufficiently closely also for neighboring ray directions. For, since in our initial formulae (3) and (6) only the cosines of these angles appeared, only a "cosine-error" appears in the reflection of neighboring rays (a second order deviation from zero reflection).

D. SOAP BUBBLES AND NEWTON'S RINGS.

The color play of thin *soap bubbles* is explained in the same manner as that of the oil spot. The only difference is that medium *III* (interior of the soap bubble) is now the same as medium *I*, namely air. Hence, we have $n_1 = n$. According to (15) this has the result that the intensity of the minima becomes zero and that, therefore, when the condition for a minimum is fulfilled for a wavelength λ_1 , the complementary color λ_2 is seen in particularly pure form. In general however, the reflected colors are mixed colors.

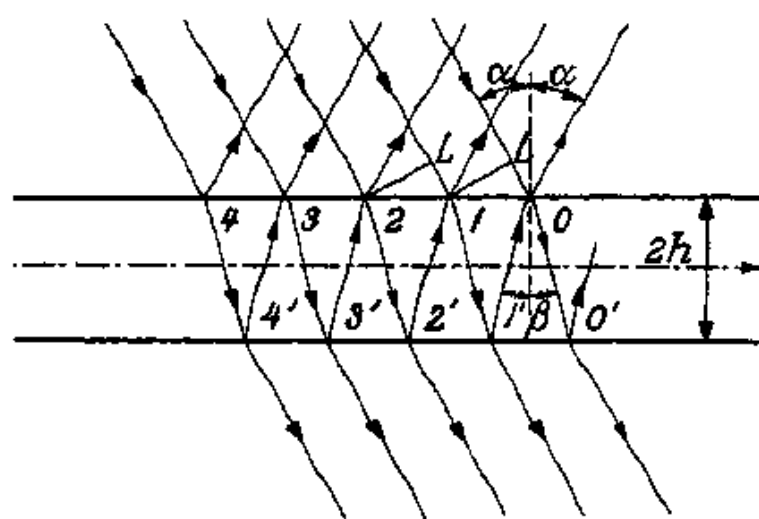


Fig. 9.

Summation process for multiple reflection in a plane-parallel plate.

We estimate the order of magnitude of the thickness of the soap bubble as equal to or smaller than some visible wavelength. This we conclude from the fact that when the bubble is strongly inflated, a dark spot appears at the top which indicates a very small thickness compared to the wavelength. Because the soap solution flows off to the bottom, the thickness of the soap film does indeed become vanishingly small at the top.

Quite similar circumstances prevail in the case of *Newton's rings*: a plano-convex lens is placed with its weakly curved convex surface on a plane glass plate, so that between the two an air gap is created which widens toward the outside. Again media *I* and *III* have the same index of refraction as compared to medium *II* which is now air. If the illumination is monochromatic, one sees a number of dark circles between bright rings. In white light a small

number of colored rings appear. These show no pure spectral colors but, rather, mixed colors. The transmitted light is colored complementarily to the reflected light.

E. COMPARISON OF METHODS: SUMMATION OR BOUNDARY-VALUE TREATMENT?

Let a plane-parallel glass plate of thickness $2h$ which at the front and back borders on air be obliquely illuminated by parallel, monochromatic light. Ordinarily one argues in the following manner (see fig. 9): at point 0 on the front face there emerges, besides the directly reflected light, also light which entered the plate at 1 and was reflected at 1'. Furthermore, light which entered at 2 and was reflected at 2', 1 and 1', etc., will also leave the plate at 0. Generally speaking, the light emerging at an arbitrary point of the front face is a sequence of components which have been refracted *twice* and reflected an *odd* number of times. Correspondingly, the light emerging from an arbitrary point on the back face is composed of a sequence of component waves which have been refracted *twice* and reflected an *even* number of times. We must calculate the differences in *phase* and *amplitude* of these various rays.

The length of the light path 11'0 measured in wavelengths λ_g in the glass (λ = wavelength in air) amounts to

$$\frac{4h/\cos\beta}{\lambda_g} = \frac{4n h/\cos\beta}{\lambda}.$$

When multiplied by 2π , this is the increase in phase which the light attains along the path 11'0

$$(17) \quad \varphi_1 = \frac{2\pi}{\lambda} \frac{4n h}{\cos\beta} = \frac{4\pi n h}{\cos\beta}.$$

But, in addition, this light is ahead in phase compared to the light incident at 0 by

$$(18) \quad \varphi_2 = 2\pi \frac{OL}{\lambda} = k \cdot OL,$$

where L is the point at which the perpendicular from 1 to OL intersects the ray, see fig. 9:

$$OL = \sin\alpha \cdot 01 = \sin\alpha \cdot 2 \tan\beta \cdot 2h.$$

The *total phase difference* against the light incident at 0 amounts, then, to

$$(18a) \quad \varphi = \varphi_1 - \varphi_2 = \frac{4\pi n h}{\cos\beta} \left(1 - \frac{\sin\alpha \sin\beta}{n} \right) = \frac{4\pi n h}{\cos\beta} (1 - \sin^2\beta) = 4\pi n h \cos\beta.$$

For perpendicular incidence ($\beta = 0$) this φ proves to be identical with the auxiliary angle φ introduced in (12) and explains the physical meaning of the latter. For the light path 22'11'0 the corresponding phase difference obviously amounts to 2φ , for the following path to 3φ , etc.

In order to determine, on the other hand, the *amplitude differences*, we use the energy coefficients r and d from (4.18) which are, as was emphasized there, the same for the front and back side of the plate. Expressed in these terms, the factor which is to multiply the amplitude in the case of one-, three-, five-, ... fold reflection and, in every case, two-fold transit across the front face amounts to

$$(19) \quad \sqrt{r}d, \quad \sqrt{r}rd, \quad \sqrt{r}r^2d, \dots$$

From (18 a) and (19) it follows that the summation of the first p rays is given by

$$(20) \quad \sqrt{r}e^{i\varphi}d + \sqrt{r}re^{2i\varphi}d + \dots + \sqrt{r}r^{p-1}e^{ip\varphi}d.$$

To this is to be added the contribution corresponding to the direct reflection at 0. This is, including the correct phase factor¹, $C/A = -\sqrt{r}$. One obtains finally:

$$(21) \quad \frac{C}{A} = -\sqrt{r} \{1 - d e^{i\varphi} (1 + r e^{i\varphi} + \dots + r^{p-1} e^{i(p-1)\varphi})\} = -\sqrt{r} \left\{1 - d e^{i\varphi} \frac{1 - r^p e^{ip\varphi}}{1 - r e^{i\varphi}}\right\}.$$

For $p = \infty$, because $r < 1$ and $r + d = 1$, this contracts to:

$$(22) \quad \frac{C}{A} = -\sqrt{r} \left\{1 - \frac{d e^{i\varphi}}{1 - r e^{i\varphi}}\right\} = -\sqrt{r} \frac{1 - e^{i\varphi}}{1 - r e^{i\varphi}}.$$

The amplitude factor for the light passing through the plate at 0', which we shall call D/A , is determined quite similarly. Because of the two-fold passing across the boundary surface (once front, once back) and zero-, two-, four-, ... fold reflection (see fig. 9), one obtains instead of (19) and (20)

$$(19 \text{ a}) \quad d, rd, r^2d, \dots$$

$$(20 \text{ a}) \quad e^{\frac{i\varphi_1}{2}} (d + r e^{i\varphi}d + r^2 e^{2i\varphi}d + \dots + r^{p-1} e^{i(p-1)\varphi}d)$$

and instead of (21) and (22), respectively

$$(21 \text{ a}) \quad \frac{D}{A} = e^{\frac{i\varphi_1}{2}} d (1 + r e^{i\varphi} + \dots + r^{p-1} e^{i(p-1)\varphi}) = e^{\frac{i\varphi_1}{2}} d \frac{1 - r^p e^{ip\varphi}}{1 - r e^{i\varphi}}$$

¹For reflection at the denser medium the sign of C/A was chosen opposite from that for reflection at the rarer medium; see, for instance, the comment after formula (5.3).

and for $\phi = \infty$

$$(22 \text{ a}) \quad \frac{D}{A} = \frac{e^{\frac{i\phi_1}{2}} d}{1 - r e^{i\phi}}.$$

We note here that the expressions (21) and (21 a) become essentially the same if we suppress in (21) the first term on the right which is due to direct reflection. For we then have

$$(23) \quad \frac{C}{A} = \sqrt{r} e^{i(\varphi - \frac{\phi_1}{2})} \frac{D}{A} \quad \text{and hence} \quad \left| \frac{C}{A} \right|^2 = r \left| \frac{D}{A} \right|^2.$$

The reflected intensity is thus equal to the intensity transmitted through the plate except for a factor r .

The use of the symbols r and d in (21), (22) already indicate that the results are valid for both polarization cases (parallel and perpendicular to the plane of incidence). r and d only stand for somewhat different expressions in the two cases.

In particular, we consider the special case of perpendicular incidence in which this difference disappears and in which a comparison of (22) with our formula (9) is possible if we specialize the latter by setting $n_1 = n$ (air also at the back face of the plate). Then we have to set in (9)

$$(24) \quad \nu_1 = 1, \quad \nu_2 = \frac{(n+1)^2}{(n-1)^2} = r.$$

In addition, the following must be taken into account: in (9), just as in our general assumption (I) on p. 41, C and A refer to the center of the plate, namely to $y = 0$ in the choice of coordinates used there, and not to the top surface of the plate $y = h$. Therefore, at the top of the plate A must be multiplied by the factor $\exp(i k h)$ and C by the factor $\exp(-i k h)$ (both for perpendicular incidence: $\alpha = 0$) if we wish to compare (9) with our present amplitude ratio C/A given by (22), since the coordinates of the latter refer to the top surface. This means that we must suppress in (9) the factor $\eta^{-2} = \exp(-2 i k_1 h)$. Thereupon (9) becomes, using (24),

$$(25) \quad \frac{C}{A} = -\sqrt{r} \frac{1 - e^{i\varphi}}{1 - r e^{i\varphi}}$$

which now indeed agrees with (22).

Thus both of our methods lead to the same result and not only in the special case of perpendicular incidence which was used here for the comparison but quite generally. Both methods have their advantages and their disadvantages. The *boundary value method* saves us the somewhat laborious phase considerations in fig. 9. The *summation method* seems to lend itself

more readily to visualization and is not limited to the assumption that $p = \infty$. The latter can also be used to treat the case of a plate of finite length and of an incident light bundle of finite width¹. This problem is inaccessible to the formal hypothesis underlying our boundary value problem which is restricted to the xy -plane. That is why the summation method is preferred in treating the problems on resolving power in Chapter VI. Another reason is that this method fits in better with the usual grating theory. As we shall see, both methods are in fact equivalent for the two types of high resolution interference apparatus which we will now discuss.

F. THE LUMMER-GEHRKE PLATE (1902).

In our discussion of total reflection in Sec. 5 the wave was incident in the denser medium and emerged into air. For angles of incidence $\alpha \sim \alpha_{tot}$ we obtained angles of emergence β nearly equal to $\pi/2$ and a reflecting power of almost 1. Lummer's original idea was to let the light impinge on the top

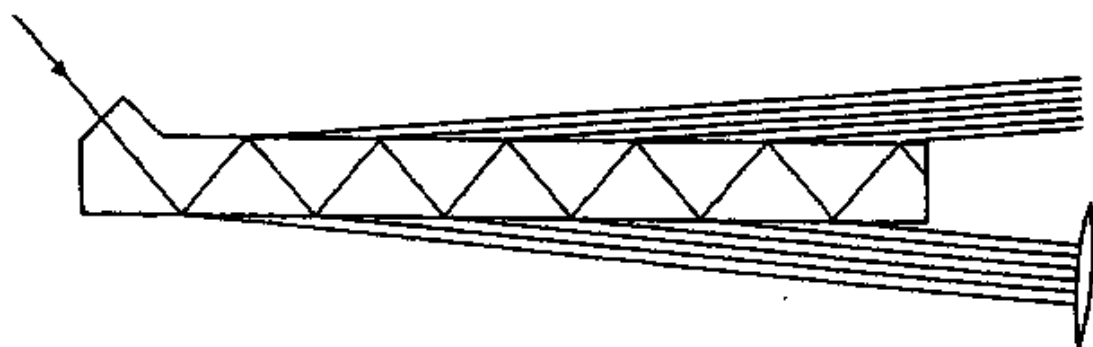


Fig. 10.
The interference of rays in the Lummer Plate.

of the plate at a grazing angle so that the refracted portion would be reflected back and forth at an angle close to α_{tot} . In this way the high reflecting power r of almost 1 would be utilized. Gehrke simplified the procedure by capping the plate with a prism of angle α_{tot} (see fig. 10). Light which is incident perpendicularly upon the face of the prism strikes the lower surface and thence

¹The fact that the summation method is only an approximation in the sense that it does not account for the diffraction phenomena at the corners of a finite plate and at the boundaries of the light bundle has hardly any practical significance.

alternately the upper and lower surfaces under the desired angle; it emerges at a grazing angle. In this way the first reflection of the incident beam is suppressed as had indeed been assumed in eq. (23).

The number p is not very large for the Lummer Plate because a perfectly homogeneous plane-parallel plate of thickness say 1 cm. cannot be made with a length much greater than 20 cm. Nevertheless we can without hesitation go to the limit $p \rightarrow \infty$ and, hence, we can use eq. (22 a) as a starting point. We will write it here in the form

$$(26) \quad \left| \frac{D}{A} \right| = \frac{1-r}{|1-r e^{i\varphi}|}.$$

This is justified because the limit $\alpha \rightarrow \alpha_{tot}$ cannot be approached arbitrarily closely. If for no other reason, this is so because we never deal with a precisely parallel incident plane wave but always with a wave bundle which has a certain angular distribution. Hence r is never exactly equal to 1, but only to a certain degree of approximation. Therefore, already for $p = 20$, r^p becomes vanishingly small and it is immaterial whether we set p equal to the maximum value occurring in the Lummer Plate or to ∞ . Thus it is evident that in spite of the finite value of p there can be no difference between the results of the summation and boundary value methods.

From eq. (26) we find immediately

$$(27) \quad \begin{cases} \left| \frac{D}{A} \right| = 1 & \text{for } \varphi = 2\pi z, \quad z = \text{integer} \\ \left| \frac{D}{A} \right| \sim 0 & \text{for all } \varphi \text{ appreciably different from } 2\pi z. \end{cases}$$

The latter expression is correct because the numerator $1-r \sim 0$, the former holds because for $\varphi = 2\pi z$ numerator and denominator of (26) become exactly equal.

In order to see what "appreciably different" means, we rewrite the denominator of (26) in the form

$$(27 a) \quad \sqrt{(1-r e^{i\varphi})(1-r e^{-i\varphi})} = \sqrt{1+r^2-2r \cos \varphi}.$$

We set $\varphi = 2\pi z - \Delta\varphi$, and hence $\cos \varphi = \cos \Delta\varphi = 1 - \frac{(\Delta\varphi)^2}{2}$. In particular, we seek those values of $\Delta\varphi$ which correspond to the so-called "half-width" of the intensity maximum 1 occurring at $\varphi = 2\pi z$, that is, those values which satisfy the condition

$$(28) \quad \left| \frac{D}{A} \right|^2 = \frac{1}{2} = \frac{(1-r)^2}{(1-r)^2 + r(\Delta\varphi)^2}.$$

Carrying out the computation one obtains directly

$$(1-r)^2 + r(\Delta\varphi)^2 = 2(1-r)^2, \quad \Delta\varphi = \frac{1-r}{\sqrt{r}} \sim \pm(1-r).$$

The half-width is twice $|\Delta\varphi|$, hence

$$(28a) \quad 2|\Delta\varphi| \sim 2(1-r).$$

As expected, this width becomes narrower as r approaches its limiting value 1. This low value of the half-width will play a decisive role for the resolving power of the Lummer Plate. We shall discuss this problem in greater detail in Chap. VI.

G. THE INTERFEROMETER OF PEROT AND FABRY (ABOUT 1900)

While Lummer attained a high reflecting power r by approaching very closely the limiting angle of total reflection, Perot and Fabry employed the surfaces of a half-silvered glass plate and used an angle of incidence almost perpendicular to the surface. The importance of their method is increased by the fact that the glass plate can be replaced by an "air plate" between two glass surfaces which have been silvered semi-transparently. These surfaces can be spaced by means of invar-steel pieces and in this way a *standard measure* ("etalon") for the exact measurement of wavelengths is created. This standard is entirely independent of temperature, index of refraction or irregularities in the glass.

It is again advantageous to use the boundary value method. But we must alter our former boundary conditions. Assuming p -polarization, that is \mathbf{E} parallel to the z -axis, we consider the z -component of the second Maxwell eq. (2.4). For the displacement current $\dot{\mathbf{D}}$ in this equation we substitute the specific conduction current σE_z in the silver layer. We integrate this equation over a rectangle lying in the x, y -plane of length 1 in the x -direction and having the very small thickness of the silver layer as its width in the y -direction. The left-hand side of the integrated equation is then equal to the total current per unit length in the silver layer. The right-hand side yields, according to Stokes' Theorem, the contour integral of \mathbf{H} around the rectangle which equals the *discontinuity* of H_x in passing through the silver layer. Instead of the previous continuity of H_x we have now a discontinuity in H_x which is proportional to E_x . We write

$$(29) \quad \text{Discontinuity of } H_x = -g \sqrt{\frac{\epsilon_0}{\mu_0}} E_x.$$

g is a factor of proportionality which depends on the conductivity and thickness of the silver layer and is dimensionless owing to the factor $(\epsilon_0/\mu_0)^{1/2}$. Because of the inertia of the electrons g is, in the visible spectrum, actually not a real but a complex number.

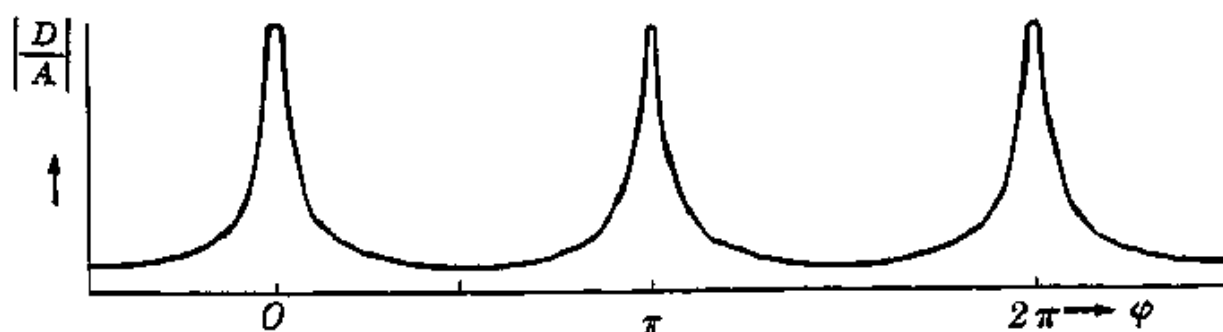


Fig. 11.

The amplitude-ratio $|D/A|$ vs. the phase difference φ for the Lummer Plate ($|D/A|_{\max} = 1$) and for the Perot-Fabry Etalon ($|D/A|_{\max} \ll 1$).

The continuity of E_z and the resulting boundary conditions (3) and (4) as well as the law of refraction (2) are unchanged by the silvering; they become somewhat simpler, however, because now $n_1 = n$ and $\gamma = \alpha$ (same conditions at the top and bottom plate surfaces). On the other hand, according to (29), the boundary conditions (5) and (6) must be modified in the following manner:

$$(30) \quad \begin{cases} (A e^{-ikh \cos \alpha} - C e^{ikh \cos \alpha}) \cos \alpha - (B e^{-in k h \cos \beta} - E e^{in k h \cos \beta}) n \cos \beta = \\ g (A e^{-ikh \cos \alpha} + C e^{ikh \cos \alpha}) = g (B e^{-in k h \cos \beta} + E e^{in k h \cos \beta}), \end{cases}$$

$$(31) \quad \begin{cases} (B e^{in k h \cos \beta} - E e^{-in k h \cos \beta}) n \cos \beta - D e^{ikh \cos \alpha} \cos \alpha = \\ = g (B e^{in k h \cos \beta} + E e^{-in k h \cos \beta}) = g D e^{ikh \cos \alpha}. \end{cases}$$

The two forms in which the right-hand sides of these equations are written correspond to the two ways of expressing the value of E_z in (29) in terms of the left- or right-hand sides of eqs. (3) and (4), respectively. Thus the four equations embodied in (30) and (31) represent the complete system of boundary conditions pertaining to the present problem.

For most practical applications only the transmitted light, represented by the quotient D/A , is of interest. By elementary, though somewhat laborious, computation, we obtain from (30) and (31)

$$(32) \quad \frac{D}{A} = \frac{e^{-2ikh \cos \alpha}}{(1 + g/\cos \alpha) \cos \frac{\varphi}{2} - \frac{i}{2} \left(\frac{(1 + g/\cos \alpha)^2}{n \cos \beta / \cos \alpha} + \frac{n \cos \beta}{\cos \alpha} \right) \sin \frac{\varphi}{2}};$$

φ represents the phase difference generated by the ray in passing back and forth in the plate and is defined as in (18 a) by

$$(32 a) \quad \varphi = 4nkh \cos \beta \quad (\text{where } n = 1 \text{ in the case of the air plate}).$$

The dependence of the absolute value of (32) on φ is shown in fig. 11, where variations in the independent variable φ may be thought of as expressing either variations in the wave number k or variations in the angle of incidence α which is connected with β by the law of refraction. α and β themselves can be considered as constants insofar as they appear explicitly in (32). But in (32 a) where $\cos \beta$ is multiplied by the very large factor kh even extremely small changes in β cause φ to change appreciably. Therefore, in spite of the practically constant β , φ can be used as an independent variable in fig. 11.

Let us check the figure using the special case of the air-étalon ($n = 1$), almost perpendicular incidence ($\alpha = \beta = 0$) and almost real g . Equation (32) then yields simply

$$(33) \quad \left| \frac{D}{A} \right|^{-2} = (1 + g)^2 \cos^2 \frac{\varphi}{2} + \frac{1}{4} ([1 + g]^2 + 1)^2 \sin^2 \frac{\varphi}{2}.$$

The extremal condition which follows from this by differentiation with respect to φ is:

$$\sin \frac{\varphi}{2} \cos \frac{\varphi}{2} = 0.$$

To $\sin \varphi/2 = 0$ corresponds:

$$(33 a) \quad \left| \frac{D}{A} \right|_{\max} = \frac{1}{1 + g}, \quad \varphi = 2z\pi,$$

to $\cos \varphi/2 = 0$:

$$(33 b) \quad \left| \frac{D}{A} \right|_{\min} = \frac{2}{(1 + g)^2 + 1}, \quad \varphi = (2z + 1)\pi.$$

In both cases z is a very large integer.

In (33 a, b) we assumed g to be large. By definition this assumption corresponds to a heavy layer of silver (a strong conduction current). Consequently the incident light is very much weakened even at the maxima. The minima, on the other hand, are weaker than the maxima by a factor of $\frac{1}{1 + g}$.

The maxima are equidistantly spaced, and so are the minima which lie halfway

between the maxima. The maxima are *sharp*; the minima are *very flat*. This is a consequence of (33) because only if the condition for a maximum, i. e. $\sin \varphi/2 = 0$, is *exactly* fulfilled is the maximum value (33 a) of order of magnitude $(1 + g)^{-1}$ attained; for all other φ the second term on the right-hand side of (33) dominates because of the fourth power of g which it contains; the resulting magnitude of $|D/A|$ is then about $[(1 + g)^2 \sin^2 \varphi/2]^{-1}$ and reaches the minimum as given by (33 b). Thus fig. 11 has been checked for the case of sufficiently large g .

We shall also compute the "half-width" of the intensity maxima. Since, according to (33 a), the latter are equal to $(1 + g)^{-2}$, we must substitute on the left-hand side of (33) the value $2(1 + g)^2$. On the right-hand side we let

$$\frac{\varphi}{2} = z\pi - \Delta\varphi, \quad \sin^2 \frac{\varphi}{2} = (\Delta\varphi)^2, \quad \cos^2 \frac{\varphi}{2} = 1 - (\Delta\varphi)^2$$

and dividing by $(1 + g)^2$, we obtain

$$2 = 1 + \frac{1}{4}(1 + g)^2 (\Delta\varphi)^2, \quad \Delta\varphi \sim \pm \frac{2}{1 + g}.$$

The half-width is therefore

$$(34) \quad 2|\Delta\varphi| = \frac{4}{1 + g}.$$

In exercise I.7 we shall explain these results concerning the positions and half-widths of the interference maxima from the point of view of the electromagnetic characteristic oscillations.

In Chap. VI we shall see that the Perot-Fabry Etalon attains its excellent resolving power only because of a large value of g . Only for large g , i. e. strong silvering, is the half-width of the maxima sufficiently small and thus the prime purpose of this interferometer, namely the resolution of fine structures, is attained. One must, therefore, accept the large loss in intensity which is engendered by heavy silver layers. The Lummer Plate, because of $r \sim 1$, is preferable from the point of view of intensity. But it cannot attain the resolving power of the Perot-Fabry Etalon and is, furthermore, experimentally less convenient than the latter.

It is to be emphasized that the general formula (32) encompasses also the Lummer Plate. The latter is described by the opposite limiting case $g = 0$ (no silvering). For $\varphi = 2\pi z$ (32) gives then immediately

$$(35) \quad \left| \frac{D}{A} \right|_{\max} = 1$$

which agrees with the first equation (27). For all other values of φ , on the other hand,

$$\left| \frac{D}{A} \right| = \left[\cos^2 \frac{\varphi}{2} + \frac{1}{4} \left(\frac{\cos \alpha}{n \cos \beta} + \frac{n \cos \beta}{\cos \alpha} \right)^2 \sin^2 \frac{\varphi}{2} \right]^{-1/2}.$$

Passing now to grazing incidence, as required by Lummer, that is allowing $\cos \alpha$ to approach 0, the coefficient of $\sin^2 \varphi/2$ tends to infinity and one obtains, in agreement with the second eq. (27)

$$(35 \text{ a}) \quad \left| \frac{D}{A} \right| \rightarrow 0.$$

(35) and (35 a) confirm our earlier assertion that the Lummer plate can also be treated by means of the boundary value method.

8. Standing Light Waves

The question of the position of the "light vector" with respect to the plane of polarization was left unanswered by the elastic theory of light. Fresnel was of the opinion that the light vector was perpendicular to the plane of polarization while F. Neumann thought it to be parallel to that plane. But the word light vector could not be clearly defined on the basis of the elastic theory. Electromagnetically we have two light vectors \mathbf{E} and \mathbf{H} (in a crystal there are even four: \mathbf{E} , \mathbf{D} and \mathbf{H} , \mathbf{B}). We saw in Sec. 4 that in the production of polarized light by reflection, the electric vector \mathbf{E} is perpendicular and the magnetic vector is parallel to the plane of incidence. Since in this case the plane of polarization is traditionally identified with the plane of incidence, we have also that \mathbf{E} is perpendicular and \mathbf{H} is parallel to the plane of polarization. Therefore, depending on whether one calls \mathbf{E} or \mathbf{H} the light vector, one decides the question in favor of Fresnel or Neumann. But even in this way only a nominal definition of the word "light vector" is achievable; physical significance, however, can be attributed to it on the force of electromagnetic evidence.

When light acts on a photographic layer, an electron is removed from a silver bromide or chloride molecule and thereby a silver atom is prepared to blacken during the development of the film. Only the electric field strength \mathbf{E} is able to accomplish this. Since, moreover, the processes occurring in the eye's retina are quite similar (both phenomena are without doubt "photoelectric effects"), we have good reason to give the name "light vector" to the field vector \mathbf{E} rather than to the magnetic vector \mathbf{H} .

The beautiful experiments by O. Wiener (Ann. d. Physik, 1890) have placed the results of these general considerations on a sound empirical basis. This was accomplished by a thorough study of the photographic process.

A. MONOCHROMATIC, LINEARLY POLARIZED LIGHT WHICH IS INCIDENT PERPENDICULARLY UPON A METAL SURFACE

A polished silver mirror is used as a reflector. The normal to this surface shall, as before, be the y -axis. Let the direction of incidence be the negative y -direction and the direction of reflection the positive y -direction. Because of the transversality of light, $E_y = 0$. There is no need to distinguish between E_x and E_z since both directions are equivalent for normal incidence. We can write for either or both of these components:

$$(1) \quad \begin{aligned} E_i &= A e^{-iky - i\omega t}, \\ E_r &= C e^{+iky - i\omega t}. \end{aligned}$$

As a good conductor ($\sigma \rightarrow \infty$) the silver mirror does not permit the existence of an electric field tangential to its surface. Any such field vanishes because of conduction. Hence, we have

$$(2) \quad E_{tan} = E_i + E_r = 0 \quad \text{for} \quad y = 0.$$

From (1), it follows that

$$(3) \quad C = -A \quad (\text{Phase change during reflection})$$

and, writing real parts and letting A be real, we have for $y \geq 0$

$$(4) \quad E = \text{Re}(E_i + E_r) = 2A \sin ky \cos \omega t.$$

This is the typical expression for a *standing wave*. The nodes are at

$$ky = n\pi, \quad y = n \frac{\lambda}{2},$$

the antinodes at

$$ky = \left(n + \frac{1}{2}\right)\pi, \quad y = \frac{\lambda}{4} + n \frac{\lambda}{2}.$$

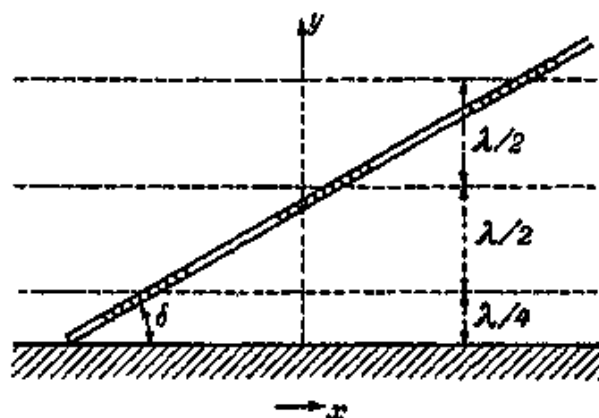


Fig. 12.

Wiener's experiment on standing light waves. The photographic plate placed at an angle δ is blackened at the antinodes of the electric vector (indicated by dotted lines).

We would expect maximum photographic blackening at the antinodes and no blackening at the nodes. The distance of the first blackening from the metal surface should then be equal to half of the spacing between succeeding blackened spots.

To prove the foregoing, Wiener used an age-old method for measuring the water level of rivers. A photographically sensitive film which was spread on the bottom surface of a glass plate was placed against a silver mirror at the extremely small angle δ , as shown in fig. 12. Distances measured perpendicular to the mirror are thus magnified on the film by a factor of $1/\delta$. The distances $\lambda/4$ and $\lambda/2$ are in this way depicted on a macroscopically measurable scale.

The result confirmed completely the expected periodic spacing of blackened spots as well as the fact that the first maximum occurred at $1/2$ of that spacing from the metal. *Thus the electric vector E is indeed photographically active and is to be considered as the light vector*¹. The magnetic vector is *not* the light vector. Its antinodes alternate with those of the electric vector, the first one being on the surface itself. Indeed, using Maxwell's relationship between H , D and E , it follows directly from (4) that we obtain for H

$$(5) \quad H = 2 A \sqrt{\frac{\epsilon_0}{\mu_0}} \cos k y \sin \omega t.$$

B. OBLIQUELY INCIDENT LIGHT.

The following experiment performed by Wiener is also very revealing. The photographic film was placed in the same position as before but the light was incident upon the silver mirror at an angle of 45° with the normal. When the light was polarized in the plane of incidence, (E perpendicular to that plane), then the film exhibited blackened stripes which were qualitatively in the same positions as in the case of perpendicular incidence. If, however, the plane of polarization was placed perpendicular to the plane of incidence and the angle of incidence was made precisely 45° , then no stripes appeared and rather the blackening was uniformly distributed over the plate.

In exercise I.8 the results of this experiment will be computed for arbitrary angles of incidence α .

¹The fact that, according to H. Jäger, Ann. d. Phys. (Lpz) 84, 280, 1939, the proof can be based directly on the photoelectric process instead of photography corresponds to the above remark concerning the identity of photographic and photoelectric action.

C. LIPPMANN'S COLOR PHOTOGRAPHY.

Lippmann arranged a very fine grained photographic film in the manner of Wiener's experiment by placing it flat on a mercury surface and shining a *spectrum* perpendicularly upon the surface. At the antinodes of the standing waves thus created, a system of Wiener-type silver layers was formed photochemically. These layers were spaced at distances of $\lambda/2$ where λ was the wavelength of the spectral region which illuminated the particular point in question.

If a film prepared in this way is developed and then illuminated perpendicularly with *white* light, every spot of the film emits the same wavelength λ to which it had been exposed during preparation. Only this wavelength (or a submultiple of it) will fit into the screen formed by the system of Wiener planes. All other λ are destroyed by interference. Thus, looking perpendicularly at the film, one sees the whole spectrum in brilliant interference colors. If the film is breathed upon, it swells and the spectrum is shifted toward the *red* because longer wavelengths fit into the expanded screen. When looked upon obliquely, the spectrum shifts toward the *violet*. This is due to the relationship

$$(6) \qquad 2d \cos \alpha = \lambda_a$$

where λ_a is the wavelength seen at the angle of reflection α and d is the spacing of the planes of the screen. Equation (6) represents the condition that a plane wave incident at the angle α shall be reflected by all planes of the system with the same phase (or phases differing by multiples of 2π). We shall encounter this equation again under the name of *Bragg's Equation* in Chap. V, Sec. 32 where it will be in a somewhat more general form and somewhat different notation. For the present it suffices to make two observations:

1. For $d = \lambda/2$ and $\alpha = 0$ (perpendicular incidence and viewing), $\lambda_a = \lambda$,
i. e. the *color is unchanged*.
2. For $d = \lambda/2$ and $\alpha \neq 0$ (oblique incidence and viewing), $\lambda_a = \lambda \cos \alpha < \lambda$,
i. e. *shift toward the violet*.

As is well known, the modern practical solution of the problem of color photography is based on entirely different principles. Nevertheless, as the earliest proposal of "photography in natural colors", the Lippmann method has great historical interest.

CHAPTER II

OPTICS OF MOVING MEDIA AND LIGHT SOURCES ASTRONOMICAL TOPICS

The fundamental optical constant is the velocity of light in vacuum. According to the Theory of Relativity this constant governs the scale of time and space. We shall, therefore, discuss the velocity of light before turning, in later chapters, to the optical properties of matter which, though apparently more elementary, are fundamentally really more complicated. We shall discover the most important facts about the velocity of light, c , not from terrestrial experiments but rather by discussing astronomical measurements.

9. Measurement of the velocity of light

The satellites of Jupiter were discovered in 1610 by Galileo. He called them Mediceic Stars in honor of his patron Duke Cosimo of Florence. These were the four bright satellites which are close to Jupiter. The periods of

their orbital motions amount to several days and are, therefore, very short compared to the period of Jupiter's orbital motion around the sun (twelve years). At the present time twelve satellites of Jupiter are known.

The periods of these satellites can be determined exactly by means of their eclipses (times at which they enter Jupiter's shadow in the sun light). The Dane Olaf Römer (1676) discovered remarkable variations in the records of measurements of these periods: they increased when the earth moved away

from Jupiter and decreased when the earth moved towards Jupiter. From this Römer concluded that it must take a finite amount of time for light to traverse the diameter of the earth's orbit. Using the radius of the earth's orbit as it was then known, he computed a fairly accurate value of the velocity of light c .

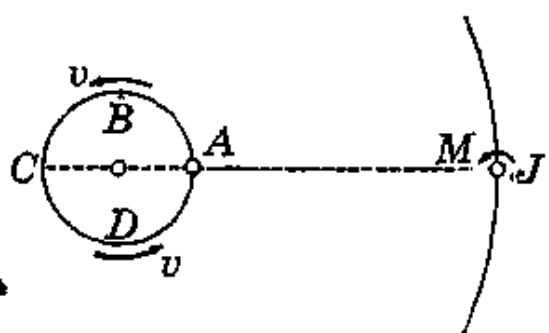


Fig. 13.

Determination of the velocity of light
by the method of Olaf Römer. J,
Jupiter; M, satellite of Jupiter;
ABCD, earth's orbit.

We can best explain this computation by comparing it with the *Doppler Effect*. This phenomenon will be treated more thoroughly in 11. For present purposes it will suffice to characterize it by the statement

$$(1) \quad \frac{\Delta \lambda}{\lambda} = \frac{\Delta \tau}{\tau} = \pm \frac{v}{c}.$$

λ = wavelength, τ = period of oscillation, v = relative velocity of observer and source. The signs \pm hold when the distance between source and observer increases or decreases, respectively. The direction of propagation of the observed light is here assumed to be parallel to v , that is in the same direction or opposite to it. Since the Doppler effect is a purely kinematic phenomenon (it is true for sound as well as for light) we can apply it to the satellites of Jupiter with their periodic eclipses. The sunlight reflected by Jupiter itself plays no part in this consideration.

In our case, see fig. 13, τ is the orbital period of a satellite of Jupiter as measured from the moving earth. One of these satellites is denoted by the letter M in fig. 13 and its true orbital period will be called τ_0 . $A B C D$ are points on the earth's orbit. At B the earth is moving away from the satellite of Jupiter and the period of the latter appears to be lengthened by $\Delta \tau = v \tau_0 / c$. At D the earth is moving toward the satellite and τ appears to be shortened by $v \tau_0 / c$. At A and C where the velocity of the earth is perpendicular to the direction of the light coming from Jupiter, $\Delta \tau = 0$ and, hence, there the true period τ_0 is observed. The extreme values of τ are found at B and D and, according to (1), they are

$$(2) \quad \tau_{max} = \tau_0 + \frac{v}{c} \tau_0, \quad \tau_{min} = \tau_0 - \frac{v}{c} \tau_0.$$

From this follows

$$(3) \quad \tau_B - \tau_D = \tau_{max} - \tau_{min} = 2 \frac{v}{c} \tau_0$$

or if the value $2\pi R/T$ is substituted for v :

$$(4) \quad \tau_B - \tau_D = 4\pi \frac{\tau_0 R}{T c},$$

where T is the time interval of one year and R is the radius of the earth's orbit. Thus, this latter value must be known for the computation of c .

This result can be represented as in fig. 13 a and in this form it has been widely used in popular lectures. As the earth moves from A to C , it moves away from the eclipse-signals coming from Jupiter so that the intervals between receptions of such signals keep increasing. The opposite happens when the earth in its travel from C to A moves toward the eclipse-signals.

The total time difference in each case is equal to the time which light takes in moving through the diameter of the earth's orbit, that is

$$(5) \quad \Sigma \Delta \tau = \frac{2 R}{c}.$$

This is fairly close to the area¹ bounded by the sine curve above $A C$ (or below $C A'$) whose growth (and decay) is plotted separately at the bottom in the step-wise form in which it can be read directly from data tables.

Not until almost 200 years later were the first successful terrestrial determinations of c made. These were done by Fizeau using a rotating toothed wheel

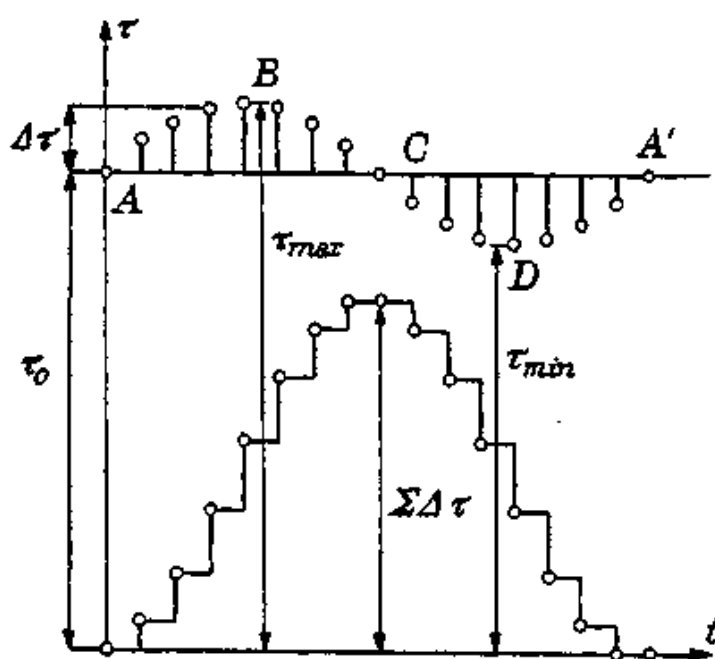


Fig. 13a.

The variations in the period of the light signals due to the earth's motion.

and upon returning is stopped by the following tooth, provided the rotation of the wheel is sufficiently fast) and by Foucault using a rotating mirror (a device used later by Michelson in his much more precise experiments).

The most important fact about the propagation of light is that it is independent of the state of motion of the emitting source, that the light velocity cannot "remember" the velocity of the source. Only wavelength and period of oscillation have such a "memory" according to eq. (1). This fact seemed to be

understandable in terms of the notion of a stationary light ether acting as the carrier of the propagation of light. It is now well known that

¹This area equals the product of the integral

$$(a) \quad \int_0^{T/2} \sin 2\pi \frac{t}{T} dt = \frac{T}{\pi}$$

and the ordinate of the sine curve at C which, according to (2), is

$$(b) \quad \Delta \tau_{max} = \frac{v}{c} \tau_0 = \frac{2\pi R}{c T} \tau_0.$$

The product of (a) and (b) divided by τ_0 does, indeed, yield the right hand side of eq. (5).

this theory did not stand up in the face of the experiments to be described in 12 and 14 and in the face of the theory of relativity.

In the following discussion it will be assumed that the reader has a certain amount of knowledge of the special theory of relativity. Such knowledge can be gained, for example, from Sec. 27 of Vol. III of this series. Without such knowledge the reader will have to omit several quantitative proofs in the following paragraphs.

10. Aberration and Parallax

By the *parallax* of a star we will here mean the so-called "yearly parallax", i. e. the solid angle of the cone which is formed by the lines of sight from different points on the earth's orbit to the star. The projection of this cone on the celestial sphere is the parallactic orbit which the star appears to describe in the course of a year. This orbit is generally a small ellipse and, in particular, it is a circle if the fixed star is at the pole of the ecliptic (as is assumed in fig. 14 a) and it is a straight line if the star lies in the ecliptic. Because of its importance as the final confirmation of the Copernican system, the proof of the existence of such an orbit was sought for a long time. While seeking this proof Bradley discovered the *aberration* of light in 1728.

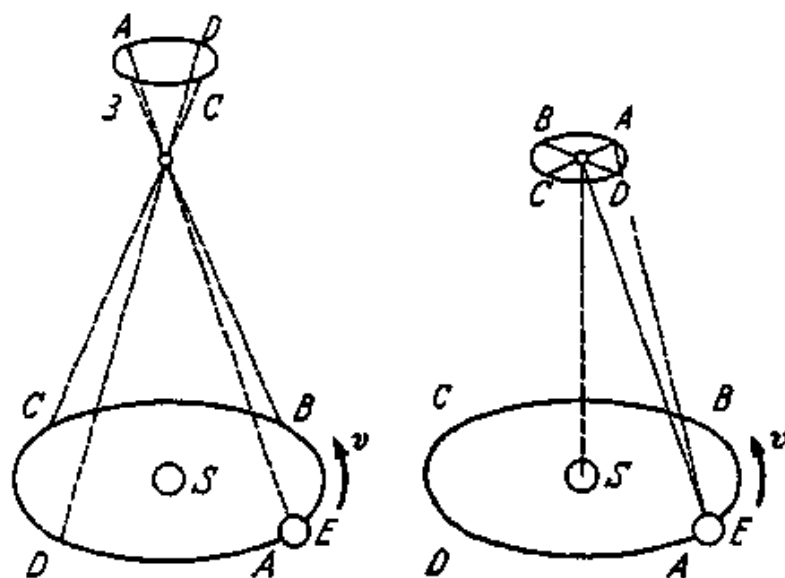


Fig. 14.

Apparent orbits of the star in the case of the North Star. a) The parallactic orbit arising from the star's finite distance from the sun. b) Orbit resulting from aberration of light. The apparent locations of the star ABCD belong to the corresponding points on the earth's orbit.

This phenomenon also causes the locus of a star to describe a small ellipse on the celestial sphere and this ellipse again degenerates into a sphere at the pole and into a straight line in the ecliptic. But the direction and magnitude

of the angular deviation are entirely different from those caused by parallax: the two directions are perpendicular to each other, see fig. 14 a, b; the magnitude of the deviation due to aberration is independent of the distance of the star and is much greater than the deviation due to parallax even for the fixed stars which are closest to the sun. The first confirmation of a real parallax of a fixed star was found 100 years later by Bessel.

Lenard's attempt to invoke aberration as a contradiction to the relativity of motion was an incomprehensible misunderstanding; especially so, as Einstein had derived aberration directly from the principle of relativity as early as 1905. Aberration does not reveal the "absolute motion" of the earth in space but rather the earth's relative motion during its yearly orbital period, that is to say, the differences in the direction of motion from one season to the next. Observatories are built (among other reasons) in order to make it possible to determine and measure these differences in motion. If these differences in direction of the earth's motion did not exist, that is, if the earth's motion were linear, then no aberration could be observed.

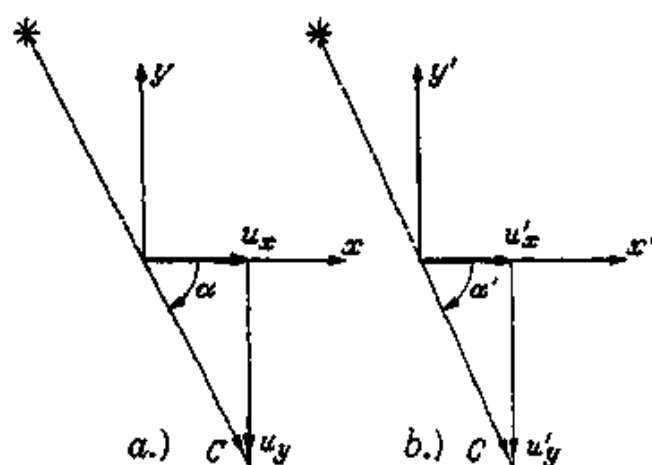


Fig. 15.

Figures for computing aberration a) heliocentric coordinate system, b) geocentric coordinate system x = direction of motion of the earth.

Still referring to fig. 14 b but removing the restriction that the star shall be at the pole of the ecliptic, we consider the plane containing the star and the direction of the earth's velocity. This is the plane of the drawing in fig. 15 a. We let the velocity of the earth be directed along the x -axis and call α the angle between the incoming ray and the x -axis. The coordinate system x, y, z , which is at rest with respect to the plane of fig. 15 a is a heliocentric system since both the fixed star and the sun are at rest with respect to it. If, however,

we move with the velocity v of the earth, we have to introduce a geocentric coordinate system x', y', t' (fig. 15 b). The fact that the time as well as the space coordinates are transformed is a characteristic feature of the theory of relativity and is made necessary by the constancy of c . The transformation between the two systems is the Lorentz transformation

$$(1) \quad x' = \frac{x - vt}{\sqrt{1 - \beta^2}}, \quad y' = y, \quad t' = \frac{t - \beta x/c}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c}.$$

Differentiating, we get

$$(2) \quad dx' = \frac{dx - v dt}{\sqrt{1 - \beta^2}}, \quad dy' = dy, \quad dt' = \frac{dt - \beta dx/c}{\sqrt{1 - \beta^2}}.$$

Hence, the geocentric velocity components

$$u_x' = \frac{dx'}{dt'}, \quad u_y' = \frac{dy'}{dt'}$$

are expressed in terms of the heliocentric components

$$u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}$$

by

$$(3) \quad u_x' = \frac{u_x - v}{1 - \beta u_x/c}, \quad u_y' = \frac{u_y \sqrt{1 - \beta^2}}{1 - \beta u_x/c}.$$

Now, instead of considering the velocities of material particles in the two systems we shall consider the velocities u and u' of the light radiated by the star as seen by observers in the respective systems. This is quite admissible. The heliocentric velocity is $u = c$ with the components (see fig. 15 a)

$$(3 a) \quad u_x = c \cos \alpha, \quad u_y = -c \sin \alpha.$$

We write the corresponding components of u' in the form (see fig. 15 b)

$$(3 b) \quad u_x' = u' \cos \alpha', \quad u_y' = -u' \sin \alpha'$$

where the magnitude of u' is left undetermined for the present. Taking the ratio of the two equations (3) and using (3 a), we obtain

$$(3 c) \quad \frac{u_y'}{u_x'} = \frac{-c \sin \alpha \sqrt{1 - \beta^2}}{c \cos \alpha - v},$$

which, according to (3 b), can be written

$$(4) \quad \tan \alpha' = \frac{\sin \alpha \sqrt{1 - \beta^2}}{\cos \alpha - \beta}.$$

Thus the angle of incidence α' as seen in the geocentric system is *different* from the angle α as seen in the heliocentric system. By squaring (3) it is seen that the geocentric velocity of light propagation u' is *equal* to the heliocentric velocity. For, substituting (3 a) we have

$$\begin{aligned} u'^2 &= \frac{c^2 \cos^2 \alpha - 2cv \cos \alpha + v^2 + c^2 \sin^2 \alpha (1 - v^2/c^2)}{(1 - \beta \cos \alpha)^2} \\ &= \frac{c^2 - 2cv \cos \alpha + v^2 \cos^2 \alpha}{(1 - \beta \cos \alpha)^2} = c^2. \end{aligned}$$

We could have written down this result without computation merely on the basis of Einstein's theorem on the addition of velocities. According to this theorem the following brief and seemingly paradoxical expression holds (see Vol. III, Sec. 27 F):

$$c + v = c.$$

Returning to the relationship between the angles α and α' , we write the first order approximation of (4) by neglecting terms of order β^2

$$(5) \quad \tan \alpha' = \frac{\sin \alpha}{\cos \alpha} \left(1 + \frac{\beta}{\cos \alpha} \right) = \tan \alpha + \frac{\beta \sin \alpha}{\cos^2 \alpha}.$$

Letting $\alpha' = \alpha + \Delta \alpha$, (5) now becomes

$$\tan (\alpha + \Delta \alpha) - \tan \alpha = \frac{\beta \sin \alpha}{\cos^2 \alpha}$$

and, expanding the left-hand side in terms of the small quantity $\Delta \alpha$, the denominator $\cos^2 \alpha$ on both sides cancels to give

$$(6) \quad \Delta \alpha = \beta \sin \alpha.$$

β is called the "constant of aberration", and is almost exactly equal to 10^{-4} if $\Delta \alpha$ is measured in radians. Hence, in degrees we have

$$\beta = 10^{-4} \frac{180^\circ}{\pi} = 20.5''.$$

If the fixed star is at the pole of the ecliptic, then $\alpha = 90^\circ$ all along the earth's orbit. Hence the aberrational orbit is, in this case, a circle about the pole of the ecliptic with radius $\Delta \alpha = \beta$. For stars in the plane of the ecliptic, which coincides with the xy -plane in fig. 15 a, α alternates between $\pm 90^\circ$ and 0. In this case also the aberration is in the plane of the ecliptic and oscillates between $\pm \beta$ passing twice through 0. For stars in general the aberrational orbit is an ellipse whose major axis is β and whose minor axis is $\beta \sin \delta$, where δ is the "celestial latitude" of the star (complement of its polar distance).

The above formulae, in particular eq. (2), make it clear that aberration is a direct consequence of the relativistic deviation of the time measure t' from the time measure t . From the point of view of classical kinematics it was difficult to reconcile aberration with the universality of c . Now we recognize it to be a necessary consequence of the fact that the light velocity is independent of the reference system. In the following paragraph aberration will come up again in connection with a more general point of view.

11. The Doppler Effect

The elementary explanation of the Doppler effect is well known. If a light source which is at rest emits waves of period τ , then the number of oscillations which meet a stationary observer during a time interval t is $N = t/\tau$. But if the observer moves toward the wave with a velocity v , thus covering

a distance vt in time t , he will meet an additional vt/λ oscillations. Therefore, altogether the moving observer encounters

$$(1) \quad N' = \frac{t}{\tau} + \frac{vt}{\lambda} = \frac{t}{\tau} \left(1 + v \frac{\tau}{\lambda} \right) = \frac{t}{\tau} \left(1 + \frac{v}{c} \right)$$

oscillations in the time t .

If, on the other hand, it is the light source which moves with velocity v toward the observer who is now at rest, then the spacing between two successive crests or troughs is no longer λ but, because the light source has progressed a distance $v\tau$ in time τ , this spacing is

$$(1a) \quad \lambda' = \lambda - v\tau = \lambda \left(1 - \frac{v}{c} \right).$$

The corresponding spacing in time is

$$(1b) \quad \tau' = \tau \left(1 - \frac{v}{c} \right).$$

Therefore, during the time t the observer at rest encounters

$$(2) \quad N'' = \frac{t}{\tau'} = \frac{t}{\tau} \frac{1}{1 - v/c} = \frac{t}{\tau} \left(1 + \frac{v}{c} + \frac{v^2}{c^2} + \dots \right)$$

oscillations. N' and N'' differ by terms of second and higher orders in $\beta = v/c$. They agree only up to terms of first order.

Against this argument can be said: nature knows no absolute motion, be it that of the light source or that of the observer. She gathers both cases (1) and (2) into the same law, which thereby becomes simpler and more beautiful. How this is accomplished we shall discover from the following consideration:

Every physical relation must be invariant with respect to the group of transformations which governs the particular domain being considered. If a relationship is expressed by means of an analytic function, then the argument of that function must be a *dimensionless scalar*. With this in mind, we consider the exponential function in the expression for a plane wave. Its argument is, aside from the factor i , the *phase* of the wave. In particular, this argument may be written in different forms representing various levels of generality as follows:

$$(3) \quad kx - \omega t, \quad \mathbf{k} \cdot \mathbf{r} - \omega t, \quad \vec{\mathbf{K}} \cdot \vec{\mathbf{R}}.$$

In the last of these expressions $\vec{\mathbf{R}}$ is the space-time four-vector

$$(4) \quad \vec{\mathbf{R}} = x_1, x_2, x_3, x_4 \quad \text{where} \quad x_4 = i c t.$$

\vec{K} is the wave number four-vector with the dimensions of an inverse length:

$$(5) \quad \vec{K} = k_1, k_2, k_3, k_4$$

$$\text{where } k_4 = \frac{i \omega}{c} = \frac{2\pi i}{\tau c} = \frac{2\pi i}{\lambda},$$

and the space components of \vec{K} are given by

$$(5a) \quad k_1, k_2, k_3 = \frac{2\pi}{\lambda} (\cos \alpha_1, \cos \alpha_2, \cos \alpha_3);$$

$\alpha_1, \alpha_2, \alpha_3$ are the angles which \vec{K} makes with the x_1, x_2, x_3 axes and

$$\cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 = 1.$$

From this it follows that the absolute value of \vec{K} is zero:

$$|k|^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2 = \frac{4\pi^2}{\lambda^2} (\cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 - 1) = 0.$$

We now view the wave from a primed coordinate system which moves with a velocity $v = \beta c$ in the x -direction with respect to the unprimed system. The Lorentz transformation (10.1) must then be applied to \vec{K} , where we must replace t by the quantity x_4/ic or, in the present case, by k_4/ic . Thus we obtain the following components of the transformed four-vector \vec{K}' :

$$(6) \quad k_1' = \frac{k_1 + i\beta k_4}{\sqrt{1-\beta^2}}, \quad k_2' = k_2, \quad k_4' = \frac{k_4 - i\beta k_1}{\sqrt{1-\beta^2}}.$$

These expressions are somewhat specialized in that we have taken $\cos \alpha_3 = 0$, that is the wave is assumed to propagate in the x_1, x_2 -plane, and we have omitted the equation $k_3' = 0$. For $\cos \alpha_3 = 0$ we get $\cos^2 \alpha_2 = 1 - \cos^2 \alpha_1 = \sin^2 \alpha_1$. If α_1', α_2' are the corresponding angles in the x_1', x_2' -plane, we also have $\cos^2 \alpha_2' = 1 - \cos^2 \alpha_1' = \sin^2 \alpha_1'$. From now on we shall write α, α' in place of α_1, α_1' .

Substituting eq. (5) in (6) and applying the definitions (5a), we get

$$(7) \quad \frac{\cos \alpha'}{\lambda'} = \frac{\cos \alpha - \beta}{\lambda \sqrt{1-\beta^2}}, \quad \frac{\sin \alpha'}{\lambda'} = \frac{\sin \alpha}{\lambda}, \quad \frac{1}{\lambda'} = \frac{1 - \beta \cos \alpha}{\lambda \sqrt{1-\beta^2}}.$$

Forming the ratio of the first two of these equations, we obtain

$$(7a) \quad \tan \alpha' = \frac{\sin \alpha \sqrt{1-\beta^2}}{\cos \alpha - \beta},$$

which is identical with the equation for the aberration (10.4), while the third of eqs. (7) represents the exact relativistic formulation of the Doppler principle.

We could have derived these equations in an even more elementary manner using the second expression (3) for the phase of the plane wave. We could, namely, require that

$$\mathbf{k}' \cdot \mathbf{r}' - \omega' t' = \mathbf{k} \cdot \mathbf{r} - \omega t,$$

write x_1', x_2', t' in terms of x_1, x_2, t by means of a Lorentz transformation, and then equate the coefficients of the terms x_1, x_2, t on both sides. We have favored the former method (using the covariance of the wave number vector instead of the invariance of the phase) because it expresses more clearly the relativistic four-dimensional origin of the Doppler equation.

For general directions of the velocity of the light source with respect to the observer or vice versa, we let $v \cos \alpha = v_n =$ projection of this velocity on the normal to the wavefront and according to (7), we get

$$(8) \quad \frac{\lambda}{\lambda'} = \frac{1 - v_n/c}{\sqrt{1 - \beta^2}}.$$

Letting $\Delta \lambda = \lambda' - \lambda$, it follows that

$$(9) \quad \frac{\Delta \lambda}{\lambda} = \frac{\sqrt{1 - \beta^2} - 1 + v_n/c}{1 - v_n/c}.$$

In the first order approximation this yields the well-known elementary expression for the Doppler effect

$$(10) \quad \frac{\Delta \lambda}{\lambda} = \frac{v_n}{c}.$$

A more detailed discussion of (9) will show that this equation contains not only the longitudinal Doppler effect in the case $v_n = \pm v$, which is a first order effect, but it also contains the *second order transverse Doppler effect*. For if $v_n = 0$, we have

$$\frac{\Delta \lambda}{\lambda} = \sqrt{1 - \beta^2} - 1 = -\frac{\beta^2}{2} + \dots;$$

recently this transverse effect has been measured accurately by means of the red shift of spectral lines (see Vol. III Sec. 27 D).

12. Fresnel's Coefficient of Drag and Fizeau's Experiment

Regarding the velocity of propagation of light in a moving transparent medium, the most obvious assumption suggested by the classical ether theory would have been that the velocity of light c/n ($n =$ index of refraction of the medium) is added to the velocity v of the medium. However, Fresnel, through ingenious reasoning, found the resulting velocity to be

$$(1) \quad u = \frac{c}{n} + v \left(1 - \frac{1}{n^2} \right).$$

The factor $(1 - 1/n^2)$ is called the "*Fresnel Coefficient of Drag*". The formula was completely confirmed by performing the *Fizeau experiment* in streaming water.

The light originating from the source L passes in two separate bundles of rays through the two pipes shown in fig. 16. In one of the pipes the light velocity is increased and in the other decreased and the resulting optical path difference can be measured at A by means of an interferometer.

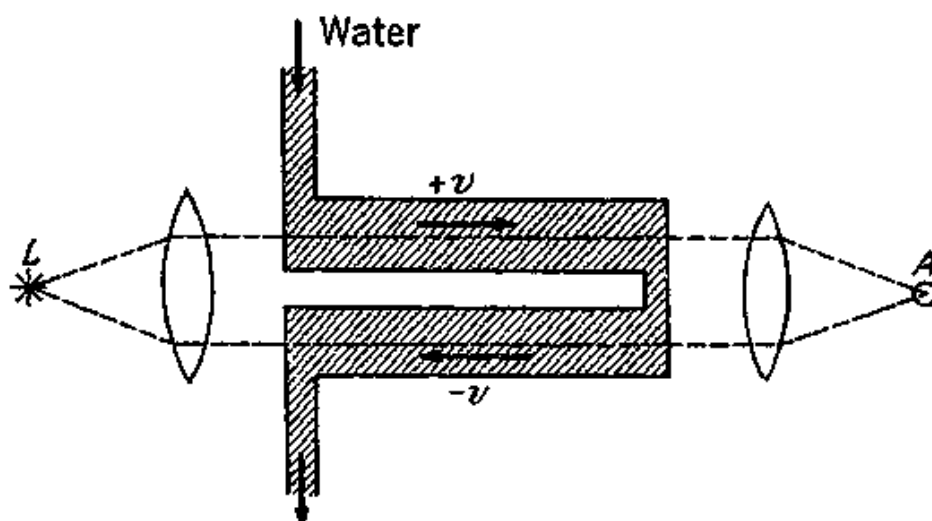


Fig. 16.

Fizeau's experiment for the determination of Fresnel's drag coefficient.

As was first noted by v. Laue¹, Eq. (1) can be explained purely phenomenologically on the basis of the velocity addition theorem as given by formula (27.15) Vol. III

$$(2) \quad u = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}$$

without making any special assumptions regarding the nature of the propagation of light in the moving medium. If in Eq. (2) v_1 is set equal to the phase velocity in water of index of refraction n , i. e. equal to c/n , and v_2 is set equal to the velocity of the water with respect to the reference system which is at rest in the laboratory, i. e. v_2 is $+v$ in the upper pipe and $-v$ in the lower one, then according to the addition theorem, the resulting velocities u are

$$(3) \quad u = \frac{\frac{c}{n} \pm v}{1 \pm \frac{v}{nc}}$$

¹Ann. d. Phys. 28, p. 989, 1907.

From this it follows in the first approximation if $v \ll c/n$, that

$$u = \frac{c}{n} \left(1 \pm \frac{vn}{c} \right) \left(1 \mp \frac{v}{nc} \right) = \frac{c}{n} \left(1 \pm \frac{vn}{c} \mp \frac{v}{nc} \right),$$

which, indeed, agrees with (1):

$$(4) \quad u = \frac{c}{n} \pm v \left(1 - \frac{1}{n^2} \right).$$

Lorentz¹ showed that this formula can be refined by combining it with the Doppler effect. In this way one obtains

$$(5) \quad u = \frac{c}{n} \pm v \left(1 - \frac{1}{n^2} - \frac{\lambda}{n} \frac{dn}{d\lambda} \right),$$

This equation is derived in the following way: n is not constant but is a function of the wavelength. One now considers a certain spectral line λ which is emitted by the light source L . As seen from a reference frame moving with the water, this spectral line is modified to $\lambda' = \lambda + \Delta\lambda$. One obtains, therefore,

$$(5a) \quad n(\lambda') = n(\lambda + \Delta\lambda) = n + \frac{dn}{d\lambda} \Delta\lambda.$$

In the upper pipe the water flows away from the light source and in the lower pipe it flows toward it. $\Delta\lambda$ is found from Eq. (11.10) by replacing c by the velocity of propagation c/n in water. Accordingly

$$\frac{\Delta\lambda}{\lambda} = \pm \frac{v}{c/n}; \quad \text{hence from (5a)} \quad n(\lambda') = n \pm \lambda \frac{dn}{d\lambda} \frac{v}{c} n,$$

and

$$\frac{c}{n(\lambda')} = \frac{c/n}{1 \pm \lambda \frac{dn}{d\lambda} \frac{v}{c}} = \frac{c}{n} \left(1 \mp \lambda \frac{dn}{d\lambda} \frac{v}{c} \right).$$

The numerator in Eq. (3) is thereby changed to

$$(6) \quad \frac{c}{n} \left(1 \mp \lambda \frac{dn}{d\lambda} \frac{v}{c} \pm n \frac{v}{c} \right).$$

The correction in the denominator would amount only to a term of second order in v/c , which may be neglected. As before, the inverse of this denominator reads

$$(6a) \quad 1 \mp \frac{v}{nc}.$$

¹Versuch einer Theorie der elektrischen und optischen Erscheinungen von bewegten Körpern, Leiden 1895, p. 101.

Multiplying (6) and (6 a) one obtains

$$(7) \quad n = \frac{c}{n} \left(1 \mp \lambda \frac{dn}{d\lambda} \frac{v}{c} \pm n \frac{v}{c} \mp \frac{v}{nc} \right),$$

which agrees with (5). Zeeman's¹ mastery of spectroscopy enabled him to verify this more exact formula experimentally.

The following general conclusion is to be drawn regarding the dragging of light in moving (ponderable and isotropic) bodies: as seen by an observer moving with the body, or at rest with respect to it the light propagates with the velocity c/n uniformly in all directions (first term of eq. (1)), regardless of whether the body is at rest or in a state of uniform motion. For an observer at rest in the laboratory with respect to which the body is moving with velocity v , a first order effect is added in the direction of the motion (this effect is given by the second term in eq. (1) or eq. (5) and is small compared to the first term by the order of magnitude v/c). As in the case of the transverse Doppler effect, an effect of second order exists in the direction perpendicular to the motion. Though not included in the Fresnel formula (1), this second order effect is easily computed by means of the addition theorem of velocities.

When $n = 1$ the first order effect disappears. A medium of index of refraction 1, however fast it may move (for instance, the so-called "ether wind"), has no effect at all on the propagation of light. This fact was once considered to be a proof that the ether was at rest and ponderable matter moved through it. According to that theory, only the charges associated with matter which find their expression in the index of refraction n were to affect the propagation. We know now that no particular assumptions need be made regarding the mechanism of the emission of light. The concepts of electron theory are, to be sure, useful for the visualization of the dragging term but they are in no sense necessary for its derivation.

13. Reflection by a Moving Mirror

The problem to be discussed in this section will serve as a preparation for the experiments to be described in 15 and will, moreover, be of help in connection with the thermodynamics of radiation which will be treated in Vol. V. (Wien's Displacement Law). We distinguish two cases: a) the mirror is moved in a direction tangential to its plane surface and b) it is moved in the direction perpendicular to its surface. In both cases the mirror will be assumed to be perfectly reflecting and its velocity will always be uniform.

¹Amsterd. Akademie Versl. 1914, p. 245 and 1915, p. 18.

a) We use the wave number four-vector defined in (11.5) referred to the "primed" system¹ which moves with the mirror, see fig. 17 a. In this system we call our vector k' and its components are

$$(1) \quad k_1', k_2' \text{ in the plane of incidence, } k_3' = 0, \quad k_4' = i\omega'/c.$$

We shall call the corresponding quantities describing the reflected ray \bar{k}' and \bar{k}_1' , etc., respectively. Since the mirror is at rest with respect to the primed coordinate system, the ordinary law of reflection holds:

$$(1') \quad \bar{k}_1' = k_1', \quad \bar{k}_2' = -k_2', \quad \bar{k}_3' = 0, \quad \bar{k}_4' = k_4'.$$

When observed from the laboratory with respect to which the mirror moves at a velocity $v = \beta c$ in the direction of the x -axis, the four-vectors describing the incident and reflected rays shall be denoted by k and \bar{k} and their components by $k_1 \dots k_4$ and $\bar{k}_1 \dots \bar{k}_4$, respectively. Inverting the transformation eq. (11.6) by replacing β by $-\beta$ and vice versa, we find for the incident wave

$$(2) \quad k_1 = \frac{k_1' - i\beta k_4'}{\sqrt{1-\beta^2}}, \quad k_2 = k_2', \quad k_4 = \frac{k_4' + i\beta k_1'}{\sqrt{1-\beta^2}}$$

and for the reflected wave, taking into account (1'),

$$(2') \quad \bar{k}_1 = \frac{k_1' - i\beta k_4'}{\sqrt{1-\beta^2}}, \quad \bar{k}_2 = -k_2', \quad \bar{k}_4 = \frac{k_4' + i\beta k_1'}{\sqrt{1-\beta^2}}.$$

If we denote the angles of incidence and reflection as measured in the laboratory system by

$$\alpha \text{ and } \bar{\alpha}, \text{ respectively,}$$

then by definition

$$(3) \quad \tan \alpha = \frac{k_1}{k_2}, \quad \tan \bar{\alpha} = -\frac{\bar{k}_1}{\bar{k}_2}.$$

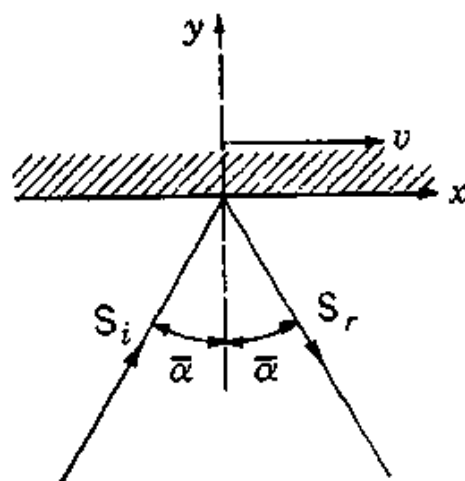


Fig. 17 a.

Reflection from a moving mirror. Direction of motion parallel to the plane of the mirror.

¹The convention by which the primed system is identified with the "moving" body and the un-primed system with the "stationary" laboratory is, though customary, entirely as arbitrary as the words "moving" and "stationary" themselves. Since we shall call the angle of incidence in the primed system α' , we must change our former notation for the angle of reflection (Chap. I). We will, therefore, distinguish the latter from the angle of incidence by a superposed bar.

From this it follows, according to eq. (2) and (2'), that

$$(4) \quad \bar{\alpha} = \alpha.$$

From the connection between ω' and k_4' given by (1) and the corresponding relationships between ω and k_4 and between $\bar{\omega}$ and \bar{k}_4 it follows that

$$(5) \quad \bar{\omega} = \omega.$$

Thus, for a mirror moving tangential to its surface the law of reflection which holds for stationary mirrors is preserved, and, seen from the laboratory system, the frequency of the light remains unchanged by reflection. However, α differs from the angle of incidence in the primed system α' by a small term of first order (which we could call the angle of aberration). Also the frequency ω differs somewhat from ω' because of the Doppler effect.

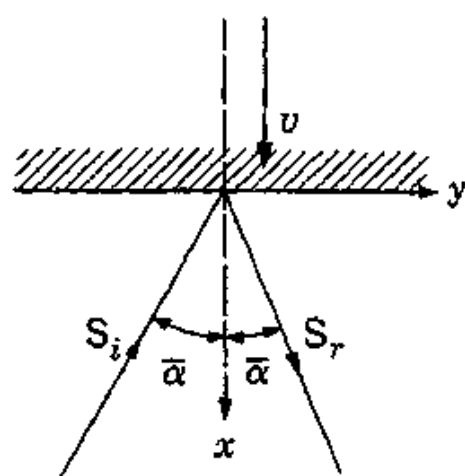


Fig. 17b.

Reflection from a moving mirror.
Direction of motion perpendicular
to the plane of the mirror.

b) Now let the mirror move in a direction perpendicular to its surface, for instance, forward against the incident light. The x -axis will again be in the direction of the velocity v , see fig. 17 b, and the y -axis will be in the plane of the mirror. In the primed coordinate system which moves with the mirror one finds, because of the interchange of indices 1 and 2, instead of (1')

$$(6) \quad \bar{k}_1' = -k_1', \quad \bar{k}_2' = k_2', \quad \bar{k}_3' = 0, \quad \bar{k}_4' = k_4' = i\omega'/c,$$

From the Lorentz transformation, taking (6) immediately into account, one obtains instead of (2) and (2')

$$(7) \quad k_1 = \frac{k_1' - i\beta k_4'}{\sqrt{1-\beta^2}}, \quad k_2 = k_2', \quad k_4 = \frac{k_4' + i\beta k_1'}{\sqrt{1-\beta^2}},$$

and

$$(7') \quad \bar{k}_1 = \frac{-k_1' - i\beta k_4'}{\sqrt{1-\beta^2}}, \quad \bar{k}_2 = k_2', \quad \bar{k}_4 = \frac{k_4' - i\beta k_1'}{\sqrt{1-\beta^2}}$$

In contrast to (3) the angles of incidence and reflection are now to be defined by

$$(8) \quad \tan \alpha = \frac{k_2}{-k_1} = \frac{k_2' \sqrt{1-\beta^2}}{-k_1' + i\beta k_4'}, \quad \tan \bar{\alpha} = \frac{\bar{k}_2}{\bar{k}_1} = \frac{k_2' \sqrt{1-\beta^2}}{-k_1' - i\beta k_4'}.$$

Thus, as observed from the laboratory, the angle of reflection differs from the angle of incidence. The same is true for the frequencies ω and $\bar{\omega}$.

In the case illustrated by fig. 17 b, $\bar{\omega} > \omega$; if v were in the opposite direction, then we would find that $\bar{\omega} < \omega$. This is readily understood by temporarily replacing the plane wave by a stationary point source at a finite distance and by considering the image of this source produced by the moving mirror. This image approaches the observer at a velocity $2v$ and hence the wavelength of the reflected light appears shortened due to the Doppler effect, and its frequency appears increased. If the mirror moves in the opposite direction, the whole situation is merely reversed. Correspondingly, $\bar{\alpha} < \alpha$ in the case illustrated by fig. 17 b, while $\bar{\alpha} > \alpha$ when the motion of the mirror is in the opposite direction (or, to speak more precisely, when the relative motions of mirror and observer are opposite).

Anticipating the corpuscular considerations of Sec. 16, we may point out a mechanical analogue: a tennis ball which falls obliquely on the racquet is reflected at a smaller angle than that at which it impinges. This is because the perpendicular component of the ball's velocity is increased by the forward motion of the racquet.

14. The Michelson Experiment

The most famous experiment in the field of optics of moving media is that of Michelson. For dates see the historical table in Sec. 1. After its repetition at Jena the negative result of this experiment can be considered as definitely established. The following will serve to indicate the degree of accuracy which was striven for: the apparatus was operated entirely automatically; in order to eliminate every possible temperature effect the apparatus was set up in a cellar of the Zeiss works and was inaccessible to the experimenter. Joos rightfully considered these precautions more important than the measures taken by D. C. Miller, another successor of Michelson. The latter placed his apparatus in a wooden shed on a high mountain in order to provide the "ether wind" with the freest possible passage through the apparatus. The apparatus used by Joos is now in the "Deutsches Museum" in Munich.

Michelson's experimental set-up is sketched in fig. 18. As in the case of other experiments of Michelson as well as of Perot-Fabry, the most important item is the semi-reflecting plate H . This plate allows the light coming from the lamp L to follow two different paths between L and B (observing telescope), namely,

$$L H S_1 H B \quad \text{and} \quad L H S_2 H B.$$

Since along both these paths the light passes once through H and is once reflected by H , the attenuation along both paths is the same and is given by

the product rd . It is, therefore, not necessary to obtain exact semi-transparency ($d = r$). Likewise, it is immaterial whether the mirrors S_1 and S_2 are precisely perpendicular to each other or not, a condition which, in any case, would be experimentally unattainable. Hence, we are here not

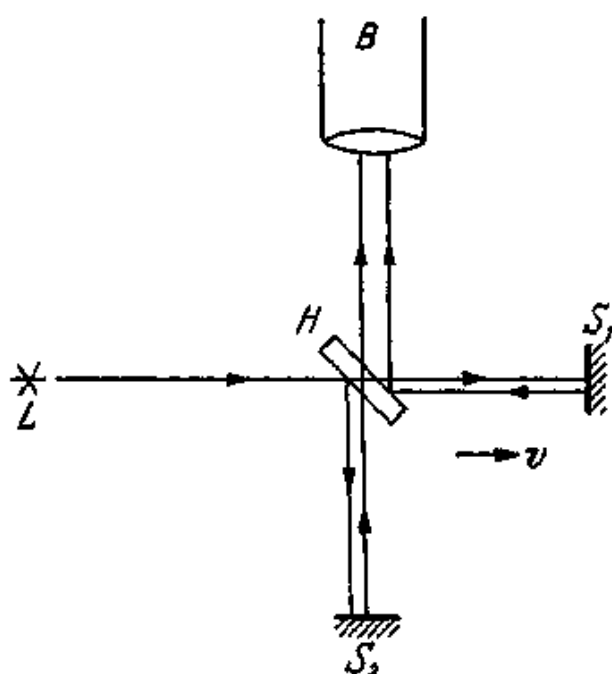


Fig. 18.

Michelson's experiment to prove that the velocity of light is independent of the earth's motion. L is a monochromatic light source. H is a semi-transparent plate. S_1 , S_2 are mirrors. B is the telescope used for observation.

concerned with interference arising from a plane-parallel air plate but rather with the kind of fringe pattern to be expected from an air space which is somewhat wedge shaped. Even the equality of the distances $l_1 = HS_1$ and $l_2 = S_2H$, though desirable, is not of critical importance (see footnote p. 78) and is never exactly attained. We shall, however, carry out all computations using $l_1 = l_2 = l$. In the experiment of Michelson and Morley the light paths were increased to eleven meters by repeated reflections. The whole apparatus floated on mercury¹. First it was oriented so that the direction LHS was parallel to the direction of the earth's motion about the sun. Then the whole apparatus was rotated through 90° and any possible shift in the

interference fringes was observed. *According to the theory of relativity no such shift of the fringes can occur.* This is due to the fact that the earth qualifies as a practically unaccelerated reference system and (in contrast to the experiments on aberration) the change of the earth's direction of motion during the duration of the experiment is negligible.

This is, however, not true from a non-relativistic viewpoint when the stationary reference system is assumed to be at rest with respect to the sun. For in this case, we would have to assert that the light always moves with a velocity c with respect to *this* reference system. The velocity of propagation of light with respect to the moving apparatus has then to be calculated. For this purpose the positions of H and S_1 for the first part of the experiment

¹In Joos' set-up the apparatus was suspended by springs. The arms l_1 and l_2 consisted of quartz glass, the light paths amounted to 21 meters and for a source the Hg-line $\lambda = 5461$ A.U. was used. See Ann. d. Phys. (Lpz) 7, p. 385, 1930.

($L S_1$ parallel to v) have been drawn in fig. 18 a in the following way. H is the position of the semi-transparent plate at the moment at which it is traversed by a certain phase of the monochromatic wave coming from L , a maximum for instance. S_1 is the simultaneous position of the mirror S_1' is the position of the mirror at the instant at which it reflects the above-mentioned phase. This reflection will occur a time t_1 later. H' is the position of H at that instant. H'' is the position of H at the time t_2 when the same phase returns to the semi-transparent plate. If we use the path lengths $S_1 S_1' = v t_1$, $H' H'' = v t_2$ shown in the figure and let $H S_1 = H' S_1' = l$, then ordinary non-relativistic kinematics yields

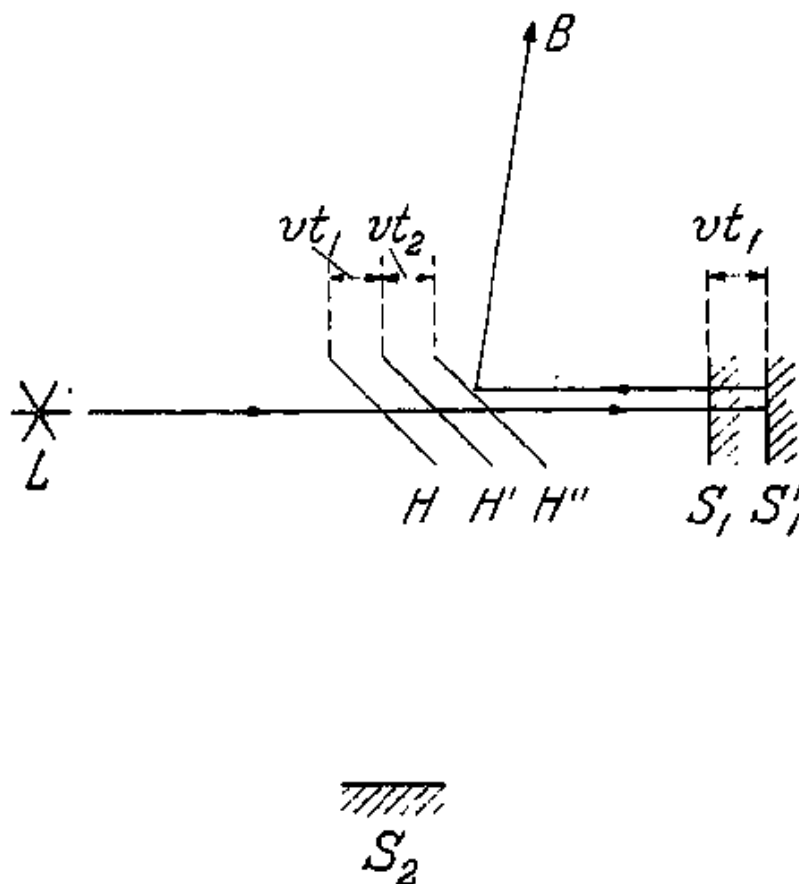


Fig. 18a.

Determination of the light path in Michelson's experiment. Ray parallel to the motion of the earth.

$$(1) \quad \begin{cases} l + v t_1 = c t_1, & t_1 = \frac{l}{c - v}, \\ l - v t_2 = c t_2, & t_2 = \frac{l}{c + v}. \end{cases}$$

Hence, the total time taken by the light is

$$(2) \quad t_1 + t_2 = \frac{l}{c - v} + \frac{l}{c + v} = \frac{2lc}{c^2 - v^2} = \frac{2l/c}{1 - \beta^2}.$$

The fact that the light actually meets the telescope B , which has itself moved forward, is due to the changed law of reflection at the moving H -mirror when in the position H'' ¹.

¹In this case the reflection is from a mirror which moves neither perpendicular to its surface as in 13a, nor parallel to it as in 13b. This mirror H is inclined at an angle of 45° with respect to its direction of motion. It is evident, however, that also in this case the event (arrival of the reflected light at B) which is observed on earth must be relativistically preserved if the reference system is moved to the sun.

For the computation of the other ray path we use fig. 18 b. Here H is the position of the semi-transparent plate at the time at which it reflects the same light phase, as before. S_2 is the position of the second mirror when this phase is reflected from it after a time t' , and H' is the simultaneous position of H . H'' is its position when the light again passes through it. Since, in this case, the reflection takes place at the mirror S which moves parallel to its surface, the ordinary law of reflection holds and

$$(3) \quad HH' = H'H'' = vt' \quad \text{and} \quad HS_2 = S_2H'' = \sqrt{l^2 + v^2 t'^2}.$$

On the other hand, in the stationary system of the sun it must be true that

$$(3a) \quad HS_2 = S_2H'' = ct'.$$

From (3) and (3a) follows $c^2 t'^2 = l^2 + v^2 t'^2$ and also

$$(4) \quad 2t' = \frac{2l/c}{\sqrt{1-\beta^2}}.$$

This time interval differs from that found in (2). The difference is only of second order in β ($\beta =$ aberration constant $= 10^{-4}$, see Sec 10), namely,

$$(5) \quad \Delta t = t_1 + t_2 - 2t' = \frac{l}{c} \beta^2$$

or, expressed as a light path length,

$$(5a) \quad c \Delta t = l \beta^2.$$

Nevertheless, this difference means that the phase under consideration reaches the observer B noticeably later when it goes by way of S_1 than when it goes by way of S_2 .

If we now rotate the apparatus through 90° , then S_2 takes the place of S_1 and vice versa. Thus the time intervals $t_1 + t_2$ and $2t'$ are interchanged¹

¹The arms l_1 and l_2 interchange in the same way even if they differ in length from one another ($l_1 = l$, $l_2 = l + \delta l$). In this case, however the term $2\delta l(1 + \beta^2/2)$ is added to (5a), and the part independent of β^2 cancels in (5a) and one obtains

$$(5c) \quad \Delta Z = \frac{2l}{\lambda} \beta^2 \left(1 - \frac{\delta l}{l}\right)$$

which agrees closely with (5b) as long as $\delta l \ll l$.

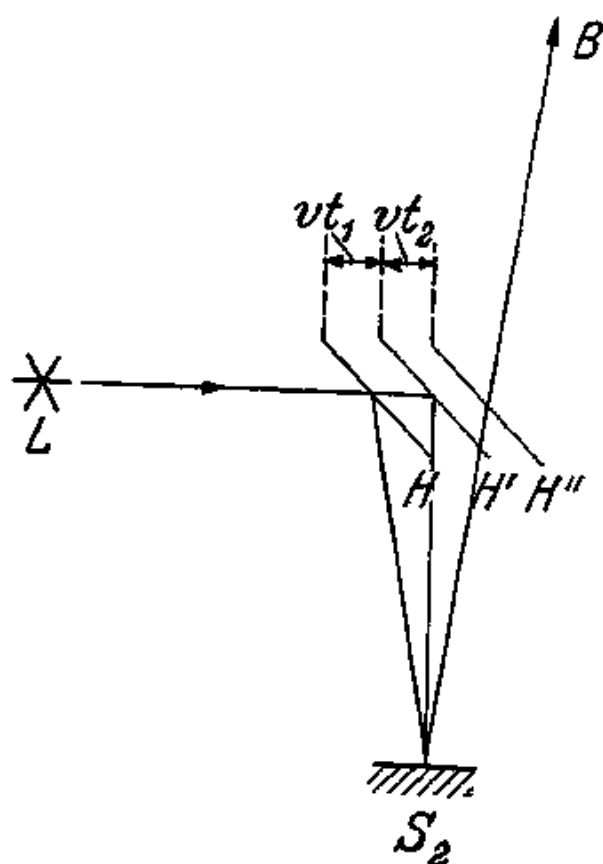


Fig. 18b.

Determination of the light path in Michelson's experiment. Ray perpendicular to the motion of the earth.

so that the expressions (5) and (5 a) change their signs. The difference in the time of arrival at B is thereby doubled. Hence B should observe twice the shift given by (5 a). Expressed in fractions of one whole fringe width, this amounts to

$$(5\ b) \quad \Delta Z = 2 \frac{c \Delta t}{\lambda} = 2 \frac{l}{\lambda} \beta^2.$$

For the above-mentioned values of $l = 21$ meters, $\lambda = 5461 \times 10^{-10}$ m, this gives

$$(6) \quad \Delta Z = 0.4.$$

In contrast to this result Joos summarized the result of the Jena experiments as follows: "We can say with a clear conscience that the upper limit of the effect due to any true ether wind which might still be possible on the basis of these experiments is 1/1000 of a fringe."

In order to reconcile this disagreement between theory and experiment, Lorentz and, independently, Fitzgerald found it necessary to introduce the following bold hypothesis: every moving body contracts in the direction of its motion by a factor $\sqrt{1-\beta^2}$. From Vol. III, Sec. 27 C we know that this "Lorentz contraction" is a general consequence of the principle of relativity and that it holds true, not only for the particular experiment here considered, but for all relative motions and for all space measurements parallel to such motion. Thus the Lorentz contraction is not an "ad hoc hypothesis". We should never have had to mention it had we used the relativistically correct kinematics immediately in (1) and had there replaced l by $l\sqrt{1-\beta^2}$. In eq. (3), where l is the length of the arm which is perpendicular to the direction of the motion, no such change would have been necessary.

15. The Experiments of Harress¹, Sagnac² and Michelson-Gale³

The negative result of Michelson's experiment has, of course, no bearing on the problem of the propagation of light in *rotating* media. To discuss this problem one must use not the special but rather the general theory of relativity with its additional terms which correspond to the mechanical centrifugal forces. However, in view of the fact that in the following experiments only velocities $v \ll c$ occur and only first order effects in v/c are important, relativity theory can be dispensed with entirely and the computations can be carried out classically.

¹Diss. Jena, 1912.

²Comptes Rendus, 1913.

³Astrophys. Journ., 1925.

The *Sagnac experiment* is the easiest to describe. As shown in fig. 19 the semi-reflecting plate H and the three mirrors S are mounted at the corners of a square which is inscribed in a disc. The plate H is mounted in a radial position, the mirrors S are mounted tangentially to the disc. The monochromatic light source L and the photographic plate Ph are, likewise, rigidly attached to the disc. The two rays emitted by L and separated by H are made to interfere at Ph . If the disc is made to rotate, then the ray going

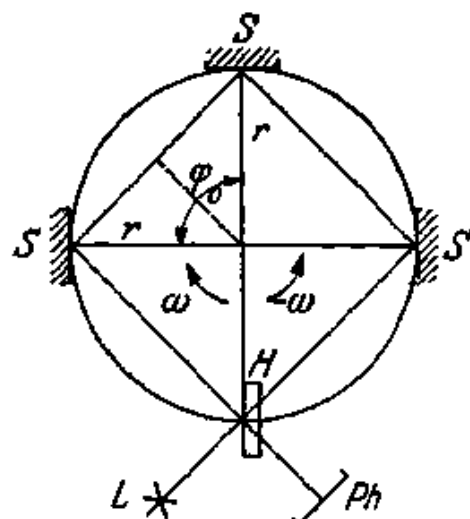


Fig. 19.

Sagnac's experiment. The interference arrangement consists of the light source L , the semi-transparent plate H , three mirrors S , and the photographic plate Ph . The entire set-up rotates with angular velocity $\pm \omega$ about a central axis perpendicular to the plane of the paper.

ω in the same direction, φ_- for ω opposite to the direction of the ray:

$$\varphi_0 = \frac{\pi}{2}, \quad \varphi_{\pm} = \frac{\pi}{2} \pm \frac{1}{4} \omega \tau_{\pm},$$

where τ_{\pm} is the time which it takes the corresponding ray to travel its path $L \rightarrow Ph$. Neglecting the distances LH and HPh which are the same for both rays, the length of this path is equal to $c\tau_{\pm}$. It is also equal to four times one side of the square, where this side has been lengthened or shortened by the rotation. Thus:

$$c\tau_{\pm} = 4 \cdot 2r \sin \frac{1}{2} \varphi_{\pm} = 8r \sin \left(\frac{\pi}{4} \pm \frac{\omega}{8} \tau_{\pm} \right).$$

in the same direction as the rotation is made to travel a longer path and the ray which is oppositely directed travels a shorter path. Thus interference fringes are formed whose positions differ for oppositely directed rotations. If this shift ΔZ in the fringes is measured, it is found to obey the following theoretical formula:

$$(1) \quad \Delta Z = 4\beta \frac{F}{r\lambda};$$

F is the surface enclosed by the ray path, i. e. a square in Sagnac's experiment; r is the radius of the disc; v is the velocity of the disc's circumference and $\beta = v/c$.

In order to prove eq. (1) we note that owing to the law of reflection at tangentially moving mirrors, the four sides of the "square" (which for a rotating disc is no longer a closed figure) subtend the same central angle; namely, φ_0 if $\omega = 0$, φ_+ for

From this one finds the time difference to be

$$\begin{aligned} \Delta \tau &= \tau_+ - \tau_- = \frac{8r}{c} \left\{ \sin \left(\frac{\pi}{4} + \frac{\omega}{8} \tau_+ \right) - \sin \left(\frac{\pi}{4} - \frac{\omega}{8} \tau_- \right) \right\} \\ (2) \quad &= \frac{16r}{c} \cos \left(\frac{\pi}{4} + \frac{\omega}{16} \Delta \tau \right) \sin \frac{\omega}{16} (\tau_+ + \tau_-). \end{aligned}$$

Neglecting small quantities on the right-hand side, one can put

$$\begin{aligned} \cos \left(\frac{\pi}{4} + \frac{\omega}{16} \Delta \tau \right) &\sim \cos \frac{\pi}{4} \\ \sin \frac{\omega}{16} (\tau_+ + \tau_-) &\sim \sin \frac{\omega}{8} \tau_0 \sim \frac{\omega}{8} \tau_0, \end{aligned}$$

where τ_0 is the time taken by the light if the disc is at rest and is given by

$$\tau_0 = \frac{8r}{c} \sin \frac{\pi}{4}.$$

Thus (2) becomes

$$(3) \quad \Delta \tau = \frac{8\omega r^2}{c^2}.$$

Since $\omega = v/r$ and $F = (r\sqrt{2})^2$, one can write instead

$$(3a) \quad \Delta \tau = 4\beta \frac{F}{rc}.$$

This expression becomes identical with (1) if one computes the shift of fringes ΔZ from the time difference $\Delta \tau$.

We could have shortened the above calculations had we started out by using the Doppler effect. The Doppler effect originates in the semi-reflecting plate H which acts as a moving light source emitting different wavelengths in the forward and backward directions (while no additional Doppler effect is caused by the tangentially moving mirrors). The shift in the fringes ΔZ is due to the difference in wavelength between the ray going with and against the direction of rotation.

In Harress' experiment a number of glass prisms were arranged along the circumference of the disc. The same formula (1) again applies, where, however, F is the area bounded by the polygon that is described by the ray in its path from prism to prism. Formula (1) was entirely confirmed by the results of both experiments.

In the experiment of Michelson and Gale the *earth* plays the role of the rotating disc. The component of the earth's angular velocity along the perpendicular at the place of observation takes the place of ω . Preliminary experiments on Mt. Wilson showed that if the necessarily long light paths were placed in free air, then even under the best atmospheric conditions the interference fringes were too unsteady to make measurements possible. Hence, it was necessary "to resort to a pipe line about one mile long and one foot in diameter which could be exhausted of air". The area F was a rectangle with sides of 340 and 610 meters. The mirrors S and the semi-transparent plate H were attached at the corners of this rectangle. In order to obtain a zero setting for the fringe shift, a comparison path enclosing only a small area was provided. A total of 269 observations yielded a mean shift of $\Delta Z = 0.230 \pm 0.005$ fringes which is again in full agreement with eq. (1).

This experiment is a beautiful analogue to *Foucault's Pendulum experiment*. While the translatory motion of the earth cannot be noticed either mechanically or optically, the earth's rotation is measurable both mechanically according to Foucault and optically according to Michelson-Gale.

16. The Quantum Theory of Light

At the end of the seventeenth century Huygens' wave theory and Newton's corpuscular theory entered upon a period of competition. Although the corpuscular theory prevailed during the eighteenth century, at the beginning of the nineteenth century the interference experiments of Thomas Young brought about the victory of the wave theory. But with the start of the twentieth century a rebirth of the corpuscular theory was brought about by the work of Einstein: *Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt*, Ann. d. Phys. (Lpz) 17, 1905. (On a heuristic viewpoint concerning the creation and conversion of light.)

This paper was much more radical than the theory of relativity which had its origin in the same year. While the latter represented the crowning achievement of classical physics, the former revolutionized it.

In 1887 Hertz discovered the photoelectric effect and soon afterwards it was measured electrostatically by Hallwachs. The explanation of this effect on the basis of electron theory as given by Lenard and J. J. Thompson led to the following results: the number of electrons ejected from a metal plate by light depends on the light *intensity* but the kinetic *energy* of these electrons is solely determined by the frequency of the incident light. Einstein, applying Planck's discovery of the quantum of action h and the quantum of energy $h\nu$,

interpreted these results in the following way: the upper limit v of the velocity spectrum of the photoelectrons is given by the energy equation

$$(1) \quad h\nu = \frac{m}{2}v^2 + A$$

where A is the minimum energy required to separate the electron from the metal. Only when $h\nu > A$ does a photoelectric effect appear. Ultraviolet light always yields a photoelectric effect while only the alkali metals, which have a small A , are photoelectrically active under red light. In 1916 Millikan confirmed the existence of such an upper bound of the velocity spectrum precisely and used it to determine h .

Thus a new elementary particle, the "photon", was introduced into physics. Its energy is

$$(1a) \quad E = h\nu.$$

Since this particle always moves with the velocity c , we must ascribe to it the rest mass $\mu_0 = 0$; otherwise its velocity mass $\mu = \mu_0/\sqrt{1 - \beta^2}$ would become infinite. From the general relation between mass and energy

$$E = (\mu - \mu_0) c^2$$

the mass is found to be $\mu = h\nu/c^2$ and the momentum

$$(2) \quad p = \mu c = \frac{h\nu}{c}.$$

In his original paper Einstein also called attention to *Stokes rule* for *fluorescence*: the frequency of the fluorescent light is always displaced toward the red with respect to that of the exciting light. This rule is also generally true for *phosphorescence bands* (delayed fluorescence) and for the *characteristic radiation frequencies* in the *X-ray spectrum*. To excite, for instance, the K radiation of an atom the exciting radiation must be harder than the hardest line of the K spectrum.

The *continuous X-ray spectrum* is produced by a kind of reversal of the photoelectric effect. While in the ordinary photoelectric effect primary photons produce secondary electrons, we now have the case of primary electrons (cathode rays of energy E) impinging on a target electrode where they excite the secondary photons of the continuous X-ray spectrum. The following Stokes rule is obeyed:

$$(3) \quad h\nu < E, \quad h\nu_{\max} = E.$$

Therefore, the continuous X-ray spectrum has a short-wave limit $\lambda_{\min} = c/\nu_{\max}$ which can again be used in connection with eq. (3) to determine h . In contrast

to this the classical computation of the radiation given off by a decelerating electron, as performed in Vol. III, eq. (19.22) or (30.11), always yields a spectrum which is continuous up to $\nu = \infty$.

The following consideration is to be added: the "accumulation period" which would be necessary if the energy provided by single cathode ray electrons were to accumulate to an X-ray energy of $h\nu$ would become very long — of the order of several hours! But actually the secondary X-rays are emitted simultaneously with the primary cathode rays just as the photoelectric effect starts immediately with the primary illumination. As a last desperate attempt to save the classical theory of radiation, Debye and the author undertook in 1913 to explain the photo effect classically by using a special hypothesis regarding the action integral¹.

Since that time it has, of course, become possible to register directly the discontinuous quantum nature of weak X-rays or ultraviolet light through the amplifying action of the *counter tube*. It is even possible to make the clicks associated with the separate discharges audible. We will not discuss here the *Compton effect* which makes the corpuscular nature of X-rays especially evident. Rather we shall limit our discussion to those effects which have already been explained wave-theoretically in this chapter. In the case of the moving mirror we have already indicated the possibility of a corpuscular explanation by using the example of the tennis ball at the end of Sec. 13. While our previous derivations of the aberration and dragging effects were ultimately based upon the velocity addition theorem, these two phenomena can also be easily explained on the basis of the corpuscular theory. But how about the Doppler effect with its expansion or crowding of wave surfaces which seems to require a definitely wave-theoretical explanation? Schrödinger² showed that this effect also can be understood from a photon point of view.

We shall assume that a radiating atom O, instead of emitting a spherical wave, sends out photons of energy $h\nu$ and momentum $h\nu/c$ in random directions. In this way such a photon will occasionally also travel in the direction of the observer P. When this is the case, the atom recoils in the direction PO. We shall assume here that the observer is at rest and the emitting atom is in motion, though we could equally well treat the reverse situation. The recoil momentum $h\nu/c$ combines with the original momentum of the atom Mv_1 to give Mv_2 . Let α be the angle between OP and v_1 and $\alpha + d\alpha$ the angle between OP and v_2 . We construct the momentum triangle

¹Ann. d. Phys. 41 (Lpz), 1913, see also: First Solvay Congress, "Theorie du rayonnement et des quanta", p. 344. These efforts had to fail, of course, because of the tremendous length of the required accumulation period.

²Physikal. ZS. 23, 301, 1922.

OAB , as shown in fig. 20, where $OA = Mv_1$, $OB = Mv_2$, $AB = h\nu/c$. We now project OB on OA and find that for the infinitesimal right triangle ABC the following relation holds:

$$(4) \quad M \Delta v = \frac{h\nu}{c} \cos \alpha.$$

This is the law of conservation of momentum. The conservation law of energy is:

$$(5) \quad \frac{M}{2} v_1^2 + E_1 = \frac{M}{2} v_2^2 + E_2 + h(\nu + \Delta \nu)$$

where $h(\nu + \Delta \nu)$ is the energy emitted by the moving atom. Quantum-theoretically the energy of the photon emitted by an atom which is at rest or moving uniformly ($v_1 = v_2$) and which experiences a change in its configuration $E_1 \rightarrow E_2$ is given by

$$(5a) \quad E_1 - E_2 = h\nu.$$

Substituting this in (5) one obtains

$$(6) \quad h \Delta \nu = \frac{M}{2} (v_1^2 - v_2^2) = M \Delta v \frac{v_1 + v_2}{2}.$$

Neglecting $(\Delta \nu)^2$ we let $(v_1 + v_2)/2 = v$ and obtain, from (4),

$$(7) \quad h \Delta \nu = h\nu \frac{v}{c} \cos \alpha.$$

Characteristically, h cancels out and we obtain the Doppler formula which is identical with (11.10)

$$(8) \quad \frac{\Delta \nu}{\nu} = \frac{v}{c} \cos \alpha$$

The reader may convince himself that the significance of the sign mentioned in connection with (11.10) is also in agreement with fig. 20.

This derivation appears to be inconsistent in that the recoil momentum was assumed to be $h\nu/c$ instead of $h(\nu + \Delta \nu)/c$. However, if we use the latter value, the result will differ from our present result only in terms of second order, i. e. terms proportional to $(v/c)^2$. If we had considered such

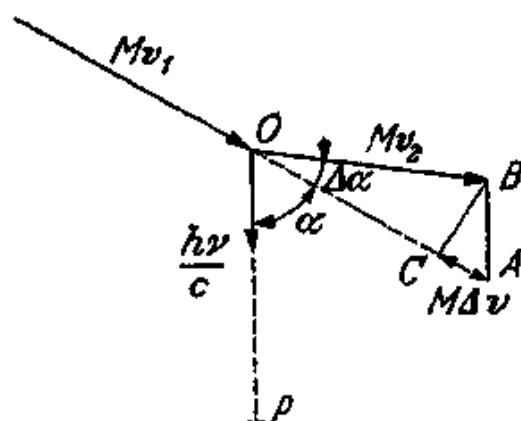


Fig. 20.
The corpuscular explanation of the Doppler effect.

terms, however, we should have had to carry out our calculations relativistically from the very beginning. In particular, the kinetic energy of the atom should have been set up differently. As Schrödinger has pointed out, we would then have obtained the relativistically rigorous Doppler formula, that is eq. (11.9).

From an epistemological point of view we are thus faced with an extremely remarkable situation: the phenomena described in this chapter, and in particular the Doppler effect, can be understood in terms of either the wave theory or the corpuscular theory. This is also true of the light pressure which was treated wave-theoretically by means of Poynting's energy flux in Vol. III, Sec. 51. According to the corpuscular theory this pressure can be described very vividly in terms of a "photon-hail". However, the wave theory completely fails to explain the photoelectric effect and the most important results of X-ray spectroscopy. On the other hand, the photon theory, at least in its present state of development, is unable to account precisely for polarization and interference phenomena. *Therefore, we are forced to adopt a dualistic conception of light: not Huygens or Newton, but Huygens and Newton.* Newtons' theory explains the coarse but nevertheless fundamental *energy* problems while Huygens' theory is of use for the much more delicate problems concerning *interference*. Light has a *dual nature*; it presents us with either its corpuscular or its wave aspect depending on the particular question which we are posing. It is wrong to ask which of these aspects is the *true* one. As far as we know today, *they are both on an equal footing* and only both aspects taken together are capable of representing the nature of light completely.

One speaks, therefore, not of a *duality* of light but, more appropriately, of its *complementarity*. This expression, which was coined by Bohr, is all the more appropriate because we know today that also *electrons and all material bodies* possess in addition to their corpuscular character and on an equal basis with it also a wave-mechanical nature. There are *matter waves* as well as *light waves*. Our previous characterization of cathode rays as corpuscles and X-rays as waves was an antiquated way of speaking. The difference between these two lies not in their wave character but rather in their velocities and charges and consequently their very different interactions with the atoms of material bodies.

It is clear that this complementarity overthrows the scholastic ontology. What is truth? We pose Pilate's question not in a skeptical, anti-scientific sense, but rather in the confidence that further work on this new situation

will lead to a deeper understanding of the physical and mental world. Indeed, the revised quantum theory and Heisenberg's uncertainty principle on which it is based demonstrate that there can be no logical contradiction between the corpuscular and wave-theoretical viewpoints.

In the following chapters we shall only rarely have occasion to return to these fundamental questions. We shall have to restrict ourselves to a development of the wave theory. However, we should always remember that, though the latter forms the most important practical part of optics, it does not reveal its full content.

CHAPTER III

THEORY OF DISPERSION

So far we have discussed only the *nature of light*. Now we shall investigate more closely the nature of *refractive media*. We have already mentioned in connection with eq. (3.4 a) that the electromagnetic explanation of the index of refraction is inadequate since it even fails to account for the decomposition of white light in a prism. We can understand such phenomena only by learning more about the optical properties of matter.

The electric composition of matter is well known: every atom consists of a positive nucleus and a shell of more or less mobile electrons. However, we need not go into electron theory in the usual sense. We can carry out our calculations using, instead of individual electrons, an *electron fluid*¹ which is spread uniformly throughout the whole body. The situation here is analogous to the theory of hydrodynamics in which a continuous density replaces the individual molecules. In the same way the electrons will be thought of as "smeared-out"² into a continuum.

We shall treat the charges of the positive ions in the same way. They will serve the purpose of neutralizing the enormous electrostatic repulsive forces which would otherwise act within the electron fluid, and conversely the latter will neutralize the electrostatic repulsion of the ions. This point of view, which is usually adopted in dispersion, is entirely justified in the optical spectrum where a cube one wavelength in dimension contains a tremendously large number of atoms; in the X-ray region, however, this mode of attack fails.

In hydrodynamics we use a volume element to define displacement and velocity; the definition of displacement in the electron fluid will be explained in eq. (17.2); the corresponding definition for ions will be found in the beginning of Sec. 18.

¹Using a currently popular word one could speak of an electron "plasma".

²I am afraid that this ugly word which I used in my lecture in 1912 has come into general usage. At that time I posed to P. Ewald, as a theme for his dissertation, the problem of explaining double refraction and dispersion in crystals in terms of their lattice structures, that is not to smear out the electrons but to assume them to be bound to the individual building stones of the crystal. There was a close causal connection between this dissertation and the ingenious idea of M. von Laue to investigate the lattice structure of crystals with X-rays. See also Chap. V, Sec. 32 C.

17. Ultraviolet Resonance Oscillations of the Electrons

Let us now investigate a transparent isotropic non-conducting material. The optical field will be described by the two light vectors \mathbf{E} and \mathbf{H} . Excluding, for the present, magnetizable materials we set $\mathbf{B} = \mu_0 \mathbf{H}$. However, we do not set $\mathbf{D} = \epsilon \mathbf{E}$ as in the case of slowly varying fields, but rather more generally we let

$$(1) \quad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}.$$

\mathbf{P} is the *polarization* vector which was introduced in Vol. III, Sec. 11. Polarization means that the field \mathbf{E} displaces the electrons from their rest positions. We shall call the displacement vector \mathbf{s} and set

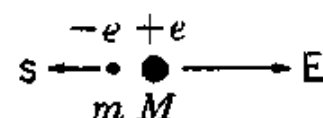
$$(2) \quad \mathbf{P} = -N e \mathbf{s}.$$

$-e$ is the charge of the electron and N is the number of dispersion electrons per unit volume. These definitions of \mathbf{P} and \mathbf{s} obviously presuppose individual electrons rather than a continuum. Equation (2), therefore, refers to the state of affairs before the "smearing-out".

Let us note that this assumption is dimensionally correct. The dimension of \mathbf{P} is, like that of \mathbf{D} and $\epsilon_0 \mathbf{E}$, Q M^{-2} (see table at the end of Sec. 2). And indeed, since N has the dimensions M^{-3} , $N\mathbf{s}$ has the dimensions M^{-2} . Let us note further that the sign in (2) is chosen correctly. Figure 21 represents the action of the field \mathbf{E} upon the electronic charge $-e$. This charge is separated from the ionic charge $+e$, with which it originally coincided, by being displaced in a direction opposite to the displacement of the positive charge. We can assume the ionic charge $+e$ which carries the ionic mass M to be stationary. Thus an electric moment $(+e, -e)$ with moment arm $|\mathbf{s}|$ and directed parallel to \mathbf{E} is created as represented in eq. (2).

Fig. 21.

Separation of the electron charge (mass m) and ion charge (mass M) in the electric field \mathbf{E} .



In fig. 21 we assume that the electrons are bound "quasi-elastically" to their rest positions, so that they will seek to return to those positions when displaced from them by a field \mathbf{E} . Therefore \mathbf{s} satisfies the following differential equation:

$$(3) \quad m \ddot{\mathbf{s}} + f \mathbf{s} = -e \mathbf{E}.$$

Here the restoring force $-f \mathbf{s}$ has been put on the left-hand side with the opposite sign. On the right-hand side is the field force acting on $-e$. Instead of (3) we shall write

$$(3a) \quad \ddot{\mathbf{s}} + \omega_0^2 \mathbf{s} = -\frac{e}{m} \mathbf{E} \quad \text{where} \quad \omega_0^2 = \frac{f}{m}.$$

ω_0 is the characteristic frequency of the electron (and therefore also of the electronfluid) which the title of this paragraph asserts to be in the far *ultraviolet*. This is particularly the case for the gases H_2 , N_2 , O_2 , etc. Written in terms of the polarization eq. (3 a) becomes:

$$(4) \quad \left(\frac{\partial^2}{\partial t^2} + \omega_0^2 \right) \mathbf{P} = \frac{N e^2}{m} \mathbf{E}.$$

Thus we have not two, but *three* light vectors, \mathbf{E} , \mathbf{H} and \mathbf{P} which are connected by three vector differential equations. For we have, besides (4), the two Maxwell equations

$$(5) \quad \mu_0 \dot{\mathbf{H}} = -\text{curl } \mathbf{E} \quad \text{and} \quad \epsilon_0 \dot{\mathbf{E}} + \dot{\mathbf{P}} = \text{curl } \mathbf{H}.$$

In the second of these equations $\dot{\mathbf{D}}$ is already expressed in terms of \mathbf{E} and \mathbf{P} by means of (1). Eliminating \mathbf{H} from the two eqs. (5), we obtain, taking account of $\text{div } \mathbf{E} = 0$ (uncharged dielectric)

$$(6) \quad \epsilon_0 \mu_0 \ddot{\mathbf{E}} + \mu_0 \ddot{\mathbf{P}} = -\text{curl curl } \mathbf{E} = \Delta \mathbf{E},$$

see Vol. III eq. (6.2). We need now only eliminate \mathbf{P} from eqs. (4) and (6) to obtain a pure differential equation for \mathbf{E} . For this purpose we operate on (6) and (4) with the operators $\partial^2/\partial t^2 + \omega_0^2$ and $\mu_0 \partial^2/\partial t^2$, respectively. Thus we get the following fourth order differential equation:

$$(7) \quad \left(\frac{\partial^2}{\partial t^2} + \omega_0^2 \right) \left(\frac{1}{c^2} \ddot{\mathbf{E}} - \Delta \mathbf{E} \right) + \frac{\mu_0 N e^2}{m} \ddot{\mathbf{E}} = 0.$$

We specialize this immediately to the case of a linearly polarized plane wave of frequency ω and wave number k

$$(8) \quad E_y = A e^{i(kx - \omega t)}, \quad E_x = E_z = 0.$$

Then (7) yields the following algebraic relation between ω and k :

$$(-\omega^2 + \omega_0^2) \left(-\frac{\omega^2}{c^2} + k^2 \right) = \frac{\mu_0 N e^2}{m} \omega^2,$$

which, when solved for k^2 , gives

$$(9) \quad k^2 = \frac{\omega^2}{c^2} \left(1 + \frac{\mu_0 c^2 N e^2 / m}{\omega_0^2 - \omega^2} \right).$$

Now, $u = \omega/k$ is the phase velocity of our plane wave (8) in the dispersive medium. The index of refraction of this medium relative to the vacuum is, according to the definition (3.4),

$$(9a) \quad n = \frac{c}{u} = \frac{ck}{\omega}.$$

Setting $\mu_0 c^2$ equal to $1/\epsilon_0$ we write instead of (9)

$$(10) \quad n^2 = 1 + \frac{N e^2 / m \epsilon_0}{\omega_0^2 - \omega^2}.$$

Thus, the *index of refraction has become frequency dependent*. This is what is meant by *dispersion*. Because

$$\omega_{\text{red}} < \omega < \omega_{\text{violet}} < \omega_0$$

the denominator $\omega_0^2 - \omega^2$ is positive in the entire visible spectrum and is *larger* at the red end than at the violet end. *Blue light is refracted more than red light*. This is normal dispersion.

Assuming ω_0 very large, we now develop (10) in powers of ω/ω_0 and retain only the first two terms:

$$(11) \quad n^2 = 1 + \frac{N e^2}{m \epsilon_0 \omega_0^2} \left(1 + \frac{\omega^2}{\omega_0^2} \right).$$

Setting $\omega = \frac{2\pi c}{\lambda}$, where λ is the wavelength in vacuum, we are led to a formula which corresponds to an old molecular elastic theory of Cauchy (ca. 1830). Abbreviating, we write

$$(12) \quad n^2 = 1 + A \left(1 + \frac{B}{\lambda^2} \right), \quad A = \frac{N e^2}{m \epsilon_0 \omega_0^2}, \quad B = \frac{4\pi^2 c^2}{\omega_0^2}.$$

A and B are called the *coefficients of refraction* and *dispersion*, respectively. We note that the ratio B/A does not contain the characteristic frequency ω_0 , so that the value of this ratio is the *universal* number

$$(12a) \quad \frac{B}{A} = \frac{4\pi^2 c^2 \epsilon_0}{N e^2 / m}.$$

Its dimension is M^2 , as can be ascertained directly from (12).

We shall now compare (12) and (12a) with very exact measurements¹ of the dispersion in hydrogen which yield

$$(13) \quad n^2 = 1 + 2.721 \times 10^{-4} + \frac{2.11}{\lambda^2} 10^{-18}.$$

This gives the values

$$B = \frac{2.11}{2.721} 10^{-14} M^2, \quad \frac{B}{A} = \frac{2.11}{(2.721)^2} 10^{-10} = 0.29 \times 10^{-10} M^2.$$

¹ J. Koch, Nova Acta Upsal. 2, 1909. The measurements refer to 0° C and 760 mm Hg. In conformity with our system of units we have written (13) so that λ is measured in meters.

We substitute this in the left-hand side of (12 a). In accordance with the table at the end of Sec. 2, we put $4\pi c^2 \epsilon_0 = 10^7 \text{ M S}^{-1} \Omega^{-1}$ on the right-hand side and obtain

$$(14) \quad \frac{N e^2}{m} = \frac{\pi}{0.29} 10^{17} = 1.1 \times 10^{18} \text{ M}^{-1} \text{ S}^{-1} \Omega^{-1}.$$

N is found from the density of H_2 which at 0° C and 760 mm Hg is almost exactly equal to $9.00 \times 10^{-2} \text{ KM}^{-3}$. Hence, the mass per unit volume is $9.00 \times 10^{-2} \text{ K}$. But the mass per unit volume is also equal to $2 N_0 m_H$ where m_H is the mass of a hydrogen atom and N_0 the number of molecules per unit volume. Hence

$$N_0 = \frac{9.00 \times 10^{-2}}{2 m_H}.$$

Furthermore, since each H_2 molecule has two electrons,

$$N = 2 N_0, \quad N e = 9.00 \times 10^{-2} \frac{e}{m_H}.$$

Now e/m_H , which is Faraday's *electrochemical equivalent* (the charge of one gram-atom in electrolysis), is also a very precisely known number, namely 9649 in abs. e. m. u. and hence $9649 \times 10^4 = 9.65 \times 10^7 \text{ QK}^{-1}$ in our Q, K units. Equation (14) becomes, therefore,

$$N e \cdot \frac{e}{m} = 9.00 \times 10^{-2} \times 9.65 \times 10^7 \times \frac{e}{m} = 1.1 \times 10^{18}$$

which yields

$$(15) \quad \frac{e}{m} \sim 1.4 \times 10^{11} \text{ QK}^{-1}.$$

This is of the same order of magnitude as the "specific charge of the electron" $e/m = 1.76 \times 10^{11} \text{ QK}^{-1}$. The resonance frequency ω_0 which is easily computed by comparing (12) and (13) confirms our basic assumption by being in the far *ultraviolet*. Clearly, our theory is still very crude in comparison to the detailed results on molecular structure and emission of light which atomic physics has provided.

We shall not justify a once much debated rule (Drude, Natanson) according to which the number of "dispersion electrons" of an ideal gas is the same as the "valence number" of the particular molecule (namely 2×2 for O_2 , 2×3 for N_2). From the point of view of our present atomic models this rule is less understandable than the number 2×1 in the case of H_2 , because in the cases of O and N the valence is not the number of electrons *present* as in the case of H, but rather it is the number of electrons *missing* from the complete shell of eight electrons. Empirically, however, this rule is satisfied

to a good degree of approximation, and this is also understandable from the point of view of atomic physics because in many respects (for instance, in the Pauli principle) present and missing electrons play the same role.

We shall now mention, though again only briefly, a refinement of the dispersion formula which yields the dependence of the index of refraction on pressure in the case of gases and which is known as the Lorenz-Lorentz formula. This formula results from a more exact calculation of the polarization P , which depends not only on the external field E , as was assumed in (2) and (3), but is also influenced by the electric moments of the neighboring molecules. We investigated this dependence more closely in Vol. III, Sec. 11 and derived there the Clausius-Mosotti formula (11.8) for the dielectric constant, in which we now have to replace ϵ_{rel} by n^2 and the molecular constant as defined there

$$N \propto \text{by } \frac{N e^2/m \epsilon_0}{\omega_0^2 - \omega^2}.$$

Thus we obtain

$$(16) \quad \frac{n^2 - 1}{n^2 + 2} = \frac{1}{3} \frac{N e^2/m \epsilon_0}{\omega_0^2 - \omega^2}.$$

When n^2 differs only slightly from 1, which is true in particular for ideal gases, then it can be seen that (16) goes over into the above eq. (10).

18. Infrared Resonance Oscillations of the Ions in Addition to Ultraviolet Electron Resonance Oscillations

If there is a considerable difference between the indices of refraction for visible light and for Hertzian waves, one must expect that other resonance oscillations besides the ultraviolet one contribute to the index of refraction. If, furthermore, the material in question is transparent so that there are no resonances in the visible spectrum, these additional resonances must be *infrared resonance oscillations* (perhaps rotational resonances). It is reasonable to ascribe these not to the mobile electrons, but rather to the much more inert ions. These, too, we shall consider to be "smeared out"; thus we shall not use individual ions, but rather a continuous ion-fluid.

The polarization P is then the sum of the two contributions P_1 (electrons) and P_2 (ions):

$$(1) \quad P = P_1 + P_2.$$

Again, as in the case of P_1 in eq. (17.2), the definition of P_2 refers to the state before the "smearing-out", namely to the dipole moments formed by the separations of the ions from ions of opposite charge.

The *resonance frequency of the electrons* which we called ω_0 in Sec. 17 will now be called ω_1 . The number of electrons per unit volume will again be N . The differential equation for the oscillations of the electrons brought about by the alternating field E is, according to the model of (17.4)

$$(2) \quad \left(\frac{\partial^2}{\partial t^2} + \omega_1^2 \right) P_1 = \frac{N e^2}{m} E.$$

where the displacements have already been replaced by the electric moment P .

The *resonance oscillation of the ions*, whose frequency will be called ω_2 , consists of a relative oscillation of oppositely charged components. If only one pair of such components is present in each molecule, and we shall here limit ourselves to this case, then the relative displacement of these components takes the place of an absolute displacement and the so-called "reduced mass" M takes the place of the individual masses M_1 and M_2 . As will be shown in exercise III. 1, the reduced mass is given by

$$(3) \quad \frac{1}{M} = \frac{1}{M_1} + \frac{1}{M_2}.$$

This mass M enters into the differential equation for the *forced* ion oscillations in place of the mass m in eq. (2). Let p denote the valence of the ion as determined by electrolysis, e. g. $p = 1$ for $\text{Na}^+ \text{Cl}^-$, $p = 2$ for $\text{Ca}^{++} \text{F}_2^{--}$ etc. Because the optical material is electrically neutral, the number of ions per unit volume is equal to the number of electrons N divided by p , while the charge of each individual ion is the electron charge e multiplied by p . In this way we have in place of $N e^2/m$

$$\frac{N (p e)^2}{p M} = \frac{N p e^2}{M}.$$

Hence, we obtain the following differential equation for P_2 :

$$(4) \quad \left(\frac{\partial^2}{\partial t^2} + \omega_2^2 \right) P_2 = \frac{N p e^2}{M} E.$$

The Maxwell relationship between P and E remains the same as in (17.6), where however, P must now be replaced by $P_1 + P_2$ in accordance with (1):

$$(5) \quad \epsilon_0 \mu_0 \ddot{E} + \mu_0 (\ddot{P}_1 + \ddot{P}_2) = \Delta E.$$

By eliminating P_1 and P_2 from (2), (4) and (5), we obtain in place of (17.7) a somewhat complicated sixth order differential equation for E . We do not need to write out this equation, but can immediately treat the pure harmonic state of a linearly polarized wave of frequency ω of the type (17.8); that is,

we assume that the electrons as well as the ions have attained their steady state in this field of frequency ω . Then, according to (2) and (4)

$$\ddot{P}_1 \text{ is proportional to } \frac{-\omega^2}{\omega_1^2 - \omega^2}, \quad \ddot{P}_2 \text{ is proportional to } \frac{-\omega^2}{\omega_2^2 - \omega^2},$$

and by (5) k^2 becomes an algebraic function of ω^2 :

$$k^2 - \frac{\omega^2}{c^2} = \mu_0 \omega^2 \left(\frac{N e^2 / m}{\omega_1^2 - \omega^2} + \frac{p N e^2 / M}{\omega_2^2 - \omega^2} \right).$$

According to the definition of the index of refraction in (17.9 a), the left-hand side equals $(n^2 - 1) \omega^2 / c^2$. Hence, we obtain immediately the following generalization of (17.10):

$$(6) \quad n^2 = 1 + \frac{N e^2 / m \epsilon_0}{\omega_1^2 - \omega^2} + \frac{p N e^2 / M \epsilon_0}{\omega_2^2 - \omega^2}.$$

Evidently when a larger number of resonance oscillations are present, whether they be in the ultraviolet or in the infrared or perhaps in the visible spectrum, a formula of the same structure is obtained. The summation on the right-hand side must then include a term for every one of the resonance oscillations.

In order to make eq. (6) more convenient for purposes of comparison with observations, we express ω in terms of the wave length in vacuum λ , and similarly we express ω_1 and ω_2 in terms of λ_1 and λ_2 :

$$\omega = \frac{2\pi c}{\lambda}, \quad \omega_1 = \frac{2\pi c}{\lambda_1}, \quad \omega_2 = \frac{2\pi c}{\lambda_2}.$$

Using the abbreviations

$$(7) \quad C_1 = \frac{N e^2}{4\pi^2 c^2 \epsilon_0 m}, \quad C_2 = \frac{p N e^2}{4\pi^2 c^2 \epsilon_0 M}$$

we obtain

$$(7 a) \quad n^2 - 1 = C_1 \frac{\lambda^2 \lambda_1^2}{\lambda^2 - \lambda_1^2} + C_2 \frac{\lambda^2 \lambda_2^2}{\lambda^2 - \lambda_2^2}$$

or, eliminating λ^2 from the numerators,

$$(8) \quad n^2 = 1 + \lambda_1^2 C_1 + \lambda_2^2 C_2 + \frac{\lambda_1^4 C_1}{\lambda^2 - \lambda_1^2} + \frac{\lambda_2^4 C_2}{\lambda^2 - \lambda_2^2}.$$

Next we consider the limiting case as $\lambda \rightarrow \infty$. Then the last two terms on the right-hand side vanish and we obtain

$$(9) \quad n_\infty^2 = 1 + \lambda_1^2 C_1 + \lambda_2^2 C_2.$$

Only in this limiting case when the actual resonance terms vanish is the Maxwell relation (3.4 a) fulfilled exactly. Thus, *the failure of the Maxwell relation to hold in the visible spectrum can be understood in terms of the existence of infrared resonance oscillations* (because $\lambda_1^2 \ll \lambda_2^2$ the term in (9) which contains λ_2^2 is clearly the decisive one). Hence the Maxwell relation should be corrected to read

$$(9a) \quad n_\infty = \sqrt{\epsilon} \quad \text{instead of} \quad n = \sqrt{\epsilon}$$

(by ϵ is meant in both cases the dielectric constant relative to vacuum).

We turn now to the visible part of the spectrum. In this range very exact measurements for some cubic halide crystals are available. (As we shall see in Chap. IV, cubic crystals, strangely enough, behave *optically isotropic*, while elastically, thermally, etc. they reveal their anisotropy). From among these we select fluorspar CaF_2 (or fluorite, from which substance the phenomenon of fluorescence received its name). According to measurements by Paschen¹ the following holds for fluorite:

$$(10) \quad n^2 = 6.09 + \frac{6.12 \times 10^{-15}}{\lambda^2 - 8.88 \times 10^{-15}} + \frac{5.10 \times 10^{-9}}{\lambda^2 - 1.26 \times 10^{-9}}.$$

Comparing this with (8) we obtain

$$\frac{C_2}{C_1} = \frac{\lambda_1^4}{\lambda_2^4} \cdot \frac{5.10 \times 10^{-9}}{6.12 \times 10^{-15}} = \left(\frac{8.88 \times 10^{-15}}{1.26 \times 10^{-9}} \right)^2 \cdot \frac{5.10 \times 10^{-9}}{6.12 \times 10^{-15}} = 4.15 \times 10^{-5}.$$

On the other hand (7) yields

$$\frac{C_2}{C_1} = \phi \frac{m}{M}.$$

Because $\phi = 2$ (the Ca^{++} ion has given two electrons to the two F ions) we conclude that

$$(11) \quad \frac{M}{m} = \frac{2 \times 10^5}{4.15}.$$

To calculate the reduced mass M we set in (3) $M_1 = 40 m_H$ (40 = atomic weight of Ca, m_H = mass of the H atom), $M_2 = 2 \times 19 m_H$ (19 = atomic weight of F, therefore 2×19 = molecular weight of the negative component F_2). Thus we obtain

$$\frac{1}{M} = \left(\frac{1}{40} + \frac{1}{38} \right) \frac{1}{m_H}, \quad \text{hence} \quad M = 19.5 m_H.$$

¹Ann. d. Phys. (Lpz.) 54, p. 672, 1895.

From this and (11) we find

$$(12) \quad \frac{m_H}{m} = 2450.$$

This value is of the same order of magnitude as that obtainable from the values for e/m and e/m_H given in Sec. 17. Our original assumption: ultra-violet resonance oscillations = electron oscillations, infrared resonance oscillations = ion oscillations, is thus confirmed.

The relationship between n_∞^2 and the dispersion constants $C_1, C_2, \lambda_1, \lambda_2$ required by eq. (9) is also tolerably well fulfilled. For, according to (9), one obtains

$$n_\infty^2 = 6.09, \quad \lambda_1^2 C_1 = \frac{6.12}{8.88} = 0.7, \quad \lambda_2^2 C_2 = \frac{5.10}{1.26} = 4.06,$$

hence

$$1 + \lambda_1^2 C_1 + \lambda_2^2 C_2 = 5.76.$$

The electric determination of the dielectric constant gives

$$\epsilon = 6.7 \quad \text{to} \quad 6.9.$$

When Drude arrived at these and many similar results around 1900, he made the incidental remark to the author: "We live in a grandiose era; we are beginning to get a glimpse of the electric composition of matter." Had he lived to witness the developments of the following decades, he would have seen his boldest hopes surpassed.

From a practical point of view the shape of the dispersion curve for glass is clearly of paramount importance to the problem of designing achromatic lenses and other optical apparatus. In exercise III. 3 we shall treat the achromatic prism and in the same connection also the direct vision prism. Exercise III. 2 serves as preparation for these problems.

19. Anomalous Dispersion

We shall now investigate the dispersion in the immediate vicinity of a resonance frequency $\omega = \omega_0$. We assume that the latter lies in the visible spectrum, because only in the visible spectrum can sufficiently precise measurements be made with which to check the theory. Hence, our body is no longer transparent as had previously been assumed, but, as we shall see, it is *colored* and the coloration depends on the value of ω_0 .

Since for $\omega = \omega_0$ the equation for the forced oscillations (17.3 a) would result in an oscillation of infinite amplitude, we must add a *damping term*. This same procedure is used in all other resonance problems of mechanics

and electrodynamics. We shall write this term in the form $g \omega_0 \dot{s}$, because it is convenient to make the term proportional to the velocity \dot{s} and because inclusion of the factor ω_0 makes the *damping constant* g a dimensionless number. In order for the resonance to be sharp, g must be $\ll 1$. Equation (17.4) is thus changed into

$$(1) \quad \left(\frac{\partial^2}{\partial t^2} + g \omega_0 \frac{\partial}{\partial t} + \omega_0^2 \right) P = \frac{N e^2}{m} E.$$

Since ω_0 is in the visible spectrum, and the oscillating particles are therefore ions, m is a reduced mass and also includes the valence number p of the ion as in (18.4).

For a field E of frequency ω and for steady harmonic oscillations of the ions, (1) gives

$$P = \frac{N e^2}{m} \frac{E}{(\omega_0^2 - \omega^2 - i g \omega_0 \omega)}$$

and, corresponding to (17.10) and (18.6), the resulting index of refraction is

$$(2) \quad n^2 = n_m^2 + \frac{N e^2 / m \epsilon_0}{\omega_0^2 - \omega^2 - i g \omega_0 \omega}.$$

n_m is the average contribution in the vicinity of $\omega = \omega_0$ of all other resonances which add to the dispersion in the visible range (also included in n_m is the contribution 1 of the pure displacement current which has heretofore been written down separately). As in the case of metallic reflection the index of refraction n has now become *complex*. As in eq. (6.2) we again replace n by $n(1 + i\kappa)$; by separating real and imaginary parts we obtain from (2)

$$(3) \quad n^2(1 - \kappa^2) = n_m^2 + \frac{N e^2}{m \epsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + g^2 \omega_0^2 \omega^2},$$

$$(4) \quad 2 n^2 \kappa = \frac{N e^2}{m \epsilon_0} \frac{g \omega_0 \omega}{(\omega_0^2 - \omega^2)^2 + g^2 \omega_0^2 \omega^2}.$$

We introduce the abbreviation

$$(5) \quad a^2 = \frac{N e^2}{m \epsilon_0 \omega_0^2} \text{ (dimensionless number)}$$

and use as variables

$$(6) \quad x = \frac{\omega^2 - \omega_0^2}{\omega_0^2}, \quad y = \frac{n^2(1 - \kappa^2)}{a^2}, \quad z = \frac{2 n^2 \kappa}{a^2}.$$

Then eqs. (3) and (4) become

$$(7) \quad y = \frac{n_m^2}{a^2} - \frac{x}{x^2 + g^2(1 + x)}, \quad z = \frac{g \sqrt{1 + x}}{x^2 + g^2(1 + x)}.$$

The extrema of y are found in the following way:

$$\frac{dy}{dx} = -\frac{1}{x^2 + g^2(1+x)} \left(1 - \frac{2x^2 + g^2x}{x^2 + g^2(1+x)} \right) = 0, \quad \text{hence} \quad x^2 = g^2,$$

$$x = +g, \quad y = y_{\min} = \frac{n_m^2}{a^2} - \frac{1}{2g + g^2},$$

$$x = -g, \quad y = y_{\max} = \frac{n_m^2}{a^2} + \frac{1}{2g - g^2}.$$

We now visualize the content of our formulae by means of a table and a graph. For the sake of clarity we shall treat g as a small number. Then we can neglect g^2 as compared to g and, where they occur together, g as compared to 1. In the first line of the table x is used as a spectral measure and in the last line ω is given.

x	-1	-g	0	+g	∞
$y = n_m^2/a^2$	+1	$1/2g$	0	$-1/2g$	0
z	0	$1/2g$	$1/g$	$+1/2g$	0
ω	0	$\omega_0 \sqrt{1-g}$	ω_0	$\omega_0 \sqrt{1+g}$	∞

In agreement with this table fig. 22 shows that the extrema of the y -curve occur at $x = \pm g$ and that the curve cuts the line $y = n_m^2/a^2$ at $x = 0$. Furthermore, the narrow bell-shaped z curve attains its maximum at $x \sim 0$ (more precisely at $x = -g^2/4 + \dots$) and its half-value width is $2g$. The scales to which the y - and z -curves are drawn in fig. 22 are not comparable. The scale of the z -curve is indicated on the right of the figure.

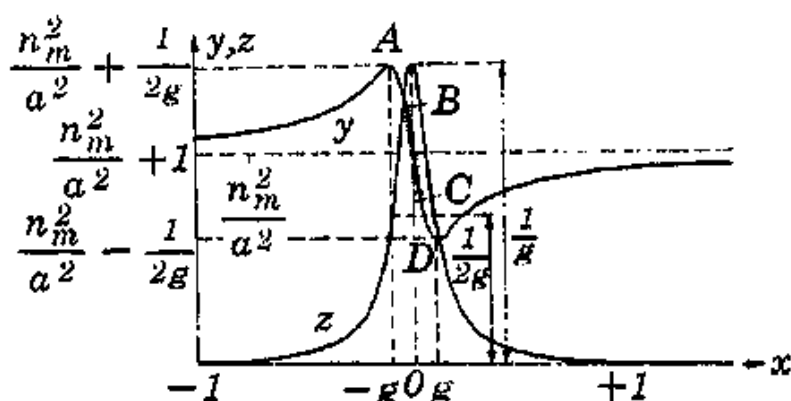


Fig. 22

Representation of anomalous dispersion.

$$x = \frac{\omega^2 - \omega_0^2}{\omega_0^2}, \quad y = \frac{n^2(1-x^2)}{a^2}, \quad z = \frac{2n^2x}{a^2}$$

The y -curve (scale on the left) indicates substantially the course of the index of refraction. The z -curve (scale on the right) indicates the coefficient of absorption.

Besides representing y and z , fig. 22 also serves as a qualitative illustration of the behavior of n^2 and κ . Only in the vicinity of $x = 0$ are the curves somewhat distorted from the exact representations of n and κ as given by eq. (3) and (4).

In fig. 22 the portion of the curve for the index of refraction lying between A and D is of greatest interest to us. While before A and after D this curve rises with increasing ω (normal dispersion), between A and D it descends with increasing x . This is called *anomalous dispersion*, i. e. *the short waves are refracted less than the longer ones*. The Dane Christiansen first observed this phenomenon in fuchsin around 1870. Almost at the same time and independently Kundt detected it in various dyes.

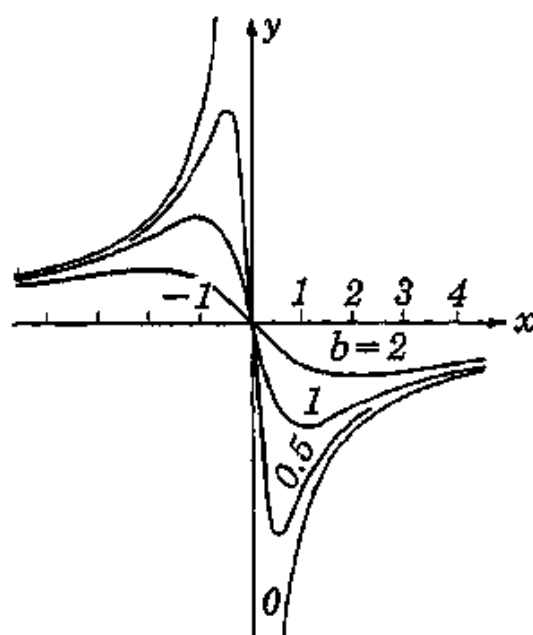


Fig. 23.

Family of y curves for increasing damping (expressed by the parameter b). $b = 0$ gives the rectangular hyperbola.

The portion BC of the descending curve AD is marked with hatch lines to indicate that in this range the spectrum is extinguished by absorption. Hence the anomalous dispersion is observable only along the short segments AB and CD . In the case of fuchsin the absorption band is in the yellow-green. The transmitted unabsorbed light therefore, has the complementary intensely red color.

The above remarks regarding dyes apply even more strongly to the vicinity of all the spectral lines of gases. By virtue of this fact Kundt's original "method of crossed prisms" has been developed into a powerful spectroscopic method.

In order to familiarize ourselves with the characteristic shape of the y -curve, we recall (fig. 23) the method of avoiding the singularity, at $x = 0$, of the hyperbola $y = -1/x$ by substituting the continuous function

$$y = -\frac{x}{x^2 + b^2},$$

which, in the limit as $b \rightarrow 0$, approaches the rectangular hyperbola $y = -1/x$. The latter function corresponds to the index of refraction for the case of an undamped resonance denominator. The continuous function corresponds to the index of refraction for the case of the damped resonance denominator in eq. (7).

Figure 22 furthermore demonstrates that in passing through the resonance from short wavelengths to longer wavelengths the y -curve (n^2/a^2) is raised by unity. We can also verify this by comparing the second and last columns

in the above table. Every resonance frequency contributes such an increase in the value of n^2/a^2 . Every time n^2 passes through a resonance it is increased by a^2 . This enables us to understand why for solids with complex molecules, e. g. glass, the indices of refraction in the optical range differ from those in the range of Hertz waves.

The particularly striking case of water ($n = 4/3$ in the visible spectrum, $n_\infty = \sqrt{\epsilon} = \sqrt{80}$ in the limiting case of electrostatics $\lambda = \infty$) does not belong in the same category, but is rather due to the polar nature of the H_2O -molecule; see Vol. III, p. 74, footnote 1. Owing to their triangular structure the water molecules possess a permanent electric moment which can follow the external field oscillations only at long wavelengths. A sort of molecular stiffness prevents them from being set into oscillation by short waves. The transition between the two regions of behavior is in the vicinity of $\lambda = 1.7$ cm; and it is here also that the main portion of the jump between the optical value of n^2 and the electrostatic dielectric constant lies.

20. Magnetic Rotation of the Plane of Polarization

One of the important contributions to the development of the electromagnetic theory of light was Faraday's discovery in 1845 (see historical table) of a connection between optics and magnetism, two up to that time entirely distinct fields. Although this connection does not apply to processes in free space, but is limited to ponderable bodies and is related to the motions of the dispersion electrons in such bodies, Faraday's discovery was, nevertheless, an impressively strong hint of the electromagnetic nature of light.

We begin with eq. (17.3) where the force of the field on the electron was described solely by $-eE$. On the other hand we know from Vol. III that the force of the field on a *moving* charge (which we shall again denote by $-e$) is given generally by the Lorentz force expression

$$(1) \quad F = -e(E + v \times B).$$

Setting $B = \mu H$ and noting that according to (2.5) the relation

$$|H| = \sqrt{\frac{\epsilon}{\mu}} |E|$$

holds for a light field, we remark that the B term in (1) differs from the E term by the factor

$$\sqrt{\epsilon \mu} v = \frac{v}{u} = n \frac{v}{c} = n \beta.$$

The magnetic force is, therefore, only a correction of first order in β and can be neglected¹ in comparison to the electric force $-e \mathbf{E}$.

But now we shall suppose that the material is placed in an external field \mathbf{B} which can be made much stronger than the optical B-field. In that case the equation of motion (17.3) must be provided with a possibly large correction term, and it then reads:

$$(2) \quad m \ddot{\mathbf{s}} + f \mathbf{s} = -e (\mathbf{E} + \dot{\mathbf{s}} \times \mathbf{B}).$$

We choose the direction of \mathbf{B} as the positive z -direction and assume that the light wave propagates in this same direction. With these assumptions \mathbf{E} and \mathbf{s} are vectors in the xy -plane¹. Separating the components of (2), dividing by m , and letting $\omega_0^2 = f/m$, we obtain the two simultaneous equations.

$$(3) \quad \begin{cases} \ddot{s}_x + \frac{e}{m} B \dot{s}_y + \omega_0^2 s_x = -\frac{e}{m} E_x, \\ \ddot{s}_y - \frac{e}{m} B \dot{s}_x + \omega_0^2 s_y = -\frac{e}{m} E_y. \end{cases} \quad \begin{matrix} 1 \\ \pm i \end{matrix}$$

Multiplying by the factors indicated on the right and adding we get

$$(4) \quad \ddot{S} \mp \frac{e}{m} i B \dot{S} + \omega_0^2 S = -\frac{e}{m} E$$

where the following abbreviations have been used:

$$(4a) \quad S = s_x \pm i s_y, \quad E = E_x \pm i E_y.$$

However, it must be emphasized that this method of deriving (4) by way of (3) is unnecessarily indirect. For (4) is nothing else than the initial eq. (2), provided the latter is interpreted in the following way: two-dimensional vectors are complex numbers of the form $a + ib$ and therefore the vectors \mathbf{s} and \mathbf{E} in (2) are identical with the complex quantities S and E as defined in (4a). Moreover, since multiplication by i means a right-handed rotation from x to y around z , and since the vector product is also defined by the right-handed screw rule, it follows that $\dot{\mathbf{s}} \times \mathbf{B}$ in (2) is simply the complex number $\mp i B \dot{S}$ which appears multiplied by $-e/m$ on the left-hand side in eq. (4).

This point of view is quite generally valid. It suggests that we treat the complex quantities E themselves as the basic field variables, rather than the components E_x, E_y . Physically this implies a *transition from linear to circular*

¹Like all quantities which we use to represent monochromatic states, \mathbf{E} and \mathbf{s} are, of course, everywhere multiplied by the same time factor $\exp(-i\omega t)$. While heretofore we could almost always omit writing this factor it improves the clarity of the calculations to retain it occasionally and we shall do so starting with eq. (5). The transition to real parts is again postponed until the final formula (14).

polarization. If originally (i. e. without magnetic field or where the light enters this field) we have a linearly polarized wave \mathbf{E} : $E_x = A \cos \omega t$, $E_y = 0$, then we decompose this wave into the two right and left circularly polarized waves, say E_+ and E_- , and in agreement with (4 a) set

$$(4 \text{ b}) \quad E_x = \frac{1}{2}(E_+ + E_-), \quad E_y = \frac{1}{2i}(E_+ - E_-).$$

Then we carry out all calculations using E_{\pm} as the simplest elements of the field and find expressions for their velocities of propagation and indices of refraction (which will differ somewhat for the two components). After the light has passed through the magnetic field, we again combine E_{\pm} into a linear oscillation \mathbf{E} which will have been turned through a certain angle χ in the xy -plane compared to the original oscillation E_x . The resulting law for χ is rather complicated and becomes clear only because of our simpler assumptions about E .

We assume the wave to be monochromatic, i. e. simply periodic in t ,

$$(5) \quad E_{\pm} = A e^{i(k_{\pm} z - \omega t)}$$

where A is a real number. We thereby fulfill our initial condition at $z = 0$ (entrance of linearly polarized light into the magnetic field):

$$(5 \text{ a}) \quad E_x = A \cos \omega t, \quad E_y = 0,$$

and at the same time we take into account the circular nature of E_{\pm} in the magnetic field:

$$(5 \text{ b}) \quad |E_{\pm}| = A |e^{i(k_{\pm} z - \omega t)}| = A,$$

since, as we shall show below, this expression describes two circular oscillations, provided only that k_{\pm} are real.

Using (5) and the corresponding expression for S we obtain from (4) for the steady state, that is, for the purely periodic state of the electron fluid

$$(6) \quad S_{\pm} = \frac{-e/m}{-\omega^2 \mp \frac{e}{m} B \omega + \omega_0^2} E_{\pm}.$$

It should be remarked that the denominator here is *real* and not complex as in the case of anomalous dispersion, eq. (19.2). This is due to the fact that the magnetic field *does no work* on the electrons. We can disregard the *absorption* which takes place when $\omega = \omega_0$ since that effect is not caused by B .

The vector \mathbf{P} which is proportional to \mathbf{s} behaves just like the latter. If we set

$$P_{\pm} = P_x \pm i P_y$$

then, according to (17.2), we get

$$(6a) \quad P_{\pm} = \frac{N e^2/m}{-\omega^2 \mp \frac{e}{m} B \omega + \omega_0^2} E_{\pm}.$$

Therefore, for the periodic state the differential eq. (17.6) gives in place of (17.9)

$$(7) \quad k_{\pm}^2 = \frac{\omega^2}{c^2} \left(1 + \frac{\mu_0 c^2 N e^2/m}{\omega_0^2 \mp \frac{e}{m} B \omega - \omega^2} \right).$$

Hence there are two different indices of refraction corresponding to the two wave numbers k_{\pm} , namely

$$(7a) \quad n_{\pm} = \frac{c k_{\pm}}{\omega}$$

and their values are given by (compare with (17.10))

$$(8) \quad n_{\pm}^2 = 1 + \frac{N e^2/m \epsilon_0}{\omega_0^2 \mp \frac{e}{m} B \omega - \omega^2}.$$

This then, is the result of our dispersion calculation in its simplest form, i. e. using E_{\pm} . We see that n_{+} and n_{-} differ from one another, and n_{+} is somewhat larger than n_{-} .

This difference is small, however, because, as was remarked in connection with (2), the middle term in the denominator is only a correction term. By neglecting the square of this correction term we get from (8)

$$(9) \quad n_{+}^2 - n_{-}^2 = \frac{N e^2}{m \epsilon_0} \cdot 2 \frac{e}{m} \frac{B \omega}{(\omega_0^2 - \omega^2)^2}$$

or, if we introduce the mean index of refraction $n = 1/2 (n_{+} + n_{-})$,

$$(9a) \quad n_{+} - n_{-} = \frac{N e^3}{n m^2 \epsilon_0} \frac{B \omega}{(\omega_0^2 - \omega^2)^2}.$$

We now turn to the method of measuring this effect. First we shall justify the title of this section: "*Rotation of the Plane of Polarization.*" Let the path of the light in the magnetic field extend from $z = 0$ to $z = l$. Let us determine the values of E_{\pm} at $z = l$. For this it is advantageous to decompose k_{\pm} into two terms which are symmetric and antisymmetric, respectively, with respect to an interchange of k_{+} and k_{-} . Thus we set

$$(10) \quad k_{\pm} = \frac{1}{2} (k_{+} + k_{-}) \pm \frac{1}{2} (k_{+} - k_{-}).$$

We also introduce the abbreviations

$$(11) \quad \varphi = \frac{l}{2} (k_+ + k_-) - \omega t, \quad \chi = \frac{l}{2} (k_+ - k_-).$$

As we shall see, φ is a *phase difference* and χ is an *angle of rotation*. We obtain then

$$(12) \quad E_{\pm} = A \exp i \left\{ \frac{l}{2} (k_+ + k_-) \pm \frac{l}{2} (k_+ - k_-) - \omega t \right\},$$

and applying (11)

$$(13) \quad E_+ = A e^{i\varphi} e^{i\chi}, \quad E_- = A e^{i\varphi} e^{-i\chi},$$

from which follows, according to (4 b),

$$(14) \quad E_x = A e^{i\varphi} \cos \chi, \quad E_y = A e^{i\varphi} \sin \chi.$$

Since E_x and E_y oscillate with the same phase, they combine into a *linear oscillation which is rotated in a positive sense (right-handed screw direction around the magnetic field B) by an angle χ with respect to the incident oscillation (5 a)*. At the same time the phase φ is changed from its original value at $z = 0$ in (5 b) by the amount

$$\frac{k_+ + k_-}{2} l.$$

The angle χ can be measured very precisely. One sets

$$(15) \quad \chi = V l H$$

where V is called *Verdet's Constant*. By (11) and the relation (7 a) between k_{\pm} and n_{\pm} this constant is

$$V = \frac{\omega}{2c} \frac{n_+ - n_-}{H}.$$

From this follows by (9 a)

$$(16) \quad V = \frac{N e^3}{2 n m^2 c} \frac{\mu}{\epsilon_0} \frac{\omega^2}{(\omega_0^2 - \omega^2)^2}.$$

At first sight it might seem that the strong rotation of the plane of polarization in ferromagnetic materials is due to the factor μ in (16). However, this is not the case. For μ plays only a formal role in (16). It appears there only because of the conventional definition (15) of χ (which postulates proportionality to H , rather than to B , which would actually be better). As a matter of fact, our theory cannot encompass ferromagnetism at all because it fails to take the electron spin¹ into account.

¹The extent to which the electron spin is able to explain the rotation of polarization in ferromagnetic media has been investigated by H. R. Hulme, Proc. Roy. Soc. London 185, 237, 1935.

Clearly, (16) is frequency dependent. Therefore, as with refraction, the magnetic rotation of the plane of polarization is connected with *dispersion*. One could develop V in powers of ω^2/ω_0^2 , which is the procedure used for n in Sec. 17, and by eliminating ω_0 from the first two coefficients of the resulting expansion one could derive a universal relationship between e , m and N . However, the accuracy with which the dispersion factor in (16) can be measured is hardly sufficient to justify this procedure. A better method is to use the first terms of the expansions of both n and V ; the resulting relation is satisfied sufficiently well for the gases H_2 , O_2 , N_2 ¹.

More important and more interesting than the magnetic rotation is the *natural rotation* of the plane of polarization in crystals which have helical structure (quartz, sodium chlorate, etc.) and in liquids which have an asymmetrically bound carbon atom (sugar solutions). This rotation of the plane of polarization is an indispensable reagent in the entire sugar industry. We shall take up these phenomena in Chap. IV.

At this stage a basic difference between natural and magnetic rotation should be emphasized: if at the end of the path l , one reflects the light ray backwards, then the *natural* rotation is cancelled while the magnetic rotation is *doubled*. The magnetic effect is due to the fact that for the return path not only k_+ and k_- are interchanged in formula (11), but also i and $-i$ must be interchanged in formulae (12) and (13). For after the reflection the vector direction of a positive rotation in the Gaussian plane is opposite to the direction of the magnetic field. Because of this Faraday was able to multiply his very minute rotation effect by repeated back and forth reflections.

21. The Normal Zeeman Effect and Some Remarks on the Anomalous Zeeman Effect

The above considerations provide us with a very simple approach to the *Zeeman effect*, even if only to the *normal* Zeeman effect in which the spin of the electron plays no essential role. Even for the hydrogen atom with its single electron the Zeeman effect is actually *anomalous*.

Strictly, the normal Zeeman effect occurs only for *singlet lines*, i. e. when the spins of the contributing electrons add up to zero. The simplest example is parahelium (two electrons with opposing spins). On the other hand, the hydrogen lines, as well as the alkali lines, are *doublet lines*. However, the Zeeman effect of the hydrogen atom approaches the normal Zeeman effect very closely even for weak magnetic fields. In the case of the alkalis, which

¹According to observations by Siertsema. See A. Sommerfeld, Ann. d. Phys. (Lpz.) 57, 513, 1917.

have the same anomalous Zeeman effect as hydrogen, this approach to the normal Zeeman effect, the so-called *Paschen-Back effect*, takes place only in very much stronger fields. Latter, we shall give the necessary field strengths for each case. H. A. Lorentz worked out the theory of the normal Zeeman effect on a classical basis. The spin of the electron, and hence the anomalous Zeeman effect, can be understood only quantum-mechanically.

The treatment of the Zeeman effect for *absorption*, that is of the interaction of an incoming light field with a magnetic field, remains entirely within the framework of the basic concepts of the theory of dispersion. Woldemar Voigt used this method very successfully to treat the *D*-lines of sodium. He called it the *method of the inverse Zeeman effect*. Experimentally one is usually concerned with the Zeeman effect in emission, the "direct Zeeman effect", which we prefer to discuss here because of its mathematically simpler theory.

We start out from the equation of motion of the electron (20.2) in which we must, however, set $E = 0$, since we are concerned only with the *magnetic* action upon the emitted radiation. This equation then reads

$$(1) \quad \ddot{\mathbf{s}} + \omega_0^2 \mathbf{s} = -\frac{e}{m} \dot{\mathbf{s}} \times \mathbf{B}.$$

ω_0 is the frequency of the light emitted by the atom without magnetic field. The term $\omega_0^2 \mathbf{s}$ is due to the retarding "quasielastic force"; see fig. 21 and eq. (17.3).

As in Sec. 20 let the z -axis lie along the direction of the magnetic field \mathbf{B} . In this direction $\dot{\mathbf{s}} \times \mathbf{B} = 0$, hence

$$(2) \quad \ddot{s}_z + \omega_0^2 s_z = 0.$$

Therefore, the z -oscillation of the electron has the original frequency ω_0 . It is *not influenced* by the magnetic field.

As before we use complex notation in the xy -plane; that is, as in (20.4 a) we set

$$(3) \quad S = s_x \pm i s_y.$$

Hence we get

$$(4) \quad \ddot{S} \mp i \frac{e}{m} B \dot{S} + \omega_0^2 S = 0$$

which corresponds to (20.4). This equation is integrated by assuming

$$(5) \quad S = a e^{i\omega t},$$

where the factor a is due to the original excitation producing the oscillation and remains, therefore, undetermined. Equation (5) implies a *circular oscillation*. Substitution of (5) in (4) yields:

$$(6) \quad -\omega^2 \pm \frac{e}{m} B \omega + \omega_0^2 = 0.$$

The middle term is small compared to the other two. Hence, assuming $\Delta \omega$ small, we set

$$\omega = \omega_0 + \Delta \omega, \quad \omega^2 = \omega_0^2 + 2\omega_0 \Delta \omega, \quad B\omega = B\omega_0,$$

and find from (6)

$$-2\omega_0 \Delta \omega \pm \frac{e}{m} B \omega_0 = 0.$$

Hence

$$(7) \quad \Delta \omega = \pm \frac{1}{2} \frac{e}{m} B.$$

We verify (7) by the following elementary considerations: the circular oscillation must be such as to maintain equilibrium between the centrifugal

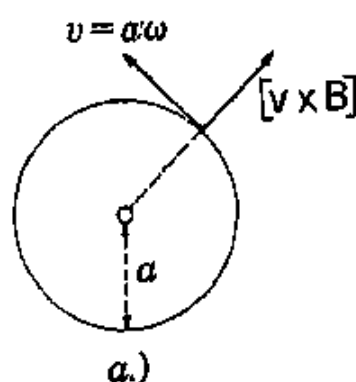


Fig. 24 a.

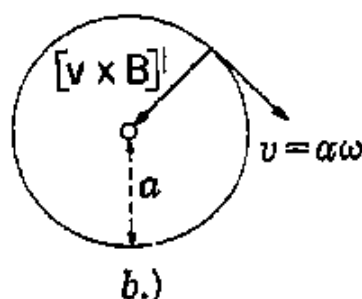


Fig. 24 b.

Direction of the Lorentz force $\mathbf{v} \times \mathbf{B}$ as centrifugal force for left-handed rotation (a), and as centripetal force for right-handed rotation (b). The vector \mathbf{B} is directed out of the paper.

inertia force on the one hand, and the sum of the centripetal quasielastic force and the magnetic force on the other. Figure 24a represents the oscillation $S_+ = s_x + i s_y$. For the radius $r = a$ and the velocity $v = a \omega$ ($\omega =$ angular frequency and also the angular velocity), the centrifugal force is

$$(8) \quad m \frac{v^2}{a} = m a \omega^2 = m a \left(\omega_0 + \frac{1}{2} \frac{e}{m} B \right)^2 = m a \omega_0^2 + a \omega_0 e B.$$

The first term in the last of these expressions is balanced by the quasielastic force. The second term is balanced by the magnetic force $-e \mathbf{v} \times \mathbf{B}$. As shown in the figure, $\mathbf{v} \times \mathbf{B}$ is directed centrifugally, hence $-e \mathbf{v} \times \mathbf{B}$ is centripetally directed like the quasielastic force.

Figure 24b shows the same for the oscillation $S_- = s_x - i s_y$ which we can describe, according to (5), by reversing the sign of i :

$$s_x - i s_y = a e^{-i\omega t}.$$

Hence, we are now dealing with a circular path with the same radius a but the opposite direction of motion and with its $\Delta \omega$ given by the lower sign in (7). The centrifugal force is now

$$(8a) \quad m \frac{v^2}{a} = m a \omega^2 = m a \left(\omega_0 - \frac{1}{2} \frac{e}{m} B \right)^2 = m a \omega_0^2 - a \omega_0 e B.$$

$\mathbf{v} \times \mathbf{B}$ is here centripetally directed and, therefore, $-e \mathbf{v} \times \mathbf{B}$ is directed centrifugally. The magnetic force acts oppositely to the quasioelectric centripetal force which it holds in equilibrium together with the now reduced inertial force.

How do the theoretically expected spectra look now? 1. *Longitudinal observation*, i. e. observation in the z -direction. The linear oscillation (2) has the magnetically uninfluenced frequency ω_0 and *does not radiate* in the z -direction; just as a radio antenna does not radiate in its own direction of oscillation. On the other hand, the two circular oscillations with their magnetically influenced frequencies, eq. (7), radiate two *circularly polarized electromagnetic waves*, one of which is *left* polarized and the other *right* polarized. We define these directions of polarization as those seen by an observer who is looking in the direction of B (that is, looking out of the picture in fig. 24). Thus we obtain in fig. 25a the picture which shows the spectrogram seen by an observer looking in the direction of the B -field: no light is seen at the position of the original spectral line. To the right and left of that position there are equally intense magnetically displaced lines.

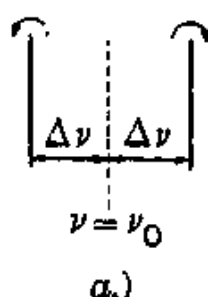


Fig. 25 a.

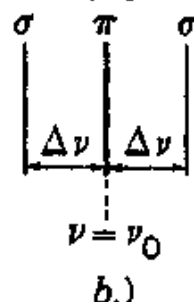


Fig. 25 b.

Normal Zeeman Effect for (a) longitudinal and (b) transverse observation.

The quantitative connection between the primary electron oscillation and the emitted radiation which has here been implicitly assumed rests upon the treatment in Vol. III, Sec. 19.

We also note that in fig. 25a, b the angular frequency ω has been replaced by the frequency $\nu = \omega/2\pi$, as is usual in spectroscopy. Thus we get instead of $\Delta\omega$ as given by (7)

$$(9) \quad \Delta\nu = \frac{\mu_0}{4\pi m} \frac{e}{\hbar} H,$$

where, following common usage, B has been replaced by $H = B/\mu_0$.

2. *Transverse observation*, i. e. observation in a direction perpendicular to the magnetic field, as, for example, in the y -direction. The components s_y of the circular oscillations do not radiate in this direction and can be omitted. The components s_x emit their strongest radiation in this direction, just as the radiation of an antenna or a Hertzian dipole has a maximum in the transverse direction. The frequency ν_0 belonging to s_x is now present in the spectrogram. The frequencies $\nu_0 \pm \Delta\nu$ of the two circular components are also present, but their intensities are only half as great¹ because only s_x contributes to

¹For statistical excitation the intensity of the linear oscillation s_x is of the same magnitude as that of each of the circular oscillations $s_x \pm i s_y$ which were denoted by a^2 in (5). Since on the average $s_x^2 = s_y^2$, it follows that $s_x^2 = a^2/2$ as stated in the text.

them. Since s_x oscillates perpendicularly to the magnetic field, the transverse E-field emitted by s_x is also perpendicular to H . On the other hand, the E-field emitted by s_z is directed parallel to H . In fig. 25b the corresponding lines are denoted by the usual symbols π (parallel) and σ (perpendicular). The intensity ratio 2:1 is indicated by the widths of the lines. The resulting picture is called a "normal Lorentz triplet". Indeed, H. A. Lorentz formulated the theory, which we have here sketched, immediately upon Zeeman's discovery of the magnetic splitting of lines in 1896.

To be sure, Zeeman's original observations were far from yielding the precise spectrograms which we have drawn here. He did not use the light of a singlet line but rather that of the (unresolved) sodium D -line doublet and, instead of obtaining discrete components, he saw only a general broadening of the spectroscopic picture. This was, nevertheless, sufficient to demonstrate the existence of a new fundamental effect; an effect, by the way, for which even Faraday had searched in vain. Furthermore, Zeeman's result sufficed to indicate a qualitative similarity to Lorentz's theory of the effect. For, the outer edges of the spot of light were *linearly polarized* with the direction of oscillation of the E-vector, *perpendicular* to H , if the observation was made transverse to the magnetic field. For longitudinal observation, on the other hand, the edges were *circularly polarized* with the rotational sense the same as that indicated in fig. 25a. This latter fact was of special significance to the electron theory which was then being formulated because it indicated the *negative* charge of the oscillating particles. Indeed, if these particles had positive charges, the sign of $\Delta\nu$ and thereby the sense of the circular oscillations would be reversed in all the above formulae and figures.

The following circumstance, which could not be known at that time, was essential to this comparison between experiment and theory: *Also in the anomalous Zeeman effects, the σ -components lie near the edges of the pattern while the π -components are nearer its center, as shown by fig. 25b. In these effects the short wave components are circularly polarized in a right-handed screw sense around the magnetic field lines while the long wave components are left circularly polarized, as in fig. 25a.*

We shall confirm this by considering the complete transverse decomposition of the two D -lines as it was later measured by Zeeman and others. The two lines are the D_2 line, $\lambda = 5896 \text{ \AA}$, fig. 26a, and the D_1 line, $\lambda = 5890 \text{ \AA}$, whose intensity is half that of the D_2 line; fig. 26b. In both drawings the distance of each component from the position $\nu = \nu_0$ is a multiple of *one third of the normal $\Delta\nu$* . In both cases the center position is unoccupied and is indicated by a broken line. The positions $\pm \Delta\nu$ are occupied by strong σ -components in fig. 26a and are unoccupied in fig. 26b. The π -components

lie nearest to the center and are displaced from it by $\frac{\Delta \nu}{3}$ in fig. 26a and by $2\Delta \nu/3$ in fig. 26 b. Instead of being normal Lorentz triplets as shown in fig. 25b, fig. 26a is a sextet and fig. 26b a quartet.

Runge's rule says: for all anomalous Zeeman effects the displacements of the components from the position of the original line, measured in wave numbers, are *rational multiples* of the Lorentz $\Delta \nu$. The denominator of these rational multiples is called Runge's denominator. It's value is 3 for the principal series of sodium and all other alkalis. A general formula which is due to Landé enables one to compute the complete splitting diagram, including the denominator, for every series character. *Preston's rule* says that spectral lines with the same series character have the same Zeeman splitting.

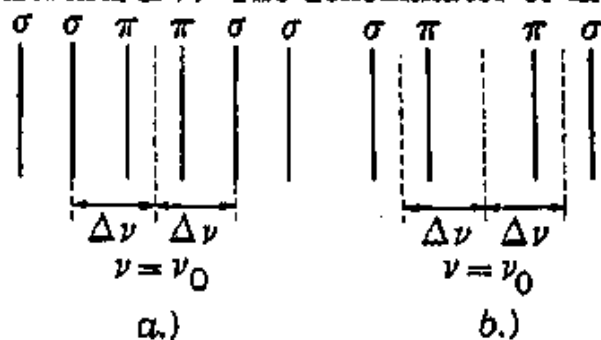


Fig. 26 a. Fig. 26 b.
Splitting of the Na D-lines under transverse observation.
 $D_1: \lambda = 5890 \text{ \AA}; \quad D_2: \lambda = 5896 \text{ \AA}.$

However, to these rules must be added the reservation "provided the magnetic field is not too strong". What is meant by "not too strong"? Paschen and Back found the answer to this question to be:

$$(10) \quad \Delta \nu \ll \Delta \nu_0.$$

where $\Delta \nu$ is the magnetic splitting arising from the normal Zeeman effect as given by (9), and $\Delta \nu_0$ is, in the case of a doublet line such as the D-lines, the spacing of the original two lines. In the case of a "multiplet" $\Delta \nu_0$ is the smallest spacing which occurs between two individual lines. If, with increasing H , $\Delta \nu$ approaches $\Delta \nu_0$ in magnitude, then $\Delta \nu$ no longer increases proportionally to H . As $\Delta \nu$ becomes much larger than $\Delta \nu_0$, the multiplet shrinks, so to speak, into a singlet compared to the strong magnetic field; the Zeeman effect becomes more and more *normal*. This degeneration phenomenon is called the *Paschen-Back effect*. One consequence of this effect is that the hydrogen lines with their extremely small doublet spacing $\Delta \nu_0$ exhibit a normal Zeeman effect even for very weak fields. Therefore these hydrogen lines as well as the helium lines (and not only the singlet lines of parahelium, but also the close triplet lines of orthohelium) were for a long time considered to be typical examples of the normal Zeeman effect.

We shall compute the critical value of H for which $\Delta \nu = \Delta \nu_0$ in the case of hydrogen. The magnitude $\Delta \nu_0$ for the hydrogen doublet is given by the formula $R\alpha^2/2^4$ where R is the "Rydberg frequency" in reciprocal seconds

and $\alpha \sim 1/137$ is the "fine structure constant". In agreement with spectroscopic results this formula yields $\Delta \nu_0 = 1.08 \times 10^{10} \text{ S}^{-1}$. Hence, according to (9), we set

$$(11) \quad \frac{\mu_0}{4\pi} \frac{e}{m} H = 1.08 \times 10^{10} \text{ S}^{-1}.$$

The factors on the left are (see table at the end of Sec. 2):

$$(11a) \quad \frac{\mu_0}{4\pi} = 10^{-7} \text{ M}^{-1} \text{ S}\Omega, \quad \frac{e}{m} = 1.76 \times 10^{11} \text{ QK}^{-1}.$$

Their product has the dimensions

$$(11b) \quad \text{M}^{-1} \text{ S}\Omega \text{ QK}^{-1} = \text{M}^{-1} \text{ S} \frac{\text{Volt}}{\text{Amp}} \text{ QK}^{-1} = \text{M}^{-1} \text{ SK}^{-1} \frac{\text{Joule}}{\text{Amp}} = \frac{\text{MS}^{-1}}{\text{Amp}}$$

Hence, eq. (11) stipulates that

$$1.76 \times 10^{11} H \frac{\text{MS}^{-1}}{\text{Amp}} = 1.08 \times 10^{10} \text{ S}^{-1}, \quad H = 5.8 \times 10^5 \frac{\text{Amp}}{\text{M}}.$$

Since (see Vol. III, Sec. 8, eq. (5a))

$$(11c) \quad 1 \frac{\text{Amp}}{\text{M}} = 4\pi \cdot 10^{-3} \text{ Oerstedt}$$

the desired value turns out to be

$$(12) \quad H = 4\pi \times 5.8 \times 10^5 = 7200 \text{ Oerstedt}.$$

This agrees well with very precise experiments made by Försterling and Hansen¹ with a Lummer plate. They observed the beginning of the Paschen-Back effect at 4000 Oerstedt and found that the π -components of the hydrogen doublet merged at 10,000 Oerstedt. In the case of the *D*-lines whose $\Delta \nu_0$ is fifty times that of the hydrogen doublet, the critical field is, instead of (12),

$$(12a) \quad H = 50 \times 7200 = 360,000 \text{ Oerstedt},$$

a field strength which even today is not easy to attain.

Before ending our brief description of the extremely interesting subject of the anomalous Zeeman effect, we wish to reproduce a photometer curve taken by Zeeman which he very kindly contributed to the fifth edition of "Atombau und Spektrallinien" Vol. I, p. 523. This result will serve to demonstrate the progress in technique which has been made in this field. The line in question is the chromium line $\lambda = 4254 \text{ \AA}$ from the septet system of that element. In agreement with Landé's theory the splitting consists of seven π -components ($\Delta \nu > \Delta \nu_{\text{norm}}$) and twice seven σ -components ($\Delta \nu \geq \Delta \nu_{\text{norm}}$).

¹Z. f. Phys. 18, p. 26, 1923. A precise comparison of these observations with the theory of the Paschen-Back effect can be found in Sommerfeld and Unsöld, *ibid.* 86, p. 268, 1926.

Runge's denominator is 4. All 21 components are beautifully discernible on the photometric curve which represents an automatic thirty-six fold enlargement of the original photographic pattern.

Returning once more to the *normal* Zeeman effect and its splitting $\Delta \nu_{\text{norm}}$, we evaluate eq. (9) numerically using the numerical values and dimensions given in eqs. (11 a, b, c). Thus we obtain

$$(13) \quad \Delta \nu_{\text{norm}} = 1.76 \times 10^4 H_{\text{Amp/M}} = 1.76 \times 10^4 \frac{10^3}{4\pi} H_{\text{Oerstedt.}}$$

where $\Delta \nu$ has the dimensions sec^{-1} , in agreement with the definition of ν as frequency. However, we want to express this result in the dimensions cm^{-1} as is customary in spectroscopy (reciprocal wave length instead of reciprocal period of oscillation). To do this, we must divide (13) by $c = 3 \times 10^{10} \text{ cm/sec}$. Then we obtain

$$(13 a) \quad \Delta \nu_{\text{norm}} = 4.67 \times 10^{-5} H.$$

Since the units Gauss and Oerstedt are defined as the basic units of the absolute cgs-system, H in (13 a) means H_{Oerst} as well as H_{abs} .

The entire discussion of this section has remained within the realm of *classical* mechanics and electrodynamics. That these results still remain valid in the *quantum theory* is due to the fact that Planck's constant h , which

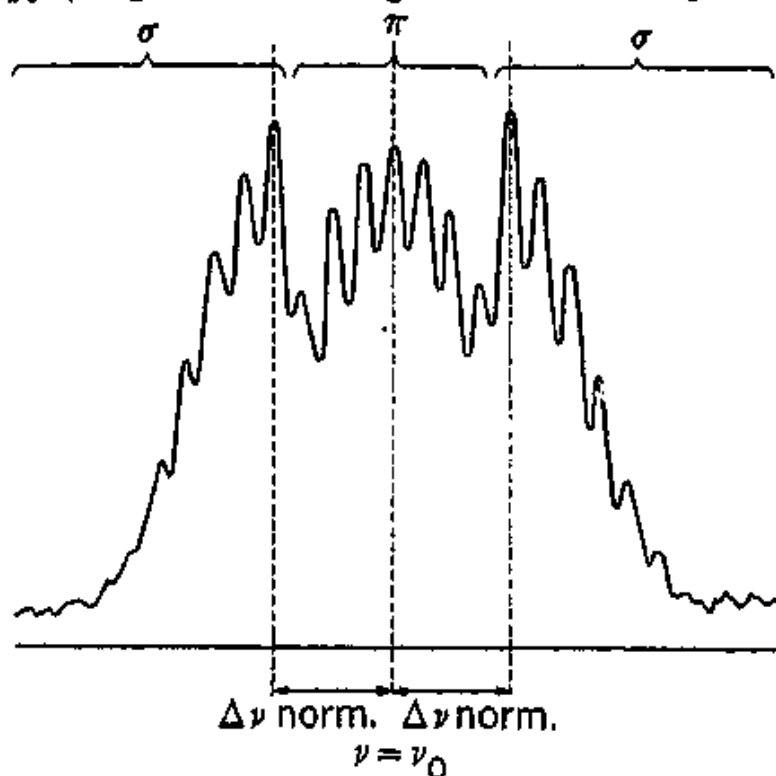


Fig. 27.

Photometer curve of the anomalous Zeeman effect for the chromium line $\lambda 4254 \text{ \AA}$.

is characteristic of the quantum theory, accidentally, so to say, drops out of the quantum conditions for magnetically influenced spectral lines. A somewhat similar statement can be made with regard to the anomalous Zeeman effect. The introduction of electron spin and the "vector model", which is constructed from the spins and the orbital angular momenta of the electrons, made it also possible to formulate a theory of the anomalous Zeeman effect even before the introduction of a definitive quantum theory. In that way Runge's rule, the Paschen-Back effect, etc. could be understood. The complete theory

of spin and the anomalous Zeeman effect, however, had to await the relativistic Dirac theory. We want to point out briefly the interesting interpretation of $\Delta \omega$ in (7) as the *Larmor frequency*. According to this point of view the additional frequency defined in (7) can be considered as the *angular velocity* of an additional rotation which the emitting atom experiences in a magnetic field which is switched on infinitely slowly (adiabatically). See exercise III. 4.

22. Phase Velocity, Signal Velocity, Group Velocity

In our discussion of dispersion we considered only the steady, purely sinusoidal state of the electrons and ions. It was made clear that without these induced oscillations no dispersion and refraction, and therefore no value of n different from 1 can exist. Hence also the phase velocity $u = c/n$ refers exclusively to purely periodic states of the light and of matter, which is to say, states that will last for all eternity and were established an infinitely long time ago.

A. FOURIER REPRESENTATION OF A BOUNDED WAVE TRAIN

This fact immediately enables us to overcome an objection to the theory of relativity which was much discussed around the year 1910¹. In a region of anomalous dispersion it may happen that $n < 1$, hence $u > c$. To see this we need only assume in fig. 22 that the medium has no infrared resonance oscillations. Then the value of n_m in the figure is equal to 1 and the γ -curve which coincides with the n -curve (except in the immediate vicinity of the absorption frequency) lies below the line $n = n_m = 1$ to the right of D . Thus u would be a *velocity greater than that of light* which cannot exist according to the theory of relativity.

However, we emphasized in Vol. III, Sec. 27 F, that this prohibition is limited to processes which can serve as a *signal* and are able to initiate material events. A monochromatic light wave without beginning or end can do no such thing. The Morse signals used in wireless telegraphy are interrupted wave trains. So far, our considerations in no way imply that the front of such a Morse signal propagates with the phase velocity u . In order to be able to apply our previous results to such a signal we must decompose the interrupted signal into a sum of purely periodic waves without beginning or end. We do this with the aid of the *Fourier Integral*.

¹ Gesellschaft der Naturforscher 1907, *Physikalische Zeitschrift* 8, p. 841, and Weber *Festschrift* 1912 (publ. by Teubner). Further discussed in *Ann. d. Phys. (Lpz.)* 44, 1914: A. Sommerfeld, p. 177, L. Brillouin, p. 203.

The resulting spectrum of partial waves is calculated in Vol. VI, exercise I.4, and is represented there in fig. 33 c. It can be described as a "fluted spectrum" which has a pronounced maximum at $\omega = 2\pi/\tau$ and a half-width which decreases as the length of the wave train contained in the signal increases. This result pertains to a signal which consists of a finite sequence of identical sine oscillations of period τ . It is noted in Vol. VI that such a wave train which is bounded on two sides can be treated as the difference between two wave trains each of which is bounded at only one end.

However, for a signal bounded on only one side, such as

$$(1) \quad f(t) = \begin{cases} 0 & t < 0 \\ \sin 2\pi t/\tau & t > 0 \end{cases}$$

the usual form of the Fourier integral fails because the latter obviously diverges since $f(t)$ does not vanish as $t \rightarrow \infty$. In Vol. VI the Fourier integral is, therefore, replaced by a converging contour integral in the *complex* plane.

We repeat¹ the same procedure here. The original path of integration shall be the upper curve in fig. 28 a:

$$(2) \quad f(t) = -\frac{1}{\tau} \int e^{-i\omega t} \frac{d\omega}{\omega^2 - (2\pi/\tau)^2}.$$

One sees immediately that for negative values of t , $-i\omega t$ has a negative real part in the upper half of the complex ω -plane, and that this real part goes to $-\infty$ as the distance from the real axis increases; that is, $\exp(-i\omega t)$ becomes vanishingly small. Since there is nothing to prevent us from shifting the path of integration to infinity in the upper half-plane (indicated by the \uparrow arrows), $f(t)$ vanishes as required by the first line in (1). For $t > 0$, however, $\exp(-i\omega t)$ vanishes at infinity when approached through the lower half-plane. If the path of integration is pushed out to infinity in the lower half plane (indicated by the \downarrow arrows), it is left hanging on the poles $\omega = \pm 2\pi/\tau$.

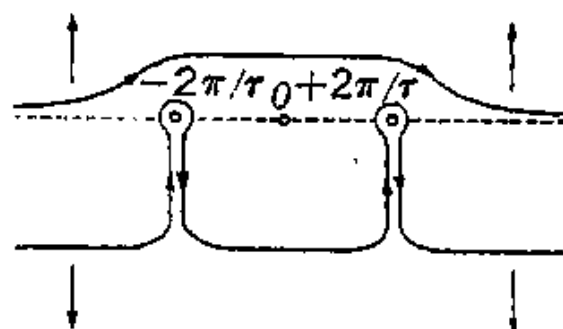


Fig. 28 a.

Representation of a wave train bounded at one end. Path of integration in the complex ω -plane.

Integration around these poles yields $-2\pi i$ times the sum of the residues at the two poles (direction of integration in a negative sense in the complex

¹In a somewhat changed form since we write $\exp(-i\omega t)$ instead of $\exp(+i\omega t)$. Correspondingly, the upper and lower half planes of the complex ω -plane are interchanged in the present figure in comparison to fig. 24 b in Vol. VI.

plane). Since the contributions to the integral of the paths from the poles to infinity and back cancel each other, it follows from (2) that

$$(2a) \quad f(t) = \frac{2\pi i e^{-2\pi i t/\tau} - e^{+2\pi i t/\tau}}{4\pi/\tau} = \sin 2\pi t/\tau$$

as required by the second line of (1).

B. PROPAGATION OF THE WAVE FRONT IN A DISPERSIVE MEDIUM

We now consider one of the individual component oscillations of (2) having the time dependence $\exp(-i\omega t)$ and complement it to form the wave $\exp i(kx - \omega t)$ which propagates in the positive x -direction. At time $t = 0$ let the front of the resulting wave train fall upon the boundary plane $x = 0$ of a dispersive medium which extends from $x = 0$ to $x = \infty$. Each component wave train knows nothing, so to speak, of its origin in the bounded wave train but behaves exactly like the plane wave in a dispersive medium which we treated in Sec. 17. Therefore, we can use for k the value obtained in Sec. 17:

$$(3) \quad k = \frac{\omega n}{c}, \quad n^2 = 1 + \frac{a^2 \omega_0^2}{\omega_0^2 - \omega^2}, \quad a^2 = \frac{N e^2}{m \epsilon_0 \omega_0^2}.$$

The abbreviation a^2 is the same as that used in eq. (19.5).

Treating all component waves of the aggregate (2) in this same way and superposing them by means of our complex integral, we obtain a possible state in the dispersive medium which has the form (2) at $x = 0$ and is, therefore, the complete solution of our problem, namely

$$(4) \quad f(x, t) = -\frac{1}{\tau} \int e^{i(kx - \omega t)} \frac{d\omega}{\omega^2 - (2\pi/\tau)^2}.$$

It remains only to discuss this expression for values of $x > 0$.

To do this we must know its singularities in the ω -plane. These are, besides the poles $\omega = \pm 2\pi/\tau$, the singularities of k . (3) yields

$$(5) \quad k = \frac{\omega}{c} \sqrt{\frac{\omega_0^2(1+a^2) - \omega^2}{\omega_0^2 - \omega^2}} = \frac{\omega}{c} \sqrt{\frac{\omega - \omega_1}{\omega - \omega_0}} \sqrt{\frac{\omega + \omega_1}{\omega + \omega_0}}, \quad \omega_1 = \omega_0 \sqrt{1+a^2}.$$

Thus k has two pairs of branch points. For small values of a ($\omega_1 \sim \omega_0$, $n \sim 1$) it is best to treat ω_0 and ω_1 together as one pair, and $-\omega_0$ and $-\omega_1$ as another. Each of these pairs of branch points is joined by a branch cut which the path of integration must not cross. Since damping was neglected in (3) so that

ω_0 and ω_1 in (5) are real, we consider the cuts to be along the real axis in fig. 28 b. In any case the upper half-plane is free of singularities. According to (3), k approaches ω/c asymptotically at infinity in the upper half-plane. Hence, there we can replace

$$(6) \quad \exp \{i(kx - \omega t)\} \quad \text{by} \quad \exp \left\{ i \omega \left(\frac{x}{c} - t \right) \right\}.$$

From this it follows that for $t < x/c$ the argument of the exponential function has a negative real part in the upper half-plane of the complex ω -plane. Hence, we can shift the integral (4) into the upper half-plane and thus obtain

$$(7) \quad f(x, t) = 0 \quad \text{for} \quad t < \frac{x}{c}.$$

The wave front penetrates to the depth x in the medium only after a time $t \geq x/c$. It certainly does not propagate with a velocity greater than that of light. If any light at all is noticeable at the time $t = x/c$, (see C.), then it must have propagated with the velocity c of light in vacuum.

This is also made clear by the following consideration: the dispersion electrons are originally at rest (their thermal agitation which is in no way related to the rhythm of the light wave can obviously be disregarded). But according to our theory, refraction and dispersion are due entirely to the induced periodic oscillations of the electrons or ions. Thus, to begin with, the medium is *optically void* like a vacuum. The propagation velocity is equal to c and the index of refraction, if one still cares to speak of one, is equal to 1.

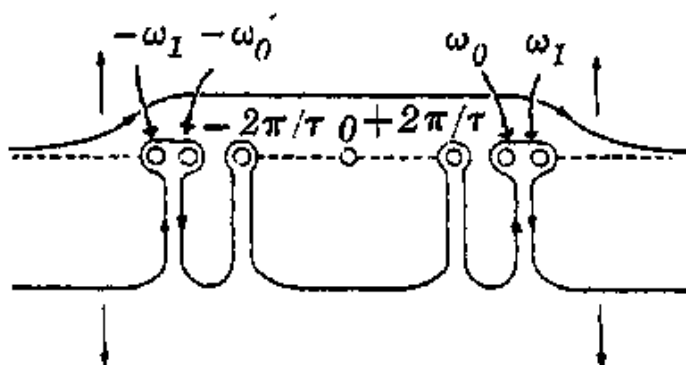


Fig. 28 b.

Propagation of the wave train in a dispersive medium. In deforming the integration path downward the poles and branch points must be taken into consideration.

So far we have assumed that the wave train falls *perpendicularly* upon the surface $x = 0$. If we let it enter obliquely, then at first it is neither *refracted* nor *reflected*. The law of refraction takes effect only as the electrons are brought into *forced oscillations*. Accordingly, on a photographic plate placed behind a dispersive medium, the gap between the light spot corresponding to the regular refraction and the rectilinear projection of the incident beam should be bridged by an extremely weak line of light.

So far we have assumed our medium to be *isotropic*. If the medium is a crystal, calcite for instance, then *no double refraction* should appear at the moment of incidence of the wave train. Such crystals also must be traversed by the initially incident wave train in an undeviated straight line.

It is obvious, however, that the above paradoxes depend on a practically unattainable degree of monochromatism, straightness of direction, and regularity of the wave train.

C. THE PRECURSORS

We shall use this name, adopted from seismology, to denote the events observed at a depth x immediately following the arrival of the initial wave front. We introduce the time interval

$$t = t - \frac{x}{c}$$

which according to the above discussion is positive and which we shall assume to be very small. We deform the original integration path of fig. 28 b into a

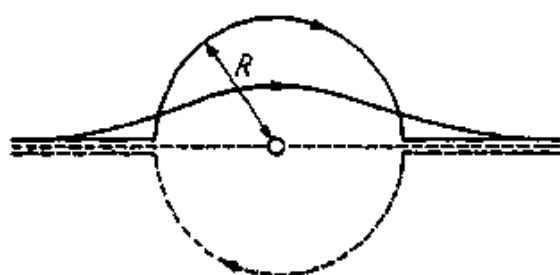


Fig. 29.

Transformation of the path of integration of fig. 28b into a very large semicircle in the upper half-plane (for the calculation of the precursors).

semicircle of very large radius R in the upper half-plane plus the segments of the real axis, as shown in fig. 29. Because of the denominator $\omega^2 - (2\pi/\tau)^2$, the integrand goes to zero as $1/\omega^2$ on the real axis. We may add the path in the lower half plane which is shown as a dotted line in the figure, for if the radius of the semicircular portion of this lower path is increased to infinity, the integrand vanishes exponen-

tially because $t > 0$. Therefore, we may replace our original path of integration by the entire circle. Expressing t in terms of t we obtain, instead of (4),

$$(8) \quad f(x, t) = -\frac{1}{\tau} \oint \exp i \left\{ \left(k - \frac{\omega}{c} \right) x - \omega t \right\} \frac{d\omega}{\omega^2 - (2\pi/\tau)^2}.$$

Now, according to (3), for large $|\omega|$

$$(8a) \quad k - \frac{\omega}{c} = (n-1) \frac{\omega}{c} = \left(\sqrt{1 - \frac{a^2 \omega_0^2}{\omega^2}} - 1 \right) \frac{\omega}{c} = -\frac{a^2 \omega_0^2}{2c\omega}.$$

Using the abbreviation

$$(9) \quad \xi = \frac{a^2 \omega_0^2}{2c} x,$$

and henceforth neglecting $2\pi/\tau$ as compared to ω , we obtain from (8)

$$(10) \quad f(x, t) = f_1(\xi, t) = \frac{-1}{\tau} \oint \exp i \left\{ -\frac{\xi}{\omega} - \omega t \right\} \frac{d\omega}{\omega^2} = \\ = \frac{-1}{\tau} \oint \exp -i \left\{ \sqrt{\xi t} \left(\frac{1}{\omega} \sqrt{\frac{\xi}{t}} + \omega \sqrt{\frac{t}{\xi}} \right) \right\} \frac{d\omega}{\omega^2}.$$

This integral can be transformed into a known form by making the substitution

$$(11) \quad \omega \sqrt{\frac{t}{\xi}} = e^{i\varphi}, \quad \frac{d\omega}{\omega} = i d\varphi, \quad \frac{d\omega}{\omega^2} = i \sqrt{\frac{t}{\xi}} e^{-i\varphi} d\varphi.$$

Then (10) becomes

$$(12) \quad f_1(\xi, t) = \frac{-i}{\tau} \sqrt{\frac{t}{\xi}} \oint \exp \{-2i \sqrt{\xi t} \cos \varphi\} e^{-i\varphi} d\varphi.$$

Taking our radius R equal to $\sqrt{\xi t}$ (since $t \ll 1$, this is indeed a very large radius), φ becomes, by (11), the central angle of our circle, and its value therefore goes from 0 to 2π along the path of integration. Now we compare (12) with the familiar integral representation of the Bessel function of order 1 (see, for instance, Vol. VI, eq. (19.8):

$$(12a) \quad J_1(\varrho) = \frac{1}{2\pi} \oint_0^{2\pi} \exp(i\varrho \cos \varphi) e^{i\left(\varphi - \frac{\pi}{2}\right)} d\varphi.$$

Because J_1 is real for real ϱ , we can change the sign of i . Then we see that (12) can simply be written as

$$(13) \quad f_1(\xi, t) = \frac{2\pi}{\tau} \sqrt{\frac{t}{\xi}} J_1(2\sqrt{\xi t}).$$

From the behavior of $J_1(\varrho)$ for small ϱ (where J_1 becomes equal to $\varrho/2$) we find the state of the signal immediately upon its arrival at the depth x to be the following: the initial amplitude is very small compared to 1, that is to the amplitude of the incident oscillation. The initial period of oscillation is extremely small compared to the incident period τ . The amplitude and period both increase with increasing t , the former because of the factor \sqrt{t} , the latter because of the positions of the roots $J_1(\varrho) = 0$ which are spaced at distances of approximately π apart. This gives the following time interval for the m^{th} half-period of the precursor:

$$\Delta t_m \sim \frac{m\pi^2}{2\xi}.$$

According to (9) this value is independent of the incident period of oscillation τ . It depends only on the depth x and on the dispersive power of the medium. For not too small values of x the first precursors are perhaps in the x -ray region. Figure 30 illustrates this phenomenon qualitatively by means of a very coarsely scaled graph.

D. THE SIGNAL IN ITS FINAL STEADY STATE

In contrast to C, we now assume t to be so large that the electrons have already attained their final state of oscillation of period τ . The process by which the electrons attain this state is, clearly, represented by the integration around the two pairs of branch points in the lower part of fig. 28 b. The

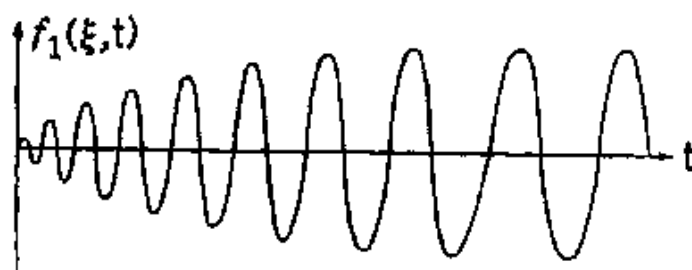


Fig. 30.

Schematic sketch of the excitation immediately upon arrival of the precursors.

positions of these branch points depend on the nature of the electron binding forces and on the resonance frequencies of the electrons. But now the previously neglected damping term must be taken into consideration, so that the resonance oscillations can die out. If we do this, the branch points which fig. 28 b

show to lie on the real axis are shifted somewhat downward into the lower half plane. This means that for very large values of t the factor $\exp(-i\omega t)$ which occurs in (8) becomes very small. Therefore, the contributions to the integral from the paths around the branch points vanish. All that is left, then, are the loop integrals around the two poles $\pm 2\pi/\tau$ on the real axis in fig. 28 b and these can be evaluated directly by the method of residues.

At the two poles we have, according to (3),

$$k = \pm \frac{2\pi n}{\tau c} = \pm \frac{2\pi 1}{\tau u}$$

where n and u are the index of refraction and the phase velocity, respectively, which belong to the period of oscillation τ . Then we get from (4)

$$\begin{aligned} f(x, t) &= \frac{2\pi i}{\tau} \left\{ \exp \left[\frac{2\pi i}{\tau} \left(\frac{x}{u} - t \right) \right] - \exp \left[-\frac{2\pi i}{\tau} \left(\frac{x}{u} - t \right) \right] \right\} \frac{1}{4\pi/\tau} = \\ (14) \quad &= \sin \left\{ \frac{2\pi}{\tau} \left(t - \frac{x}{u} \right) \right\}. \end{aligned}$$

But this is precisely the wave pattern that results from the incident wave (2 a) when it is displaced towards increasing x with phase velocity u .

In fig. 31 we have plotted t horizontally to the right and the depth x in the medium divided by c vertically downward. The straight line $t = x/c$ makes an angle of 45° with the horizontal and it marks for each depth x the time of arrival of the precursors. The line $t = x/u$ makes a smaller angle with

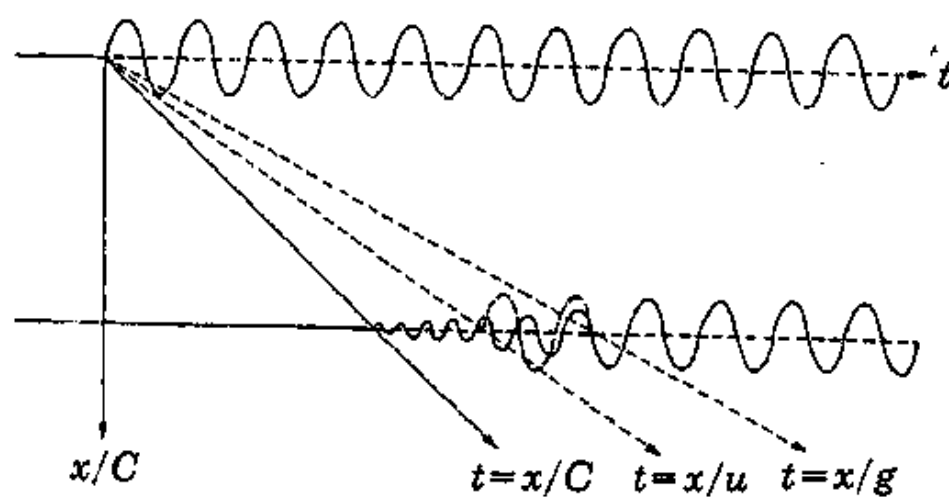


Fig. 31.

Scheme of propagation of a wave to a depth x in a dispersive medium. Transition of the precursors to the steady state.

the horizontal because $u < c$ and it indicates how the amplitude and phase are transmitted to the depth x : The wave train starting at time $t = 0$ on the surface reproduces itself identically at the depth x , its phase being merely shifted by x/u . Any other result would be disastrous to the theory of interference phenomena which rests upon the precise transmission of the phase through dispersive media.

To be sure, the validity of eq. (14) is assumed only if a sufficiently long time $t = t - x/c$ has elapsed. This condition is not necessarily fulfilled at $t = x/u$. Therefore, we have shown the beginning of the wave train which starts at $t = x/u$ as a dotted curve and have drawn it as a solid curve only after a later time $t = x/g$. The locus of points $t = x/g$ is drawn in the figure as a dotted straight line (just like the locus of points $t = x/u$). If $g < u$, this line makes a smaller angle with the horizontal than the line $t = x/u$.

E. GROUP VELOCITY AND ENERGY TRANSPORT

The concept of group velocity is familiar from hydrodynamics, Vol. II, Sec. 26. This velocity refers not to the propagation of the phase but rather to the propagation of energy (or amplitude). We denote¹ the group velocity by g and compare its formal definition with that of the phase velocity;

$$(15) \quad g = \frac{d\omega}{dk}, \quad u = \frac{\omega}{k}.$$

Because $\omega = uk$, $d\omega = u dk + k du$, it follows that

$$g = u + k \frac{du}{dk}.$$

If we take into consideration that $k = 2\pi/\lambda$, $dk/k = -d\lambda/\lambda$, then we can write

$$(16) \quad g = u - \lambda \frac{du}{d\lambda}.$$

For *normal* dispersion

$$\frac{dn}{d\lambda} < 0, \quad \text{and hence } \frac{du}{d\lambda} > 0 \quad \text{because } u = \frac{c}{n}.$$

Therefore, as was emphasized already in Vol. II, loc. cit.,

$$(16 a) \quad g < u.$$

This fact accounts for the smaller slope of the line $t = x/g$ in fig. 31. In the case of anomalous dispersion this line would be *steeper* than the line $t = x/u$.

As in hydrodynamics we expect that the full amplitude 1 of the incident wave will be attained not at $t = x/u$, but at $t = x/g$. This has been confirmed by L. Brillouin in the paper referred to on p. 114. The previously neglected contour integrals around the branch points in fig. 28 b are here essential and are discussed precisely by Brillouin by means of the saddle-point method (Vol. VI, Sec. 19 E and Sec. 21 D). The evaluation of these integrals is not too simple and will be omitted here. The result shows that the precursors are followed by a *transition state* which corresponds to the gradual building up of the electron oscillations to the point where these oscillations correspond to the incoming frequency and amplitude. Thus the ultimate steady state of amplitude 1 is reached not at the time $t = x/u$ but rather (for normal dispersion) at the later time $t = x/g$. In this final state the free oscillations of the electrons have been damped out and only the forced oscillation of period τ remains. The wave train which is drawn dotted in fig. 31 is to be replaced by

¹In Vol. II we wrote V , U instead of the present notation u , g .

this transition state¹. Beyond $t = x/g$ the solid wave train correctly represents the final amplitude and phase. The formula $t = x/g$ also gives the time required for the transport of light energy through a distance x in the dispersive medium.

These results can also be derived directly by the *method of stationary phase*². The phase of the exponential function under the integral in (4)

$$\varphi(\omega) = kx - \omega t$$

is "stationary" with respect to a displacement along the real ω -axis if

$$(17) \quad \frac{d\varphi}{d\omega} = \frac{dk}{d\omega} x - t = 0.$$

While neighboring oscillations generally cancel each other when integrated with respect to ω because of the changes in sign of the exponential function, this is not the case for the "stationary" ω given by (17). In the vicinity of this ω the contributions to the integral have the same sign and add. Therefore, the energy propagation is determined essentially by the stationary ω , that is, by eq. (17). According to the definition (15) of g , (17) does indeed yield

$$t = \frac{x}{g}.$$

We conclude with a remark, due to Lord Rayleigh, concerning the measurement of the velocity of light c . Fizeau's toothed wheel, as well as Foucault's mirror, uses cut-off wave trains of the type we have considered here. Therefore, it is the group velocity and not the phase velocity which determines the time interval necessary for the light to pass through the required distances in air. Thus, these experiments really measure g , and not u or c . It is only because of the small dispersion and refraction in air that $g \sim u$ and that c can be computed from u by applying a small correction.

23. The Wave-Mechanical Theory of Dispersion

So far we have not made use of an atomic model. We shall now show how according to Schrödinger³ we can obtain a deeper understanding of the theory of dispersion if we replace our previous rough assumption of a "quasielastic binding" by well-defined wave-mechanical binding energies.

¹According to fig. 20 in Brillouin, loc. cit., the transition from $t = x/u$ to $t = x/g$ is by no means as simple as we have sketched it in fig. 31.

²See Vol. II Sec. 27 under 3, where we also used this method as a substitute for the mathematically precise saddle-point method.

³In his fourth communication, Ann. d. Phys. (Lpz.) 81, 1927.

Of course, we cannot here develop the formalism of wave mechanics with any degree of completeness. Rather, we must limit ourselves to describing the progress made by the wave-mechanical treatment as compared to our previous discussion. This will be done in subsection A. In subsection B we shall only indicate how the dispersion formula used in A may be derived.

We shall use the particularly simple case of the Na spectrum as an example (qualitatively the same results will hold for the other alkalis). We consider, therefore, a gas of Na atoms mixed with one of the noble gases (the latter will not further enter into our considerations). If we illuminate this gas with the prismatically decomposed continuous spectrum of a hot flame, then in the light which has passed through the gas, the *principal series* of the Na atom appears as an absorption spectrum. The first of these lines is the yellow *D*-line¹. The spectrum has a series limit in the near ultraviolet where the lines of higher frequency converge. By means of extremely refined resolving apparatus spectroscopists have been able to find and measure more than 50 lines in the principal series.

We denote the angular frequencies of these series lines

$$\omega_1 \text{ (D-line), } \omega_2, \omega_3, \dots, \omega_\infty.$$

Wave mechanics associates with these frequencies the energy levels of the atom

$$W_0, W_1, W_2, \dots, W_\infty.$$

Let W_0 be the energy of the atom in its ground state and let W_1, W_2, \dots be the energies of excited states when the valence electron of the Na-atom is lifted out of its original orbit² into a higher orbit which is more distant from the atom. $W_\infty = W_I$ is the ionization energy needed to separate the electron from the atom and leave a Na^+ ion behind. Above the series limit there is a continuous spectrum of ω -values, or energy levels W , which we will not need to go into further here. The connection between ω_j and W_j is

$$(1) \quad \omega_j = \frac{W_j - W_0}{\hbar}$$

where \hbar is Planck's quantum of action divided by 2π . See fig. 32.

¹We can here neglect the doublet nature of the *D*-line.

²Abbreviation for "eigenfunction".

A. COMPARISON OF THE OLDER DISPERSION FORMULA WITH THE WAVE-MECHANICAL FORMULA

If we take into account not just a single resonance frequency ω_0 but a series of such frequencies $\omega_1, \dots, \omega_j, \dots$, our dispersion formula (17.10) becomes

$$(2) \quad n^2 - 1 = \frac{e^2}{m \varepsilon_0} \sum_j \frac{N_j}{\omega_j^2 - \omega^2}.$$

N_j is the number of electrons per unit volume which have a resonance frequency ω_j . The numbers N_j themselves are unknown. But they must collectively satisfy

$$(2 a) \quad \sum_j N_j = N$$

where N is the number of Na-atoms and therefore also (for single valency Na) the total number of valence electrons per unit volume.

In place of this, wave mechanics yields

$$(3) \quad n^2 - 1 = \frac{e^2}{m \varepsilon_0} \sum_j \frac{N f_j}{\omega_j^2 - \omega^2}.$$

f_j is called the "transition probability", or the "oscillator strength", and is a definite number which can be calculated from the atomic model by wave-mechanical methods. It is subject to the "sum rule" which is analogous to the requirement (2 a), namely,

$$(3 a) \quad \sum f_j = 1.$$

The difference between eq. (2) and (3) lies not so much in the greater definiteness which distinguishes the latter because the f_j occurring in it can be computed. The difference lies principally in the meaning of the ω_j . In eq. (2) which we have taken over from Sec. 17 the ω_j are *resonance frequencies* of different electrons with different binding energies. In (3) the ω_j are *frequencies of transition* of one and the same valence electron from the excited state W_j to the ground state W_0 . In (2) the oscillations ω_j occur side by side and independently of one another. In (3) the transitions take place one after the other, depending on the excitation just preceding each, so that they mutually exclude one another. Thus, in spite of their formal similarity, the meanings of eq. (2) and (3) are quite different. This new interpretation of the ω_j as energy differences is equivalent to Ritz's *combination principle* which, since Bohr, has been the foundation of the theory of spectral lines. The ideas on

which eq. (3) is based are represented schematically in fig. 32. The energy levels W_i are plotted vertically upward from the ground state W_0 to the ionization energy W_J . The transition frequencies ω_i with their limiting value ω_∞ are plotted horizontally to the right.

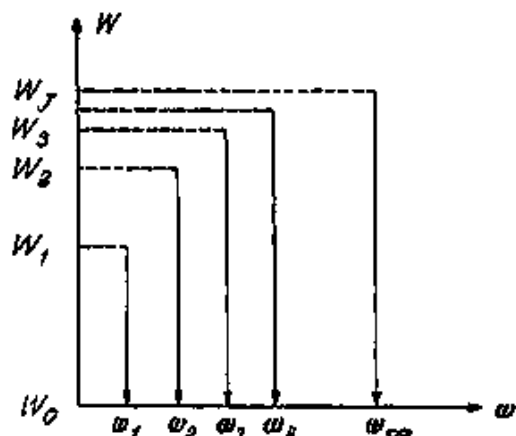


Fig. 32.

Correspondence between emission frequencies $\omega_1, \omega_2, \dots$ and energy level W_1, W_2, \dots and the ground level W_0 .

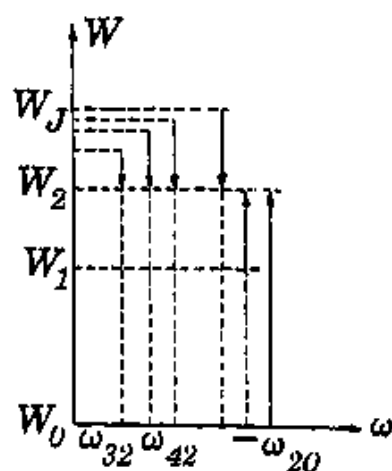


Fig. 32 a.

Dispersion of light due to an excited atom. Besides the positive dispersion terms also negative terms appear which correspond to transitions from deeper lying levels.

discrete set of values $W = W_0, W_1, \dots, W_k, \dots$. These solutions are the "eigenfunctions" of the atom. The k^{th} of these functions, completed so as to include its time dependence, is:

The wave-mechanical scheme can be extended considerably. Instead of starting with the ground state, we could investigate the dispersion formula for any desired excited state of energy W_k . Then we would have to draw the arrows originating above W_k only down to W_k . Arrows pointing up from the levels below W_k would also enter the picture and these would contribute negative dispersion terms. See fig. 32 a in which we have chosen $k = 2$. In such a case the transition frequencies must be provided with double indices ω_{jk} . To be consistent the ω_i in fig. 32 should, then, be denoted as ω_{j0} .

B. OUTLINE¹ OF THE DERIVATION OF EQ. (3)

The Schrödinger equation for the wave function ψ of our valence electron reads

$$(4) \quad \Delta \psi + \frac{2m}{\hbar^2} (W - V) \psi = 0.$$

V is the potential of the force field and takes into account not only the attraction of the nucleus but also the mean repulsions of the remaining atomic electrons. Equation (4) has continuous, normalizable solutions only for the

¹For further details see any text book on wave mechanics, e. g. "Atombau und Spektrallinien", Vol. II, p. 360.

$$(5) \quad u_k = \psi_k \exp\left(-\frac{i W_k t}{\hbar}\right).$$

Let the atom be perturbed by an incident light wave of angular frequency ω which propagates in the x -direction and is polarized in the y -direction. We represent the space-time dependence of this wave by

$$e^{i\omega\left(t-\frac{x}{c}\right)} + e^{-i\omega\left(t-\frac{x}{c}\right)}.$$

The perturbed state u of the valence electron satisfies the time-dependent Schrödinger equation

$$(6) \quad \Delta u + \frac{2im}{\hbar} \frac{\partial u}{\partial t} - \frac{2m}{\hbar^2} V u = a \left\{ e^{i\omega\left(t-\frac{x}{c}\right)} + e^{-i\omega\left(t-\frac{x}{c}\right)} \right\} \frac{\partial u}{\partial y}.$$

a is a constant factor which is proportional to the amplitude of the incident wave. The reader may convince himself that for the case of no perturbation ($a = 0$, $u = u_k$) eq. (6) reduces to eq. (4). The factor $\partial u / \partial y$ on the right corresponds to the term $(\mathbf{A} \cdot \text{grad } u)$ in the general time dependent Schrödinger equation and takes that form because the "vector potential" \mathbf{A} has the direction of the light vector \mathbf{E} which, according to our assumption, is the y -direction.

Owing to the perturbation the state (5) becomes

$$(7) \quad u = u_k + a \left\{ w_+ \exp\left(-\frac{i}{\hbar} W_k t - i \omega t\right) + w_- \exp\left(-\frac{i}{\hbar} W_k t + i \omega t\right) \right\}$$

where the perturbation factors w_{\pm} must satisfy a time-independent differential equation which is derived from (6), namely

$$(7a) \quad \Delta w_{\pm} + \frac{2m}{\hbar^2} (W_k \pm \hbar \omega - V) w_{\pm} = \frac{\partial \psi_k}{\partial y} e^{\pm \frac{i \omega x}{c}}.$$

This equation can be integrated by the general methods of perturbation theory. The right-hand side of (7a) which is to be considered as a known function of the position x, y, z must be expanded into a series in terms of the complete system of eigenfunctions ψ_j ; that is, it is written in the form¹

$$(8) \quad \sum_j A_j \psi_j.$$

¹Actually we should have denoted the coefficients A_j of the series (8) by A_j^{\pm} , corresponding to the \pm sign on the right-hand side of eq. (7a). But because the wave length $\lambda = 2\pi c/\omega$ of the incident light is very large compared to the size of the atom, the exponent $\pm i \omega x/c$ is very small for all values of x which come into consideration. Therefore, the \pm sign can be suppressed for A_j .

In the same way we write on the left-hand side of (7 a)

$$(8 \text{ a}) \quad w_{\pm} = \sum_i B_i^{\pm} \psi_i$$

and obtain

$$(9) \quad \sum_i B_i^{\pm} \left\{ \Delta \psi_i + \frac{2m}{\hbar^2} (W_k \pm \hbar \omega - V) \psi_i \right\} = \sum_i A_i \psi_i.$$

If we substitute here the value of $\Delta \psi_i$ as given by (4), the position dependent quantity V is eliminated from the left-hand side and (9) simplifies to

$$(9 \text{ a}) \quad \frac{2m}{\hbar^2} \sum_i B_i^{\pm} (W_k - W_i \pm \hbar \omega) \psi_i = \sum_i A_i \psi_i.$$

By comparing coefficients one finds

$$B_i^{\pm} = \frac{\hbar^2}{2m} \frac{A_i}{W_k - W_i \pm \hbar \omega}.$$

Applying eq. (1) and recalling the remarks made in connection with fig. 32 a, we may write this as

$$(10) \quad B_i^{\pm} = -\frac{\hbar}{2m} \frac{A_i}{\omega_{ik} \mp \omega}.$$

Thus the values of the *transition frequencies* ω_{ik} (which in (1) are defined for the ground state) result automatically from the perturbation calculation and take the place of the *oscillation frequencies* ω in the dispersion formula (2). With this result we have attained the essential purpose of our wave-mechanical considerations. The following remarks will serve only to show how this result leads to eq. (3) which is analogous to the classical dispersion formula (2).

It follows from (7) that together with w_{\pm} the function u , which describes the perturbed state, may also be represented as a series in terms of the ψ_i . We need not go into the calculation of the coefficients A_i of this series which is performed in the Fourier manner and requires that the system of eigenfunctions ψ_i be known. From u one obtains the density distribution $\rho = u u^*$ and from it is found P_y , the component of the electric moment of this distribution in the direction of polarization y of the incident wave. From \bar{P} which is the average of P_y taken over all possible orientations of the atom the value of $n^2 - 1$ is found. In this way one obtains precisely eq. (3) with a definite expression for f which takes the form of a space integral over the eigenfunctions.

CHAPTER IV

CRYSTAL OPTICS

So far, we have assumed all optical media to be isotropic. But the complete range of optical refinement is revealed only by anisotropic media. The interference patterns of crystal plates in polarized light are among the most beautiful and splendidly colored phenomena of nature. They indicate the regular structure of crystals even more clearly than the outward shape does. Moreover, calcite, mica, and quartz are essential components of some of the most important optical apparatus.

However, we shall in general not deal with the atomistic structure of anisotropic media but shall treat these only from a phenomenological point of view, just as we have done with isotropic media. The simplest assumptions regarding directional dependence and symmetry suffice for a fairly complete description of the phenomena. The condition required by such methods, i. e. that "the wave length of the light shall be large compared to the inter-atomic distances", is certainly fulfilled in the visible spectrum.

24. Fresnel's Ellipsoid, Index Ellipsoid, Principal Dielectric Axes

A medium is electrically anisotropic if the relationship between the excitation \mathbf{D} and the field strength \mathbf{E} is determined by a "linear vector function"

$$\begin{aligned} D_1 &= \epsilon_{11} E_1 + \epsilon_{12} E_2 + \epsilon_{13} E_3, \\ D_2 &= \epsilon_{21} E_1 + \epsilon_{22} E_2 + \epsilon_{23} E_3, \\ D_3 &= \epsilon_{31} E_1 + \epsilon_{32} E_2 + \epsilon_{33} E_3 \end{aligned} \quad (1)$$

rather than by the simple proportionality $\mathbf{D} = \epsilon \mathbf{E}$ as in the isotropic case. 1, 2, 3 in (1) are three mutually perpendicular coordinate directions which are fixed in the crystal in some particular way. The dielectric constant is now not a scalar but a symmetric tensor of rank two. The symmetry condition

$$(1a) \quad \epsilon_{ik} = \epsilon_{ki}$$

follows from the requirement that the work done per unit volume ($\mathbf{E} \cdot d\mathbf{D}$) in building up a field must be a total differential. See Vol. III, eq. (5.6 d) and footnote 2. Only if this symmetry condition is fulfilled does there exist

an "electric energy per unit volume" as a variable of state which is independent of previous events, namely

$$(2) \quad W_e = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D}) = \frac{1}{2} \sum_i \sum_k \varepsilon_{ik} E_i E_k.$$

Because of (1), \mathbf{E} and \mathbf{D} are not parallel but have, in general, different directions.

We have met linear vector functions before on various occasions. There was, for instance, the relationship between the angular velocity and the angular momentum of a rigid body in Vol. I, eq. (24.9). This relationship led to Poinso't's construction: to find the angular momentum \mathbf{M} corresponding to a rotation ω , place a plane tangential to the inertia ellipsoid ($I = \text{const.}$) through the tip of the ω vector, which is in that ellipsoid. Then draw a perpendicular to this tangential plane from the center of the ellipsoid. This perpendicular is in direction and magnitude the desired angular momentum. The same thing written as formulae¹ reads:

$$(3) \quad M_1 = \frac{\partial I}{\partial \omega_1}, \quad M_2 = \frac{\partial I}{\partial \omega_2}, \quad M_3 = \frac{\partial I}{\partial \omega_3}, \quad I = \frac{1}{2} \sum_i \sum_k I_{ik}' \omega_i \omega_k$$

where x_1, x_2, x_3 are the rectangular coordinates of the tip of ω as measured in our 1, 2, 3 system. We may also say: " \mathbf{M} is the normal to that polar plane of the inertia ellipsoid which belongs to the ω -direction."

We have already emphasized in Vol. I that this same construction is valid for any linear vector function which is derived from a symmetric tensor. Indeed, one obtains our eq. (1) for \mathbf{D} from the eq. (3) for \mathbf{M} by replacing in the latter the tensor (I_{ik}') by (ε_{ik}) and the vector ω by \mathbf{E} . Then the characteristic "tensor surface" to be used in the construction

$$(4) \quad \sum_i \sum_k \varepsilon_{ik} x_i x_k = \text{const.}, \quad \text{const.} = 2 W_e$$

is called *Fresnel's ellipsoid*. That this surface is in fact an ellipsoid and not a general second order surface follows, as in the case of the inertia ellipsoid, from the fact that the left-hand side of (4) represents an energy. This left-hand side must therefore be a positive definite quadratic form.

The polar planes belonging to the three principal axes of the ellipsoid are perpendicular to these axes. Hence along these axes and only along them are \mathbf{D} and \mathbf{E} parallel. We call these axes "principal dielectric axes" (in contrast

¹ The I_{ik}' are related to the usual products of inertia I_{ik} of Vol. I in the following way: $I_{ii}' = I_{ii}$, $I_{ik}' = -I_{ik}$. This changed notation is clearly convenient for the comparison with the ε_{ik} in eq. (1) and (2).

to the "optic axes" to be introduced later). If these axes are chosen as the coordinate axes, one obtains instead of (1)

$$(5) \quad D_1 = \varepsilon_1 E_1, \quad D_2 = \varepsilon_2 E_2, \quad D_3 = \varepsilon_3 E_3.$$

These ε_i are called "*principal dielectric constants*". Just as in mechanics where the products of inertia I_{ik} vanish in the coordinate system of the principal moments of inertia, so here the mixed ε_{ik} vanish and Fresnel's ellipsoid assumes the form

$$(6) \quad \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \varepsilon_3 x_3^2 = \text{const.}, \quad \text{const.} = 2 W_e.$$

Making use of Maxwell's relationship for non-magnetic media $n = \sqrt{\varepsilon/\varepsilon_0}$, we can write instead of (6)

$$(6a) \quad n_1^2 x_1^2 + n_2^2 x_2^2 + n_3^2 x_3^2 = \text{const.}, \quad \text{const.} = \frac{2 W_e}{\varepsilon_0}$$

The n_i which are here introduced are called the "*principal indices of refraction*". Equation (6a) shows that *the lengths of the principal axes of Fresnel's ellipsoid are the reciprocals of the three principal indices of refraction*. For later use we also define the three "*principal light velocities*" as

$$(6b) \quad u_i = \frac{c}{n_i} = \frac{1}{\sqrt{\varepsilon_i \mu_0}}.$$

We now take the opposite viewpoint and assume D to be given and express E as a linear vector function of D . This is done by solving eq. (1). We write this solution as

$$(7) \quad \begin{aligned} E_1 &= \eta_{11} D_1 + \eta_{12} D_2 + \eta_{13} D_3, \\ E_2 &= \eta_{21} D_1 + \eta_{22} D_2 + \eta_{23} D_3, \\ E_3 &= \eta_{31} D_1 + \eta_{32} D_2 + \eta_{33} D_3. \end{aligned}$$

The η are the minors of the previous ε , divided by the determinant of ε :

$$(7a) \quad \eta_{mn} = \frac{|\varepsilon_{ik}|_{m \ n}}{|\varepsilon_{ik}|}.$$

The symmetry of the η -tensor follows from that of the ε -tensor as expressed in eq. (1a) (our present notation obviously has nothing to do with the electric susceptibility in Vol. III, Sec. 11 C).

Using (7), eq. (2) can be rewritten

$$(8) \quad W_e = \frac{1}{2} (D \cdot E) = \frac{1}{2} \sum_m \sum_n \eta_{mn} D_m D_n.$$

The corresponding tensor surface becomes

$$(9) \quad \sum \sum \eta_{mn} x_m x_n = \text{const.}$$

This equation is different from (4), but because of its connection with the electric energy it also represents an ellipsoid. Transformed to its principal axes system this equation assumes the form

$$(9 a) \quad \eta_1 x_1^2 + \eta_2 x_2^2 + \eta_3 x_3^2 = \text{const.}$$

The principal axes of the η -tensor (9 a) are in the same direction as those of the ϵ -tensor (6) because both are determined by the condition that \mathbf{D} and \mathbf{E} shall be parallel.

Furthermore, it can easily be shown that

$$(10) \quad \eta_i = \frac{1}{\epsilon_i}.$$

For by substituting into (7 a) the principal axis values for the ϵ_i , one obtains

$$\eta_1 = \left| \begin{array}{cc} \epsilon_2 & 0 \\ 0 & \epsilon_3 \end{array} \right| \div \left| \begin{array}{ccc} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{array} \right| = \frac{1}{\epsilon_1} \text{ etc.}$$

If we now express the ϵ_i in terms of the principal indices of refraction n_i , as in (6 a), then we obtain from (9 a) and (10)

$$(11) \quad \frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} = \text{const.}, \quad \text{const.} = 2 W_e \epsilon_0.$$

Therefore, the lengths of the principal axes of our present ellipsoid are *equal* to the principal indices of refraction and not to their reciprocals as in the Fresnel ellipsoid. (11) is, therefore, called the *index ellipsoid* (also Fletcher's ellipsoid or "reciprocal ellipsoid").

The positions of the principal dielectric axes in the crystal change a little with temperature and also differ somewhat for different frequencies. Therefore, one speaks of a "dispersion of the principal axes". Only the *symmetry* of the crystal lattice, if it exists, which controls all physical phenomena, completely fixes the principal axes. We shall go further into this in Sec. 28.

All these considerations have been concerned only with *electrically* anisotropic bodies. But there are also *magnetic* crystals. We mentioned the most important ferromagnetic ones in Vol. III at the beginning of Sec. 12. These are, however, of no interest in optics because the magnetization cannot follow the rapid optical oscillations, but dies out in the far infrared. For this reason we can henceforth set $\mu = \mu_0$, that is, treat only magnetically isotropic media. Correspondingly, our earlier considerations in Sec. 3 C in which the distinction between μ and μ_0 was important, referred not to optical but to centimeter waves.

25. The Structure of the Plane Wave and its Polarization

As we know (Vol. III, Sec. 4), Maxwell's equations are valid in crystals as well as in isotropic media. Since we can set $\mu = \mu_0$, these equations contain the three quantities E , D , and H where D and E are connected by eq. (24.1). Thus, if we assume the crystal to be non-conducting, we have

$$(1) \quad \mu_0 \frac{\partial H}{\partial t} = -\text{curl } E, \quad \frac{\partial D}{\partial t} = \text{curl } H.$$

From $\text{div curl} = 0$, it follows that $\text{div } H$ and $\text{div } D$ are constant with respect to time. Both of these constants are to be set equal to zero; the first one because the magnetic force lines are free of sources, the second because we can assume the crystal to be free of charges and because the charge density is generally to be defined by $\text{div } D$. Hence

$$(2) \quad \text{div } H = 0, \quad \text{div } D = 0.$$

The condition

$$(2a) \quad \text{div } D = \frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2} + \frac{\partial D_3}{\partial x_3} = 0$$

holds in every cartesian coordinate system, not merely in the principal axis system of the dielectric. If one were to replace D by E by means of (24.1), a rather unwieldy formula would result. Only in the coordinate system of the principal axes does this substitution have the relatively simple form

$$(2b) \quad \epsilon_1 \frac{\partial E_1}{\partial x_1} + \epsilon_2 \frac{\partial E_2}{\partial x_2} + \epsilon_3 \frac{\partial E_3}{\partial x_3} = 0.$$

We mention this mainly in order to show that

$$(2c) \quad \text{div } E = \frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}$$

cannot vanish simultaneously with $\text{div } D$; this occurs neither in the principal axis system nor in a general cartesian coordinate system.

Limiting ourselves to the case of the plane wave, we will make the following consistent assumptions as to the forms of D and E :

$$(3) \quad D = A \exp i \{k \cdot r - \omega t\}, \quad E = B \exp i \{k \cdot r - \omega t\}.$$

These are valid in any cartesian coordinate system. In this way we express the fact that the space and time dependences of both these vectors are the same, but that their amplitudes and directions will differ. As in the isotropic case of Sec. 2, these assumptions express an *ideal* state which is completely monochromatic (single frequency ω) and directed completely parallel (single wave vector k). We have already discussed in Sec. 2 how such a state can be approximated with natural light by using a monochromator and collimator.

The D-wave is *transverse*, i. e. the vector \mathbf{D} is *perpendicular* to the wave vector \mathbf{k} . This follows from (2 a). For, according to (3),

$$(4) \quad \operatorname{div} \mathbf{D} = i (A_1 k_1 + A_2 k_2 + A_3 k_3) \exp i \{ \mathbf{k} \cdot \mathbf{r} - \omega t \} = i \mathbf{k} \cdot \mathbf{D} = 0.$$

Hence, \mathbf{D} has no component in the direction of \mathbf{k} . As a result of the remarks accompanying (2 b, c), this is *not* true of the field vector \mathbf{E} .

We now inquire into the connection between ω , \mathbf{k} , and the phase velocity u of our plane wave. In the isotropic case this connection was given by

$$(5) \quad u = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon \mu}}.$$

The first of these equalities is valid also for crystals. To prove this we need only differentiate the phase $\varphi = \mathbf{k} \cdot \mathbf{r} - \omega t$ with respect to t and follow the progress of a certain phase value by setting $d\varphi/dt$ equal to zero:

$$(5 a) \quad \frac{d\varphi}{dt} = \mathbf{k} \cdot \dot{\mathbf{r}} - \omega = 0$$

here $\dot{\mathbf{r}}$ is nothing else but the vector \mathbf{u} which has the same direction as the wave vector \mathbf{k} so that $(\mathbf{k} \cdot \dot{\mathbf{r}}) = |\mathbf{k}| u = k u$ and

$$(5 b) \quad \omega = k u, \quad u = \frac{\omega}{k}.$$

The second equality in (5) for isotropic media resulted from the wave equation which, written in terms of \mathbf{D} instead of \mathbf{E} and for $\mu = \mu_0$, reads

$$(6) \quad \epsilon \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} = \Delta \mathbf{D}.$$

We must now see what takes the place of this equation in the anisotropic case. For this purpose we eliminate \mathbf{H} from the two eqs. (1) by applying the curl operator to the first and the operator $\mu_0 \partial/\partial t$ to the second. Thus we obtain

$$(6 a) \quad \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} = -\operatorname{curl} \operatorname{curl} \mathbf{E},$$

or using a well-known, actually only symbolic, vector relation (see Vol. III, eq. (6.2))

$$(6 b) \quad \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} = \Delta \mathbf{E} - \operatorname{grad} \operatorname{div} \mathbf{E}.$$

This present wave equation is considerably more complicated than eq. (6). The last term on the right-hand side does not vanish, as was remarked in connection with (2 c); nor can $\Delta \mathbf{E}$ be written in vector form as a sum of deriva-

tives of \mathbf{D} . Therefore, we shall not discuss (6 b) further but shall return to eq. (6 a). Performing the indicated differentiations on the expressions (3) we can write

$$\frac{\partial^2 \mathbf{D}}{\partial t^2} = -\omega^2 \mathbf{D}, \quad \text{curl } \mathbf{E} = i [\mathbf{k} \times \mathbf{E}], \quad \text{curl curl } \mathbf{E} = -[\mathbf{k} \times [\mathbf{k} \times \mathbf{E}]].$$

Equation (6 a) yields then

$$(7) \quad -\mu_0 \omega^2 \mathbf{D} = [\mathbf{k} \times [\mathbf{k} \times \mathbf{E}]] = \mathbf{k} (\mathbf{k} \cdot \mathbf{E}) - k^2 \mathbf{E}.$$

Using (5 b) to express ω in terms of u and dividing by k^2 , this becomes

$$(8) \quad -\mu_0 u^2 \mathbf{D} = \frac{\mathbf{k}}{k^2} (\mathbf{k} \cdot \mathbf{E}) - \mathbf{E}.$$

We now decompose this vector equation into its components along the three principal axis directions. In the principal axis system we get from (3) and the connection (24.5) between \mathbf{D} and \mathbf{E}

$$B_j = \frac{A_j}{\epsilon_j}, \quad j = 1, 2, 3.$$

Using the principal light velocities u_j defined in (24.6 b) and cancelling the common exponential factor, (8) can be rewritten

$$(9) \quad (u^2 - u_j^2) A_j = k_j K$$

with the abbreviation

$$(9a) \quad K = -\frac{1}{k^2} \sum_i u_i^2 k_i A_i.$$

Formula (9) is a system of homogeneous linear equations for the A 's which is solvable only if its determinant vanishes. Instead of setting up this determinant, it is simpler to do the following: we multiply (9) by $\frac{k_j}{(u^2 - u_j^2)}$ and sum over j . This gives

$$(9b) \quad \sum_j k_j A_j = K \sum_j \frac{k_j^2}{u^2 - u_j^2}.$$

The left-hand side of this equation vanishes because by (4)

$$\mathbf{k} \cdot \mathbf{A} = \sum_j k_j A_j = 0.$$

The factor K on the right-hand side of (9 b) generally does not vanish (the principal dielectric axes with their special values of A_j and k_j form an exception). Therefore we conclude from (9 b) that

$$(10) \quad \frac{k_1^2}{u^2 - u_1^2} + \frac{k_2^2}{u^2 - u_2^2} + \frac{k_3^2}{u^2 - u_3^2} = 0.$$

This is a quadratic equation for u^2 , as can be seen by multiplying it by the product of the denominators. Therefore, to every direction k there correspond two, generally different, values of u^2 . The fact that each of these still yields the two u -values $\pm u$ means, of course, that the same value of $|u|$ corresponds to the two directions $\pm k$.

We denote the two roots belonging to any k by u'^2 and u''^2 . We shall call the corresponding dielectric displacements D' , D'' and their amplitude coefficients A'_j, A''_j . We now assert that D' and D'' are *mutually perpendicular*, that is, that

$$(11) \quad D' \cdot D'' = 0.$$

This follows from the two equations

$$A'_j = K' \frac{k_j}{u'^2 - u_j^2}, \quad A''_j = K'' \frac{k_j}{u''^2 - u_j^2}$$

which are contained in (9). Multiplying these and summing gives

$$(11 a) \quad \sum_j A'_j A''_j = K' K'' \sum_j \frac{k_j^2}{(u'^2 - u_j^2)(u''^2 - u_j^2)} = \\ = \frac{K' K''}{u''^2 - u'^2} \left\{ \sum_j \frac{k_j^2}{u'^2 - u_j^2} - \sum_j \frac{k_j^2}{u''^2 - u_j^2} \right\}.$$

Because of (10), the last two summations over j vanish so that $\sum_j A'_j A''_j$ and $D' \cdot D''$ also vanish.

These calculations of u' , u'' and the resulting facts concerning D' , D'' can be illustrated and made more definite by means of the geometrical construction in fig. 33. We begin with the *index ellipsoid* (24.11). If we replace the n_i by the principal light velocities u_i , the equation of that ellipsoid reads

$$(12) \quad u_1^2 x_1^2 + u_2^2 x_2^2 + u_3^2 x_3^2 = C, \quad C = 2 W_e \epsilon_0 c^2 = \frac{2 W_e}{\mu_0}.$$

We place a plane perpendicular to k through the center of the ellipsoid. Its equation is

$$(13) \quad k_1 x_1 + k_2 x_2 + k_3 x_3 = 0.$$

We now construct the ellipse which forms the intersection between the plane and the ellipsoid. We assert that the principal axes of this ellipse are (aside from a common factor) equal to the reciprocal values of u' , u'' , and that their directions coincide with the directions of D' , D'' .

We find these principal axes by computing the extrema of $x_1^2 + x_2^2 + x_3^2$, subject to the subsidiary conditions (12) and (13). Using the Lagrangian multipliers λ_1 and λ_2 , we write

$$(14) \quad \delta \{x_1^2 + x_2^2 + x_3^2 + \lambda_1 (u_1^2 x_1^2 + u_2^2 x_2^2 + u_3^2 x_3^2) + \lambda_2 (k_1 x_1 + k_2 x_2 + k_3 x_3)\} = 0.$$

After introducing λ_1 and λ_2 , the variations δx_j of the coordinates x_j belonging to the vertices can be considered as independent of one another. Hence, the coefficients of δx_j resulting from (14) must individually be equal to zero. Thus we obtain three conditions for the x_j :

$$(14a) \quad 2x_j(1 + \lambda_1 u_j^2) + \lambda_2 k_j = 0.$$

To determine λ_1 ¹ we multiply (14a) by x_j and sum over j . Applying the conditions (12) and (13), we obtain

$$\sum x_j^2 + \lambda_1 C = 0$$

where $\sum x_j^2$ means a^2 or b^2 (a and b are the major and minor axes of the ellipse). If we introduce the noncommittal abbreviation C/u^2 to cover both of these possibilities, then

$$(15) \quad \lambda_1 = -\frac{1}{u^2}.$$

Equation (14a) reads then

$$(15a) \quad \frac{2x_j}{u^2}(u^2 - u_j^2) = -\lambda_2 k_j \quad \text{or} \quad \frac{k_j}{u^2 - u_j^2} = -\frac{2x_j}{\lambda_2 u^2}.$$

If we multiply the last equation by k_j and sum over j , then the right-hand side vanishes because of (13), and we obtain as in (10)

$$(16) \quad \sum \frac{k_j^2}{u^2 - u_j^2} = 0.$$

Here u has the same meaning as in (10), and our two velocities of propagation u' , u'' are, aside from the factor C as defined in (12), equal to the reciprocals of the two principal axes a and b as was claimed.

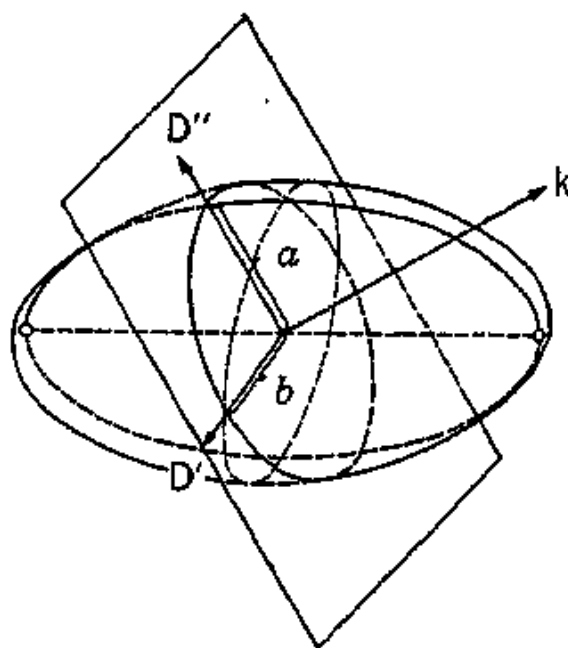


Fig. 33.

Index ellipsoid and construction of the D-vectors belonging to the wave number vector k .

¹We shall not need the value of λ_2 . But we could determine it also from (14a) by multiplying by k_j and summing over j .

To determine the *directions* of the two principal axes, we form from (15 a) the ratios

$$(17) \quad x_1 : x_2 : x_3 = \frac{k_1}{u^2 - u_1^2} : \frac{k_2}{u^2 - u_2^2} : \frac{k_3}{u^2 - u_3^2}.$$

According to (9), these same ratios also hold for the coefficients $A_1 : A_2 : A_3$ of our \mathbf{D} -vectors. Thus the *directions of oscillation of the two \mathbf{D} -vectors coincide with the directions of the principal axes of our intersectional ellipse.*

The directions of oscillation of the \mathbf{H} -vectors are now also fixed. As a result of the first eq. (1), \mathbf{H} oscillates *transversely*, i. e., perpendicularly to the wave vector \mathbf{k} . In addition, \mathbf{H} is perpendicular to \mathbf{D} as is easily proved from the second eq. (1). Hence, we always have

$$(18) \quad \mathbf{H} \cdot \mathbf{k} = 0, \quad \mathbf{H} \cdot \mathbf{D} = 0.$$

In particular this means, since we know the position of \mathbf{D} , that \mathbf{H} oscillates in the direction of b if \mathbf{D} has the direction of a , and vice versa.

We collect all of this information in the following two tables in which the second lines give the directions of oscillation and the third lines the velocities of propagation which are common to the pair of vectors \mathbf{D} , \mathbf{H} :

$$(19) \quad \begin{array}{cc|cc} \mathbf{D}' & \mathbf{H}' & \mathbf{D}'' & \mathbf{H}'' \\ a & b & b & a \\ \hline u' = \sqrt{C}/a & & u'' = \sqrt{C}/b & \end{array}$$

In connection with (19) we must still discuss the physical meaning of the principal light velocities u_1, u_2, u_3 which were introduced only formally in (24.6 b). We consider, for example, a wave vector \mathbf{k} in the direction of the first principal axis. To it belong two sets of vectors \mathbf{D} , \mathbf{H} whose respective velocities u', u'' are, according to (19), the reciprocals of the principal axes of the intersection ellipse formed by the plane perpendicular to \mathbf{k} . Hence u', u'' are equal to u_2, u_3 , respectively.

Therefore, the principal light velocities u_2 and u_3 are the velocities of the two waves which propagate in the direction of the first principal axis of the index ellipsoid. A corresponding statement holds, with cyclic interchange of the indices, for the other two principal axes.

The most important result of this section is that *all monochromatic plane waves propagating in a crystal are completely linearly polarized in directions which are determined by the crystalline structure.*

How does this result compare with the optical behavior of isotropic media? To make a valid comparison we must, of course, recall the properties of *monochromatic and parallel*, i. e. perfectly collimated, light in an isotropic medium, and not those of completely unpolarized *natural* light. In Sec. 2 we

saw that such light must necessarily be *elliptically polarized*. Therefore the distinction between isotropic and anisotropic media does not lie in the fact of polarization as such; the distinction lies in the type of polarization of the light. The crystal structure of an anisotropic medium permits two waves with different linear polarizations and different velocities to propagate in any given direction. In an isotropic medium this completely specified type of polarization is smeared out¹ into the oscillatory pattern of an ellipse whose orientation and size of axes remain unspecified. This is quite understandable, because in an isotropic medium all waves have the same velocities and all directions are equivalent and, as we know, the elliptic polarization can be considered as a superposition of two perpendicular plane oscillations with different phases. In contrast to the waves in crystals, any two such oscillations propagate with equal velocities in isotropic media and are therefore indistinguishable. On the other hand, it is just the fact that these two oscillations are distinguishable in anisotropic media which gives crystals their importance as principal components of polarization apparatus (calcite, mica, etc., see Sec. 29).

So far we have characterized the state of polarization of the light wave by the state of the excitation \mathbf{D} , whereas previously we have usually considered the field strength as the actual light vector. But since \mathbf{D} and \mathbf{E} are connected by the *unique* linear relationship of Sec. 24, it is clear that if \mathbf{D} is linearly polarized, so is \mathbf{E} . The next section will deal with the directions of oscillation imposed on \mathbf{E} by the crystal structure.

26. Dual Relations ², Ray Surface and Normal Surface, Optic Axes

The calculations of Sec. 25 can immediately be transposed from the index ellipsoid to Fresnel's ellipsoid. They then yield information about the *field vector* \mathbf{E} and the propagation of the *ray*

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

¹Using an expression common in wave mechanics we could say: the two linearly polarized oscillations in a crystal degenerate into elliptically polarized oscillations in isotropic media.

² For a complete treatment of this dual relationship we refer to the excellent textbook by T. Liebisch, *Physikalische Kristallographie*, Leipzig 1891. What is involved here is the same duality which exists in projective geometry between the coordinate spaces of points and planes. If one considers the components of \mathbf{E} as point coordinates, then the components of \mathbf{D} are plane coordinates. From this point of view the Fresnel and index ellipsoids represent the same surface, one in point coordinates, the other in plane coordinates. The elementary-geometric method of exercises IV.1 and IV.2 adheres closely to Liebisch's textbook.

We want to derive the equation which is the dual of this one; that is, the equation which expresses the coplanar position of \mathbf{E} , \mathbf{D} and \mathbf{s} . First we write, using two undetermined coefficients p and q ,

$$(4) \quad p \mathbf{E} = \mathbf{D} - q \mathbf{s}.$$

Because $\mathbf{s} \cdot \mathbf{E} = 0$ and $\mathbf{s} \cdot \mathbf{s} = 1$, it follows that $q = \mathbf{s} \cdot \mathbf{D}$ and hence

$$(4a) \quad p \mathbf{E} = \mathbf{D} - (\mathbf{s} \cdot \mathbf{D}) \mathbf{s}.$$

Now we form the scalar products of (3) with \mathbf{s} and of (4a) with \mathbf{n} . Because $\mathbf{s} \cdot \mathbf{E} = 0$ and $\mathbf{n} \cdot \mathbf{D} = 0$, these result in

$$\begin{aligned} \mu_0 u^2 (\mathbf{s} \cdot \mathbf{D}) &= -(\mathbf{n} \cdot \mathbf{E}) (\mathbf{s} \cdot \mathbf{n}), \\ p (\mathbf{n} \cdot \mathbf{E}) &= -(\mathbf{s} \cdot \mathbf{D}) (\mathbf{s} \cdot \mathbf{n}). \end{aligned}$$

If we multiply the right- and left-hand sides of these two equations and remember that $\mathbf{s} \cdot \mathbf{D}$ and $\mathbf{n} \cdot \mathbf{E}$ do not vanish, then

$$(5) \quad \mu_0 u^2 p = (\mathbf{s} \cdot \mathbf{n})^2.$$

According to fig. 34 and eq. (1)

$$\mathbf{s} \cdot \mathbf{n} = \cos \sigma = \frac{u}{v}.$$

Hence, according to (5)

$$(6) \quad p = \frac{1}{\mu_0 v^2}.$$

Thus eq. (4a) becomes

$$(7) \quad \frac{1}{\mu_0 v^2} \mathbf{E} = \mathbf{D} - (\mathbf{s} \cdot \mathbf{D}) \mathbf{s}.$$

This equation has precisely the same form as eq. (3); it is its "dual".

We now recall the plane wave expression (25.3) with the coefficients A_j and B_j . By writing this in the principal dielectric axis system and setting $B_j = A_j/\epsilon_j$, we were able to derive in eq. (25.9) a linear system of equations for the coefficients A_j . We shall now similarly compute the coefficients B_j from (7) by setting $A_j = \epsilon_j B_j$. First, we get from (7)

$$(8) \quad \frac{1}{\mu_0 v^2} B_j = \epsilon_j B_j - s_j \sum_i s_i \epsilon_i B_i;$$

then, multiplying by μ_0 and rearranging, we obtain

$$\left(\epsilon_j \mu_0 - \frac{1}{v^2} \right) B_j = \mu_0 s_j \sum_i s_i \epsilon_i B_i.$$

Making use of the principal light velocities $u_i = (\epsilon_i \mu_0)^{-1/2}$, we have

$$(9) \quad \left(\frac{1}{u_i^2} - \frac{1}{v^2} \right) B_i = s_i K',$$

$$(9a) \quad K' = \sum_i s_i \frac{B_i}{u_i^2}.$$

These equations correspond precisely to eq. (9) and (9a) of Sec. 25. The same is true of the following equation which is derived from the two previous ones just as in Sec. 25:

$$(9b) \quad \sum_i s_i B_i = K' \sum_i \frac{s_i^2}{\frac{1}{u_i^2} - \frac{1}{v^2}}.$$

The left-hand side is zero because $\mathbf{s} \perp \mathbf{E}$. Since $K' \neq 0$, it follows, as in (25.10), that

$$(10) \quad \frac{\frac{s_1^2}{v^2 - \frac{1}{u_1^2}}}{\frac{1}{v^2} - \frac{1}{u_1^2}} + \frac{\frac{s_2^2}{v^2 - \frac{1}{u_2^2}}}{\frac{1}{v^2} - \frac{1}{u_2^2}} + \frac{\frac{s_3^2}{v^2 - \frac{1}{u_3^2}}}{\frac{1}{v^2} - \frac{1}{u_3^2}} = 0.$$

Equation (10) is quadratic in v^2 just as eq. (25.10) was quadratic in u^2 . Therefore, to every ray direction \mathbf{s} there correspond two values v' and v'' (if we leave the \pm sign out of consideration). From a construction on Fresnel's ellipsoid analogous to that of fig. 33, one sees that the corresponding field vectors \mathbf{E}' and \mathbf{E}'' are mutually perpendicular.

We now summarize the transformation from \mathbf{D} , \mathbf{n} , u to \mathbf{E} , \mathbf{s} , v which we have thus developed in the form of the useful "transformation rule":

$$(11) \quad \mathbf{D}, \mathbf{E}, \mathbf{n}, \frac{u_i}{c}, \frac{u}{c} \rightleftharpoons \epsilon_0 \mathbf{E}, \frac{\mathbf{D}}{\epsilon_0}, \mathbf{s}, \frac{c}{v_i}, \frac{c}{v}.$$

The reader may convince himself that this rule does indeed transform eqs. (3) and (7) into each other, both as far as their general form and their coefficients are concerned. The same is also true of the expressions (10) and (25.10). We have departed from the usual formulation of this rule only insofar as we have throughout related to each other only quantities with the same dimensions.

A. DISCUSSION OF THE RAY SURFACE

We shall now construct a complete picture of the distribution of the ray velocities v' , v'' for all possible spatial directions of \mathbf{s} . For this purpose we plot these velocities as radius vectors in the direction of \mathbf{s} from the origin of an orthogonal coordinate system ξ_1, ξ_2, ξ_3 . In this way we obtain a two-sheeted

surface in our ξ_1, ξ_2, ξ_3 space, one sheet of which corresponds to v' , the other one to v'' . The points on this surface have the coordinates

$$(12) \quad \xi_i = s_i v.$$

Equation (10) can be written in terms of these as

$$(13) \quad \sum_i \frac{\xi_i^2 u_i^2}{u_i^2 - v^2} = 0.$$

Because $v^2 = \sum \xi_i^2$, this equation appears to be of the sixth order in the ξ_i ; but upon multiplication by the product of the denominators, one finds that it reduces to the following fourth order equation:

$$(13a) \quad \xi_1^2 u_1^2 (u_2^2 - v^2) (u_3^2 - v^2) + \xi_2^2 u_2^2 (u_3^2 - v^2) (u_1^2 - v^2) + \xi_3^2 u_3^2 (u_1^2 - v^2) (u_2^2 - v^2) = 0$$

or, grouped according to powers of v :

$$(13b) \quad v^4 (u_1^2 \xi_1^2 + u_2^2 \xi_2^2 + u_3^2 \xi_3^2) - v^2 \{ \xi_1^2 u_1^2 (u_2^2 + u_3^2) + \xi_2^2 u_2^2 (u_3^2 + u_1^2) + \xi_3^2 u_3^2 (u_1^2 + u_2^2) \} + u_1^2 u_2^2 u_3^2 (\xi_1^2 + \xi_2^2 + \xi_3^2) = 0.$$

Since the last term contains the factor $v^2 = \sum \xi_i^2$, a factor v^2 can be cancelled, and the equation indeed represents a surface of only the fourth order.

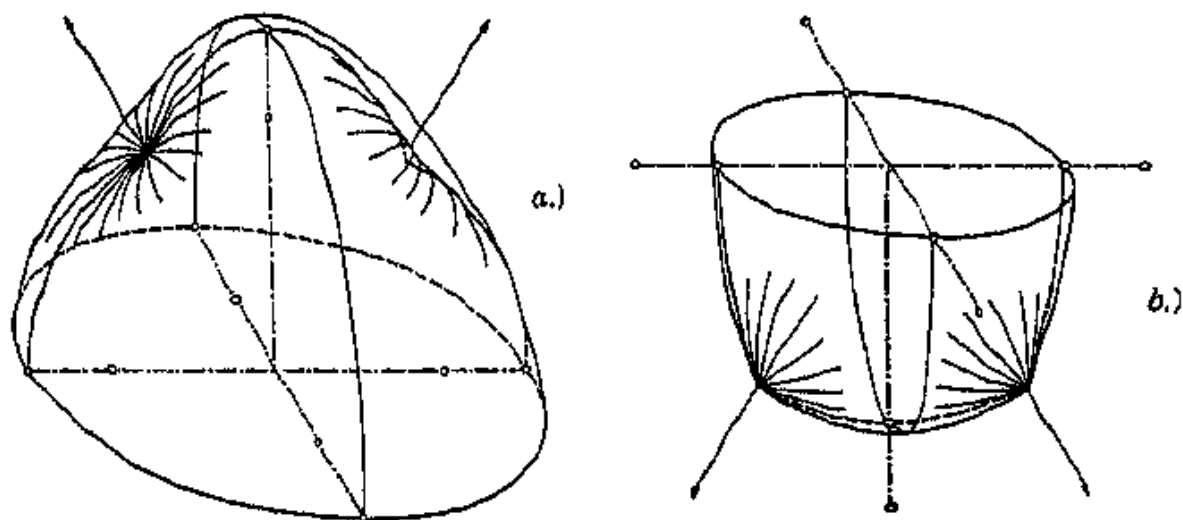


Fig. 35 a, b.

Ray surface: a) upper half of the outer sheet, b) lower half of the inner sheet. The directions of the arrows indicate the two optic axes.

We shall call this surface the *ray surface*. It used to be commonly called "Fresnel's wave surface". Our name indicates the origin of the surface from the ray velocity v . There are beautiful plaster models of the ray surface which can be taken apart and so reveal the way in which the two sheets are connected. Figure 35 represents the upper half of the outer sheet and the

lower half of the inner sheet. The missing half of each surface is the mirror image of the half which is shown. We shall study only the principal sections of the ray surface in greater detail, that is, its traces in the planes $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$. We may assume for this purpose that

$$(14) \quad u_1 > u_2 > u_3.$$

For $\xi_1 = 0$, we obtain from (13), by multiplying by the product of the two remaining denominators and by cancelling the factor $(\xi_2^2 + \xi_3^2)$,

$$(15) \quad \frac{\xi_2^2}{u_3^2} + \frac{\xi_3^2}{u_2^2} = 1.$$

This is an ellipse with the principal axes u_3 and u_2 . There is, however, yet another solution of (13) which is obtained by setting both ξ_1 and the

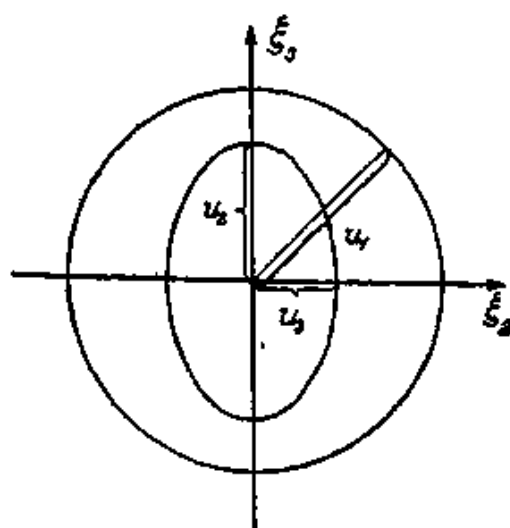


Fig. 36 a.

Intersection of the ray surface and the plane $\xi_1 = 0$.

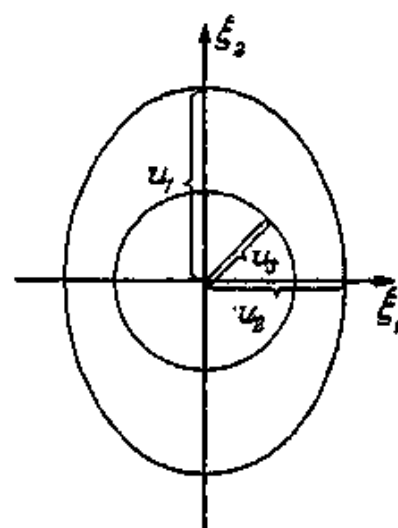


Fig. 36 b.

Intersection of the ray surface and the plane $\xi_3 = 0$.

denominator $u_1^2 - v^2$ equal to zero. The indeterminate expression $0/0$ which is thus introduced does indeed enable us to satisfy (13). Thus the second solution for $\xi_1 = 0$ becomes

$$(15 a) \quad \xi_2^2 + \xi_3^2 = u_1^2.$$

This is a circle of radius u_1 . Because of (14) this circle encloses the ellipse (15), see fig. 36 a.

These two solutions (15), (15 a) can obviously be obtained also from the complete expression (13 a) for the ray surface. For, if one sets $\xi_1 = 0$, then each of the remaining terms contains the factor $(u_1^2 - v^2)$. If this factor is taken out, an expression equivalent to (15) remains.

Next we consider the principal section $\xi_3 = 0$ which is in the plane perpendicular to the smallest axis of Fresnel's ellipsoid. Again the section consists of a circle and an ellipse, but now the circle lies inside the ellipse. For, from (13) we find

$$(16) \quad \frac{\xi_1^2}{u_2^2} + \frac{\xi_2^2}{u_1^2} = 1 \quad \text{and} \quad \xi_1^2 + \xi_2^2 = u_3^2, \text{ respectively;}$$

(see fig. 36 b.)

The principal section $\xi_3 = 0$ is more interesting. It yields

$$(17) \quad \frac{\xi_1^2}{u_3^2} + \frac{\xi_2^2}{u_1^2} = 1, \quad \xi_1^2 + \xi_2^2 = u_2^2.$$

(see fig. 36 c.) Now the radius u_2 of the circle is smaller than the major axis u_1 but larger than the minor axis u_3 of the ellipse. The circle, therefore, intersects the ellipse. At the points of intersection the two branches of the ray surface interpenetrate. What is the significance of the two axes which join the diametrically opposite points of intersection?

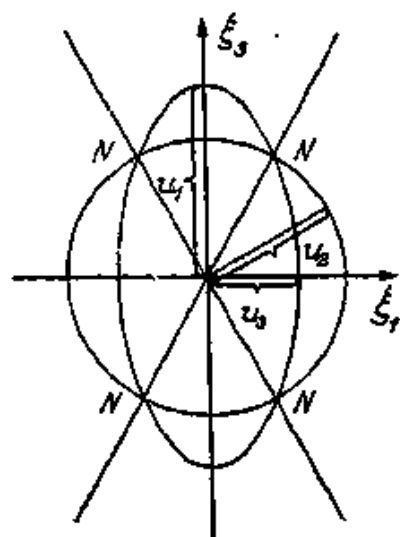


Fig. 36 c.
Intersection of the ray surface and the plane $\xi_3 = 0$.

B. THE OPTIC AXES

In these axes the two ray velocities v' and v'' are identical just as in isotropic media. They are, therefore, called the *axes of isotropy* or the *optic axes*. The latter name indicates that these axes are even more important in crystal optics than are the principal axes of the Fresnel, or index ellipsoid which were called "principal dielectric axes".

As we have seen, the ray velocities v' , v'' may be determined from the principal axes of an elliptic section of Fresnel's ellipsoid, and in the special case $v' = v''$ this elliptic section degenerates into a circle. Therefore, we see that the optic axes are perpendicular to the planes which intersect the ellipsoid in circles. There are well-known cardboard models of triaxial ellipsoids which consist of two sets of parallel circular discs which are fitted into each other and still have a certain degree of mobility. These models provide an interesting and complete representation of the surface of a triaxial ellipsoid. The points at which the normals to these circular discs intersect the surface of the ellipsoid are known as the *umbilical points* (German: "Nabelpunkte", hence the notation NN in fig. 36 c.) Fig. 36 d shows the

positions of both pairs of umbilical points on Fresnel's ellipsoid and their connecting lines which are the optic axes; the relation of these lines to the principal axes 1 and 3 is also shown. If we call the lengths of the principal axes of Fresnel's ellipsoid u_1, u_2, u_3 , as before, and denote the angle between the two optic axes by $2\delta_s$, then

$$(18) \quad \tan \delta_s = \frac{u_1}{u_3} \sqrt{\frac{u_2^2 - u_3^2}{u_1^2 - u_2^2}}.$$

This expression agrees with the value of ξ_3/ξ_1 which is obtained from the two eqs. (17) for the intersection points of the circle and ellipse.

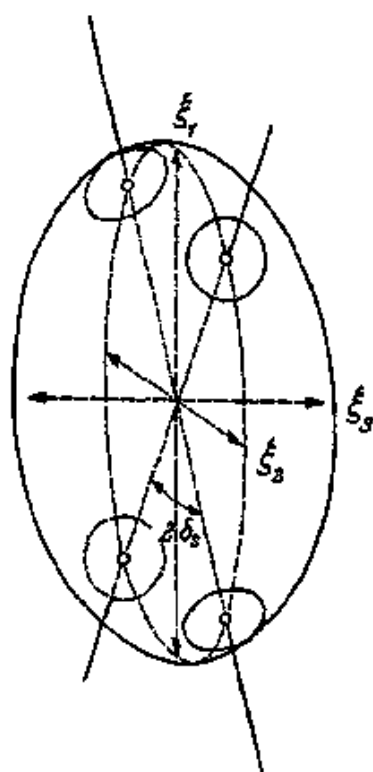


Fig. 36 d.

Construction of the optic axes as perpendiculars to the centers of the circular sections of Fresnel's ellipsoid.

If we define the *polarization of the ray* by the direction of the E-vector (in Sec. 25 we correspondingly defined the *polarization of the wave* by the direction of the D-vector), we can say: the polarization is linear for all ray directions. The planes of polarization of the two rays propagating in any one direction are perpendicular to one another. The optic axes form the only exception. Because of the circular shape of the sections belonging to these axes, no direction of polarization is preferred over any other. This gives further motivation to the name "axes of isotropy".

C. THE NORMAL SURFACE

We generate this surface by plotting in every wave number direction k the two phase velocities u', u'' of the waves propagating in that direction. If we describe this locus again by means of the rectangular coordinates ξ_1, ξ_2, ξ_3 , then we must write in place of (12)

$$(19) \quad \xi_i = \frac{k_i}{k} u, \quad \sum \xi_i^2 = u^2$$

and instead of (13)

$$(19a) \quad \sum \frac{\xi_i^2}{u^2 - u_i^2} = 0$$

because of eq. (25.10). Multiplying by the product of the denominators we obtain instead of (13 b)

$$(19\ b) \quad u^6 - u^2 [\xi_1^2 (u_2^2 + u_3^2) + \xi_2^2 (u_3^2 + u_1^2) + \xi_3^2 (u_1^2 + u_2^2)] + \\ + \xi_1^2 u_2^2 u_3^2 + \xi_2^2 u_3^2 u_1^2 + \xi_3^2 u_1^2 u_2^2 = 0.$$

Since the last set of terms does not contain a factor u^2 , this equation represents a sixth order surface. By (19 a) the principal section $\xi_1 = 0$ is found to consist of a circle

$$\xi_2^2 + \xi_3^2 = u_1^2$$

and an "oval"

$$(\xi_2^2 + \xi_3^2)^2 - u_3^2 \xi_2^2 - u_2^2 \xi_3^2 = 0,$$

which is a fourth order curve (containing also its center $\xi_2 = \xi_3 = 0$ as an isolated point of the curve). The other two principal sections consist of similar curves. These sections can again be illustrated by figs. 36 a, b, c, where, however, the ellipses must be replaced by slightly differently shaped ovals.

As in fig. 36 c there are two pairs of intersection points in the principal section $\xi_2 = 0$. They correspond to the umbilical points of the index ellipsoid and the lines connecting them define the "optic normal axes". The angle between these axes, which we shall call $2\delta_n$, is only slightly larger than the angle $2\delta_s$ defined in (18). Its magnitude is determined by

$$(20) \quad \tan \delta_n = \sqrt{\frac{u_2^2 - u_3^2}{u_1^2 - u_2^2}}.$$

In view of the connection of these axes with the circular sections of the index ellipsoid, they are the "isotropy axes" of the D-vector. This vector is not necessarily linearly polarized for propagation along these and only these axes.

A simple geometrical connection exists between the normal surface and the ray surface: the normal surface is the pedal surface to the ray surface. The section in the 1 - 3 plane yields, for example, the picture shown in fig. 26 e. This figure also demonstrates that the oval of the normal surface section (dotted) differs only slightly from the ellipse of the ray surface section (full line). It has already been shown in fig. 34 that the ray surface is the *envelope* of the wave planes (planes of equal phase).

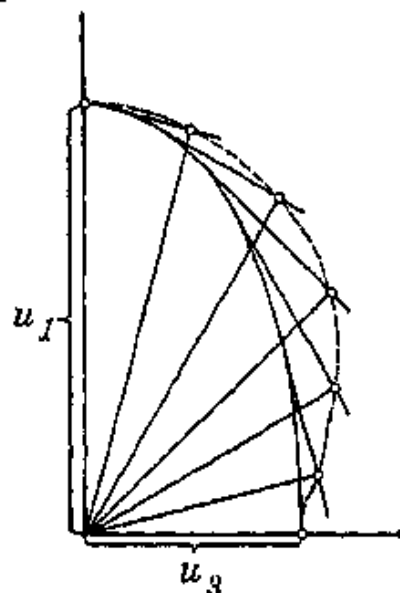


Fig. 36 e.

The normal surface as the pedal surface of the ray surface and the ray surface as the envelope of the normal surface.

27. The Problem of Double Refraction

If we place a slab of calcite over a sample of writing on a piece of paper, then on looking through the calcite the writing appears double. One image is displaced parallel with respect to the other. If we consider, for simplicity, a negative picture, that is white writing on a black background, then we can say the following: the waves emitted by the writing reach the eye by two paths which have different directions in the calcite and after the refraction have also different directions in the air. The eye extrapolates the perceived directions wrongly by projecting them in straight lines to the bottom of the calcite and therefore the impression is created that the writing is double.

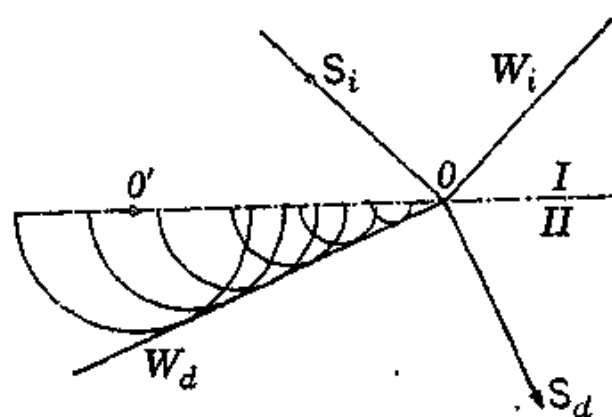


Fig. 37.

Construction of the refracted wave by Huygens' principle in the isotropic case.

The beautiful construction based on Huygens' principle for the refracted wave in an isotropic medium (see fig. 37) is well known. A plane wave propagating in the direction S_i is incident upon another medium. At the instant that the wave front W_i reaches the point O on the boundary plane, all previous positions O' of the wave have already radiated waves (which are drawn as hemispheres in fig. 37) into the second medium with the velocity

of light in that medium. Upon drawing from O the envelope of this system of hemispheres, one obtains a straight line which is a wave front W_d of the refracted ray. The normal S_d to W_d is the direction of propagation of the refracted light.

A. DOUBLE REFRACTION ACCORDING TO HUYGENS' PRINCIPLE

Huygens himself¹ extended this construction with ingenious foresight to the case of the (optically uniaxial) calcite crystal. He assumed the surface of propagation of the light in the crystal to be not a sphere but a certain combination of a sphere and an oblate ellipsoid of rotation (compare with fig. 39 b below). Thus he obtained two envelopes and thereby two wave fronts, one for the system of spheres and another for the system of ellipsoids. Here we shall describe the general (optically biaxial) case by replacing the combina-

¹The complete title of Huygens' book is: *Traité de la lumière, où sont expliquées les causes de ce qui lui arrive dans la réflexion et dans la réfraction et particulièrement dans l'étrange réfraction du cristal d'Islande.* Leiden 1690.

tion sphere + ellipsoid by our two-sheeted ray surface (26.13); see fig. 37 a. The envelopes of the two sheets again yield two wave surfaces W_d' and W_d'' of the refracted light and the lines connecting the center O' of the ray surface with the points of contact of the envelope give two ray directions S_d' and S_d'' . The perpendiculars from O' upon the two envelopes (in the figure these

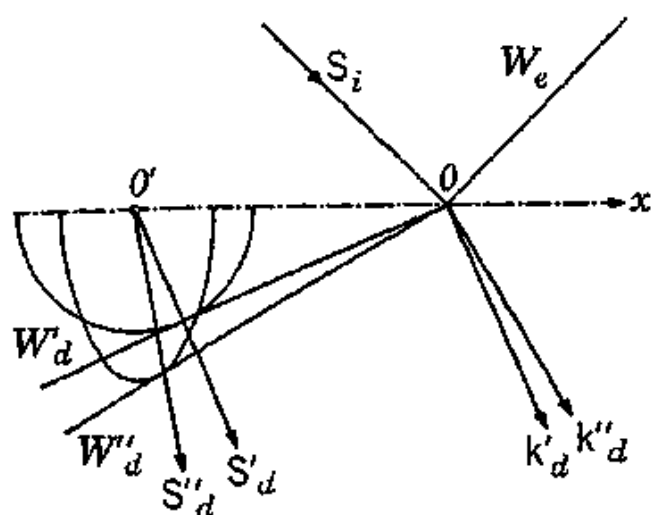


Fig. 37 a.

Construction of the two refracted waves by Huygens' principle in the anisotropic case.

perpendiculars are erected at O) represent the propagation vectors (the "wave normals") k_d' , k_d'' .

This construction illustrates well the origin of double refraction, and it is therefore widely used in the literature; however it gives an incomplete description in several respects.

1. The construction presupposes that a diverging bundle of rays which originates from a point source behaves in the same way as a system of mutually independent plane waves. It is the latter type of wave which

we have considered in Sec. 26 and whose velocities of propagation we symbolized purely geometrically by means of our ray surface. Lamé¹ was the first to recognize that this presents a mathematical problem which is by no means simple. He posed the problem of representing the complex of waves sent out by a concussion center (which is analogous to the spherical wave in an isotropic medium) in a precise mathematical manner by computing the three displacement components. He was, indeed, led (upon excluding the longitudinal waves) to the form of the ray surface. These results were criticized and extended by V. Volterra².

2. Figure 37a as it stands gives no information about the polarization of either the D or the E wave. In this respect our description would have to be completed by means of the results obtained in Secs. 25 and 26.

3. The construction leaves the question of the amplitude ratios between the different waves unanswered.

¹In his *leçons sur la théorie mathématique de l'élasticité*, Paris 1852. The differential equations integrated by Lamé agree with the differential equations for the *magnetic* field components in electromagnetic optics.

²*Acta Mathematica*, Vol. 16, p. 153, 1892.

B. THE LAW OF REFRACTION AS A BOUNDARY VALUE PROBLEM

The preceding remarks indicate that a complete and quantitative theory of double refraction can be obtained only by considering the boundary conditions, following the method of Chap. I. The reasoning remains the same; we operate again with infinitely extended plane waves as the only known strict solutions of the optical differential equations, and not with the limited rays which usually form the objects of experimentation. We can do without additional hypotheses, such as Huygens' principle or the construction of envelopes. Let us, however, first reconsider the origin of the ordinary law of refraction and reflection which was derived from the boundary conditions in Sec. 3. This time we will not specify the boundary conditions beyond the fact that they exist and do not contain the time explicitly. This will lead us to a reinterpretation of the oldest geometrical construction for the angles of reflection and refraction, that of Snell (before 1637).

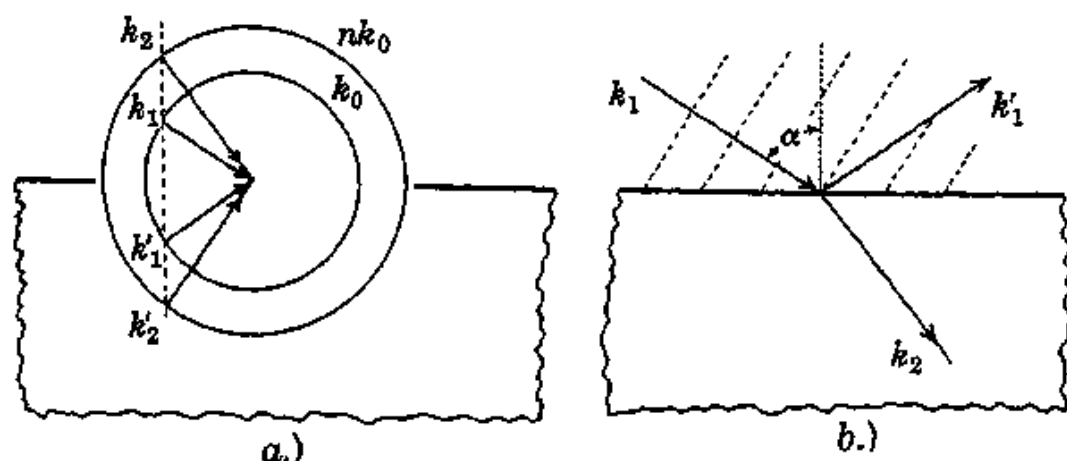


Fig. 38. Boundary condition for combination of plane waves. (a) Snell's construction; the surface of the body is indicated outside of the circles for propagation in free space (k_0) and in the body ($n k_0$). (b) The planes of equal phase are shown for the incident wave (dotted lines); the same construction applied to the reflected and refracted waves gives the same trace pattern along the surface.

For any wave of frequency ω the wave vector has a prescribed length, namely $|k_0| = k_0 = \omega/c$ in free space, and $|k| = \omega/u = n k_0$ in the body of refractive index n (assumed > 1 in the figures). This fact can be expressed geometrically by drawing two concentric spheres of radii k_0 and $n k_0$; any vector drawn from the surface of one of the spheres to its center is then a possible wave vector of a plane wave in the corresponding medium. Outside of the spheres we indicate in the figure the position of the body and its surface. We assume an incident wave to fall on the surface and show its wave vector k_1 of length k_0 in fig. 38 a. In fig. 38 b the incident wave itself is drawn by indicating its planes of equal phase. These planes travel along the arrow with

velocity c ; their trace forms a wave pattern on the surface which travels along the surface from left to right with velocity $c/\sin \alpha$ to which corresponds a wave vector tangential to the surface and of length $|k_1| \sin \alpha$.

We may then say that the surface wave obtained as the intersection of the incident wave with the surface is described by a wave vector which is the tangential component of k_1 . The same will hold for any other wave, in free space or in the medium: the field produced at the surface is a surface wave the wave vector of which is the component of the spatial wave vector along the surface.

Now a time-independent relation on the surface can exist only between waves which give surface traces of the same mode of propagation. Otherwise wave amplitudes fulfilling the boundary conditions at selected points or times would fall out of step between these points or times.

Thus, going back to fig. 38 a, only three other wave vectors can be combined with k_1 , namely those having the same tangential components as k_1 . They are easily obtained by drawing the normal to the surface through the tail point of k_1 . Of these vectors k_1' is the wave vector of the reflected wave (in free space), k_2 that of the refracted wave, and its mirror image k_2' would be a second internal wave compatible with the boundary conditions. This last wave exists in a plate with parallel surfaces: it is the internally reflected refracted wave which is essential for obtaining the interference pattern of the Lummer-Gehrke plate and for calculating the corresponding optical field directly from the boundary conditions instead of by the method of summing repeated reflexions and refractions as in section 7 F. If we wish to deal with a single surface and a single wave incident on it — and this is the assumption which leads to the Fresnel formulae — we have to omit the wave of wave vector k_2' . It is easily seen from the construction that the directions of k_1 , k_1' and k_2 give the geometrical laws of reflection and refraction, in fact fig. 38 a is Snell's construction.

We can now generalize this construction to the case of a doubly refracting crystal. It was pointed out in the discussion following eq. (25.10) that to each direction of k belong two values of n , viz. n' , n'' . Since $|k| = \omega/n$ there are thus in each direction (except those of the optic axes) two vectors k of different length. These correspond, according to fig. 33, to two linearly polarized waves of which the D -vectors are in fixed, mutually orthogonal directions. In Snell's construction this means that the outer sphere has to be replaced by a double surface. This surface is actually the dual counterpart of the ray-surface of paragraph 26 A and fig. 35, a, b and it can be obtained from it by the translation rule 26 (11). Without going into more detail, however, we see that Snell's construction now yields two refracted

waves progressing at different angles of refraction β' and β'' and polarized at right angles to one another. Each of the directions satisfies the law of refraction

$$\frac{\sin \alpha}{\sin \beta'} = \frac{c}{u'}; \quad \frac{\sin \alpha}{\sin \beta''} = \frac{c}{u''}.$$

It is worth noting, however, that this law gives the angles β' , β'' implicitly only, because u' , u'' vary with these angles.

C. THE AMPLITUDES OF REFLECTED AND REFRACTED RAYS

Let us limit the calculation to the case that the incident ray lies in the plane containing the greatest and the smallest axis of the index ellipsoid (eq. (24.11)), i. e. that, as in Sec. 26 B, it also lies in the plane of the optic axes. Then we know, by symmetry, that either \mathbf{E} and \mathbf{D} both lie in the plane of incidence and \mathbf{H} at right angles to it, or, for the other case of polarization, \mathbf{E} and \mathbf{D} are both normal to the plane (and thus parallel to one another) and \mathbf{H} is in the plane. In the latter case the boundary conditions are simpler, and we restrict the derivation to it.

a) Plane of Polarization Parallel to Plane of Incidence

In medium I (air, $y > 0$), we assume, as in eq. (3.1), cf. fig. 3 a,

$$(1) \quad \mathbf{E} = E_x = A e^{i k_0 (x \sin \alpha - y \cos \alpha)} + C e^{i k_0 (x \sin \alpha' + y \cos \alpha')}$$

and in the crystal

$$(2) \quad \mathbf{E} = E_x = B e^{i(k \cdot r)}, \quad B_x = B_y = 0; \quad B_z \equiv B.$$

Together with the (suppressed) time factor $\exp(-i \omega t)$, these terms represent the incident, reflected, and refracted waves, respectively. Snell's construction gives

$$(3) \quad \alpha' = \alpha \quad \text{and} \quad k \sin \beta = k_0 \sin \alpha.$$

Continuity of the tangential component of \mathbf{E} at the surface $y = 0$ is expressed in the equation

$$(4) \quad A + C = B.$$

Since differences of permeability are neglected, the second boundary condition may be expressed as: \mathbf{H} continuous. Now in the first medium at the boundary ($y = 0$)

$$(5) \quad \begin{aligned} H_x &= -\frac{1}{i \omega} \frac{\partial E_x}{\partial y} = -\frac{k_0}{\omega} (-A \cos \alpha + C \cos \alpha') e^{i k_0 x \sin \alpha}, \\ H_y &= +\frac{1}{i \omega} \frac{\partial E_x}{\partial x} = \frac{k_0}{\omega} (A \sin \alpha + C \sin \alpha') e^{i k_0 x \sin \alpha}, \end{aligned}$$

and in the second medium

$$(6) \quad \begin{aligned} H_x &= -\frac{1}{i\omega} \frac{\partial E_z}{\partial y} = -\frac{k}{\omega} (-B \cos \beta) e^{ikx \sin \beta}, \\ H_y &= +\frac{1}{i\omega} \frac{\partial E_z}{\partial x} = \frac{k}{\omega} (B \sin \beta) e^{ikx \sin \beta}. \end{aligned}$$

The boundary condition thus is, using (27.3),

$$(7) \quad \begin{aligned} A - C &= \frac{\tan \alpha}{\tan \beta} B, \\ A + C &= \frac{\sin \alpha}{\sin \beta} B \frac{\sin \beta}{\sin \alpha} = B. \end{aligned}$$

The second of these conditions is the same as that for E . In terms of the incident amplitude we obtain the refracted and reflected waves

$$(8) \quad \begin{aligned} B/A &= \frac{2}{1 + \frac{\tan \alpha}{\tan \beta}}, & C/A &= \frac{\tan \beta - \tan \alpha}{\tan \beta + \tan \alpha}, \\ &= \frac{2 \sin \beta \cos \alpha}{\sin (\beta + \alpha)}, & &= \frac{\sin (\beta - \alpha)}{\sin (\beta + \alpha)}. \end{aligned}$$

This result is, of course, the same as that obtained in Sec. 3 A for the reflected and refracted waves of the same polarization in the case of an isotropic body (3.12).

b) Plane of Polarization Normal to Plane of Incidence

In this case we cannot expect to obtain the same result as for an isotropic body, which is expressed in (3.16). Whereas in the isotropic body the field vectors E and D (electric force and dielectric displacement) have the same direction, shown by the amplitude vector B in fig. 3 b, their directions differ in an anisotropic body. The boundary conditions are: continuity of the tangential component of E and of the normal component of D , as well as continuity of the magnetic vector H . Now the decomposition of E and D into tangential and normal components means their decomposition according to the axial system of fig. 3 b (x parallel, y normal to surface), but the anisotropic relation between the two vectors finds its simple expression if they are decomposed according to the principal axes of the dielectric tensor or index ellipsoid (conf. Sec. 24). This double decomposition complicates the derivation and expression of the amplitude ratios for this case of polarization.

28. The Optical Symmetry of Crystals

So far we have concerned ourselves only with the general form of the dielectric tensor ϵ_{ik} . This tensor is defined by 6 parameters (because $\epsilon_{ik} = \epsilon_{ki}$). We can find the number of parameters not only from the tensor scheme but also from its geometrical interpretation, namely Fresnel's ellipsoid. The ellipsoid is defined by its 3 principal axes and by their positions in space which in turn are defined by 3 angle parameters. A crystal without symmetry properties is called *triclinic*. Such a crystal is described by the general (ϵ_{ik}) -tensor.

If the crystal has symmetry, the number of independent parameters is reduced. By *symmetry direction* let us understand either an axis of rotational symmetry or the direction normal to a mirror plane, as the case may be. Then if a crystal has a single symmetry direction it must coincide with one of the three symmetry directions of the Fresnel ellipsoid. If the crystal has two symmetry directions at right angles to one another, there exists a third one which is orthogonal to both, and the symmetry directions of the Fresnel ellipsoid must coincide with those of the crystal. Furthermore, the existence of a symmetry axis¹ of higher order than 2 necessitates equal magnitude of the axes of the Fresnel ellipsoid at right angles to its direction; thus the ellipsoid will be one of revolution. Again, if there are, as in the cubic system, four threefold axes of symmetry (along the body diagonals of a cube), then the Fresnel ellipsoid degenerates into a sphere, and a single constant is left over from the ϵ_{ik} scheme.

Let us consider some cases in more detail. If a crystal has only one symmetry direction, it is called *monoclinic*. One of the principal axes of the Fresnel ellipsoid being fixed by this direction, there remain four parameters. These can be found from the scheme of the ϵ_{ik} . Assume, for instance, that there is a mirror plane normal to the direction of symmetry (index 2), so that $+x_2$ and $-x_2$ are symmetrically equivalent directions. If, for convenience, we write x_i, y_i as the variables instead of E_i, D_i , the general scheme

$$\begin{aligned} (1) \quad y_1 &= \epsilon_{11} x_1 + \epsilon_{12} x_2 + \epsilon_{13} x_3, \\ y_2 &= \epsilon_{21} x_1 + \epsilon_{22} x_2 + \epsilon_{23} x_3, \\ y_3 &= \epsilon_{31} x_1 + \epsilon_{32} x_2 + \epsilon_{33} x_3 \end{aligned}$$

must remain unchanged if we substitute $-x_2$ for x_2 and simultaneously $-y_2$ for y_2 . Thus we have also (reversing all signs in the second line),

¹A rotational symmetry axis of order n is one about which rotation of the crystal by $2\pi/n$ will bring it to the nearest covering position.

$$\begin{aligned}
 (1 \text{ a}) \quad y_1 &= \epsilon_{11} x_1 - \epsilon_{12} x_2 + \epsilon_{13} x_3, \\
 y_2 &= -\epsilon_{21} x_1 + \epsilon_{22} x_2 - \epsilon_{23} x_3, \\
 y_3 &= \epsilon_{31} x_1 - \epsilon_{32} x_2 + \epsilon_{33} x_3.
 \end{aligned}$$

Eq^s. (1) and (1 a) are consistent only if $\epsilon_{12} = \epsilon_{21} = 0$ and $\epsilon_{23} = \epsilon_{32} = 0$. The tensor scheme for this crystal therefore reduces to the four parameters

$$(2) \quad \begin{pmatrix} \epsilon_{11} & 0 & \epsilon_{13} \\ 0 & \epsilon_{22} & 0 \\ \epsilon_{13} & 0 & \epsilon_{33} \end{pmatrix}.$$

Next consider a crystal with two symmetry directions. There follows automatically a third such direction, but we assume that these three directions are not further related to one another. This is the case

- (i) in the *orthorhombic* crystal system where the directions are either twofold axes of rotation or normals of mirror planes,
- (ii) in the *rhombohedral* crystal system where one direction is that of threefold rotation symmetry,
- (iii) in the *hexagonal* system, where one direction is a sixfold axis.

Since now three angular parameters of the ellipsoid are fixed, the number of free parameters is reduced to 3. Assume, for instance, that direction 3 is that of a twofold rotation axis, and direction 2, as before, corresponds to a mirror plane. The rotation about 3 means that no change is brought about by the simultaneous substitutions $(x_1, x_2) \rightarrow (-x_1, -x_2)$, $(y_1, y_2) \rightarrow (-y_1, -y_2)$. Applying this to the equations condensed in the scheme eq. (2), we see that the addition of the twofold axis to the mirror plane makes necessary the vanishing of ϵ_{13} . The tensor scheme thus reduces to the terms of the principal diagonal.

If we assume a fourfold rotation axis in direction 3 (together with the mirror plane normal to 2), then also a rotation by 90° should produce no change; this is given by the substitution

$$(x_1, x_2) \rightarrow (-x_2, x_1), \quad (y_1, y_2) \rightarrow (-y_2, y_1).$$

Since the repetition of the 90° rotation produces a rotation by 180° , which is that of a twofold axis, we may use the previous simplified scheme for reducing it further by this substitution. In this way we obtain the following sets of relations

$$\begin{array}{ll}
 y_1 = \epsilon_{11} x_1 & -y_2 = -\epsilon_{11} x_2 \\
 y_2 = \epsilon_{22} x_2 & y_1 = \epsilon_{22} x_1 \\
 y_3 = \epsilon_{33} x_3 & y_3 = \epsilon_{33} x_3.
 \end{array}$$

From these follows $\epsilon_{11} = \epsilon_{22}$, and this leaves only two constants undetermined.

Crystals which have three symmetry directions of which two are symmetrically equivalent are called *tetragonal*. If all three directions become equivalent, they are called *cubic*. If one of the symmetry directions contains a threefold or sixfold axis of rotation, then the Fresnel ellipsoid has full rotational symmetry.

Summarizing we have the following optical properties in the seven crystal systems:

Cubic crystals are optically isotropic, i. e. optically they do not differ from amorphous bodies, glasses or liquids.

Tetragonal, hexagonal and rhombohedral crystals have an ellipsoid of rotation. This implies that they are optically uniaxial and have two principal indices of refraction.

Orthorhombic, monoclinic, and triclinic crystals have three principal indices of refraction and are optically biaxial. The directions of the principal axes of refraction are fixed (for all wavelengths and temperatures) in the orthorhombic crystals, but the angle of the optic axes may vary, since it depends on the values of the principal refractive indices. In the monoclinic system only one, and in the triclinic system none, of the directions of principal index of refraction is fixed.

The reason why optically only three groups, the biaxial, uniaxial and isotropic groups, are distinct lies in the fact that eqs. (1) relate two vector quantities (field strength and excitation) to one another. In the theory of elasticity two tensor quantities, stress and strain, are related, and this leads to the distinction of many more groups than by the crystal-optical behavior. Electrostriction and piezoelectricity connect a vector (field strength) with a tensor (deformation) and give again a different classification.

The fact that in optics we are only concerned with the Fresnel or index ellipsoids, i. e. with surfaces of a very restricted type, makes it understandable that the more elaborate symmetry properties need not be discussed in this connection. They are systematically enumerated in the 32 *crystal classes* which are unequally distributed over the seven crystal systems. These classes contain a complete description of all geometrically possible symmetry relations between directions in space which pass through one point. According to the modern conception of crystal structure, crystals are bodies with an internal three-dimensional periodicity of atomic arrangement; the complete enumeration of the symmetry-types compatible with this periodicity led Barlow, Schoenflies and Fedorow to 230 different *space groups*. Crystal optics cannot probe into these; only X-ray analysis is able to do this (see Sec. 32).

In the uniaxial case the familiar constructions which we have used in the biaxial case are considerably simplified. Letting $\epsilon_1 = \epsilon_2 \neq \epsilon_3$, we introduce the notation

$$(3) \quad \begin{array}{ll} u_1 = u_2 = u_o & \text{ordinary wave velocity,} \\ u_3 = u_e & \text{extraordinary wave velocity,} \end{array}$$

and correspondingly, v_o the ordinary and v_e the extraordinary ray velocity. Then one of the principal axes a , b of the elliptic section in fig. 33 lies in the equatorial plane of the index ellipsoid; the other principal axis lies in the meridian plane through the optic axis. The planes of polarization of the respective waves are that meridian plane and the plane perpendicular to it which passes through the direction of propagation but not through the optic axis. The same is true for the construction of the ray directions from Fresnel's ellipsoid.

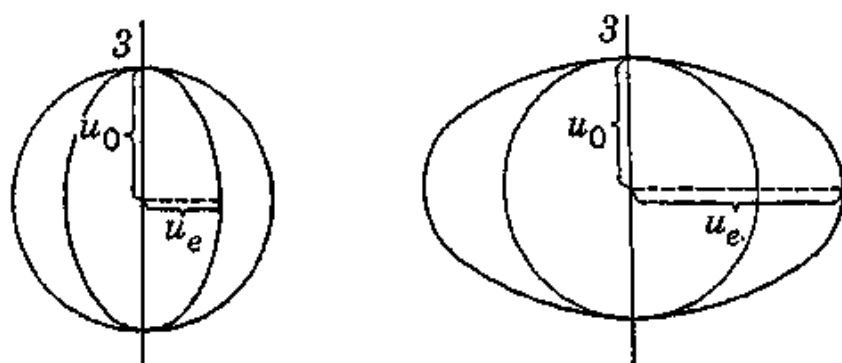


Fig. 39.

Ray surfaces of optically uniaxial crystals

a) $u_o > u_e$ uniaxial positive; example, quartz.

b) $u_o < u_e$ uniaxial negative; example, calcite.

The shape of the *ray surface* is now particularly simple. Not only can the factor v^2 be removed in eq. (26.13 b) for this surface, but since $u_1 = u_2 = u_o$, also a factor $u_o^2 - v^2$ cancels out. Thus the equation becomes

$$\{\xi_1^2 + \xi_2^2 + \xi_3^2 - u_o^2\} \{(\xi_1^2 + \xi_2^2) u_o^2 + \xi_3^2 u_e^2 - u_o^2 u_e^2\} = 0$$

which separates into a sphere of radius u_o and an ellipsoid of revolution

$$(4) \quad \frac{\xi_1^2 + \xi_2^2}{u_o^2} + \frac{\xi_3^2}{u_e^2} = 1.$$

These two surfaces touch at the points $\xi_3 = \pm u_o$. This verifies the form of the radiation surface which Huygens predicted for calcite.

Depending on the way in which the sphere and ellipsoid touch, one distinguishes positive and negative optically uniaxial crystals, see fig. 39 a, b.

The wave surface (normal surface) looks less simple. It separates into a sphere of radius u and a fourth order surface of revolution which is called an "ovaloid", namely

$$(5) \quad (\xi_1^2 + \xi_2^2 + \xi_3^2) (\xi_1^2 + \xi_2^2 + \xi_3^2 - u_e^2) = \xi_3^2 (u_o^2 - u_e^2).$$

In the limiting case of isotropy $n_o = n_e$, (4) becomes a sphere of radius n_o and hence the ray surface consists of this sphere, counted twice. At the same time (5) degenerates into a sphere of radius zero (isolated double point) and a sphere of radius n_o .

29. Optically Active Crystals and Fluids

By means of the structure theory of crystals it is possible to confirm the purely phenomenological discussion of this chapter, beginning with the relation (24.1). For, in the summation over the lattice structure of elementary constituents, different directions in the crystal would give dielectrically different results. In the same way the theory of dispersion of Chap. III could be extended to the crystalline state. However, "optical activity", i. e. the rotating power of certain crystal classes, seems to demand some use of structure theory, at least to the extent to which it is discussed in Secs. 75 and 84 of the textbook by Max Born which we mentioned on p. 40. We will, however, show in parts A and B how a much shorter phenomenological discussion can describe even this phenomenon. Only occasionally shall we require structural considerations in order to illustrate our results.

In subsection C we shall give only a very cursory presentation of optical activity in fluids and optically isotropic crystals even though this subject is of enormous theoretical importance for stereochemistry and of considerable practical importance in industry. A more precise treatment would require a more profound discussion of molecular structure than is compatible with the scope of these lectures. For such treatments we refer the reader to Secs. 84 and 99 of Born's book and to the articles by Born, C. W. Oseen, and W. Kuhn which are cited there.

A. THE GYRATION VECTOR OF SOLENOIDAL CRYSTAL STRUCTURES

We can write our linear vector function (24.1) in the abbreviated form

$$(1) \quad D_j = \epsilon_{jk} E_k$$

where the ϵ_{jk} form a real *symmetric tensor*. We now discard the reality condition for the ϵ and replace

$$\epsilon_{jk} \text{ by } \epsilon_{jk} + i\gamma_{jk}.$$

The additional terms γ , which are supposed to be *small*, are not to have an ohmic, dissipative character as in the complex dielectric constant of metal optics. Rather, they shall be conservative, which means that the γ_{jk} must not contribute to the electric energy density $W_e = 1/2 (\mathbf{D} \cdot \mathbf{E})$ as computed

from Maxwell's equations. This latter condition is fulfilled if they form an *antisymmetric tensor* as do the conservative gyroscopic terms in mechanics (see Vol. I, Sec. 30).

In order to explain this in greater detail, we write instead of (1)

$$(2) \quad D_j = \epsilon_{jkh} E_k + i \gamma_{jh} E_h$$

and find

$$\mathbf{D} \cdot \mathbf{E} = \sum_j \sum_h \epsilon_{jkh} E_j E_k + i \sum_j \sum_h \gamma_{jh} E_j E_h.$$

In order that the contribution of the γ term vanish for arbitrary values of the \mathbf{E} components, every γ need not be identically zero, but it suffices if the γ fulfill the following conditions:

$$\gamma_{ii} = 0, \quad \gamma_{jh} + \gamma_{hj} = 0.$$

These are precisely the conditions for the antisymmetry of the γ -tensor. We see now also why the γ -tensor must be purely imaginary¹. For if it had a real part, this would add to the ϵ -tensor and disturb its symmetry, but we know that the ϵ -tensor must be symmetric for general reasons of energy.

An antisymmetric tensor (γ_{jh}) can always be replaced by a vector γ with the components

$$\gamma_1 = \gamma_{23} = -\gamma_{32}, \quad \gamma_2 = \gamma_{31} = -\gamma_{13}, \quad \gamma_3 = \gamma_{12} = -\gamma_{21}.$$

Thus we obtain

$$\sum_j \gamma_{1jh} E_h = \gamma_3 E_2 - \gamma_2 E_3 = -[\gamma \times \mathbf{E}]_1 \text{ etc.}$$

Then (2) becomes

$$(3) \quad D_j = \sum_k \epsilon_{jkh} E_k - i [\gamma \times \mathbf{E}]_j.$$

We call γ the *gyration vector*. It is not a polar but an *axial* vector like the angular velocity ω in Vol. I, Sec. 22. We quote from there:

"Axial vectors are properly represented by an axis provided with a sense of rotation and a magnitude of rotation." "The signs of their components do not change under inversion of the coordinate system (interchange of

¹Physically, the factor i is due to the fact that the value of \mathbf{D} at any given point depends not only on the value of \mathbf{E} at that particular point but also on the behavior of \mathbf{E} in the vicinity of this point; that is to say, \mathbf{D} depends also on the local derivatives of \mathbf{E} . Owing to the wave character of \mathbf{E} , these derivatives contain the factor i . From an atomistic viewpoint this is due to the influence of neighboring ions which are in a field differing from that at the point under consideration. In order that these effects shall not cancel one another, the lattice has to have a certain amount of asymmetry which will be determined presently.

x, y, z with $-x, -y, -z$." "The vector product of an axial and a polar vector is a polar vector." "Under inversion a right-handed coordinate system becomes a left-handed coordinate system." But thereby also the sense of rotation $+i$ of the complex plane is replaced by $-i$.

Let us consider a crystal which has a symmetry center. For such a crystal eq. (3) must be invariant under inversion. Upon inversion the components of the polar vectors \mathbf{D} and \mathbf{E} change sign as do the components of $\boldsymbol{\gamma} \times \mathbf{E}$. But since i also changes its sign, the sign of the last term in (3) remains unchanged. Hence after an inversion (3) reads

$$(3a) \quad -D_j = -\sum_k \epsilon_{jk} E_k - i[\boldsymbol{\gamma} \times \mathbf{E}]_j.$$

This is consistent with (3) only if $\boldsymbol{\gamma} \times \mathbf{E} = 0$. Therefore our invariance condition for a centrally symmetric crystal requires that $\boldsymbol{\gamma} = 0$. Only a crystal without a center of symmetry can possess a gyration vector. There are examples of such acentric crystals among each of the seven crystal systems. The most common example is quartz (silicon dioxide, SiO_2).

Writing eq. (3) in the principal axes of the dielectric, 1, 2, 3, we obtain

$$(4) \quad \begin{aligned} D_1 &= \epsilon_1 E_1 + i\gamma_3 E_2 - i\gamma_2 E_3, \\ D_2 &= \epsilon_2 E_2 - i\gamma_3 E_1 + i\gamma_1 E_3, \\ D_3 &= \epsilon_3 E_3 + i\gamma_2 E_1 - i\gamma_1 E_2. \end{aligned}$$

We see here that in general, in a triclinic acentric crystal, for instance, the $\boldsymbol{\gamma}$ -direction is by no means determined by the principal dielectric axes. However, in the case of quartz with its rotational dielectric symmetry, the gyration vector must also submit to this symmetry and must be parallel to the principal axis. Hence $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = \gamma$, and (4) becomes

$$(4a) \quad \begin{aligned} D_1 &= \epsilon_1 E_1 + i\gamma E_2, \\ D_2 &= \epsilon_2 E_2 - i\gamma E_1, \\ D_3 &= \epsilon_3 E_3. \end{aligned}$$

B. THE ROTATION OF THE PLANE OF POLARIZATION IN QUARTZ.

We proceed exactly as in Sec. 26 except that the above eq. (4a) is used in place of $D_j = \epsilon_j E_j$. As a result, a number of correction terms which differ for the different directions $j = 1, 2, 3$ have to be added in eq. (26.8). For $j = 1$, we obtain in place of (26.8)

$$(5) \quad \frac{1}{\mu_0 v^2} B_1 = \epsilon_1 B_1 + i\gamma B_2 - s_1 \left\{ \sum_i s_i \epsilon_i B_i + i\gamma (s_1 B_2 - s_2 B_1) \right\}.$$

Introducing the abbreviation,

$$(5a) \quad K = \sum s_j \varepsilon_j B_j + i\gamma (s_1 B_2 - s_2 B_1),$$

multiplying (5) by μ_0 , and introducing the ordinary and extraordinary principal light velocities $u_o = (\varepsilon_1 \mu_0)^{-1/2} = (\varepsilon_2 \mu_0)^{-1/2}$, $u_e = (\varepsilon_3 \mu_0)^{-1/2}$, one obtains in place of (26.9)

$$(6) \quad \left(\frac{1}{u_o^2} - \frac{1}{v^2} \right) B_1 + i\mu_0 \gamma B_2 = \mu_0 s_1 K.$$

Correspondingly, we find for $j = 2$ and 3

$$(7) \quad \left(\frac{1}{u_o^2} - \frac{1}{v^2} \right) B_2 - i\mu_0 \gamma B_1 = \mu_0 s_2 K,$$

$$(8) \quad \left(\frac{1}{u_e^2} - \frac{1}{v^2} \right) B_3 = \mu_0 s_3 K.$$

We now multiply eqs. (6), (7), and (8) by the factors

$$\frac{s_1}{\left(\frac{1}{u_o^2} - \frac{1}{v^2} \right)}, \quad \frac{s_2}{\left(\frac{1}{u_o^2} - \frac{1}{v^2} \right)}, \quad \frac{s_3}{\left(\frac{1}{u_e^2} - \frac{1}{v^2} \right)}$$

respectively, and add the three resulting equations, obtaining

$$(9) \quad \sum s_j B_j + i\mu_0 \gamma \frac{s_1 B_2 - s_2 B_1}{\frac{1}{u_o^2} - \frac{1}{v^2}} = \mu_0 K \left\{ \frac{s_1^2 + s_2^2}{\frac{1}{u_o^2} - \frac{1}{v^2}} + \frac{s_3^2}{\frac{1}{u_e^2} - \frac{1}{v^2}} \right\}.$$

The first term on the left-hand side vanishes because $s \perp E$. But the second term also vanishes to an order higher than the first, for the presence of the factor γ allows us to approximate B_1 and B_2 by means of eq. (26.9). Since in this equation B_1 and B_2 are proportional to s_1 and s_2 respectively, $s_1 B_2 - s_2 B_1$ is at least of the first order in γ . Hence the right-hand side of (9) must vanish at least like γ^2 . Since $K \neq 0$, we find that to the same degree of accuracy

$$(9a) \quad \frac{s_1^2 + s_2^2}{\frac{1}{u_o^2} - \frac{1}{v^2}} + \frac{s_3^2}{\frac{1}{u_e^2} - \frac{1}{v^2}} = 0.$$

This is our former eq. (26.10) specialized to the case of an uniaxial crystal. Therefore, *except for differences of the second order in γ , the ray surface of an optically active crystal agrees with that of an inactive crystal.*

But this is true only "generally", namely only so long as the denominator of the second term on the left side of eq. (9) does not itself become as small as γ ; but for a ray in the approximate direction of the optic axis one has $v^2 \sim u_o^2$.

For such a ray s_1 and s_2 are small of the first order, and so is B_3 , because of the transversality condition. Therefore, according to (5), K also becomes small of the first order. This means that the right-hand sides of eqs. (6) and (7) are second order quantities in s_1 and s_2 and can be neglected, while eq. (8) is automatically satisfied in the first order. Equations (6) and (7) thus become

$$(10) \quad \begin{cases} \left(\frac{1}{u_o^2} - \frac{1}{v^2} \right) B_1 + i\mu_o \gamma B_2 = 0 \\ \left(\frac{1}{u_o^2} - \frac{1}{v^2} \right) B_2 - i\mu_o \gamma B_1 = 0 \end{cases} \quad \begin{matrix} 1 \\ \pm i \end{matrix}$$

Multiplication by the factors indicated on the right and addition gives rise to two equations which must be fulfilled simultaneously:

$$(11) \quad \begin{cases} \left(\frac{1}{u_o^2} - \frac{1}{v^2} + \mu_o \gamma \right) (B_1 + i B_2) = 0, \\ \left(\frac{1}{u_o^2} - \frac{1}{v^2} - \mu_o \gamma \right) (B_1 - i B_2) = 0. \end{cases}$$

If we satisfy the first equation by choosing v^2 so as to make the first factor zero, then we must satisfy the second equation by setting its second factor equal to zero, and vice versa. Thus there are two solutions for v^2 which correspond to the two branches of the ray surface. As before, we shall denote the two solutions by v' and v'' , and the corresponding values of B will also be distinguished by primes and double primes. Then our two solutions read

$$(11a) \quad \frac{1}{u_o^2} - \frac{1}{v'^2} + \mu_o \gamma = 0, \quad B_1' - i B_2' = 0,$$

$$(11b) \quad \frac{1}{u_o^2} - \frac{1}{v''^2} - \mu_o \gamma = 0, \quad B_1'' + i B_2'' = 0.$$

These expressions represent two opposite circularly polarized waves which propagate with the respective velocities

$$(12) \quad \left. \begin{matrix} v' \\ v'' \end{matrix} \right\} = u_o \left(1 \mp \frac{g}{2} \right), \quad g = \mu_o \gamma u_o^2$$

and whose directions of rotation are determined by the two non-vanishing complex quantities $B_1' + i B_2'$ and $B_1'' - i B_2''$.

This situation is illustrated in fig. 40. However, in contrast to fig. 39 a, this figure refers not to linearly but to circularly polarized waves. Where the optic axis passes through them, the two branches of the ray surface are now separated by the small distance

$$(13) \quad v'' - v' = u_o g$$

instead of being tangent as before. For all other ray directions for which the two branches were separated anyway in fig. 39 a, the additional separation is a second order correction and can be neglected. In these other directions, in particular normally to the optic axis, ordinary double refraction of linearly polarized waves takes place.

As in Sec. 20 we interpret the difference between v' and v'' in terms of a *rotation of the plane of polarization*. Let us consider a linearly polarized wave which is normally incident on a quartz plate whose surfaces are cut perpendicular to the optic axis. We decompose the linear polarization into two equal and oppositely directed circular polarizations. In passing through the plate of thickness l the phase of one of these waves lags behind that of the other. When they emerge from the crystal, we again combine the two waves into one linearly polarized wave. Its plane of polarization differs from that of the incident wave. The plane of polarization has been turned through an angle χ which is proportional to the thickness l and to the difference $k_+ - k_-$. Hence χ is also proportional to the difference $v'' - v'$ which along the optic axis is identical with the difference $u'' - u'$ between the wave velocities. The difference between magnetic and "natural" rotation of the plane of polarization, which we discussed at the end of Sec. 20, is due to the fact that if we reverse the direction of the ray, our present gyration vector, which depends on the structure of the quartz, does not change sign, while in Sec. 20 the magnetic field strength does change its sign under reversal of the direction of propagation.

The absence of a center of symmetry as a "*conditio sine qua non*" is indicated by the outward shape of quartz. There are "right-handed and left-handed quartzes" which are distinguished by the enantiomorphic trapezoidal faces which truncate their hexagonal prisms to the right or to the left. The rotating power of cinnabar HgS is several times stronger than that of quartz. Optical activity has also been found to exist in the axis directions of optically biaxial crystals (cane sugar, Rochelle salt) (Voigt, Pocklington). If cubic crystals, in which every direction is a principal axis and an optic axis, are optically active at all, they are optically active in every direction, e. g. NaClO_3 . Optical activity is not due to the crystal lattice, i. e., to the mere internal periodicity of the crystal, but to its structure, that is,

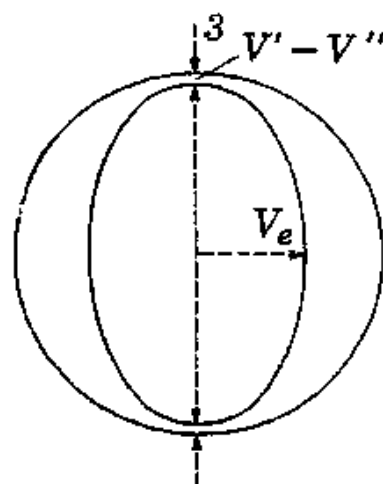


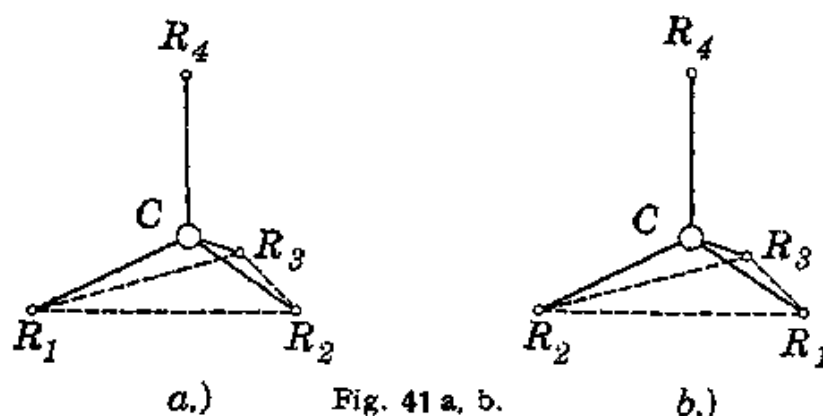
Fig. 40.

Ray surface of an optically active uniaxial crystal. (In contrast to fig. 39 a, this refers to a circularly polarized wave.)

to the symmetry of arrangement of the elements of structure, the atoms, within the unit of periodic repeat. In molecular crystals, like cane sugar, the part of the rotatory power that resides in the molecule is retained when the crystal is dissolved in a liquid; in atomic crystals, like NaClO_3 , rotation resides entirely in the crystal structure, and the solution, in which the molecule has dissociated into ions, is optically inactive.

C. OPTICALLY ACTIVE FLUIDS.

Here we are not concerned with a rigid structure as in crystals but rather with *fluid molecules* whose spatial positions and orientations are statistically distributed. If we average over all possible orientations of these molecules, the gyration vector γ which was introduced in the assumption (3) reduces to a "gyration constant" γ . The degree of asymmetry necessary for activity is now greater than for crystals. Not only must the molecule have no *center of symmetry* but also it must not have any *plane of symmetry*. These conditions are satisfied in molecules which contain an asymmetrical carbon atom, i. e. one



The two enantiomeric forms of an optically active molecule. These two forms cannot be brought into coincidence by any space rotation.

whose four valences are attached to four different atoms or radicals. There exist two mutually enantiomorphic arrangements of these four substituents which are related to one another like image and mirror image or like right- and left-handed screws. In fig. 41a the sequence $R_1 R_2 R_3 \rightarrow R_4$ forms a right-handed screw; in fig. 41b this sequence forms a left-handed screw. These two forms cannot be brought into coincidence by any motion in three-dimensional space. Examples of this kind of molecule are the two sugar types, grape sugar, dextrose, and fruit sugar, levulose. The rotating power of solutions and mixtures of these sugars can be determined with extreme accuracy. A balanced mixture of right- and left-rotating molecules is called a "racemic" state. We have already mentioned at the end of Sec. 20 the great importance of activity measurements in the sugar industry.

Lindman¹ devised a macroscopic model which serves to illustrate the molecular process responsible for optical activity. The arrangement is the following: a cardboard box contains several hundred wire spirals which are about 2 cm. in diameter and consist of about two windings, all having the same rotational sense. The spirals are wrapped individually in tissue paper and the box is shaken so that the spirals assume arbitrary positions. Linearly polarized electromagnetic dipole radiation of about 10 cm. wavelength falls on the box. This radiation can be thought of as decomposed into two waves, right- and left-circularly polarized. One of these two waves is accelerated in its propagation by the metal spirals while the other wave is retarded. Behind the box the two circularly polarized waves again combine into one linearly polarized wave whose direction of oscillation is, however, rotated with respect to that of the incoming wave. This rotation of the plane of polarization can be detected by rotating a linear antenna which is tuned to the primary radiation and is connected to a receiver. The rotation of the plane of polarization can be cancelled by inserting a second box which is identical to the first and contains an equal number of spirals which are, however, twisted in the opposite rotational sense. The two boxes together constitute a racemic mixture.

The description of this attractive model experiment will have to serve as a substitute for the treatment of the actual molecular theory of optical activity which, unfortunately, cannot be taken up here.

30. Nicol's Prism, Quarter Wave Plate, Tourmaline Tongs, and Dichroism

A. NICOL'S PRISM

Looking at a model of the structure of calcite CaCO_3 , one gains the impression that the constituents Ca^{++} and CO_3^{--} would assume a cubic arrangement (as do the constituents Na^+ and Cl^- of rock salt) if it were not for the fact that the plane triangular radical CO_3 (in contrast to the spherical Cl^- ion) imposes special lateral conditions on the spatial arrangement. These requirements force the cubic structure which characterizes rock salt into a rhombohedral structure. The transition between these two types of structures can be imagined as a lateral stretching or, alternatively, as a longitudinal compression of the cubic model. In this process one of the threefold symmetry axes formed by the diagonals of the cubic structure becomes the threefold principal axis and at the same time the optic axis of the rhombo-

¹K. F. Lindman, *Ann. d. Phys.* 68, p. 621, 1920 and 69, p. 270, 1922.

hedral crystal. The fundamental cell of such a lattice is a rhombohedron bounded by $3 + 3$ rhombuses. This is easily confirmed because the crystals can be split along the face planes of the cells. The double refraction of the famous crystal clear "Iceland Spar" was discovered by Bartholinus and studied by Huygens.

A Nicol prism (which is really not a prism but a parallelepipedon) is made from a cleavage rhombohedron which is about three times as long as it is wide. The end surfaces AB , CD are then cut so that they form an angle of 68° with the long edge; see fig. 42 (the natural surfaces AB' , CD' which are drawn dotted in the figure make an angle of $70^\circ 52'$ with the long edges). Finally, the resulting parallelepiped is cut in two along a plane perpendicular

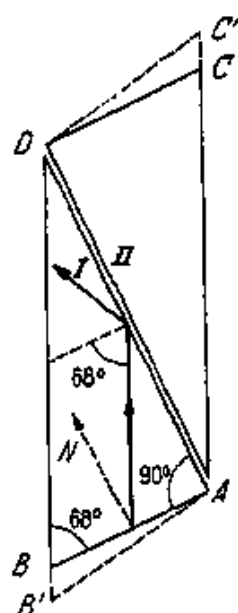


Fig. 42.

Section parallel to the longitudinal edges of a Nicol's prism. Geometrical description of the prism.

to the end surfaces AB , CD , and the two halves I and II are glued together with Canada balsam. The index of refraction of Canada balsam is 1.55. The two principal indices of refraction of calcite are

$$(1) \quad n_o = 1.66, \quad n_e = 1.49.$$

For the ordinary ray the Canada balsam is the rarer medium; for the extraordinary ray it is the denser medium. The ordinary ray can enter the balsam only if its angle of incidence is less than the limiting angle of total reflection. According to Sec. 5 the latter is given by

$$(2) \quad \sin \alpha_{\text{tot}} = \frac{1.55}{1.66}, \quad \alpha_{\text{tot}} = 69^\circ 10'.$$

If a ray parallel to the longitudinal edges is incident on one of the end surfaces, the ordinary ray is refracted at that surface so that it falls on the balsam at an angle of incidence of about 77° . This ray is therefore totally reflected at the balsam layer and does not enter the second half II of the crystal. It is deflected towards the side face BD . The same is also true for a certain easily calculated interval of neighboring directions of the incident ray. The face BD is blackened so that it will absorb these ordinary rays.

The extraordinary ray for which the balsam is the denser medium cannot be totally reflected (see fig. 42a). Furthermore, because $n_e < n_o$, this ray is less strongly refracted toward the normal N upon entering the calcite. After passing through the balsam, the ray traverses the crystal II in a direction parallel to that in I and it emerges from the Nicol prism with a direction parallel to that of the incoming ray. In the figure this direction is parallel to

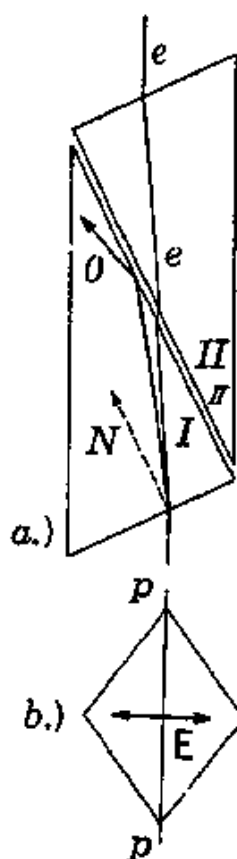
Fig. 42 a, b.

- a) Ray paths in the Nicol prism. The section is the same as that drawn in fig. 42. e is the extraordinary ray passing through the prism. o is the totally reflected ordinary ray.
- b) The Nicol prism viewed end on. The position of the plane of polarization is indicated by pp . The direction of oscillation is shown by E .

the longitudinal edges of the prism. Its plane of polarization is that of the extraordinary ray in calcite, namely, see Sec. 28, parallel to the optic axis. This plane of polarization is indicated by the longer diagonal of the end surface of the prism as shown in fig. 42 b.

Therefore the Nicol prism produces *linearly polarized light* whose direction of oscillation is known. Because the light oscillating perpendicularly to that direction is suppressed by total reflection, the polarization is *complete*.

When two Nicol prisms which can be rotated about their longitudinal axes are placed one behind the other in the path of a light ray, then the first prism is called a *polarizer* and the second an *analyzer*. If the analyzer is oriented perpendicular to the polarizer and there is no birefringent or optically active material between the two, then no light leaves the analyzer. If the analyzer is now rotated, a gradually increasing amount of light passes through it which attains its maximum intensity when the analyzer is oriented parallel to the polarizer. In Sec. 31 we shall discuss the interesting intensity and color patterns which result if a double refracting crystal plate is placed between the polarizer and the analyzer and if the incident light is parallel. We shall also treat the still more interesting patterns produced by *converging* light.



B. THE QUARTER-WAVE PLATE AND THE BABINET COMPENSATOR

Mica (alkali-aluminum silicate) is a monoclinic crystal with an extraordinarily pronounced cleavage parallel to the base plane. For optical purposes the transparent potassium mica, $\text{KH}_2\text{Al}_3(\text{SiO}_4)_3$, called *muscovite* is of particular interest. The twofold crystallographic symmetry axis is identical with our dielectric principal axis 2; the base plane is identical with the principal axis plane 12, see fig. 43. The plane perpendicular to 2 is the crystallographic symmetry plane. This plane contains the dielectric principal axis 3 and the two optic axes as well as the crystallographic¹ axis 3' (which is drawn dotted

¹The angle between the two crystallographic axes 1' and 3' is $\beta = 95^\circ 5'$ and hence differs little from $\pi/2$. For this reason mica used to be thought of as orthorhombic or hexagonal because of the frequently occurring hexagonal shape of the base plane which is shown in fig. 43.

in the figure). The other two crystallographic axes 1' and 2' are identical with 1 and 2, respectively. The structural model² of mica illustrates the stratified structure of the crystal and its prominent cleavage parallel to the base plane.

Taking advantage of this cleavage property we make a very thin mica plate. A light ray which has been polarized by a Nicol prism is allowed to fall perpendicularly, that is in the direction 3, on this plate. The Nicol prism

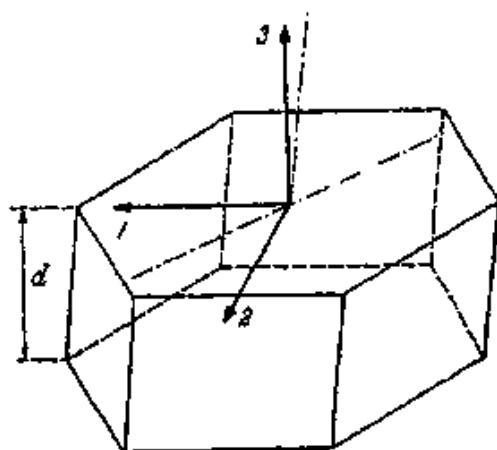


Fig. 43.

Crystallographic model of mica.

is oriented so that the trace of the polarization plane on the mica surface bisects the angle between the axes 1 and 2 (drawn as a dot-dash line in the figure). For convenience the directions of these axes can be marked on the frame holding the mica plate. In the crystal the ray is decomposed into two linearly polarized waves of equal amplitude and direction of propagation (neither wave is refracted, see Sec. 27 B), which oscillate in the directions 2 and 1, respectively. These two waves propagate with the principal

light velocities u_1 and u_2 . In the yellow part of the spectrum (D-line) the corresponding indices of refraction are

$$(3) \quad n_1 = 1.5941, \quad n_2 = 1.5997.$$

Since one wave propagates faster than the other, a phase difference develops between them which at a depth x amounts to

$$(4) \quad (k_2 - k_1) x = k (n_2 - n_1) x = \frac{2\pi}{\lambda} (n_2 - n_1) x;$$

k and λ are the wave number and wavelength in air. At the rear surface of the plate, $x = d$, the two waves emerge again without refraction, so that they still propagate in the same direction and are polarized in mutually perpendicular directions. While originally their phases were the same, they now differ. Combination of the two waves yields *elliptically polarized light*.

If we make the phase difference (4) for $x = d$ equal to $\pi/2$, that is, if we set

$$(5) \quad \frac{\pi}{2} = \frac{2\pi}{\lambda} (n_2 - n_1) d; \quad d = \frac{\lambda/4}{n_2 - n_1},$$

²This was first designed by Lawrence Bragg during an extended visit in Munich and was constructed by Karl Selmayr, the skillful mechanic of the Institute of Theoretical Physics.

then we obtain circular polarization. From (3) and (5) we find for $\lambda = 5.9 \times 10^{-4}$ mm.

$$(5a) \quad d = \frac{5.9 \times 10^{-4}}{4 \times 0.0056} \text{ mm.} \sim 0.026 \text{ mm.}$$

The name "quarter-wave plate" is somewhat misleading (especially for examination candidates!): its thickness is not $\lambda/4$, but fortunately it is larger than $\lambda/4$ by a factor of one over the small quantity $n_2 - n_1$. We could, of course, use any odd multiple of $\pi/2$ instead of $\pi/2$ in (5). Then d would be three times, five times, ... as large as the thickness given by (5a). However, because of their stronger dispersion, these thicker plates are optically less advantageous than the actual quarter-wave plate.

The same is true for every other birefringent crystal. Mica is used only because of its extremely pronounced cleavage. Formula (4) for the phase difference can be applied to any arbitrary thickness. If we cut a crystal such as quartz in the shape of a wedge and view the light passing through the varying thickness of the wedge, we observe a whole range of phase differences emerging from the crystal. Since we are here interested in double refraction and not in rotation of the plane of polarization, the light must pass through the quartz wedge in a direction *perpendicular to the optic axis*. The horizontal hatched lines in the lower part of the front surface in fig. 44 indicate the position of the optic axis. The latter is therefore perpendicular to the edge of the wedge which passes through O .

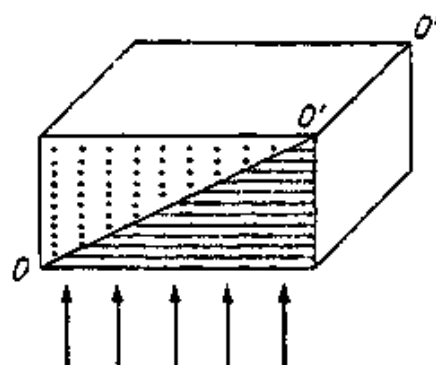


Fig. 44.
Babinet Compensator

We now cut a second quartz wedge which is outwardly congruent to the first one, but in which the optic axis is *parallel* to the edge $O'O'$ of the wedge. We place the two wedges against each other so that together they form a plane parallel plate. The position of the optic axis in the upper wedge is indicated by the dots in fig. 44. Let us consider a point on the crystal where the upper wedge has a thickness x_2 and the lower wedge a thickness x_1 . Then by (4) phase differences for a ray passing through the two wedges at that point are

$$\frac{2\pi}{\lambda} (n_e - n_o) x_1 \quad \text{and} \quad -\frac{2\pi}{\lambda} (n_e - n_o) x_2.$$

We have written here n_e and n_o (since quartz is uniaxial) instead of n_1 and n_2 (as in the case of the biaxial mica). The negative sign is due to the reversed positions of the two wedges. The total phase difference is therefore

$$(6) \quad \Delta = \frac{2\pi}{\lambda} (n_e - n_o) (x_1 - x_2).$$

This combination of quartz wedges when placed between crossed Nicol prisms is called a *Babinet Compensator*. (In the figure the polarizer will be below the plate and the analyzer above it. The polarization planes of the Nicols must be placed at an angle of 45° with respect to the wedge edges.) If such a compensator is illuminated with monochromatic light, then for $x_2 = x_1$ a dark fringe appears because there $\Delta = 0$. Thus in the center of the plate we observe complete extinction just as though no birefringent medium were present between the crossed Nicols. The same is true for the points where $\Delta = \pm 2\pi, \pm 4\pi, \dots$. One obtains therefore a system of equidistant dark fringes. If one wedge is shifted with respect to the other by means of a micrometer screw, the system of fringes moves also. The same effect is achieved if any doubly refracting plate is inserted between the wedge plate and one of the Nicols in such a way that the new optic axis also makes an angle of 45° with the planes of the Nicols. The resulting shifted fringes can be returned to their original positions by displacing the wedges with respect to each other (hence the name compensator). From this shift, which is read on the micrometer, the amount of double refraction of the inserted sample (its $n_e - n_o$ or $n_1 - n_2$) can be determined by means of (6).

If the compensator is illuminated with white light, it again produces a dark center fringe at $x_2 = x_1$. To the right and left of this line the Newtonian colors of thin plates appear.

We need not go into the many existing modifications of this apparatus.

C. TOURMALINE AND THE POLARIZATION FILTER

Tourmalines are boron silicates of various chemical compositions. Their crystalline structure belongs to a class of the hexagonal system without symmetry center and with a "polar principal axis". The latter is the reason for the pyroelectric property¹ of tourmaline. A plate made from suitable material and cut parallel to the principal axis has a transparent green appearance, while a plate cut perpendicularly to the principal axis looks

¹See Vol. III, 11 E. The permanent electric moment of tourmaline, which is ordinarily compensated by a surface charge, becomes apparent if the temperature is changed.

almost black. This property of tourmaline is called *dichroism*. It is a special form of *pleochroism* (multicoloredness).

Thus the absorptivity of tourmaline depends on direction, and this dependence must, of course, conform to the symmetry of the crystal structure. This is true for all absorbing crystals. In tourmaline the ordinary ray is almost completely absorbed, while the extraordinary ray is absorbed only weakly. The light emerging from a plate which is cut parallel to the principal axis consists almost entirely of the extraordinary ray. Therefore this light is almost completely linearly polarized. The long familiar instrument known as the "tourmaline tongs" is based on this fact.

Modern commercial "*polarization filters*" are made from impregnated plastic materials which are subjected to a strong tension. The absorbing pigment is thereby given an anisotropic arrangement which causes *complete polarization* of any light passing through the material. The same effect can be obtained with strongly dichroic dyes (methylene blue) which are crystallized on glass in thin layers like "frost flowers".

In Sec. 6 we described the absorption in isotropic metals by adding to the dielectric constant the conduction term $i\sigma/\omega$. In this way we obtained the complex dielectric constant of (6.1). For a crystal we are led to the complex tensor

$$(7) \quad \epsilon'_{jk} = \epsilon_{jk} + i \frac{\sigma_{jk}}{\omega}$$

which consists of the dielectric and the conductivity tensors.

The nature of the imaginary part of (7) clearly differs from that in (29.2). The tensor γ_{jk} of Sec. 29 was non-dissipative and therefore had to be anti-symmetric. Our present imaginary tensor, on the other hand, is dissipative, as in the case of metallic reflection, and it can therefore be assumed to be symmetric. Its principal axes need not agree with those of the ϵ_{jk} tensor. However, the symmetry rule which we used in Sec. 29 still holds: if the principal axes of one of the tensors are completely determined by the crystallographic structure, then the axes of the other tensor must also be completely determined, and the two principal axes systems must be identical. By this rule which applies to tourmaline because of its hexagonal structure, the calculations of Sec. 24 ff. can be formally extended without change to the absorbing crystals. This leads to complex principal dielectric constants and therefore the principal light velocities defined in (24.6 b) also become complex. The wave velocities corresponding to a given direction of the wave number vector are

again defined by the quadratic equation (25.10). Since the roots u' , u'' of this equation are now complex, the components of the corresponding electric vectors D' , D'' are also complex. This means that D' , D'' now describe not linearly, but *elliptically* polarized oscillations. Thus it is seen that the quantitative theory of anisotropic absorption requires, at least for sufficiently symmetric crystals, no essentially new mathematical development. Since absorption, like double refraction, depends in general on the wavelength, we have thus obtained a general scheme for the explanation of the *pleochroism* of crystals.

31. Interference Phenomena Due to Crystal Plates in Parallel and Converging Polarized Light

Consider a thin crystal plate of the type customarily used in petrography which is placed between two, usually crossed, Nicol prisms. We shall assume the light to be monochromatic except where we expressly state it to be white. For observations with parallel light the rays shall fall perpendicularly on the plate. By "converging light" we mean an arrangement of converging lenses in front of and behind the plate (see below) which enables us to observe simultaneously all bundles of parallel rays which pass through the plate in arbitrary directions which, however, do not differ very much from the normal to the plate.

There are two "principal directions of oscillation" in the crystal plate. These are the principal axes of the ellipse formed by the intersection of the plate surface (which is also a wave surface) with Fresnel's ellipsoid (or the index ellipsoid). The plate is assumed to be in the "diagonal position" between the Nicols; this means that the two principal directions of oscillation are bisected by the plane of polarization of the polarizer (and also by that of the perpendicularly placed analyzer). In this position the amplitudes of the two components of the *incident* light along the principal directions are equal, and so are their phases, since they arise from one linearly polarized oscillation in the polarizer. Since we assume the crystal to be transparent (not dichroic), the amplitudes of these components of the light *emerging* from the plate are also equal. But, as we shall presently show, their *phases differ*. Therefore when the two components are recombined, the resultant emerging intensity differs from the incident intensity. The value of the former varies between maximum brightness and complete darkness as the crystal is rotated out of its diagonal position. We can neglect the small intensity changes which take place when the light enters and emerges from the crystal.

A. PARALLEL LIGHT

The wave velocities u' , u'' of the two principal oscillations H_1 , H_2 (fig. 45) determine the two indices of refraction $n_1 = c/u'$, $n_2 = c/u''$. These in turn determine the phase difference which arises between the two component waves as the light passes through a crystal plate of thickness d . As in (30.4) this phase difference is given by

$$(1) \quad \Delta = \frac{2\pi}{\lambda} (n_2 - n_1) d.$$

If we call a the amplitude of the wave incident from the polarizer (whose polarization plane is indicated by PP in the figure), then the initial amplitudes of the principal components of this wave are $a/\sqrt{2}$ if the crystal plate is in the diagonal position. After passage through the plate, these oscillations are represented by

$$\frac{a}{\sqrt{2}} \begin{pmatrix} e^{\frac{2\pi i}{\lambda} n_1 d} \\ e^{\frac{2\pi i}{\lambda} n_2 d} \end{pmatrix} e^{-i\omega t}$$

which we shall write in the form

$$(2) \quad \frac{a}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\Delta} \end{pmatrix} \exp \left(\frac{2\pi i}{\lambda} n_1 d - i\omega t \right).$$

We now project this oscillation onto the polarization plane of the analyzer (AA in fig. 45). The signs of these projections are determined from the solid lines in the figure, and the amplitude of the resultant oscillation behind the analyzer is found to be

$$(3) \quad \frac{a}{2} |1 - e^{i\Delta}|.$$

Now we use

$$|1 - e^{i\Delta}|^2 = (1 - e^{i\Delta})(1 - e^{-i\Delta}) = 2 - 2 \cos \Delta = 4 \sin^2 \frac{\Delta}{2}$$

which gives for (3) simply

$$(4) \quad a \sin \frac{\Delta}{2}.$$

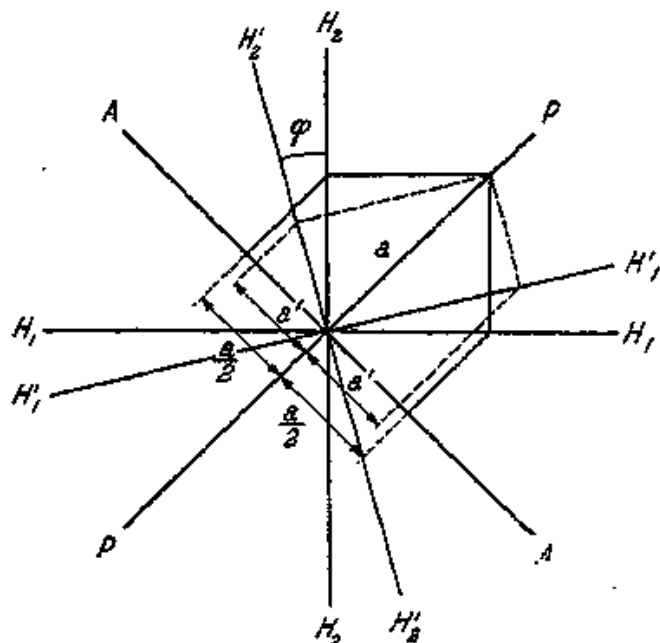


Fig. 45.

Crystal plate in normally incident parallel light for different positions H_1 , H_2 and H_1' , H_2' of the principal oscillation directions. The plate is between crossed Nicols whose oscillation directions are PP and AA , respectively.

If the plate is rotated into the position H_1' , H_2' making an angle φ with the diagonal position (see the dotted lines in fig. 45), then the initial amplitudes are, instead of $a/\sqrt{2}$

$$(5) \quad a_1 = a \cos \left(\frac{\pi}{4} - \varphi \right), \quad a_2 = a \cos \left(\frac{\pi}{4} + \varphi \right).$$

The projections of a_1 and a_2 on the analyzer plane are given by $a \cos (\pi/4 + \varphi)$ and $a \cos (\pi/4 - \varphi)$, respectively. Therefore both projections are equal except for their signs. The magnitude of both is

$$a \cos \left(\frac{\pi}{4} - \varphi \right) \cos \left(\frac{\pi}{4} + \varphi \right) = \frac{a}{2} (\cos^2 \varphi - \sin^2 \varphi) = \frac{a}{2} \cos 2\varphi.$$

It follows that the resultant amplitude behind the analyzer is given by

$$(6) \quad a' = a |\cos 2\varphi| \sin \frac{\Delta}{2}$$

instead of by (3) and (4). The observed intensity is therefore

$$(7) \quad J = J_0 \cos^2 2\varphi \sin^2 \frac{\Delta}{2}$$

where J_0 is the intensity of the light incident on the plate.

According to (7), for one complete revolution of the plate the intensity observed behind the analyzer changes four times between maximum brightness at the diagonal positions

$$(8) \quad \varphi = 0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}$$

and complete darkness whenever H_1' , H_2' coincide with P or A , that is, when

$$(8a) \quad \varphi = \frac{\pi}{4}, \quad \frac{3\pi}{4}, \quad \frac{5\pi}{4}, \quad \frac{7\pi}{4}.$$

If the illumination is monochromatic, the plate appears of varying but uniform brightness in its entirety.

If white light is used, the positions (8a) again yield darkness. In the intermediate positions the entire plate is uniformly colored with a *mixed color*.

Only for very thin or very thick plates does the color remain white. For very thin plates this is true because there is no wavelength for which $\Delta/2$ attains the value π . For very thick plates there are very many points distributed over the whole spectrum for which $\Delta/2$ is a multiple of π . In this case the spectrum (not to be confused with the appearance of the plate!) has a large number of *dark lines* but retains its white character. For moderately thin or moderately thick plates there are only *one or a few such dark lines*. The missing wavelengths and the intensity variations in the remaining portion of the spectrum cause

the deviation of the color from white and determine the character of the mixed color which is seen by the eye. If the observation is made with parallel instead of crossed Nicols, precisely the complementary mixed color is seen, and full illumination takes the place of darkness at the positions (8 a).

The pattern becomes much more interesting if, instead of a single crystal, a mosaic of crystal fragments is used; for instance, granite which consists of feldspar, quartz, mica, hornblende, etc. In that case each constituent yields under white illumination a different color, which is determined by the material and its orientation with respect to the surface of the plate. The principal directions H_1 , H_2 belonging to the individual crystal fragments are distributed at random in the plate. Therefore, when a thin plate of this type is rotated, its various components extinguish the light at different angular positions. By the same token the various pieces exhibit different intensities at different positions of the plate. Petrographic investigations depend to a large extent upon such observations.

We shall not investigate the appearance of the plate when the polarizer and analyzer are in an intermediate position, i. e. neither crossed nor parallel.

B. CONVERGING LIGHT

As was stated at the beginning of this section we are in fact again considering bundles of parallel rays, which now, however, in passing through the plate, assume all possible directions in the neighborhood of the normal to the plate; they are then *simultaneously* focused at the eye (at O) by means of the converging lens L' in fig. 46, B' being the focal plane of L' on which the eye is focused (with the help of a lens or microscope). B is the focal plane of the converging lens L . The light source is an *extended* luminous surface which is placed below B . We need follow the rays originating from that surface only

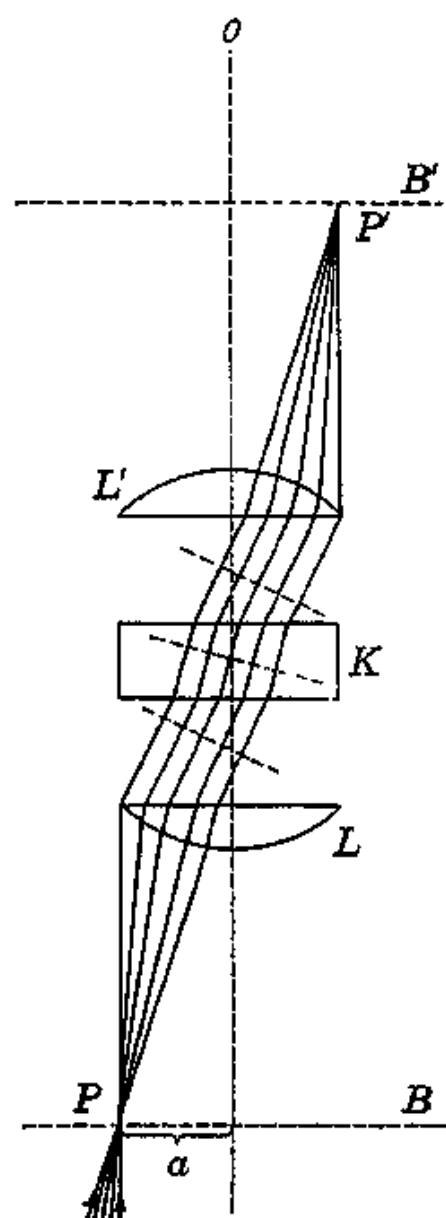


Fig. 46.

Crystal plate illuminated by "converging" light, that is by parallel light bundles which form a finite solid angle around the normal to the plate.

after they have passed through B . L converts the originally divergent rays, e. g. the rays emerging from P , into parallel light. L' converts the parallel rays emerging from the crystal plate K into converging light, e. g. into light focused at P' . The polarizer is below B and the analyzer is between B' and O .

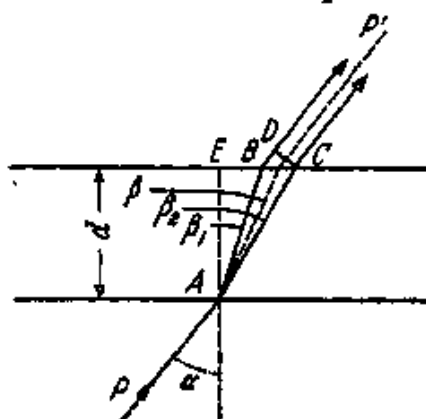


Fig. 47.

Construction for the computation of the phase difference between the two rays ABD (angle β_1) and AC (angle β_2) which are produced by double refraction.

The rays drawn in fig. 46 which are parallel inside K cannot interfere because they originate from different points on the luminous surface. Therefore, the *intensities* of these rays add. The phenomena at P' are made quantitatively observable by the fact that we have parallel ray bundles of considerable width, as is indicated in fig. 46.

Interference takes place only between any two rays which are created by double refraction in K and originate from the same ray coming from P . Each such pair of interfering rays is parallel as it leaves the plate K . Figure 47 indicates that the directions of the interfering ray pairs which have angles of refraction β_1, β_2 in the plate can (for β_1, β_2 not too large) be approximated by one ray with the average direction β . This ray is drawn dotted in the figure. It is easily verified, see exercise IV. 3, that the phase difference Δ between the two waves which propagate in this direction is given by an expression similar to eq. (1):

$$(9) \quad \Delta = \frac{2\pi n_2 - n_1}{\lambda \cos \beta} d.$$

The ray β is focused on the same point P' in the focal plane B' as the rays β_1, β_2 (see figure), and this point is characterized by the value of Δ given by (9). Therefore, the intensity observed from O at the point P' is, in analogy to (7),

$$(10) \quad J = J_0 \cos^2 2\varphi \sin^2 \frac{\Delta}{2}$$

where J_0 is the intensity incident at A and the angle φ depends on the orientations of the Nicols with respect to the principal oscillation directions of the plate and also on the ray direction β .

According to (10) we have

$$(11) \quad J = 0, \text{ extinction, if } \Delta/2 = g\pi \quad (g = \text{integer}).$$

By (9) this condition means that

$$(12) \quad \cos \beta = \frac{d}{\lambda g} (n_2 - n_1).$$

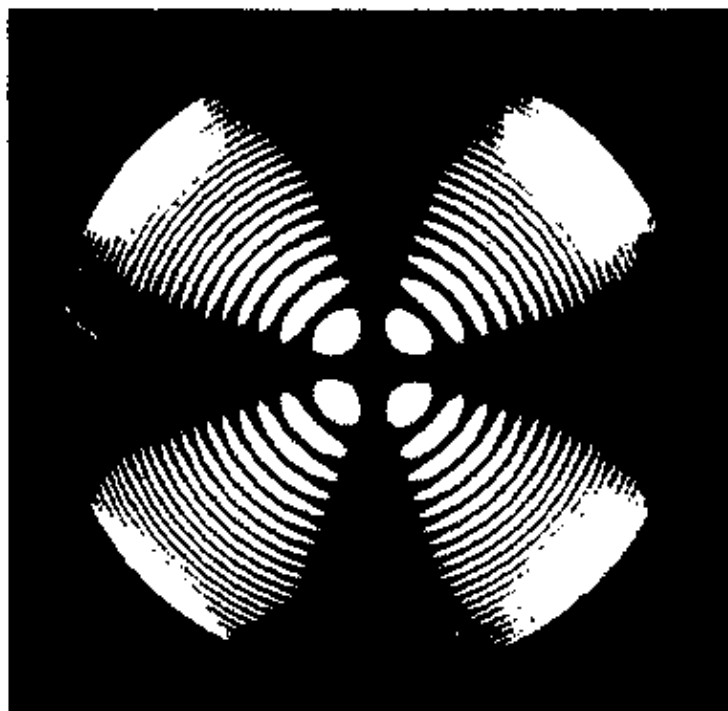


Fig. 48.

Calcite plate, cut perpendicularly to the optic axis, in sodium light between crossed Nicols.

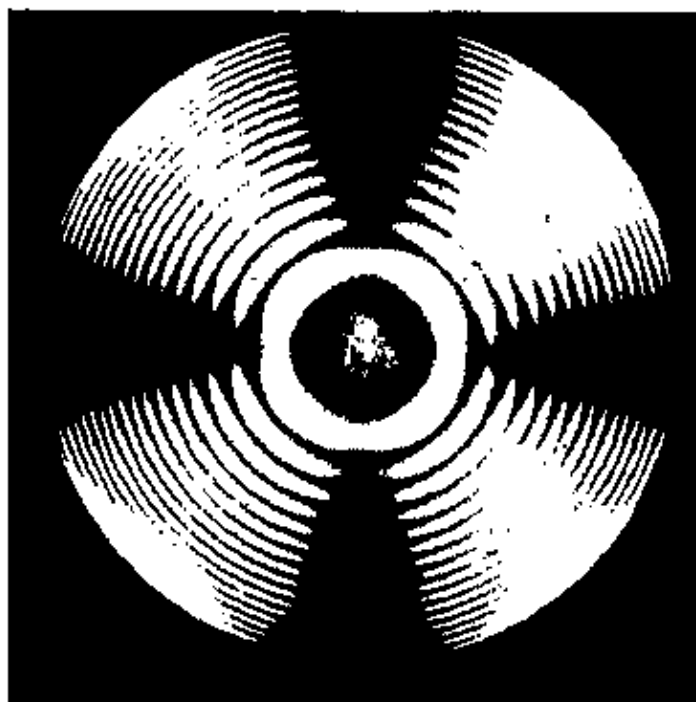


Fig. 49.

Quartz plate, cut perpendicularly to the optic axis, in sodium light between crossed Nicols. Note the bright center.

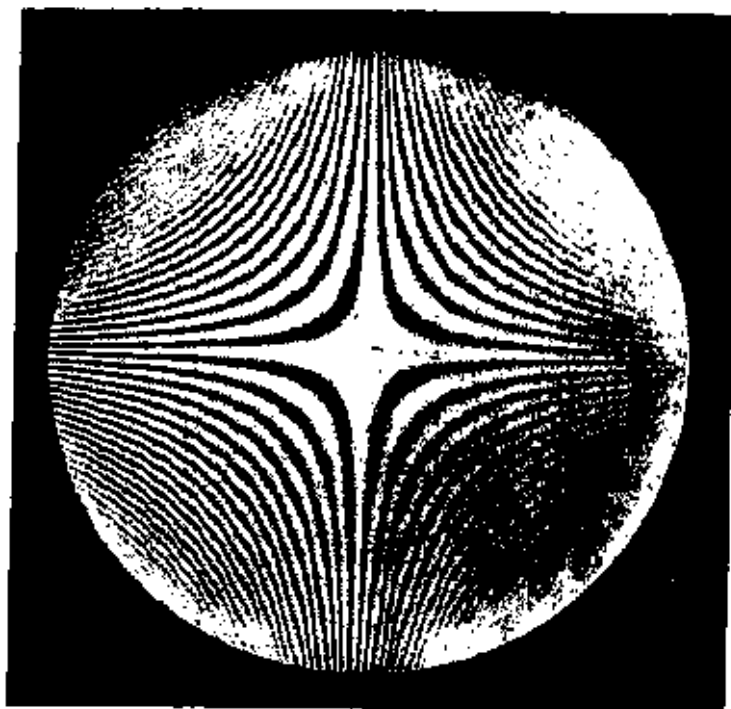


Fig. 50.

Calcite plate, cut parallel to the optic axis, in white light.
Diagonal position.

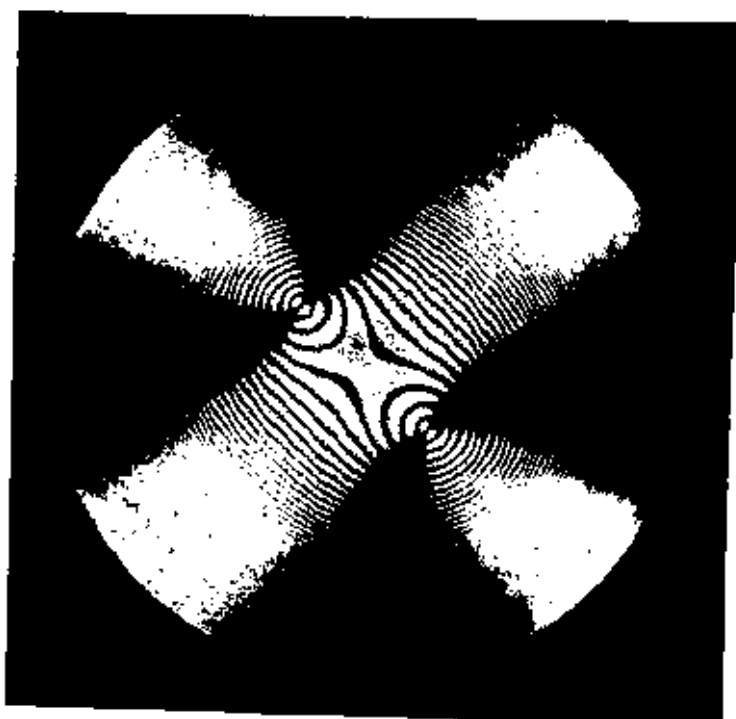


Fig. 51.

Cerussite, biaxial, cut perpendicularly to the bisector of the angle between the
optic axes. Diagonal position.

Now

$$n_1 = \frac{c}{u'}, \quad n_2 = \frac{c}{u''}$$

where u', u'' are the two wave velocities belonging to the direction β . Therefore, we have also

$$(13) \quad \cos \beta = \frac{d}{\lambda g} \left(\frac{c}{u''} - \frac{c}{u'} \right) = \frac{c d u' - u''}{\lambda g u' u''}.$$

This formula suggests that we find those directions $\beta = \beta_g$ on the Fresnel wave surface for which the above equation is satisfied. In the uniaxial case in which one branch of the wave surface is the sphere $u = u_0$ and the other the ovaloid of rotation about the optic axis (28.5), this becomes a relatively simple algebraic calculation which, however, we shall not carry out here.

Rather, we shall immediately turn to the completely symmetric situation:

crystal plate \perp optic axis.

In this case the angle of refraction lies in a meridian plane of the ovaloid, and every cone $\beta = \text{constant}$ intersects the ovaloid in a *circle*. Because of the condition (13) one obtains, therefore, in the focal plane B' a family of concentric circles

$$(14) \quad \beta = \beta_1, \beta_2, \dots, \beta_g, \dots$$

at which the intensity vanishes. By (13) their radii depend on the ratio d/λ . The difference between successive radii decreases steadily with increasing g .

But according to eq. (10), we have extinction not only for $\sin \Delta/2 = 0$ which led to (14) but also for $\cos 2\varphi = 0$. The latter condition indicates extinction along the two mutually perpendicular directions

$$(15) \quad \varphi = \pm \frac{\pi}{4}.$$

The figures 48 to 51 are reproduced from the famous collection of photographic plates "Interferenzerscheinungen im polarisierten Licht" by H. Hauswaldt, Magdeburg, 1902 and 1904. Figure 48 is obtained with calcite (1/2 mm thick) in sodium light between crossed Nicols. The system of concentric circles represents the β -values (14). The dark cross which coincides with the polarization planes of the Nicols represents the φ -values (15). With white light illumination the pattern is colored and fewer interference circles are discernible. The curves $\Delta = \text{constant}$ are called *isochromatics* since each of them is characterized by its own mixed color.

Figure 49 is obtained with a quartz plate (7 mm thick because the double refraction of quartz is less than that of calcite) which is photographed in the same manner. The pattern differs from that in fig. 48 in that the center is bright. This indicates the rotation of the plane of polarization for rays parallel to the optic axis of quartz. The center bright spot illustrates the gap between the two branches of the ray surface shown in fig. 40.

Figure 50 shows calcite which is cut parallel to the optic axis and is placed diagonally between the Nicols. The black cross is missing. The isochromatics are rectangular hyperbolas. It can be proved generally that for arbitrarily cut slabs of uniaxial crystals the circles in fig. 48 become conic sections. Indeed, these conic sections are formed by the intersections of the circular cones $\beta = \text{constant}$ with the plate surface.

Figure 51 is obtained with a biaxial orthorhombic crystal, namely cerussite, PbCO_3 , cut perpendicularly to the bisector of the angle between the two optic axes and placed in diagonal position. The cross of extinction of figures 48 and 49 is here pulled apart so that two branches meet at each of the optic axes. The isochromatics are not conic sections as for uniaxial crystals but are fourth degree curves (lemniscates). It is wonderful to observe with a polarization microscope how nature traces these varied patterns with geometric precision and colors them so brilliantly.

CHAPTER V

THE THEORY OF DIFFRACTION

Any deviation of light rays from rectilinear paths which cannot be interpreted as reflection or refraction is called diffraction. Reflection and refraction, clearly, occur only when the bodies causing the deviations of the rays from straight lines have surfaces whose radii of curvature are everywhere large compared to the wavelength of the light.

The phenomenon of the shadow, which seemed to pose difficulties for the elementary wave theory, is explainable only by the theory of diffraction. According to the latter, the border of a shadow is shown to be diffuse and composed of diffraction bands. The conflict between geometrical and wave optics is resolved by the theory of diffraction.

Geometrical optics is the limiting case of wave optics as $\lambda \rightarrow 0$. In this limiting case there is no diffraction. Hence, in contrast to ordinary refraction, rays of greater wavelength are diffracted more strongly than rays of shorter wavelength. Diffraction, then, generally deflects the red end of the spectrum more strongly away from the geometrical direction of the rays than the violet end, which is just the opposite of prismatic refraction. The *coronae* around the sun and the moon are *diffraction phenomena* which are caused by water droplets randomly distributed in a layer of haze and are especially strong when the droplets are of approximately uniform size. Their outer rims are colored red. The *halos* around the sun and the moon, on the other hand, are caused by *refraction* in the ice crystals of thin cirrus clouds. The sequence of colors, when visible, is the opposite in this phenomenon: red inside and violet outside. It is well known that Descartes had already explained the principal features of the *rainbow* by refraction and reflection in raindrops. The complete treatment of the rainbow, however, involves also a difficult diffraction problem.

Because of their low intensities and small dimensions, diffraction phenomena are, in daily life, generally not noticeable to the naked eye, but there are exceptions. If we view a distant light source through a fine fabric (an opened umbrella, for instance) we see the beautiful colored figures of a Fraunhofer cross grating. When we squint with nearly closed eyes at a distant candle, the eyelashes act like a (very distorted) line grating and decompose the candle light into its natural spectrum.

In Secs. 32 and 33 we shall treat those phenomena in which the difficulty of low intensity is overcome by the use of a large number of diffracting elements. Such devices are: the regular gratings in which the *amplitudes* of the oscillations add by interference, and random distributions of diffraction centers in which case the *intensities* add. In these devices the diffraction of single grating elements or particles plays as yet no essential role. This latter problem will be studied in various degrees of approximation in Sec. 34 ff. For the visual observation or photographic recording of this diffraction of single elements, a telescope or a lens is required.

While up to this point it has been sufficient to operate with a *plane* wave, the *spherical* wave will now come into its own. Only in the two following paragraphs, and later on in the treatment of Fraunhofer diffraction, shall we continue to use plane waves. The classical theory of diffraction, which is based upon *Huygens' principle*, operates essentially with the scalar spherical wave.

32. Theory of Gratings

A. LINE GRATINGS

The first diffraction gratings, made by Fraunhofer, consisted of parallel, stretched thin wires. Later, he used a glass plate covered with lampblack which he scored with a ruling engine in such a way that a system of equidistant transparent lines appeared on the glass. Fraunhofer's original gratings are preserved in the "Deutsches Museum" in Munich.

Famous, and hardly excelled even today, are *Rowland's reflection gratings*. They consist of up to 1800 ruled lines per millimeter on a metallic mirror surface; altogether some 100,000 lines. Faultless uniformity of the spacing¹ of the lines over the whole length of the grating is important.

We take the y -axis along the direction of the grating lines and assume that the lines are spaced at intervals of length d measured along the x -axis. Let the total number of lines be N . Let the plane of incidence be the xz -plane; and let $z = 0$ be the plane of the grating. Let the incident light be white and its rays be made parallel by an ideal collimator. The vectorial wave numbers k of the monochromatic components of the white light have the direction cosine α_0 with the positive x -axis. (α_0 is now not the cosine of the angle of incidence, but rather the cosine of the so-called "glancing angle" which is the complement of the angle of incidence.) We now assume a cylindrical

¹Periodically repeated irregularities of the ruling engine produce "ghosts", i. e. false lines in the diffraction spectrum.

wave sent out from each grating line. This process we call, as is commonly done, "diffraction", although we could instead use the more general term "scattering". The waves emerging from different grating lines are capable of interference because they originate from the same incident wave.

We now look at fig. 52 and recall the similar figs. 9 and 47 in which we were also concerned with the phase differences between neighboring rays. The ray emerging from O having direction cosine α with the x -axis covers a distance which exceeds that covered by the ray emerging from P by the path difference

$$OQ - RP = \alpha d - \alpha_0 d,$$

$$d = OP = \text{"grating constant"}.$$

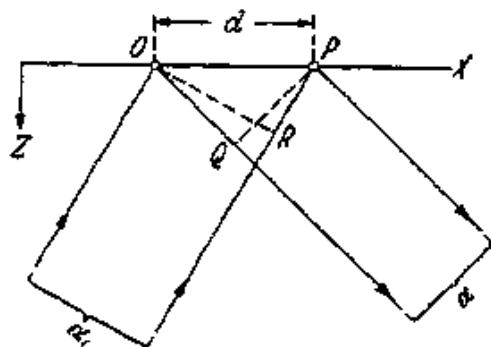


Fig. 52.
Determination of the phase difference in a reflection grating
 $OP = d = \text{grating constant}.$

The phase difference between these two rays is, therefore,

$$(1) \quad \Delta = kd(\alpha - \alpha_0), \quad k = \frac{2\pi}{\lambda}.$$

This must be an even multiple of π in order that, sufficiently far from the grating and in the direction indicated by the rays, these two waves shall show maximum reinforcement owing to interference. The condition for this is

$$(2) \quad \alpha - \alpha_0 = h \frac{\lambda}{d}, \quad h = \text{positive or negative integer}.$$

$h = 0$ corresponds to ordinary reflection $\alpha = \alpha_0$; $h = \pm 1$ corresponds to the first order spectrum on the right or left of the regularly reflected light; $h = \pm 2$ corresponds to the second order spectrum, etc.

If we collect the cylindrical waves emerging from all the different grating lines with the same phase difference, we obtain, at a distance large compared to the grating constant, a *plane* wave. In the case given by (2) this wave has its maximum amplitude; its amplitude is zero when Δ is an odd multiple of π . For a given λ the dependence of the amplitude of this wave upon Δ is shown by the intensity curve in fig. 53 which will be calculated below. For its observation one employs, following Fraunhofer, a telescope focused at infinity.

That the grating actually does produce spectra in this way, i. e. that it separates the colors, follows from the fact that, according to (2), $\alpha - \alpha_0$ depends on the wavelength. Hence, the different colors are diffracted in different directions. Since $\alpha - \alpha_0$ increases with λ , red is diffracted more strongly than violet, as mentioned in the introduction to this chapter. The dispersion, that

is, the separation of the various colors, is directly proportional to λ . Thus, the grating yields a spectrum of the incident white light, whose scale of wavelengths is normal and quantitatively correct. Furthermore, the dispersion is directly proportional to the order number h . In the second order spectrum the dispersion is twice that in the first order spectrum. For this reason the second or third order spectra are preferred for exact wavelength measurements. Finally, according to (2), the dispersion is inversely proportional to d ; hence, the close spacing of the lines of the Rowland gratings. An exception is the order $h = 0$, for which $\alpha - \alpha_0$ is independent of color. The spectrum of zero order produces white light.

There is a critical limiting value, not necessarily integer, $h = h_c$, which corresponds to the value $\alpha = 1$. If h_c happens to be integer the diffracted light ray is parallel to the plane of the grating, just like the reflected wave in the limiting case of total reflection¹. In any case, even for $h > h_c$ a diffracted wave runs parallel to the plane of the grating, however, not as a regular wave, but as an inhomogeneous wave. This is also analogous to total reflection.

We now wish to show that the grating produces practically pure spectral colors. In order to do this, we must estimate the width of the maximum computed in (2). We do this by considering monochromatic light with a given λ instead of the white light employed until now.

We define the radiation emitted by an arbitrary grating groove in the direction α by means of an amplitude factor $f(\alpha)$, which is the same for all grooves and can be considered as a function which varies slowly as α varies between the limits $\alpha = \pm 1$ (positive and negative x -axis). Consecutive grating grooves shall have the abscissas

$$x_0, \dots, x_n, \dots, x_{N-1}, \quad x_n = x_0 + n d.$$

Omitting the time factor, we write for the oscillations emitted by the n^{th} groove in the direction α the expression

$$(3) \quad u_n = f(\alpha) \exp \{i k (\alpha x + \gamma z) + i n \Delta\}$$

regardless of whether we are concerned with the vectors \mathbf{E} , \mathbf{D} , or \mathbf{H} . For the significance of $f(\alpha)$ and a systematic development of (3), we refer the reader to Sec. 36. Δ is the phase difference defined in (1). Superposition of the effects of all the lines gives

$$(4) \quad u = \sum u_n = f(\alpha) S \exp \{i k (\alpha x + \gamma z)\}$$

¹Lord Rayleigh, *Phil. Mag.* 14, 60, 1907 and *Proc. Roy. Soc.* 79, 399, 1907; W. Voigt, *Göttinger Nachr.* 40 (1911); also U. Fano, *Ann. d. Phys.* 82, 393, 1938, *Phys. Rev.* 80, 921, 1948.

with

$$(4a) \quad S = \sum_{n=0}^{N-1} e^{in\Delta} = \frac{1-e^{iN\Delta}}{1-e^{i\Delta}} = \frac{e^{iN\frac{\Delta}{2}} \sin N\frac{\Delta}{2}}{e^{i\frac{\Delta}{2}} \sin \frac{\Delta}{2}}.$$

Changing from the amplitude to the intensity, we get

$$(5) \quad J = |u|^2 = f^2(\alpha) |S|^2 = f^2(\alpha) \frac{\sin^2 N\frac{\Delta}{2}}{\sin^2 \frac{\Delta}{2}}.$$

This expression is composed of two parts. The first factor corresponds to the intensity arising from a single groove and varies slowly with α . The second factor arises from the sequence of many grooves, and is a very rapidly varying function of $\alpha - \alpha_0$.

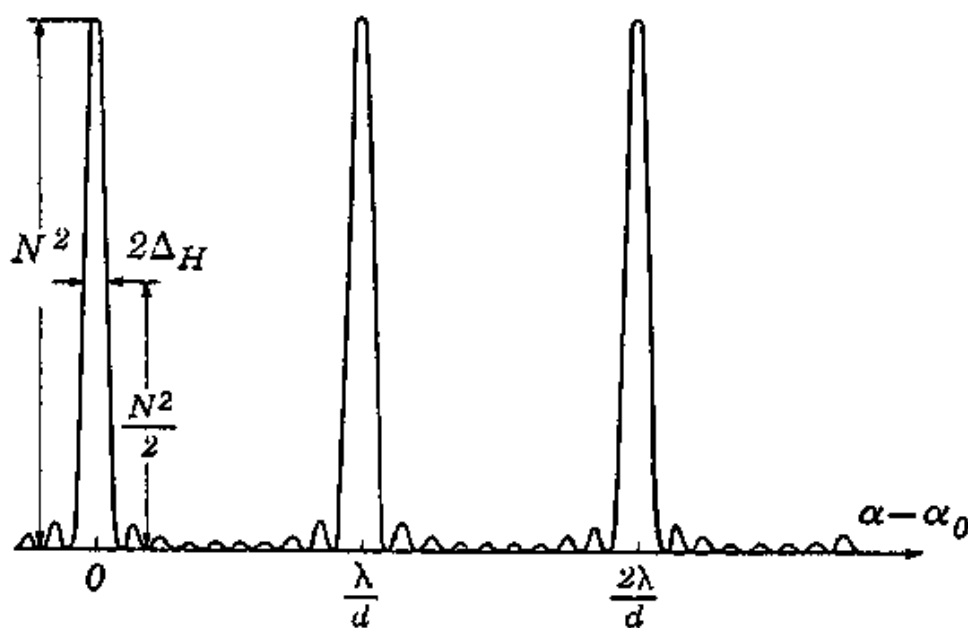


Fig. 53.

The function $\frac{\sin^2 N\Delta/2}{\sin^2 \Delta/2}$ plotted against $\alpha - \alpha_0 = \frac{\lambda \Delta}{2\pi d}$.

The incident primary intensity is contained in the first factor of (5). Figure 53 shows the dependence of the second factor on $\alpha - \alpha_0$. Its *principal maxima* lie, in accordance with (1) and (2), at

$$(6) \quad \frac{\Delta}{2} = \pi h.$$

These maxima are given by (5) as $0/0$. The value in the limit, as computed in the well-known manner (de l'Hôpital's rule), is the same for all h , namely N^2 .

In addition, there are *subsidiary maxima* corresponding to the rapid fluctuations of the numerator. Since the denominator varies slowly, the positions of these maxima are given sufficiently accurately by the maxima of the numerator. Thus, in the vicinity of the h^{th} principal maximum, the positions of the subsidiary maxima are given by

$$(6a) \quad \frac{\Delta}{2} = \pi \left(h + \frac{\nu}{2N} \right), \quad \nu = (1), 3, 5.$$

The value $\nu = 1$ is put in parentheses because the corresponding peak is masked by the flank of the principal maximum. The heights of the subsidiary maxima are

$$\frac{1}{\sin^2 \frac{\nu \pi}{2N}} \sim \frac{4N^2}{\pi^2 \nu^2}.$$

Hence, the first subsidiary maximum to be taken into account is $4/(9\pi^2) \approx 1/22$ times as small as the principal maximum, whose height was N^2 . The second subsidiary maximum is $4/(25\pi^2) \approx 1/62$ times as small, etc. These maxima follow one another at the very small interval of $\lambda/(Nd)$; between them the intensity always drops to zero.

We shall now compute $2\Delta_H$, the *width of the principal maximum at half intensity*, as illustrated on the left of fig. 53. This width is determined by the equation

$$(6b) \quad \frac{\sin^2 N \frac{\Delta_H}{2}}{\sin^2 \frac{\Delta_H}{2}} = \frac{N^2}{2}$$

where the right-hand side corresponds to half the maximum intensity. Since Δ_H must certainly be very small, we may replace, in the denominator on the left-hand side, the sine function by its argument. Thereby, we get

$$(6c) \quad \sin^2 x = \frac{x^2}{2}, \quad \text{where} \quad x = N \frac{\Delta_H}{2}.$$

The solution of the equation $\sin x = x/\sqrt{2}$ is found from the table of sines to be

$$x \sim 80^\circ = 1.38, \quad \text{hence} \quad \Delta_H = \frac{2 \times 1.38}{N}.$$

Since N is a very large number the width at half intensity which is twice this Δ_H , namely $5.5/N$, is extremely small. It follows from this that the principal maxima belonging to distinct colors of our spectrum fall next to one another, so that no appreciable overlapping of colors occurs.

That does not, of course, exclude the possibility that the ends of spectra of *different* orders may overlap. Since the dispersion increases with h , mixed colors can be produced in this way. In fact, one sees immediately that the red end of the second order spectrum, for instance, will overlap the violet end of the third order spectrum since

$$2 \lambda_{\text{red}} > 3 \lambda_{\text{violet}} \quad \text{because} \quad \lambda_{\text{red}} \sim 2 \lambda_{\text{violet}}.$$

Finally, we must multiply the second part of the intensity expression (5), as illustrated in fig. 53, by the factor f^2 . In general this factor will decrease steadily with increasing $|\alpha|$ and will, therefore, attenuate the spectra with larger values of h in comparison with those which have smaller h values. However, this is true only "in general". In each particular case the form of f depends entirely on the shape of the groove (i. e. on the shape of the diamond of the ruling machine). Nor need f be an even function of α . The spectra with $h > 0$ can, for instance, be enhanced over those with $h < 0$. It can even happen that most of the intensity is thrown into a single spectrum. Under certain circumstances this may be especially desirable. More details are given in Sec. 36 D.

B. CROSS GRATINGS

Two systems of grating lines which intersect at right (or oblique) angles are called a cross grating. We have, then, on the plane of the grating a set of dark rectangles (or parallelograms) extending in two directions. Or we could, instead, consider a two-dimensional system of bright rectangles (as in the above-mentioned example of the opened umbrella) or of arbitrarily shaped bright spots (circles for instance) on a dark background. These also will be called cross gratings.

As in the case of the line grating, let the grating plane be the xy -plane. For convenience we will consider our cross grating to be oriented along the x - and y -axes, that is, to be rectangular. Here we must replace the summation over n in eq. (4) by a double sum over n_1 and n_2 :

$$(7) \quad S = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \exp \{i n_1 \Delta_1 + i n_2 \Delta_2\},$$

$$\Delta_1 = 2\pi d_1 \frac{(\alpha - \alpha_0)}{\lambda},$$

$$\Delta_2 = 2\pi d_2 \frac{(\beta - \beta_0)}{\lambda}.$$

Performing the summation and computing the intensity, we obtain instead of (5)

$$(8) \quad J = f^2(\alpha, \beta) \frac{\sin^2 N_1 \frac{\Delta_1}{2} \sin^2 N_2 \frac{\Delta_2}{2}}{\sin^2 \frac{\Delta_1}{2} \sin^2 \frac{\Delta_2}{2}}.$$

According to eqs. (1) and (2), principal maxima will now occur if

$$(8a) \quad \Delta_1 = 2\pi h_1 \text{ and simultaneously } \Delta_2 = 2\pi h_2$$

where h_1 and h_2 are arbitrary positive or negative integers. The directions α, β of the deflected rays corresponding to these principal maxima are given, according to (7), by

$$(9) \quad \alpha - \alpha_0 = h_1 \frac{\lambda}{d_1}, \quad \beta - \beta_0 = h_2 \frac{\lambda}{d_2}.$$

The corresponding intensity is proportional to $N_1^2 N_2^2$. If only one of the two conditions (8a) is fulfilled, then the intensity is only proportional to N_1^2 , or N_2^2 , and is, therefore, imperceptibly small compared to the intensity of the principal maxima. Also the subsidiary maxima, described by (6), can be disregarded when compared to the principal maxima. Since, according to (9), a definite pair of values α, β is associated with every λ , each of the cases characterized by (9) constitutes a complete color spectrum. The spectra extend parallel to the direction of the x -axis when $h_2 = 0$, and in the direction of the y -axis when $h_1 = 0$. In the general case $h_1 \neq 0$ and $h_2 \neq 0$, the spectra are directed radially, i. e. towards the central point α_0, β_0 . Only at this latter point is the light not spectrally decomposed but white. Again, owing to the factor $f^2(\alpha, \beta)$ in (8) the outer spectra of this manifold and colorful display are generally strongly attenuated.

C. SPACE GRATINGS

We ask next how a three-dimensional grating could be produced. Neither a ruling machine nor stacked layers of the sheerest fabric produce an optically useful grating. Max von Laue had the ingenious idea that nature herself offers us an ideal space grating in the form of a flawless, non-absorbing crystal. Though useless in the field of optics, such crystals find application in the much more interesting spectral range of X-rays. In this connection it is to be noted that this range was not even known in 1912 but was determined quantitatively only by means of Laue's discovery. For optical purposes the mesh of a crystal lattice is far too fine, but for the analysis of X-rays its order of magnitude is just right. In fact, the spacing

between atoms in a crystal is approximately the same as the wavelength of soft X-rays (several Å units, $1 \text{ Å} = 10^{-8} \text{ cm}$), just as the spacing of the lines in the Rowland grating agrees approximately with the wavelength of red light ($1/2 \mu$, $1 \mu = 10^{-4} \text{ cm}$).

In order to make the formulae as clear as possible, we shall restrict ourselves here to the special case of an orthorhombic crystal, but it is to be emphasized that if an oblique coordinate system is used, the general triclinic crystal presents no difficulties. Let the sides of the fundamental orthorhombic cell have lengths a , b and c (this would correspond to d_1 , d_2 and d_3 in our previous notation). Rewriting eq. (9) in three dimensions, we obtain immediately *Laue's Fundamental Equations*:

$$(10) \quad \alpha - \alpha_0 = h_1 \frac{\lambda}{a}, \quad \beta - \beta_0 = h_2 \frac{\lambda}{b}, \quad \gamma - \gamma_0 = h_3 \frac{\lambda}{c}.$$

The special cases of the tetragonal and cubic systems ($b = a$, $c = b = a$), are, of course, contained in (10).

In Laue's experimental arrangement the rays are made to pass through thin crystal slabs. In particular, for the first photographs by Friedrich and Knipping, made in the spring of 1912, thin slabs of zinc blende, ZnS , sliced perpendicularly to the fourfold or threefold axis of symmetry, were used. The crystal acts here not as a reflection grating, but as a transmission grating. The rays emerging from the crystal produce the surprisingly beautiful "Laue Diagrams" on a photographic plate placed beyond the crystal. The original pictures are preserved in the "Deutsches Museum" in Munich and have been reproduced in countless textbooks.

The intensity is computed by the properly extended formula (8). The number N of the contributing lattice elements is determined by the thickness of the crystal slab and by the cross-section of the incident X-ray beam. The "atomic form factor", much discussed in the theory of crystal analysis, takes the place of $f(\alpha, \beta)$ in eq. (8).

The difference between this theory and the theory of cross gratings arises from the fact that, because of the condition $\alpha^2 + \beta^2 + \gamma^2 = 1$, the three eqs. (10) are not compatible for any arbitrarily given values of λ . While the cross grating produces complete spectra containing all λ , the space grating is *selective*. To every Laue spot corresponds a characteristic λ . (However, for reasons of symmetry several spots may correspond to the same λ ; as, for instance, in the fourfold symmetrical picture of zinc blende in which each λ occurs, in general, in eight spots.) The polychromatic character of the cross grating spectra occurs again in the Laue diagram in the sense that every Laue spot selects its own special "color" from the incident "X-ray light".

We confirm this analytically by squaring and adding the values of α , β , γ as determined by (10), and by taking into account the condition $\alpha_0^2 + \beta_0^2 + \gamma_0^2 = 1$. Cancelling a common factor λ , we thus obtain:

$$(11) \quad \lambda = -2 \frac{\alpha_0 \frac{h_1}{a} + \beta_0 \frac{h_2}{b} + \gamma_0 \frac{h_3}{c}}{\frac{h_1^2}{a^2} + \frac{h_2^2}{b^2} + \frac{h_3^2}{c^2}}.$$

Therefore, once the order numbers h_1 , h_2 , h_3 of the interference are determined for every Laue spot, the wavelength producing each such spot is (for a known crystal lattice) also determined. At the same time we now see that in contrast to Bragg's method, Laue's arrangement uses the continuous X-ray spectrum (the so-called "white X-ray light" or "Bremsstrahlung").

Before discussing Bragg's experiment, we draw yet another conclusion from eq. (10). We form the sum of the squares of the left-hand sides of (10)

$$(12) \quad (\alpha - \alpha_0)^2 + (\beta - \beta_0)^2 + (\gamma - \gamma_0)^2 = 1 - 2(\alpha\alpha_0 + \beta\beta_0 + \gamma\gamma_0) + 1 \\ = 2 - 2 \cos 2\vartheta = 4 \sin^2 \vartheta.$$

2ϑ is here the angle between the incident ray $\alpha_0, \beta_0, \gamma_0$ and the diffracted ray α, β, γ (see fig. 54). The plane E bisects the angle 2ϑ between these two rays. Next, we compute the sum of the squares of the right-hand sides of (10), namely,

$$(13) \quad \lambda^2 \left\{ \left(\frac{h_1}{a} \right)^2 + \left(\frac{h_2}{b} \right)^2 + \left(\frac{h_3}{c} \right)^2 \right\} = \frac{\lambda^2}{D^2}$$

where D is a length of the order of magnitude of the sides a, b, c . We can make the definition of D , which is implicit in (13), more precise if we rid the integers h of a possible common factor and write

$$(14) \quad h_1 = n h_1^*, \quad h_2 = n h_2^*, \quad h_3 = n h_3^*,$$

$$(14 a) \quad D = \frac{d}{n}, \quad d = \left\{ \left(\frac{h_1^*}{a} \right)^2 + \left(\frac{h_2^*}{b} \right)^2 + \left(\frac{h_3^*}{c} \right)^2 \right\}^{-\frac{1}{2}}.$$

By equating (12) and (13) and applying (14) and (14 a), there results *Bragg's equation*:

$$(15) \quad 2 d \sin \vartheta = n \lambda.$$

We have encountered this equation before as eq. (8.6) in connection with Wiener's standing light waves. There the length d denoted the distance between two neighboring layers of the "screen" which is produced by standing light waves and is utilized in Lippmann's color photography. We must now investigate the significance of d in the case of the crystal lattice.

For this purpose we construct the equation of the plane E in fig. 54. We use a system of rectangular coordinates x, y, z whose axes are parallel to the crystal axes a, b, c and whose origin lies in the plane E at the lattice point O . This is the same point which in fig. 54 was considered as the point of origin of the diffracted ray. From O we mark off the segments OP along the extension of the incoming ray direction and OQ along the direction of the diffracted ray and we let

$$OP = OQ = 1.$$

The coordinates of the points P and Q are then

$$\alpha_0, \beta_0, \gamma_0 \quad \text{and} \quad \alpha, \beta, \gamma.$$

respectively. We can now define the plane E as the locus of all points equidistant from P and Q :

$$\begin{aligned} (x - \alpha_0)^2 + (y - \beta_0)^2 + (z - \gamma_0)^2 &= \\ &= (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2. \end{aligned}$$

This simplifies to

$$(\alpha - \alpha_0)x + (\beta - \beta_0)y + (\gamma - \gamma_0)z = 0.$$

Substituting (10) and (14) into this yields

$$(16) \quad \frac{h_1^*}{a}x + \frac{h_2^*}{b}y + \frac{h_3^*}{c}z = 0.$$

This plane E is a *lattice plane* of the crystal, which means that in an unbounded crystal the plane E contains an infinite number of lattice points. (If the plane contains three lattice points, then because of the periodicity of the lattice it must contain an infinite number of lattice points.) The numbers h^* are called the *Miller indices* of the lattice plane. (Their magnitudes determine the density of lattice points on the plane; small values of the h^* imply a large density; large values of the h^* imply a small density. Only planes with small h^* values occur as natural boundary planes of a crystal.) A plane parallel to our lattice plane intersects the crystal axes a, b, c at coordinates which have the ratios

$$(17) \quad \frac{a}{h_1^*} : \frac{b}{h_2^*} : \frac{c}{h_3^*}.$$

This was the original definition of the Miller indices as used in macroscopic crystallography where one did not speak of lattice planes but only of the natural crystal faces. The a, b, c were then defined only as relative lengths (for instance, setting $b = 1$). Since we are here dealing with the microscopic theory of structure, we can introduce the lengths of the edges of the orthorhombic unit cell as absolute values of a, b, c . Therefore the quantities given

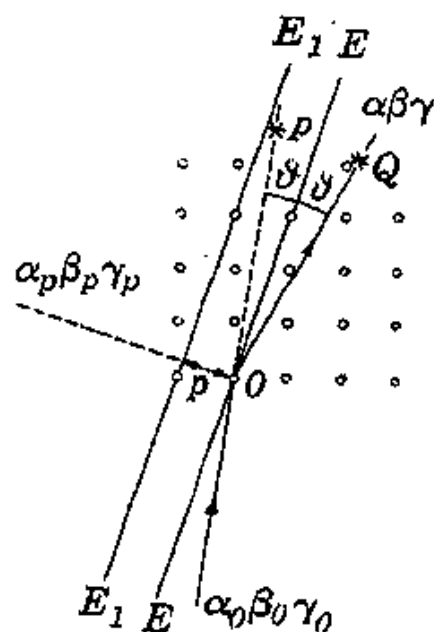


Fig. 54.

Diffraction of X-rays in a space lattice (description in the text).

in (17) also become the absolute lengths of the axis segments, and the lattice plane E which is parallel and nearest to (16) has the equation

$$(18) \quad \frac{h_1^*}{a} x + \frac{h_2^*}{b} y + \frac{h_3^*}{c} z = 1.$$

If we replace the 1 in (18) by any integer n , we obtain another lattice plane which is parallel to E . This will be the n^{th} lattice plane E_n which intersects the axes at the coordinates na/h_1^* , nb/h_2^* , nc/h_3^* . Non-integer values of n do not give lattice planes; they are in contradiction to the periodicity of the crystal.

In its so-called normal form eq. (18) reads

$$(19) \quad \cos \alpha_p x + \cos \beta_p y + \cos \gamma_p z = p,$$

$$\cos \alpha_p = \frac{h_1^* p}{a}, \quad \cos \beta_p = \frac{h_2^* p}{b}, \quad \cos \gamma_p = \frac{h_3^* p}{c},$$

$$p = \left\{ \left(\frac{h_1^*}{a} \right)^2 + \left(\frac{h_2^*}{b} \right)^2 + \left(\frac{h_3^*}{c} \right)^2 \right\}^{-\frac{1}{2}}.$$

As we know, p denotes the length of the perpendicular from O to E_1 or, as we may also say, it equals the distance between the planes E and E_1 . The $\alpha_p, \beta_p, \gamma_p$ are the direction cosines of p (see fig. 54). But according to (19), p

is identical with the length d which was introduced in (14 a). Hence our former quantity d is the spacing of that system of parallel lattice planes whose Miller indices are equal to our interference numbers h (divided by any common divisor which they may contain).

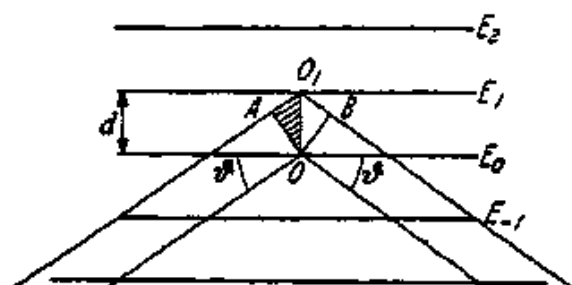


Fig. 55.

Direct derivation of Bragg's equation.
Reflection at the lattice planes E .

Bragg's eq. (15) may be visualized as "reflection on the lattice planes", and not merely on a single such plane but on the entire system of parallel

lattice planes. This is seen immediately and independently from Laue's theory by means of fig. 55. In order that the wave reflected at E_0 be reinforced by the wave reflected at E_1 , that is, in order that the amplitude be doubled, the path difference of the two rays must be a multiple of λ . This path difference is $A O_1 + O_1 B$. From the shaded triangle with the hypotenuse $O O_1 = d$ in the figure it follows that

$$A O_1 = d \sin \theta = O_1 B.$$

Hence, the above condition becomes, in agreement with (15),

$$2 d \sin \theta = n \lambda.$$

This same condition also guarantees the enhancement of the reflections from the lattice planes $E_2, E_3, \dots, E_{-1}, E_{-2}$ and leads to Laue's amplification factor N^2 for the intensities.

This derivation shows that in the present case, only the regularity of the sequence of lattice planes E is important and not the periodic arrangement of the crystal atoms within the planes E . Even if these atoms were distributed completely at random, as are the silver grains in the Wiener layers, the interference effect would not be disturbed. In that case we would have, so to say, a one-dimensional crystal. The three-dimensional crystal lattice differs from it by producing simultaneous interference on many sets of parallel lattice planes.

Immediately after Laue, William Bragg and his son, Lawrence Bragg, (successor of Rutherford at the Cavendish Laboratory) were the first to determine most of the now known simpler crystal structures (rock salt, diamond, fluorite, pyrite, etc.). Later they also determined the structures of some highly complicated organic and inorganic crystals (see, for instance, the remarks on mica on p. 167). They observed the "glancing angles" ϑ for various crystal faces and from them determined the lattice plane spacings d by means of eq. (15). Instead of a continuous X-ray spectrum they used for this purpose a known characteristic line such as the Cu K_α -line $\lambda = 1.537 \text{ \AA}$. In addition to the directions of the reflected rays, their intensities were of essential importance in these structure determinations. Particularly significant were observations on the extinction of even or odd orders of possible reflections.

Debye gave a general explanation of the extent to which thermal motion in crystals influences the intensities of reflections. C. G. Darwin studied the effect of disorientations found in most crystals (the so-called mosaic structure of crystals). P. P. Ewald in his "dynamic theory" of X-ray interference accomplished a profound extension of Laue's original theory. This theory takes into account the attenuation of the primary radiation during its passage through the lattice and the mutual radiation passing from lattice point to lattice point. The valuable notion of the "reciprocal lattice" is also due to Ewald.

33. Diffraction Arising from Many Randomly Distributed Particles

We shall here consider a glass plate which is covered with condensed fog droplets or has been dusted with lycopodium powder. The light source shall be as small as possible and very distant, and we shall observe this source through the glass plate with the eye focused at infinity. We shall assume that the droplets or spore grains are of uniform size and circular in shape. By means of a filter we select a small spectral range of wave number k from the source.

The glass plate shall lie in the xy -plane whose origin $x = 0$, $y = 0$ will be taken to lie on the straight line connecting the eye with the source. The centers of the small circular diffracting discs will be described by the coordinates x_n , y_n . We write the formula for the radiation reaching the eye from each particle and having the direction of propagation α , β , γ , in the form

$$(1) \quad u_n = f(\alpha, \beta, \gamma) \exp \{i k [\alpha (x - x_n) + \beta (y - y_n) + \gamma z]\}.$$

The factor $f(\alpha, \beta, \gamma)$ will be further explained in Sec. 36. As in (32.4) we find for the total amplitude

$$(2) \quad \left| \sum u_n \right| = f(\alpha, \beta, \gamma) S,$$

$$(3) \quad S = \left| \sum_{n=1}^N \exp [-i k (\alpha x_n + \beta y_n)] \right|,$$

where N is the total number of particles. We have here put the factors containing x , y , and z in front of the summation sign and have taken into account the fact that they disappear when the absolute value is taken.

Because the particle coordinates x_n , y_n are unknown, the summation cannot, of course, be performed algebraically as was done in Sec. 32. Therefore, we must apply a statistical procedure. The value of k in eq. (3) is given; α , β are arbitrary but are to be chosen and then held fixed; the x_n , y_n in the summation assume completely random values. Equation (3) tells us to add N unit vectors of random directions in the complex plane and to determine the length of the resultant vector. A theorem in the theory of probability says: if all directions have *equal probabilities*, the length of the resultant vector is \sqrt{N} . This theorem is used, for instance, in the theory of Brownian motion where, as in our diffraction problem one is interested, in the addition of a large number of, on the average, equal impulses which the observed colloidal particles receive by collisions with the molecules of the surrounding fluid.

To prove this theorem we set the exponents in (3) (reduced modulo 2π) equal to $i\varphi_n$ and obtain

$$S = \left| \sum_{n=1}^N e^{i\varphi_n} \right|, \quad S^2 = \sum_n e^{i\varphi_n} \sum_m e^{-i\varphi_m}.$$

We now find the statistical mean value \bar{S} of S , which we define as the square root of the average value of S^2 :

$$(4) \quad \bar{S} = \sqrt{\bar{S}^2}, \quad \bar{S}^2 = \frac{1}{2\pi} \int_0^{2\pi} d\varphi_1 \frac{1}{2\pi} \int_0^{2\pi} d\varphi_2 \dots \frac{1}{2\pi} \int_0^{2\pi} d\varphi_N S^2.$$

Thus we average S with respect to each φ_n over its whole range of values from 0 to 2π . The assumption of equal probabilities of all angles for each φ_n and the mutual independence of all φ_n which is implied in (4) is motivated by our complete lack of knowledge of the values of x_n and y_n in eq. (3) which defines the φ_n .

First we compute only the integral with respect to φ_1 occurring in (4):

$$(4a) \quad \frac{1}{2\pi} \int_0^{2\pi} d\varphi_1 \left(e^{i\varphi_1} + \sum_{n=2}^N e^{i\varphi_n} \right) \left(e^{-i\varphi_1} + \sum_{m=2}^N e^{-i\varphi_m} \right).$$

The product of the two parentheses in this integral gives

$$1 + \dots + \dots + S_1^2, \quad S_1^2 = \sum_{n=2}^N e^{i\varphi_n} \sum_{m=2}^N e^{-i\varphi_m}.$$

The two middle terms which have not been written down contain the factors $\exp(i\varphi_1)$ and $\exp(-i\varphi_1)$ and therefore vanish on integrating over φ_1 . The two other terms are independent of φ_1 and therefore integrate to $1 + S_1^2$.

Next, we calculate

$$(4b) \quad \frac{1}{2\pi} \int d\varphi_2 (1 + S_1^2) = 1 + 1 + S_2^2, \quad S_2^2 = \sum_{n=3}^N e^{i\varphi_n} \sum_{m=3}^N e^{-i\varphi_m}.$$

Continuation of this procedure leads to

$$(5) \quad \overline{S^2} = 1 + 1 + 1 + \dots = N, \quad \overline{S} = \sqrt{N}.$$

This proves our probability theorem. As was to be expected from the symmetry of the arrangement, \overline{S} is independent of α, β .

Returning now to eq. (2), we find for the intensity of the diffraction pattern

$$(6) \quad J = N J_0, \quad J_0 = f^2(\alpha, \beta, \gamma).$$

J_0 is the intensity arising from one individual diffracting disc. *For random distributions of the diffracting elements, the intensities add and not the amplitudes as was the case in the grating theory.* Instead of the amplification factor N^2 of fig. 53, eq. (6) contains the factor N .

In the case of our circular diffracting discs J_0 does not, of course, depend on α, β and γ individually but only on the radial angular distance $s = (\alpha^2 + \beta^2)^{1/2} = (1 - \gamma^2)^{1/2}$. As we shall see in Sec. 36, J_0 has a flat maximum in the center of the diffraction pattern and vanishes for the first time at a

sharply defined position $s = s_1$. A considerably weaker maximum follows and then a less sharp zero, etc. According to Sec. 36, s_1 is given by

$$(7) \quad s_1 = 0.61 \frac{\lambda}{a}$$

where a is the radius of the diffracting discs. Hence, the radial extension of the diffraction pattern increases as the ratio a over λ decreases.

If we use *white* light instead of monochromatic light, then the center of the diffraction pattern appears white because all colors have a maximum there. The outer rim of this center disc is colored red because at the distance

$$(7a) \quad s_1 = 0.61 \cdot \frac{\lambda_{\text{blue}}}{a}$$

the blue component of the light is missing. At approximately twice this distance we expect a bluish tint because there the red component is extinguished. As we proceed outward from the center, the coloration and the intensity become progressively weaker. If the diffracting particles are *not circular* in shape, then of course the intensity J_0 depends on both α and β . However, the intensity J resulting from all N particles retains its circular symmetry so long as not only the positions but also the orientations of the particles are random, for then a summation over all possible orientations must be added to the summation over all positions in the expression (3) for S .

If the particles are *not of uniform size* but are, for example, water droplets of various radii, then according to (7) the rings of zero amplitude become diffuse under monochromatic illumination. Under white light illumination the color pattern becomes less distinct, but the white coloration of the center portion of the pattern remains. The size of this white disc can be estimated from eq. (7a) if a is replaced by an average radius \bar{a} .

Our statement about the statistical average value of S is only approximately valid. Under monochromatic illumination with a carefully limited source the diffraction patterns exhibit a certain "granulation", namely a *radial fiber structure*. This is due to fluctuations about the statistical average which are stronger in the radial direction than in the azimuthal direction perpendicular to it. These fluctuations have been investigated in detail by M. von Laue¹ both experimentally and theoretically.

We now turn to the *meteorological* applications of this theory. Because of the sizes of the light sources involved (sun or moon), and because of the white nature of the light, the above-mentioned fluctuations obviously do not enter into consideration. The real *coronae* about the sun and the moon are due to

¹Preußische Akademie 1914, p. 1144.

diffraction at cloud particles, hence primarily at water droplets. Because of the different sizes of the droplets, the color phenomena are rather weak. The sun and the moon are surrounded by a white or bluish-white field. As was remarked in the introduction, a frequently observed reddish rim indicates the diffractive nature of this phenomenon. From the angular radius of this rim, which differs for different observations, the average droplet diameter $2\bar{a}$ is found by means of (7 a) to vary from 0.01 to 0.03 mm. After the eruption of Krakatoa, a much larger red-brown sun ring was observed. This was due to the volcano's dust particles which had drifted as far as Europe. The angular radius of this ring amounted to 20° to 25° , which corresponds to the much smaller particle diameter of 0.002 mm.

The situation is different with the ice crystals of cirrus clouds. It can be assumed that they, too, contribute to the diffraction phenomenon of coronae, but the characteristic phenomena resulting from these crystals are the *halos* which are not due to diffraction but are caused by *refraction*. This fact is proved by the color arrangement in halos: violet outside, red inside. Moreover, the halos have definite radii which do not depend on the (varying) sizes of the particles but rather on the crystalline structure of these particles. The most frequently occurring angular radius is 22° , which corresponds to refraction of the light in the hexagonal cylinders of ice crystals (edge angle 60°). If because of gravity the ice crystals are oriented mostly vertically, then the light of the halo concentrates at the two points on its circumference which are at the same elevation as the sun. This is the origin of the two *parhelia*. In addition, a halo with about 45° angular radius occurs.

34. Huygens' Principle

Huygens' principle may be visualized as follows: the future shape of any given wave surface can be determined by assuming that each point of this surface emits a *spherical wave* and by constructing the *envelope* of all these spherical waves. In a homogeneous medium this construction yields a surface which is parallel to the original wave surface (possible boundaries of the original surface form an exception). We have already seen in fig. 37 that this procedure leads to the usual refraction at plane interfaces. The usual reflection is also obtained in this way.

Kirchhoff proved that Huygens' principle is an *exact* consequence of the differential equations of optics. This principle constitutes the foundation of the classical theory of diffraction, which has proved its fruitfulness in countless problems. Nevertheless, this theory is only an *approximation which is valid only for sufficiently small wavelengths*. This is so because the *boundary condi-*

tions which must be used in conjunction with Huygens' principle are not known precisely. The classical theory, moreover, does not take the vectorial character of the optical field into account. This deficiency will not be discussed until Sec. 38 et seq.

A. THE SPHERICAL WAVE

We are familiar with the *scalar* spherical wave of acoustics from Vol. II, Sec. 13. Like the plane wave representation, that of the spherical wave is a solution of the wave equation $\Delta u + k^2 u = 0$. If it is assumed that u depends only on the coordinate x , one obtains (disregarding a constant complex factor)

$$(1) \quad u = e^{\pm ikx}.$$

If, on the other hand, it is assumed that u is a function only of the distance r from the origin of the coordinate system, then one finds

$$\Delta u = \frac{1}{r} \frac{d^2(ru)}{dr^2}, \quad \frac{d^2(ru)}{dr^2} + k^2 ru = 0, \quad ru = e^{\pm ikr}$$

and hence

$$(2) \quad u = \frac{1}{r} e^{\pm ikr},$$

where the time dependence is assumed to be of the form $\exp(-i\omega t)$. (With a time factor of $\exp(+i\omega t)$ one would obtain an incoming rather than an outgoing wave.)

The *vectorial* spherical wave of electrodynamics is not so simple. The expression for this wave assumes its most convenient form if the *Hertz vector* is introduced as the characteristic function u . In particular, the special case of the Hertz vector which represents the radiation emitted by a linearly oscillating dipole, see Vol. III, Sec. 19 B, is useful here. Though the analytic expression of this vector is again given by (2), that is, by a spherically symmetric expression, this is not true of the field which is derived from the Hertz vector. The magnetic field lines are circles about the direction of oscillation, while the electric field lines are in the meridian planes of that direction. Only the phase of the field is spherically symmetric. Its amplitude depends on direction. The electric amplitude, for instance, vanishes in the direction of oscillation of the dipole at distances large compared to the wavelength.

A physical light source (point-like bulb or candle) contains all possible directions of oscillation. Such a source emits an average field in which no directions are preferred. The field intensity is therefore spherically symmetric. If we represent such a field by (2), we must realize that we thereby relinquish the possibility of representing the finer details of the light, such as its polarization.

B. GREEN'S THEOREM AND KIRCHHOFF'S FORMULATION OF HUYGENS' PRINCIPLE

Green's theorem suffices for the integration of the scalar wave equation; the reader is referred to the first introduction of this theorem in Vol. II, eq. (3.15) and to its repeated use in Vols. III and VI:

$$(3) \quad \int (u \Delta v - v \Delta u) d\tau = \int \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\sigma.$$

Let u be the function (2) of the spherical wave and let v be the desired solution of the equation $\Delta v + k^2 v = 0$. The surface σ separates space into two regions. One of these regions will be called the interior of σ and the other region its exterior. If, as will usually be the case, σ extends to infinity, then the point at infinity belongs both to the interior and to the exterior. For the region of integration on the left-hand side of (3) we choose the exterior of σ . The source of the wave u is assumed to lie at the point P in the exterior and is to be excluded from the region of integration; this may be accomplished by means of a sphere K of arbitrarily small radius; see fig. 56. Then, because of the differential equation which u and v satisfy, the left-hand side of (3) vanishes. The integral on the right-hand side must be evaluated over the two boundary surfaces σ and K ¹; dn is the normal to these surfaces directed into the interior. The same considerations as in Vol. II, Sec. 20, 1 a yield for the integral over K the value $-4\pi v_P$, where v_P is the value of v at the center of K . From eq. (3) it follows, therefore, that

$$(4) \quad 4\pi v_P = \int_{\sigma} \left(\frac{\partial v}{\partial n} \frac{e^{ikr}}{r} - v \frac{\partial}{\partial n} \frac{e^{ikr}}{r} \right) d\sigma.$$

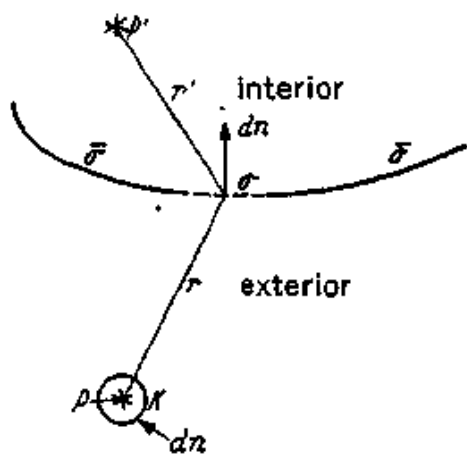


Fig. 56.

Regions of integration for Green's theorem. The surfaces σ and K together form a closed surface.

¹Actually a third boundary surface should be added, namely a sphere of very large radius with its center at P which excludes the point at infinity. Denoting the surface element by $d\sigma = r^2 d\omega$ and combining the r^2 with the integrand, the integral over this surface becomes

$$\int \left\{ r \left(\frac{\partial v}{\partial n} - i k v \right) + v \right\} e^{ikr} d\omega.$$

Because of the radiation condition, see Sec. 38, eq. (1 d), and because v vanishes as $r \rightarrow \infty$, the bracket $\{ \}$ vanishes. Hence, the integral also vanishes.

It should be emphasized that in this calculation, as in the previous applications of Green's theorem in potential theory, the spherical wave u plays the role of a *mathematical auxiliary function*. It is, so to speak, a "probe" which we use for investigating the optical field v . This "virtual" spherical wave has nothing to do with the real spherical wave which we shall introduce in (4 b) as the source function of the optical field. In eq. (4) we have in a sense removed the probe from the field v under investigation by substituting for u and $\partial u/\partial n$ their values as given by (2). From now on we shall forget the origin of these quantities and shall rather regard them as describing spherical waves which are radiated by the surface elements $d\sigma$ and which arrive at the point P at the distance r from $d\sigma$. Only with this interpretation of eq. (4) have we gained the basis of Huygens' principle.

Formula (4) allows us to calculate v at every point P of the exterior if we know the boundary values of v and $\partial v/\partial n$ on σ (or more appropriately, if we know these boundary values!). Let us assume that σ consists of an opaque portion $\bar{\sigma}$ and an aperture which we shall henceforth call σ . The latter is indicated by a dotted line in fig. 56. It is reasonable to assume that as we approach $\bar{\sigma}$ from the exterior, we shall find there the values

$$(4 a) \quad v = 0, \quad \frac{\partial v}{\partial n} = 0.$$

Therefore, eq. (4) retains its validity even with our new definition of σ . Again, it is reasonable to assume that v has the same values in the aperture as those it would have if $\bar{\sigma}$ were absent. If, for instance, v were due to the radiation from a luminous point P' of strength A , then the boundary values in the aperture would be

$$(4 b) \quad v = A \frac{e^{ikr'}}{r'}, \quad \frac{\partial v}{\partial n} = A \frac{\partial}{\partial n} \frac{e^{ikr'}}{r'}$$

where r' is defined in the figure.

But strictly speaking the assumptions (4 a, b) are mathematically inadmissible. A well-known theorem in Riemann's theory of functions says that if a two-dimensional potential v vanishes together with its normal derivative along a finite curve segment s , then v vanishes identically in the whole plane. This theorem can be extended to cover solutions of the two-dimensional wave equation¹. It is also true that any solution of the three-dimensional potential or wave equation vanishes in the whole space if the condition (4 a) is satisfied on any finite surface element σ . Therefore, (4 a) would seem to imply that $v = 0$ everywhere.

¹Heinrich Weber, Mathem. Ann. Vol. 1, 1869, p. 1.

Applying this theorem, on the other hand, to the difference $w = v - v'$ between any two analytic solutions v and v' of the three-dimensional wave equation, it follows that v and v' must be identical in the whole space if the conditions $v = v'$ and $\partial v/\partial n = \partial v'/\partial n$ are satisfied on any finite surface element σ . Therefore, the assumptions (4 a) and (4 b) not only contradict the known physical situation but also contradict one another.

As a matter of fact, we would not even obtain the boundary values (4 a) or (4 b) if we calculated them from (4) by placing P on $\bar{\sigma}$ or σ . Thus eq. (4) gives the *correct* values of v_P only if we know the *correct* boundary conditions v and $\partial v/\partial n$.

C. GREEN'S FUNCTION, SIMPLIFIED FORMULATION OF HUYGENS' PRINCIPLE.

This mathematical contradiction is avoided by substituting for the spherical wave u in the original eq. (3) the *Green's function belonging to our surface*. This function is defined by the following conditions¹:

$$\begin{aligned} (5 \text{ a}) \quad & \Delta G + k^2 G = 0 \quad \text{in } \tau, \\ (5 \text{ b}) \quad & G = 0 \quad \text{on } \sigma, \\ (5 \text{ c}) \quad & G \rightarrow u \quad \text{as } r \rightarrow 0, \\ (5 \text{ d}) \quad & r \left(\frac{\partial G}{\partial n} - i k G \right) \rightarrow 0 \quad \text{as } r \rightarrow \infty. \end{aligned}$$

As before, r is the distance from the point P and (5 d) is the so-called radiation condition of Vol. VI, Sec. 28. Equation (5 c) states that, like u , G shall have a singularity only at the point P and shall be continuous everywhere else in the exterior. G differs from u because of the additional condition (5 b). As a result of this condition the term containing $\partial v/\partial n$ in eq. (4) disappears and that equation becomes²

$$(6) \quad 4\pi v_P = - \int_{\sigma} v \frac{\partial G}{\partial n} d\sigma.$$

¹ See Vol. VI, Secs. 10 E and F. In the nomenclature used there the spherical wave u is not a Green's function but is the "principal solution" of the differential equation $\Delta u + k^2 u = 0$.

² The spherical surface which excludes infinity (see footnote 1, p. 197) contributes now

$$\int \left(G \frac{\partial v}{\partial n} - v \frac{\partial G}{\partial n} \right) r^2 d\omega = \int r \left(\frac{\partial v}{\partial n} - i k v \right) r G d\omega$$

which again vanishes.

Now we need to prescribe only the boundary values of v itself. As in (4 a, b) it is reasonable to assume that

$$(6 a) \quad v = 0 \quad \text{on} \quad \bar{\sigma},$$

$$(6 b) \quad v = A \exp \frac{ikr'}{r'} \quad \text{on} \quad \sigma.$$

These assumptions are mathematically consistent. Furthermore, according to the theory of the Green's function, the boundary values (6 a, b) are actually assumed by the function v_P as computed by (6) when the point P is placed on the screen or in the aperture.

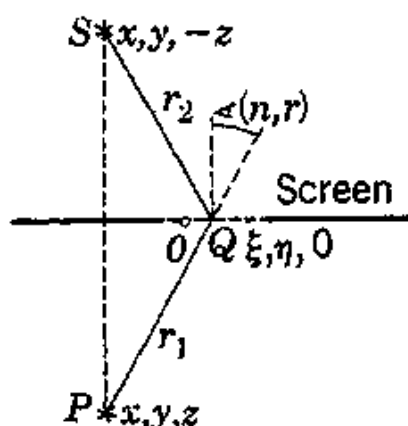


Fig. 57.

Construction of the Green's function for a plane screen.

The question remains, are these assumptions also physically justifiable? The answer again is that *they are only approximations for sufficiently small wavelengths*¹. The field does not vanish completely behind the screen, nor is the field in the aperture entirely unaffected by the presence of the screen, at least not within distances of the order of magnitude of a wavelength from the edge of the screen.

The introduction of the Green's function therefore involves no final justification of the method but it has the practical advantage of leading to the simpler form of the integral (6) as compared to (4). However, the applicability of the Green's function method is restricted to the special case of the *plane screen*. This is the only case for which the Green's function can be conveniently expressed, namely by means of the elementary *method of images*.

In fig. 57 we construct the mirror image S of the point P with respect to the plane of the screen $z = 0$. For an arbitrary point $Q = \xi, \eta, \zeta$ where $\zeta > 0$, we form

$$(7) \quad G = \frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_2}}{r_2}, \quad \begin{aligned} r_1^2 &= (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 \\ r_2^2 &= (\xi - x)^2 + (\eta - y)^2 + (\zeta + z)^2. \end{aligned}$$

x, y, z and ξ, η, ζ are measured from the same origin O which lies in the plane of the screen. This function of ξ, η, ζ satisfies all the conditions (5 a) to (5 d). It should be noted that the singularity of G at the image point S does not violate these conditions because S lies on the other side of the screen $\zeta = 0$.

¹We use here and in the following the word "wavelength" and the notation λ even though these are really only defined for plane waves and lose their simple meaning for the more complicated wave types encountered in diffraction. However, we can always interpret λ as the length $2\pi c/\omega$ which is defined for all monochromatic radiation processes and which for plane waves is identical with the actual wavelength.

From (7) we calculate

$$(7a) \quad \frac{\partial G}{\partial \zeta} = \frac{d}{dr_1} \left(\frac{e^{ikr_1}}{r_1} \right) \frac{\partial r_1}{\partial \zeta} - \frac{d}{dr_2} \left(\frac{e^{ikr_2}}{r_2} \right) \frac{\partial r_2}{\partial \zeta}.$$

If we now place Q on the screen, see fig. 57, we have

$$r_1 = r_2 = r, \quad \frac{\partial r_1}{\partial \zeta} = -\frac{\partial r_2}{\partial \zeta} = \cos(n, r)$$

and hence

$$(8) \quad \frac{\partial G}{\partial n} = -\frac{\partial G}{\partial \zeta} = 2 \frac{\partial}{\partial r} \left(\frac{e^{ikr}}{r} \right) \cos(n, r).$$

This can be further simplified for all positions of P which are not too close to the screen. For then we have $kr = \frac{2\pi r}{\lambda} \gg 1$ and therefore

$$(8a) \quad \frac{\partial}{\partial r} \left(\frac{e^{ikr}}{r} \right) = ik \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr} \right) \sim \frac{2\pi i}{\lambda} \frac{e^{ikr}}{r}.$$

Substituting (8a) in (8) and (8) in (6) we obtain

$$(9) \quad i\lambda v_P = \int_{\sigma} \frac{e^{ikr}}{r} \cos(n, r) v d\sigma.$$

Thus we have gained an expression which is equivalent to Huygens' principle and which formulates it exactly. *A light wave falling on the aperture σ propagates as if every element $d\sigma$ emitted a spherical wave the amplitude and phase of which are given by that of the incident wave v .* The factor $\cos(n, r)$ which multiplies $d\sigma$ is of interest. It corresponds to Lambert's law of surface brightness and was used earlier by Fresnel in his qualitative considerations. The factor λ on the left-hand side of (9) is understandable because of the dimensions of the right-hand side ($d\sigma/r = \text{length}$).

If one substitutes for v the value given in (4b) which corresponds to illumination by a point source, then (9) becomes

$$(10) \quad i\lambda v_P = A \int e^{ik(r+r')} \frac{\cos(n, r)}{rr'} d\sigma.$$

D. FRAUNHOFER AND FRESNEL DIFFRACTION

Let the dimensions of the diffraction aperture in the screen be small compared to the distances r and r' of observer and light source. Then the factor $\frac{\cos(n, r)}{rr'}$ varies but little inside the opening. Hence we may place this factor in front of the integral sign, setting it equal to the value it

assumes at the origin O of our integration variables ξ, η . Calling R and R' , respectively, the values of r and r' at O , we thus obtain instead of (10)

$$(11) \quad i \lambda v_P = \frac{A}{R R'} \cos(n, R) \int e^{i k(r+r')} d\xi d\eta.$$

To simplify the remaining portion of the integrand, which because of the magnitude of k is a *rapidly varying function*, we first develop r in powers of ξ and η :

$$\begin{aligned} r &= \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2} = \sqrt{R^2 - 2(x\xi + y\eta) + (\xi^2 + \eta^2)} \\ &= R - \frac{x}{R}\xi - \frac{y}{R}\eta + \frac{\xi^2 + \eta^2}{2R} - \frac{(x\xi + y\eta)^2}{2R^3} = R - \alpha\xi - \beta\eta + \frac{\xi^2 + \eta^2 - (\alpha\xi + \beta\eta)^2}{2R}, \end{aligned}$$

where α and β are the direction cosines of the diffracted ray $O \rightarrow P$ with respect to the ξ - and η -axes. If we call the direction cosines of the incident ray $P' \rightarrow O$ α_0 and β_0 (hence the direction cosines of $O \rightarrow P'$ are $-\alpha_0$ and $-\beta_0$), then we find correspondingly

$$r' = R' + \alpha_0\xi + \beta_0\eta + \frac{\xi^2 + \eta^2 - (\alpha_0\xi + \beta_0\eta)^2}{2R'}.$$

From this follows

$$(12) \quad e^{i k(r+r')} = e^{i k(R+R')} e^{-i k\Phi}$$

with the abbreviation

$$\begin{aligned} (13) \quad \Phi &= (\alpha - \alpha_0)\xi + (\beta - \beta_0)\eta - \left(\frac{1}{R} + \frac{1}{R'}\right) \frac{\xi^2 + \eta^2}{2} + \frac{(\alpha\xi + \beta\eta)^2}{2R} \\ &\quad + \frac{(\alpha_0\xi + \beta_0\eta)^2}{2R'}, \end{aligned}$$

and formula (11) becomes

$$(14) \quad i \lambda v_P = \frac{A}{R R'} \cos(n, R) e^{i k(R+R')} \int e^{-i k\Phi} d\xi d\eta.$$

The expansion (13) clearly presupposes that the linear dimensions of the diffraction opening are small compared to R and R' .

The evaluation of the remaining integral in (14) is simplest to perform for the case of *Fraunhofer diffraction*

$$(14a) \quad R \rightarrow \infty, \quad R' \rightarrow \infty$$

which obtains for the meteorological phenomena and which can also be best realized experimentally. In this case only the *linear* terms in Φ remain, and we have only to deal with a superposition of *plane waves*.

If one or both of the conditions (14 a) are not satisfied, one speaks of *Fresnel diffraction*. By proper choice of the origin O (for details see Sec. 37) one can then make $\alpha = \alpha_0$ and $\beta = \beta_0$, so that the linear terms of Φ vanish. The integration of the quadratic terms (Fresnel's integrals) provides us with a complete picture of the entire diffraction field behind the screen, while in the Fraunhofer case we confine ourselves to the limit of the diffraction field at large distances from the screen.

Fresnel and Fraunhofer diffraction are also called *microscopic* and *telescopic diffraction*, respectively. For, in the Fresnel case a magnifying glass may be used to project the field at a given point P onto an observation screen where the intensity can then be measured. In the Fraunhofer case insufficient intensity would reach the naked eye if it were placed at a very large distance from the screen (especially in the case of a single small diffraction opening). Therefore, all parallel ray bundles emerging from the diffraction opening are focused by means of a lens¹ L on a point P in the focal plane E of L . For similar reasons a point source P' at a finite distance from the screen is used, and its rays are sent through the opening as parallel ray bundles by means of a lens L' (collimator lens). P' must, of course, lie in the focal plane E' of L' ; see fig. 58.

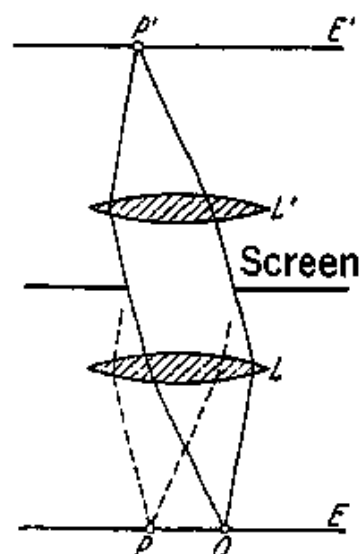


Fig. 58.

The Fraunhofer arrangement for diffraction observations.

If O is the image of P' according to geometrical optics, then the coordinates of P in the plane E (with O as the origin of the coordinate system) are proportional to the quantities $\alpha - \alpha_0$ and $\beta - \beta_0$. According to (14) and (13) the intensity at P depends only on these two quantities. (The factor in front of the integral in (14) is a constant since $|\exp\{i k (R + R')\}| = 1$ and since A must be thought of as tending to infinity like $R R'$.) L is the objective lens used in this "telescopic" observation; the eyepiece through which the diffraction pattern produced in E is viewed is not shown in fig. 58. The diffracted rays which are focused at P are represented in the figure by dotted lines which, in accordance with the notion of Huygens' principle, are drawn as if they originated in the diffraction opening².

¹We may disregard the diffraction phenomena caused by the rims of the lenses L and L' .

²Fresnel diffraction can also be observed with a telescope if the eyepiece is not focused on the focal plane E but on an arbitrary extrafocal plane. Instead of observing the patterns on these planes with the eye, one can of course record them by means of a photographic plate.

When there is a very large number N of diffraction openings as in Secs. 32 and 33, the telescopic arrangement becomes unnecessary. The amplification factor N^2 in Sec. 32 and N in Sec. 33 make it possible to observe the diffraction patterns with the naked eye even at large distances.

E. BABINET'S PRINCIPLE

Two arrangements of diffraction openings 1 and 2 are called "complementary" if the opening of 1 is congruent to the screen of 2 and vice versa. Let us calculate v_1 and v_2 for the same primary illumination and form their sum. We assert that "within the framework of Huygens' principle"

$$(15) \quad v_1 + v_2 = v,$$

where v is the undisturbed primary illumination at the point of observation when both diffraction screens are absent.

We shall prove this theorem for the general case of an arbitrary (possibly curved) screen by starting from eq. (4). When forming the desired sum, we have to replace v_P on the left-hand side of that equation by

$$(v_1 + v_2)_P.$$

On the right-hand side of (4) v has the same meaning in both summands, namely the undisturbed primary illumination. In contrast to (4) the integration must now be carried out over the *entire* surface σ because every point on σ belongs to the diffraction opening of either 1 or 2. But we obtain precisely the same integral if we apply (4) to the primary illumination v with no screens present. In that case we have just this v on the left-hand side and the integration over the whole surface σ on the right-hand side with the meanings of the symbols v and $\partial v / \partial n$ in the integrand the same as before. Equation (15) has thus been proved: $v_1 + v_2$ and v are equal because both are equal to the same $\int \dots d\sigma$.

Equation (15) is valid for all points P of the exterior and therefore encompasses both Fresnel and Fraunhofer diffraction. It is called "*Babinet's principle*". The above proof was based on Huygens' principle. In Sec. 38 F we shall discuss how Babinet's principle must be modified when it is treated from the more precise viewpoint of the boundary value problem.

In the older literature¹ Babinet's principle occurs only in a much narrower form, which restricts its applicability to Fraunhofer diffraction. The reason is that the complete functional dependences of v_1 , v_2 (including their phases)

¹ See, for instance, Kirchhoff, *Vorlesungen über Optik*, p. 96.

are not accessible to observation, but only the amplitudes $|v_1|$, $|v_2|$ or, what is equivalent, only the intensities are observable. The latter, of course, do *not* satisfy the equation

$$(15\ a) \quad J_1 + J_2 = J.$$

Rather, when forming the absolute value of (15), the left-hand side of (15 a) is found to contain the additional terms

$$(15\ b) \quad v_1 v_2^* + v_2 v_1^*.$$

Only in the case of Fraunhofer diffraction can a simple statement be made about the intensities: *the two complementary screens produce diffraction patterns of equal intensity*:

$$(16) \quad J_1 = J_2.$$

To prove this we consider the focal plane E in fig. 58. For ideal lenses the primary light v is concentrated at the point O and is zero everywhere else. Excluding the point O , at which the singularity of the diffraction pattern renders observation impossible in any case, it follows from (15) that

$$v_1 = -v_2, \quad |v_1| = |v_2|, \quad \text{hence indeed} \quad J_1 = J_2.$$

In Secs. 35 C and D we shall discuss a very elementary problem of Fresnel diffraction for which there exists no simple relationship at all between J_1 and J_2 , but we shall convince ourselves that our formulation (15) of Babinet's principle is valid.

F. BLACK OR REFLECTING SCREEN

In the theory of diffraction it is customary to speak of a *black* screen. However, in actual diffraction experiments one finds that the physical nature of the screen in general does not affect the results noticeably. Thus a piece of tin foil into which a narrow slit has been scratched yields the same diffraction pattern regardless of whether the foil has been left reflecting or whether it has been blackened. Therefore we need only describe the screen as *opaque* in order to specify that in spite of arbitrary thinness it shall transmit no light. In the Maxwell theory such a screen would have to be defined as a material possessing an *infinite conductivity*. Such a screen would not be black but would be *perfectly reflecting*; its reflecting power would be $r = 1$. On the other hand, black, that is completely *non-reflecting* material, cannot even be defined in the Maxwell theory; blackening is not a property of the material but is a property of the surface. We shall take this into account in Sec. 38 where we shall try to describe the property "black" mathematically. Our presentation of Huygens' principle shows that this property is not essential to the theory of diffraction. Only very refined experiments can reveal the nature of the material of which the diffracting screen is composed.

The material composition of the screen, of course, affects the light field only in the immediate vicinity of the edge of the opening, that is, only within a distance of a few wavelengths from the edge. If the opening is fairly large, this edge zone is negligible compared to the rest of the aperture. This explains why the crude assumptions (4 a, b) or (6 a, b), which can of course be valid only outside the edge zone, have been so eminently successful. Deviations from Huygens' principle are to be expected with the usual methods of observation only for extremely small openings which are of the order of magnitude of a wavelength in size (or for experimental arrangements which correspond to such small openings in accordance with the similarity law of Sec. 35 E).

G. TWO GENERALIZATIONS

So far we have restricted ourselves to those consequences of Huygens' principle which are directly applicable to the problems under discussion in this chapter. We shall now present two closely allied results which will be useful later.

1. Instead of the Green's function (7) which satisfies the boundary condition $G = 0$ at $z = 0$, we now form

$$(17) \quad G = \frac{e^{ikh_1}}{r_1} + \frac{e^{ikh_2}}{r_2}.$$

This is a function which satisfies the boundary condition $\partial G / \partial z = 0$ at $z = 0$. Substituting it for u in eq. (3) we obtain in place of (6)

$$(18) \quad 4\pi v_P = + \int \frac{\partial v}{\partial n} G \, d\sigma.$$

However, this v_P is identical with the v_P of eq. (6) only if the integral in (18) is taken not merely over σ but also over the entire screen which contains the aperture σ [in eq. (6) this was not necessary because we assumed $v = 0$ on the opaque screen]. With this understanding of the integration in (18), and given the exact boundary values of v and $\partial v / \partial n$ in the opening and of $\partial v / \partial n$ on the screen, we have

$$(19) \quad \int \frac{\partial v}{\partial n} G_+ \, d\sigma = - \int v \frac{\partial G_-}{\partial n} \, d\sigma,$$

where G_+ is the Green's function defined in eq. (17), G_- is that defined in eq. (7). If we substitute for these Green's functions their values on the plane $z = 0$, we obtain

$$(20) \quad \int \frac{\partial v}{\partial n} \frac{e^{ikr}}{r} \, d\sigma = - \int v \frac{\partial}{\partial n} \frac{e^{ikr}}{r} \, d\sigma.$$

2. If the screen is not a plane but a curved (e. g. spherical) surface and has an aperture σ , then doubtless there are again two functions G_- and G_+ which satisfy the conditions $G_- = 0$ and $\partial G_+ / \partial n = 0$ on σ . Every continuous solution v of the wave equation can be represented in two ways by means of these functions. Therefore, the remarkable identity (19) is valid in the sense described above also for curved surfaces σ .

However, even for the simplest case of the sphere, the analytic representations of G_{\pm} lead to infinite series containing the eigenfunctions of the sphere. Therefore, the simplification which resulted from the introduction of the Green's function in the case of the plane screen becomes illusory for curved screens, not to mention the fact that the requirement concerning the knowledge of the exact boundary values is not fulfilled in either case.

35. The Problem of the Shadow in Geometrical and in Wave Optics

Geometrical optics constitutes our day-to-day guide to the outside world; it is the basis for the construction of the image-forming devices (spectacles, telescopes, photographic lenses). We shall here treat geometrical optics as the limiting case of wave optics as $\lambda \rightarrow 0$; see also the introduction to Sec. 34.

A. THE EIKONAL

As in Sec. 34 we start with the scalar wave equation

$$(1) \quad \Delta u + k^2 u = 0, \quad k = \sqrt{\epsilon \mu} \omega = \frac{2\pi}{\lambda},$$

but now we do not assume ϵ to be constant but rather to be a (continuously or discontinuously varying) function of position. As $\lambda \rightarrow 0$, and hence $k \rightarrow \infty$, this differential equation degenerates. In order nevertheless to be able to draw quantitative conclusions from the equation, we make the following assumption¹ as to the form of the solution:

$$(2) \quad u = A e^{ik_0 S}, \quad k_0 = \sqrt{\epsilon_0 \mu_0} \omega = \frac{2\pi}{\lambda_0}.$$

A is an amplitude factor. We call S the *eikonal*, an expression introduced by H. Bruns. While u is a rapidly varying function of position (because $k_0 \rightarrow \infty$), we consider A and S as slowly varying functions of the coordinates x, y, z which do not go to infinity with k_0 . By differentiating (2) we get

¹After P. Debye, in a paper Ann. d. Phys. 85 (1911) by Sommerfeld and Iris Runge.

$$\begin{aligned}\frac{\partial u}{\partial x} &= i k_0 u \frac{\partial S}{\partial x} + u \frac{\partial \log A}{\partial x}, \\ \frac{\partial^2 u}{\partial x^2} &= -k_0^2 u \left(\frac{\partial S}{\partial x} \right)^2 + 2 i k_0 u \left(\frac{1}{2} \frac{\partial^2 S}{\partial x^2} + \frac{\partial \log A}{\partial x} \frac{\partial S}{\partial x} \right) + \dots \\ \Delta u + k^2 u &= -k_0^2 u \left[\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 - \frac{k^2}{k_0^2} \right] \\ &\quad + 2 i k_0 u \left[\frac{1}{2} \Delta S + \text{grad} \log A \cdot \text{grad} S \right] + \dots\end{aligned}$$

where the terms indicated by do not become infinite as $k_0 \rightarrow \infty$.

Hence eq. (1) is satisfied approximately if S and A satisfy the differential equations

$$(3) \quad D(S) = n^2, \quad n = \frac{k}{k_0},$$

$$(4) \quad \text{grad} \log A \cdot \text{grad} S = -\frac{1}{2} \Delta S.$$

D is the notation for the "first differential parameter"

$$D = \left(\frac{\partial}{\partial x} \right)^2 + \left(\frac{\partial}{\partial y} \right)^2 + \left(\frac{\partial}{\partial z} \right)^2$$

which has already been used in Vol. II, eq. (3.9 c); n is the usual index of refraction; Eq. (3) which is the *differential equation of the eikonal*, is an *inhomogeneous equation of first order and second degree*. Once (3) has been integrated, (4) yields the component of the gradient of $\log A$ in the direction of the gradient of S . Equation (4) makes no statement about the gradient of A in a direction perpendicular to the gradient of S . Therefore (4) permits discontinuities of A in these directions.

According to the definition (2) the surfaces $S = \text{constant}$ are surfaces of constant phase of u . Hence they represent *wave surfaces*. The normals to these surfaces are given by the gradient of S and represent the *ray directions*. In general, if n varies in space, the rays are *curved*. In an optically inhomogeneous medium the integration of (3) is the simplest method for determining the wave surfaces and the ray directions.

In an optically homogeneous medium with $n = \text{constant}$, one obtains as the simplest solution of (3) the *linear function*

$$(5) \quad S = n(\alpha x + \beta y + \gamma z) \quad \text{where} \quad \alpha^2 + \beta^2 + \gamma^2 = 1.$$

This function contains two arbitrary constants, e. g. α and β . The *wave surfaces* determined by this solution are *planes*, the *rays* are *parallel straight lines* in the direction $\alpha:\beta:\gamma$, since, indicating the three components,

$$(5a) \quad \text{grad} S = n(\alpha, \beta, \gamma).$$

For constant n the simplest solution with one singular point is the *spherical wave*

$$(6) \quad S = nr, \quad r = \sqrt{x^2 + y^2 + z^2}, \quad \text{grad } S = \frac{n}{r}(x, y, z).$$

The simplest solution with one singular straight line corresponds to the *cylindrical wave*

$$(7) \quad S = n\rho, \quad \rho = \sqrt{x^2 + y^2}, \quad \text{grad } S = \frac{n}{\rho}(x, y).$$

In both these cases, and quite generally in homogeneous media, the rays are *straight lines*.

The general solution is obtained by starting with any arbitrary surface and constructing a family of parallel surfaces to it (surfaces of constant infinitesimal spacing).

We have thus obtained the simplest mathematical scheme for the understanding of the formation of shadows. We consider the light source as given. From it there emerge rectilinear rays. A screen shall be called opaque if it absorbs all rays falling on it and does not itself emit any rays. Then the shadow behind the screen is bounded by straight ray directions which emerge from the light source. In the direction perpendicular to the limit of the shadow A decreases discontinuously to zero which, as we have seen, is compatible with eq. (4). *In the limiting case as $\lambda \rightarrow 0$ there is no diffraction.* The rays which do not meet the screen continue unobstructed along straight lines. If several light sources are present, then there are, of course, half-shadow regions.

Geometrical optics has become second nature with us to such an extent that we suppose it to be valid even in cases where we know the rays to be curved. Thus we see the sun over the horizon for about 5 minutes after it has actually set. For we project the sun's rays, which because of the inhomogeneity of the earth's atmosphere are curved, along straight lines tangential to the directions of the rays as they meet the eye. The situation is similar in the case of certain diffraction phenomena; see Sec. 38 D. The edge of a screen appears to us as a luminous line because we extrapolate rectilinearly backwards the rays of the cylindrical wave which meet our eye; but in reality the field in the vicinity of the edge of the screen is continuous.

With eqs. (1) to (7) we have made the transition from wave optics to geometrical optics. Schrödinger proceeded in the opposite direction when, guided by the comprehensive ideas of Hamilton, he accomplished the transi-

tion from classical mechanics to wave mechanics. As described in Vol. I, Sec. 44, Hamilton started with the theory of optical instruments and several years later applied it to general dynamics. The differential equation (3) of the eikonal

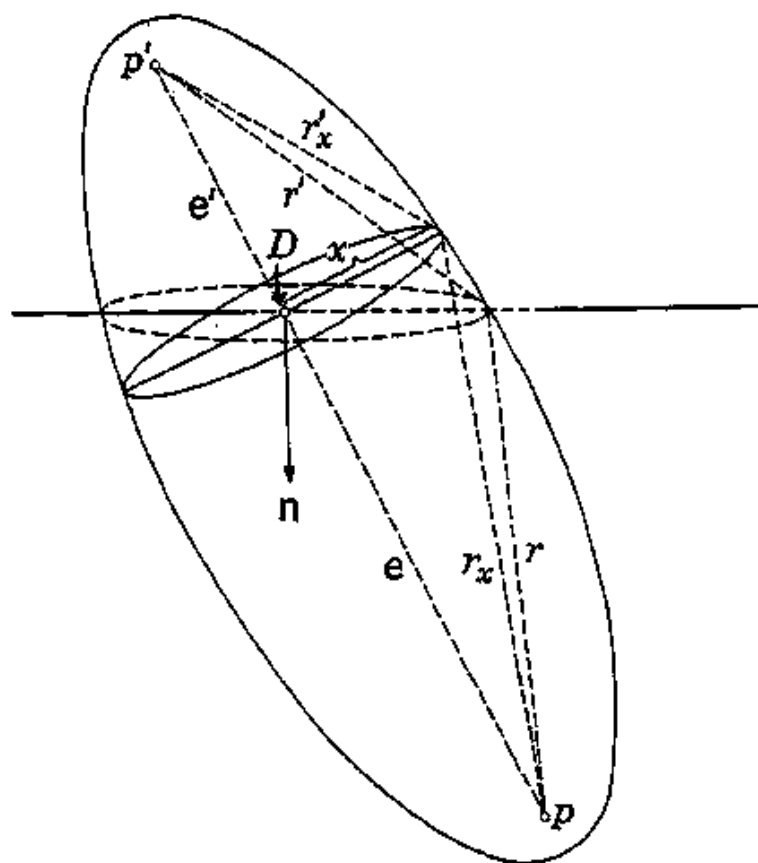


Fig. 59.

Construction of surfaces of constant phase.

is a very simple case of Hamilton's partial differential equations of dynamics. In the same sense our eq. (5 a) is a very simple specialization of Hamilton's momentum equation $\dot{p}_k = \partial S / \partial q_k$. Of course the way was cleared for Schrödinger's theory only after the discovery of the quantum of action by Planck. It should also be noted that the useful W.K.B. method (Wentzel-Kramers-Brillouin approximation) in which the same hypothesis as in (2) is made, also corresponds to the transition from wave optics to geometrical optics.

B. THE ORIGIN OF THE SHADOW ACCORDING TO WAVE OPTICS

We must now seek a solution of the problem of the shadow by means of *wave optics* instead of, as before, by its asymptotic form, namely *ray optics*. For this purpose we turn again to Huygens' principle.

Let us consider the expression under the integral sign in eq. (34.10). We construct the surfaces of constant phase

$$(8) \quad r + r' = \text{const.}$$

These are ellipsoids of rotation with the common focal points P (point of observation) and P' (light source). We shall call the focal distance $\rho + \rho'$ where, see fig. 59,

$$\rho = PD, \quad \rho' = P'D,$$

and where D is the point where the focal line intersects the plane of the screen. Let x be the radius of that circular section of the ellipsoid under consideration

which passes through D ; the distances of a point on this circle from P and P' shall be r_x and $r_{x'}$, respectively. Provided P and P' are sufficiently far away, it is seen from the figure that

$$(9) \quad r_x^2 = \rho^2 + x^2, \quad r_{x'}^2 = \rho'^2 + x^2,$$

$$(9a) \quad r_x = \rho \left(1 + \frac{1}{2} \frac{x^2}{\rho^2} + \dots \right), \quad r_{x'} = \rho' \left(1 + \frac{1}{2} \frac{x^2}{\rho'^2} + \dots \right).$$

From this and (8) it follows that

$$(10) \quad r + r' = r_x + r_{x'} = \rho + \rho' + \phi$$

where

$$(10a) \quad \phi = \left(\frac{1}{\rho} + \frac{1}{\rho'} \right) \frac{x^2}{2} + \dots, \quad d\phi = \left(\frac{1}{\rho} + \frac{1}{\rho'} \right) x dx,$$

and where ϕ is the parameter of the system of ellipses formed by the intersection of the system of ellipsoids of rotation and the plane of the screen. To the circular rings

$$(10b) \quad d\sigma_h = 2\pi x dx$$

there correspond in the plane of the screen the elliptical rings $d\sigma_e$, whose areas are proportional to $d\phi$. We write

$$(11) \quad d\sigma_e = f d\phi.$$

These $d\sigma_e$ are the proper area elements to be used in the integration with respect to $d\sigma$ in eq. (34.10).

We call $\varphi(\phi)$ the fraction of the elliptical ring $d\sigma_e$ which falls inside the aperture and distinguish the two cases illustrated in fig. 60 a and b:

a) D lies in the screen,

b) D lies in the aperture.

In case a) the integration over ϕ extends from ϕ_1 to ϕ_2 . For ϕ_1 and ϕ_2 , $\varphi(\phi) = 0$. In case b) the integration starts at $\phi = 0$. Between $\phi = 0$ and $\phi = \phi_1$, $\varphi(\phi) = 1$. From ϕ_1 to ϕ_2 , $\varphi(\phi)$ decreases from 1 to 0.

We combine the factor φ and the factor f in (11) with the factor $\frac{\cos(n, r)}{r r'}$ in (34.10) and call their product $F(\phi)$. Then, using (10) and (11), (34.10) becomes for the case a)

$$(12) \quad i \lambda v_P = A e^{ik(\rho + \rho')} \int_{\phi_1}^{\phi_2} F(\phi) e^{ik\phi} d\phi.$$

Integration by parts¹ gives

$$(12 a) \quad \int_{p_1}^{p_2} F(p) e^{ikp} dp = \frac{1}{ik} F(p) e^{ikp} \Big|_{p_1}^{p_2} - \frac{1}{ik} \int_{p_1}^{p_2} F'(p) e^{ikp} dp.$$

For the case illustrated in fig. 60 a the first term on the right-hand side of (12 a) vanishes because F contains the factor φ and $\varphi(p_1) = \varphi(p_2) = 0$. As k goes to infinity, the integral in the second term likewise goes to zero. Therefore the right-hand side of (12) vanishes even after division by $i\lambda = 2\pi i/k$ and from (12),

$$(13) \quad v_P \rightarrow 0, \text{ shadow.}$$

The shadow is brought about by interference of the waves originating at the surface elements $d\sigma_s$.

In case b) we obtain in place of (12)

$$(14) \quad i\lambda v_P = A e^{ik(\rho+\rho')} \int_0^{p_2} F(p) e^{ikp} dp.$$

Integration by parts yields instead of (12 a)

$$(14 a) \quad \int_0^{p_2} F(p) e^{ikp} dp = \frac{1}{ik} F(p) e^{ikp} \Big|_0^{p_2} - \frac{1}{ik} \int_0^{p_2} F'(p) e^{ikp} dp.$$

The second term on the right-hand side again vanishes for the case illustrated in fig. 60 b as $k \rightarrow \infty$. The first term vanishes at its upper limit because $\varphi(p_2) = 0$. At the lower limit we have, according to fig. 59

$$r = \rho, \quad r' = \rho', \quad \frac{\cos(n, r)}{r r'} = \frac{(\cos n, \rho)}{\rho \rho'}, \quad \varphi(0) = 1,$$

$$d\sigma_s = \frac{d\sigma_k}{\cos(n, \rho)} = f dp,$$

and according to eqs. (10 a, b)

$$(14 b) \quad \begin{aligned} dp &= \left(\frac{1}{\rho} + \frac{1}{\rho'} \right) x dx = \frac{1}{2\pi} \frac{\rho + \rho'}{\rho \rho'} d\sigma_k, \\ f &= \frac{1}{\cos(n, \rho)} \frac{d\sigma_k}{dp} = \frac{2\pi}{\cos(n, \rho)} \frac{\rho \rho'}{\rho + \rho'}. \end{aligned}$$

Hence

$$F(0) = \varphi(0) f \frac{\cos(n, \rho)}{\rho \rho'} = \frac{2\pi}{\rho + \rho'}.$$

¹Although the derivative $F'(p)$ which occurs in (12 a) can under certain circumstances become infinite at the limits of integration, a closer investigation in Sec. 36 D will show that the convergence of the integral is nevertheless preserved.

This lower limit therefore yields for the value of the right-hand side of (14 a)

$$-\frac{2\pi}{i k} \frac{1}{\rho + \rho'} = \frac{i \lambda}{\rho + \rho'}.$$

Dividing (14) by $i \lambda$ we obtain therefore

$$(15) \quad v_P = A \frac{e^{i k(\rho + \rho')}}{\rho + \rho'}.$$

This is the incident spherical wave at the distance $\rho + \rho'$ from the light source.

Equations (15) and (13) contain the Fresnel theory of the phenomenon of "light and shadow"; they make it understandable from the optical point of view that light "in general" propagates along straight lines.

"In general" means that there are exceptions, as we shall see below under C and D, and in particular in Sec. 36 D, where we shall investigate Fraunhofer diffraction caused by screens with straight edges.

C. DIFFRACTION BEHIND A CIRCULAR DISC

The result derived in (13) suffers an exception if a finite portion of the edge of the screen coincides with one of the two bounding ellipses $p_1 = \text{constant}$ or $p_2 = \text{constant}$. For then $\varphi(p_1)$ or $\varphi(p_2)$ is not zero and the first term on the right-hand side of eq. (12 a) does not vanish. Therefore (13) is no longer valid. *There is no shadow; we may speak of diffraction at the elliptically curved portion of the edge of the screen.*

A particular example of this situation is a screen which consists of a circular disc with the points P and P' lying on the perpendicular through the disc's center. Then the point D is at the center of the disc. The ellipses $p = \text{constant}$ become the circles $x = \text{constant}$ (notation the same as in fig. 59). The diffraction opening consists of the whole exterior of the circular disc $a < x < \infty$ ($a = \text{radius of the disc}$). Equation (34.10) becomes then

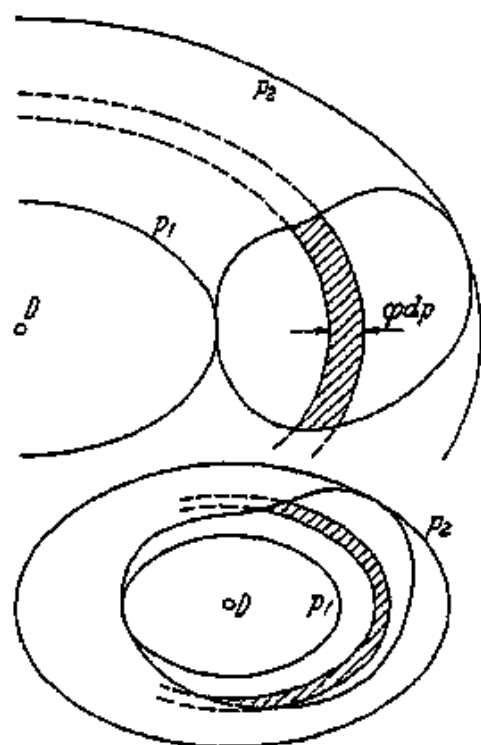


Fig. 60 a, b.

Intersection of the surfaces of constant phase with the plane of the screen.

- a) The point D defined in fig. 59 lies in the screen; upper figure.
- b) D lies in the opening, lower figure.

In each case the irregular curve represents the edge of the diffraction opening; P_1 and P_2 are, respectively, the parameters of the smallest and largest ellipses which touch the edge of the opening.

$$(16) \quad i \lambda v_P = A \int_a^\infty e^{ik(r+r')} \frac{\cos(n, r)}{r r'} 2\pi x dx.$$

For purposes of a later application we let $\rho' = \rho$ and $r' = r$, which will also simplify the evaluation of (16). We must emphasize, however, that this specialization does not affect the result. It would be just as convenient to choose $\rho' = \infty$, that is, to have an incident plane wave instead of a spherical wave; cf. footnote ¹ on p. 125.

If $\rho' = \rho$, we find (see fig. 61)

$$r^2 = \rho^2 + x^2 = r'^2, \quad x dx = r dr, \quad \cos(n, r) = \frac{\rho}{r};$$

hence according to (16)

$$(16a) \quad i \lambda v_P = 2\pi A \rho \int_a^\infty \frac{e^{2ikr} dr}{\sqrt{\rho^2 + a^2}}.$$

Integrating by parts one obtains

$$(16b) \quad i \lambda v_P = \frac{2\pi A \rho}{2ik} \left\{ \frac{e^{2ikr}}{r^2} \Big|_a^\infty + 2 \int_a^\infty \frac{e^{2ikr} dr}{r^3} \right\}.$$

If the second term in the bracket $\{ \}$ is again integrated by parts, it will contain the factor $\frac{1}{2ikr}$, which shows that this term is almost completely eliminated by interference. Disregarding, therefore, the second term, we obtain from (16a)

$$(16c) \quad i \lambda v_P = -\frac{2\pi A \rho}{2ik} \frac{e^{2ik\sqrt{\rho^2 + a^2}}}{\rho^2 + a^2}.$$

If we introduce the following notation for the primary excitation at the edge of the disc

$$v_{P'} = A \frac{e^{ikr'}}{r'} = A \frac{e^{ik\sqrt{\rho^2 + a^2}}}{\sqrt{\rho^2 + a^2}},$$

then we can simplify (16c), after cancelling the factor $i \lambda$, as follows

$$v_P = \frac{1}{2} \frac{\rho}{\sqrt{\rho^2 + a^2}} e^{ik\sqrt{\rho^2 + a^2}} v_{P'}.$$

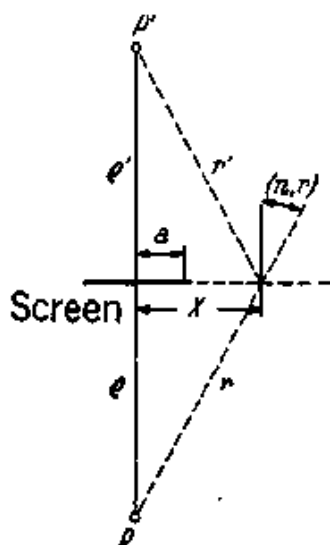


Fig. 61.
Diffraction behind a
circular disc.

Writing this in terms of the intensities $J = |v_P|^2$ and $J_0 = |v_{P'}|^2$, we obtain

$$(17) \quad J = \frac{1}{4} \frac{\rho^2}{\rho^2 + a^2} J_0.$$

This paradoxical result is represented graphically in fig. 62. *There is no darkness anywhere along the central perpendicular behind an opaque circular disc (except immediately behind the disc).* The relative intensity increases with increasing distance between the light source and the point of observation. For very large distances the intensity at the point of observation approaches one-fourth of the intensity at the edge of the disc¹. The primary light waves pass around the edge of the disc along its whole circumference, and because of the symmetry of the arrangement, they meet along the central perpendicular with equal phases. The result is in striking contradiction with the rectilinear ray paths postulated by geometrical optics and with the shadow boundary to be expected according to the latter. We must note, however, that the intensity given by (17) is to be expected only in the immediate vicinity of the central perpendicular, because only there do the lines $\phi = \text{constant}$ coincide with the edge of the disc. At a small distance from this central line the complete shadow predicted by (13) will be observed.

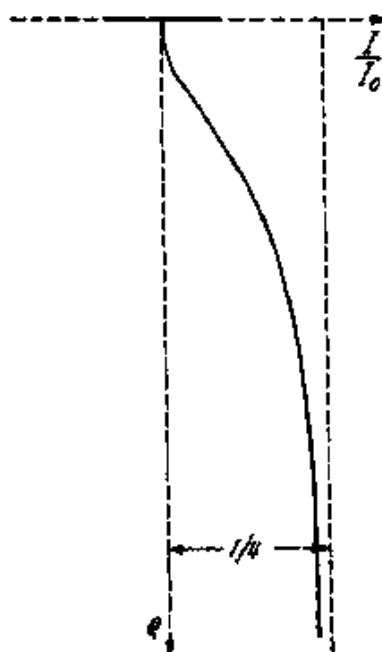


Fig. 62.

Relative light intensity J/J_0
along the axis behind a
circular disc.

Poisson predicted the brightness along the axis as a consequence of Fresnel's theory of the shadow and cited it as an *objection* to that theory². Therefore, or perhaps nevertheless, this phenomenon is called *Poisson diffraction*. It takes place behind an opaque sphere as well as behind a circular disc. In the case of radio waves it has been possible to detect an increase in the strength of the signal at the point of the earth which is antipodal to the primary antenna.

¹If we illuminate the disc with a plane wave [$\rho' \rightarrow \infty$ in eq. (16)] instead of with a spherical wave, then the factor $1/4$ in (17) disappears. Therefore $J \rightarrow J_0$ as $\rho \rightarrow \infty$. At a sufficiently large distance the disc cannot be seen; the primary light wave appears undisturbed.

²The crucial experiment was performed by Arago and Fresnel. One therefore often speaks of an *Arago spot* instead of a *Poisson spot*. The reader is also referred to the experiment of W. Kossel who obtained stronger intensities and whose experimental arrangement had a deeper significance, Z. f. Naturforschung, Vol. 8a, p. 496 (1948).

"A photographic objective can be replaced by a steel sphere." This conclusion was drawn by R. W. Pohl in his *Einführung in die Optik* and proved by means of his illustration 185. We are fortunate in being able to show in fig. 63 a photograph taken by our colleague E. von Angerer. A sheet metal disc 50 mm. in diameter served as a "lens". The distance of the object and the plate from the disc was 35 m. each.

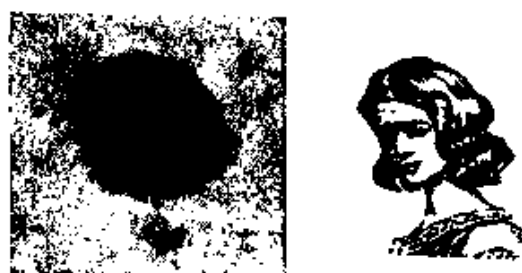


Fig. 63.

"Photograph" by means of a circular sheet metal disc.

The object used is considerably more complex and rich in detail than the simple monogram which had been used by Pohl. Though contrasts are considerably weakened in the picture, it is surprisingly true to the original; v. Angerer found that in order to obtain a sharp picture it was essential that the edge of the disc should be a precise circle (theoretically even to within the order of magnitude of

a wavelength!). It is surprising that the circular disc should perform as well as Pohl's sphere, since the disc can depict only the central ray precisely while the sphere presents a circular cross section to all rays.

D. THE CIRCULAR OPENING AND FRESNEL ZONES

We shall now consider the complementary arrangement, namely the circular opening. Retaining all of the above assumptions and notations and changing only the limits of integration in eq. (16) from $x = a$ and $x = \infty$ to $x = 0$ and $x = a$, we obtain in place of eq. (16 a)

$$(18) \quad i \lambda v_P = 2 \pi A \rho \int_{\rho}^{\sqrt{\rho^2 + a^2}} e^{2ikr} \frac{dr}{r^2}.$$

If we integrate this by parts and retain only the first order term, we get

$$(18 a) \quad i \lambda v_P = \frac{2 \pi A \rho}{2 i k} \frac{e^{2ikr}}{r^2} \Big|_{\rho}^{\sqrt{\rho^2 + a^2}} \\ = -\frac{2 \pi A}{2 i k \rho} e^{2ik\rho} \left\{ 1 - \frac{\rho^2}{\rho^2 + a^2} e^{2ik(\sqrt{\rho^2 + a^2} - \rho)} \right\}.$$

In order to put this expression into a more convenient form, we shall neglect a^2 in comparison with ρ^2 in the factor $\frac{\rho^2}{\rho^2 + a^2}$. The exponent, however,

which contains the factor k must of course be evaluated more precisely. Accordingly, we put in the exponent

$$\sqrt{\rho^2 + a^2} - \rho = \rho \left\{ \sqrt{1 + \frac{a^2}{\rho^2}} - 1 \right\} = \frac{1}{2} \frac{a^2}{\rho}.$$

After dividing, as before, by the factor $i\lambda$, (18 a) becomes

$$(18 \text{ b}) \quad v_P = \frac{A}{2\rho} e^{2ikh\rho} \left\{ 1 - e^{\frac{ikh a^2}{\rho}} \right\} \\ = \frac{A}{2\rho} e^{2ikh\rho} e^{\frac{ikh a^2}{2\rho}} \left\{ -2i \sin \frac{kh a^2}{2\rho} \right\}.$$

Introducing again the primary intensity calculated

at the edge of the screen $J_0 = \frac{A^2}{\rho^2 + a^2} \sim \frac{A^2}{\rho^2}$,

we find the intensity $J = |v_P|^2$ to be

$$(19) \quad J = J_0 \sin^2 \frac{kh a^2}{2\rho}.$$

The relative intensity J/J_0 is plotted in fig. 64. This quantity has an infinite number of maxima and minima which have their limit in the vicinity of the screen. All maxima have the magnitude unity, and all minima are zero. Thus the paradox represented by fig. 62 has been aggravated. *While the central axis behind a circular screen is nowhere dark, the central axis behind a circular opening has an infinite number of dark places.*

This last statement is of course only valid when the illumination is monochromatic. If white light is used, the central axis appears colored, the color alternating along its length.

The fundamental difference between the formulae (17) and (19) immediately shows that there exists no simple relation between the *intensities* of the two complementary cases of the disc, J_1 eq. (17), the opening, J_2 eq. (19), and the primary intensity J_0 . However, the general relation (34.15) between the *amplitudes* v_1 , v_2 , and v_0 which we called the "Babinet principle" is valid even on the very singular central axis of our diffraction problem. For, when forming the sum of v_1 as given by eq. (16 a) and v_1 as given by (18), we obtain

$$i\lambda(v_1 + v_2) = 2\pi A \rho \int_0^\infty e^{2ikr} \frac{dr}{r^2}.$$

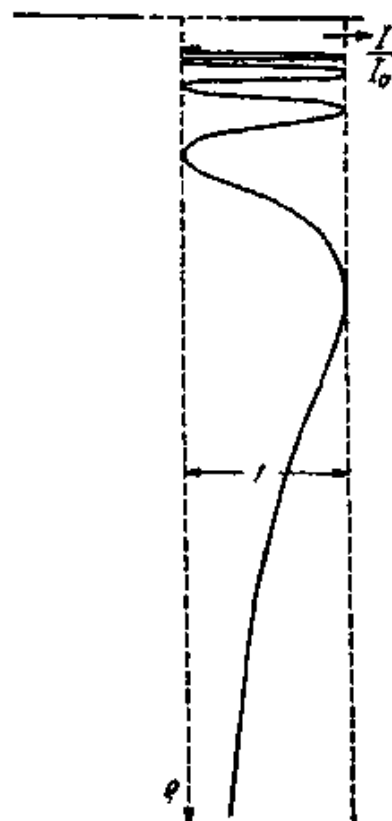


Fig. 64.
Relative intensity behind a
circular opening.

On the other hand, for our special arrangement ($\rho' = \rho$) the primary amplitude at the point P is given by

$$v_0 = A \frac{e^{2ik\rho}}{2\rho}.$$

Hence, according to eq. (34.15) the following equality should hold:

$$\int_{\rho}^{\infty} e^{2ikr} \frac{dr}{r^2} = \frac{i\lambda}{4\pi} \frac{e^{2ik\rho}}{\rho^2}.$$

That this is indeed true can be shown by differentiating with respect to ρ , which yields

$$-\frac{e^{2ik\rho}}{\rho^2} = -\frac{2k\lambda}{4\pi} \frac{e^{2ik\rho}}{\rho^2} + \dots$$

The coefficient of the first term on the right-hand side is equal to 1. The dots indicate a second term which vanishes as $1/k$ compared to the first term. We would have obtained the precise equality demanded by (34.15) if we had not already neglected the corresponding higher order terms in eqs. (16 a) and (18).

The construction of the *Fresnel zones* provides us with a pictorial, though only qualitative, understanding of these results. About the light source P' as a center we construct a set of spheres which intersect the plane of the screen in a set of circles $K_1, K_2, \dots, K_n, \dots$. We choose the radii of the spheres in such a way that the light paths from P' via K_n to the point of observation P and from P' via K_{n+1} to P differ by $\lambda/2$. The distances of P' and P from the plane of the screen (which we called ρ' and ρ before) shall be a and b , respectively; r_n' and r_n shall be the light paths $P'K_n$ and K_nP , respectively. The straight line of length $a + b$ which connects P' with P intersects the plane of the screen in a point K_0 (circle of radius 0), which is also the common center of the family of circles K_n . According to Fresnel's procedure the following equalities characterize the circles K_1, K_2, \dots :

$$r_1' + r_1 = a + b + \frac{\lambda}{2}, \quad r_2' + r_2 = r_1' + r_1 + \frac{\lambda}{2}, \dots$$

By adding the first n of these equations we have for K_n :

$$(20) \quad r_n' + r_n - a - b = n \frac{\lambda}{2}.$$

The radius x_n of this n^{th} circle is calculated as in (9), (9 a) in the following way:

$$\begin{aligned} r_n'^2 &= a^2 + x_n^2, & r_n^2 &= b^2 + x_n^2, \\ r_n' &= a + \frac{1}{2} \frac{x_n^2}{a} + \dots, & r_n &= b + \frac{1}{2} \frac{x_n^2}{b} + \dots \\ r_n' + r_n - a - b &= \frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right) x_n^2 + \dots \end{aligned}$$

Hence, according to (20)

$$(21) \quad x_n = \sqrt{n \lambda f} \quad \text{where} \quad \frac{1}{f} = \frac{1}{a} + \frac{1}{b}.$$

This expression f (which agrees with the definition of the f used in (14 b) except for the factor 2π) reminds us of the well-known formula for the focal length f of a lens. For the present, however, we shall consider f only as a convenient abbreviation.

Figure 65 shows a system of Fresnel zones which consist of the sequence of circular rings K_n, K_{n+1} . These rings are alternately denoted with the signs $+$ and $-$. If we consider the phase in the central zone to be positive, then because of the path difference $\lambda/2$ the phase in the second zone is negative, and so on. All waves falling on the central zone reinforce one another; they are attenuated by the waves falling on the second zone, reinforced by the third zone, and so on.

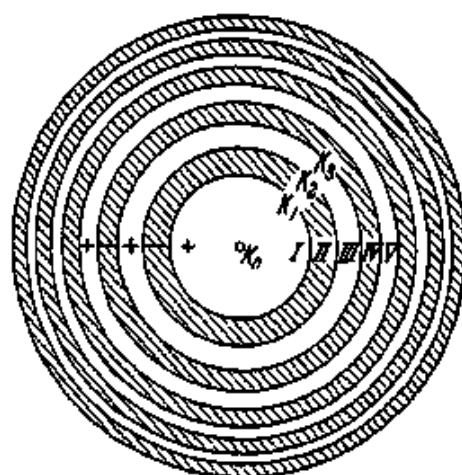


Fig. 65.
The Fresnel zones.

We now compare this process with our formula (19) where we replace a by x_n . This formula yields a *maximum* for

$$\frac{2\pi x_n^2}{\lambda 2\rho} = n \frac{\pi}{2} \quad \text{for odd } n,$$

hence for

$$(21 a) \quad x_n = \sqrt{\frac{n \lambda \rho}{2}}.$$

This result agrees with (21) because in (19) we had assumed that $\rho = \rho'$, and therefore in our present notation $a = b = \rho$ and hence $f = \rho/2$. The same result is obtained for the *minima* if n is even. The fact that all the maxima have the same value and that all the minima are zero can also be checked, though not without some arbitrariness, by summing the contributions of successive zones.

The following consequence was considered particularly surprising in Fresnel's time: a diaphragm which consists only of the central zone yields the same intensity as a very large opening; that is, it gives the full intensity of the incident light. If the opening does not coincide with any of the circles K_n , or if its shape is not circular, then the contributions from partial zones must, of course, be taken into account.

Figure 65 also shows that a "zone plate" (J. L. Soret, 1875) acts like a lens. In order to illustrate this the negative zones have been shaded. If these negative zones are covered up or blackened, then all the remaining positive zones act to reinforce one another and produce an intensity which is four times as large as the incident intensity. This resulting zone plate has the focal length f . Since f , like the zones themselves, depends on the wavelength, our "lens" has a strong "chromatic aberration". The submultiples f/n of f are also focal lengths.

E. THE SIMILARITY LAW OF DIFFRACTION

Let us compare two objects (openings or screens) which can be mapped into each other by a similarity transformation. We arrange the source and point of observation so that both objects contain the same number of zones and possible fractional zones. Then the diffraction patterns caused by the two objects will also be geometrically similar. According to eq. (21) the necessary and sufficient condition for this is that the dimensionless quantity

$$(22) \quad \frac{x}{\sqrt{\lambda f}} \quad (x = \text{an arbitrary linear dimension of the object})$$

shall have the same numerical value for both arrangements. This will be called the *similarity law of diffraction*.

It is often said that diffraction phenomena are noticeable only for very small objects. However, the similarity law says: the same diffraction phenomena observed with a small object are also observed with an object magnified by a similarity transformation, provided only that the distances of the source and the point of observation from the object are correspondingly magnified. To a magnification factor q of the linear dimensions of the object there corresponds a magnification factor q^2 of these distances. Conversely, if one wishes to observe the diffraction phenomena due to a large object at large distances in the laboratory where the distances are reduced by a factor q , then the dimensions of the object need be reduced only by a factor \sqrt{q} . On the basis of this law W. Arkadiew¹ performed a set of very

¹Physikal. ZS. Vol. 14, 1913, p. 832.

interesting model experiments. As an example let us consider the following macroscopic object: a dinner plate of ordinary size held by a hand. In a laboratory at Moscow a distance $a + b = 40$ m. between the light source and the photographic plate was available. At that distance the picture of the shadow (suitably reduced to the dimensions of the photographic plate) shows, of course, no diffraction pattern but corresponds to the shadow of geometrical optics.



Fig. 66 a, b, c.

Illustration of the similarity law of diffraction. Photographs by Arkadiew.

a) $a + b = 7$ km, b) $a + b = 29$ km, c) $a + b = 235$ km.

We now inquire about the appearance of the shadow at a distance $a + b = 7$ km. In order to discover this pattern in the laboratory, we must use the reduction factor

$$q = \frac{40}{7000}, \quad \sqrt{q} \sim \frac{1}{13},$$

where q applies to the distances a , b and hence also to f ; \sqrt{q} applies to all linear dimensions of the object. Arkadiew cut a model of the macroscopic object reduced by $1/13$ out of thin sheet metal. The photographic plate showed the image pictured in fig. 66 a: the plate has received a hole (Poisson spot) and a white edge; the wrist contains bright fringes; the sleeve below the wrist is fringed.

Pictures of the shadow for $a + b = 29$ km. and 235 km. are produced by models with the reduction factors

$$\sqrt{q} = \sqrt{\frac{40}{29,000}} \sim \frac{1}{27}, \quad \sqrt{q} = \sqrt{\frac{40}{235,000}} \sim \frac{1}{77}, \quad \text{respectively.}$$

In fig. 66 b the whole arm contains diffraction fringes. Figure 66 c shows only slight similarity to the original: the Poisson spot in the center of the plate has become enlarged and a second bright spot has appeared in the sleeve.

36. Fraunhofer Diffraction by Rectangles and Circles

To observe Fraunhofer diffraction (fig. 58) one looks through a diffracting aperture at an infinitely far removed light source with the aid of a telescope which is focused at infinity. As shown in fig. 58 such a source can be realized by placing a point source or an illuminated slit in the focal plane of a collimator lens. The position of the diffraction opening with respect to lens and telescope is in principle immaterial. However, in practice the opening is placed directly in front of the objective of the telescope so that waves which are diffracted at large angles will also enter the telescope. The eyepiece is focused on the focal plane of the objective (E in fig. 58). Every point P on this plane corresponds to a plane wave emerging from the diffraction opening. A corresponding plane wave enters the eye through the eye piece. (As was remarked in connection with fig. 58, the visual observer may be replaced by a photographic plate in the focal plane E .)

Because all ray bundles entering and leaving the opening are parallel, we must set $R = R' = \infty$ in eq. (34.13). As has already been noted in connection with (34.14 a), the phase Φ then reduces to the linear expression

$$(1) \quad \Phi = a\xi + b\eta, \quad a = \alpha - \alpha_0, \quad b = \beta - \beta_0$$

and the evaluation of the integral in (34.14) becomes elementary.

Let ξ, η be the cartesian coordinates of an arbitrary point in the diffraction opening (which we shall assume to be plane as before); α, β, γ are the direction cosines of a diffracted bundle of rays; $\alpha_0, \beta_0, \gamma_0$ are the direction cosines of the incoming rays which are all parallel because of the collimator lens. If, in particular, the light source lies in the direction normal to the plane of the opening, then $\alpha_0 = \beta_0 = 0, \gamma_0 = 1$.

A. DIFFRACTION BY A RECTANGLE

Let the sides of the rectangle be $2A$ and $2B$ in length. The coordinates of the center shall be $\xi = 0, \eta = 0$. Then we can write for (34.14)

$$(2) \quad v = Ck \int_{-A}^{+A} e^{-ik a \xi} d\xi \int_{-B}^{+B} e^{-ik b \eta} d\eta,$$

where C is a complex constant which is proportional to the amplitude of the incident light and is independent of the angle between the central ray and the direction of observation. The factor k outside the integral corresponds to the factor λ on the left-hand side of (34.14), which has now been transferred to the right. Performing the integrations we obtain

$$(3) \quad v = Ck\Delta \frac{\sin x}{x} \frac{\sin y}{y}, \quad \begin{cases} \Delta = 4AB = \text{Area of the rectangle,} \\ x = kaA, \quad y = kbB. \end{cases}$$

From this we obtain for the intensity $J = |v|^2$

$$(3a) \quad \frac{J}{J_0} = \left(\frac{\sin x}{x} \right)^2 \left(\frac{\sin y}{y} \right)^2.$$

As is evident from (3), $J_0 = (C h \Delta)^2$ is the intensity at the center $a = 0$, $b = 0$ of the diffraction pattern. At that point also $x = 0$, $y = 0$.

The behavior of the function

$$(4) \quad X = \left(\frac{\sin x}{x} \right)^2$$

is well known: its principal maximum lies at $x = 0$ and has the value 1. The minima of magnitude $X = 0$ are located equidistantly at the points $x = \pm \pi$, $\pm 2\pi$, $\pm 3\pi$, ... There are subsidiary maxima at the points where $\tan x = x$, that is, at points which approach $\pm 3/2\pi$, $\pm 5/2\pi$, ... more and more closely as x increases. The values of the subsidiary maxima are in the same order

$$X = 0.047, 0.017, 0.008, \dots$$

(4a)

Measured in terms of x the distance between the principal maximum and the first minimum is equal to π . We shall now express this distance in terms of the angular measure $a = \alpha - \alpha_0$. From eq. (3) we obtain

$$(5) \quad \pi = k a A, \quad \text{hence} \quad a = \frac{\lambda}{2A}.$$

The smaller the side $2A$ of the rectangle is, the larger does the angular distance a become. The same is, of course, true for b and B .

At the left of fig. 67 we have drawn a diffraction opening in the shape of an upright rectangle, $2A < 2B$. The diffraction pattern on the right is subdivided into rectangles which are geometrically similar to the above but which are in an oblong position. Four of these elementary rectangles make up the rectangular field of the central principal maximum, which is bounded by lines of zero intensity; two each belong to the fields which are

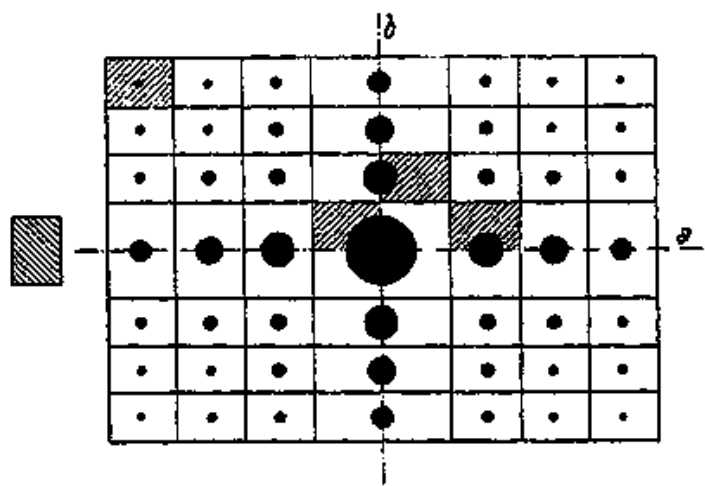


Fig. 67.

Distribution of light resulting from diffraction by a rectangle. The maxima lying between the lines of minimum intensity are symbolized by black circles. Actually they would look more like rectangles than like circles. The shape and orientation of the diffraction opening is shown at the left (not to the same scale, because the dimensions of the diffraction pattern and the opening are not commensurable).

bisected by the axes a and b (these axes are shown dotted because they do not form part of the system of lines of zero intensity). A single such rectangle is also indicated in the upper left-hand corner bounded by the two systems of equidistant minimum lines whose spacings are π in the scales both along the x and y directions. In accordance with the equation $\tan x = x$ the subsidiary maxima lie approximately in the centers of their respective fields.

The principal maximum exceeds by far all subsidiary maxima in intensity. It forms an extended intensity maximum in the center of the diffraction pattern. The ratios of the intensities of the subsidiary maxima on the axes a and b to the principal maximum are given by the sequence (4 a). The remaining subsidiary maxima are usually barely visible because the ratios of their intensities to that of the principal maximum are given by products of the already small numbers in (4 a).

B. DIFFRACTION BY A SLIT

Our rectangle $2A, 2B$ becomes a slit if we increase B until it is very much larger than A . As we increase B , the diffraction pattern parallel to the b -axis will contract more and more. We shall assume that the light source is a distant luminous line whose separate line elements emit *incoherent* light. Therefore, we shall have to add the *intensities* resulting from different line elements. Since the direction of the incoming rays is given by α_0, β_0 , we shall have to perform an integration with respect to $b = \beta - \beta_0$ between certain limits $\pm b_1$ which correspond to the length of the collimator slit. Hence, according to (3), we must write

$$\int_{-b_1}^{+b_1} \left(\frac{\sin y}{y} \right)^2 db = \frac{1}{k B} \int_{-\gamma_1}^{+\gamma_1} \left(\frac{\sin y}{y} \right)^2 dy \quad \text{where} \quad \gamma_1 = k b_1 B.$$

Because of the large value of k , the limits of integration $\pm \gamma_1$ may be treated as very large numbers, even though the values of B and b are experimentally limited. Therefore, except for terms which vanish as $1/\gamma_1$, the above integral can be replaced by¹

$$\int_{-\infty}^{\infty} \left(\frac{\sin y}{y} \right)^2 dy = \pi.$$

¹ This value is most easily obtained by the method of complex integration; see Vol. VI, exercise I.5, where Dirichlet's discontinuous factor $\int (\sin y/y) dy$ is treated by this method.

We see, therefore, that the intensity of the diffraction pattern is only a function of $x = k a A$ and is identical with the expression X in (4) except for a constant factor. Thus the slit likewise produces a principal maximum at $x = 0$ and almost equidistant subsidiary maxima, the intensities of which are scarcely noticable in comparison with the intensity of the principal maximum.

We shall now use this result to fill in a gap in Sec. 32. In eq. (5) of that paragraph we separated the intensity of the grating spectrum into two factors, the second of which was calculated from the sequence of grating lines. The first factor $f^2(\alpha)$ which resulted from the width and shape of each individual grating line was left undetermined. At least in certain very simple cases this factor is given just by our expression X in (4); by definition of x , X is clearly a function of $a = \alpha - \alpha_0$. We shall now investigate the influence of this factor on the intensity distribution of a grating (we may now set the intensity of the incident light equal to 1).

For this purpose we write down the more complete form of eq. (32.5):

$$(6) \quad J = \frac{\sin^2 x}{x^2} \frac{\sin^2 N \frac{\Delta}{2}}{\sin^2 \frac{\Delta}{2}} \begin{cases} x = \frac{2\pi a A}{\lambda} \\ \Delta = \frac{2\pi a d}{\lambda} \end{cases}$$

The quantity $2A$, which above was the width of the slit, is now the width of each individual grating line; d is again the spacing of the grating lines. In the gratings which Fraunhofer originally made, d was very large compared to $2A$. If this is so, then according to (6) x increases only by the small quantity A/d while Δ changes by 1, and the first factor on the right-hand side of eq. (6) varies slowly compared to the second factor. As was mentioned at the end of Sec. 32 A , this first factor has the effect of weakening the grating spectra of higher orders in comparison with the first order spectrum. The intensity pattern given by the second factor in (6) as represented in fig. 53 remains qualitatively unchanged.

Just as the results on diffraction of a slit serve to complete our previous theory of the line grating, so the results on diffraction of a rectangle as given by eq. (3) yield the function $f(\alpha, \beta)$ which was left undetermined in the theory of the cross grating, eq. (32.8).

C. THE CIRCULAR APERTURE

The circular opening is obviously of tremendous importance to the theory of the telescope, the microscope, and the photographic lens, as well as to the process of vision.

Clearly, we must introduce polar coordinates to replace the rectangular coordinates ξ, η and a, b of (1). We set

$$\begin{aligned}\xi &= r \cos \varphi, & a &= s \cos \psi, \\ \eta &= r \sin \varphi, & b &= s \sin \psi.\end{aligned}$$

r is the distance from the center of the opening; s is the sine of the deflection angle between the diffracted ray and the perpendicularly incident ray. Denoting the radius of the aperture by the, again available, letter a , we obtain instead of (2)

$$(7) \quad v = C k \int_0^a r dr \int_{-\pi}^{+\pi} e^{-i k r s \cos(\varphi - \psi)} d\varphi.$$

The φ integral cannot be evaluated by elementary methods, but it is well known to us from Vol. II, Sec. 27 and Vol. III, Sec. 22 as the Bessel function J_0 . For further details see Vol. VI, Chap. IV. We recall here the formulae

$$\begin{aligned}(8) \quad J_0(\rho) &= 1 - \frac{1}{(1!)^2} \left(\frac{\rho}{2}\right)^2 + \frac{1}{(2!)^2} \left(\frac{\rho}{2}\right)^4 - \frac{1}{(3!)^2} \left(\frac{\rho}{2}\right)^6 + \dots \\ &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{\pm i \rho \cos \alpha} d\alpha,\end{aligned}$$

$$(8a) \quad J_1(\rho) = \frac{\rho}{2} \left(1 - \frac{1}{1!2!} \left(\frac{\rho}{2}\right)^2 + \frac{1}{2!3!} \left(\frac{\rho}{2}\right)^4 - \dots \right) = -\frac{d}{d\rho} J_0(\rho)$$

and the differential equation

$$(8b) \quad \frac{d}{d\rho} \left(\rho \frac{dJ_0}{d\rho} \right) + \rho J_0 = 0$$

from which the following relation is obtained:

$$(8c) \quad \int_0^\rho \rho' J_0(\rho') d\rho' = \rho J_1(\rho).$$

We are also acquainted with the asymptotic representations for large ρ

$$(8d) \quad J_0(\rho) = \sqrt{\frac{2}{\pi\rho}} \cos\left(\rho - \frac{\pi}{4}\right), \quad J_1(\rho) = \sqrt{\frac{2}{\pi\rho}} \sin\left(\rho - \frac{\pi}{4}\right).$$

Using these results, eq. (7) can be written simply as

$$(9) \quad v = 2\pi C k \int_0^a J_0(k r s) r dr = \frac{2\pi C k}{k^2 s^2} \int_0^{k s a} J_0(\rho') \rho' d\rho' \\ = \frac{2\pi C a}{s} J_1(k s a).$$

For $s = 0$ (center of the diffraction pattern, $\alpha = \alpha_0$, $\beta = \beta_0$) (8 a) yields

$$(10) \quad v = \pi a^2 C k.$$

The zeros of v are given by the zeros of $J_1(\rho)$. The first of these is at

$$(11) \quad \rho_1 = 3.95 = 0.61 \times 2\pi, \quad s_1 = 0.61 \frac{\lambda}{a}.$$

This zero and all the more so the following zeros ρ_2, ρ_3, \dots are given with sufficient precision by the asymptotic formula (8 d):

$$\sin\left(\rho - \frac{\pi}{4}\right) = 0, \quad \rho_n = \left(n + \frac{1}{4}\right)\pi, \\ (11 a) \quad s_n = \left(n + \frac{1}{4}\right) \frac{\lambda}{2a}.$$

The corresponding graph of v is shown in fig. 68. The resulting intensity pattern $|v|^2$ has again a towering maximum at the center which is surrounded by almost equidistant dark rings. Between the dark rings are weaker maxima which rapidly decrease in intensity.

This central intensity maximum which is bounded by the first zero-ring determines the size of the central field produced by the droplets of Sec. 33. Indeed, we used the above expression (11) to calculate the sizes of the coronae about the sun and moon. Because of their weak intensities, the outer maxima indicated by fig. 68 generally do not affect these results.

In Chap. VI we shall discuss the fundamental importance of eq. (11) for the theory of the microscope.

In principle it is not difficult to predict the diffraction patterns of other, particularly polygonally bounded, openings. This was first done by Schwerd¹ in an exemplary fashion.

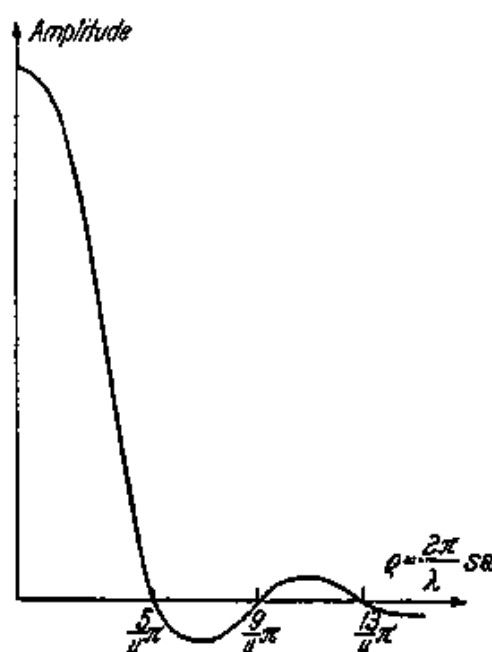


Fig. 68.

The amplitude v behind a circular opening as a function of $k s a$.

¹F. M. Schwerd, Die Beugungserscheinungen aus den Fundamentalgesetzen der Undulationstheorie analytisch entwickelt, Mannheim 1835. Schwerd was a high school teacher in Speyer. He painstakingly colored all the figures in the whole edition of his book by hand.

D. PHASE GRATINGS

In formula (32.3) we assumed the grating lines to be linear light sources which radiated in all directions when excited by an incident wave. The nonuniform distribution of this radiation for the various directions α was taken into account by the function $f(\alpha)$ which remained undetermined there. Huygens' principle has now enabled us to calculate this function [eq. (6)] for slit openings of arbitrary widths by setting the field excitation in

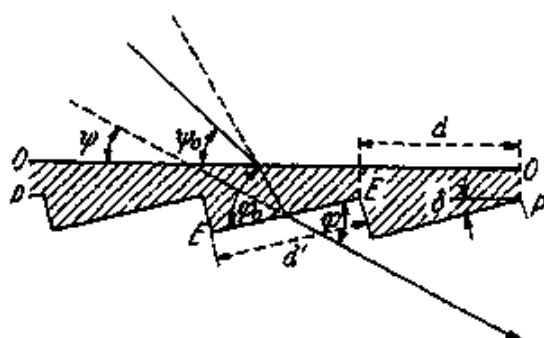


Fig. 69.

Step or echelette grating which has been ruled on the lower side of a plane glass plate. The figure shows the ray incident on the upper surface OO' , the refracted ray in the glass plate, and the wave which has been diffracted at the angle ϕ with respect to EE' .

the opening equal to the unperturbed incident wave. In this way it has been possible to determine the diffraction field of wire gratings or of gratings which are ruled on a silver layer deposited on a glass plate. Fraunhofer produced such gratings which shall be called "*amplitude gratings*", because for perpendicular incidence the phase is constant over the plane of the grating — the plane over which the integration is to be extended, according to Huygens' principle. On this plane only the amplitude varies between zero at points which are on the metal and some constant value at points on the glass. The situation is different with the modern very closely

ruled gratings. In these, groove follows upon groove in such a way that one cannot speak of a plane surface. These gratings are illuminated fairly *uniformly*, that is, with essentially constant amplitude, over their whole extent. But the *phase varies* because the points of the grating surface penetrate to varying depths into the optical field in the medium of different index of refraction. A plane wave-surface which is perpendicularly incident on the grating meets these different points at *different* times. Therefore, different points on the grating surface *radiate their elementary Huygens' waves with different phases*. Devices of this type are called "*phase gratings*". The general grating properties derived in Sec. 32 remain unchanged for phase gratings, but the directional distribution of the radiation from any one grating element, that is, the function $f(\alpha)$, can be affected in many different ways. It is, for instance, possible to divert the major part of the incident energy into a single spectrum of a given order on one side and to suppress almost completely all other spectra, in particular that of zeroth order.

We shall now calculate the function $f(\alpha)$ for several shapes of grating surfaces. Since the *surfaces are no longer plane*, and we therefore do not know their Green's functions, we are forced to use Huygens' principle in the old Kirchhoff formulation of Sec. 34 B.

Kirchhoff's assumption that the incident wave proceeds *unperturbed* up to the surface of the grating limits our calculations to *large grating constants* $d \gg \lambda$, to grating elements which are *not too deeply cut*, and to moderate angles of incidence and diffraction. Otherwise the radiation proceeding from one part of each grating element to another would affect the results of the calculation. In order to be able to apply previous results, we shall also limit ourselves to grating elements which are bounded by *plane* surfaces.

First we consider a step profile PP which has been cut into the lower surface of a plane glass plate; see fig. 69. OO shall be the upper surface of the glass plate; EE is the grating element under consideration. We illuminate the plate from above and observe the light below the plate. Just as Kirchhoff used the unperturbed *unbounded* wave as an approximation for the wave in the bounded diffraction opening, so we must use an unperturbed plane wave emerging from an unbounded glass plane to approximate the wave originating from our closely bounded plane surface EE (of width d'). As a result we shall use in the Kirchhoff formula (34.4) the values v and $\partial v / \partial n$ of the *refracted* wave emerging from the glass plate. The ray which is refracted upon incidence on the upper surface OO of the glass plate determines the angle of incidence φ_0 on the step EE . The angle of diffraction with respect to the step surface shall be φ . We call the direction cosines of the incident and diffracted waves

$$a_0 = \cos \varphi_0, \quad a = \cos \varphi,$$

and φ_1 the angle at which the refracted ray (not drawn in the figure) would emerge if the plane EE were infinite: $\cos \varphi_1 = n a_0$.

Referring to our very first treatment of the problem of refraction in Sec. 3, we rewrite eq. (3.1a) in our present notation $\frac{\cos}{\sin} \varphi_1$ in place of $\frac{\sin}{\cos} \beta$; because the ray emerges into air, we write k instead of k_2 ; the x -axis lies now in the plane EE ; hence $y = 0$ on EE :

$$E = B e^{i k (x \cos \varphi_1 - y \sin \varphi_1)}.$$

Identifying v and $\partial v / \partial n$ with E and $\partial E / \partial y$, we obtain at $y = 0$

$$v = B e^{i k n a_0 x} \quad \text{and} \quad \frac{\partial v}{\partial n} = -\sin \varphi_1 i k v.$$

The other wave function u which occurs in Green's theorem as a "probe" can be written for the Fraunhofer mode of observation [limit as $r \rightarrow \infty$ in (34.1)] as

$$u = e^{-ikax} \quad \text{and} \quad \frac{\partial u}{\partial n} = + \sin \varphi i k u.$$

Substituting this in (34.4), one obtains for the relative amplitude distribution at infinity

$$\begin{aligned} f(a) &= 4\pi v_P = -ikB(\sin \varphi_1 + \sin \varphi) \int_{-\frac{d'}{2}}^{+\frac{d'}{2}} e^{ik(na_0 - a)x} dx \\ &= ikd'B(\sin \varphi_1 + \sin \varphi) S, \quad S = \frac{\sin \left\{ k(na_0 - a) \frac{d'}{2} \right\}}{k(na_0 - a) \frac{d'}{2}}. \end{aligned}$$

The factor in front of the sine quotient S is slowly varying and causes a moderate attenuation for large angles of diffraction; we can disregard this

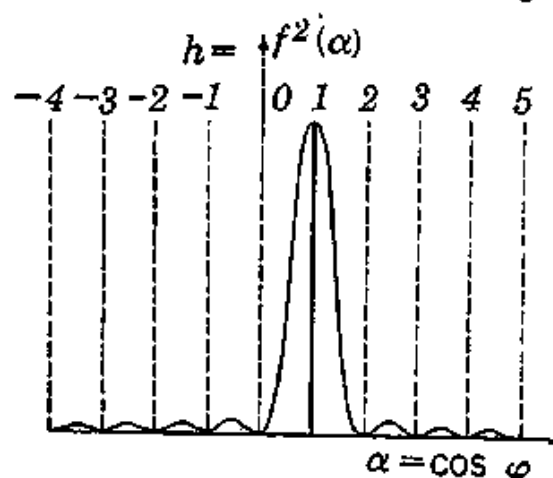


Fig. 69 a.

Intensity distribution for the step grating shown in fig. 69. The curve represents the diffraction pattern due to of a single step. The ordinate at $h = 1$ also gives the intensity of the first order grating spectrum which is the only spectrum emitted by the step grating.

factor. The function $f(a)$ is, therefore, essentially given by S . The curve for S is similar to that of diffraction by a slit (36.6). Its principal maximum is at $a = na_0$ or, what is the same, at $\varphi = \varphi_1$, which is precisely the direction of the refracted ray as determined by geometrical optics. The zeros are arranged symmetrically around the principal maximum at

$$a = na_0 \pm \nu \frac{\lambda}{d'}, \quad \nu = \text{integer}.$$

On the other hand, let us now consider the grating spectrum which is produced according to (32.4) by the regular sequence of such grating elements at a spacing d . With the definition of d given in fig. 69 we must write in (32.1)

$$\alpha = \cos \varphi = \cos(\varphi - \delta) \quad \text{and} \quad \alpha_0 = \cos \varphi_0 = n \cos(\varphi_0 - \delta).$$

The grating maxima (which, because of the large number N of grating elements, are very sharp) are at the positions $\Delta/2 = 0 \pm h\pi$, thus at $\alpha = \alpha_0 \pm h\lambda/d$.

Their amplitudes are given by the function $f(\alpha)$, which differs from $f(a)$ only because the origin of the angles φ, φ_0 is shifted by δ with respect to the origin of ψ and ψ_0 . First we see that for $d \sim d'$, that is, for small angles ψ_0, ψ , and δ , the spacing of these spectra is the same as the spacing of the zeros of $f(\alpha)$. Therefore, if by suitable choice of δ (or if δ is given by suitable choice of ψ_0), one causes the first order maximum, $h = +1$, for instance, to coincide with the principal maximum of $f(\alpha)$, then all other grating spectra coincide with zeros of $f(\alpha)$ and are completely suppressed, including the zero order spectrum. This is illustrated in fig. 69 a

where the *intensity* of the spectra, that is, the *square* of $f(\alpha)$, is plotted.

A reflection grating which is ruled on metal can also be treated by means of the above formulae if we formally set $n = -1$. Then the principal maximum lies, independently of wavelength, in the direction of the geometrically

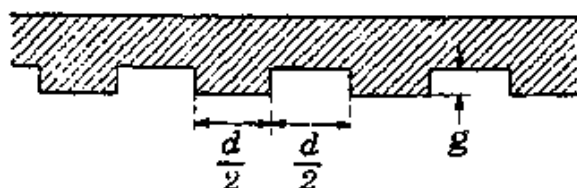


Fig. 70.

A so-called "laminary profile" with grating constant d and depth of slits g .

reflected ray. Such gratings were used for the analysis of long infrared waves for which no suitable refracting material is available. Their grating constants amount to fractions of a millimeter. With these spacings the desired step profile is quite easily attainable; these are the so-called echelette gratings. Even in gratings suitable for visible light it is possible to favor one order quite strongly over all others by using a cutting diamond of suitable shape.

As a second example we choose the *rectangular laminary profile*, fig. 70. This profile is produced by evaporating a transparent substance on a plane plate and ruling regular slits into the layer, so that the deposited substance is removed along equidistantly spaced lines. Let us call the thickness of the layer g and its index of refraction n . Then for small angles of incidence the wave falling on *half* of the grating element d is retarded by $2\Theta = (n-1)gk$. At a large distance the amplitude distribution $f(a)$ arising from a grating element extending from $-d/2$ to $+d/2$ is given by

$$f(a) = \frac{1}{d} \int_{-\frac{d}{2}}^0 e^{i(\Phi x - \Theta)} dx + \frac{1}{d} \int_0^{+\frac{d}{2}} e^{i(\Phi x + \Theta)} dx = \frac{\sin\left(\Phi \frac{d}{2} + \Theta\right) - \sin \Theta}{\Phi \frac{d}{2}}$$

where $\Phi = k(n\alpha_0 - \alpha)$, and where all nonessential terms have been omitted. This is the diffraction pattern of a *single* step element. Because the step elements are parallel, see fig. 70, the distinction between a, a_0 and α, α_0 becomes meaningless. Therefore we shall henceforth write $f(\alpha)$ in place of the

above $f(\alpha)$. The asymmetry of this function with respect to the direction $\Phi = 0$ averages out in the spectrum produced by the whole grating; for, with perpendicular incidence, the directions $\Phi d/2 = \pm h\pi$ which go with the two spectra of equal order ($\pm h$) always produce the same contribution

$$f(\alpha_{\pm h}) = \frac{1}{\pi h} [(-1)^h - 1] \sin \Theta.$$

(For $h = 0$ the limit as $\Phi \rightarrow 0$ must be taken which yields $f(\alpha) = \cos \Theta$.) The curve $f^2(\alpha)$ is shown in fig. 70 a, and from it the intensities of the grating spectra of different orders can be determined.

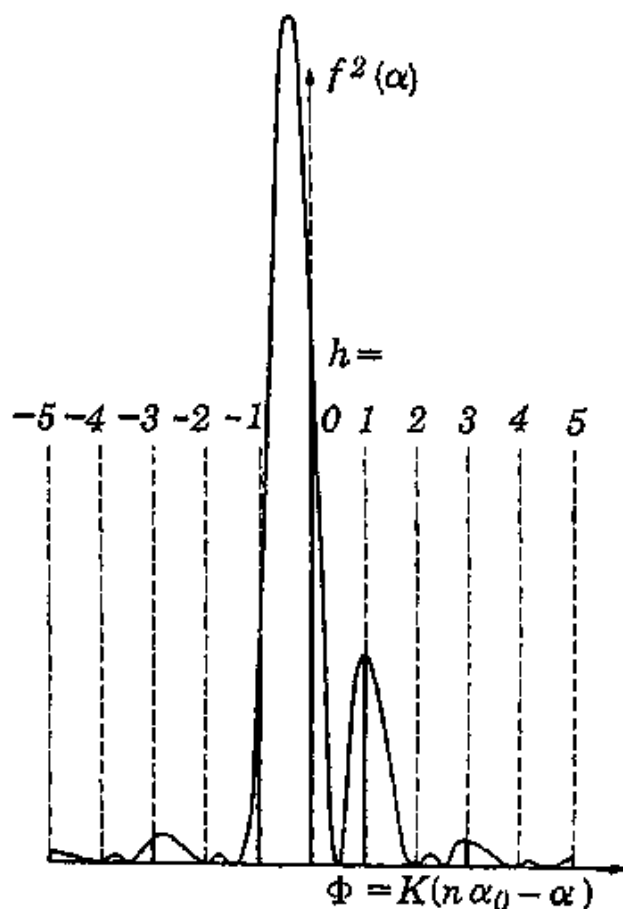


Fig. 70 a.

Intensity distribution for the laminary grating shown in fig. 70. The curve represents the diffraction pattern due to a single grating element. The heavily drawn ordinates indicate the intensities of the grating spectra of orders h which are emitted by the entire array of grating elements.

increasing or decreasing the phase difference between the zeroth order spectrum and all higher order spectra by $\pi/2$.

This method can be used to replace the formerly very important staining process which used to be necessary in the microscopic observation of trans-

Because of its application to the microscope, a similarly designed *amplitude grating* is also of interest. Such a grating is obtained if, instead of a transparent material, an absorbing metallic layer is applied to the plane base. Mathematically this means that we must set $\Theta = i\Theta'$ (Θ' is real). The ratio of the transmissivities of the two halves of a grating element is then $e^{4\Theta'}$, and Θ' is directly proportional to the thickness of the absorbing layer. Thus we obtain for the zeroth order spectrum $f(\alpha) = \cosh \Theta'$, and for the higher order spectra $f(\alpha_{\pm h}) = [(-1)^h - 1] \cdot \frac{i}{\pi h} \sinh \Theta'$. The factor i in this equation means that the light in the higher order spectra differs in phase by $\pi/2$ from the zeroth order spectrum. Therefore, the diffraction pattern of a phase grating can be changed into that of an amplitude grating simply by

parent tissues whose constituents differ by very little in index of refraction but absorb, owing to their chemical difference, different amounts of dye. It is seen that actually the staining method also amounts to *changing a phase grating into an amplitude grating*.

E. SUPPLEMENT TO SECTION 35 B. LIGHT FANS ARISING FROM POLYGONALLY BOUNDED APERTURES

What is the relationship between the special results derived in A and B of this section and the general theory of the *shadow* which was discussed in Sec. 35 B? In order to answer this question we must first of all specialize eq. (35.12) to the present case of Fraunhofer observation with perpendicular incidence. In that case the point D in fig. 60a lies infinitely far away (if we disregard the center point $\alpha = \beta = 0$). The former surfaces of constant phase are now planes, and the former intersection ellipses have become a family of parallel straight lines. The former parameter p of the system of ellipses is now proportional to the spacing of these straight lines as measured from one of them, for instance, the one passing through the center of the diffraction opening.

Let us first consider a *rectangular* opening and determine the shadow boundary in the direction perpendicular to one of the sides of the rectangle. The straight lines $p = \text{constant}$ are parallel to this side, and therefore the segments cut out of these lines by the rectangle are all equal. Hence the fraction $\varphi(p)$ which was introduced in fig. 60 a becomes independent of p . The same is true of the function $F(p)$ occurring in (35.12). Thus $F'(p) = 0$ and the second term on the right-hand side of (35.12 a) vanishes. If, for convenience, we normalize p so that $p_2 = -p_1 = \bar{p}$, then the first term yields

$$(12) \quad \frac{F}{i k} (e^{i k \bar{p}} - e^{-i k \bar{p}}) = \frac{\lambda F}{\pi} \sin k \bar{p}.$$

Eq. (35.12) (the factor λ on both sides cancels) leads to the result that in the direction defined above a *light fan of finite intensity* is radiated and *no shadow* appears. It should be emphasized that the sinusoidal variations in the amplitude which are indicated by (12) are blurred out if the light source is extended and not monochromatic. Our name "light fan" is descriptive of this fact. The same is obviously true for the light fans in the directions perpendicular to the other sides of the rectangle; it is *not* true, however, in any other direction which is inclined with respect to these sides. For such directions the values of p_1 and p_2 in (32.12 a) correspond to the corners of the rectangle at which the fraction $\varphi(p)$ continuously decreases to zero. Now the first term on the right-hand side of (32.12 a) vanishes; the second term has

a finite but small value. This is illustrated in our (very schematic) fig. 71 a by the quadrants which are left unhatched; these quadrants indicate *shadows*. In the singly hatched strips a finite part of the boundary of the diffraction opening coincides with an effective wave zone, and therefore a *noticeable intensity of diffracted light* is present. This situation is similar to that of the circular opening in fig. 68. Figure 71a should be compared with fig. 67 where the same results are expressed more precisely for monochromatic light.

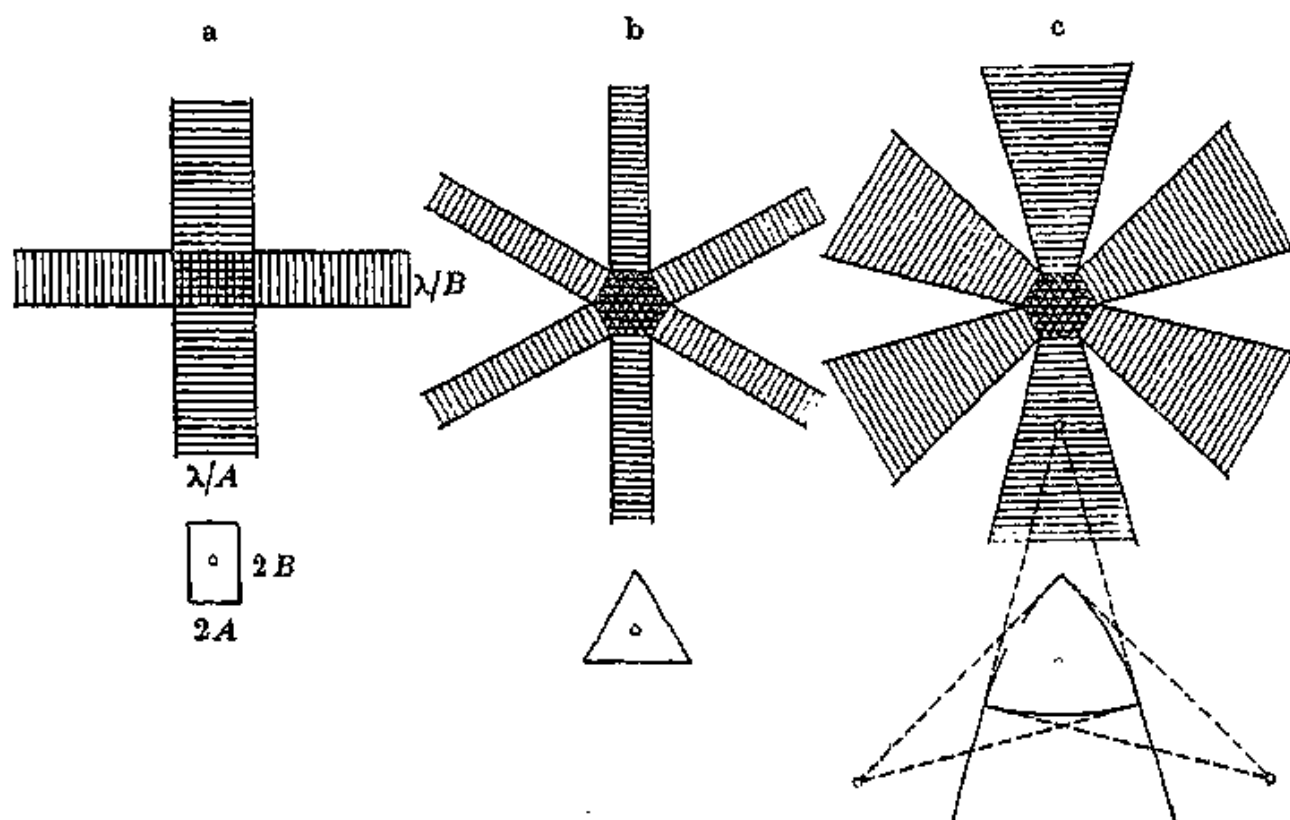


Fig. 71.

Light fans for polygonal apertures (indicated as shaded regions). In each case the apertures themselves are indicated underneath.
a) rectangle. b) triangle. c) curvilinear triangle.

These results are independent of wavelength; this is true in particular for the intensities along the axes *a*, *b* in our figure. This seems to be in contradiction to geometrical optics, i. e. the limiting case as $\lambda \rightarrow 0$, according to which only the central field should be illuminated. The apparent contradiction is resolved by the fact that in this limiting case our light fans become infinitely narrow. This is indicated by the widths λ/A and λ/B which are taken from eq. (5) and are shown in the figure. Therefore in the limit also the energy taken from the central field and radiated in these directions becomes infinitely small. The situation is here the same as in the case of the Poisson spot which also remains present in the limiting case of geometrical optics but is reduced in size to a geometrical point.

If the opening is a parallelogram, then the two bright strips are not perpendicular to each other but are perpendicular to the sides of the parallelogram. In the case of a triangular opening there are three strips which are perpendicular to the three sides of the triangle, thus there are six light fans altogether, as shown in fig. 71b. A rectilinear polygonal opening with n sides has in general $2n$ such ray directions.

If one looks at a light source through a small hole shaped like a parallelogram, and if the eye is not focused on the source (or the source is not sufficiently small), then the source appears as a star with four rays. Under the same conditions, a triangular opening yields a six-cornered star. An irregular opening usually yields a many-cornered star. It is to be noted that the diffraction pattern is very sensitive to small irregularities in the shape of the opening. It is because of small irregularities in the iris of the eye that a star seen in the night sky does not appear to be surrounded by circular rings, which would be the case with an ideal circular aperture; rather, the stars appear to us stellate and thus they have traditionally been represented in the art of all ages. The five-cornered star which is preferred in heraldry and in depicting the Christmas star is, incidentally, wave-optically impossible because the light fans must, necessarily, occur in pairs.

So far we have limited ourselves to diffraction openings which are bounded by straight lines. What happens with curved boundaries? To treat these we must again return to fig. 60a, but for Fraunhofer diffraction we must replace the elliptic arcs in that figure by a system of parallel straight lines. For every position of the (infinitely distant) point D , there are two straight lines p_1, p_2 of the system which are tangent to the edge of the diffraction opening. Thus there are now no *finite* line segments along which the edge and a wave front coincide, as there were in the case of rectilinear edges. With curved openings there are only infinitesimal points of coincidence, i. e. points of *tangency*. For this reason the intensities of the light fans are of a smaller order of magnitude than those produced by piecewise rectilinear edges. Let us estimate this order of magnitude.

As was noted following eq. (35.12) $\varphi(p_1) = \varphi(p_2) = 0$ at the points of tangency; therefore the first term on the right-hand side of (35.12 a) vanishes. In order to estimate the second term we replace the boundary curve at the point of tangency by its circle of curvature (radius ρ). We denote by 2ψ the central angles of the arcs cut out of this circle by the system of p -lines. We can choose the parameter p so that it measures the distance from the center of the circle; then $p = \rho \cos \psi$. The length of each chord is then

$$(13) \quad \varphi(p) = 2\rho \sin \psi = 2\sqrt{\rho^2 - p^2}$$

and therefore

$$(13a) \quad \varphi'(\rho) = -2\rho(\rho^2 - \rho^2)^{-\frac{1}{2}}.$$

At the point of tangency ($\rho = \rho$) $\varphi'(\rho)$ and therefore also $F'(\rho)$ become infinite, but slowly enough so that the integral in question remains finite. An approximate evaluation shows that the intensity is proportional to $\lambda\rho$ and thus vanishes in the limiting case of geometrical optics. On the other hand, we saw in (12) that for rectilinear edges the diffracted intensity is of the same order of magnitude as the intensity of the incident light. Though an opening with a curved edge also radiates diffracted light in a direction perpendicular to the tangent of the edge, its intensity is of a smaller order of magnitude than that of the diffracted light from a rectilinear opening. The diffracted intensity decreases as the curvature $1/\rho$ of the edge at the point in question increases. If we consider a curvilinear polygon¹, see fig. 71 c, then each corner E of the polygon has an adjoining shadow region; the intensities radiated in directions which are within these shadows are of the same order of magnitude as the intensities in the shadow regions of rectilinear polygons, and they decrease to zero with decreasing wavelength.

Summarizing and completing the quantitative relationships we can say: every diffracting aperture produces a diffraction pattern which fans out from the central image; this pattern consists of light fans separated by shadows. If the angular separation from the central image is denoted by the (dimensionless) number a , and if A is used to denote the length of one side of the diffraction opening in the case of a rectilinearly bounded opening, and the same letter A denotes the radius of curvature (formerly ρ) in the case of a curvilinear edge, then the diffracted intensity is

in the light fans of rectilinear edges	$\frac{A^2}{a^2}$	$\frac{\lambda^2}{a^2 A^2}$
in the light fans of curvilinear edges	$\frac{A \lambda}{a^3}$	$\frac{\lambda^3}{a^3 A^3}$
in the shadow regions of rectilinear or curvilinear edges	$\frac{\lambda^2}{a^4}$	$\frac{\lambda^4}{a^4 A^4}$

¹The diffraction opening in fig. 71c is a curvilinear triangle formed by three circular arcs. The three centers (centers of curvature) are indicated in the figure. The tangents which are drawn at the three corners are used to construct the boundaries of the light fans which appear in the upper drawing.

The expressions in the first column are the intensities expressed as fractions of the light energy incident per unit area of the opening; they have, therefore, the dimensions of a length squared. The expressions in the second column are the intensities relative to the intensity in the middle of the central image; they are, therefore, dimensionless.

We have neglected the interference fringes which traverse the diffraction image. As we have said before, these fringes are blurred if the light source is not a point or is not monochromatic.

37. Fresnel Diffraction by a Slit

The course to be followed in deriving the theory of pure Fresnel diffraction has already been indicated in Sec. 34 D. The procedure is somewhat cumbersome and does not always lead to its intended goal. The points in the opening are described by coordinates ξ, η ,

the origin of which lies at the point D at which the straight line connecting the light source P' to the point of observation P intersects the plane of the screen S ; see fig. 72. Then $\alpha = \alpha_0$, $\beta = \beta_0$, $\gamma = \gamma_0$ ($\alpha_0, \beta_0, \gamma_0 = \text{direction } P'D$; $\alpha, \beta, \gamma = \text{direction } DP$), and the linear terms in the expression (34.13) for the phase Φ vanish. This results, however, in the following difficulty: or very eccentric positions of the point P [indicated by (P) in

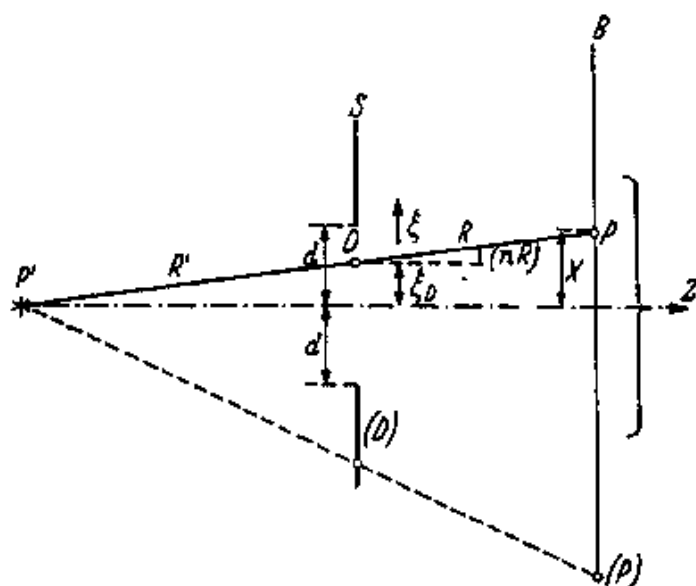


Fig. 72.

Fresnel diffraction of a slit.

fig. 72], D will lie outside the opening, which must be assumed to be small, and then the coordinates ξ, η are no longer small as required in the series expansion of Φ . We must therefore restrict the position of P to a region (indicated by the curly bracket in fig. 72) which does not extend too far into the geometrical shadow. Outside that region it is impossible to represent Φ by the quadratic terms alone. The pattern produced by the quadratic terms must then be supplemented by a diffraction pattern to be calculated in the Fraunhofer manner. A further inconvenience is that, even with this restriction on P , the position of D varies with that of P so that every separate position of P requires its separate coordinate system ξ, η .

The particular problem to be treated now is the diffraction pattern of a narrow rectangle (called a slit in the heading of this section). The plane of fig. 72 passes through the center of the rectangle and is parallel to its short side $2d$; the long side $2h$ is perpendicular to the plane of the drawing. The coordinates ξ and η are measured parallel to the sides of the rectangle. The observation screen which receives the diffraction pattern is parallel to the diffraction screen S ; both are perpendicular to the plane of the drawing. We shall limit ourselves to points of observation P which are in the plane of the drawing. The light source P' will be assumed to lie directly in front of the center of the rectangle. Then the line $P'P$ lies in the plane of the drawing, and since the γ -axis and also the η -axis are perpendicular to that plane, we have

$$(1) \quad \beta - \beta_0 = 0, \quad \alpha^2 = \alpha_0^2 = 1 - \gamma^2;$$

as indicated in the figure, γ is equal to $\cos(n, R)$, a fact which will be used in eq. (4). By means of (1), the expression (34.13) reduces to

$$(2) \quad \Phi = -\frac{1}{2} \left(\frac{1}{R} + \frac{1}{R'} \right) \{ \xi^2 + \eta^2 - \alpha^2 \xi^2 \} = -\frac{1}{2} \left(\frac{1}{R} + \frac{1}{R'} \right) (\gamma^2 \xi^2 + \eta^2).$$

Using the abbreviations

$$(3) \quad k\Phi = -\Phi_\xi \xi^2 - \Phi_\eta \eta^2, \quad \begin{cases} \Phi_\xi = \frac{1}{2} \gamma^2 k \left(\frac{1}{R} + \frac{1}{R'} \right) \\ \Phi_\eta = \frac{1}{2} k \left(\frac{1}{R} + \frac{1}{R'} \right) \end{cases}$$

eq. (34.14) becomes

$$(4) \quad i \lambda v_P = \frac{A \gamma}{R R'} e^{i k (R + R')} \int_{-d - \xi_D}^{+d - \xi_D} \exp(i \Phi_\xi \xi^2) d\xi \int_{-h}^{+h} \exp(i \Phi_\eta \eta^2) d\eta.$$

Let us remember the meanings of R and R' : R = distance DP , R' = distance DP' , where D , being the point where $P'P$ pierces the plane S , itself depends on the position of P ; according to (3) the abbreviations Φ_ξ , Φ_η therefore also depend on the position of P . This dependence also affects the limits of integration given in (4). Since ξ is to be measured not from the center of the slit but from the point D which has the coordinate ξ_D , these limits are not $\pm d$ but $\pm d - \xi_D$. Because of symmetry this dependence on P does not affect the η -integral in (4).

A. FRESNEL'S INTEGRALS

In order to conform to the historically established notation we consider the integral

$$(5) \quad F(w) = \int_0^w e^{i \frac{\pi}{2} \tau^2} d\tau.$$

We call it *Fresnel's integral*. Ordinarily this name is reserved for the two real integrals

$$(5a) \quad C(w) = \int_0^w \cos\left(\frac{\pi}{2} \tau^2\right) d\tau, \quad S(w) = \int_0^w \sin\left(\frac{\pi}{2} \tau^2\right) d\tau$$

which evidently form the real and imaginary parts of F :

$$(5b) \quad F = C + iS.$$

We wish to emphasize, however, that the separation of F into real and imaginary parts is absolutely of no advantage [we did not separate the plane wave $\exp(i k x)$ into a cosine and a sine either!]. The two integrals in (4) can be reduced to F by simple substitutions. One obtains

$$\begin{aligned} \int_{-d-\xi_D}^{+d-\xi_D} \exp(i \Phi_\xi \xi^2) d\xi &= \sqrt{\frac{\pi}{2\Phi_\xi}} [F(w_2) - F(w_1)], & \left. \begin{matrix} w_2 \\ w_1 \end{matrix} \right\} &= \sqrt{\frac{2\Phi_\xi}{\pi}} (\pm d - \xi_D) \\ \int_{-h}^{+h} \exp(i \Phi_\eta \eta^2) d\eta &= \sqrt{\frac{2\pi}{\Phi_\eta}} F(W), & W &= \sqrt{\frac{2\Phi_\eta}{\pi}} h. \end{aligned}$$

For a slit $h \gg d$ and $W \gg w_{2,1}$. We shall now convince ourselves that W may be set equal to infinity¹. To show this it is convenient to introduce the quasi-focal length f which was introduced in connection with the similarity law, Sec. 35 E. Thus, in our case we set

$$(6) \quad \frac{1}{f} = \frac{1}{R} + \frac{1}{R'}, \quad \Phi_\eta = \frac{\pi}{\lambda f}, \quad W = \sqrt{\frac{2}{\lambda f}} h \gg \sqrt{\frac{2}{\lambda f}} d.$$

In accordance with the similarity law, if the magnitude of d is just barely suitable for diffraction experiments, then W is so large that it does not produce

¹In the beginning of Subsection C we shall return to the question of the admissibility of such limit processes.

an appreciable diffraction effect of its own, and we can therefore go to the limit $\lambda \rightarrow 0$, $W \rightarrow \infty$ as in geometrical optics. Introducing at the same time the value of Φ_η from (6), we thus set

$$(6 \text{ a}) \quad \int_{-h}^{+h} e^{i\Phi_\eta \eta^2} d\eta = \sqrt{2\lambda f F(\infty)}.$$

Correspondingly, we find

$$(6 \text{ b}) \quad \int_{-d-\xi_D}^{+d-\xi_D} \exp(i\Phi_\xi \xi^2) d\xi = \frac{1}{\gamma} \sqrt{\frac{\lambda f}{2}} \{F(w_2) - F(w_1)\}, \quad \left. \begin{matrix} w_2 \\ w_1 \end{matrix} \right\} = \gamma \frac{\pm d - \xi_D}{\sqrt{\frac{\lambda f}{2}}}.$$

Hence, according to (4)

$$i v_P = f A \frac{e^{ik(R+R')}}{R R'} \{F(w_2) - F(w_1)\} F(\infty).$$

This expression simplifies if we immediately introduce the value $F(\infty) = \frac{1+i}{2}$, a result which will be derived later, and return to the original definition (6) of f . For then

$$(7) \quad v_P = \frac{1-i}{2} A \frac{e^{ik(R+R')}}{R+R'} \{F(w_2) - F(w_1)\},$$

where

$$A \frac{e^{ik(R+R')}}{R+R'} = v_0$$

is the optical field amplitude which would be observed at the point P if the intervening screen were removed entirely. Thus we can write for (7)

$$(7 \text{ a}) \quad v = \frac{1-i}{2} v_0 \{F(w_2) - F(w_1)\}.$$

If we disregard the first factor which is of no interest for the present, we can say:

The pattern produced on the screen B differs from the primary undiffracted field by a factor which is equal to the difference between the Fresnel integrals $F(w_2)$ and $F(w_1)$.

We can be brief in our description of the analytic properties of the function $F(w)$. They correspond entirely to those of the Gaussian error integral

$$F(x) = \int_0^x e^{-\tau^2} d\tau.$$

a) $F(w)$ is an entire transcendental function of w ; by its definition (5), $F(w)$ can therefore be expanded in the following series which converges everywhere in the finite plane

$$(8) \quad F(w) = w \left(1 + \frac{i}{1!} \frac{\pi}{3} \frac{w^2}{2} - \frac{1}{2!} \frac{1}{5} \left(\frac{\pi}{2} w^2 \right)^2 - \frac{i}{3!} \frac{1}{7} \left(\frac{\pi}{2} w^2 \right)^3 + \dots \right).$$

This expansion follows directly from the exponential series. From it one obtains the respective series for $C(w)$ and $S(w)$.

b) Of greater importance is the divergent (so-called asymptotic) series development of $F(w)$ which yields a sufficiently exact approximation of the function for large values of w , provided that only a limited number of the terms of this series are summed. We obtain this development by setting

$$F(w) = F(\infty) - \int_w^{\infty} e^{\frac{i\pi\tau^2}{2}} d\tau = F(\infty) - \int_w^{\infty} \frac{d}{d\tau} \left(e^{\frac{i\pi\tau^2}{2}} \right) \frac{d\tau}{i\pi\tau}.$$

Upon integrating by parts this becomes

$$\begin{aligned} F(w) &= F(\infty) + \frac{e^{\frac{i\pi}{2}w^2}}{i\pi w} - \int_w^{\infty} e^{\frac{i\pi}{2}\tau^2} \frac{d\tau}{i\pi\tau^2} \\ &= F(\infty) + \frac{e^{\frac{i\pi}{2}w^2}}{i\pi w} - \int_w^{\infty} \frac{d}{d\tau} \left(e^{\frac{i\pi}{2}\tau^2} \right) \frac{d\tau}{(i\pi)^2 \tau^3}, \end{aligned}$$

and continuing the process of integrating by parts we get

$$(8a) \quad F(w) = F(\infty) + \frac{e^{\frac{i\pi}{2}w^2}}{i\pi w} \left(1 + \frac{1}{i\pi w^2} + \frac{1 \cdot 3}{(i\pi w^2)^2} + \frac{1 \cdot 3 \cdot 5}{(i\pi w^2)^3} + \dots \right).$$

From this follow the respective asymptotic series for $C(w)$ and $S(w)$.

c) In order to calculate $F(\infty)$ we recall the well-known Laplace integral

$$\int_0^{\infty} e^{-\alpha\tau^2} d\tau = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}.$$

We need only set $\alpha = -\frac{i\pi}{2}$ to find

$$(8b) \quad F(\infty) = \frac{1}{2} \sqrt{\frac{2}{-i}} = \frac{1}{\sqrt{-2i}} = \frac{1}{1-i} = \frac{1+i}{2}.$$

This result can be checked by considering the integral in the complex plane of the variable τ which we shall, however, omit here.

B. DISCUSSION OF THE DIFFRACTION PATTERN

We now investigate the intensity extrema (maxima and minima) in the diffraction pattern. That is, we seek those points on the observation screen which under monochromatic illumination will correspond to bright and dark fringes. These points are defined by the condition $\frac{d|v|^2}{dx} = 0$, where x is the distance of a point on the observation screen from the center of the screen. According to fig. 72, x is related to the coordinate ξ_D which measures distance from the center on the diffraction screen S . Therefore we may discuss $\frac{d|v|^2}{d\xi_D} = 0$ instead of $\frac{d|v|^2}{dx} = 0$. Since ξ_D occurs only in the limits of integration w_2 and w_1 in eq. (6 b), and since $\frac{dw_2}{d\xi_D} = \frac{dw_1}{d\xi_D} = -\gamma \left(\frac{\lambda f}{2}\right)^{-1/2}$, the following condition for the extrema¹ results:

$$(9) \quad \frac{d}{d\xi_D} \{F(w_2) - F(w_1)\} = -\frac{\gamma}{\sqrt{\lambda f/2}} \{F'(w_2) - F'(w_1)\} = 0,$$

so that

$$\exp\left(\frac{i\pi}{2} w_2^2\right) = \exp\left(\frac{i\pi}{2} w_1^2\right),$$

$$(10) \quad \frac{\pi}{2} (w_2^2 - w_1^2) = -2\pi g, \quad (w_2 - w_1)(w_2 + w_1) = -4g$$

where g is a (positive or negative) integer. Now, according to (6 b) we have

$$(10a) \quad w_2 - w_1 = \frac{2\gamma d}{\sqrt{\lambda f/2}}, \quad w_2 + w_1 = \frac{-2\gamma \xi_D}{\sqrt{\lambda f/2}}.$$

From (10) and (10 a) follows

$$\frac{2\gamma^2 \xi_D d}{\lambda f} = g, \quad \xi_D = \frac{\lambda f g}{2\gamma^2 d},$$

and the distance between two successive extrema is

$$(10b) \quad \Delta \xi_D = \frac{\lambda f}{2\gamma^2 d}.$$

¹Using temporarily the abbreviation $f(x) = F(w_2) - F(w_1)$, then

$$|v|^2 = C f f^*, \quad C = \frac{1}{2} |v_0|^2, \quad \frac{d}{dx} |v|^2 = C \left(f \frac{df^*}{dx} + f^* \frac{df}{dx} \right)$$

where f^* is the complex conjugate of f . In eq. (9) we have satisfied the condition $df/dx = 0$, but at the same time also the condition $df^*/dx = 0$ is fulfilled (interchange of $+i$ with $-i$ and $-g$ with $+g$). Therefore the condition $d|v|^2/dx = 0$ is fulfilled as well. Hence eq. (9) is not only the extremal condition for the amplitude v but also for the intensity $|v|^2$.

The separation between extrema decreases with increasing d and increases with increasing λ and f . The same is true of the separation Δx of the fringes on the observation screen.

The discussion of the diffraction pattern can be well illustrated by means of *Cornu's spiral* which is constructed by the following mapping process:

We interpret $F = C + iS$ as a point on the complex F -plane, that is, as the point with the cartesian coordinates C and S . In addition we consider a complex w -plane in which, however, only the real axis is of interest. The equation $F = F(w)$ represents a conformal (angle-preserving) mapping of the w -plane onto the F -plane. The real axis of the w -plane, which is the only part of that plane which will enter into consideration, is mapped onto a certain curve in the F -plane. We claim that this mapping is length-preserving. For we have

$$(11) \quad \frac{dF}{dw} = e^{\frac{i\pi}{2}w^2}, \quad \text{hence} \quad \left| \frac{dF}{dw} \right| = 1, \quad |dF| = |dw|.$$

Hence the w -axis and the F -curve are mapped on each other without stretching. We already know three points of this map, see eqs. (8) and (8 b):

$$\begin{aligned} w = 0, \quad w = \infty, \quad w = -\infty, \\ F(0) = 0, \quad F(\infty) = \frac{1+i}{2}, \quad F(-\infty) = -\frac{1+i}{2}. \end{aligned}$$

The length of the F -curve between the two end points $F(\pm\infty)$ is infinite as is the length of the w -axis. The curve is symmetrical with respect to the origin of the F -plane; for by eq. (8)

$$F(-w) = -F(w).$$

The tangent at the origin is horizontal; the curve has an inflection point there; for according to (8), we have at $w = 0$

$$\frac{dF}{dw} = 1, \quad \frac{d^2F}{dw^2} = 0.$$

For $w = \pm\infty$ the direction of the tangent is indeterminate, according to (8 a). Asymptotically the curve approaches these points as a spiral. The entire curve is plotted in fig. 73.

Not only does this curve illustrate the whole range of values which F assumes (for real w), but at the same time it also represents all of the amplitude ratios $|v|/|v_0|$ in the diffraction pattern. For from (7 a)

$$(12) \quad \sqrt{2} |v|/|v_0| = |F(w_2) - F(w_1)|,$$

that is, the amplitude ratio (times $\sqrt{2}$) equals the *length of the chord* which connects the two points representing w_2 and w_1 on Cornu's spiral. By (10 a) the difference between the two w -values is

$$w_2 - w_1 = \frac{2\gamma d}{\sqrt{\lambda f/2}} = \text{const.}$$

hence it is independent of both ξ_D and of the coordinate x of the point of observation. $w_2 - w_1$ is a certain segment on the real w -axis. The *arc of Cornu's spiral* between the endpoints of our chord has this same constant length.

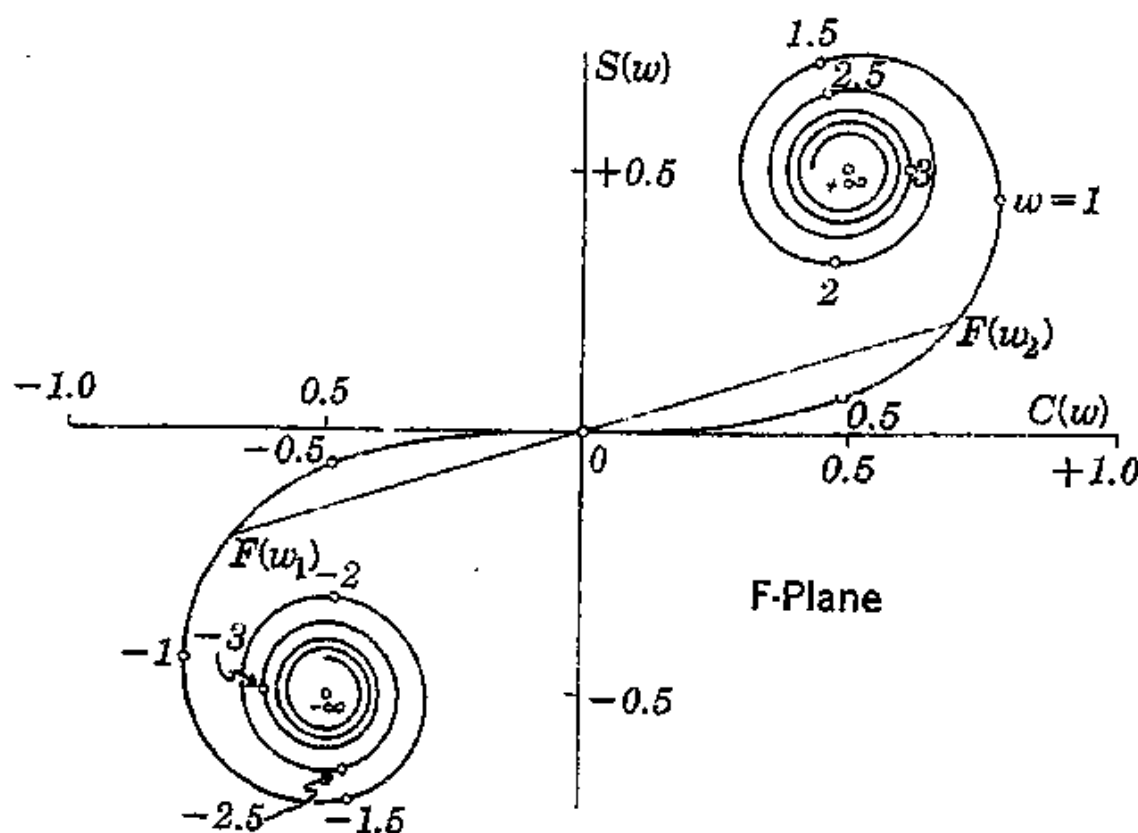


Fig. 73.
Cornu's Spiral.

In fig. 73 we have drawn the chord which corresponds to the point $x = 0$ on the diffraction pattern. This chord passes through the origin of the F -plane and ends at two diametrically opposite points on the spiral which belong to the arguments $w_2 = \frac{d}{\sqrt{\lambda f/2}}$, $w_1 = -\frac{d}{\sqrt{\lambda f/2}}$. If we shift the starting point of the chord by a certain distance, then we must shift the end point by so much that the arc of the spiral has the same length as before. In this way the length of the chord is changed. This change implies a changed amplitude $|v|$ at the new point of observation x which corresponds to the new position of

the chord. If we approach the upper limit point of the spiral with the starting point of the chord, then the end point of the chord will also approach this limit; the chord becomes progressively smaller, and so does the amplitude $|v|$ which, in the process, goes through an infinite number of extrema of continually decreasing magnitudes.

C. DIFFRACTION BY A STRAIGHT EDGE

If we make the slit infinitely wide ($d \rightarrow \infty$) by keeping one of the edges, e. g. the right-hand one, fixed and moving the left-hand edge off to infinity, then we have the simpler problem of the straight edge. To begin with it is to be noted that the various limiting processes pile up and seem to exclude one another; in the series development (34.13) we had assumed the opening to be "small". In treating the slit we assumed $h \gg d$ and put $h = \infty$. Now we also let $d \rightarrow \infty$. In order to be mathematically precise, we would have to conduct a careful appraisal of these limit processes. However we shall omit this here, because the problem of the half-plane will be treated again with all desirable accuracy in Sec. 38.

We further simplify the problem by letting the incident wave be plane instead of spherical; that is, we move P' to infinity. We are, however, still dealing with Fresnel diffraction (see p. 206) if we observe the pattern on an observation screen B which is placed at a finite distance a from the diffraction screen. In this case $f = a$ (because $b = \infty$ and $1/f = 1/a + 1/b$). If the light is perpendicularly incident the coordinates ξ on S equal the coordinates x on B and $\gamma = 1$. If we place the origin of x on the boundary of the geometrical shadow, then

$$(13) \quad d - \xi_D = \xi = x, \quad w_2 = \frac{x}{\sqrt{\lambda a/2}}, \quad w_1 = -\infty.$$

Setting $w_2 = w$, we obtain instead of (7 a)

$$(13 a) \quad \left| \frac{v}{v_0} \right| = \frac{1}{\sqrt{2}} |F(w) - F(-\infty)|.$$

In the Cornu spiral construction the starting point of the chord is now fixed at the lower limit point of the spiral. Only the end point of the chord changes with x . In the region of the geometrical shadow ($-\infty < x < 0$) the length of the chord increases steadily, as indicated by the sequence of chords ending at the points a, b, c, d, e in fig. 74. The point d corresponds to the boundary of the geometric shadow. At that point $w = 0$ and $F(w) = 0$ and

$$(13 b) \quad \left| \frac{v}{v_0} \right| = \frac{1}{\sqrt{2}} |F(-\infty)| = \frac{1}{\sqrt{2}} \left| -\frac{1+i}{2} \right| = \frac{1}{2}, \text{ according to eq. (8 b).}$$

From there on the length of the chord keeps increasing up to the first maximum which is attained at the point f in the figure. Then the chord decreases to the first minimum at the point g , and after that the chord oscillates between alternating extrema of decreasing heights. The asymptotic value of $|v/v_0|$ for $w = \infty$ is twice its value (13 b) on the boundary of the geometric shadow; it is given by

$$(13\ c) \quad \frac{1}{\sqrt{2}} |F(+\infty) - F(-\infty)| = 1,$$

which corresponds to the full intensity of the incident light. The intensity at the shadow boundary is one-fourth of the incident intensity. The variations in the amplitude are shown in fig. 75.

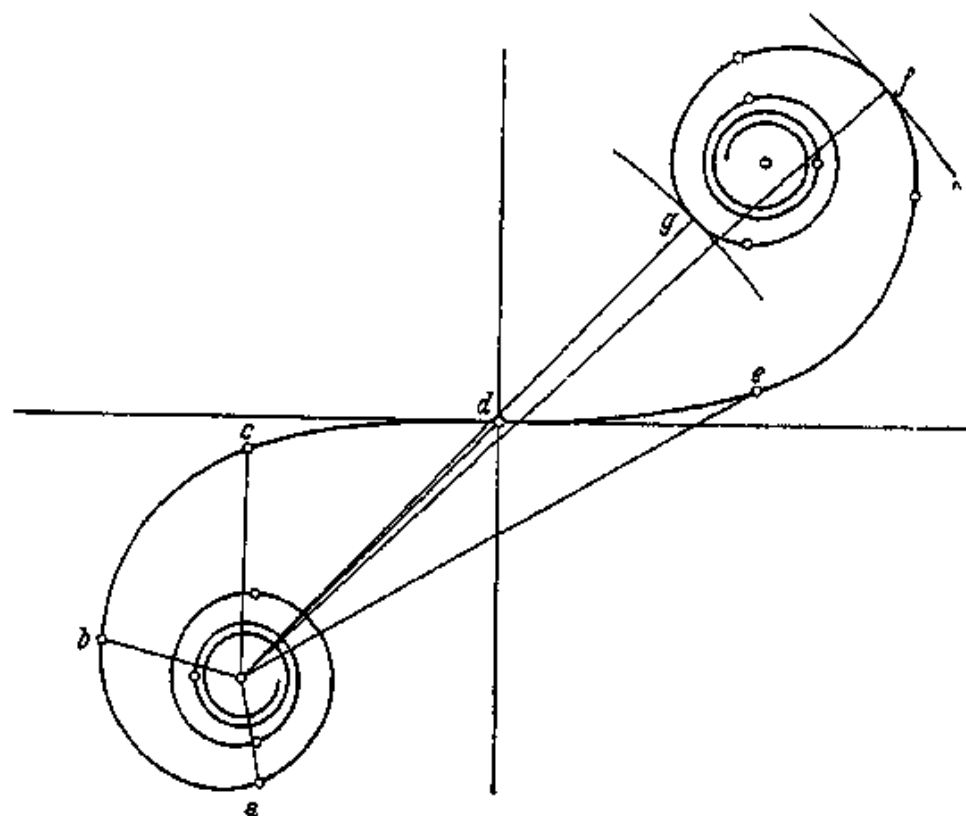


Fig. 74.

Determination of the diffraction pattern of the straight edge by means of Cornu's spiral.

We have assumed the diffraction screen to be infinitely thin and at the same time opaque. Therefore these results cannot be realized experimentally. Under a microscope even the edge of a razor looks more like a parabolic cylinder than like a sharp half-plane. However, it is very remarkable that the patterns on precise diffraction photographs (see for instance, Arkadiew loc. cit. p. 225) exhibit almost no dependence on the material and shape of

the diffraction edge. Even a bent glass plate whose radius of curvature is several meters and which may or may not be blackened yields essentially the same diffraction fringes as the edge of a razor. In each case the pattern is that shown in fig. 75.

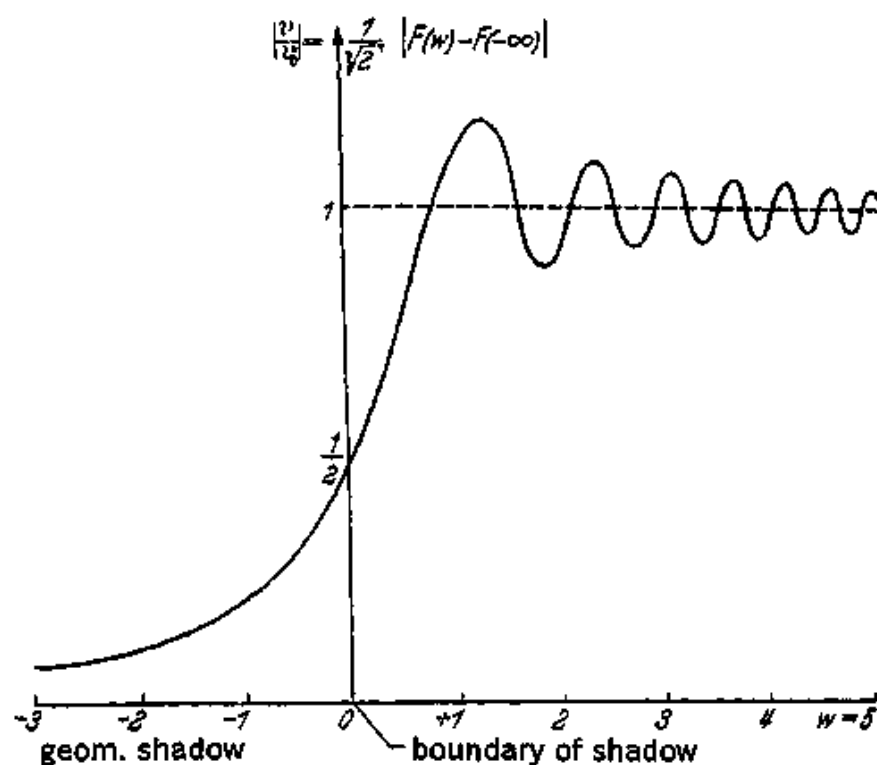


Fig. 75.

Amplitude $|v|$ behind a straight edge.

38. Rigorous Solutions of Certain Diffraction Problems

We shall call a solution of a diffraction problem exact only if it satisfies Maxwell's equations both outside and inside the diffracting object and if it satisfies the proper boundary conditions on the surface of that object. The solution must, furthermore, correspond to a given type of excitation (plane wave or point source). Such a solution can be found only for special shapes of diffracting objects, and certainly only if the wave equation can be "separated" in a coordinate system which is suited to the shape of the object.

The simplest example of such an object is a sphere. The field outside a sphere can be represented by series of spherical harmonics and Bessel functions of half-integer indices. These series have been discussed by G. Mie¹ for colloidal particles of arbitrary compositions. But even there a mathematical difficulty develops which quite generally is a drawback of this "method of series

¹Ann. d. Phys. 25, p. 377. 1908.

development": for fairly large particles ($ka > 1$, a = radius, $k = 2\pi/\lambda$) the series converge so slowly that they become practically useless. Except for this difficulty we could in this way obtain a complete solution of the problem of the rainbow¹, the difficulty of which was pointed out on p. 179.

What is true for the sphere is also true for a cylindrical wire of circular cross section. In that case the field can be represented by series of trigonometric functions and Bessel functions of integer indices. These series are entirely satisfactory in the domains of acoustic waves and Hertz waves² but they fail in the domain of optics. Debye overcame this difficulty by means of his famous asymptotic representation of the Bessel functions. Epstein³ reduced the problem of the parabolic cylinder to Hermite's functions.

The problem which includes all of these cases and which is in principle still separable is that of diffraction by a triaxial ellipsoid. This problem in its most general form leads to Lamé functions. The special case of the ellipsoid of rotation leads to products of a trigonometric function of the cylinder angle and two "spheroidal functions" which can be considered specialized Lamé functions or generalized spherical or Bessel functions.

The circular disc and the complementary plane screen with a circular opening are special degenerate cases of the oblate ellipsoid of revolution. In order that it be at all possible for a disc or screen of vanishing thickness to influence the light field, the material must, of course, be assumed to be opaque (perfectly conducting). The general Maxwell boundary conditions reduce then to the requirement that $E_{\text{tangential}} = 0$ and consequently that $B_{\text{perpendicular}} = 0$. With these boundary conditions the treatment of the problem can still be made mathematically rigorous, but it is no longer rigorous in the physical sense as defined above, for the diffracting material is no longer physically realizable. The solution to such a problem can be considered physically rigorous only in the case of acoustic waves⁴ or Hertz waves⁵ (wavelengths large compared to the thickness of the diffracting object).

The series of spheroidal functions which appear in these solutions again converge sufficiently well only if the radius a of the disc or opening is not too large compared to the wavelength. Even the case $ka \sim 1$ can be computed

¹The two papers which come closest to solving this problem are those by B. van der Pol and H. Bremmer, *Phil. Mag.* 24, p. 191 and 825, 1937 and by H. Buerius, *Optik*, Vol. I, p. 181, 1946. Debye had previously treated the two-dimensional rainbow (diffraction by a glass rod), *Phys. Zeitschr.* 9, p. 775, 1908.

²Schaefer-Grossmann, *Ann. d. Phys. (Leipzig)* 81, p. 454, 1910. Experimental verification with undamped waves: Schaefer-Merzkirch, *Z. f. Phys.* 18, p. 166, 1922 and Schaefer-Wilmsen, *ibid.* 24, p. 345, 1924.

³P. S. Epstein, Dissertation, Munich, 1914.

⁴O. J. Bouwkamp, *Proefschrift*, Groningen, 1941.

⁵J. Meixner, *ZS. f. Naturf.* Vol. 8a, p. 506, 1948.

numerically only with the aid of tables; here again asymptotic formulae of the type of Debye's formula for the Bessel functions are needed for an approximative evaluation of the result.

The problem of the slit and the complementary problem of the strip leads to Mathieu functions and has been solved numerically by Morse and Rubenstein¹ with the aid of tables of Mathieu functions.

It is impossible to discuss these function-theoretical details here; they belong to the chapter "Eigenvalues and Eigenfunctions" of Vol. VI.

A. THE PROBLEM OF THE STRAIGHT EDGE

This problem also is physically not rigorous because we shall assume the screening half-plane to be infinitely thin but nevertheless opaque. We shall obtain a mathematically rigorous solution of the problem which will even be in closed form and easily applicable to all wavelength domains. With this problem it was first demonstrated² that Fresnel diffraction constitutes a well-defined mathematical boundary value problem. (Fraunhofer diffraction cannot be treated directly by this method but only as a limiting case of Fresnel diffraction.)

We let the edge of the screen be the z -axis of a cylindrical coordinate system r, φ, z ; the front and rear surfaces of the screen shall be the surfaces $\varphi = 0$ and $\varphi = 2\pi$, respectively. We assume that in the r, φ -plane a monochromatic plane wave is incident on the front surface of the screen at an angle α (the angle of incidence measured against the normal to the screen is then $\pi/2 - \alpha$). The wave shall be linearly polarized in such a way that the electric field is directed parallel to the z -axis. Then the diffracted electric field will also be parallel to the z -axis and the problem becomes two-dimensional; only processes in the r, φ -plane are involved. Therefore we can use a scalar function u ; the part of this function which describes the incident wave will be

$$(1) \quad u_0 = A e^{-i k r \cos(\varphi - \alpha)}.$$

The negative sign in the exponent is due to the fact that we think of the time dependence as given by $\exp(-i\omega t)$ as usual, and that the wave propagates in the direction of the half-ray $\varphi = \pi + \alpha$; see fig. 76 (the arrows originating from O pertain to the discussion in the later section D). The field u as modified by the presence of the screen must satisfy the following conditions:

$$(1a) \quad \text{the wave equation } \Delta u + k^2 u = 0, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2},$$

¹Phys. Rev. 54, p. 895, 1938.

²A. Sommerfeld, Mathem. Ann., Vol. 47, p. 317, 1896. A simplified presentation is to be found in chapter 20 of Vol. II of "Differentialgleichungen der Physik", edited by Frank and von Mises, second edition 1934, first edition 1927 (Vieweg, Braunschweig).

- (1 b) the boundary conditions $u = 0$ for $\varphi = \begin{cases} 0 \\ 2\pi \end{cases}$, (corresponds to $E_{\tan} = 0$),
 (1 c) the condition u is finite and continuous everywhere, including the edge of the screen.

To these must be added the radiation condition¹ at infinity. Specialized to our case this condition must be formulated differently for the "illuminated" region $I + II$ and for the "shaded" region III of the r, φ plane (the words "illuminated" and "shaded" refer to the geometrical optics point of view). These conditions are

$$(1 d) \quad \lim_{r \rightarrow \infty} r \left(\frac{\partial v}{\partial r} - i k v \right) = 0, \quad v = \begin{cases} u - u_0 & \text{for } 0 < \varphi < \pi + \alpha \\ u & \text{for } \pi + \alpha < \varphi < 2\pi; \end{cases}$$

or expressed in words: in the illuminated region the incident portion of the field is given precisely by u_0 and the difference $u - u_0$ (reflected + diffracted wave) has the radiative character as required by (1 d); in the shaded region u itself is the radiative field.

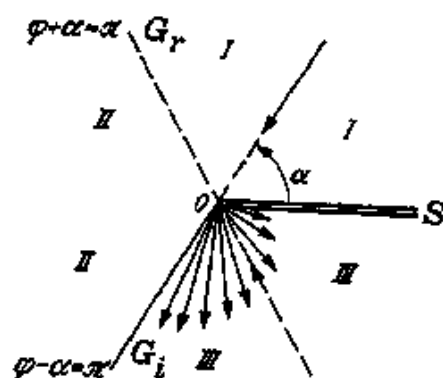


Fig. 76.

The diffraction screen S with the shadow boundary G_i of the incident ray and G_r of the reflected ray.

Finally, we must complete our requirement (1 c) with a statement about the behavior of $r \text{ grad } u$ at the edge of the screen, namely that

$$(1 e) \quad r \text{ grad } u \rightarrow 0 \quad \text{as} \quad r \rightarrow 0.$$

Accordingly, $\text{grad } u$ can become infinite at $r = 0$ but only "weakly" so. In the limit, $r \text{ grad } u$ must vanish. We shall see in section C below that when this condition is satisfied, the edge of the screen neither radiates nor absorbs energy. Therefore we can characterize the requirements (1 d) and (1 e) as additional energy conditions which suffice to make the problem physically unique².

¹This condition is fully discussed in Vol. VI, Sec. 28. The requirement is equivalent to demanding that if all light sources are situated in the finite regions of space, then the field at infinity must behave like an *outgoing spherical wave*, $\exp(i k r)/r$. This expression fulfills (1 d) everywhere when $v = u_0$. Separate formulations have to be given in (1 d) for the two regions because the incident wave is a plane wave originating at infinity.

²J. Meixner, ZS. f. Naturforsch., Vol. 8, p. 506 established a more general condition (that the energy density at the edge of the screen shall be integrable with respect to space). In our case this condition becomes equivalent to (1 e). In three-dimensional cases where our condition (1 c) on the finiteness of u cannot be imposed, Meixner's "edge condition" is not only necessary but also *sufficient*.

The problem obviously cannot be solved by means of the usual method of images. For, if we were to add to the incident wave (1) a reflected wave

$$u_0' = -A e^{-ikr \cos(\alpha + \varphi)}$$

(direction of incidence $\varphi = 2\pi - \alpha$), our condition (1 d) would be violated. Furthermore, the resulting solution would vanish not only on the half-ray $\varphi = \begin{cases} 0 \\ 2\pi \end{cases}$ but on the whole ray $\varphi = \begin{cases} 0 \\ \pi \end{cases}$, which would certainly be wrong.

However, the method of images can be retained if one uses, instead of the ordinary plane wave $u_0(r, \psi)$ of period 2π (where ψ stands for $\varphi - \alpha$), a function $U(r, \psi)$ which has the period 4π in the variable ψ and which satisfies the conditions (1 a) and (1 c) for all $-2\pi < \psi < 2\pi$ and the condition (1 d) with $v = U - u_0$ for $|\psi| < \pi$ and with $v = U$ for $|\psi| > \pi$. In the language introduced by Riemann for algebraic functions this means the following: U is a solution of our wave equation on a two-sheeted *Riemann surface* which has simple branch points at $r = 0$ and $r = \infty$. U is uniquely determined by its behavior at infinity (incident wave only in the sheet $|\psi| < \pi$, no incident wave in the sheet $|\psi| > \pi$) and by the requirement that it shall be everywhere continuous.

The usual model of this Riemann surface is familiar. It consists of two flat sheets which lie one on top of the other and are joined along the half-rays $\psi = \pm \pi$, for instance. Instead, we shall represent the surface by a single plane defined by the angle $\varphi/2$; see fig. 77. Every quadrant of this plane is, of course, a half-plane in the variable φ . The arrow¹ drawn as coming from the direction $+\alpha/2$ corresponds to the incident plane wave $U(r, \varphi - \alpha)$. The image-wave $U(r, \varphi + \alpha)$ is represented by the half-ray $-\alpha/2$ in the fourth quadrant. Since these two waves are symmetric with respect to α , they cancel one another on the symmetry lines $\varphi/2 = 0$ and $\varphi/2 = \pm \pi$ which

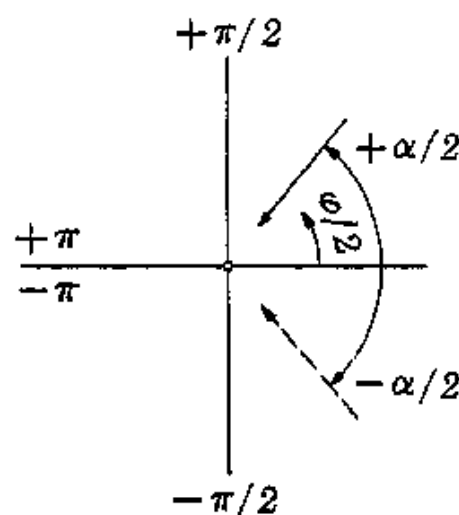


Fig. 77.

Symbolic representation of the method of images applied to the half-plane.

¹In connection with fig. 77 it should be noted that the two straight arrows drawn there represent only the rays passing through the origin correctly; rays parallel to these would in the figure have to be drawn as parabolic curves (coordinates $r, \varphi/2$ instead of r, φ). Further, it should be noted that the arrows refer only to the incident waves at infinity; our schematic figure does not represent the diffraction produced by the screen at all.

represent the two surfaces of our diffraction screen $\varphi = 0$ and $\varphi = 2\pi$. Therefore the solution of our diffraction problem is given by the formula

$$(2) \quad u = U(r, \varphi - \alpha) - U(r, \varphi + \alpha).$$

We now turn to the diffraction problem for the other polarization, that is, for light whose electric field is not parallel but perpendicular to the edge of the diffraction screen. All other geometrical conditions will be kept the same. In this case the magnetic vector \mathbf{H} is parallel to the edge of the screen not only for the incident component but also for the reflected and diffracted components of the field. We now denote the magnetic vector by u and ask for the correct boundary condition which again follows from the requirement that $E_{\text{tangential}} = 0$. If we introduce, temporarily, the cartesian coordinates x, y in place of r, φ , then we must require that in going from air into the screen

$$E_x = 0 \quad \text{and} \quad E_z = 0.$$

The latter condition is automatically satisfied because of the given polarization of the \mathbf{E} -component. The first condition requires, according to Maxwell, that on both sides of the screen

$$\text{curl}_x \mathbf{H} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = 0$$

and hence, because $H_y = 0$, $H_z = u$:

$$\frac{\partial u}{\partial y} = 0 \quad \text{for} \quad \varphi = \begin{cases} 0 \\ 2\pi \end{cases}.$$

For this we can write also

$$(3) \quad \frac{\partial u}{\partial n} = 0$$

where n stands for the normals on both sides of the screen. We can satisfy this condition immediately by means of the sum

$$(4) \quad u = U(r, \varphi - \alpha) + U(r, \varphi + \alpha)$$

which is analogous to (2). We wish to recall here that the two Green's functions G_- and G_+ of Sec. 34 G were formed by a method of images quite analogous to (2) and (4).

Fundamentally, our method has an even wider applicability. It can be extended without difficulty to the problem of the slit. In that case a Riemann surface with two branch points at the traces of the two slit edges would have to be used in place of the surface with a single branch point in the finite plane and another at infinity; besides, the ordinary polar coordinates would have to be replaced by bipolar coordinates. The method could even be extended, at least in the scalar (acoustic) case, to cover the problem of

an arbitrarily bounded plane screen or a complementary opening. Here the two-sheeted Riemann surface would be replaced by a "Riemann double-space", the two "sheets" of which would have a common "branch line" on the bounding curve of the screen or opening. The difficulty with these generalizations lies in the mathematical construction of the branched solutions. It has been possible to construct these solutions only for the simplest case of the half-plane. As we shall see presently, even here very special mathematical devices are required.

B. CONSTRUCTION OF BRANCHED SOLUTIONS

The factor A in eq. (1) can be considered as an arbitrary function of the angle of incidence α . Replacing this α by a variable of integration β and integrating the expression with respect to β , we obtain the "wave bundle"

$$(5) \quad u = \int A(\beta) e^{-ikr \cos(\varphi - \beta)} d\beta.$$

This expression is a solution of the differential equation (1a) for any arbitrary, possibly complex, path of integration. If the integration path is complex, then u represents an "inhomogeneous" wave of the type, for instance, which we have encountered in total reflection. Let us first choose a closed path in the complex β -plane which encloses the point $\beta = \alpha$. If we see to it that $A(\beta)$ has a pole of the first order with the residue $1/2 \pi i$ at α , then by Cauchy's residue theorem (5) becomes the solution u_0 given by (1), normalized so that $A = 1$. (Let u_0 henceforth denote this normalized solution.) In particular, we shall choose $A(\beta)$ to be a periodic function of β with the period 2π , namely

$$(6) \quad A(\beta) = \frac{1}{2\pi} \frac{e^{i\beta}}{e^{i\beta} - e^{i\alpha}}.$$

Thus we obtain

$$(7) \quad u_0 = \frac{1}{2\pi} \oint \frac{e^{i\beta}}{e^{i\beta} - e^{i\alpha}} e^{-ikr \cos(\varphi - \beta)} d\beta$$

where \oint indicates contour integration around a closed path.

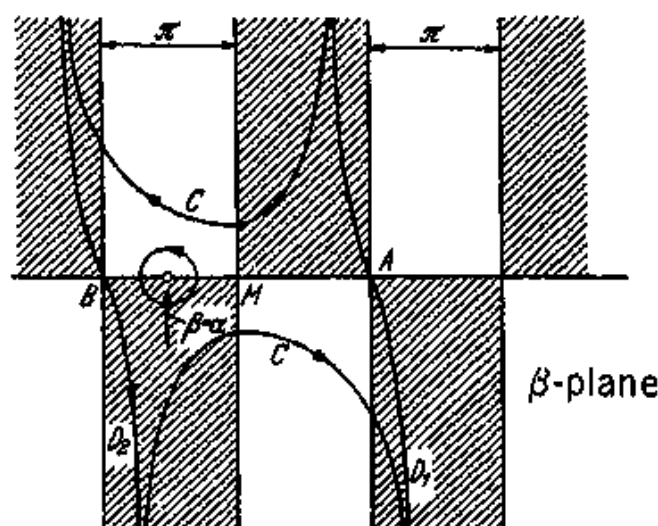


Fig. 78.

Path of integration in the β -plane used in the representation of u_0 .

We can deform the path around the pole $\beta = \alpha$ in an arbitrary manner as long as it does not cross any other singularities of the integrand (that is, none of the points $\beta = \alpha \pm 2\pi, \alpha \pm 4\pi, \dots$). If we wish to deform the path so that it goes to infinity, then we must make sure that the integrand vanishes in the limit along the path. In fig. 78 the regions in which $\cos(\varphi - \beta)$ has a negative imaginary part have been shaded. These regions are bounded by straight lines. For positive values of kr the real part of $-ikr \cos(\varphi - \beta)$ goes to $-\infty$ as β goes to infinity inside the shaded regions; therefore the integrand in (7) becomes vanishingly small. At the corners A, B and M of the resulting pattern we have

$$(7a) \quad \beta = \begin{cases} \varphi - \pi & \dots A \\ \varphi + \pi & \dots B \\ \varphi & \dots M \end{cases}$$

The path of integration which has been drawn in fig. 78 consists of the two loops C and the connecting paths D_1 and D_2 . The latter two have been chosen so that they are brought into congruence by a displacement of 2π . Because of this and because the directions of integration are opposite for these two paths, their contributions to the integral cancel. Thus we need only integrate (7) along the two loops C ; the path of integration will still be equivalent to the original circuit around $\varphi = \alpha$ and the integral (7) will still be identical to the plane wave u_0 .

With this preparation we can immediately find the desired function U on the Riemann surface. To do this we give the arbitrary function A the period 4π (instead of 2π) but still insist that it possess a pole with residue $1/2\pi i$ at the point $\beta = \alpha$. Thus instead of (6) we set

$$(8) \quad A(\beta) = \frac{1}{4\pi} \frac{e^{i\beta/2}}{e^{i\beta/2} - e^{i\alpha/2}}.$$

Then we obtain in place of (7)

$$(9) \quad U = \frac{1}{4\pi} \int_C \frac{e^{i\beta/2}}{e^{i\beta/2} - e^{i\alpha/2}} e^{-ikr \cos(\beta - \varphi)} d\beta,$$

where the path of integration is to be taken along the loops C (without the connecting paths D). This function is obviously also a solution of the wave equation because, like (7), it consists of a superposition of ordinary plane waves.

The shaded pattern in fig. 78 depends on the value of the angle φ , as is evident from (7a). The whole pattern together with the integration paths shifts when φ is varied. This is inconvenient in the calculations to follow and can be avoided by replacing the integration variable β with

$$(10) \quad \gamma = \beta - \varphi,$$

which is advisable because the angles φ and α can then be combined as before:

$$(10 a) \quad \psi = \varphi - \alpha.$$

Writing (9) in terms of γ and ψ , we obtain

$$(11) \quad U = \frac{1}{4\pi} \int_C \frac{e^{i\gamma/2}}{e^{i\gamma/2} - e^{-i\psi/2}} e^{-ikr \cos \gamma} d\gamma.$$

This representation of U immediately shows us that U has the period 4π in ψ and is therefore double-valued on the simple r, ψ plane; but U is single-valued on our Riemann surface. U also satisfies the wave equation because it is, after all, only the function (9) written in a different form.

We shall now explain fig. 79. The points marked

$$\gamma = +\pi, -\pi, \text{ and } 0$$

correspond, according to the eqs. (10) and (7 a), to the points A, B , and M in fig. 78. The branches D_1, D_2 of the path of integration drawn in fig. 79 should be disregarded for the present. The pole $\beta = \alpha$ in fig. 78 lies now at $\gamma = \alpha - \varphi = -\psi$. It has not been drawn in fig. 79 because we shall begin by considering the case $|\psi| > \pi$ for which the pole lies outside the segment $-\pi < \gamma < +\pi$. Since the loops C go to infinity inside the shaded regions,

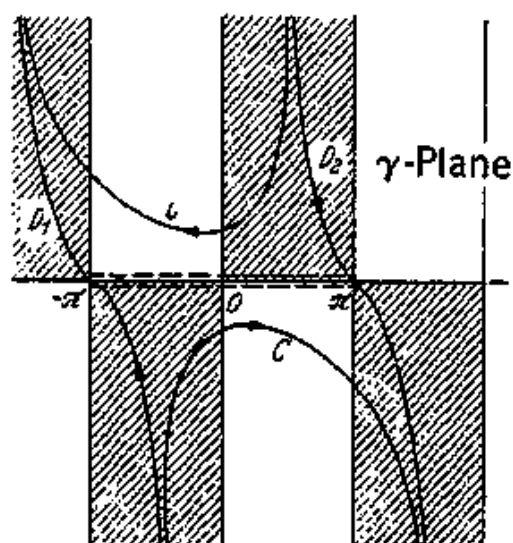


Fig. 79.

Path of integration in the γ -plane used in the representation of U .

U is certainly finite and continuous for all $r > 0$. Only the point $r = 0$ requires special consideration. For here the factor $\exp(-ikr \cos \gamma)$ which insures convergence of the integral becomes equal to 1. Nevertheless, the integral converges because for $r = 0$ it becomes, except for a finite factor $-2i$,

$$(12) \quad \int \frac{dz}{z - \zeta} = \log(z - \zeta) \Big|_{C_1}^{C_2} = \log \frac{z_2 - \zeta}{z_1 - \zeta}$$

where $z = e^{i\gamma/2}$, $\zeta = e^{-i\psi/2}$. C_1 and C_2 indicate the end points at infinity of the (upper and lower) loops C ; z_1 and z_2 are the values of z at those points. But since $z = e^{i\gamma/2}$, their values are $z_1 = z_2 = 0$ for the upper loop and $|z_1| = \infty$, $|z_2| = \infty$ for the lower loop. Therefore on the lower loop we can neglect ζ in comparison to z . Thus we obtain

$$(12 a) \quad \log \frac{z_2 - \zeta}{z_1 - \zeta} = \begin{cases} \log 1 = 0 & \text{on the upper loop,} \\ \log z_2/z_1 = i\pi & \text{on the lower loop.} \end{cases}$$

The last value follows from the fact that according to fig. 79 the γ -values of C_1 and C_2 differ by 2π . Thus (12) and (12 a) constitute the proof for the convergence of the integral (11) at $r = 0$.

Next, we must investigate the behavior of U as $r \rightarrow \infty$. As $r \rightarrow \infty$, the integrand of (11) vanishes *everywhere* in the shaded regions and not merely in their infinitely distant portions. In the case $|\psi| > \pi$, which is represented by fig. 79, the two loops C can be deformed so that they lie entirely in the shaded regions. For instance, the upper loop can be made to coincide with the two previously used by paths D_1 and D_2 plus the segment of the real axis between $-\pi$ and $+\pi$; the lower loop can be made to coincide with the lower portions of D_1 and D_2 and the same segment $+\pi$ to $-\pi$ of the real axis traversed in the opposite direction. Then the sum of the integrals over the two loops reduces to the sum of the integrals along the connecting paths D_1 and D_2 , with the direction of integration indicated by the arrows in the figure (opposite to that in fig. 78). The integrals along these two paths no longer cancel as they did before because the period of γ is 4π and not 2π . However, the integrals along both paths are individually zero and therefore

$$(13) \quad U = 0 \quad \text{for} \quad r \rightarrow \infty \quad \text{and} \quad |\psi| > \pi.$$

If, however, $|\psi| < \pi$, then the pole of the integrand of (11) at $\gamma = -\psi$, lies on the segment between $-\pi$ and $+\pi$ in fig. 79. Therefore, if the loops C are again deformed so as to lie in the shaded regions, a positive circulation of the pole must be added. Because the residue is 1 at this pole the integral now becomes

$$(13 a) \quad U = e^{-ikr \cos \psi} \quad \text{for} \quad r \rightarrow \infty \quad \text{and} \quad |\psi| < \pi$$

instead of (13).

Expressed in the language of the two-sheeted Riemann surface, we have an "upper sheet" $|\psi| < \pi$ which is illuminated by the plane wave u_0 and a "lower sheet" $|\psi| > \pi$ which lies in the shadow. These sheets are connected along the "shadow boundary" $\psi = \pm \pi$. The contradistinction of "light and shadow" here finds its simplest mathematical formulation. For finite values of r the transition of U from one sheet to the other is continuous, and this transition constitutes the diffraction phenomenon. Actually the transition is also continuous for $r \rightarrow \infty$ in spite of the apparent discontinuity expressed by (13) and (13 a); (the transition region shrinks to zero only when measured in terms of the angular scale ψ).

However, we are not finished with our investigation of the infinitely distant point. We must carry our analysis a step further and show that not only the conditions (13), (13 a), but also the more stringent condition (1 d) is

fulfilled. This condition must be satisfied by $v = U - u_0$ in the illuminated sheet and by $v = U$ in the shadow sheet. Only then can we be sure that the function U which we have constructed is actually the unique solution of the diffraction problem demanded by nature.

Hence, we must find an asymptotic approximation of the integral (11) for large r . In the shadow sheet we obtain from the difference between D_2 and D_1

$$(14) \quad 4\pi U = \int_{D_1} e^{-ikr \cos \gamma} \Phi(\gamma) d\gamma,$$

$$(14a) \quad \Phi(\gamma) = \frac{e^{i\gamma/2}}{e^{i\gamma/2} - e^{-i\gamma/2}} - \frac{e^{i\gamma/2}}{e^{i\gamma/2} + e^{-i\gamma/2}}.$$

In the second fraction which is due to D_1 , the sign of $\exp(i\gamma/2)$ is reversed because γ is shifted by 2π with respect to its value in the first numerator. This has the effect of changing the sign of $\exp(-i\gamma/2)$. The negative sign in front of the second fraction corresponds to the opposite directions of integration on D_1 and D_2 in fig. 79. The general method of evaluating (14) is the method of saddle-points, see Vol. VI, Sec. 19 E. We do not need to carry this out here, because we shall immediately develop a more convenient and even more precise method of evaluating this integral. We shall only make the following remark on the saddle-point method: from fig. 79 we see that in our case the critical saddle-point lies at $\gamma = \pi$, because at that point the path D_2 passes closely by two unshaded regions. For $\gamma = \pi$ the first factor of the integrand of (14) becomes $\exp(ikr)$; this factor can be taken out of the integral. The remaining integral which is to be computed only in the vicinity of the saddle-point yields the factor $\frac{1}{\sqrt{k r}}$, aside from a constant which does not interest us at present. Therefore, one finds that

$$(15) \quad U = \frac{C}{\sqrt{k r}} e^{ikr}.$$

This expression does indeed satisfy our radiation condition (1 d). For,

$$\frac{\partial U}{\partial r} - ikU = \frac{ikC}{\sqrt{k r}} \left(e^{ikr} - \frac{1}{2ikr} e^{ikr} - e^{ikr} \right),$$

and this vanishes, even when multiplied by r , in the limit as $r \rightarrow \infty$. The difference $U - u_0$ in the illuminated sheet behaves in the same manner.

We shall postpone the verification of the condition (1 e) as well as the complete proof of (15) until the end of the next section.

C. REPRESENTATION OF U BY A FRESNEL INTEGRAL

Our purpose in this section is to bring the above formulae into a form which is comparable to the expressions derived in Sec. 37 C. Unfortunately this transformation is of a somewhat lengthy and largely formal character.

The expression (14 a) can be rewritten

$$\Phi(\gamma) = \frac{2 e^{1/2 i (\gamma - \psi)}}{e^{i\gamma} - e^{-i\psi}}.$$

We introduce the new variable of integration η by setting

$$\gamma = \pi + \eta \quad \text{and} \quad \gamma = \pi - \eta$$

on the upper and lower half of D_2 , respectively. The sum of the values of Φ at points with equal $|\eta|$ is found to be

$$\Phi(\pi + \eta) + \Phi(\pi - \eta) = \frac{-4i \cos \frac{\psi}{2} \cos \frac{\eta}{2}}{\cos \psi + \cos \eta}.$$

If we substitute this in (14), the latter becomes

$$(16) \quad \pi U = -i \cos \frac{\psi}{2} \int e^{ikr \cos \eta} \frac{\cos \frac{\eta}{2}}{\cos \psi + \cos \eta} d\eta,$$

where the range of integration is to be taken from $\eta = 0$ to a value $i\infty - \eta'$ and η' can be any arbitrary real number $< \pi$ (see fig. 79).

The expression (16) suggests that, instead of U , the quantity

$$(16 a) \quad V = \frac{U}{u_0}$$

be considered and that the factor $1/u_0 = \exp(+i k r \cos \psi)$ be placed under the integral, for then the denominator of (16) disappears after differentiation with respect to r . Performing this differentiation, one obtains

$$(17) \quad \pi \frac{\partial V}{\partial r} = k \cos \frac{\psi}{2} \int e^{i k r (\cos \psi + \cos \eta)} \cos \frac{\eta}{2} d\eta.$$

Now it is possible to perform the integration with respect to η . Since $\cos \eta = 1 - 2 \sin^2 \eta/2$, the integral to be evaluated is

$$(17 a) \quad \int e^{-2i k r \sin^2 \frac{\eta}{2}} \cos \frac{\eta}{2} d\eta,$$

which, because $\cos \psi = 2 \cos^2 \psi/2 - 1$, is still to be multiplied by the factor

$$(17 b) \quad \exp\left(2i k r \cos^2 \frac{\psi}{2}\right),$$

which is independent of η . Upon making the substitution

$$\sin \frac{\eta}{2} = \sqrt{\frac{\pi}{4kr}} \tau$$

(17 a) becomes an integral of the Fresnel type. For, introducing the above limits of integration, (17 a) is equal to

$$\sqrt{\frac{\pi}{kr}} F^*(\infty), \quad F^*(\infty) = \int_0^\infty e^{-\frac{i\pi}{2}\tau^2} d\tau = \frac{1-i}{2}, \quad \text{see (37.8 b).}$$

Multiplying now by the factor (17 b), we obtain for the value of (17)

$$(18) \quad \frac{\partial V}{\partial r} = \frac{1-i}{2} \sqrt{\frac{k}{\pi r}} \cos \frac{\psi}{2} \exp \left(2i kr \cos^2 \frac{\psi}{2} \right).$$

The right-hand side can be written as the differential quotient with respect to r of an expression which we can again write in the form of a Fresnel integral¹. That is, (18) is equivalent to

$$(18 a) \quad \frac{\partial V}{\partial r} = \frac{1-i}{2} \frac{\partial}{\partial r} \int_{-\infty}^{\rho} e^{\frac{i\pi}{2}\tau^2} d\tau, \quad \rho = 2 \sqrt{\frac{kr}{\pi}} \cos \frac{\psi}{2}.$$

This becomes, by integrating with respect to r ,

$$(18 b) \quad V = \frac{1-i}{2} \int_{-\infty}^{\rho} e^{\frac{i\pi}{2}\tau^2} d\tau,$$

and because of (16 a)

$$(19) \quad U = u_0 \frac{1-i}{2} \int_{-\infty}^{\rho} e^{\frac{i\pi}{2}\tau^2} d\tau.$$

Because of the definition of ρ in (18 a), this representation of U is an *analytic function of ψ with the period 4π* . Therefore, it is valid not only on the shadowed sheet, for which (19) was derived, but also on the illuminated sheet of our two-sheeted Riemann surface. On the latter we obtain for $r \rightarrow \infty$

$$\begin{aligned} U &= u_0 \frac{1-i}{2} \int_{-\infty}^{\infty} e^{\frac{i\pi}{2}\tau^2} d\tau = u_0 (1-i) \int_0^{\infty} e^{\frac{i\pi}{2}\tau^2} d\tau \\ &= u_0 (1-i) F(\infty) = u_0 \frac{(1-i)(1+i)}{2} = u_0, \end{aligned}$$

as required.

¹If we were to choose the lower limit of the integral as zero, the ordinary Fresnel integral $F(\rho)$ would result. But then an (easily evaluated) integration constant would have to be added to the right-hand side of (18 b). With our choice of $-\infty$ as the lower limit this constant becomes zero.

We can discuss the representation (19) conveniently for both large and small values of $|\rho|$ with the help of the approximation formulae for Fresnel's integral which were derived in Sec. 37.

a) For large $|\rho|$ in the region of the geometrical shadow ($\rho < 0$) we write

$$\int_{-\infty}^{\rho} = \int_{-\infty}^0 - \int_{-\rho}^0 = F(\infty) - F(|\rho|),$$

and hence, according to (37.8 a)

$$U = -u_0 \frac{1-i}{2} \frac{e^{\frac{i\pi}{2}\rho^2}}{i\pi\rho} \left(1 + \frac{1}{i\pi\rho^2} + \dots \right).$$

Introducing the expressions for u_0 and ρ we find

$$u_0 e^{\frac{i\pi}{2}\rho^2} = \exp \left\{ -ikr \cos \psi + 2ikr \cos^2 \frac{\psi}{2} \right\} = \exp(ikr),$$

and obtain

$$(20 \text{ a}) \quad U = \frac{1+i}{4\sqrt{\pi kr} \cos \frac{\psi}{2}} e^{ikr} \left(1 + \frac{1}{i\pi\rho^2} + \dots \right).$$

If the correction terms inside the parentheses are neglected, this is of the same form as the asymptotic behavior which we predicted in (15); the factor C , which was there undetermined, now turns out to be a function of ψ .

b) For large ρ in the illuminated region ($\rho > 0$) we substitute in (19)

$$\int_{-\infty}^{\rho} = \int_{-\infty}^0 + \int_0^{\rho} = F(\infty) + F(\rho) = 2F(\infty) - [F(\infty) - F(\rho)],$$

and consider that

$$\frac{1-i}{2} 2F(\infty) = \frac{(1-i)(1+i)}{2} = 1.$$

Using the same expansion as in a), we obtain from (19)

$$(20 \text{ b}) \quad U = u_0 - \frac{1+i}{4\sqrt{\pi kr} \cos \frac{\psi}{2}} e^{ikr} \left(1 + \frac{1}{i\pi\rho^2} + \dots \right).$$

This asymptotic expression is also in agreement with the remarks made following formula (15).

c) For small ρ (illuminated or shaded sheet) we set

$$\int_{-\infty}^{\rho} = \int_{-\infty}^0 + \int_0^{\rho} = F(\infty) + F(\rho)$$

and by means of (37.8) we obtain from (19)

$$(20\ c) \quad U = \frac{u_0}{2} \left\{ 1 + (1-i) \rho \left(1 + \frac{i\pi}{6} \rho^2 + \dots \right) \right\}.$$

At the branch point itself $\rho = 0$, $u_0 = 1$ and therefore

$$(20\ d) \quad U = \frac{1}{2} \quad \text{and} \quad \frac{\partial U}{\partial r} = \frac{1-i}{2} \frac{\partial \rho}{\partial r} = \frac{1-i}{2} \sqrt{\frac{k}{\pi r}} \cos \frac{\psi}{2}.$$

Thus $\partial U / \partial r$ goes to ∞ as $r \rightarrow 0$ but only so weakly that the product of r and the gradient remains finite as required by (1 e):

$$(20\ e) \quad r \text{ grad } U \rightarrow 0 \quad \text{as} \quad r \rightarrow 0.$$

Thus we have finally shown that our branched solution U satisfies all the conditions as postulated in section A.

D. THE DIFFRACTION FIELD OF THE STRAIGHT-EDGE

Returning now to fig. 76 and the representations (2) and (4), we shall describe the field in the region of observation $0 < \varphi < 2\pi$. The plane is divided into three sectors I, II, III by the screen S , the shadow boundary G of the reflected wave and the shadow boundary G_i of the incident wave. These three sectors have the central angles $\pi - \alpha$, 2α , $\pi + \alpha$, respectively. I is illuminated by the incident wave $U(r, \varphi - \alpha)$ and by the reflected wave $U(r, \varphi + \alpha)$, II belongs to the illuminated sheet of the incident wave and to the shaded sheet of the reflected wave, while III is in the shadow of both the incident and reflected waves. The direction of incidence of the reflected wave does not lie in the region of observation. We must think of this direction as lying on a Riemann sheet which is connected to the region of observation along S .

Let us first consider sector III. Since we are only interested in distances $r \gg \lambda$, we need to consider only large values of kr and large values of ρ (except in the immediate vicinity of the shadow boundary G_i where $\cos \psi/2 = 0$). Therefore, we may use the approximation formula (20 a) and thus obtain by (2) and (4)

$$(21) \quad u = \frac{1+i}{4\sqrt{\pi kr}} e^{ikr} \left(\frac{1}{\cos \frac{\varphi - \alpha}{2}} \mp \frac{1}{\cos \frac{\varphi + \alpha}{2}} \right).$$

The upper sign corresponds to the case where \mathbf{E} oscillates parallel to the edge of the screen; the lower sign corresponds to the case where \mathbf{E} oscillates perpendicular and \mathbf{H} parallel to the edge of the screen (in this latter case u represents not \mathbf{E} but \mathbf{H}). The second term in the parentheses takes on appreciable values only at the shadow boundary G , and can therefore be neglected¹ in comparison to the first term.

Since the value of the expression in the parentheses decreases slowly with increasing φ , we conclude that *the light is diffracted far into the region of the geometrical shadow*. The infinitely large value which this expression assumes on the shadow boundary is, of course, illusory and is due to the fact that our asymptotic approximation is not valid there. In this region the exact representation (19) should be used instead of the approximation (21); see below.

Even more interesting than the φ -dependence of eq. (21) is its r -dependence $\frac{e^{ikr}}{\sqrt{r}}$ which has the character of a cylindrical wave emitted by the edge of the screen.

We have indicated this by the arrows emerging from O in fig. 76. A. Kala-schnikow² showed that these ray directions can be photographed. He inserted pins into a photographic plate which he then placed at an angle to the ray directions. After a sufficiently long exposure, radially directed shadows of these pins appeared on the plate.

If one focuses the eye on the edge of the screen, this edge appears as a *thin luminous line*. This effect was described very early by Grimaldi, the father of all diffraction discoveries. The explanation is that the eye performs an inadmissible extrapolation. It infers from the asymptotic field, which is correctly represented by (21), that the field becomes infinite at $r = 0$, which is not true. In fact, the energy radiated into an angular region $\delta\varphi$ per unit length of the edge per unit time is, depending on whether \mathbf{E} is parallel or perpendicular to the edge,

$$(22) \quad \delta S = S_r r \delta\varphi = r \delta\varphi \begin{Bmatrix} -E_z H_\varphi \\ +E_\varphi H_z \end{Bmatrix} = \begin{Bmatrix} \frac{i}{\omega \mu_0} E_z r \frac{\partial E_z}{\partial r} \delta\varphi, \\ \frac{i}{\omega \epsilon_0} H_z r \frac{\partial H_z}{\partial r} \delta\varphi. \end{Bmatrix}$$

In the upper line ($E = E_z$) we have used the equation $\dot{\mathbf{B}} = -\text{curl } \mathbf{E}$, and in the lower line ($H = H_z$) we have used $\dot{\mathbf{D}} = \text{curl } \mathbf{H}$. The factors $\partial/\partial r$ in both lines become infinite for $r = 0$ but only so slowly that $\delta S = 0$; see (20 e). Hence the "luminous edge" is not real.

¹ Retention of this term would yield a small difference between the cases \mathbf{E}_\parallel and \mathbf{E}_\perp ; and hence a small polarization effect.

² Journal of the Russ. Phys. Soc. 44, p. 133, 1912.

The factor

$$(23) \quad 1 + i = \sqrt{2} e^{i\frac{\pi}{4}}$$

in (21) is also of some interest. It shows that the phases of the diffracted and incident waves, the former extrapolated to $r = 0$, do not agree. The phase of u_0 is

$$-i k r \cos \psi - i \omega t, \quad \text{hence equal to } -i \omega t \text{ at } r = 0,$$

while that of (21) is

$$\frac{i\pi}{4} + i k r - i \omega t, \quad \text{hence equal to } \frac{i\pi}{4} - i \omega t \text{ at } r = 0.$$

Such a "phase jump" always takes place when light passes through a focal point (or focal line in the case of our cylinder wave); see Sec. 45. But as in the case of the luminous edge, the phase jump is only a result of the extrapolation and is not real. Actually the phase, like the amplitude, is continuous at the origin, as far as it is at all permissible to talk of a phase of the complicated oscillations in that vicinity.

Let us now turn to sector II. At some distance from the shadow boundaries G_1 and G_2 we can set

$$U(r, \varphi - \alpha) \sim u_0, \quad U(r, \varphi + \alpha) \sim 0;$$

that is, we can disregard diffraction and obtain the pure field $u = u_0$ of the incident wave.

We must proceed differently in the vicinity of G_1 . Here we set

$$(24) \quad \varphi - \alpha = \pi - \delta, \quad \cos \frac{\varphi - \alpha}{2} = \sin \frac{\delta}{2},$$

and call δ the "diffraction angle" which shall be reckoned positive in the direction towards II and negative in the direction towards III. Then

$$(24a) \quad \rho = 2 \sqrt{\frac{k r}{\pi}} \sin \frac{\delta}{2}$$

is finite even for large $k r$, provided δ is correspondingly small. Therefore $U(r, \varphi - \alpha)$ also has a finite value in comparison to which we can neglect $U(r, \varphi + \alpha)$. Using the exact representation (19) for $U(r, \varphi - \alpha)$, we obtain

$$(25) \quad U = u_0 \frac{1-i}{2} \{F(\infty) + F(\rho)\}$$

for both cases (2) and (4). Calculating the ratio U/u_0 we obtain

$$(25a) \quad \left| \frac{U}{u_0} \right| = \frac{1}{\sqrt{2}} |F(\infty) + F(\rho)|.$$

Formally, this agrees exactly with (37.13). For, since Cornu's spiral is symmetrical about the origin, $F(\infty) = -F(-\infty)$. There is, however, a difference in meaning between our present variable w and the previous variable

$$(26) \quad w = \frac{x}{\sqrt{\lambda a/2}},$$

as defined in (37.6 b). There, x was the distance of the point of observation from the shadow boundary, which we must now denote by $r \sin \delta$; a was the distance between the observation screen and the diffraction screen for perpendicular incidence; this we may now denote by r . Thus we obtain from (26)

$$(26 a) \quad w = \frac{\sqrt{r \sin \delta}}{\sqrt{\lambda/2}} = \sqrt{\frac{k r}{\pi}} \sin \delta.$$

A comparison with (24 a) shows that (26 a) contains the factor $\sin \delta$ in place of our previous factor $2 \sin \delta/2$. For small values of δ , which are the only ones of interest in the vicinity of the shadow boundary, this represents only a difference of the third order. Therefore we can still use fig. 75 to represent the results of our present more rigorous theory. This figure correctly exhibits the positions and amplitudes of the diffraction maxima and minima on the illuminated side of the shadow boundary, as well as the monotonically decreasing intensity in the region of the geometrical shadow. The intensity value $1/4$ on the shadow boundary itself is also in agreement with the present theory. Leaving all numerical considerations aside, we nevertheless wish to point out that the occurrence of $\sin \delta/2$ in (24 a) reflects a typical feature of our theory, namely that the diffraction angle has the period 4π .

We add one critical remark regarding the use of Huygens' principle. Let us consider in greater detail the half-plane $\varphi = \pi$ which is left open by the screen and plays the role of the "diffraction opening" in Huygens' principle. According to our prescription of Sec. 34 C the "boundary values" are chosen as the values u_0 given by the unperturbed incident wave; if, for simplicity, we assume perpendicular incidence ($\alpha = \pi/2$), then $u_0 = 1$. In contrast to this assumption eqs. (19) and (18 a) yield for $\varphi = \pi$ and $\alpha = \pi/2$

$$U = \frac{1-i}{2} \int_{-\infty}^{\rho} e^{-\frac{i\pi}{2}\tau^2} d\tau, \quad \rho = \sqrt{\frac{2kr}{\pi}}.$$

This expression varies from the value $U = 1/2$ at $r = 0$ to $U = 1$ at $r = \infty$ and oscillates in between. These values are in greatest contrast to the assumed boundary value $u_0 = 1$ which was used in applying Huygens' principle. A corresponding result holds for the reflected wave U , in which case α must

be replaced by $-\pi/2$ and ρ by $-\sqrt{(2kr)/\pi}$, and therefore the contradiction applies also to the superposition of the two waves. Thus we can say that the boundary values used in Huygens' principle differ from the (in our case) exact boundary values not only in the vicinity of the edge of the screen but even at *large numerical distances* kr from that edge. It is amazing that the classical diffraction theory nevertheless yields for all practical purposes satisfactory results.

The sector I belongs, as we know, to the illuminated regions of both the incident and reflected waves. At the boundary G_r of the latter region there occur, of course, diffraction phenomena which can be calculated in the same manner as those at the boundary G_i . But these diffraction fringes are masked by the full intensity of the incident wave in that region; they have been investigated experimentally only in the case of "Fresnel's mirror" (two half-planes inclined at a very shallow angle with respect to each other) and have been introduced in the calculations as very small perturbations added to the ordinary interference of the two reflected waves.

E. GENERALIZATION

It is easy to make the transition from the two-sheeted to an n -sheeted Riemann surface. It is only necessary to generalize eq. (8) to

$$(27) \quad A(\beta) = \frac{1}{2\pi n} \frac{e^{i\beta/n}}{e^{i\beta/n} - e^{i\alpha/n}}.$$

Analogously to (9) this leads to a function U of period $2\pi n$ with the help of which image problems in a sectorial space of central angle $2\pi n/m$ ($m = \text{integer}$) can be solved. Among these problems is, of course, that of the exterior of a rectangular wedge for which $n = 3$, $m = 4$. The representation in terms of Fresnel's integral which we treated in C is limited to the case where $n = 2$. W. Pauli¹ has shown that for arbitrary (even non-integer) values of n , Fresnel's integral is replaced by a confluent hypergeometric function.

The limiting case $n = \infty$ is of particular interest. It leads to the infinitely many-valued function

$$(27a) \quad U = \frac{1}{2\pi i} \int e^{-ihr \cos \gamma} \frac{d\gamma}{\eta - \gamma},$$

which is here expressed in terms of the integration variable γ of eq. (11). We regard this function as the best possible representation for the case of the conventional "black screen": the wave which is incident at an angle $\varphi = \alpha$

¹Phys. Rev. 54, p. 924, 1938.

on the front of the screen $\varphi = 0$ enters the screen and loses itself among the infinitely many sheets $\varphi < 0$. None of its energy is returned to the physical space via the infinite number of sheets $\varphi > 2\pi$ through the back side of the screen $\varphi = 2\pi$. To understand this one should recall the experimental realization of a black body used in heat radiation measurements, that is, a cavity which is kept at a constant temperature and has a small hole. All radiation entering the hole from the outside is reflected back and forth inside the cavity without ever again leaving it. The hole absorbs completely and acts, therefore, like a black surface. But the property "black" cannot be defined by boundary conditions within the realm of Maxwell's theory. Therefore diffraction by a black screen cannot be formulated as a boundary value problem. Our formulation (27 a) is by no means unique or devoid of arbitrariness.

We shall indicate only briefly other possible generalizations of our method. First, there is the case of the *cylindrical wave* (luminous line in the finite region and parallel to the edge of the screen) for which our method also yields a complete solution of the diffraction problem in closed form¹. The generalization to three dimensions is directly possible only for scalar (acoustic) problems. For these the diffraction of a spherical wave or a plane wave which is incident not perpendicularly but at an angle with respect to the edge of the screen can be treated by our method.

F. BASIC REMARKS ON BRANCHED SOLUTIONS

In electrodynamics there are two general types of problems: the summation problems and the boundary value problems; see Vol. III, Sec. 7 and Sec. 9. When the distribution of charges throughout space was given, then we only needed to sum over all these charges in the proper way in order to obtain the complete electrostatic field. The same was true in the magnetostatic case when the magnetisation was everywhere known. However, when material bodies, such as conductors, dielectrics, or magnetizable matter with *unknown* charges and magnetizations were present, certain boundary conditions had to be fulfilled. We were then faced with the mathematically much more complicated boundary value problems. An obvious requirement on the correct formulation of these boundary value problems was that of *uniqueness*.

Huygens' principle attempts to solve diffraction problems by the *summation method*. Since the boundary values to be prescribed in the diffraction opening are fundamentally unknown, and a certain plausible yet arbitrary

¹For details see Frank-Mises, loc. cit., p. 826.

choice must be made regarding them, the correctness and uniqueness of the solutions obtained in this way may properly be doubted. In the case of the rigorously solvable diffraction problems which we formulated at the beginning of this section, the boundary conditions were clear on the basis of Maxwell's theory. Supplemented by the radiation condition at infinity, these boundary conditions insured the uniqueness of our problem. In the conventional case of an *infinitely thin* screen one must go to the limit of infinite conductivity and use the corresponding limiting form of the boundary conditions. We have already seen that the *black screen* which is preferred in theory and in experiment cannot be described in terms of boundary conditions, and therefore the diffraction caused by it cannot be described by a uniquely defined boundary value problem.

For a perfectly reflecting plane screen of arbitrary shape the method of images leads to the problem of constructing *branched solutions* of the wave equation such that the edge of the screen is the branch line. In two-dimensional problems (slit, parallel strip, half-plane) the range of values of the solutions is represented on a two-sheeted Riemann surface. In three-dimensional problems the solution is defined on a Riemann double space. The mathematical construction of the branched solutions is possible only for the case of the half-plane. Nevertheless, our method of the Riemann double space leads to quantitative results also for the problem of an arbitrary plane screen. In order to see this we need further preparation.

It has been known since Euler that functions which are symmetric in the n roots of any n^{th} degree *algebraic equation* are rational functions of the coefficients of that equation. The same is true for the branches of an *algebraic function*, i. e. for the roots of an n^{th} degree equation whose coefficients are entire functions of a complex variable z . Such an algebraic function is defined on an n -sheeted Riemann surface. If we denote an algebraic function by $w(z)$ and its n branches by w_1, w_2, \dots, w_n , then all symmetric functions of the w_1, w_2, \dots, w_n are single-valued in z and are rational functions of the coefficients of the defining equation.

This theorem is used repeatedly in two-dimensional potential theory, as for instance in the mapping problems of hydrodynamics. If the velocity potential $u(x, y)$ and the stream function $v(x, y)$ are combined in the form $w = u + i v$, one obtains a function of the complex variable $z = x + i y$ whose real and imaginary parts satisfy Laplace's equation $\Delta \begin{Bmatrix} u \\ v \end{Bmatrix} = 0$. If w is multiple-valued, then the symmetric functions of its branches w_1, \dots, w_n are single-valued in z just as in the case of algebraic functions. From these

single-valued functions w , and therefore also u and v , can be calculated algebraically.

Here we are interested in two-valued solutions U of the wave equation where, in contrast to the solutions of Laplace's equation, there is no function conjugate to U . Of the symmetric functions only the linear combination $U_1 + U_2$ needs to be considered. This sum is a single-valued solution of the same differential equation and can therefore be regarded as known. (The symmetric product $U_1 U_2$ is not a solution of the wave equation; otherwise the two branches U_1 and U_2 could each be calculated algebraically, and the construction of the branched solution would be simple.) Limiting ourselves to scalar problems, we consider in particular a plane wave in the Riemann double space, the branch line of which coincides with the edge of the screen. $U_1(P)$ shall refer to the first branch, $U_2(P)$ to the second branch of this space; the two branches are connected in the plane of the screen. We form

$$(28) \quad U_1(P) + U_2(P) = u_0(P).$$

Then $u_0(P)$ is a single-valued solution of the wave equation in the simple space and is identical with our previous function u_0 which represented a plane wave with no screen present. This result is rigorous because the solutions of the wave equation are uniquely determined by the continuity conditions which must always be imposed and by the condition imposed on the behavior of the solution at infinity. (The same eq. (28) is obviously valid for an incident spherical or cylindrical wave as well as for a plane wave.)

We shall first confirm eq. (28) by applying it to the explicit formulae for our double space with a straight branch line. It is convenient to start with eq. (19). Denoting by ρ the quantity (18 a) which is positive on the illuminated sheet, and hence denoting by $-\rho$ the corresponding quantity on the shaded sheet, we obtain

$$(29) \quad U_1(P) + U_2(P) = u_0 \frac{1-i}{2} \left\{ \int_{-\infty}^{\rho} + \int_{-\infty}^{-\rho} \right\} e^{\frac{i\pi}{2}\tau} d\tau.$$

We see immediately that by changing the sign of τ in the second integral both integrals can be combined into

$$\int_{-\infty}^{\infty} e^{\frac{i\pi}{2}\tau} d\tau = 2 \int_0^{\infty} e^{\frac{i\pi}{2}\tau} d\tau = 2F(\infty) = 1+i,$$

and (29) becomes identical with (28). It should be emphasized that this proof depends in no way on the transformed form (19) but can be performed equally well using the original form (9). In this latter formulation the path

of integration for U_2 is obtained by displacing the integration path for U_1 by 2π . The two paths, see fig. 78, combine then into two loops C which span a distance 4π instead of 2π . Because of the periodicity of the integrand, the two loops can be transformed into a circuit around the pole $\varphi = \alpha$, so that the integral again yields u_0 . In the same way it can also be seen that for the generalization embodied in (27) (n arbitrary instead of $n = 2$), the statement (28) is generalized to

$$U_1(P) + U_2(P) + \dots + U_n(P) = u_0(P).$$

Let us now compare (28) with our earlier formulation of Babinet's principle (34.15). The formal similarity between these two expressions suggests that U_1 and U_2 be associated with the diffraction patterns of two *complementary* screens I and II¹. This is however permissible only for the case of *black* screens which, like the branch cut of our Riemann surface, can absorb light but cannot reflect it. In addition, this association is subject to the same lack of uniqueness which the definition of the black body suffers. Therefore we seek a formulation of Babinet's principle which is valid for the well-defined *reflecting* screen and which can be considered as a more *precise formulation* of the principle. By way of preparation we again consider the simple case of the half-plane.

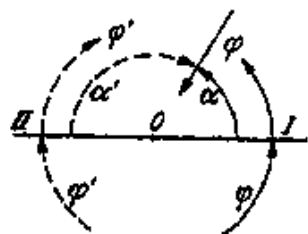


Fig. 80.

Babinet's principle: directions to be used for measuring the angles φ and φ' for the cases of the original screen $O I$ and the complementary screen $O II$.

We compare the diffraction by the original half-plane

$$(30) \quad u_I = U(r, \varphi - \alpha) \mp U(r, \varphi + \alpha), \quad 0 < \varphi < 2\pi$$

with that by the complementary half-plane which, in the corresponding notation, reads

$$(31) \quad u_{II} = U(r, \varphi' - \alpha') \mp U(r, \varphi' + \alpha'), \quad 0 < \varphi' < 2\pi.$$

If the direction of incidence of the plane wave is the same for both screens, then we must make

$$(31a) \quad \alpha' = \pi - \alpha.$$

The relationship between φ' and φ can be inferred from fig. 80. It leads to the following compilation:

Front surface of the complementary screen	$\varphi' = 0, \quad \varphi = \pi$	} (31b) $\varphi' = \pi - \varphi,$
Front surface of the original screen	$\varphi' = \pi, \quad \varphi = 0$	
Rear surface of the original screen	$\varphi' = \pi, \quad \varphi = 2\pi$	} (31c) $\varphi' = 3\pi - \varphi.$
Rear surface of the complementary screen	$\varphi' = 2\pi, \quad \varphi = \pi$	

¹Compare with the presentation by the author in Frank-Mises, Vol. II, Chap. XX, Sec. 1, eq. (15).

Equation (31 b) is the desired relation for the front surface of both screens; eq. (31 c) is the relation which applies to the rear surface.

Substituting (31 a, b) in (31) we obtain for the *front surface*

$$(32) \quad u_{II} = U(r, -\varphi + \alpha) \mp U(r, 2\pi - \varphi - \alpha).$$

Because of the property (28) of our branched solution

$$(32 a) \quad U(r, 2\pi - \varphi - \alpha) = u_0(-\varphi - \alpha) - U(r, -\varphi - \alpha)$$

and because of the right-left symmetry of both the branched and unbranched waves

$$(32 b) \quad U(r, -\varphi - \alpha) = U(r, \varphi + \alpha), \quad U(r, -\varphi + \alpha) = U(r, \varphi - \alpha), \\ u_0(-\varphi - \alpha) = u_0(\varphi + \alpha).$$

Substituting (32 a, b) in (32) it follows that

$$(33) \quad u_{II} = U(r, \varphi - \alpha) \pm U(r, \varphi + \alpha) \mp u_0(\varphi + \alpha).$$

It should be noted that the sign of $U(r, \varphi + \alpha)$ is now *opposite* to that in (30). Since, as we know, this sign is determined by the polarization of the incident wave (\mp indicates that \mathbf{E} is parallel or perpendicular to the edge of the screen, respectively), (33) tells us that we must compare the diffraction pattern of the *complementary* screen illuminated by *parallel* polarized incident light ($\mathbf{E}_{||}$) with the diffraction pattern of the *original* screen illuminated by *perpendicular* polarized light (\mathbf{E}_{\perp}). Furthermore, the term $\mp u_0(\varphi + \alpha)$ shows that for the complementary screen we must omit the reflected light wherever it is present for the original screen and that we have to add reflected light where it is missing for the original screen. This is understandable from the viewpoint of geometrical optics.

A corresponding calculation using (31 a, c) and (31) yields for the *rear surface*

$$u_{II} = U(r, 2\pi - \varphi + \alpha) \mp U(r, 4\pi + \varphi + \alpha),$$

and applying the correspondingly modified eqs. (32 a, b) we obtain

$$(34) \quad -u_{II} = U(r, \varphi - \alpha) \pm U(r, \varphi + \alpha) - u_0(\varphi - \alpha).$$

Thus we have the same interchange of polarizations as in (33); in addition, the incident wave must now be omitted for the complementary screen where it was originally present, i. e. behind the screen, and the incident wave must be added where it was missing with the original screen, i. e. in the shadow of the latter which by geometrical optics is the illuminated region of the complementary screen.

The problem of the half-plane shares with other *two-dimensional* problems (slit, grating, ...) the feature that it can be treated as a *scalar* problem. This is not the case for the *three-dimensional* problems of *optics* (e. g. circular disc

or circular opening). To solve these problems *vector* calculations are necessary (or suitably defined potentials may be used). This is not so in acoustics where the scalar pressure (or the velocity potential) is treated. In the dissertation which was referred to on p. 248, Bouwkamp set up the rigorous form of Babinet's principle for the scalar three-dimensional problem of an arbitrarily bounded rigid plane screen and its coplanar complementary screen. The statement of the principle is the same as in our two-dimensional problems, and the proof again relies upon the relation (28) for branched functions. The above method can be extended to the three-dimensional scalar case by considering the straight screen edge and the cylindrical surfaces surrounding it to be deformed in the manner of topology into the given arbitrarily shaped edge curve and the corresponding toroidal surfaces surrounding it. These surfaces can be distinguished by assigning to them a parameter φ which can be chosen so as to increase by 2π as it passes from one sheet of the double space to the other. If this is done, eqs. (30) to (34) can be interpreted directly as the expression of Babinet's principle for the scalar three-dimensional case.

The completely general, rigorous formulation of Babinet's principle for the three-dimensional optical case has been given by J. Meixner¹. Since in the incident wave E_{\perp} implies H_{\parallel} and (retaining a right-handed system) E_{\parallel} implies $-H_{\perp}$, we can replace our sign inversion $\mp \rightarrow \pm$ in going from the original to the complementary screen by the following interchange rule:

$$(E, H) \rightarrow (H, -E).$$

Therefore, according to Meixner, one obtains the diffraction field E, H of the complementary screen from the diffraction field $H, -E$ of the original screen (provided the incident or reflected wave is cancelled or added at the front or rear of the screens in a precisely specified manner). The proof holds for arbitrary distributions of light sources, not merely for a wave coming from infinity. It is based solely on the symmetry properties of Maxwell's equations. We wish to note in this connection that fundamentally our representation in terms of branched solutions also implies a symmetry property of Maxwell's equations.

Finally, we shall establish the connection between our energy condition (1 e) and the theory of functions of a complex variable z . We shall again limit ourselves to the scalar case. Specifically, (1 e) is related to the Puiseux expansion of a function about a branch point which replaces the otherwise valid Taylor expansion. If the number of sheets connected in the branch point $z = 0$ is n , and if this branch point is not at the same time a pole of the function $w(z)$, then

¹Z. f. Naturforschung, Vol. 3 a, p. 508, 1948.

$$w(z) = \sum_{m=0}^{\infty} C_m z^{\frac{m}{n}} = \sum_{m=0}^{\infty} C_m r^{\frac{m}{n}} e^{i \frac{m}{n} \varphi}.$$

The corresponding expansions of solutions u of the wave equation are in terms of Bessel functions with fractional indices

$$(35) \quad u(r, \varphi) = \sum_{m=0}^{\infty} C_m J_{\frac{m}{n}}(kr) e^{i \frac{m}{n} \varphi}.$$

Since $J_{\frac{m}{n}}(\rho)$ is proportional to $\rho^{\frac{m}{n}}$ for small values of ρ , this expansion yields at the branch point $r = 0$

$$(36) \quad u = C_0 = \text{finite value, but } \frac{\partial u}{\partial r} = k \sum_{m=1}^{n-1} C_m J'_{\frac{m}{n}}(kr) e^{i \frac{m}{n} \varphi}.$$

$\partial u / \partial r$ becomes infinite at $r = 0$ but only weakly, so that

$$(36 \text{ a}) \quad \lim_{r \rightarrow 0} r \text{ grad } u = 0.$$

This condition was verified explicitly for $n = 2$ in eq. (20 e). It is reasonable to postulate this same condition also for the case of a spatial branch line¹, in which case r would mean the shortest distance from the branch line. As a result our condition (1 e) turns out to be a *mathematical consequence* of the condition that u be everywhere continuous (also at the edge of the screen) and therefore (1 e) can be omitted as a special requirement.

¹Compare A. Sommerfeld, Proc. London Math. Soc., Vol. 28, particularly p. 405, 1897. "Branched Potentials in Space" are treated there.

CHAPTER VI

ADDENDA, CHIEFLY TO THE THEORY OF DIFFRACTION

39. Diffraction By a Very Narrow Slit

When the dimensions of the diffracting aperture become small compared to the wavelength or even only a few times larger, Huygens' principle becomes meaningless. For in our applications of this principle (Sec. 34) we used only the unperturbed wave incident on the opening and *neglected the effect of the edge zones* entirely. Therefore our present problem in which the aperture consists, so to speak, mainly of edge zones belongs definitely to the category of *boundary value problems*; it is now necessary to determine the state of the field in the opening from the continuity conditions imposed on the total solution. Thus the distinction between the incident and diffracted wave breaks down.

Lord Rayleigh¹ was the first to tackle this problem. With masterful brevity and clarity he reduced the problem to known solutions of hydrodynamic or electrostatic problems; he did this in particular for the case of a circular opening of radius $a \ll \lambda$ or for a sufficiently narrow slit.

Bethe² treated the problem of the small circular aperture independently from Rayleigh from an electromagnetic point of view and obtained substantially the same result. The work of Levine and Schwinger³ is aimed at the more difficult goal of bridging the gap between the limiting cases $a \ll \lambda$ (Lord Rayleigh) and $a \gg \lambda$ (Huygens-Kirchhoff) by means of a variational principle. For the time being this work is restricted to the scalar acoustic case.

We shall treat the experimentally important problem of the slit in which we may operate with scalar equations (see p. 277) by considering separately the two cases E and H parallel to the slit edges.

¹On the Passage of Waves through Apertures in Plane Screens and Allied Problems, Phil. Mag. 43, p. 259, 1897. Scientific Papers, Vol. IV, p. 283.

²H. A. Bethe, Phys. Rev. 66, p. 163, 1944.

³H. Levine and J. Schwinger, Phys. Rev. 74, p. 958, 1948, 75, 1423, 1949.

A. THE BOUNDARY VALUE PROBLEM OF THE SLIT

Let the slit lie in the xy -plane and let the slit edges be parallel to the y -axis. The screen shall be thought of as infinitely thin and perfectly conducting. The width of the slit shall be $2a$, and the edges are to be given by $x = \pm a$, $z = 0$. A plane wave coming from the negative z direction is incident perpendicular to the plane of the slit. The problem is entirely independent of y and is therefore two-dimensional.

The incident wave will be represented by $A \exp(i k z)$. We shall assume first that \mathbf{E} oscillates parallel to the edges of the slit, i. e. $\mathbf{E} \rightarrow E_y$. If there were no slit in the screen, the field would be given by

$$(1) \quad \begin{cases} v = A (e^{i k z} - e^{-i k z}) & \text{for } z < 0, \\ v = 0 & \text{for } z > 0. \end{cases}$$

Because of the presence of the slit, (1) must be changed to

$$(2) \quad \begin{aligned} v &= A (e^{i k z} - e^{-i k z}) + u_- & \text{for } z < 0, \\ v &= u_+ & \text{for } z > 0. \end{aligned}$$

We denote the value of v in the slit opening by \bar{u} . Because of the continuity of the field v

$$(3) \quad u_+ = u_- = \bar{u} \quad \text{for } z = 0.$$

We call the x -coordinate of a point in the slit ξ and write therefore $\bar{u} = \bar{u}(\xi)$. If $\bar{u}(\xi)$ were known, then we could compute u_+ and u_- rigorously for all points $z \geq 0$ by the general method of Green's function introduced in Sec. 34 C. However, in the Green's function (34.7)

$$(4) \quad G = \frac{e^{i k r}}{r} - \frac{e^{i k r'}}{r'}, \quad \begin{cases} r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2, \\ r'^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta + z)^2, \end{cases}$$

we must now replace the spherical wave $e^{i k r}/r$ by the cylindrical wave $H(kr)$, where $H = H_0^{(1)}$ is the Hankel function of the first kind of index zero. We must also interpret r in the two-dimensional sense (because of the nature of our light source, the integration over the coordinate y has, so to speak, already been performed implicitly in $H(kr)$). Thus we obtain in place of (4)

$$(4a) \quad G = \frac{i\pi}{2} (H(kr) - H(kr')), \quad \begin{cases} r^2 = (\xi - x)^2 + (\zeta - z)^2, \\ r'^2 = (\xi - x)^2 + (\zeta + z)^2. \end{cases}$$

Equation (34.6) yields then

$$(5) \quad 2\pi u_{\pm} = - \int_{-a}^{+a} \bar{u}(\xi) \frac{\partial G}{\partial n} d\xi.$$

The factor 2π replaces the 4π of (34.6) because in the two-dimensional Green's theorem the left-hand side of (5) arises from the integration over a circle of vanishingly small radius instead of over the surface of a sphere as in the three-dimensional theorem. $\partial G/\partial n$ means the derivative with respect to the outward normal, as before; in our case $\partial n = -\partial\zeta$ for u_+ , $\partial n = +\partial\zeta$ for u_- and by (4 a)

$$\frac{\partial G}{\partial n} = \mp \frac{i\pi}{2} \frac{\partial}{\partial \zeta} (H(kr) - H(kr')) = \pm \frac{i\pi}{2} \frac{\partial}{\partial z} (H(kr) + H(kr')).$$

In the slit $\zeta = 0$ and hence $r = r'$ and

$$(5a) \quad \frac{\partial G}{\partial n} = \pm i\pi \frac{\partial}{\partial z} H(kr_0), \quad r_0^2 = (\xi - x)^2 + z^2.$$

Equation (5) becomes

$$(6) \quad u_{\pm} = \mp i \frac{\partial}{\partial z} \int_{-a}^{+a} \bar{u}(\xi) H(kr_0) d\xi.$$

Since $\bar{u}(\xi)$ is actually not known, eq. (6) contains no information. This equation must be supplemented by the requirement that $\partial v/\partial z$ be continuous at the slit; the continuity of v is already guaranteed by (3). This continuity condition of $\partial v/\partial z$ now takes the place of our original boundary value problem which has so far been solved only incompletely.

According to eq. (2)

$$\begin{aligned} \frac{\partial v}{\partial z} &= 2ikA + \frac{\partial u_-}{\partial z} & \text{as } z \rightarrow 0 & \text{from } z < 0, \\ \frac{\partial v}{\partial z} &= \frac{\partial u_+}{\partial z} & \text{as } z \rightarrow 0 & \text{from } z > 0. \end{aligned}$$

Therefore we must require that

$$(7) \quad \frac{\partial u_+}{\partial z} - \frac{\partial u_-}{\partial z} = 2ikA \quad \text{for } z = 0.$$

From this together with eq. (6) it follows that

$$(8) \quad -\frac{\partial^2}{\partial z^2} \int_{-a}^{+a} \bar{u}(\xi) H(kr_0) d\xi = kA.$$

This condition must be fulfilled for $z = 0$ and for all values $-a < x < +a$. It is to be noted here that according to (5 a), r_0 depends on z and therefore the limit value

$$(8a) \quad r_0 = |\xi - x|$$

may be substituted only after the twofold differentiation indicated in (8) has been performed.

We can simplify the mathematical situation if we take into account¹ the fact that the cylindrical wave $H(kr_0)$ satisfies the two-dimensional wave equation $\Delta H + k^2 H = 0$. Therefore also the integral in (8) satisfies this equation and we have

$$(8\ b) \quad -\frac{\partial^2}{\partial x^2} \int \dots = \left(\frac{\partial^2}{\partial x^2} + k^2 \right) \int \dots$$

where we are now permitted to go to the limit (8 a) on the right-hand side and may also write d^2/dx^2 in place of $\partial^2/\partial x^2$. Thus we obtain from (8)

$$(9) \quad \left(\frac{d^2}{dx^2} + k^2 \right) X = k A$$

with the abbreviation

$$(9\ a) \quad X = \int_{-a}^{+a} \bar{u}(\xi) H(k|\xi - x|) d\xi.$$

We integrate (9) using the rule for the integration of inhomogeneous differential equations. A particular integral of (9) is $X = A/k$; because of the symmetry of the problem, only that part of the general solution of the homogeneous equation which is even in x , that is $B \cos kx$, is to be used. Hence

$$(9\ b) \quad X = A/k + B \cos kx.$$

In order to determine the constant of integration B we set $x = 0$ and find using (9 a, b), that

$$B = -A/k + \int_{-a}^{+a} \bar{u}(\xi) H(k|\xi|) d\xi.$$

Substituting this value on the right-hand side of (9 b) and using on the left-hand side for X its value (9 a), one obtains

$$(10) \quad \int_{-a}^{+a} \bar{u}(\xi) \{H(k|\xi - x|) - \cos kx H(k|\xi|)\} d\xi = \frac{A}{k} (1 - \cos kx).$$

This is a linear integral equation for the unknown function $\bar{u}(\xi)$ which must be satisfied for all values $-a < x < +a$. The "Kernel" of the integral equation is the expression inside the curly brackets in (10). A general rule is thus confirmed: the solution of a boundary value problem can be reduced to the solution of an integral equation. The solution of the latter can always be obtained numerically, but of course only for special values of the

¹After the manner of Levine and Schwinger, loc. cit. eq. (A 3) for the analogous case of the circular hole.

parameters occurring in the equation (in this case the values of ka and kx). This is of no help to us. In order to obtain a general solution one must in each case invent suitable approximation methods. In our case these have to arise from the assumption $ka \ll 1$. Furthermore, we note that the kernel in (10) is unsymmetric in x and ξ while mathematical theory commonly deals with symmetric kernels.

Before we proceed to solve the integral equation, we must briefly consider the other case of polarization. We now denote by v the magnetic vector H which, because $E_x = 0$, must satisfy $\partial v / \partial z = 0$ on the screen. Calling the magnetic amplitude of the incident wave $A'^{1/2}$, we set in place of (2)

$$\begin{aligned} v &= A' (e^{ikz} + e^{-ikz}) + u & \text{for } z < 0, \\ v &= u_+ & \text{for } z > 0. \end{aligned}$$

and obtain in place of (3)

$$\frac{\partial u_+}{\partial z} = \frac{\partial u_-}{\partial z} = \omega,$$

where $\omega = \omega(\xi)$ is now the unknown function to be determined. As the Green's function we have to take

$$(11) \quad G = \frac{i\pi}{2} \{H(kr) + H(kr')\}.$$

From it one finds, in contrast to (5) (compare with Sec. 34 G):

$$(11a) \quad 2\pi u_{\pm} = \mp \int_{-a}^{+a} \omega(\xi) G d\xi = \mp i\pi \int_{-a}^{+a} \omega(\xi) H(kr_0) d\xi;$$

hence in the slit opening

$$(11b) \quad 2\pi u_{\pm} = \mp i\pi \int_{-a}^{+a} \omega(\xi) H(k|\xi - x|) d\xi.$$

Because of the continuity of v at the slit

$$\bar{u}_+ - \bar{u}_- = 2A';$$

from this and (11b) it follows that

$$(12) \quad \int_{-a}^{+a} \omega(\xi) H(k|\xi - x|) d\xi = iA'.$$

The derivation and form of this integral equation is somewhat simpler than in the preceding case; also the "Kernel" $H(k|\xi - x|)$ of (12) is symmetric².

¹The notation is the same as in Sec. 2; A and A' have different dimensions and differ by the "wave resistance".

²This integral equation was first obtained by G. Jaffé, Phys. Zeitschr. 22, p. 578, 1921.

B. SOLUTION OF THE INTEGRAL EQUATIONS (10) AND (12)

It will be necessary to make a hypothesis as to the form of the function $\bar{u}(\xi)$ in (10) such that it contains an infinite number of undetermined coefficients and then to attempt to determine these coefficients from (10). The choice of form is limited by the following considerations:

1. $\bar{u}(\xi)$ must vanish at $\xi = \pm a$ because the solution must approach continuously the value $v = 0$ prescribed on the screen.

2. Because of the symmetry of the problem, $\bar{u}(\xi)$ must be an even function of ξ .

3. In view of our treatment of branched wave functions and their representation at the end of Sec. 38, $u(x, z)$ must change its sign for every complete circuit around one of the two branch points $x = \pm a$. Together with the requirements 1. and 2., this leads to the form

$$(13) \quad \bar{u}(\xi) = \sum_{n=1}^{\infty} C_n \left(1 - \frac{\xi^2}{a^2}\right)^{n-1/2}.$$

The C_n are infinitely many available complex coefficients.

In an earlier acoustic work by the author¹, which was also used as a starting point by Levine and Schwinger (see p. 273), a result analogous to (13) was obtained by a protracted calculation. This calculation at the same time provided the values of the C_n in the form of numerically definite power series in the parameter ka , which is the only characteristic parameter. It turned out that the series for C_{n+1} starts with a power of a which is greater by one than the power of the first term in the series for C_n . Here we have been able to write down the form of the solution (13) directly on the basis of function theory so that it is valid for a slit of arbitrary width. For a very narrow slit $a \ll 1$, the above-mentioned result shows that C_1 is larger by one order of magnitude than all the other C_n . Therefore we shall specialize (13) to

$$(13a) \quad \bar{u}(\xi) = C_1 \left(1 - \frac{\xi^2}{a^2}\right)^{1/2}.$$

Then the integral equation (10) reads

$$(13b) \quad C_1 \int_0^a \left(1 - \frac{\xi^2}{a^2}\right)^{1/2} K(x, \xi) d\xi = \frac{A}{k} (1 - \cos kx) \sim A k x^2/2.$$

¹Die frei schwingende Kolbenmembran, Ann. d. Phys. (Lpz.) 42, 389, 1943.

In contrast to (10) the interval of integration has here been restricted to $0 < \xi < a$; the interval $-a < \xi < 0$ must be taken into account by the following modification of the kernel in which x may be assumed to be positive:

$$(13\ c) \quad K(x, \xi) = H(k|\xi - x|) + H(k(\xi + x)) - 2 \cos kx H(k\xi).$$

We decompose this kernel into two parts

$$(13\ d) \quad K_I = H(k(\xi + x)) + H(k|\xi - x|) - 2 H(k\xi).$$

$$(13\ e) \quad K_{II} = 2(1 - \cos kx) H(k\xi) \sim k^2 x^2 H(k\xi).$$

Since $ka \ll 1$, the arguments of all the H -functions are small in the whole interval of integration, and therefore we can use everywhere the approximate formula from Vol. III, eq. (22.5):

$$(13\ f) \quad H_0(\rho) = \frac{2i}{\pi} \log \frac{\gamma \rho}{2i}; \quad \log \gamma = 0.5772 = \text{Euler-Mascheroni constant.}$$

Then

$$(13\ g) \quad K_I = \frac{2i}{\pi} \log \frac{|\xi^2 - x^2|}{\xi^2}, \quad K_{II} = \frac{2i}{\pi} k^2 x^2 \log \frac{\gamma k \xi}{2i}.$$

The logarithm in K_I must be expanded differently depending upon whether $\xi < x$ or $\xi > x$:

$$\log \frac{|x^2 - \xi^2|}{\xi^2} = \begin{cases} \log \frac{x^2}{\xi^2} - \left(\frac{\xi^2}{x^2} + \frac{1}{2} \frac{\xi^4}{x^4} + \dots \right) & \text{for } \xi < x, \\ - \left(\frac{x^2}{\xi^2} + \frac{1}{2} \frac{x^4}{\xi^4} + \dots \right) & \text{for } \xi > x. \end{cases}$$

Correspondingly, the integral (13 b) must be decomposed into two parts:

$$(14) \quad \int_0^a \dots = \int_0^x \dots + \int_x^a \dots = J_1 + J_2,$$

$$(14\ a) \quad J_1 = \int_0^x \left\{ \log \frac{x^2}{\xi^2} - \left(\frac{\xi^2}{x^2} + \frac{1}{2} \frac{\xi^4}{x^4} + \dots \right) \right\} d\xi,$$

$$(14\ b) \quad J_2 = - \int_x^a \left(1 - \frac{\xi^2}{a^2} \right)^{1/2} \left(\frac{x^2}{\xi^2} + \frac{1}{2} \frac{x^4}{\xi^4} + \dots \right) d\xi.$$

Since C_1 is independent of x , it is permissible to choose x so small compared to a that the factor $(1 - \xi^2/a^2)^{1/2}$ in J_2 may be replaced by 1. This has already been done in (14 a).

By elementary integration one obtains

$$(15) \quad J_1 = 2x - x \left(\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 7} \dots \right).$$

Thus J_1 is proportional to x , while the right-hand side of our integral equation (13 b) was proportional to x^2 . In Appendix 1 we shall show that J_1 is cancelled by the contribution from the lower limit of the integral J_2 . It will also be shown there that the contribution from the upper limit of J_2 yields, except for terms of higher power in x/a ,

$$(15 a) \quad \frac{\pi x^2}{2 a}.$$

According to (13 g) and (14) the entire value of K_I on the left-hand side of (13 b) becomes now

$$(16) \quad C_1 \frac{2 i}{\pi} \cdot \frac{\pi x^2}{2 a} = i C_1 \frac{x^2}{a}.$$

The contribution of K_{II} to (13 b) follows from (13 g):

$$(16 a) \quad C_1 \frac{2 i}{\pi} k^2 x^2 \int_0^a \left(1 - \frac{\xi^2}{a^2}\right)^{\frac{1}{2}} \log \frac{\gamma k \xi}{2 i} d\xi.$$

If we introduce the substitution (25) from Appendix 1 and put

$$(16 b) \quad q = \frac{4}{\pi} \int_0^{\pi/2} \cos^2 \varphi \log \sin \varphi d\varphi,$$

we obtain in place of (16 a)

$$(17) \quad \frac{1}{2} C_1 i k^2 a x^2 \left(\log k a + \log \frac{\gamma}{2 i} + q \right).$$

The sum of (16) and (17) is

$$i C_1 \frac{x^2}{a} \left\{ 1 + \frac{1}{2} / (k a)^2 \left(\log k a + \log \frac{\gamma}{2 i} + q \right) \right\}.$$

Because $k a \ll 1$, the second term in the curly brackets can be neglected; thereupon the integral equation (10) yields

$$(18) \quad i C_1 \frac{x^2}{a} = A k \frac{x^2}{2}, \quad C_1 = -i A k a / 2.$$

We now turn to the integral eq. (12). As at the end of section A, v and u_{\pm} are magnetic vector components parallel to the y -axis, and ω is the value of $\partial u_{\pm} / \partial z$ at $z = 0$ inside the slit. First we seek the form of ω which is analogous to (13). We assert that ω is again given by (13), except that *the lower limit in the summation must be replaced by $n = 0$* . Thus in the first approximation

$$(19) \quad \omega(\xi) = \frac{C_0}{a} / \left(1 - \frac{\xi^2}{a^2}\right)^{\frac{1}{2}}.$$

To justify this statement it is only necessary to remark that near the branch points $x = \pm a$ the vector H behaves like the square root of the distance from the branch points, and therefore the gradient of H behaves like the inverse square root of that distance. We shall indeed see that (19) leads to the unique solution of our problem. The factor a in the denominator of (19) has been included in order that C_0 (as C_1 before) have the same dimension as our present u . $\omega(\xi)$ is, as $\bar{u}(\xi)$ was before, an even function of ξ .

We now rewrite the integral eq. (12) in the form

$$(20) \quad \int_0^a \omega(\xi) \{H(k|\xi-x|) + H(k|\xi+x|)\} d\xi = i A'.$$

The $\{\dots\}$ is again called the *kernel* of the integral equation. In order to be able to use the above calculations we again separate this kernel into two parts

$$(20 \text{ a}) \quad K_I = H(k|\xi+x|) + H(k|\xi-x|) - 2H(k\xi),$$

$$(20 \text{ b}) \quad K_{II} = 2H(k\xi).$$

The first part is identical with K_I in (13 d). Therefore eqs. (14 a, b) are *mutatis mutandis* again valid; these yield now only terms proportional to x , the sum of which is zero; see appendix 2. Equation (20) therefore simplifies to

$$(21) \quad \int_0^a \omega(\xi) K_{II} d\xi = \frac{2C_0}{a} \int_0^a \left(1 - \frac{\xi^2}{a^2}\right)^{-1/2} H(k\xi) d\xi = i A'$$

and one obtains, see appendix 2:

$$(22) \quad C_0 = \frac{1}{2} A' / p; \quad p = \log \frac{\gamma k a}{4i} = \log k a - 0.81 - i\pi/2.$$

C. DISCUSSION

In figures 81 a, b the distributions of $\bar{u}(x)$ and $\omega(x)$ are shown in comparison to the amplitudes of the incident waves A and A' , respectively. From (13 a) and (18) or (19) and (22), respectively, one obtains for $\xi = 0$:

$$\frac{\bar{u}(0)}{A} = \frac{C_1}{A} = -\frac{i}{2} k a; \quad \frac{\omega(0)}{A'} = \frac{C_0}{A'} = \frac{1}{2p}.$$

$|\bar{u}(x)|$ is a very flat ellipse. $|\omega(x)|$ is the corresponding reciprocal curve which has a much larger value than $|\bar{u}(x)|$ at the center of the slit and goes to infinity at the edges.

From \bar{u} and ω one can compute $u_+(x, z)$ by means of formulae (6) and (11 a), and thereby one obtains for the diffraction field v :

$$\pi v = -C_1 \frac{\partial}{\partial z} \int_{-a}^{+a} \left(1 - \frac{\xi^2}{a^2}\right)^{1/2} H(k r_0) d\xi,$$

and

$$\pi v = -\frac{C_0}{a} \int_{-a}^{+a} \left(1 - \frac{\xi^2}{a^2}\right)^{-1/2} H(k r_0) d\xi,$$

with

$$r_0^2 = (x - \xi)^2 + z^2.$$

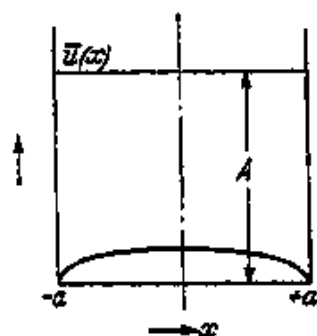


Fig. 81 a.

Electric vector \mathbf{E} parallel to the slit edges. Graph of the amplitude of $\mathbf{E} = \bar{u}(x)$ in the slit opening; $ka = 1/10$.

$$\bar{u}(x) = \frac{i}{2} ka A \sqrt{1 - x^2/a^2}.$$

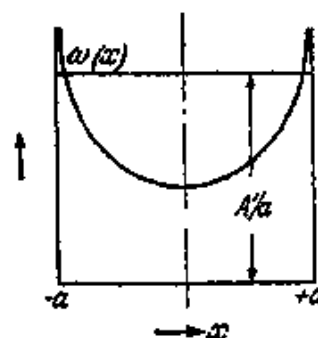


Fig. 81 b.

Magnetic vector \mathbf{H} parallel to the slit edges. Graph of the amplitude of $\partial \mathbf{H} / \partial z = |\omega(x)|$ in the slit; $ka = 1/10$

$$\omega(x) = \frac{1}{2p} \frac{A'/a}{\sqrt{1 - x^2/a^2}}.$$

Since the point of observation x, z is at a distance of many wavelengths from the slit, we may use for H its asymptotic formula [see, for instance, Vol. III, eq. (22.7)], and we may take $r_0 = r = \sqrt{x^2 + z^2}$ independent of ξ in the integration. In this way we obtain

$$\pi v = -C_1 \frac{\partial}{\partial z} \sqrt{\frac{2}{\pi k r}} e^{i(kr - \pi/4)} \int_{-a}^{+a} \left(1 - \frac{\xi^2}{a^2}\right)^{1/2} d\xi,$$

and

$$\pi v = -\frac{C_0}{a} \sqrt{\frac{2}{\pi k r}} e^{i(kr - \pi/4)} \int_{-a}^{+a} \left(1 - \frac{\xi^2}{a^2}\right)^{-1/2} d\xi.$$

Making the substitution $\xi = a \sin \Phi$, as in the appendix, one obtains for the values of the above integrals $a\pi/2$ and $a\pi$, respectively. Then, if the differentiation with respect to z is performed only in the exponent, one finds

$$(23) \quad v = -i a k C_1 \frac{z}{r} \frac{1}{\sqrt{2\pi k r}} e^{i(kr - \pi/4)},$$

and

$$(23 a) \quad v = -C_0 \sqrt{\frac{2}{\pi k r}} e^{i(kr - \pi/4)}.$$

Thus we have obtained two cylindrical waves of different amplitudes which originate from the slit (or rather from its center line). The amplitude of the first of these waves contains the cosine factor

$$\cos \delta = \frac{z}{r}, \quad (\delta = \text{angle of diffraction}).$$

In order to make eq. (23 a) dimensionally commensurate with (23) (the former represents H_y and not E_y as does the latter), we compute the electric components E_x , E_z belonging to H_y . We use Maxwell's equation $\vec{D} = \text{curl } H$

$$\begin{aligned} -i\omega\epsilon_0 E_z &= -\frac{\partial H_y}{\partial z} = -ik \frac{z}{r} H_y, & E_z &= \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{z}{r} H_y, \\ -i\omega\epsilon_0 E_x &= +\frac{\partial H_y}{\partial x} = +ik \frac{x}{r} H_y, & E_x &= -\sqrt{\frac{\mu_0}{\epsilon_0}} \frac{x}{r} H_y \end{aligned}$$

from which

$$E_{\perp} = \sqrt{E_x^2 + E_z^2} = \sqrt{\frac{\mu_0}{\epsilon_0}} H_y.$$

With this wave resistance factor $(\mu_0/\epsilon_0)^{1/2}$ the amplitude A' occurring in C_0 has the same dimensions as the A occurring in C_1 . Therefore equal intensity of illumination for both cases implies not $A' = A$, but rather $A' (\mu_0/\epsilon_0)^{1/2} = A$.

We are now able to calculate the expected *polarization* of the diffracted light if the incident light is composed of equal intensities of the two modes of polarization. The polarization of the diffracted light is characterized by the quotient of (23) and (23 a) which, taking (18) and (22) into account, is

$$(24) \quad \frac{1}{2} a k \left| \frac{C_1}{C_0} \right| \cos \delta = \frac{1}{2} (a k)^2 |\rho| \cos \delta \sqrt{\mu_0/\epsilon_0}.$$

Therefore for small values of ka much less light with E_{\parallel} passes through the slit than light with E_{\perp} . The oscillations E_{\parallel} are suppressed by the slit because of the boundary condition $E_y = 0$. The oscillations E_{\perp} induce charges on the slit edges, and these charges travel along the screen in the manner of

Hertz waves in wires; in this way they are able to overcome the curvature of the slit edges. This polarization effect is well known from Hertz's experiments with gratings.

The intensities for the two states of polarization exhibit notably different dependences on wavelength. For, according to (23) and (18)

$$J_{||} \text{ is proportional to } \left(\frac{a^2 k^2}{\sqrt{k r}} \right)^2 = \left(\frac{2\pi a}{\lambda} \right)^3 \frac{a}{r},$$

while according to (23 a) and (22)

$$J_{\perp} \text{ is proportional to } \left(\frac{1}{\sqrt{k r |p|}} \right)^2 = \frac{\lambda}{2\pi a} \frac{a}{r} \frac{1}{|\log \lambda/a + \dots|^2}.$$

In both cases the behavior is different from Rayleigh's λ^{-4} law for the blue of the sky. The latter is based on openings which are small in *all* their dimensions (or on correspondingly small discs). Our slit which is narrow in only *one* dimension yields also a totally different behavior for different directions of polarization of the incident light.

Our expressions (23) and (23 a) for the diffraction field are in complete agreement with eqs. (53) and (47) of Lord Rayleigh, loc. cit. Rayleigh also remarked that for the complementary case of the metallic strip eqs. (53) and (47) interchange their roles. This, as we know, is the precise statement of Babinet's principle; see Sec. 38 F.

Despite its complexity our solution of the problem has the following advantages:

1. Unlike Rayleigh's method it does not require any previous knowledge of electrostatics or hydrodynamics.

2. Our method is capable of being *generalized*.

One only needs to extend our one-term expressions (13 a) and (19) by adding terms with C_2, C_3, \dots or C_1, C_2 , respectively, in order to extend the results to wider and wider slits. How this may be done will be shown in appendix 3.

Appendix 1

To evaluate the integral J_2 in (14 b) the following substitution is made.

$$(25) \quad \xi = a \sin \Phi, \quad \left(1 - \frac{\xi^2}{a^2} \right)^{1/2} = \cos \Phi, \quad d\xi = a \cos \Phi d\Phi.$$

If, at the same time, we put

$$(25 a) \quad x = a \sin \psi; \quad \psi \sim \frac{x}{a} \ll 1$$

in the lower limit of the integral, we are led to the following auxiliary expressions

$$(26) \quad j_{2n} = - \int_{\psi}^{\pi/2} \frac{\cos^2 \Phi}{\sin^{2n} \Phi} d\Phi \quad \text{for } n = 1, 2, 3, \dots$$

Calling the corresponding indefinite integrals \bar{j}_{2n} , it is easily verified by differentiation that

$$(26 \text{ a}) \quad \bar{j}_2 = \Phi + \cot \Phi, \quad \bar{j}_4 = \frac{1}{3} \cot \Phi; \quad \bar{j}_6 = \frac{1}{5} \cot^5 \Phi + \frac{1}{3} \cot^3 \Phi, \dots$$

From this and a recursion formula for the \bar{j}_{2n} , one concludes that all j_{2n} vanish at the upper limit of the integral (26) with the exception of j_2 which there assumes the value $\pi/2$. Except for higher powers of the arbitrarily small quantity ψ , the j_{2n} assume the following values at the lower limit:

$$-\frac{1}{2n-1} \cot^{2n-1} \psi \sim -\frac{1}{2n-1} \psi^{-2n+1} = \frac{-1}{2n-1} \left(\frac{a}{x}\right)^{2n-1}.$$

Hence

$$j_2 = \frac{\pi}{2} - \frac{a}{x}, \quad j_{2n} = -\frac{1}{2n-1} \left(\frac{a}{x}\right)^{2n-1}, \quad n \geq 2.$$

Rewriting eq. (14 b) in terms of the j_{2n} and substituting the above values, we obtain

$$J_2 = j_2 \frac{x^2}{a} + \sum_{n=2}^{\infty} \frac{1}{n} j_{2n} \frac{x^2 n}{a^{2n-1}} = \frac{\pi}{2} \frac{x^2}{a} - x \sum_{n=1}^{\infty} \frac{1}{n(2n-1)}.$$

The first term agrees with (15 a) while the second term combines with (15) to give

$$x \left\{ 2 - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{2n+1} + \frac{1}{2n-1} \right) \right\}.$$

The { } can be rewritten in the form

$$(26 \text{ b}) \quad 2 - 4 \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = 4 \left(\frac{1}{2} - \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} - \frac{1}{5 \cdot 7} - \dots \right) = 0,$$

see, for instance, Vol. VI, solution to exercise I.3. Thus our statements following eq. (15) have been proved.

Appendix 2

In the second (magnetic) case eqs. (14 a) and (15) for J_1 are unchanged because the kernel K_I remains unchanged, and the difference between the assumed forms for ω and \bar{u} is insignificant in the interval $0 < \xi < x$. The equations for J_2 simplify because upon making the substitutions (25) and (25 a), one finds that

$$\left(1 - \frac{\xi^2}{a^2}\right)^{-1/2} d\xi = a d\Phi.$$

Hence in terms of Φ eq. (14 b) becomes:

$$J_2 = - \int_0^{\pi/2} \left(\frac{1}{\sin^2 \Phi} \frac{x^2}{a} + \frac{1}{2 \sin^4 \Phi} \frac{x^4}{a^3} + \frac{1}{3 \sin^6 \Phi} \frac{x^6}{a^5} + \dots \right) d\Phi.$$

Since

$$\begin{aligned} \int \frac{d\Phi}{\sin^2 \Phi} &= -\cot \Phi, & \int \frac{d\Phi}{\sin^4 \Phi} &= -\frac{1}{3} \cot \Phi \left(\frac{1}{\sin^2 \Phi} + 2 \right), \\ \int \frac{d\Phi}{\sin^6 \Phi} &= -\frac{1}{5} \cot \Phi \left(\frac{1}{\sin^4 \Phi} + \frac{4}{3} \frac{1}{\sin^2 \Phi} + \frac{2}{3} \right), \dots \end{aligned}$$

all terms in J_2 vanish at the upper limit $\Phi = \pi/2$. In the approximation $\psi \ll 1$, $x \ll a$ the lower limit contributes

$$-x \left(1 + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{4 \cdot 7} + \dots \right)$$

which combines with J_1 in (15) to give zero; see (26 b).

Hence, only K_{II} contributes significantly to the integral equation which, according to (20) and using the approximation (13 f) for H , now reads

$$(27) \quad \frac{4}{\pi} C_0 \left\{ \int_0^{\pi/2} \log \frac{\gamma k a}{2i} d\Phi + \int_0^{\pi/2} \log \sin \Phi d\Phi \right\} = A'.$$

The second of these integrals has the value

$$-\frac{\pi}{2} \log 2.$$

This combines with the value of the first integral to give

$$\frac{\pi}{2} p; \quad p = \log \frac{\gamma k a}{4i} \quad \text{as in eq. (22).}$$

Hence, according to (27)

$$(27 \text{ a}) \quad 2i C_0 p = -\pi A'.$$

Numerically, one obtains

$$(27\ b) \quad \rho - \log k a = \log \frac{\gamma}{4} - \frac{i\pi}{2} = 0.577 - 1.386 - i \frac{\pi}{2} = -.81 - i \frac{\pi}{2}.$$

Our assertion (22) is thus verified by (27 a, b).

Appendix 3

In order to demonstrate how our method may be generalized, we shall first consider the simpler case discussed in appendix 2. We extend (19) to

$$(28) \quad \omega(\xi) = \frac{C_0}{a} \left(1 - \frac{\xi^2}{a^2}\right)^{-1/2} + \frac{C_1}{a} \left(1 - \frac{\xi^2}{a^2}\right)^{+1/2}.$$

We must also use a more exact approximation of the kernel $K = K_I + K_{II}$ of (20 a, b) by retaining the terms in $k^2 x^2$ and $k^2 \xi^2$. The integration is again elementary and may be carried out in the manner of Appendix 2. By equating the coefficients of x^0 and the coefficients of x^2 on the left- and right-hand sides, one obtains the two conditions

$$\begin{aligned} C_0 \left[\rho \left(2 - \frac{k^2 a^2}{4}\right) + \frac{k^2 a^2}{8} \right] + C_1 \left[\rho \left(1 - \frac{k^2 a^2}{16}\right) - \frac{1}{2} + \frac{3}{64} k^2 a^2 \right] &= A' \\ -C_0 \frac{k^2 a^2}{2} \left[\rho + \frac{1}{2} \right] + C_1 \left[1 - \rho \frac{k^2 a^2}{4} \right] &= 0. \end{aligned}$$

Solving these [in the second equation the approximation (27 a) for C_0 may be used], one finds

$$(29) \quad \begin{aligned} C_1 &= \frac{i\pi A'}{4} \frac{1}{\rho} k^2 a^2 \left(\rho + \frac{1}{2} \right), \\ C_0 &= \frac{i\pi A'}{2} \frac{1}{\rho} \left\{ 1 + \frac{k^2 a^2}{8} (1 - 2\rho) \right\}, \end{aligned}$$

which obviously agrees with the result of Appendix 2 when $k^2 a^2$ is neglected.

In the case treated in Appendix 1, (13 a) must be extended to

$$(30) \quad \bar{u}(\xi) = C_1 \left(1 - \frac{\xi^2}{a^2}\right)^{1/2} + C_2 \left(1 - \frac{\xi^2}{a^2}\right)^{3/2}.$$

In (13 b) and in the expansion of the kernel (13 c), not only $k^2 x^2$ but also $k^4 x^4$ must be retained. Equating the resulting coefficients of x^2 and x^4 on the right- and left-hand sides of eq. (13 b), one obtains the two conditions

$$\begin{aligned} C_1 \left[1 - \frac{k^2 a^2}{4} (1 - \rho) \right] + C_2 \left[\frac{3}{2} - \frac{15}{64} k^2 a^2 \left(1 - \frac{4}{5} \rho \right) \right] &= \frac{1}{2i} A k a \\ -C_1 k^2 a^2 + C_2 \left[12 - \frac{5}{2} k^2 a^2 \right] &= +i A k^3 a^3. \end{aligned}$$

In the second equation C_1 may again be replaced by its first approximation (18), and the second order approximation of C_1 is found from the first equation. Thus one obtains

$$(31) \quad \begin{aligned} C_2 &= \frac{i}{24} A k^3 a^3, \\ C_1 &= \frac{1}{2i} A k a \left\{ 1 + \frac{k^2 a^2}{8} (3 - 2p) \right\} \end{aligned}$$

as the improved eq. (18). For the calculations leading to these results the author is indebted to Dr. E. Ruch.

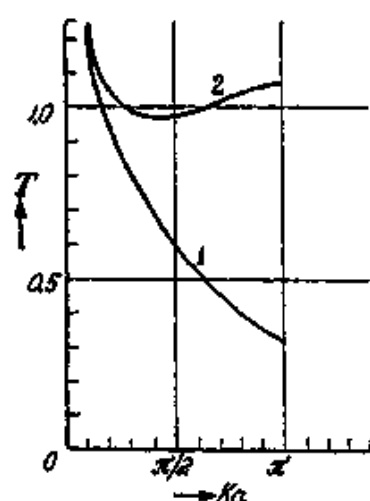


Fig. 82.

The transmission factor T as a function of the ratio of slit width to wavelength in the first and second order approximations.

Figure 82 illustrates the so-called transmission factor T in the first and second order approximations for the polarization $H=H_y$. The transmission factor is defined as the ratio of the energy which light of finite wavelength actually carries through the slit to the energy which would pass through the slit in the limiting case of geometrical optics ($\lambda \rightarrow 0$). In both cases T is measured by the energy flow across a half-cylinder of infinite radius centered at the center line of the slit. One obtains in the first and second order approximations [corresponding to eq. (22) for C_0 , eq. (29) for C_1 and C_0 , respectively]

$$(31a) \quad T_1 = \frac{1}{4 k a |p|^2}, \quad T_2 = \frac{1}{4 k a |p|^2} \left(1 + \frac{1}{4} (k a)^2 \right).$$

Curve 1 is valid only for extremely narrow slits ($ka < 1/4$). For larger values of ka curve 2 separates from 1 and exhibits a tendency to approach the value of geometrical optics ($T = 1$); curve 2 can be checked by comparing it with the work of P. M. Morse and J. Rubenstein¹ in which the problem of the slit is treated numerically and graphically by the theory of Mathieu functions with the help of tables of these functions. Our curve 2 agrees with the corresponding curve of the above authors sufficiently well for $ka < 2$. J. W. Miles² obtained about the same results as Morse and Rubenstein by means of a variational method. K. Schwarzschild³ devised an approximation starting from the opposite limiting case, that of our solution for the half-

¹Phys. Rev. 54, p. 895, 1938; see in particular the top curve denoted by 90° in fig. 4.

²Phys. Rev. 75, p. 695, 1949.

³Mathem. Ann. 55, p. 177, 1902.

plane, by a method of alternating successive approximations. Despite repeated attempts, the author has been unable to apply the method of Sec. 38 directly to the problem of the slit. But it should be noted again that the basic hypothesis (13) as to the form of the boundary values (which was also used by Levine-Schwinger) was prescribed by the method of Sec. 38.

40. The Resolving Power of Optical Instruments

The purpose of all spectroscopic apparatus is to obtain an increased resolving power. In spectroscopy "resolution" means the separation of two closely neighboring spectral lines. In the case of the microscope one is interested in distinct images of the structure of very fine tissues, while with a telescope one wishes to separate double stars, star clusters, discover new satellites, etc.

A. THE RESOLVING POWER OF LINE GRATINGS

According to Lord Rayleigh two spectral lines 1 and 2 can be considered resolved if the principal maximum of the diffraction pattern of 2 (wavelength $\lambda + \delta\lambda$) coincides with the first zero of the diffraction pattern of 1 (wavelength λ). The density of blackened grains on the photographic plate corresponds to a superposition of the intensity contours of 1 and 2. This sum of the two intensities has a depression between the two principal maxima which is sufficient to enable the eye to see the separate lines 1 and 2; see fig. 83 (we shall discuss fig. 83a later on). We shall now show that the ratio $\delta\lambda/\lambda$ measured in this way has a fixed value depending only on the nature and method of use of the grating. The resolving power is defined as the reciprocal of this ratio. Two lines are considered resolved if their $\lambda/\delta\lambda$ is less than the resolving power as defined.

We refer back to eq. (32.5). The zeros of the diffraction pattern are obtained by setting the numerator equal to zero. Hence, the value of $N\Delta/2$ at the first zero exceeds the value of $N\Delta/2$ at the principal maximum by π . At the principal maximum

$$\Delta = 2\pi h, \quad \text{hence} \quad N\Delta/2 = N\pi h,$$

where h is the order of the grating spectrum in which the observation is being made. Therefore the value of $N\Delta/2$ at the first zero is

$$(1) \quad N\pi h + \pi = N\pi \frac{d}{\lambda} (\alpha - \alpha_0).$$

The right-hand side of this equation follows from (32.1) and determines the angular deflection $\alpha - \alpha_0$ of the point in the spectrum under consideration which is the first zero of the diffraction pattern of the line 1. Now it is required that the principal maximum of the line 2 shall fall in the same direction $\alpha - \alpha_0$, which means that

$$(2) \quad N\pi h = N\pi \frac{d}{\lambda + \delta\lambda} (\alpha - \alpha_0).$$

Dividing the left- and right-hand sides of (1) and (2) by each other one obtains

$$\frac{Nh + 1}{Nh} = \frac{\lambda + \delta\lambda}{\lambda}$$

and hence

$$(3) \quad \frac{\lambda}{\delta\lambda} = Nh.$$

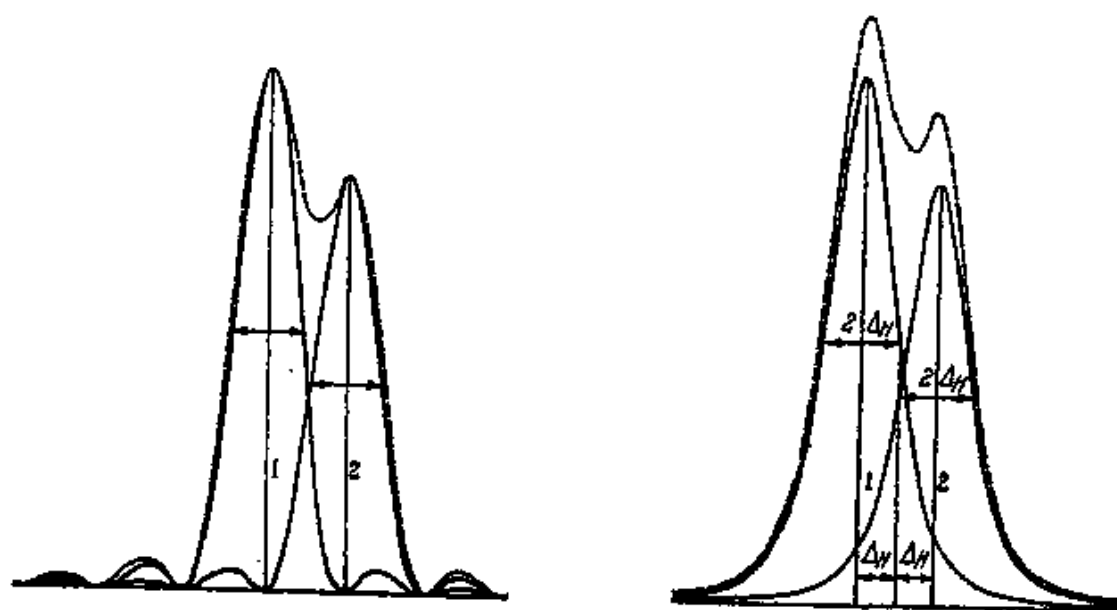


Fig. 83 and 83 a.

Rayleigh's criterion for the resolution of two spectral lines. Fig. 83 a illustrates an almost equivalent criterion.

In the second order spectrum, $h = 2$, the resolving power is twice that in the first order spectrum, a result which is used a great deal by spectroscopists. The resolving power depends only on the total number of grating lines N and not on the line spacing d . The close spacing of lines in the Rowland gratings is needed in order to put a sufficient number of lines within the width of the incident light bundle. A closer spacing of lines also increases the dispersion, that is, the angular separation of different spectral lines, but the spacing has nothing to do with the sharpness of the lines, i. e. with the resolving power.

This is the usual formulation of the theory of resolution for gratings. It is, however, valid only for spectra of *low orders* such as those which are used with Rowland gratings. The *greatest possible resolution* is attained in the spectrum of the highest possible order h_{\max} , that is, when one observes at a very small grazing angle with respect to the grating surface. This order was denoted by h_c on p. 182. For it $\alpha \sim 1$, and according to eq. (32.2) $h_{\max} \sim d/\lambda$ for perpendicular incidence. From (3) follows therefore

$$(4) \quad \frac{\lambda}{\delta\lambda} = h_{\max} N \sim \frac{Nd}{\lambda}.$$

Hence, the *maximum* resolving power of the grating depends on the *total width* Nd and not on the number of lines N or, more precisely, it depends on the *path difference* Nd/λ between the rays coming from the first and last grating lines. In what follows we shall recognize this fact to contain the most general formulation of resolving power, a formulation which is valid for all spectral apparatus.

When observed at a *grazing angle*, a grating with 10 lines spaced 1 cm apart resolves just as well as a Rowland grating with 100,000 lines which are 1μ apart. While in the case of the latter one might observe in the second order spectrum, with the former the twenty-thousandth order would have to be observed.

But observations of spectra of high orders have the serious disadvantage that with increasing order, spectra of *neighboring* orders overlap more and more. To show this we express the wavelength domain which can be observed without overlapping, say $D\lambda$, in terms of the wavelength, the order number h , and the angle of deflection, or rather of its cosine, α :

$$(\alpha - \alpha_0) d = \lambda h = (\lambda + D\lambda) (h - 1).$$

From this follows

$$\frac{D\lambda}{\lambda} = \frac{1}{h-1} \sim \frac{1}{h}.$$

This $D\lambda$ is also the wavelength interval $\delta\lambda$ between neighboring lines which is just measurable without overlapping. Hence in the twenty-thousandth order it is just barely possible to observe the structure of one narrow multiplet, and all other light must be removed by *pre-decomposition* in a prism spectrograph.

The grating with few lines has another and more serious disadvantage; the amplitudes produced by the 10 line and 100,000 line gratings are in the ratio of $1 : 10^4$; hence the intensity obtained from the former is only one 10^8 th of that produced by the latter. Moreover, the ruling of the 10 line grating would have to be just as precise as that of the 100,000 line grating and would therefore be no simpler to manufacture.

B. ECHELON GRATINGS AND INTERFERENCE SPECTROSCOPY

In our method of treatment the grating lines play the role of secondary *light sources* which are excited by the incident light wave and which, owing to their positions, have fixed phase differences with respect to one another. It is possible to obtain the same intensity with a smaller number of secondary light sources, provided that these sources act as *directional radiators* which send most of their energy into a spectrum of high order. In other words, the function $f(\alpha)$ in eq. (32.3) must have a pronounced maximum in the direction of the desired high order spectrum. A very narrow slit or a finely ruled line will not do this. Instead, one must use a sequence of narrow parallel mirrors or narrow prisms which are placed so that by geometrical optics they reflect or refract the light in the desired direction. The almost insurmountable difficulties of the manufacture of such a grating were overcome by Michelson in an elegant manner. He stacked glass plates on top of one another so as to form a series of steps, see Fig. 84a. In making a grating of this type the

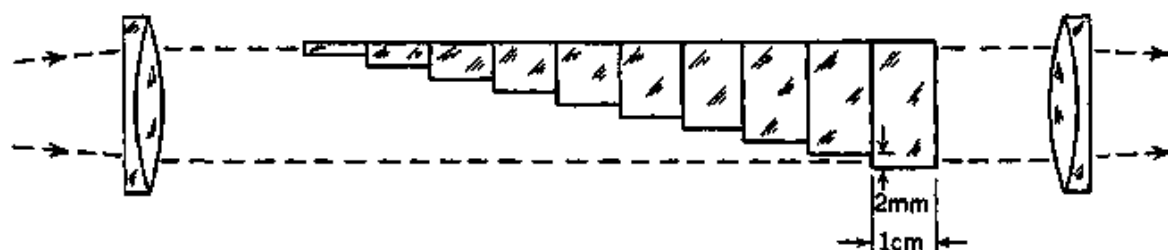


Fig. 84a. Ten-element echelon consisting of glass plates 1 cm thick, offset by 2 mm steps.

plates are cut from a single plane parallel plate, the thickness of which is everywhere constant to within a fraction of a wavelength. The steps are about 2 mm wide and perhaps 1 cm high; they are the grating elements of a "phase grating". With the help of a slit and collimator lens the grating is illuminated in a direction perpendicular to the surfaces of the glass plates, and the spectrum is observed through a telescope in the same direction. Thus the light rays form a *very small angle with the surface of the step grating*. If all grating elements except one are covered, one observes the very bright image of the slit which is, however, widened by diffraction and looks like the diffraction pattern of a slit 2 mm wide. When all the grating elements are uncovered, the image of the slit contracts to the image of the spectral line in one or two orders. As with all gratings, the resolving power of this "echelon grating" is given by the difference between the phase of the first and last ray, which is

$$(5) \quad (n-1) N d / \lambda; \quad \begin{array}{l} n = \text{index of refraction of the glass} \\ N = \text{number of steps.} \end{array}$$

In an echelon strips of the wave surface lying side by side interfere with one another. Therefore the wave surface must be made coherent throughout the entire extent of the grating by means of a collimator. As in the case of the line grating, the positions of the spectral lines depend on the direction of the incident light. If this direction is changed, then the phase differences between our secondary sources are changed; thus the positions of the sharp interference fringes are shifted. A wide collimator slit acts like the sum of many adjacent narrow slits; hence a wide slit causes the spectral lines to spread out to the width of the slit's image. This is the qualitative geometrical interpretation of the blurring of interference lines caused by insufficient coherence (see, for example, fig. 2).

This effect does not exist in "interference spectroscopy", by which we mean the spectroscopes of Perot-Fabry and Lummer. With these spectroscopes the positions of the interference fringes depend *only* on the wavelength and the thickness of the plate. The phase difference between interfering waves (though not their intensity) is *independent* of the position of the light source; in other words, the source may be extended without disturbing the interference, provided only that the source is sufficiently intense.

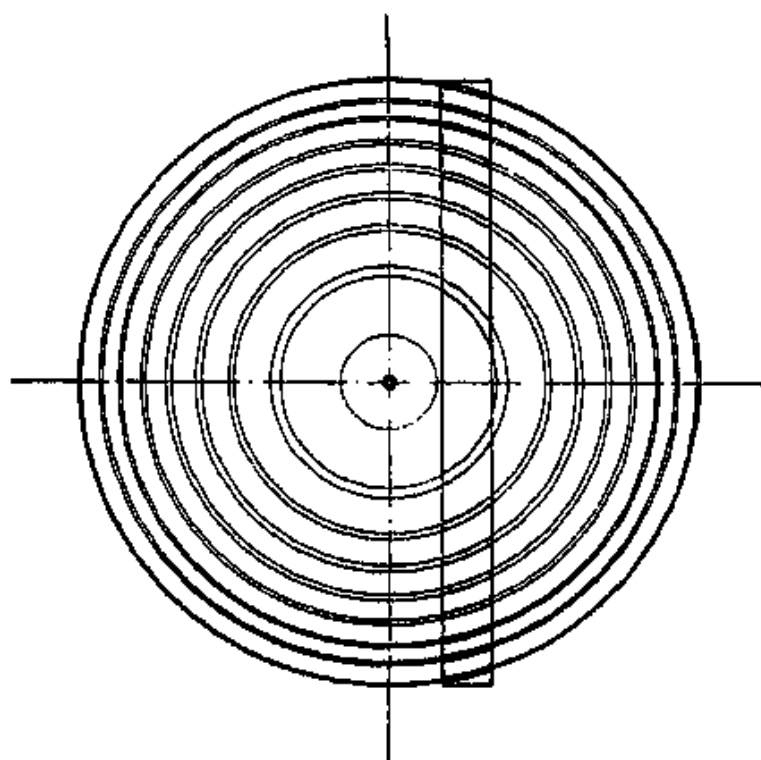


Fig. 84b.

Section from the field of view of a Perot-Fabry air plate.

As in the case of gratings the phase difference produced by interference spectroscopes changes as the angle of observation varies. To each angle between the wave normal and the plate surface corresponds a definite phase difference and therefore a definite wavelength. Thus, in the case of the Perot-Fabry interferometer the wave normals of a given spectral line lie on a narrow cone about the normal to the plate surface. This cone projects as a circle in the camera or on the retina. The different orders form concentric circles which are, however, visible only within the image of the extended source (all other wave normals are not excited). A prism spectroscope may

be used for the necessary pre-decomposition. The interference spectroscope is placed between the prism and the telescope. The system of rings is bounded by the image of the slit which is made as wide as possible without causing disturbances due to neighboring spectral lines; see fig. 84b which is a schematic drawing of a Perot-Fabry spectrogram. In the case of the Lummer plate the wave normals of the spectral lines lie on a set of very wide cones about the normal to the plate and therefore appear on the photographic plate as a set of very flat hyperboolas which are almost straight lines.

While in gratings the amplitudes of the various interfering rays are equal, the amplitudes of the interfering rays of an interference spectroscope decrease exponentially. Hence the intensity distribution is no longer given by (32.5), but rather by (7.33). The small periodicity belonging to $N\Delta$ in eq. (32.5) disappears, and only the long period which is determined by the Δ of eq. (32.1) remains because the number N of interfering rays is, so to speak, infinite. Yet, because of the exponential decrease in amplitude, only a finite number of rays is "effective", the remaining rays being too weak; therefore the resolving power remains finite.

Since no zero intensities occur in the fringe system of an interference spectroscope, see fig. 11, we define the resolving power in terms of the half-width $2\Delta_H$ of the interference fringe; that is, the resolving power is that wavelength interval within which the intensity is greater than half the intensity at the maximum. A comparison of figures 83 and 83a shows that this definition is practically equivalent to Lord Rayleigh's definition of the resolving power of gratings.

The half-width for the *Lummer plate* was computed in (7.28 a). In order to convert the φ -scale used there to the scale of wavelengths λ , we note that according to the definition (7.18a) φ is proportional to h , hence inversely proportional to λ . Therefore

$$(6) \quad \frac{d\varphi}{\varphi} = -\frac{d\lambda}{\lambda}.$$

If we substitute for $d\lambda$ the wavelength difference $\delta\lambda$ between the two spectral lines 1 and 2, then according to fig. 83a we must use for $d\varphi$ the half-width $2|\Delta\varphi| = 2(1-r)$ [eq. (7.28 a)], and we must substitute for φ the phase $2\pi z$ at the intensity maximum (see Sec. 7). Thus we obtain from (6) (the negative sign is immaterial)

$$(6a) \quad \frac{\lambda}{\delta\lambda} = \frac{\pi}{1-r} z.$$

z represents the very high order of the interference fringe and corresponds to the order number $h = 1, 2, 3, \dots$ of gratings. A comparison of (6a)

and (3) shows that the number of lines N of a grating is to be compared with the expression $\pi/(1-r)$. The orders of magnitude of the two factors contributing to the resolving power are thus interchanged for the two types of spectroscopes:

for a grating: N is very large, h is moderately large

for a plate: $\pi/(1-r)$ is moderately large, z is very large.

In order to complete the numerical comparison we recall the meaning of z in (7.28 a) and (7.18 a). If we disregard all insignificant factors in these formulae, then z is twice the plate thickness divided by the wavelength; hence for a 1 cm plate $z \sim 4 \times 10^4$. If $r \sim 0.9$, then $\pi/(1-r) \sim 30$. According to (6 a) the resolving power of such a Lummer plate is then about $30 \times 4 \times 10^4 \sim 10^6$. According to (3) this is the number of lines N in a grating with the same resolving power in the first order ($h = 1$). This means that if such a grating had 1000 lines per mm it would have to be 1 meter wide!

For the Perot-Fabry etalon one obtains similarly from the half-width (7.34) the resolving power

$$(7) \quad \frac{\lambda}{\delta\lambda} = \frac{1}{2} (1 + g) z.$$

z is again the order number of the interference fringe and is therefore a very large number. The first factor, on the other hand, is only a moderately large number since the required light intensity limits the amount of silvering that can be applied to the surfaces. If we estimate g to be 9, then the first factor becomes 5. z being twice the spacing of the plates divided by the wavelength becomes 2×10^5 if we assume the plates to be 5 cm apart. The product of these two factors is 10^6 , the same as for the Lummer plate considered above. *The resolving power of both plates exceeds that of the Rowland grating.* Because of its greater simplicity of operation, the Perot-Fabry etalon seems superior to the Lummer plate.

41. The Prism. Basic Theory of Resolving Power

We shall assume that the collimator lens provides completely parallel and monochromatic light. The telescope and collimator lenses will be assumed to be larger than the projections of the prism in the directions of the incident and refracted rays. Then the size of the ray bundle is limited by the size of the face of the prism through which the light emerges, see fig. 85. This surface is a rectangle which is perpendicular to the plane of the drawing and makes an oblique angle with the direction of the emerging ray. Using the notation of Sec. 36 A, the height of the rectangle is $2B$ and its width $2A$. This width

equals the side 2-3 of our prism cross section and is also equal to 1-3. The height $2B$ of the rectangle cannot be shown in the figure and is also immaterial to the following considerations. The Fraunhofer diffraction pattern of this rectangle appears in the focal plane of the telescope. If the prism is illuminated through a slit which is parallel to the refracting edge, then the intensities of the patterns due to the elements of the slit add up along the direction of the height of the rectangle $2B$. The intensity distribution in the direction of the

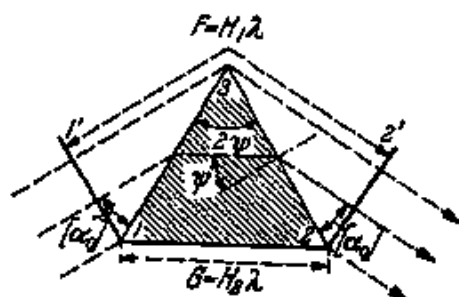


Fig. 85. Cross section of an isosceles prism, parallel to its base, with symmetrical light path. The emerging rays are drawn in the sense of geometrical optics without consideration for the diffraction at the edges of the prism. $[\alpha_0]$ means $\cos^{-1} \alpha_0$ for both the incident and emerging rays.

width $2A$ of the rectangle is given in Sec. 36 by eqs. (1), (2), (3). We note that this distribution almost agrees with that of a grating of width $Nd = 2A$, because in the vicinity of the principal maximum the factor $\sin \Delta/2$ in the grating formula (32.5) can be replaced by $\Delta/2$. The remaining calculations are similar to those in Sec. 40 for gratings. The positions of the first zeros to the right and left of the principal maximum are given by

$$(1) \quad 2\pi A (\alpha_{1,2} - \alpha_0)/\lambda = \pm \pi.$$

α_0 is now the direction cosine of the emerging ray with respect to the surface of emergence of the prism; $\alpha_{1,2}$ are the direction cosines for the first zeros to the right and left of the ray α_0 .

Let us now consider a second ray whose wavelength differs by $\delta\lambda$ from that of the previous ray. Because of dispersion this new ray will have a different index of refraction n' and a different direction of emergence α'_0 . We want to know the value of $\delta\lambda$ for which the principal maximum of the second ray (direction α'_0) will coincide with one of the two zeros α_1, α_2 . α_0 and α'_0 are determined by the law of refraction.¹ From exercise (III.2) we know that for symmetric path in a prism of refracting angle 2ψ

$$(2) \quad \alpha_0 = n \sin \psi.$$

This relation holds for both refracting surfaces 1-3 and 2-3. The ray with wavelength $\lambda + \delta\lambda$ and with the same incident direction is no longer refracted

¹Note the changed notation: The refracting angle which previously was φ is now 2ψ ; previously $\alpha, \beta, \alpha', \beta'$ were the angles of incidence and refraction, now $\alpha_0, \alpha'_0, \alpha$ are the direction cosines of the emerging (or diffracted) rays.

exactly symmetrically. It forms a small angle ε with the symmetry line so that on the surface 1-3

$$(2 a) \quad \alpha_0 = n' \sin (\psi + \varepsilon)$$

and at the surface 2-3

$$(2 b) \quad \alpha_0' = n' \sin (\psi - \varepsilon).$$

From (2 a) and (2 b) one finds

$$\alpha_0 + \alpha_0' = 2 n' \sin \psi \cos \varepsilon;$$

subtracting from this equation twice the expression (2) for α_0 and calling the change in the index of refraction resulting from dispersion $\frac{dn}{d\lambda} \delta\lambda$, we obtain, neglecting terms of order ε^2

$$\alpha_0' - \alpha_0 = 2 (n' \cos \varepsilon - n) \sin \psi \sim 2 \frac{dn}{d\lambda} \delta\lambda \sin \psi.$$

According to Rayleigh's criterion for the resolving power, this difference must now agree with the value of $\alpha_{1,2} - \alpha_0$ as given by (1). It follows that

$$\frac{\lambda}{2A} = 2 \frac{dn}{d\lambda} \delta\lambda \sin \psi$$

or

$$(3) \quad \frac{\lambda}{\delta\lambda} = 4A \frac{dn}{d\lambda} \sin \psi = G \frac{dn}{d\lambda}.$$

$G = 4A \sin \psi$ (see figure) is the *base line* of our prism cross section. Only this base line and the *dispersion* of the glass $dn/d\lambda$ affect the resolving power. The larger the refracting angle 2ψ , the smaller may we make the height of the triangle and the diameter of the lenses without decreasing the resolving power. As we approach the limiting angle of total reflection $\sin \psi = 1/n$, the resolving power becomes

$$(3 a) \quad \frac{\lambda}{\delta\lambda} \sim 2A \frac{2}{n} \frac{dn}{d\lambda}.$$

The contribution $2A$ of the width of the prism surface is analogous to the contribution Nd of the width of a grating. The length $l = n/2 \, d\lambda/dn$ now takes the place of the wavelength λ occurring in eq. (40.4) for gratings. For green light ($\lambda = 0.5 \mu$) and heavy flint glass one has $n = 1.77$, $dn/d\lambda = 0.23 \mu^{-1}$, $l = 3.84 \mu$. Thus l is about eight times the corresponding wavelength $\lambda = 0.5 \mu$. Therefore, with equal surfaces of emergence a prism attains only $1/8$ the resolution attained by a comparable grating; but the prism is free of the superposition of the spectra of higher orders. This superposition, as well as the zeroth order spectrum, causes a considerable intensity loss in gratings. Therefore a prism spectrograph produces greater intensities than a comparable

grating. Theoretically a prism could even have better resolution than a grating if it were possible to approach a characteristic frequency of the prismatic material sufficiently closely; for in the vicinity of such a frequency $dn/d\lambda$ becomes very large. Unfortunately, the strong absorption in the vicinity of a characteristic frequency prevents the utilization of such a region. An indication of this increased resolving power is already apparent in the violet ($\lambda = 0.41\mu$) where for the above-mentioned glass $l = 1.8\mu$; the resolution is here almost twice as good as in the green. If, instead of glass, quartz or rock salt is used, l is diminished still more, until in the far ultraviolet a rock salt prism becomes as good as a grating.

A. GENERAL CONSIDERATIONS REGARDING RESOLVING POWER

Let us compare the two following limiting light rays: one ray which when going from slit to crosshairs of the spectrograph passes through the vertex 3, and on the other hand, one which goes along the base 1-2. It suffices to measure the light path lengths from the wave surface 11' which passes through the front edge of the prism to the wave surface 22' which passes through the rear edge. We may limit our considerations to these portions of the rays because all rays from the slit to 11' have the same path lengths and the same is true for all rays from 22' to the cross hairs and therefore these portions of the paths do not contribute to the path difference between the rays. The extreme ray paths between the wave surfaces 11' and 22' are denoted by F and G in fig. 85. Their lengths measured in wavelengths shall be H_1 and H_2 , respectively, and their difference shall be H . For the rays drawn in the figure which belong to the wavelength λ the value of H is, of course, zero because all rays have the same optical path length between two wave surfaces:

$$(4) \quad H = n \frac{G}{\lambda} - \frac{F}{\lambda} = 0.$$

The same holds true for the ray paths belonging to the wavelength $\lambda + \delta\lambda$ (not drawn in the figure) which terminate on another wave surface which is inclined with respect to 22'. But if we consider the path difference H for the changed wavelength along the *original* ray paths instead of along the changed paths (the geometrical paths F and G are kept fixed), then by varying λ we obtain from (4):

$$(4a) \quad \delta H = \left(\frac{dn}{d\lambda} \frac{G}{\lambda} - \frac{nG}{\lambda^2} + \frac{F}{\lambda^2} \right) \delta\lambda = \frac{dn}{d\lambda} G \frac{\delta\lambda}{\lambda}.$$

Hence the above expression (3) for the resolving power is equivalent to the statement

$$(5) \quad \delta H = 1.$$

These considerations, first of all, explain the presence of the factor G in eq. (3) which may have seemed surprising at first glance: the larger G is, the easier it becomes for the dispersive power of the glass to produce the optical path difference between neighboring wavelengths which is necessary for resolution. Furthermore, the reasoning which led us to the criterion (5) can now be visualized and also *generalized* in the following way: $\delta H = 1$ means that the two extreme rays F and G , which arrive simultaneously at 2 and 2' if they have the wavelength λ , have an optical path difference of exactly *one* wavelength if their wavelength is $\lambda + \delta\lambda$; and this path difference is a linear function of position along the original wave surface 22'. This means that in the focal plane of the telescope the rays with wavelength $\lambda + \delta\lambda$ are extinguished where the rays with wavelength λ which have equal phases along 22' produce their diffraction maximum and vice versa. In other words, the criterion (5) is equivalent to Rayleigh's criterion.

B. APPLICATIONS TO GRATINGS AND INTERFERENCE SPECTROSCOPES

The applicability of the above criterion to line gratings will be tested by means of fig. 86. 12 represents the trace of the grating on the plane of the drawing. For a plane wave incident from the left, 1 is the extreme left grating line and 2 is the extreme right grating line. 11' is again the plane of constant phase for the incident wave and 22' the plane of constant phase for the diffracted wave under consideration. $[\alpha_0]$ and $[\alpha]$ are the direction cosines of these planes with respect to the plane of the grating. The optical path lengths of the extreme left- and right-light paths between the two phase planes are

$$F = 12' = \alpha Nd, \quad G = 1'2 = \alpha_0 Nd$$

where Nd is the width of the grating. From this it follows that

$$H = H_2 - H_1 = \frac{G - F}{\lambda} = \frac{Nd}{\lambda} (\alpha_0 - \alpha)$$

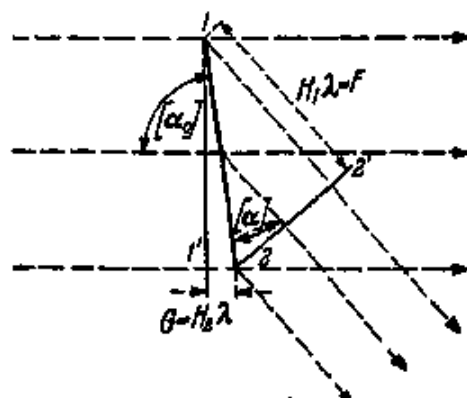


Fig. 86.

Cross section through a line grating (heavy line). The figure shows the wave incident from the left at the angle $[\alpha_0]$ and its phase plane 11'. The wave is diffracted into the angle $[\alpha]$; the phase plane of the diffracted wave is 22'.

and (the directions α , α_0 are being kept fixed!)

$$\delta H = \frac{dH}{d\lambda} \delta\lambda = \frac{Nd}{\lambda^2} (\alpha - \alpha_0) \delta\lambda.$$

But according to the basic grating formula (32.2): $\alpha - \alpha_0 = h\lambda/d$ (h = order of the spectrum) and therefore

$$\delta H = Nh \frac{\delta\lambda}{\lambda}.$$

From this and our criterion (5) follows indeed the resolving power of the grating

$$(6) \quad \frac{\lambda}{\delta\lambda} = Nh \quad \text{as in (40.3).}$$

Regarding interference spectroscopy a few words about the Perot-Fabry plate will suffice. We shall consider the ray which passes only once through the plate and then emerges as the "first ray". It follows the path F . We call the ray which traverses the plate $2p + 1$ times, $p + 1$ times in the forward direction and p times in the reverse direction, the "last ray" and the path it follows we call G . The number p depends on the amount of silvering and the weakening of the light caused by it. With our former notation of $z\lambda$ for the length of one forward and back path through the plate one obtains

$$F = \frac{1}{2} z \lambda, \quad G = \left(p + \frac{1}{2}\right) z \lambda, \quad H = pz$$

and for fixed F and G

$$\frac{\delta z}{z} = -\frac{\delta\lambda}{\lambda}, \quad |\delta H| = p |\delta z| = pz \frac{\delta\lambda}{\lambda}.$$

From this, and according to (5), the resolving power becomes:

$$(7) \quad \frac{\lambda}{\delta\lambda} = pz.$$

This result is to be compared with eq. (40.7) where the quantity $g + 1$ takes the place of our present $2p$. According to the discussion following eq. (7.29), g is a measure of the conductivity and the thickness of the silver layer, hence g is a measure of the reflecting power of the surface. On the other hand $2p$ is the number of reflections which can be observed without excessive weakening of the light [the notation is the same as in eqs. (7.20) and (7.21)]. Thus g and $2p$ mean qualitatively the same thing. Therefore, our present statement (7) agrees qualitatively with the statement (7) in Sec. 40.

42. The Telescope and the Eye. Michelson's Measurements of the Sizes of Fixed Stars

Let us assume that a telescope is directed at a pair of stars 1, 2 in such a way that the axis of the telescope points to 1. Then 1 produces in the focal plane a diffraction pattern of the type shown in fig. 68. According to eq. (36.11) the position of the first diffraction minimum is given by

$$(1) \quad s_1 = 0.61 \frac{\lambda}{a},$$

where a is the radius of the objective, λ is a mean wavelength of the star's light, and s is defined as in (36.7) as the angle between the ray under observation and the direction of the principal maximum. The number 0.61 corresponds to the first root of the Bessel function J_1 and is approximately equal to $5/8$; see (36.11 a).

If we agree that star 2 is clearly distinguishable from star 1 (either visually or on the photographic plate) only if its principal maximum is further away from the principal maximum of 1 than the first minimum of 1, then we see that the formula (1) also contains a *measure of the resolving power of the telescope*. The smaller s_1 is, the larger is the resolving power. Therefore we shall define the resolving power of the telescope as the *reciprocal* of the smallest resolved angular distance between 1 and 2 as determined by (1); that is, as the dimensionless number

$$(2) \quad \frac{1}{s_1} = \frac{a}{0.61 \lambda}.$$

From this we conclude: *the resolving power is proportional to the size of the objective*. This fact is the reason for the giant telescopes on Mt. Wilson and for the large mirror at the Palomar observatory; in the case of the latter, $2a = 200$ in. ~ 5 meters! We note further that the resolution is somewhat better at the short wavelength end of the spectrum than at the long wavelength end.

In the *eye* the pupil takes the place of the rim of the objective; its diameter $2a$ varies between 1 mm and 8 mm depending on the brightness. It follows that for a medium value of λ equal to 5×10^{-4} mm

$$10^{-3} > \frac{\lambda}{a} > 1.2 \times 10^{-4},$$

thus

$$6 \cdot 10^{-4} > s_1 > 0.7 \times 10^{-4}$$

or in degrees instead of radians

$$2' > s_1 > 15''.$$

Therefore, quite aside from the cellular structure of the retina, diffraction imposes an upper limit on the resolving power of the eye. With strong illumination (small pupil) only differences in direction that are of the order

of magnitude of one minute or more of arc can be perceived.

Equation (2) can also be understood from the general point of view of optical path differences as formulated in eq. (41.5). Returning to the telescope we consider the two "limiting rays" which pass through diametrically opposite points at the rim of the objective. In fig. 87 these rays are drawn as full lines for star 1 and as dotted lines for star 2. P is the image of 1 in the focal plane; P' is the image of 2. The significant path lengths of the limiting rays from star 1 are

$$G = WP, \quad F = YX + XP,$$

where, of course, $F = G$ because, being the image of 1, P is that point where all the light from 1 arrives with the same phase. Therefore

$$(3) \quad YX + XP - WP = 0.$$

But we are interested in the light paths of the rays from the star 2 to P which are

$$G = WP, \quad F = ZX + XP$$

and so taking (3) into account

$$\begin{aligned} (3a) \quad F - G &= ZX + XP - WP \\ &= ZX - YX + (YX + XP - WP) = ZX - YX. \end{aligned}$$

The right triangles WZX and WYX show that

$$ZX = 2a \sin \alpha; \quad YX = 2a \sin \alpha_0.$$

Therefore, by (3 a)

$$(3b) \quad F - G = 2a (\sin \alpha - \sin \alpha_0)$$

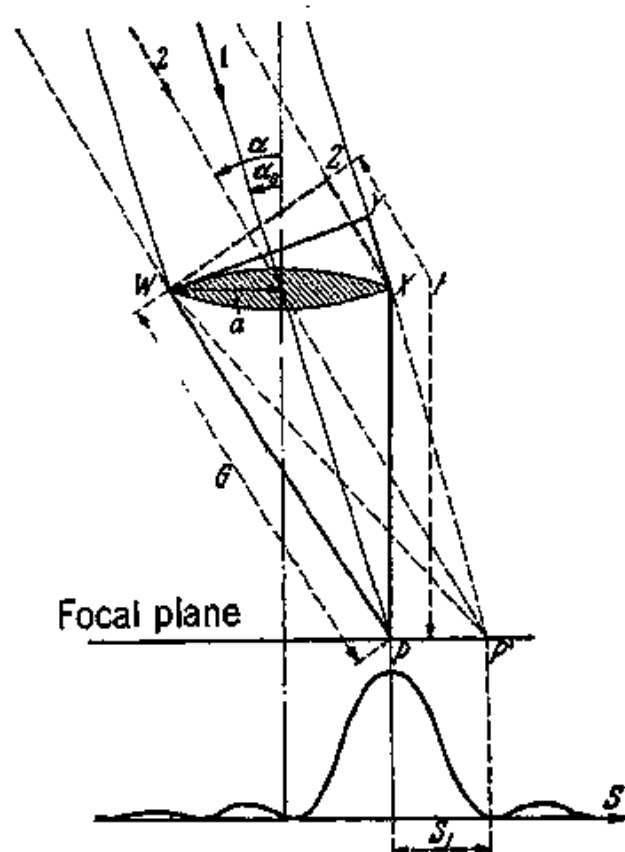


Fig. 87.

Diffraction of the light from a binary star 1, 2 when Rayleigh's criterion is fulfilled. Construction for the calculation of the difference $F - G$ between limiting rays.

and

$$(3\ c) \quad H = \frac{F-G}{\lambda} = \frac{2a}{\lambda} (\sin \alpha - \sin \alpha_0).$$

This H -value must now be varied by varying the position of the object, that is by varying α_0 and not by varying λ as was done in the case of spectroscopic apparatus. This variation yields

$$(3\ d) \quad |\delta H| = \frac{2a}{\lambda} \delta \sin \alpha_0.$$

According to our condition $|\delta H| = 1$ two objects would be resolved or not resolved depending on whether

$$\delta \sin \alpha_0 \gtrless \frac{\lambda}{2a}.$$

The resolving power is therefore

$$(4) \quad \frac{1}{\delta \sin \alpha_0} = \frac{2a}{\lambda}.$$

The difference between this and our definition (2) for the resolving power is trivial because it consists only of a numerical factor of $2 \times 0.61 = 1.22$. (The same trivial factor would appear in the resolving powers of spectroscopic devices such as gratings, prisms, etc. if the diffraction opening were bounded by a circle instead of by a rectangle. As an alternative we could replace the condition $\delta H = 1$ by $\delta H = 1.22$.) The intensity curve for the light of star 1 drawn at the bottom of fig. 87 indicates that our construction using limiting rays is equivalent to Rayleigh's condition: the image of star 2 coincides with the first minimum of the diffraction pattern of star 1.

The situation is different when "resolution" does not require (almost separate) images of the two stars but only some indication of whether or not a binary star is under observation at all. In that case it is possible to attain a much larger path difference of the limiting rays without having to require precision to within a wavelength for all rays over the entire surface of an objective. This leads to an arrangement of mirrors which had already been proposed by Fizeau but was first constructed successfully by Michelson; see fig. 88.

The two outer mirrors S, S' are a distance $b + b'$ of several meters apart; the spacing of the two inner mirrors lies within the diameter $2a$ of an ordinary telescopic objective, i. e. is of the order of a few inches.

Let us first consider only the component 1 of the twin stars and the light reaching the telescope from that star by way of S, s . The resulting diffraction pattern is determined by the cross section of the light bundle (which, in turn,

is determined by the sizes of the mirrors S , s and by the diameter $2a$ of the objective). This diffraction pattern consists of a system of rings of the type described in Sec. 36 C. The same is true for the light reaching the telescope from the component 1 by way of S' , s' . Since this light originates from the same source 1 as the previously considered light, the amplitudes of both the central spot and the system of rings would be doubled provided the

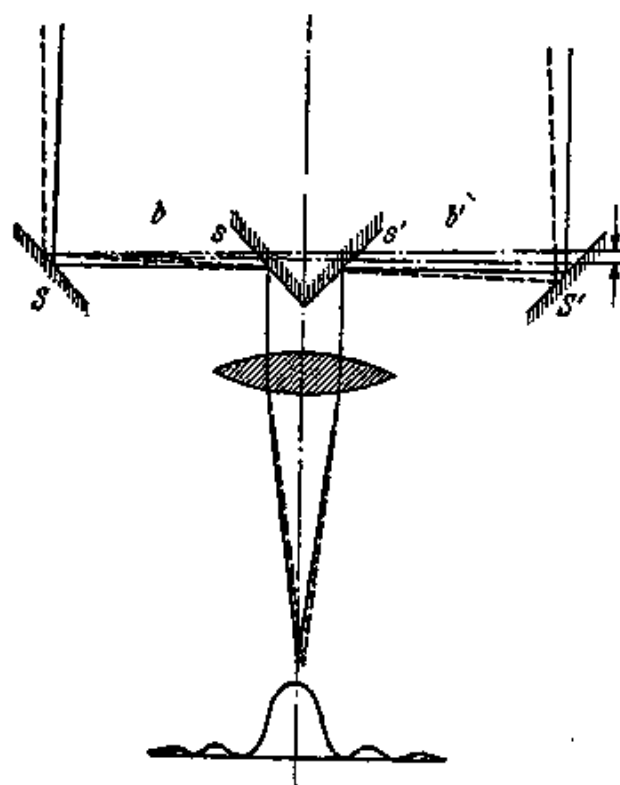


Fig. 88.
Michelson's Mirror Experiment.

arrangement of mirrors were perfectly symmetrical. Actually, the two mirrors S , S' are never exactly symmetrical with respect to the axis of the telescope and are never inclined at precisely 45° to that axis. Therefore there is also present a system of equidistant rectilinear interference fringes of the type which is already known to us from the Michelson experiment of Sec. 14 (these fringes are lines of equal optical path difference). Because of the lack of symmetry of the arrangement, we have denoted the distance $S's'$ in the figure by b' to distinguish it from $Ss = b$. The position of the system of fringes depends on the quantity

$$(b - b')/\lambda.$$

Depending on whether this ratio (for a given λ and for a given position in the ray bundle between Ss and $S's'$) is an integer or half-integer, we have a bright or a dark fringe.

Let us now turn to component 2 of the binary star. It yields a diffraction pattern of the same type as that produced by component 1. There are again a central spot, diffraction rings, and linear interference fringes. The central spots and rings of the two component stars coincide because it is assumed that the cross section of the ray bundle, or the diameter $2a$ of the objective, is not sufficiently large to resolve the double star. The central spot and diffraction rings of 2 are superimposed on those of 1 (of course they add intensity-wise because of the difference of the light sources). But the position of the system of rectilinear fringes from component 2 differs from that due to 1. The path difference responsible for these fringes is affected not only by the positions of

the mirrors but also by the direction of the incident light. As in (3 d) the difference in the directions of incidence of the light of components 1 and 2 produces a path difference of

$$(5) \quad \frac{b + b'}{\lambda} \delta \sin \alpha_0.$$

Hence in general the two systems of fringes do not coincide but are displaced from one another by the amount given in (5). The mirrors S , S' are mounted on a rigid support and can be moved parallel to themselves so that the total distance $B = b + b'$ between the mirrors can be changed. In this way the shift between the fringe systems is also changed. Let us assume that at a given value of $B = B_n$ corresponding to a path difference of n wavelengths, the two systems of fringes coincide. Then

$$(5 a) \quad \frac{B_n}{\lambda} \delta \sin \alpha_0 = n.$$

If the mirrors are now shifted until the next coincidence of fringes is observed at $B = B_{n+1}$, then

$$(5 b) \quad \frac{B_{n+1}}{\lambda} \delta \sin \alpha_0 = n + 1.$$

Subtracting these two equations and letting $\Delta B = B_{n+1} - B_n$, one obtains

$$\frac{\Delta B}{\lambda} \delta \sin \alpha_0 = 1$$

and thus

$$(6) \quad \delta \sin \alpha_0 = \lambda / \Delta B.$$

ΔB can be measured precisely; a mean wavelength must, of course, be substituted for λ . *Owing to the enlarged scale B of the interference phenomenon, the existence of a binary star can be ascertained and the angular distance between the two components can be measured even if the resolving power of the telescope being used is insufficient.*

The same method can be applied to measure the size of a *single fixed star of exceptional size* which in the mirror arrangement behaves no longer like a point source but rather like a small disc. The stars which can be measured by this method are the so-called red giants (low temperature, hence the red color, but nevertheless great brightness because of the enormous luminous surface); see examples below. Such a small disc can be thought of as divided into a left, a right and a middle third, and the two outer thirds can be treated like a binary system; the light from the middle third will weaken the contrasts of the interference fringes belonging to the left and right thirds, but they will

not be extinguished. A study of the fringe coincidences as described by eq. (6) leads to an estimate of the angular distance between the edges of the star. Michelson obtained the following values in seconds of arc:

Betelgeuse	0.047",
Antares	0.040",
Arcturus	0.022".

Since the distances of these stars from the solar system (their parallaxes) are known from other measurements, their linear diameters can be calculated. These diameters turn out to be of the order of 10^8 km, which is about one hundred times the diameter of our sun and about equal to the diameter of the earth's orbit.

43. The Microscope

Helmholtz¹ treated the resolution of the microscope in the same way as that of the telescope. For an object let us take two luminous points a distance d apart and located in the lower focal plane F_1 of the objective; see fig. 89.

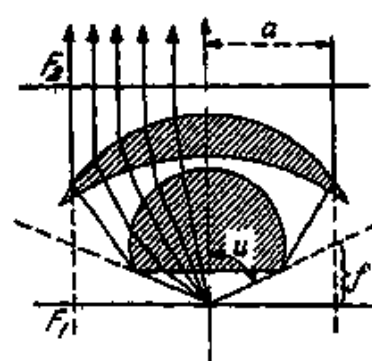


Fig. 89.

Paths of rays in a microscope. F_1 and F_2 are the front and rear focal planes.

The objective produces an image of these points at infinity because the spherical waves emitted by the two points leave the objective as plane waves forming two bundles of parallel rays, the directions of which differ by an angle α . If the medium on both sides of the objective is air, then this angle is the same as the angle between the two central rays from the objects to the optical center of the objective. The value of this angle is d/f , where f is the lower focal length of the objective. If a medium of greater index of refraction $n > 1$ occupies the space between the object and the objective (immersion in oil), then for small angles of incidence the law of refraction gives

$$(1) \quad \alpha = n d/f.$$

We assume that the objective is, from the point of view of geometrical optics, a perfectly corrected system of lenses. The rim of the objective and all other diaphragms are projected as geometrical images (real or virtual) by the succeeding lenses. *The smallest of these images is called the "exit pupil"*

¹Die theoretische Grenze für die Leistungsfähigkeit der Mikroskope, Ann. d. Physik, 1874. Fraunhofer stated much earlier (Bayerische Akademie June 14, 1823) that the limit of effectiveness of a microscope depends on diffraction.

of the lens system and it limits the size of our ray bundles. In general the exit pupil is the virtual image of the rim of the front lens. Let the radius of the exit pupil be a .

$$(2) \quad A = \frac{a n}{f}$$

is called the "numerical aperture"; from the elementary geometrical optics of lens systems (sine condition, Sec. 48) it follows that $a = f \sin u$, where $2u$ is the angle of opening of the object ray cone. Hence, the definition (2) of A becomes

$$(2a) \quad A = n \sin u.$$

Each of our two ray bundles produces at infinity the Fraunhofer diffraction image of the exit pupil as described by eq. (36.9). Hence the microscope reproduces each of our luminous points in the form of a diffraction pattern (central field plus diffraction rings). In order to render these patterns observable at a finite distance, the eyepiece contains a converging lens on whose focal plane the diffraction patterns are reproduced. This image is observed through the eyepiece which acts as a magnifying glass.

This projection of the image upon a finite plane is of no concern to the theoretical investigation of resolution. Instead, the original diffraction pattern at infinity can be treated directly. As in the case of the telescope, eq. (42.1), we are then led to the result $n d/f = 0.61 \lambda/a$, which, according to the definition (2) of A , we can also write in the form

$$(3) \quad d = 0.61 \frac{\lambda}{A}.$$

Two luminous points are resolved only if the distance between them is greater than that given by (3).

Recalling the meaning (2a) of A , we note that while the resolving power of the telescope depends on the *size of the objective*, the resolving power of the microscope depends on the *angle u subtended by the objective at the specimen*.

A. ABBE'S THEORY OF THE MICROSCOPE

With Helmholtz's theory the question of the resolving power of the microscope is essentially settled. What remains to be explained is the remarkable effect which the manner of illumination (bright or dark field) has on the resolution of different tissue structures even though it does not influence the resolution of two-point objects. This is where Abbe's theory is of importance. Abbe regarded the object as a diffraction grating (amplitude

or phase grating). Because of the thinness of the specimens observed and the low depth of focus of powerful objectives, these gratings can be considered plane and their extent as to depth can be neglected. If we illuminate the object with coherent light parallel to the axis of the microscope, then plane waves emerge from the object in the directions of the grating spectra of various orders. Those orders which are emitted inside the angle of opening u are collected in the form of a Fraunhofer diffraction pattern on the upper focal plane F_2 of the objective which is located above the objective and very close to it. This pattern is easily observed with the eye piece removed. But the rays continue, and at infinity or on the focal plane of the collector lens they combine into a more or less faithful image of the object grating.

If the spectra on the upper focal plane of the objective are stopped down even further than this is done by the exit pupil, or if an objective of smaller aperture is used, the image becomes less distinct. If with oblique incidence, for instance, at least two spectra are present, then one only sees a sinusoidal structure without other details. If only *one* spectrum is admitted, the image disappears in a uniformly illuminated surface. It is also possible to simulate a structure that does not exist. If, for instance, the first order spectrum is stopped but the second order spectrum is allowed to pass, a grating with twice the actual number of lines appears. For the correct grating period to be just visible, at least both first order spectra must appear at the edge of the exit pupil of the objective; this means that the first order spectra may emerge from the objective at angles of at most $\pm u$. From the formula for the grating spectrum (32.2) we obtain for air and perpendicular incidence¹

$$(4) \quad \sin u \geq \alpha = \frac{\lambda}{d}.$$

If, on the other hand, the object and the first lens of the objective are embedded in a medium of index of refraction n (immersion), then λ must be replaced by the smaller wavelength $\lambda' = \lambda/n$; the grating spectra then crowd closer together, which explains the significance of immersion from the point of view of Abbe's theory. The condition (4) becomes now

$$(4a) \quad \sin u \geq \alpha = \frac{\lambda'}{d} = \frac{\lambda}{n d}.$$

By (2a) this means

$$(5) \quad A = n \sin u \geq \frac{\lambda}{d}.$$

¹ α and α_0 continue to stand for direction cosines; see footnote 1, p. 808.

We can now explain the advantage of *oblique* illumination. Let us assume, for instance, that the zeroth order spectrum falls on one edge of the exit pupil and that the first order spectrum falls on the opposite edge; this situation still suffices to show the existence of a structure and to reproduce its correct grating constant but no other details. The angle between the two spectra can now be twice as large as before or the spacing of grating lines can be half that of before. In place of eqs. (4 a) and (5) we obtain

$$(6) \quad 2 \sin u = \alpha - \alpha_0 = \frac{\lambda'}{d} = \frac{\lambda}{n d},$$

$$(6 a) \quad A = n \sin u = \frac{1}{2} \frac{\lambda}{d}, \quad \text{hence} \quad d = 0.5 \frac{\lambda}{A},$$

which involves only the small improvement in the ratio 0.5 : 0.61 over (3). If, instead of the spectra of zeroth and first order, the first and second order spectra or spectra of two still higher orders are used, then one speaks of "dark field illumination". The direct light (zeroth order spectrum) does not enter the objective and with no object in place the field of vision remains dark. For $n \sim 1.6$, $\sin u \sim 1$, $A \sim 1.6$ a numerical estimate yields according to eq. (6 a)

$$d \sim \frac{\lambda}{3}.$$

Smaller spacings can be resolved only by the use of shorter wavelengths: an ultraviolet microscope equipped with quartz and fluorite lenses which is effective down to $\lambda = 0.2 \mu$ or an electron microscope. In the case of the latter the use of hard cathode rays makes the resolving power theoretically almost infinite.

These considerations are valid not only for the one-dimensional gratings which we have had in mind so far, but also for arbitrary plane structures which, according to the Fourier theorem for two dimensions, can always be considered as a superposition of cross gratings. The structure of the object and the structure of the diffraction image in the objective focal plane are "reciprocally" related to one another; one is the "Fourier-transform" of the other; see Vol. VI, eq. (4.13). This "reciprocity" means that the diffraction image of the second of the above structures again yields the structure of the original object. Also in the case of two-dimensional structures any loss of diffraction spectra resulting from stopping down reduces the similarity between the final image and the structure of the object.

B. SIGNIFICANCE OF PHASE GRATINGS IN MICROSCOPY

As an example, we shall consider the "laminary profile" of fig. 70. Like any pure phase grating, it is completely invisible if the image is perfect, for neither the retina nor the photographic plate can perceive phase differences. We would like to be able to see this grating as a set of bright and dark fringes, that is, as an *amplitude grating*. To accomplish this it suffices, according to the concluding remarks in Sec. 36 D, to shift the phase of the zeroth order spectrum by $\pi/2$ with respect to the phases of the spectra of higher orders. F. Zernicke¹ produced this phase shift in the following way: a glass plate is placed in the focal plane of the objective. A thin layer of transparent material is attached to the center of this plate where for axis-parallel illumination the spectrum of zeroth order appears. While the spectra of higher orders are unaffected, the phase of the zeroth order spectrum is changed by an amount which depends on the thickness of the thin layer and its index of refraction relative to the surrounding medium. In order for the phase change to be $\pi/2$, the path difference must be $\lambda/4$ and the thickness

$$d = \frac{\lambda}{4(n-1)}.$$

This is a real "quarter wave plate"; the layer is less than 1μ thick in contrast to the "quarter wave plate" of crystal optics, see Sec. 30 B, where the thickness was determined by the small difference $n_2 - n_1$ between the indices of refraction in the two principal directions of oscillation rather than by the much larger difference $n - 1$ between the indices of refraction of the plate and its surroundings.

This *phase contrast method* of Zernicke is perhaps the most sensitive way of making very weak phase structures visible. Previously microscopists had been obliged to use more or less oblique illumination. The resulting loss of several spectra caused blurred images of the object. The Zernicke method, on the other hand, fully utilizes the tissue structure and makes it visible to the eye in the same way as an ideal staining method would do.

C. LUMINOUS AND ILLUMINATED OBJECTS

Because of the great successes of Abbe's theory, it was thought for a long time that Helmholtz's theory applied only to *luminous sources*, while in the treatment of *illuminated objects* only Abbe's theory was believed to hold².

¹Z. f. techn. Physik, Vol. 11, 1935.

²Such as in the review article by O. Lummer and F. Reiche: Die Lehre von der Bildentstehung im Mikroskop von Ernst Abbe, 1910.

However, Laue¹ proved by means of a simple hypothetical experiment that the image of any small luminous object, for instance a glowing Wollaston wire, is perfectly complementary to the image of the same object when illuminated by an external source. If the wire is in a cavity of constant temperature, then according to the laws of radiation it is entirely invisible. The radiation emitted by the wire and that originating from the walls of the cavity and absorbed and re-emitted by the wire combine to produce the same radiation density as that of the background. The same holds for observations with a microscope: the images resulting from self-luminous objects and those resulting from homogeneous illumination of equal brightness of the same objects must structurally complement one another entirely. The resolution of neighboring objects must be the same in both cases.

To be sure, the illumination is not homogeneous in the case of a microscope. With the usual arrangement of a lower illuminating mirror, the illumination is merely as uniform as possible within the aperture. However, the illumination from above is missing, but this cannot make much difference as long as the object is not (or only slightly) reflecting. We may think of this illumination from above as added or we may just omit it. The rays which do not enter the aperture are equally insignificant. Therefore Laue was justified in applying the law of black-body radiation to the microscope.

The superiority of Abbe's point of view becomes apparent only with oblique illumination of the type used in dark field observations. For, Helmholtz's theory assumes that all points of the object radiate uniformly in all directions, and this is generally not true for the elements of a tissue structure.

44. On Young's Interpretation of Diffraction

Even before Fresnel, Thomas Young² attempted to find a wave-theoretical explanation for the diffraction phenomena which had been discovered by Grimaldi. Young assumed that the incident light undergoes "a kind of reflection" at the edges of the diffraction opening and he explained the diffraction fringes in terms of the principle of interference which he had discovered as caused by the interaction between these edge rays and the incident light rays. In this way he achieved a qualitative understanding of the diffraction pattern of the slit in particular. However, Fresnel in his prize

¹M. von Laue, *Zur Theorie der optischen Abbildung*, Ann. d. Phys. Lpz. 43, 1914.

²Phil. Trans. Roy. Soc. London 20, 1802.

essay of 1818 showed that Young's assumption did not suffice for a quantitative explanation, and as a result Young's theory was forgotten for a long time.

In this connection we wish to recall our treatment of the half-plane in Sec. 38. The light entering the geometrical shadow is a cylindrical wave which appears to originate on the edge of the screen; the diffraction fringes in the illuminated region were calculated from the interaction of this cylindrical wave with the incident light. The cylindrical wave does not, of course, radiate uniformly in all directions; rather its intensity depends in a definite way on the angle of diffraction. Furthermore, the edge of the screen is not an actual light source with infinite amplitude but only appears as such to a sufficiently distant observer because of a representation of the light field which is valid only asymptotically at large distances. From this we see the following: if we wish to talk of a reflection of the incident light as Thomas Young did, then the "kind of reflection" is very specialized and must be defined precisely.

The question arises whether and in what way Young's interpretation may be extended to arbitrary diffraction screens. This question was answered conclusively by A. Rubinowicz¹.

A. REFORMULATION OF KIRCHHOFF'S SOLUTION OF THE PROBLEM OF DIFFRACTION

We shall make the following assumptions:

1. The screen shall be arbitrarily bounded. We apply the Kirchhoff formula (34.4); we cannot use the simplified representation (34.6) in terms of the Green's function of the half-space because (even in the case of a plane screen) we will have to integrate not only over the diffraction opening but also over the surface of a cone.

2. Let the light source be a luminous point at a finite distance as in (34.4 b). We shall change the notation from r' (distance of a point on the surface of integration from the light source) to ρ . We shall not carry out the specialization to an incident *plane* wave ($\rho \rightarrow \infty$) because this would complicate rather than simplify the presentation.

3. We shall regard the light field as *scalar*, just as Kirchhoff did. Thus we shall actually discuss the diffraction problem of *acoustics* rather than the vectorial problem of *optics*. However, this will suffice to point out the essential features of Young's interpretation.

¹Ann. d. Phys. Lpz. 58, 1917 and 78, 1924.

4. Upon applying eqs. (34.4 b) in which we can set $A = 1$, the integrand in Kirchhoff's formula (34.4) becomes

$$(1) \quad J = \frac{e^{ikh\rho}}{r} \frac{\partial}{\partial n} \frac{e^{ikh\rho}}{\rho} - \frac{e^{ikh\rho}}{\rho} \frac{\partial}{\partial n} \frac{e^{ikh\rho}}{r},$$

where r is the distance of the point of integration from the point of observation P . First we perform the integration in the manner of Kirchhoff over a surface σ which in some way spans the diffraction opening. This surface together with the diffraction screen S will separate a region of space containing the light source P' from a portion which contains P . Equation (34.4) then reads

$$(2) \quad 4\pi v_P = \int_{\sigma} J d\sigma.$$

The following calculations serve only to transform this Kirchhoff formula and will not improve it or change its essence.

We adopt the following point of view: the surface σ over which (2) is to be integrated is entirely arbitrary; its choice is limited only by the requirement that σ shall pass through the curve s which forms the boundary of the diffraction opening. Hence (2) depends only on s and not on σ . Therefore it must be possible to transform the surface integral $\int d\sigma$ into a line integral $\int ds$. To accomplish this we construct the cone formed by the rays emitted by P' and passing through the boundary of the diffraction aperture, see fig. 90. We call the surface of this cone f and its surface elements df . We now consider the space bounded by σ and f and apply Kirchhoff's eq. (34.4) to this region. The boundary values to be used on f are

$$v = \frac{e^{ikh\rho}}{\rho} \quad \text{and} \quad \frac{\partial v}{\partial n} = \frac{\partial}{\partial n} \frac{e^{ikh\rho}}{\rho} = 0,$$

the latter because dn is perpendicular to $d\rho$ and $e^{ikh\rho}/\rho$ is only a function of ρ . The integrand of (1) simplifies to

$$(3) \quad J' = -\frac{e^{ikh\rho}}{\rho} \frac{\partial}{\partial n} \frac{e^{ikh\rho}}{r}.$$

For this region bounded by σ and f we obtain in place of eq. (2) a value differing from v_P :

$$(4) \quad 4\pi v_{P'} = \int_{\sigma} J d\sigma + \int_f J' df.$$

However, we know the exact value of $v_{P'}$. If P lies inside our truncated ray cone, then

$$(4a) \quad v_{P'} = \frac{e^{ik\rho}}{\rho};$$

if P lies outside this region and also outside the region which is directly illuminated by P' , then

$$(4b) \quad v_{P'} = 0.$$

Indeed we have incorporated into (4) the exact boundary values on both σ and f of the solution $u = e^{ik\rho}/\rho$ of the wave equation $\Delta u + k^2 u = 0$. Since by Green's theorem Kirchhoff's equation follows rigorously from this wave equation when the exact boundary values are known, $v_{P'}$ agrees exactly with the solution $u = e^{ik\rho}/\rho$ inside the truncated cone and vanishes outside it. If we substitute (4a, b) and (2) in (4), then for interior points

$$(5a) \quad v_P = \frac{e^{ik\rho}}{\rho} - \frac{1}{4\pi} \int J' df,$$

and for exterior points

$$(5b) \quad v_P = -\frac{1}{4\pi} \int J' df.$$

This completes the first step in transforming the Kirchhoff integral (2).

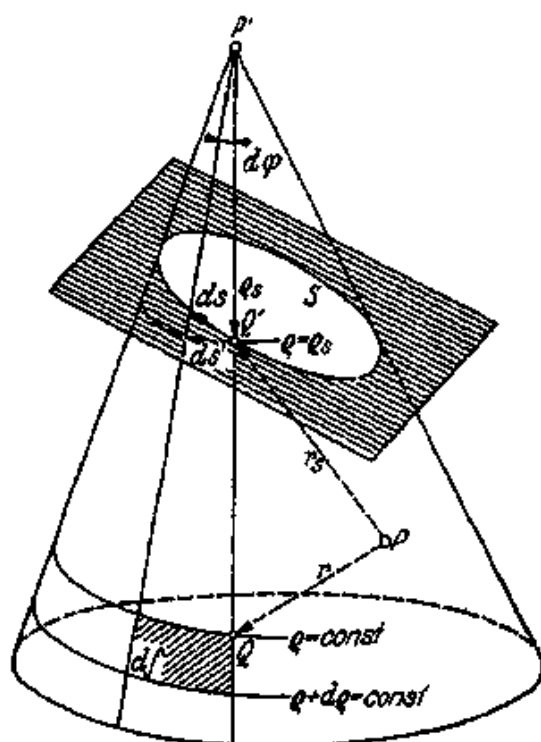


Fig. 90.

Concerning Thomas Young's theory of diffraction. Transformation of the surface integral into a line integral by the method of A. Rubinowicz.

B. REDUCTION OF THE SURFACE INTEGRAL OVER THE CONE TO A LINE INTEGRAL OVER THE BOUNDARY OF THE DIFFRACTION OPENING. SHARPENING OF YOUNG'S THEORY

In fig. 90 we have drawn two neighboring generators of the cone f which enclose an angle $d\varphi$. In the lower part of the figure we have drawn the intersections of the spheres $\rho = \text{constant}$ and $\rho + d\rho = \text{constant}$ with the cone surface. The shaded surface element df in the figure is then

$$df = \rho d\rho d\varphi.$$

We now replace $d\varphi$ by the element of arc ds' which the sphere $\rho = \rho_s$ intercepts at the point Q' on the diffraction edge. Next we express ds' in terms of the line element ds of the boundary curve:

$$ds' = \rho_s d\varphi = ds \cos(ds', ds) = ds \sin(\rho_s, ds).$$

Thus we obtain

$$(6) \quad df = \frac{\rho}{\rho_s} \sin(\rho_s, ds) d\rho ds.$$

We then evaluate the differential quotient occurring in (3) at the point Q on df

$$(7) \quad \frac{\partial}{\partial n} \frac{e^{ikh} r}{r} = \frac{\partial}{\partial r} \frac{e^{ikh} r}{r} \cos(n, r) = \left(\frac{ikh}{r} - \frac{1}{r^2} \right) e^{ikh} r \cos(n, r),$$

where n is the direction perpendicular to df at the point Q and r is again the distance of the element of integration from the point of observation P . Besides this distance r , we have also indicated in the figure the distance r_s of the point of observation from Q' , that is, from the vicinity of the boundary element ds . It is seen then that

$$(8) \quad r \cos(n, r) = r_s \cos(n, r_s);$$

indeed, the left- and right-hand sides of this equation are both equal to the shortest distance from the point of observation to the surface of the cone; it should be noted here that the normal n to the cone surface at Q is parallel to the normal to the cone at Q' .

From (3), (6), (7) and (8) we finally obtain

$$(9) \quad \frac{1}{4\pi} \int J' df = \frac{1}{4\pi} \int ds \sin(\rho_s, ds) \cos(n, r_s) \frac{r_s}{\rho_s} \int_{\rho_s}^{\infty} e^{ikh(\rho+r)} \left(\frac{ikh}{r^2} - \frac{1}{r^3} \right) d\rho.$$

The integrand of the first integral on the right-hand side consists of factors which depend only on the boundary curve s . The second integrand contains all the factors which are functions of the element df and of ρ ; among these is also r because of the following relation which is derived from the triangle $Q P Q'$:

$$(10) \quad r^2 = r_s^2 + (\rho - \rho_s)^2 + 2 r_s (\rho - \rho_s) \cos(r_s, \rho_s).$$

From this there follows by differentiation with respect to ρ , keeping ρ_s and r_s fixed (shifting Q with unchanged positions of P and Q')

$$r \frac{dr}{d\rho} = \rho - \rho_s + r_s \cos(r_s, \rho_s).$$

Hence also

$$(10\ a) \quad r \left(1 + \frac{dr}{d\rho} \right) = r + \rho - \rho_s + r_s \cos(r_s, \rho_s).$$

We now claim that the integrand of the ρ -integral in (9) is a perfect differential quotient, specifically, that

$$(11) \quad e^{ik(\rho+r)} \left(\frac{ik}{r^2} - \frac{1}{r^3} \right) = \frac{d}{d\rho} \left\{ e^{ik(\rho+r)} / r [] \right\},$$

where $[]$ symbolizes the right-hand side of (10 a). For, upon carrying out the differentiation with respect to ρ on the right-hand side of (11), one obtains the following three terms, which can immediately be simplified by applying eq. (10 a):

$$\begin{aligned} (a) \quad & ik \left(1 + \frac{dr}{d\rho} \right) e^{ik(\rho+r)} / r [] = \frac{ik}{r^2} e^{ik(\rho+r)}, \\ (b) \quad & -e^{ik(\rho+r)} r^{-2} \frac{dr}{d\rho} / [] = -\frac{1}{r^3} e^{ik(\rho+r)} \frac{dr/d\rho}{1 + dr/d\rho}, \\ (c) \quad & -e^{ik(\rho+r)} r^{-1} \left(\frac{dr}{d\rho} + 1 \right) / []^2 = -\frac{1}{r^3} e^{ik(\rho+r)} \frac{1}{1 + dr/d\rho}. \end{aligned}$$

The sum (a) + (b) + (c) indeed equals the left-hand side of (11). Thus the value of the ρ -integral in eq. (9) becomes

$$(12) \quad \left\{ e^{ik(\rho+r)} r^{-1} []^{-1} \right\}_{\rho_s}^{\infty} = -\frac{e^{ik(\rho_s+r_s)}}{r_s^2 (1 + \cos(r_s, \rho_s))}.$$

As a result the right-hand side of (9) becomes a single integral over the edge of the diffraction opening:

$$(13) \quad \frac{1}{4\pi} \int_s ds \frac{e^{ik\rho_s} e^{ikr_s}}{\rho_s r_s} \frac{\cos(n, r_s)}{1 + \cos(r_s, \rho_s)} \sin(\rho_s, ds).$$

The first factor in the integrand represents the phase and amplitude of the wave incident on the edge; the second factor corresponds to the phase at the point of observation of the spherical wave reflected by the edge; the third factor determines the rather complicated angular dependence of the reflected wave. (r_s, ρ_s) is the angle of reflection at the edge; (n, r_s) is, so to speak, the angle of reflection at the surface of the cone; (ρ_s, ds) is the angle of incidence at the curve element of the edge.

Returning to eqs. (5 a, b), we can say in agreement with Thomas Young: according to eq. (5 a) the diffraction fringes in the illuminated region result from the interference of the incident light with a wave which is reflected by the edge; according to eq. (5 b) only this edge wave is present in the shadow

region. We have improved Young's qualitative statement by defining the kind of reflection at the edge quantitatively in such a way that in both regions the resulting excitation v_p agrees exactly with that found by means of the Kirchhoff-Huygens theory. We have not, however, gone beyond the limits of validity of the Kirchhoff theory. Therefore the new formulae which have been adapted to Young's point of view are valid only if the wavelength is small compared to the diffraction opening and if the vectorial character of the electromagnetic problem does not come into play; see Sec. 46.

C. DISCUSSION OF THE CONTOUR INTEGRAL

Rubinowicz approximates the contour integral (13) by the method of *stationary phase* (the saddle-point method, simplified and adapted to a real domain): only those points on the boundary curve yield a substantial contribution to the integral at which the phase is stationary with respect to translation along the curve; the contributions of all other portions of the curve are small of higher order because of interference with the contributions of neighboring line elements. According to (13) the phase of the integrand on the boundary curve is

$$i k (\rho_s + r_s).$$

This remains constant under translation along the contour when

$$(14) \quad \frac{d\rho_s}{ds} = -\frac{dr_s}{ds},$$

or, what is the same, when the "reflection condition"

$$(14 a) \quad \cos(\rho_s, ds) = -\cos(r_s, ds)$$

is satisfied. There are in general a finite number of points $s = s_1, s_2, \dots$ on the curve which satisfy this condition. Each of these points radiates a substantial intensity to the point of observation P , and the line integral may be evaluated with sufficient accuracy as the sum of these radiations. The locus of points P which receive radiation from any one point s , on the edge is a circular half-cone with the apex at s , and with the axis ds . This has been proved experimentally by E. Maey¹ for the simple case of the half-plane, for which there exists only one such point s_1 .

Thus Young's point of view, when analytically formulated, also leads to a quantitative explanation of diffraction phenomena — however, only within the same limits of applicability which already restrict Kirchhoff's method.

¹Ann. d. Phys. (Lpz.) 49, p. 93, 1893.

It is to be noted that the discontinuity of the diffraction field at the shadow boundary, which a comparison of (5 a) and (5b) seems to indicate, does not in fact exist. This apparent discontinuity is exactly compensated by a jump in the value of our line integral, which is due to the fact that the denominator $1 + \cos(\tau_s, \rho_s)$ in eq. (13) vanishes as one passes through the shadow boundary.

Finally, we recall our consideration of "light fans" in Sec. 36 E, where these were explained from the point of view of Fresnel's zones. We now see that this phenomenon can be understood particularly well in terms of Young-Rubinowicz reflections. Each point s_i on the diffraction edge radiates a conical light fan; when there are not merely discrete points s_i but continuous sequences of such points, the light fans become particularly strong. This is the case for rectangles and more generally for polygonally bounded diffraction apertures, for then our cone surface / consists of plane portions which satisfy the reflection condition (14 a) along finite line segments. The specific intensity in these light fans is of the same order of magnitude as that of the incident light; the phenomenon of shadow formation thus disappears.

45. Diffraction Near Focal Points

From daily life we are well acquainted with the variously shaped caustics (focal lines) which appear within a teacup illuminated by a point source. These curves can be constructed by geometrical optics as the envelopes of pencils of rays. A more exact investigation of the vicinities of such lines leads to diffraction problems which have been treated particularly by Airy.

According to geometrical optics a focal point is an infinite concentration of rays. Wave optics resolves this (physically obviously inadmissible) singularity into a strong light concentration of finite amplitude and finite extent. In passing through a focal point a phase jump of magnitude π occurs. This jump has been studied experimentally by Gouy and Sagnac, among others. In the case of a focal line of the type which results, for instance, from the convergence of the rays of a cylindrical wave, the phase jump is $\pi/2$ instead of π . Rubinowicz¹ used his line integral (Sec. 44) as a starting point for the theoretical explanation of these phase jumps. He considered a ray bundle selected from a converging spherical or cylindrical wave by means of a diaphragm and treated its further course as a diffraction problem.

¹A. Rubinowicz, Phys. Rev. 54, 931, 1938; see also C. J. Bouwkamp, Physica 7, 485, 1940.

We shall adopt a simpler method which is due to Debye². He removed the diaphragm to infinity and in that way obtained a solution of the differential equation of optics which is valid in the whole space and exactly describes not only the phase jump but also the diffraction patterns in the vicinity of the focal point (or line). Debye's method is not limited to Kirchhoff's approximation but is based on the fundamentals of wave optics. His solution can claim the same degree of exactness as, for instance, our treatment of the problem of the straight edge in Sec. 38. There we had assumed as given the incident light at infinity in one half-space (plane wave) and had required that in the other half-space the radiation condition be fulfilled (hence no incident light). Correspondingly, Debye prescribes incident light (as a converging spherical wave) in one portion of infinity and requires that everywhere else at infinity no light shall be incident but shall only emerge.

A. THE HYPOTHESIS OF DEBYE

The expression

$$(1) \quad u = e^{-ikr \cos \Theta}, \quad \cos \Theta = \cos \vartheta \cos \vartheta_0 + \sin \vartheta \sin \vartheta_0 \cos (\varphi - \varphi_0)$$

represents a plane wave coming from infinity, which is incident from the direction $\vartheta = \vartheta_0$, $\varphi = \varphi_0$ and which, after passing through the point $r = 0$, radiates toward infinity in the direction $\vartheta = \vartheta_0 + \pi$, $\varphi = \varphi_0$. As always, the time factor $\exp(-i\omega t)$ is to be thought of as added. $r \cos \Theta$ is a linear function of the coordinates

$$x = r \sin \vartheta \cos \varphi, \quad y = r \sin \vartheta \sin \varphi, \quad z = r \cos \vartheta$$

having the coefficients

$$\alpha = \sin \vartheta_0 \cos \varphi_0, \quad \beta = \sin \vartheta_0 \sin \varphi_0, \quad \gamma = \cos \vartheta_0,$$

the sum of the squares of which equals 1. Hence u satisfies the wave equation $\Delta u + k^2 u = 0$ in the whole x, y, z space including the point $r = 0$ which is not a singular point of u .

The same is true for the wave packet

$$(2) \quad U = \iint e^{-ikr \cos \Theta} d\Omega, \quad d\Omega = \sin \vartheta_0 d\vartheta_0 d\varphi_0,$$

which represents *incident* waves only within the (arbitrarily defined) solid angle Ω . However this same expression (2) yields divergent waves for all ϑ, φ outside the solid angle Ω (and not merely in the solid angle which is diametrically opposite to Ω). This is due to the manner in which (2) was constructed as a superposition of the waves (1). Since it is an exact solution of the wave equation, U contains the answer to all questions regarding the behavior of the wave bundle in the vicinity of the focal point $r = 0$.

²Das Verhalten von Lichtwellen in der Nähe eines Brennpunktes oder einer Brennpunktlinie. Ann. d. Phys. (Lpz.) 80, 755, 1909.

It should be noted that no *boundary conditions* of any kind need to be satisfied. It is just this requirement, that boundary conditions be satisfied, which in other problems makes solutions in closed form impossible. Thus in contrast to other diffraction problems, Debye's formulation of the problem of focal point diffraction involves a simple *summation method*, as is brought out clearly by the form of (2). To be sure, we have performed only a scalar summation in (2) and not a vector summation as required by the directional character of the electromagnetic light field. But Debye has shown that his expression can be applied without change to describe the rectangular components of the Hertz vector from which the vectorial optical field can be derived.

B. THE DIFFRACTION FIELD IN THE NEIGHBORHOOD OF THE FOCAL POINT

First we shall show that the singularity of the light field at the focus which results from geometrical optics does not really exist, that according to wave optics the field is entirely regular. For the sake of brevity we shall make reference to several formulae from Vol. VI. According to eq. (VI. 22.35)

$$(3) \quad e^{-i\rho \cos \Theta} = \sum_{n=0}^{\infty} (2n+1) (-i)^n \psi_n(\rho) P_n(\cos \Theta),$$

where $\rho = k r$. The P_n are the Legendre polynomials

$$(4) \quad P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \dots, x = \cos \Theta;$$

according to Vol. VI, eq. (21.11) the ψ_n are the modified Bessel functions

$$(5) \quad \begin{aligned} \psi_n(\rho) &= \sqrt{\frac{\pi}{2\rho}} J_{n+\frac{1}{2}}(\rho) = \frac{\rho^n}{1 \cdot 3 \dots (2n+1)} \left(1 - \frac{\rho^2}{2(2n+3)} + \dots \right) \\ \psi_0(\rho) &= \frac{\sin \rho}{\rho}, \quad \psi_1(\rho) = \frac{\sin \rho - \rho \cos \rho}{\rho^2}, \dots \end{aligned}$$

If we neglect in (3) all powers of ρ higher than the second and perform the integration with respect to Ω indicated in (2), then we obtain, by using (4) and (5)

$$(6) \quad U = \left(1 - \frac{\rho^2}{6}\right) \int d\Omega - i\rho \int \cos \Theta d\Omega - \frac{\rho^2}{3} \int P_2(\cos \Theta) d\Omega.$$

For convenience we define the solid angle Ω as the interior of a circular cone with its apex at the focus. Thus the integration is to be extended over

$$0 < \vartheta_0 < \alpha, \quad -\pi < \varphi_0 < +\pi.$$

Then

$$\int d\Omega = 2\pi \int_0^\alpha \sin \vartheta_0 d\vartheta_0 = 2\pi (1 - \cos \alpha) = \Omega,$$

and recalling the meaning of $\cos \Theta$ from eq. (1) [the term containing $\cos (\varphi - \varphi_0)$ vanishes in the integration with respect to φ]

$$\begin{aligned} \int \cos \Theta d\Omega &= 2\pi \cos \vartheta \int_0^\alpha \cos \vartheta_0 \sin \vartheta_0 d\vartheta_0 = \pi \cos \vartheta (1 - \cos^2 \alpha) \\ &= \frac{1}{2} \cos \vartheta (1 + \cos \alpha) \Omega; \end{aligned}$$

furthermore, using the addition theorem for spherical harmonics [Vol. VI, eq. (22.36)]

$$\begin{aligned} \int P_2(\cos \Theta) d\Omega &= 2\pi P_2(\cos \vartheta) \int_0^\alpha P_2(\cos \vartheta_0) \sin \vartheta_0 d\vartheta_0 \\ &= \pi P_2(\cos \vartheta) \int_0^\alpha (3 \cos^2 \vartheta_0 - 1) \sin \vartheta_0 d\vartheta_0 = \pi P_2(\cos \vartheta) \cos \alpha (1 - \cos^2 \alpha) \\ &= \frac{1}{2} P_2(\cos \vartheta) \cos \alpha (1 + \cos \alpha) \Omega. \end{aligned}$$

Substituting this in (6) one obtains

$$(7) \quad \frac{U}{\Omega} = 1 - \frac{\rho^2}{6} - i \frac{\rho}{2} \cos \vartheta (1 + \cos \alpha) - \frac{\rho^2}{6} P_2(\cos \vartheta) \cos \alpha (1 + \cos \alpha).$$

For $\rho = 0$, the finite value

$$(8) \quad U = \Omega$$

is obtained (for our particular normalization of the incident amplitude). Hence, *in contrast to geometrical optics there is no singularity*. Going on from U to determine $|U|^2$, one finds, consistently disregarding higher powers of ρ than the second,

$$(9) \quad |U|^2 / \Omega^2 = 1 - a_1 \rho^2 + a_2 \rho^2 \cos^2 \vartheta, \\ a_1 = \frac{1}{3} - \frac{1}{6} \cos \alpha (1 + \cos \alpha), \quad a_2 = \frac{1}{4} (1 - \cos^2 \alpha), \quad a_1 - a_2 = \frac{1}{12} (1 - \cos \alpha)^2.$$

We are interested in the region of large amplitude surrounding the origin which represents the wave-optical spreading of the focal point of geometrical optics. We consider the first surface of extinction $U = 0$ as the outer limit of this region. Calculating it from the approximation (9) and setting

$$\rho^2 = k^2 (x^2 + y^2 + z^2), \quad \rho \cos \vartheta = k z,$$

we obtain

$$(10) \quad k^2 a_1 (x^2 + y^2) + k^2 (a_1 - a_2) z^2 = 1.$$

This is the equation of an ellipsoid of revolution which is elongated in the direction of incidence. The smaller α is, the larger are the principal axes $1/k\sqrt{a_1}$, $1/k\sqrt{a_1 - a_2}$. This means that the focal region becomes more extended as the incident bundle of rays becomes narrower. It decreases in size with decreasing wavelength (increasing k).

C. AMPLITUDE AND PHASE ALONG AND NEAR THE AXIS OF THE LIGHT CONE

On the axis of the light cone $\vartheta = 0$ in front of the focus and $\vartheta = \pi$ beyond the focus. Hence on the axis $\cos \Theta = \pm \cos \vartheta_0$. For these values the integration in (2) can be carried out by elementary methods and yields for $\vartheta = 0$

$$(11 a) \quad \frac{U}{2\pi} = \int_0^\alpha e^{-i k r \cos \vartheta_0} \sin \vartheta_0 d\vartheta_0 = \frac{e^{-i k r} - e^{-i k r \cos \alpha}}{-i k r},$$

and for $\vartheta = \pi$

$$(11 b) \quad \frac{U}{2\pi} = \int_0^\alpha e^{+i k r \cos \vartheta_0} \sin \vartheta_0 d\vartheta_0 = \frac{e^{+i k r} - e^{+i k r \cos \alpha}}{+i k r}.$$

Both expressions agree for $r = 0$ with the value of U given by (8). They remind us of the elementary representation of the diffraction pattern along the central axis of a circular disc or opening which was derived in Sec. 35 C and D. Like the latter expressions, eqs. (11 a, b) are valid only on the axis of symmetry of the light bundle. We know from Sec. 35 C that the "Poisson spot" disappears only a short distance off the axis. The same is true of the interference patterns given by (11 a, b).

We shall prove this in a somewhat indirect way by differentiating eq. (2) with respect to the angle α . In the case of a circular light cone which we are now considering, α appears in (2) as an upper limit of integration. Thus the integration with respect to ϑ_0 is to be omitted and the integral with respect to φ_0 can be written in terms of the Bessel function J_0 :

$$\begin{aligned} \frac{\partial U}{\partial \alpha} &= e^{-i k r \cos \vartheta \cos \alpha} \int_0^{2\pi} e^{-i k r \sin \vartheta \sin \alpha \cos (\varphi - \varphi_0)} \sin \alpha d\varphi_0 \\ &= 2\pi \sin \alpha e^{-i k r \cos \vartheta \cos \alpha} J_0(k r \sin \vartheta \sin \alpha). \end{aligned}$$

For all values of r which are physically of interest the argument of J_0 is a very large number except when $\vartheta = 0$ or π (or also when $\alpha = 0$ or π). We know, however, that J_0 vanishes for large real arguments; see, for instance Vol. VI, eq. (20.57). Therefore

$$\frac{\partial U}{\partial \alpha} = 0 \quad \text{for all } \vartheta \text{ except } \vartheta = 0 \quad \text{and} \quad \vartheta = \pi.$$

Except on these two half-rays U is, at infinity, independent of the angle α . From this we conclude that on the right-hand sides of (11 a) and (11 b) the second term in each of the numerators is to be neglected. Thus we obtain

$$(12 \text{ a}) \quad \frac{U}{2\pi} = \frac{e^{-ikr}}{-i k r} \quad \text{for} \quad 0 < \vartheta \leq \pi/2,$$

$$(12 \text{ b}) \quad \frac{U}{2\pi} = \frac{e^{+ikr}}{+i k r} \quad \text{for} \quad \pi/2 \leq \vartheta < \pi.$$

In both cases the interferences have disappeared; at large distances in front of and behind the focal point the light propagates as a spherical wave just as in geometrical optics.

We are mainly interested in the phase factors in eqs. (12 a, b), namely

$$-\frac{1}{i} = e^{+i\pi/2} \quad \text{in (12 a) and} \quad +\frac{1}{i} = e^{-i\pi/2} \quad \text{in (12 b).}$$

The phase jumps by π at the focal point as stated at the beginning of this paragraph. Since this phase change is in no way connected with the amplitude pattern at the focus but refers to the state of the spherical wave at large distances from the focus, we are justified in considering it as a property of geometrical optics; see Rubinowicz, loc. cit.

D. THE CYLINDRICAL WAVE AND ITS PHASE JUMP

The two-dimensional analogue of the above phenomenon concerns a ray bundle propagating in the r, φ -plane

$$(13) \quad U = \int_{-\alpha}^{+\alpha} e^{-ikr \cos(\varphi - \varphi_0)} d\varphi_0,$$

which, coming from infinity within the range $-\alpha < \varphi_0 < +\alpha$, proceeds toward the origin. In the language of three dimensions this origin is a geometrical focal line which is perpendicular to the r, φ -plane. We shall see again that wave-optically no singularity appears. For the proof we use formula (21.2 b) of Vol. VI:

$$(14) \quad e^{-i\rho \cos \psi} = J_0(\rho) + 2 \sum_{n=1}^{\infty} (-i)^n J_n(\rho) \cos n\psi.$$

The J are ordinary Bessel functions with integer indices. Neglecting higher powers of ρ , we approximate these Bessel functions by

$$J_0 = 1 - \left(\frac{\rho}{2}\right)^2, \quad J_1 = \frac{\rho}{2}, \quad J_2 = \frac{1}{2} \left(\frac{\rho}{2}\right)^2$$

and obtain from (13) and (14)

$$(15) \quad \frac{U}{2\alpha} = 1 - \frac{\rho^2}{4} - i\rho \cos \varphi \frac{\sin \alpha}{\alpha} - \frac{\rho^2}{4} \cos 2\varphi \frac{\sin 2\alpha}{2\alpha}.$$

At $\rho = 0$, U has (in contrast to geometrical optics) the *finite* value

$$(15a) \quad U = 2\alpha.$$

In order to find the intensity distribution for small ρ , we calculate $|U|^2$, limiting ourselves consistently to quadratic terms in ρ . Using $kx = \rho \cos \varphi$, $ky = \rho \sin \varphi$, we find

$$(16) \quad |U|^2/4\alpha^2 = 1 - k^2 a_1 x^2 - k^2 a_2 y^2, \\ a_1 = \frac{1}{2} \left(1 + \frac{\sin 2\alpha}{2\alpha} - 2 \frac{\sin^2 \alpha}{\alpha^2} \right), \quad a_2 = \frac{1}{2} \left(1 - \frac{\sin 2\alpha}{2\alpha} \right).$$

Setting $|U| = 0$ we obtain a measure of the size of the focal spot in the x, y -plane. According to (16), $|U| = 0$ on an ellipse whose major axis $1/k\sqrt{a_1}$ lies in the direction of incidence.

While in the three-dimensional case we obtained incident and radiating *spherical waves* (12 a, b) at large distances in front of and behind the focus, we now obtain for $kr \gg 1$ unperturbed convergent and divergent *cylindrical waves* in the region of the light bundles. These are represented by Hankel functions of the second and first kind, respectively,

$$(17a) \quad H_0^2(kr) \cong \sqrt{\frac{2}{\pi kr}} e^{-i(kr - \pi/4)},$$

and

$$(17b) \quad H_0^1(kr) \cong \sqrt{\frac{2}{\pi kr}} e^{+i(kr - \pi/4)}.$$

Here we have restricted ourselves to the asymptotic representations of Vol. VI, eqs. (19.55) and (19.56), which are the only ones of interest to us. From these we conclude that in passing through the focus the phase jumps from $\exp(+i\pi/4)$ in eq. (17 a) to $\exp(-i\pi/4)$ in eq. (17 b). *The phase jump in a cylindrical wave amounts to $\pi/2$.*

Actually the phase, like the amplitude, varies continuously in the vicinity of the focus; the "phase jump" appears only because in (12 a, b) and (17 a, b) we have compared the phases at points very far in front of and behind the focus.

46. The Huygens' Principle of the Electromagnetic Vector Problem

We have at various times pointed out the difference between the scalar acoustic and the vectorial optical problems. By limiting ourselves to a two-dimensional case in deriving the exact solutions of Sec. 38 and 39, we were able to reduce the vector problem to that of two scalar problems which described two different polarizations of the incident light. Huygens' principle is entirely *scalar* from the very beginning, so is the Fresnel-Kirchhoff application of Huygens' principle. We shall here, very briefly, discuss the *vectorial* formulation of this principle.

The problem is to calculate the vector fields \mathbf{E} , \mathbf{H} behind a diffraction opening σ in an opaque screen when the tangential components of \mathbf{E} and \mathbf{H} in the opening are given. We shall denote the latter, considered as vectors, by \mathbf{E}_0 and \mathbf{H}_0 . If these were known *exactly*, then it would be possible to calculate \mathbf{E} and \mathbf{H} exactly. If, instead of the exact values, the unperturbed \mathbf{E} , \mathbf{H} -values of the incident wave are used, one obtains only a first approximation which is valid for small wavelengths, just as in the Fresnel-Kirchhoff treatment.

First, we shall state the formulation of the vectorial Huygens' principle which is due to W. Franz¹:

$$(1) \quad 4\pi \mathbf{E} = \text{curl} \int_{\sigma} [d\sigma \times \mathbf{E}_0] \frac{e^{i\hbar R}}{R} - \frac{1}{i\omega\epsilon} \text{curl} \text{curl} \int_{\sigma} [d\sigma \times \mathbf{H}_0] \frac{e^{i\hbar R}}{R},$$

$$(2) \quad 4\pi \mathbf{H} = \text{curl} \int_{\sigma} [d\sigma \times \mathbf{H}_0] \frac{e^{i\hbar R}}{R} + \frac{1}{i\omega\epsilon} \text{curl} \text{curl} \int_{\sigma} [d\sigma \times \mathbf{E}_0] \frac{e^{i\hbar R}}{R}.$$

The following will explain the notation:

a) $d\sigma$ is an element of area with which is associated the direction perpendicular to the surface σ ; the vector product of $d\sigma$ and \mathbf{E}_0 is therefore a vector lying in the tangential plane of σ . The normal component of \mathbf{E}_0 does not enter into the calculation of this vector product. The same holds for the vector product of $d\sigma$ and \mathbf{H} .

b) R is the distance from the point of observation x, y, z to the point of integration ξ, η, ζ . \mathbf{E} , \mathbf{H} are functions of x, y, z ; \mathbf{E}_0 , \mathbf{H}_0 are functions of ξ, η, ζ .

¹ZS. f. Naturforschung, Vol. 8a, 500, 1948; we cannot here discuss the resulting corrections to the Kirchhoff diffraction calculations. See also Stratton and Chu, Phys. Rev. Vol. 56, 99, 1939, as well as the book by J. A. Stratton, *Electromagnetic Theory*, Internat. Series in Pure and Appl. Physics, New York, 1941.

c) The eqs. (1) and (2) therefore represent superpositions of spherical waves radiating from the points ξ, η, ζ just as in the elementary Huygens' principle; these waves interact at the point x, y, z , but now they combine vectorially.

d) The operation curl is everywhere to be performed with respect to the coordinates of the point of observation x, y, z . If cartesian coordinates are chosen, the identity

$$(3) \quad \text{curl curl} = \text{grad div} - \Delta$$

may be used, as is done in the following.

e) In Franz's work the arrow of $d\sigma$ points toward the back of σ , that is, into the region where \mathbf{E} and \mathbf{H} are to be determined.

f) The values of ε, μ are everywhere constant and can be identified with the values in vacuum.

We now show that the \mathbf{E} and \mathbf{H} as defined in (1) and (2) satisfy Maxwell's equations. We write the latter for a purely periodic state of frequency ω :

$$(4) \quad \text{curl } \mathbf{E} = i\omega\mu \mathbf{H}, \quad \text{curl } \mathbf{H} = -i\omega\varepsilon \mathbf{E}.$$

Next we form curl \mathbf{E} from eq. (1). Then the first term on the right-hand side becomes identical with the second term of eq. (2) multiplied by $i\omega\mu$. Using cartesian coordinates temporarily and applying (3), the second term of curl \mathbf{E} becomes

$$-\frac{1}{i\omega\varepsilon} \text{curl} (\text{grad div} - \Delta) \int_{\sigma} [d\sigma \times \mathbf{H}_0] \frac{e^{ikR}}{R}.$$

Curl grad vanishes and there remains

$$(5) \quad \frac{1}{i\omega\varepsilon} \text{curl} \int [d\sigma \times \mathbf{H}_0] \Delta \frac{e^{ikR}}{R} = -\frac{k^2}{i\omega\varepsilon} \text{curl} \int [d\sigma \times \mathbf{H}_0] \frac{e^{ikR}}{R},$$

since $u = e^{ikR}/R$ satisfies the wave equation $\Delta u + k^2 u = 0$. Moreover

$$k^2 = \omega^2/c^2 = \varepsilon\mu\omega^2, \quad \text{hence} \quad -k^2/i\omega\varepsilon = i\omega\mu.$$

Accordingly, (5) is identical with the first term on the right side of (2) multiplied by $i\omega\mu$. Thus we have proved that eqs. (1) and (2) satisfy the first Maxwell eq. (4). Quite analogously it can be verified that the second eq. (4) is also fulfilled.

But as Franz points out, this representation satisfies the boundary conditions

$$(6) \quad \mathbf{E} \rightarrow \mathbf{E}_0, \quad \mathbf{H} \rightarrow \mathbf{H}_0 \quad \text{as} \quad x, y, z \rightarrow \sigma$$

only approximately.

In contrast to this we now show that the integral representation

$$(7) \quad 2\pi \mathbf{E} = \text{curl} \int_{\sigma} [d\boldsymbol{\sigma} \times \mathbf{E}_0] \frac{e^{i k R}}{R},$$

$$(8) \quad 2\pi \mathbf{H} = \text{curl} \int_{\sigma} [d\boldsymbol{\sigma} \times \mathbf{H}_0] \frac{e^{i k R}}{R}$$

satisfies the boundary conditions (6) but not the differential eqs. (4).

We shall merely make this statement plausible by assuming that $\mathbf{E}_0, \mathbf{H}_0$ are differentiable functions of position everywhere in the opening (including its boundary!) and by restricting ourselves to the case of a *plane* screen and hence a *plane* opening σ . We choose as the origin of a cartesian ξ, η, ζ -coordinate system the point O on σ , which is the point that P is to approach. The ζ -axis is to be perpendicular to $d\boldsymbol{\sigma}$, and its positive direction is to coincide with the direction of $d\boldsymbol{\sigma}$. The x, y, z -system in which the point of observation P is defined shall be parallel to the ξ, η, ζ -system. With this choice of coordinate system the three components of the vector product $[d\boldsymbol{\sigma} \times \mathbf{E}_0]$ have the values

$$(9) \quad (-E_{0y}, E_{0x}, 0) d\xi d\eta.$$

We now form

$$(9a) \quad \text{curl}_x \left\{ [d\boldsymbol{\sigma} \times \mathbf{E}_0] \frac{e^{i k R}}{R} \right\} = \left\{ [d\boldsymbol{\sigma} \times \mathbf{E}_0]_x \frac{\partial}{\partial y} - [d\boldsymbol{\sigma} \times \mathbf{E}_0]_y \frac{\partial}{\partial z} \right\} \frac{e^{i k R}}{R}.$$

According to (9) this is equal to

$$= -d\xi d\eta E_{0x} \frac{\partial}{\partial z} \frac{1}{R},$$

where, in this last expression, we have set $e^{i k R} \sim 1$ in the neighborhood of O . We now calculate the right-hand side of the x -component of (7) and obtain:

$$(10) \quad - \iint d\xi d\eta E_{0x} \frac{\partial}{\partial z} \frac{1}{R} = \iint E_{0x} \frac{z}{R} \frac{d\xi d\eta}{R^2}.$$

On the right, z/R is the cosine of the angle between the directions z and R ; hence $d\xi d\eta z/R$ is the projection of $d\xi d\eta$ onto the sphere of radius R centered at P . Divided by R^2 this yields the solid angle which $d\xi d\eta$ subtends at P . As $P \rightarrow O$ this solid angle approaches 2π in the vicinity of O ; in the more distant parts of the ξ, η -plane it goes to zero. Hence (10) becomes equal to $2\pi E_{0x}$. In the same way the y -component of (7) may be checked.

The same calculation, applied to (8) shows that the boundary condition (6) is satisfied also for \mathbf{H} . If one substitutes the expressions (7) and (8) into the differential eqs. (4), and if one performs suitable integrations by parts and

applies eq. (34.20), the resulting surface integrals on both sides are found to be equal. But there also appear line integrals over the boundary of σ ("magnetic currents" as Stratton calls them, loc. cit.) which do not compensate one another.

The above remarks do not, of course, solve the problem posed in this paragraph; they only describe its general aspects. In any case, an exact solution of the diffraction problem could be obtained only if the exact boundary values E_0 and H_0 (or more correctly E_0 or H_0) were known. This is, of course, not the case. Rather, these boundary values can only be found simultaneously with the solution of the diffraction problem. Even in the case of the circular opening, the two auxiliary functions (potentials) which must be introduced to solve the problem are coupled in a complicated manner¹. *The vectorial Huygens' principle is no magic wand for the solution of boundary value problems*, but it is of interest as a generalization of the time-honored idea of Christian Huygens.

47. Cerenkov Radiation

According to the theory of relativity a material body cannot possibly move with a velocity v greater than the speed of light c . However, we know that in a medium of index of refraction n , light propagates with the phase velocity $u = c/n < c$; see Sec. 2. Hard cathode rays and Compton electrons produced by very hard γ -rays can attain velocities in the range

$$u < v < c.$$

What happens in this velocity range?

We might expect to find phenomena which are known to us from the field of ballistics; see Vol. II, fig. 45 a, b: The projectile overtakes the pressure wave which it produces and thus causes a Mach cone with the characteristic angle $\sin \vartheta = c/v$ (c = velocity of sound). While this phenomenon can be derived only with some difficulty from the non-linear equations of aerodynamics, the corresponding electro-optical phenomenon follows simply and rigorously from Maxwell's equations.

The following discussion will adhere closely to an early paper by the author which was communicated to the Amsterdam Academy² by H. A. Lorentz; the only difference being that where that paper (which preceded the theory of relativity!) deals with velocities below and above that of light, $v < c$ and $v > c$, we must now substitute $v < u$ and $u < v < c$, respectively. For velocities below that of light the electron carries its own field with it and no

¹ J. Meixner, ZS. f. Naturforschung, Vol. 8a, 506, 1948.

² Proc. Nov. 26, 1904, particularly p. 359 where older papers by Heaviside and Des Coudres are referred to; see also Göttinger Nachrichten 1905, p. 201.

energy is radiated; see Vol. III, Sec. 30 A. However for velocities above that of light the electron leaves its field behind in the shape of a Mach cone. The field radiates in directions perpendicular to the surface of the cone, and because of the nature of dispersion, this radiation consists mainly of visible light. Since the electron loses energy in the form of radiation, its velocity decreases rapidly to the value of the light velocity $v = u$. The radiated light is polarized so that the electric vector lies in the plane passing through the trajectory of the electron.

This radiation, then an unheard-of optical phenomenon, was first observed in 1934 by P. A. Cerenkov¹. At first he used Compton electrons and later many different types of cathode rays. These observations were soon thereafter explained by Frank and Tamm² in the manner here indicated and were compared quantitatively with the theory.

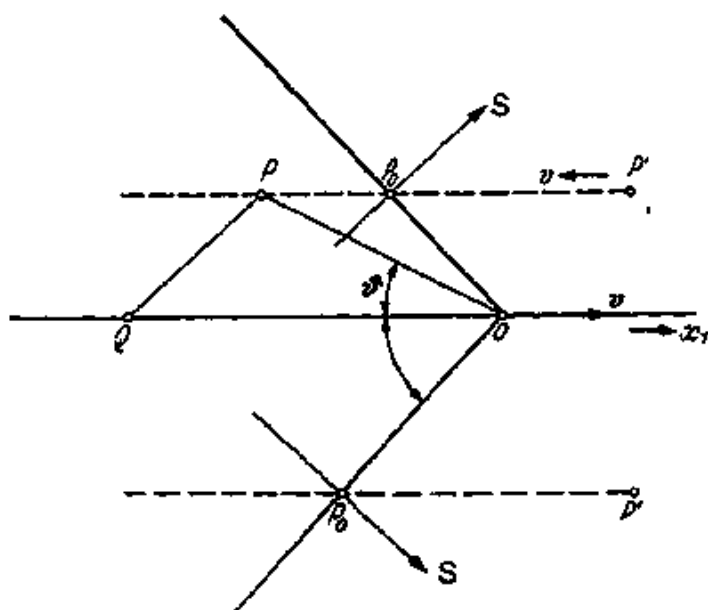


Fig. 91.
The Mach cone of the Cerenkov electron and its radiation vector S .

A. THE FIELD OF THE CERENKOV ELECTRON

Let us assume that an electron moves with a velocity v (to be thought of as constant) through a medium of refractive index $n > 1$ such that

$$(1) \quad u < v < c, \quad u = c/n.$$

At the time $t = 0$ let the electron be at the point O , and let Q be the position of the electron at the earlier time $\tau < 0$. We are to find the field of the electron at an arbitrary point P at the time $t = 0$. Let

$$(2) \quad r = \text{distance } OP, \quad \vartheta = \text{angle } QOP$$

(see fig. 91). Choosing the x -axis in the direction of motion of the electron, we denote the space-time coordinates of P by

$$(3) \quad x_1 = -r \cos \vartheta, \quad x_2 = r \sin \vartheta \cos \varphi, \quad x_3 = r \sin \vartheta \sin \varphi, \quad x_4 = i c t = 0$$

¹Ac. Sci. USSR., 2, 451, 1934, 3, 414, 1936, 20 et seq. Phys. Rev. 52, 378, 1937.

²C. R. Ac. Sci. USSR. 14, 109, 1937; Tamm, Journ. of Sci. USSR. 1, 409, 1939.

and those of Q by

$$(3a) \quad \xi_1 = v\tau < 0, \quad \xi_2 = \xi_3 = 0, \quad \xi_4 = ic\tau.$$

As we saw in Sec. 2, in a medium of refractive index n the light velocity c occurring in the wave equations for E and H is to be replaced by the phase velocity $u = c/n$. The same substitution must be made in the differential equations for the retarded potentials in Vol. III, Sec. 19. Since the velocities v in question are close to c , it is natural to use the four-dimensional potential Ω of Vol. III, Sec. 29 instead of the three-dimensional retarded potentials. The symbol \square which was introduced there must now be amended to mean

$$(4) \quad \square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + n^2 \frac{\partial^2}{\partial x_4^2};$$

for, indeed, the fourth term in this expression is

$$n^2 \frac{\partial^2}{\partial x_4^2} = -\frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} = -\frac{1}{u^2} \frac{\partial^2}{\partial t^2},$$

as it should be according to our present differential equations for the retarded potentials.

Correspondingly, we must continue to replace c wherever it occurs by $u = c/n$. Thus the "current density four-vector Γ " introduced in Vol. III, eq. (28.16) is now defined as

$$(4a) \quad \Gamma = (\rho v, i u \rho) = (\rho v, 0, 0, i u \rho),$$

where ρ is the charge density of the electron. The connection between Ω and the retarded potentials A and ψ which in vacuum was $\Omega = (A, i \psi/c)$ by eq. (26.4), Vol. III is now to be changed to

$$(4b) \quad \Omega = (A, i \psi/u).$$

With these changes in the definitions, the differential equation of our problem reads exactly as in Vol. III, Sec. 26:

$$(4c) \quad \square \Omega = -\mu_0 \Gamma.$$

In order to integrate (4c) we require, as in Vol. III, Sec. 29, a solution of the differential equation $\square U = 0$ which is infinite at $P = Q$. This solution is now defined by

$$(5) \quad U = \frac{1}{R^2}, \quad R^2 = (\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + (\xi_3 - x_3)^2 + \frac{1}{n^2} (\xi_4 - x_4)^2.$$

The reader may convince himself that this function satisfies the differential equation $\square U = 0$ in both sets of quadruple variables except at $P = Q$.

Applying Green's theorem, one obtains an expression identical with eq. (29.6), Vol. III except that the factor c occurring in ξ_4 must again be replaced by $u = c/n$. This results in the occurrence of the denominator n in the following equation:

$$4\pi^2 \Omega/\mu_0 = \int \Gamma \frac{d\xi_1 d\xi_2 d\xi_3 d\xi_4}{n R^2}.$$

If the electron is considered to be a point and the integration with respect to ξ_1, ξ_2, ξ_3 is carried out, one obtains

$$(6) \quad 4\pi^2 \Omega_1/\mu_0 = \frac{e v}{n} \int \frac{d\xi_4}{R^2}, \quad \Omega_2 = \Omega_3 = 0,$$

$$4\pi^2 \Omega_4/\mu_0 = \frac{i e u}{n} \int \frac{d\xi_4}{R^2}.$$

As shown in the text accompanying fig. 41, Vol. III, the integration with respect to ξ_4 must be performed by evaluating the contour integral around the negative-imaginary ξ_4 -axis because the position of the electron is given only for $\tau < 0$. This integral is non-vanishing only if there are points on this semi-axis at which the denominator R^2 vanishes. According to (3), (3 a) and (5) the condition for this is

$$(7) \quad R^2 = (v^2 - u^2) \tau^2 + 2 v r \tau \cos \vartheta + r^2 = 0.$$

The roots τ_{\pm} of this quadratic equation are given by

$$(7 a) \quad (v^2 - u^2) \tau_{\pm} = -v r \left(\cos \vartheta \pm \sqrt{\frac{u^2}{v^2} - \sin^2 \vartheta} \right).$$

We see, therefore, that for

$$(8) \quad \vartheta > \vartheta_M, \quad \text{where } \vartheta_M = \text{Mach angle, } \sin \vartheta_M = \frac{u}{v},$$

no real roots of (7) exist. Therefore, in this region the integrals in (6) vanish and hence not only $\Omega_2 = \Omega_3 = 0$ but also $\Omega_1 = \Omega_4 = 0$. *The field is zero everywhere outside the Mach cone.*

On the other hand, for all points P within the Mach cone, (7 a) has *two real negative solutions* (also τ_- is negative because $v > u$); *both* points for which $R = 0$ lie on the negative imaginary ξ_4 -axis and hence contribute to the integral. These contributions are, moreover, equal when the integration is performed in *opposite* directions around the two points¹. *An electromagnetic field exists everywhere in the interior of the Mach cone.*

¹If the directions of integration were the *same*, the sum of the two contributions would be zero. Thus we would not obtain an actual solution of the differential eq. (1). Therefore the path must be defined as a lemniscoid loop about the two points τ_{\pm} ; this is certainly permissible.

Before determining this field, we shall consider the connection between the times τ_{\pm} calculated in (7 a) and the so-called relaxation time of the retarded potentials, the "retardation" time of Vol. III, eq. (19.13 c)

$$(8 a) \quad \tau = r_{PQ}/c.$$

Here we have used r_{PQ} instead of the letter r of Vol. III in order to emphasize that what is meant is the distance between the position Q of the electron at time $t = \tau$ and the point P at which the field is being observed at the time $t = 0$. Using the notation of fig. 91 this distance is, according to the theorem of Pythagoras (r means here the distance PO):

$$r_{PQ} = \sqrt{v^2 \tau^2 + r^2 + 2 v r \tau \cos \vartheta}.$$

This, substituted in (8 a), where however c is replaced by u , yields indeed eq. (7). We note in this connection that *both* light points L and L' of fig. 41 of Vol. III are given by one and the same quadratic equation, namely our eq. (7); the only difference is that while the two points had different signs in Vol. III, they are now both negative. We also note that for $u = c$ the Mach angle (8) becomes imaginary and that is why there was *no* Mach cone in the problems in Vol. III.

We return now to the representation (6) of the four-vector potential. In order to evaluate the integrals involved we rewrite the expression (7) for R in the form

$$R^2 = (v^2 - u^2) (\tau - \tau_+) (\tau - \tau_-).$$

In evaluating the loop integral around $\tau = \tau_+$ we can replace the factor $\tau - \tau_-$ by $\tau_+ - \tau_-$. If we also write $d\xi_4 = i c d\tau$, we obtain

$$(9) \quad \oint \frac{d\xi_4}{R^2} = \frac{i c}{(v^2 - u^2) (\tau_+ - \tau_-)} \oint \frac{d\tau}{\tau - \tau_+} = \frac{-2\pi c}{(v^2 - u^2) (\tau_+ - \tau_-)}.$$

According to (7 a) the denominator of this fraction is equal to

$$-2 v r \left(\frac{u^2}{v^2} - \sin^2 \vartheta \right)^{1/2}$$

and therefore (9) becomes

$$\oint \frac{d\xi_4}{R^2} = \frac{\pi c}{v r} \left(\frac{u^2}{v^2} - \sin^2 \vartheta \right)^{-1/2}.$$

If we carry out the same calculation for the zero $\tau = \tau_-$, then $\tau_- - \tau_+$ takes the place of $\tau_+ - \tau_-$ in the denominator of (9). Therefore, with opposite directions of integration the two contributions are equal as stated above. One obtains therefore from (6)

$$(10) \quad 2\pi \Omega_1 / \mu_0 e u v = 2\pi \Omega_1 / \mu_0 i e u^2 = \frac{1}{v r} \left(\frac{u^2}{v^2} - \sin^2 \vartheta \right)^{-1/2} \\ = (u^2 r^2 - v^2 r^2 \sin^2 \vartheta)^{-1/2},$$

or written in terms of the coordinates x_1, x_2, x_3 of the point of observation

$$(10a) \quad 2\pi\Omega_1/\mu_0 \epsilon u v = 2\pi\Omega_1/\mu_0 i \epsilon u^2 = \{u^2 x_1^2 - (v^2 - u^2)(x_2^2 + x_3^2)\}^{-1/2}.$$

The equipotential surfaces are hyperboloids; they take the place of the Heaviside ellipsoids of Vol. III, Sec. 242, footnote 1. The field is finite and continuous everywhere inside the Mach cone; the reason why according to (10) the field becomes infinite on the surface of the cone ($\sin \vartheta = u/v$) is that we have assumed the electron to be a point charge; for an electron of finite radius a the field would attain only a maximum of the order of magnitude $1/a^1$.

We have derived the existence of the Mach cone from the phenomenological theory of a continuum. Physically, the cone, like the refractive index is caused by the molecular structure of matter. We need only consider that the electron moving with the velocity $v > u$ arrives at the molecules in its path sooner than the radiation emitted by the previously excited molecules. If dispersion is taken into account (see below), then the difference between the *phase* and *group velocity* affects the size of the Mach cone in an interesting way: The velocity of the wave fronts emitted by the previously excited molecules is not equal to the phase velocity u , but according to Sec. 22, it equals the group velocity $g < u$. Therefore the Mach angle is somewhat smaller than the value given by (8) because u must be replaced by g . A precise measurement of the angle of the Cerenkov waves (see fig. 92 below) should show this².

B. THE RADIATION OF THE CERENKOV ELECTRON

So far we have considered only the instantaneous picture of the field produced by the electron which we were able to characterize arbitrarily by the time coordinate $t = 0$. Therefore the field has apparently been independent of t . But for an observer in the laboratory, the field obviously depends on t . If the observer is at the point P' in fig. 91, he sees nothing until the surface of the Mach cone reaches him. For convenience fig. 91 has been drawn as if the observer were travelling into the cone along the dotted straight line $P'P$

¹See the above-mentioned paper in the Proc. of the Amsterdam Academy. There it will also be seen that the Mach cone is surrounded by a *marginal zone* of width $2a$ in which the field decreases continuously from its maximum on the cone to zero on the outside.

²A full discussion of this statement has recently been given by H. Motz and L. I. Schiff, Am. Journ. Phys. 21 (1953), p. 258. This paper also shows that the discussion given in the following sections B.-D. cannot be accepted without modification. (Translator).

with a velocity opposite to v . At the point P_0 on the surface of the cone the observer perceives the maximum field (in our approximation, actually an infinite field). Also at all interior points P a field is observed which becomes weaker with increasing distance from P_0 .

Mathematically, we obtain this time dependence simply by replacing in the above formulae

$$x_1 \quad \text{by} \quad x - vt.$$

That is, we consider the point O in fig. 91 not as fixed but as moving with time. In addition we shall write y, z instead of x_2, x_3 and express Ω in terms of the real potentials A, ψ in the manner of eq. (4 b). Then eq. (10 a) becomes

$$(11) \quad 2\pi A_x / \mu_0 e u v = 2\pi \psi / \mu_0 e u^3 = \{u^2 (x - vt)^2 - (v^2 - u^2) (y^2 + z^2)\}^{-1/2}.$$

From this it is easy to establish the validity of the relationship

$$(12) \quad \frac{\partial}{\partial t} A_x = -\frac{v^2}{u^2} \frac{\partial}{\partial x} \psi.$$

We use this to calculate the electric field E from the relation in Vol. III, eq. (19.7) $E = -\dot{A} - \nabla \psi$ and obtain

$$(13) \quad \begin{aligned} E_x &= \left(\frac{v^2}{u^2} - 1 \right) \frac{\partial \psi}{\partial x} = \frac{\mu_0 e u^3}{2\pi} (v^2 - u^2) (x - vt) \left\{ \right\}^{-3/2}, \\ E_y &= - \frac{\partial \psi}{\partial y} = \frac{\mu_0 e u^3}{2\pi} (v^2 - u^2) y \left\{ \right\}^{-3/2}, \\ E_z &= - \frac{\partial \psi}{\partial z} = \frac{\mu_0 e u^3}{2\pi} (v^2 - u^2) z \left\{ \right\}^{-3/2}, \end{aligned}$$

where $\{ \}$ stands for the expression inside the curly brackets of eq. (11). Thus the electric field has the direction of the vector

$$x - vt, \quad y, \quad z$$

which points from the instantaneous position of the electron to the point of observation. *If, in particular, the latter has the position P_0 of fig. 91, then the direction of the electric field coincides with a generator of the Mach cone.*

To calculate H the relation of Vol. III, eq. (19.6) $B = \text{curl } A$ is used. According to (11) this yields besides $H_x = 0$

$$(14) \quad \begin{aligned} H_y &= \frac{1}{\mu_0} \frac{\partial A_x}{\partial z} = \frac{e u v}{2\pi} (v^2 - u^2) z \left\{ \right\}^{-3/2}, \\ H_z &= -\frac{1}{\mu_0} \frac{\partial A_x}{\partial y} = -\frac{e u v}{2\pi} (v^2 - u^2) y \left\{ \right\}^{-3/2}. \end{aligned}$$

The magnetic field lines therefore form circles around the trajectory of the electron. *At the point P_0 the direction of H is tangential to a circular cross-section of the Mach cone.*

The directions of \mathbf{E} and \mathbf{H} determine the direction of the ray $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. At the point P_0 the ray direction is perpendicular to the Mach cone. The radiated light is polarized; the electric vector lies in the plane passing through the trajectory of the electron. Thus we have proved our initial statements about the character of the Cerenkov radiation. We still have to show that the spectrum lies principally in the *visible frequency range*. We note also that the radiation proceeds almost like a shock wave because the factor $\{\}^{-3}$ which occurs in the product of \mathbf{E} and \mathbf{H} is large only in the immediate vicinity of the surface of the Mach cone.

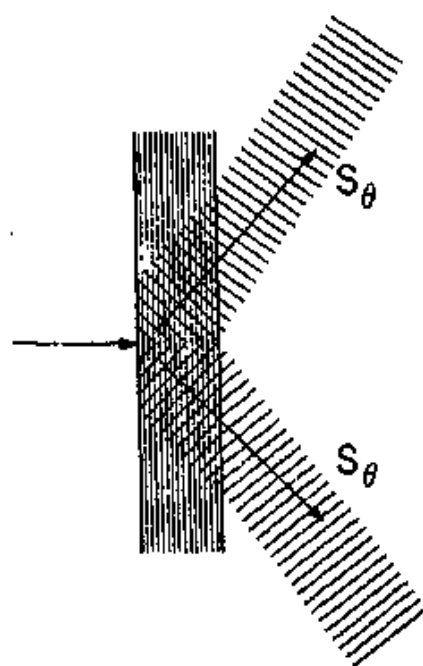


Fig. 92.
Observation of Cerenkov radiation behind a plate of large dielectric constant.

To produce Cerenkov radiation it is best to use a thin resin plate on which the electrons impinge perpendicularly. Thus only a small portion of the Mach cone appears. The emitted radiation fills a thin annular cone perpendicular to that portion; see fig. 92. The exposed area on a photographic plate placed behind the resin plate renders the annular trace of this cone visible.

C. CERENKOV RADIATION WITH DISPERSION TAKEN INTO ACCOUNT

So far we have carried out all calculations as if the index of refraction were a fixed number. Actually the refractive index is a function of frequency which rapidly approaches the value 1 in the far ultraviolet. But as n approaches 1, the interval $u < v < c$ in which the Cerenkov effect can take place shrinks to zero.

Therefore our previous treatment does not yet suffice for a quantitative analysis of experimental observations. In order to complete the calculations we would have to decompose the time dependence of the radiation field into its Fourier components in terms of ωt , and each of these components would have to be provided with its own $n(\omega)$. We would then find that only those Fourier components which lie in the visible spectrum make a noticeable contribution to the Cerenkov effect, while the ultraviolet spectrum is unable to excite Cerenkov radiation. Therefore we can state what has already been indicated in the introduction: *the Cerenkov effect makes electrons visible*.

However, owing to the singularity of the field on the Mach cone, the Fourier analysis involves certain formal difficulties. These can be avoided if, instead of a point-electron, an electron of finite extent (radius a) is used (see

footnote 1 on p. 341), for then the singularity is smoothed out into a transition zone with finite field strength. We cannot present these somewhat involved calculations here. Tamm (loc. cit.) overcame the above-mentioned difficulties by using so-called δ -functions¹; but these calculations, too, are complicated.

It is interesting to note that because in quantum mechanics one is forced to decompose the electromagnetic field into its Fourier components from the very start, the quantum-mechanical treatment of the Cerenkov effect² leads directly to a representation of the field in which dispersion can be taken into account, and therefore the visible character of the Cerenkov light is directly put in evidence.

Finally we wish to mention that in the author's work of 1904 a simple relation was derived for the force \mathbf{F} which, because of the radiation, acts to decelerate the electron. If c is replaced by the phase velocity u , assuming again a constant refractive index, and if the electron is treated like a rigid sphere with uniform spatial charge distribution, this force is given by

$$(15) \quad 4\pi |\mathbf{F}| = \frac{1}{4} \frac{e^2}{\epsilon_0 a^2} \left(1 - \frac{u^2}{v^2} \right).$$

An equivalent expression was derived by Tamm (loc. cit.). For an interesting application of eq. (15) to ballistics see F. Klein and A. Sommerfeld, *Theorie des Kreisels*, IV, p. 925, Leipzig, 1910.

D. A FINAL CRITICAL REMARK

We have used the formalism of the special theory of relativity to achieve a simple integration of the wave equation for the case of the Cerenkov electron. But in this formalism we have everywhere replaced the light velocity c by the phase velocity u . This change must also be made in the equation $\text{Div } \Omega = 0$, Vol. III, eq. (26.7) which therefore becomes

$$(16) \quad \text{div } \mathbf{A} + \frac{1}{i} \frac{\partial}{\partial t} \frac{i\psi}{u} = \text{div } \mathbf{A} + \psi/u^2 = 0.$$

This form of the equation was implicit in the above calculations.

However, we must note that the four-component quantities Ω , \mathbf{F} which we have used are not really four-vectors; they are not relativistic covariants. Rather, they are based specifically on the system of the stationary dielectric medium in which the electron is moving. In order to change from this reference system to another (such as the rest system of the electron, for instance), one should not use the usual Lorentz transformation but rather Minkowski's electrodynamics of moving media (Vol. III, chap. IV).

¹Similarly in G. Beck, *Phys. Rev.* 74, 795, 1948.

²K. M. Watson and J. M. Jauch, *Phys. Rev.* 75, 1249, 1949.

48. Supplement on Geometrical Optics. Curved Light Rays, Sine Condition, Lens Formulae, Rainbow

In Sec. 35 A we based geometrical optics on the existence of the *eikonal*, that is, a system of surfaces

$$S(x, y, z) = \text{constant}$$

the orthogonal trajectories of which are *rays*. According to (35.3) the equation of the eikonal is

$$D(S) = (\text{grad } S \cdot \text{grad } S) = n^2, \quad (n = \text{refractive index}).$$

The unit vector¹ in the ray direction, hence the normal to the eikonal surface passing through the point in question, is given by

$$(1) \quad s = \frac{\text{grad } S}{\sqrt{D(S)}} = \frac{1}{n} \text{grad } S.$$

From this follows

$$(2) \quad \text{curl}(ns) = 0.$$

This condition is equivalent to the existence of the eikonal. *All ray bundles (straight or curvilinear) realized in geometrical optics are normals to surfaces and are distinguished from more general systems of curves in that they satisfy the condition (2).*

Parallel rays and rays diverging from a luminous point source, which are the types of rays usually considered, obviously have the property of being *normals to surfaces*. A theorem which had already been stated by Malus says that this property is preserved under arbitrary reflections and under refractions in arbitrary lens systems. This theorem is self-evident to us because of the existence of the eikonal before and after every reflection or refraction, and it is also expressed by (2).

The following integral requirement is equivalent to the differential condition (2):

$$(2a) \quad \oint n(s_x dx + s_y dy + s_z dz) = 0.$$

We can also abbreviate this as follows

$$(2b) \quad \oint n s \cdot \vec{ds} = 0.$$

As a consequence

$$(2c) \quad \int_1^2 n s \cdot \vec{ds} = S_2 - S_1.$$

¹On its use is based the work by Sommerfeld and Iris Runge cited on p. 201. It will also simplify and clarify the following presentation.

That is, the line integral from a point 1 in the field to a point 2 is independent of the path and is equal to the difference between the values of the eikonal at the two points. Later we shall call the starting point 1 of the integral the "object" and the end point 2 the "image".

A. THE CURVATURE OF LIGHT RAYS

At the point P under consideration we construct the osculating plane to the curved light ray. Next we construct the tangent unit vectors s at P and s' at a neighboring point P' lying on the same ray. The curvature of the ray is defined as the angle between these two vectors divided by the distance $PP' = ds$. Since $|s| = 1$ the above angle equals the vector difference $s' - s$ which we call ds . Hence we define the curvature as

$$(3) \quad \vec{K} = \frac{ds}{ds}.$$

The absolute value of this vector gives the magnitude of the curvature; the direction of ds in (3) gives the position of the radius of curvature in the osculating plane.

For clarity we temporarily use cartesian coordinates¹ to transform the right-hand side of (3):

$$\frac{ds}{ds} = \frac{\partial s}{\partial x} \frac{dx}{ds} + \frac{\partial s}{\partial y} \frac{dy}{ds} + \frac{\partial s}{\partial z} \frac{dz}{ds}.$$

The factors $dx/ds \dots$ are nothing else but the components of the unit vector s . Hence we have

$$(3a) \quad \frac{ds}{ds} = \frac{\partial s}{\partial x} s_x + \frac{\partial s}{\partial y} s_y + \frac{\partial s}{\partial z} s_z.$$

In addition, it follows from $|s|^2 = 1$ for every direction of the gradient:

$$(3b) \quad 0 = \frac{1}{2} \text{grad } |s|^2 = s_x \text{grad } s_x + s_y \text{grad } s_y + s_z \text{grad } s_z.$$

Subtracting this from (3a) one obtains

$$(3c) \quad \frac{ds}{ds} = s_x \left(\frac{\partial s}{\partial x} - \text{grad } s_x \right) + s_y \left(\frac{\partial s}{\partial y} - \text{grad } s_y \right) + s_z \left(\frac{\partial s}{\partial z} - \text{grad } s_z \right).$$

The x -component of this vector equation (put $s = s_x$, $\text{grad} = \partial/\partial x$) reads

$$\begin{aligned} \frac{ds_x}{ds} &= s_y \left(\frac{\partial s_x}{\partial y} - \frac{\partial s_y}{\partial x} \right) + s_z \left(\frac{\partial s_x}{\partial z} - \frac{\partial s_z}{\partial x} \right) \\ &= -s_y \text{curl}_z s + s_z \text{curl}_y s = (\text{curl } s \times s)_x, \end{aligned}$$

¹Otherwise we would have to use tensor calculus which we wish to avoid.

and corresponding results hold for the y - and z -components of (3 c). Combining these into a single vector relation we have

$$(4) \quad \frac{ds}{ds} = \text{curl } s \times s.$$

This is true for every unit vector s and not merely for the surface normals of optics.

We now use the fundamental eq. (2) which characterizes the light vector s and which may be written in the form

$$n \text{ curl } s - s \times \text{grad } n = 0$$

or

$$(4 a) \quad \text{curl } s = \frac{1}{n} s \times \text{grad } n.$$

We substitute this in (4) and form the absolute values of both sides. We note that according to (4 a) $\text{curl } s$ is perpendicular to s and that therefore the absolute value of the vector product in (4) is equal to the product of $|\text{curl } s|$ and $|s|$. Thus we obtain

$$(5) \quad \left| \frac{ds}{ds} \right| = \frac{1}{n} |s \times \text{grad } n|.$$

According to (3) this is equal to the curvature

$$(6) \quad K = \frac{1}{n} |s \times \text{grad } n| = \frac{1}{n} |\text{grad } n| \sin \alpha, \quad \alpha = \text{angle } (s, \text{grad } n).$$

For the direction of the radius of curvature we obtain from (4) and (4 a):

$$\vec{K} = \frac{1}{n} [s \times \text{grad } n] \times s,$$

or, using a well-known vector theorem,

$$(6 a) \quad n \vec{K} = \text{grad } n - s (s \cdot \text{grad } n).$$

From this we see that the principal normal \vec{K} , the tangent s , and the gradient of n all lie in one plane, namely the osculating plane. Or, to say it better, if we consider as given not the osculating plane but the gradient of n and the direction s of the light ray, then the osculating plane passes through s and the gradient of n . The principal normal \vec{K} also lies in this plane. According to (4 a) the binormal has the direction of the (axial) vector $\text{curl } s$.

Equation (6) contains a theorem which was already used in 35 A: *in a homogeneous medium ($n = \text{constant}$) the light rays are straight lines ($K = 0$).*

As an example of an optically inhomogeneous medium, let us consider the earth's atmosphere at sunset. The refractive index n of the air decreases

with altitude; therefore the gradient of n points toward the center of the earth. The plane of the drawing in fig. 93 is the osculating plane of the curved light ray shown; this plane is perpendicular to the earth's surface. The direction of \vec{K} is essentially given by the first term on the right-hand side of (6 a) because

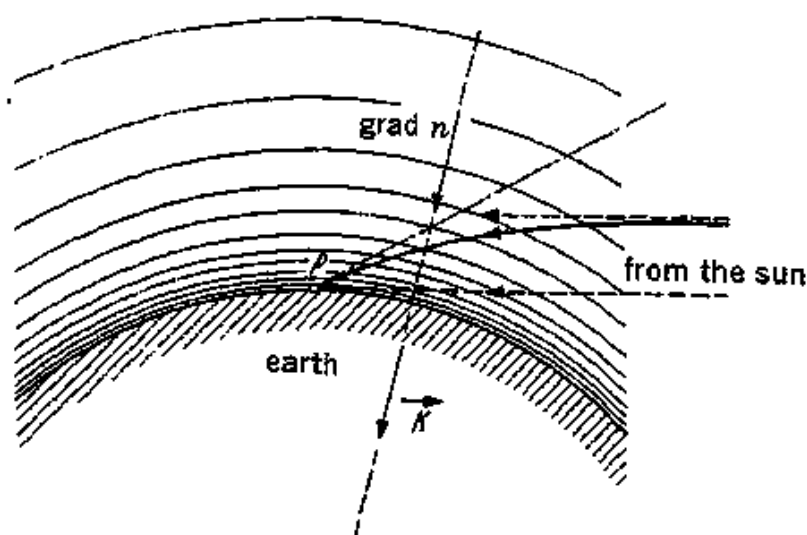


Fig. 93.

Curvature of the sun's rays in the atmosphere of the earth.

the second term is directed almost horizontally. Hence the light ray is curved concavely toward the earth. In a more elementary way of speaking the light ray is "refracted" in the direction of increasing air density. It follows that the setting sun appears *elevated* as indicated by the dash-dot tangent at the point P in the figure.

In the present example of a medium with parallel

strata, the equation of the curved ray can be written explicitly directly from the law of refraction in the form

$$(6b) \quad n \sin \alpha = \text{const.}$$

It is easy to show (by logarithmic differentiation and evaluation of $K = d\alpha/ds$) that this equation agrees with the generally valid eq. (6).

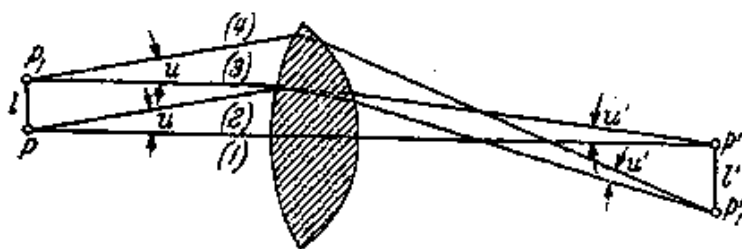


Fig. 94.

On the sine condition

B. ABBE'S SINE CONDITION

Let us consider a system of lenses which is symmetrical about an axis PP' and which, by means of bundles of rays of *finite* opening, maps the points

on the plane PP_1 in the object space as (almost) faultless images onto the plane $P'P'_1$ in the image space. Let u and u' be the angular apertures of the bundles; let n and n' be the indices of refraction in front of and behind the system of lenses; $n > n'$ implies immersion.

Let (1) be the axial ray from P to P' and (2) the limiting ray of the bundle emitted by P (because of the rotational symmetry only one meridian plane

need be considered, see fig. 94). The object point P_1 is at a distance l from P ; the rays which are emitted by P_1 and are parallel to (1) and (2) shall be called (3) and (4). Let the distance of the image P_1' from P' be l' . We are not concerned with the (possibly curved) ray paths inside the lens system; outside the lenses the rays are straight.

According to eq. (2 c) two line integrals $\int n s \cdot \vec{ds}$ between the same beginning and end points are equal to one another and to the difference between the respective values of the eikonal

$$(7) \quad \begin{aligned} (1) &= (2) = S(x', 0) - S(x, 0), \\ (3) &= (4) = S(x', l') - S(x, l), \end{aligned}$$

where x and x' are the abscissae of object and image measured along the central axis. We wish to find the difference (3) - (1) and the equal difference (4) - (2).

From (7) there follows for sufficiently small l' and l

$$(8) \quad \begin{aligned} (3) - (1) &= S(x', l') - S(x', 0) - \{S(x, l) - S(x, 0)\} \\ &= l' \frac{\partial}{\partial y'} S(x', y') - l \frac{\partial}{\partial y} S(x, y) \quad \text{with} \quad y' = y = 0. \end{aligned}$$

The two differential quotients, originating as they do from a Taylor expansion, are to be evaluated on the path 1. The first derivative is to be evaluated in the image space at the point P' , the second in the object space at the point P . On the other hand, because S is the line integral of $n s$, these two differential quotients are equal to

$$(8 a) \quad n' s_y \text{ in } P' \quad \text{and} \quad n s_y \text{ in } P, \text{ respectively}$$

Since they are evaluated on the path (1) where $s_y = 0$, they are both zero. Therefore

$$(8 b) \quad (3) - (1) = 0.$$

If we calculate (4) - (2) by the same method, we again obtain from the Taylor expansion the right-hand side of (8) where, however, the derivatives with respect to y are to be evaluated on the path (2) in the image and object space, respectively; on the other hand, by the definition of S as a line integral, these derivatives are again given by the values (8 a) but are now taken for the path (2). These are, see figure,

$$n' s_y = n' \sin u' \quad \text{and} \quad n s_y = n \sin u, \text{ respectively.}$$

Hence one obtains

$$(9) \quad (4) - (2) = l' n' \sin u' - l n \sin u.$$

But according to (7), $(4) - (2) = (3) - (1)$ and since $(3) - (1) = 0$ by (8 b), also $(4) - (2) = 0$. This is *Abbe's sine condition* which we had anticipated on p. 307:

$$(10) \quad n' l' \sin u' = n l \sin u.$$

As was shown by Straubel¹, this theorem is contained in a general reciprocity theorem of geometrical optics.

A similar theorem can be derived by considering two points P and P_1 which are *axially* displaced with respect to one another instead of being displaced *transversely* as before. The assumption now is that light bundles of finite opening project these points onto two image points P' , P'_1 which are also axially displaced with respect to one another. The only difference in this arrangement is that in all the above formulae the derivatives of the eikonal are to be taken with respect to x and x' instead of y and y' . The result is

$$\begin{aligned} (3) - (1) &= -n l + n' l', \\ (4) - (2) &= -n l \cos u + n' l' \cos u', \end{aligned}$$

which yields the relation

$$(11) \quad n' l' (1 - \cos u') = n l (1 - \cos u).$$

The incompatibility of this expression with the sine condition (10) makes it clear that no optical system can simultaneously produce sharp images of transversally and axially neighboring points by means of ray bundles with wide angles of opening.

C. ON THE STRUCTURE OF RECTILINEAR RAY BUNDLES

Ray bundles in homogeneous media and consisting of straight lines are, of course, of special interest. Kummer investigated the most general, not necessarily normal, bundles of straight lines. We shall limit ourselves solely to bundles which are possible in optics, that is, to those which for $n = \text{constant}$ satisfy the condition [eq. (2)]

$$(12) \quad \text{curl } s = 0.$$

We call one ray the central ray and consider only those rays which deviate a very slight distance from the central ray. That is, we consider an infinitesimally thin bundle. We construct a plane E perpendicular to the central ray and mark the points at which the rays of our infinitesimally small bundle intersect the plane. We do the same with a parallel plane E' which is located at the (small) distance δ from E . The corresponding points on the planes E and E' are related to one another by an affine transformation.

¹R. Straubel, *Physikal. ZS.* 4, 114, 1902.

According to the fundamental theorem of the kinematics of continuous media (see Vol. II, Sec. 1, specialized to a two-dimensional continuum), this transformation is composed of a deformation in two mutually perpendicular directions (symmetrical transformation coefficients) and a rotation about the axis perpendicular to these directions (antisymmetric transformation). A small circle drawn about the intersection of the central ray on the plane E is transformed into an ellipse by the deformation, and this ellipse is rotated by the rotational transformation through an angle given by $\frac{1}{2} \text{curl } s$. This rotation is analogous to the angular velocity in hydrodynamics. Our condition (12) applied to the central ray states that the rotational component of the transformation vanishes and that therefore the principal axes of the deformation ellipses are parallel for all planes E, E' . These axes lie in two fixed mutually perpendicular planes which are the *symmetry planes of the structure of the ray bundle*.

As such they must contain the two degenerate cases of the deformation ellipse in which one of the two principal axes shrinks to zero. These degenerate ellipses are called the *focal lines of the ray bundle*. They are perpendicular to one another and to the central ray (*Sturm's theorem*). The points where the focal lines intersect the central ray are called the *focal points* of the ray bundle. The distance between the two focal points is called the *astigmatic difference* d . (In the case of general Kummer rays the two focal lines are not necessarily perpendicular; non-optical ray complexes have no symmetry planes but, since $\text{curl } s \neq 0$, they have a positive or negative sense of rotation.)

From p. 208 we know that to every optical ray bundle there corresponds a system of parallel surfaces, the surfaces $S = \text{constant}$. In the simplest case of the spherical wave (and its special case, the plane wave) these surfaces are concentric spheres (or parallel planes). In this case the positions of the symmetry planes and of the perpendicular focal lines are indeterminate. The two focal points coincide, forming only *one* focal point; d becomes zero and the ray bundle converges to the focal point. This situation is characterized as *anastigmatism*.

D. ON THE LENS FORMULA

In high school one learns the lens formula

$$(13) \quad \frac{1}{a} + \frac{1}{b} = \frac{1}{f},$$

a = distance from object to lens

b = distance from lens to image

f = principal focal length of the lens; $1/f = (n-1) (1/R_1 + 1/R_2)$.

We shall show that this formula can be proved by the above method without recourse to trigonometry, to the law of refraction or to special graphical constructions.

The object point P emits spherical waves. According to (35.6) their eikonal for $n = 1$ (air) and the origin at P is

$$S = \sqrt{x^2 + y^2 + z^2}.$$

We consider the anastigmatic bundle which falls perpendicularly on the lens and place the z -axis along the central ray of this bundle. Near the lens ($z = a + \zeta$) we have, neglecting higher powers of x , y and ζ ,

$$(14) \quad S = z \left(1 + \frac{1}{2} \frac{x^2 + y^2}{z^2} \right) = a + \zeta + \frac{x^2 + y^2}{2a}.$$

We now continue the bundle into the interior of the lens. Let the front surface of the lens be a sphere of radius R_1 with its center at $x = y = 0$, $z = a + R_1$. Its equation is therefore

$$(14a) \quad x^2 + y^2 + (z - a - R_1)^2 = R_1^2.$$

The point where the central ray intersects this sphere shall be called T_1 .

We assume that the perpendicularly incident bundle remains anastigmatic after being refracted. (This is not the case for oblique incidence.) The proof that this assumption is correct lies in the fact that it permits us to satisfy the boundary condition (continuity of S in passing through the spherical surface R_1). We think of the refracted bundle as rectilinearly extended backward through the boundary surface to its point of convergence Q_1 . The distance $Q_1 T_1$ shall be ρ_1 . The eikonal of the ray bundle emerging from Q_1 is analogous to (14), where however a is replaced by ρ_1 :

$$(14b) \quad S = n \left(\rho_1 + \zeta + \frac{x^2 + y^2}{2\rho_1} \right) + S_0.$$

The factor n takes into account the fact that this ray bundle must be thought of as propagating in glass not only inside the lens but also outside it, since the bundle was constructed by a rectilinear extension across the boundary surface. The term S_0 is the integration constant available to us in integrating the differential equation of the eikonal.

We require that the eqs. (14) and (14b) connect continuously with each other on the sphere (14a) not only at the point T_1 on the central ray ($\zeta = 0$, $x = y = 0$) but also at points in its vicinity. If, as before, we set $z = a + \zeta$, these latter points are, according to (14a), characterized by

$$(14c) \quad \zeta = \frac{x^2 + y^2}{2R_1}.$$

Substituting this in (14) and (14 b) one obtains

$$(15) \quad S = \begin{cases} a + \frac{x^2 + y^2}{2 R_1} + \frac{x^2 + y^2}{2 a}, \\ n \rho_1 + S_0 + n \frac{x^2 + y^2}{2 R_1} + n \frac{x^2 + y^2}{2 \rho_1}. \end{cases}$$

The available constant S_0 is chosen such that the constant terms in both lines are equal. A comparison of the variable terms yields

$$\frac{x^2 + y^2}{2 R_1} + \frac{x^2 + y^2}{2 a} = n \frac{x^2 + y^2}{2 R_1} + n \frac{x^2 + y^2}{2 \rho_1},$$

and hence

$$(16) \quad \frac{n-1}{R_1} = \frac{1}{a} - \frac{n}{\rho_1}.$$

Next we consider a situation where the image point P' is the source of spherical waves which are refracted at the rear lens surface of radius R_2 . The point where the central ray intersects this spherical surface shall be T_2 . If R_1, a, ρ_1 are replaced by R_2, b, ρ_2 , then eq. (15) is again valid. ρ_2 is now the distance between T_2 and the convergence point Q_2 of the bundle refracted at the rear surface of the lens. Equation (16) becomes then

$$(16 a) \quad \frac{n-1}{R_2} = \frac{1}{b} - \frac{n}{\rho_2}.$$

But since P' is to be the image point of P , the rays emerging from P' must coincide inside the lens with the previously considered rays emitted by P . Hence Q_2 and Q_1 must coincide, and if we write henceforth $Q_1 = Q_2 = Q$, we obtain

$$(16 b) \quad Q T_2 = Q T_1 + T_1 T_2.$$

From it follows for $T_1 T_2 = d =$ thickness of the lens and with the same meanings of ρ_1 and ρ_2 , that

$$(17) \quad -\rho_2 = \rho_1 + d.$$

A negative sign appears on the left because with respect to the rear surface of the lens the convergence point Q has the opposite position from that with respect to the front surface.

For a "thin" lens ($d \ll \rho_1, \rho_2$) one has $\rho_2 = -\rho_1$. The sum of the two eqs. (16) and (16 a) yields directly the lens formula (13) which was to be proved. In order for the latter to be valid, it is not at all necessary for d to be small in any absolute sense. Our derivation requires only that d be small compared to the radii of curvature R_1, R_2 (or, what is approximately the same, with respect to ρ_1, ρ_2).

In order to progress somewhat beyond the usual school curriculum we shall also find the generalization for a lens of finite thickness.

For this purpose we calculate from (16)

$$\frac{\rho_1}{n} = 1 / \left(\frac{1}{a} - \frac{n-1}{R_1} \right)$$

and from (16 a) and (17)

$$\frac{-d - \rho_1}{n} = 1 / \left(\frac{1}{b} - \frac{n-1}{R_2} \right).$$

Adding these two equations as we previously added eqs. (16) and (16 a), we obtain as a generalization of (13)

$$(18) \quad \frac{d}{n} = 1 / \left(\frac{n-1}{R_1} - \frac{1}{a} \right) + 1 / \left(\frac{n-1}{R_2} - \frac{1}{b} \right).$$

We have restricted ourselves to the case of a single lens (same n in front and back). The result for a system of lenses is, of course, less simple than eq. (18). Furthermore, we have considered only the perpendicularly incident bundle. As was remarked above, an obliquely incident bundle does not remain anastigmatic after refraction; the resulting astigmatic ray structure has two focal lines, one of which lies in the plane of incidence and the other in the perpendicular plane. We wish to remark that the astigmatic bundle is decidedly the *normal case*: only with special axially symmetric arrangements is the anastigmatic character of the incident plane or spherical wave preserved.

As was noted above at eq. (13), it has not been necessary to make explicit use of the law of refraction in this entire section C. This is because the law of refraction by its very derivation (see beginning of Sec. 3) guarantees the equality of the phases of the incident and refracted waves. On the other hand, the eikonal, see Sec. 35 A, represents nothing but the phase of the wave; thus our requirement that the eikonal be continuous at the boundary surfaces of the lens represents an entirely valid substitute for the law of refraction. Indeed, the law of refraction can be obtained directly from our present eq. (2 b). It is only necessary to apply the latter to a shallow rectangle which surrounds and cuts across an element of surface of the refractive medium. Then (2 b) reads

$$(18 a) \quad n_1 s_1 = n_2 s_2,$$

where s_1 and s_2 are the tangential components of our unit vector and hence $s_1 = \sin \alpha$, $s_2 = \sin \beta$. Thus eq. (18 a) is indeed Snell's law of refraction.

E. PRODUCTION OF CURVED LIGHT RAYS BY DIFFUSION AND A REMARK ON THE THEORY OF THE RAINBOW

By means of a diffusion experiment it is possible to construct an optically inhomogeneous medium whose index of refraction varies in space in a simple and easily represented manner. A narrow tall parallel-walled glass tank is filled in its lower half, $x < 0$, with glycerine and in its upper half, $x > 0$, with water. While initially ($t = 0$) there exists a sharp plane of separation between the two media ($x = 0$), this separation becomes less and less distinct owing to diffusion. At $x = 0$ the concentration u always retains the value $u = 1/2$ which corresponds to a perfect mixture. At this point also the concentration curve $u(x)$ has an inflection tangent whose slope gradually increases with time. (Curves 1 and 2 in the figure.) At large positive values of x the curve $u(x)$ approaches the value 0 while at large negative values of x it approaches 1. As $t \rightarrow \infty$ (perfect mixture if the very small effect of gravity is neglected) $u \rightarrow 1/2$ (the line ∞ in the figure).

We may assume that the refractive index n behaves qualitatively like u , so that fig. 95 may also be used to represent n . Then we need only to read the figure with the following new designations:

the straight line $u = 0$ becomes $n = n_{\text{Water}} = n_1$,

the straight line $u = 1$ becomes $n = n_{\text{Glyc.}} = n_2$,

the straight line $u = \frac{1}{2}$ becomes $n = \frac{1}{2}(n_1 + n_2)$.

The significant point of this $n(x)$ diagram is the inflection point

$$(19) \quad x = 0, \quad n(0) = \frac{n_1 + n_2}{2} \quad \text{where } n'(0) = \text{maximum of } n'(x).$$

At this point the optical inhomogeneity of the medium, as represented by the gradient of n , is greatest. The inhomogeneity vanishes only at $t = \infty$.

Let us now paste tinfoil on the front surface of the tank so that this surface becomes opaque except for a narrow slit which is inclined 45° to the horizontal. Let us also illuminate the rear surface with perpendicularly

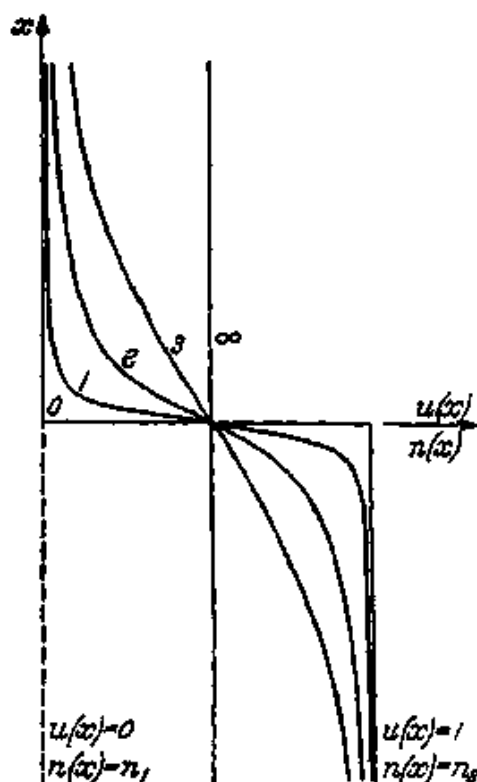


Fig. 95.
Variation of the concentration u and the refractive index n in the diffusion experiment.

incident light (arc lamp with collimator). One might think that a *rectilinear* image of the 45° slit should appear on an observation screen placed in front of the slit because each incident light ray at its position x passes through a horizontal layer with constant $n = n(x)$. But this is not so; the light ray is curved because $n'(x) \neq 0$, and because of the direction of the gradient of n , the center of curvature of the ray lies in the lower part of the trough; the center lies higher where the gradient of n is greater. Since the tank is very

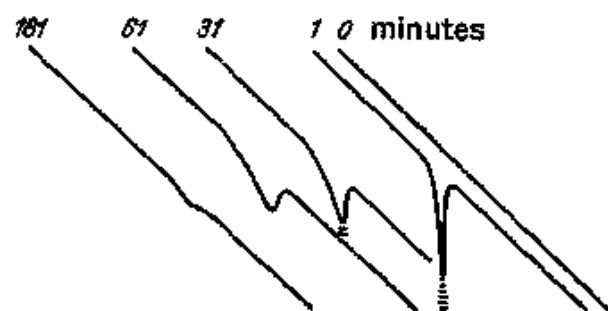


Fig. 96.

The light deflection in the diffusion experiment. The dashed lines indicated below the points of maximum light deflection should be imagined as colored in the rainbow colors.

narrow, the curved light path is very short, and we may approximate it by a circular arc with a horizontal tangent at the rear wall of the trough and a downward sloping tangent at the front wall. If d is the inside width of the trough and R the radius of the circular path, the slope of the tangent at the front wall is $\gamma = d/R$. According to eq. (6)

$$(19a) \quad \frac{1}{R} = \frac{1}{n} \left| \frac{dn}{dx} \right|,$$

because the angle α in eq. (6) is equal to $\pi/2$ (the gradient of n is perpendicular to the approximately horizontal light path). On emerging from the trough the slope γ is increased by a factor n (refraction at the interface with the thinner medium air). Hence the slope of the emerging light rays is given by

$$(19b) \quad \gamma' = n\gamma = d \left| \frac{dn}{dx} \right|.$$

The image point on the observation screen is deflected downward a distance corresponding to this angle. The deflection is greatest at the level $x = 0$; for large positive and negative values of x the deflection goes to zero so that the upper and lower ends of the image curve form a straight line inclined at 45° .

Figure 96 shows a sketch of the time variation of the image curve which was prepared during a seminar demonstration by H. Ott, who was at that time an assistant at our institute. In this figure 1 is the shape of the curve immediately after the start of the diffusion process; 31 is its appearance 30 minutes later, 61 is its appearance another 30 minutes later and 181 is the curve two hours later when the mixing process is almost complete. This experiment was first performed by O. Wiener¹ and is described by

¹O. Wiener, Ann. d. Phys. (Lpz.) 49, 105, 1893.

Kohlrausch in "Praktische Physik" as a method of measuring the diffusion constant k . In order to explain this application we must digress very briefly into the theory of diffusion.

In the one-dimensional case the differential equation of diffusion is

$$(20) \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

If the following substitution, which suggests itself for dimensional reasons, is made

$$(20 \text{ a}) \quad u = f(\xi), \quad \xi = \frac{x}{\sqrt{k t}},$$

then

$$\frac{\partial u}{\partial t} = -\frac{\xi}{2} f'(\xi), \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{k t} f''(\xi)$$

and by (20)

$$f''(\xi) = -\frac{\xi}{2} f'(\xi).$$

It follows that

$$f'(\xi) = A e^{-\xi^2/4},$$

where A is a constant of integration. Hence from (20 a)

$$(20 \text{ b}) \quad \frac{\partial u}{\partial x} = \frac{1}{\sqrt{k t}} A e^{-x^2/4 k t},$$

$$(20 \text{ c}) \quad \frac{\partial^2 u}{\partial x^2} = -\frac{x}{2 (k t)^{3/2}} A e^{-x^2/4 k t}.$$

As a consequence of (20 b) u can be represented by the Gaussian error integral, a representation which is similar to that derived in Vol. VI, eq. (13.19) for an analogous diffusion problem. For, if B is a second constant of integration,

$$(20 \text{ d}) \quad \left\{ \begin{array}{l} u = \frac{1}{\sqrt{k t}} A \int_0^x e^{-x^2/4 k t} dx + B = \sqrt{\pi} A \Phi(x/2 \sqrt{k t}) + B, \\ \Phi(p) = \frac{2}{\sqrt{\pi}} \int_0^p e^{-\alpha^2} d\alpha. \end{array} \right.$$

Rewriting the above equations in terms of the refractive index n in place of u , we conclude from the initial conditions for n

$$\begin{array}{llll} n = n_1 & \text{for} & t = 0 & \text{and} & x > 0, \\ n = n_2 & \text{for} & t = 0 & \text{and} & x < 0 \end{array}$$

and from (20 d) that

$$(21) \quad \begin{aligned} n_1 &= \sqrt{\pi} A + B, & n_2 &= -\sqrt{\pi} A + B, \\ A &= \frac{n_1 - n_2}{2\sqrt{\pi}}, & B &= \frac{n_1 + n_2}{2}. \end{aligned}$$

From this it follows according to (20 d) that for $x = 0$ and arbitrary t

$$(21 a) \quad n(0) = \frac{n_1 + n_2}{2},$$

which agrees with our assertion (19). Next we calculate from (20 b) and (21), again at $x = 0$,

$$(21 b) \quad \frac{\partial n}{\partial x} = \frac{n_1 - n_2}{2\sqrt{\pi k t}}.$$

From this the maximum angle of emergence γ' of the light ray is found by (19 b) to be

$$(21 c) \quad \gamma' = \frac{(n_2 - n_1) d}{2\sqrt{\pi k t}},$$

which determines the maximum deflection of the light ray on the curves in fig. 96. This deflection decreases with time as $1/\sqrt{t}$. Lastly, we conclude from (20 c) that

$$(21 d) \quad \frac{\partial^2 n}{\partial x^2} = \frac{x}{4(k t)^{3/2}} \frac{n_2 - n_1}{\sqrt{\pi}} e^{-x^2/4kt} \quad \text{for } x = 0 \text{ and all } t.$$

The curve of the index of refraction plotted as a function of x has a permanent inflection point at $x = 0$.

Wiener used eq. (21 c) to calculate the diffusion constant k from the measured deflections γ' . In order to obtain a sharp deflection curve with a well-defined maximum deviation, he used *monochromatic* light. However we are interested in a phenomenon which appears when *white* light is used: the lowest point of the deflection curve, particularly of curve 1 in fig. 96, is decomposed into a horizontally narrow but vertically well-separated spectrum, which is indicated in the figure by seven horizontal lines (the "seven" colors of the rainbow starting with red at the top and ending with violet at the bottom). This spectrum rapidly contracts as the diffusion process proceeds in time and is just barely recognizable in curve 2 of fig. 96. At a small distance from the point of maximum deflection the spectrum is quite indistinct even in curve 1. All other points on curves 1, 2, etc. appear white. Let us interpret this phenomenon on the basis of the wave theory.

By (21 c) the angle γ' depends on wavelength, because of the dispersion of n_1 and n_2 . Therefore the various colors are deflected in different directions and appear on the observation screen in sequence from red at the top to violet

at the bottom. We can no longer restrict ourselves to the layer at $x = 0$ as before, but in order to obtain a finite color intensity on the screen we must also consider the neighboring layers. If we denote by $n(x, \lambda)$ the dispersion of the ray passing through the layer at x , then not only $n(0, \lambda)$ but also the behavior of $n(x, \lambda)$ in the vicinity of $x = 0$ is of importance. n is *stationary* at $x = 0$ and *only* there. The curve $n(x, \lambda)$ has, according to (21 d), an *inflection point* there. This means that in the immediate vicinity of $x = 0$ the deflection angle γ' [see eq. (19 b)] is the same as at $x = 0$. Here the color effect is amplified while at points $x \neq 0$ it is extinguished owing to superposition. Only at the very beginning of the diffusion process does the resulting spectrum show considerable extension. The denominator $\sqrt{k t}$ in (21 c) causes not only the deflection but also the dispersion to decrease rapidly. The latter is already considerably reduced in curve 2 of fig. 96 as compared to curve 1.

We have treated this example in some detail in order to illustrate the essential idea of the *theory of the rainbow*. In order for the rays which are reflected, refracted and dispersed in the water droplets to enter the eye as *parallel rays* with sufficient intensity, the wave front (or rather its trace on the plane of incidence) must have an *inflection point*. (Since we are here dealing with geometrical optics, we should really say "eikonal surface" instead of "wave front".) This necessary and sufficient condition fixes the radius of the principal rainbow (single reflection in the interior of the droplet) at about 41° and that of the secondary rainbow (double reflection) at about $51^\circ 20'$. It is therefore clear that the rays contributing to the rainbow have an *extremal deviation* compared to all the other diverging rays which emerge from the droplets; for, indeed, the deviation of the inflection tangent perpendicular to the rainbow is an extremum compared to that of all other tangents. (In our diffusion experiment this was shown by the fact that the color spectrum appeared only at the lowest point of the deflection curve.) From this extremal position of the effective rays it follows that for the principal rainbow the diverging rays form a smaller angle with the incident rays of the sun than do the parallel rays and that therefore the diverging rays reach the eye from below the rainbow; it further follows that for the secondary rainbow the diverging rays form a larger angle with the incident radiation than do the parallel rays and that therefore the diverging rays reach the eye from above the bow; therefore the zone between the two rainbows appears darker than the regions below the principal and above the secondary rainbow.

We have emphasized before (p. 179) that, strictly speaking, the rainbow represents a difficult diffraction problem whose character changes from one case to the next depending on the sizes of the drops. As for the wave-

theoretical treatment of the rainbow, we shall limit ourselves to a result which is directly connected with the existence of the inflection point. If we wish to investigate the propagation of a cylindrical wave surface, for example by the saddle-point method, and represent the trace of the wave surface by $y = S(x)$, then we must look for the point $x = x_0$ where $S'(x_0) = 0$. To do this we develop $S(x)$ in a Taylor series about this point:

$$y = S(x_0) + \frac{(x-x_0)^2}{2} S''(x_0) + \frac{(x-x_0)^3}{3!} S'''(x_0) + \dots$$

The radiation is in general determined by the *curvature* of the trace of this wave surface as given by $S''(x_0)$. Since we are concerned with the approximation of an integral of the form

$$\int \exp(i k y) dx,$$

we are led to a *Fresnel* integral

$$\int \exp(i k a \tau^2) d\tau, \quad \tau = x - x_0, \quad a = \frac{1}{2} S''(x_0).$$

But if the trace has an inflection point $S''(x_0) = 0$, that is, if its curvature vanishes, this approximation breaks down; in place of the Fresnel integral one is then led to an *Airy* integral of the form

$$\int \exp(i k a \tau^3) d\tau, \quad \tau = x - x_0, \quad a = \frac{1}{6} S'''(x_0).$$

This could also be called a "rainbow integral", for it is characteristic of the quantitative investigation of the color distribution in the rainbow and of all similarly degenerate wave problems. One of these latter problems is discussed in detail in connection with the asymptotic representations of the Hankel functions in Vol. VI, Sec. 21 D; see in particular the final remarks in that section. Here we have only been interested in showing that the principal features of that most impressive of celestial phenomena, the rainbow, can be understood at least qualitatively in terms of geometrical optics and that, moreover, geometrical optics provides us with a hint for the quantitative treatment of the problem.

49. On the Nature of White Light. Photon Theory and Complementarity

In the historical chart of Sec. 1 we have listed an important achievement of Lord Rayleigh in the field of optics: his theory of natural white light as a *completely random process*. We shall first quote a remark made by Rayleigh¹ which concerns the opposite of white light, that is to say, regular monochromatic waves.

¹Phil. Mag. 50, 135, 1900, Sci. Papers, Vol. IV, p. 486.

"To suppose, as is sometimes done in optical speculations, that a train of simple waves may begin at a given epoch, continue for a certain time and ultimately cease, is a contradiction in terms. A like contradiction is involved if we speak of unpolarized light as homogeneous, really homogeneous light being necessarily polarized."

Regarding the last point in this quotation we refer to our Sec. 2 where the elliptic polarization of an ideal monochromatic and plane wave was proved to be a mathematical consequence of Maxwell's equations. The first sentence quoted has been taken into account in Sec. 22 where a wave train of the type described by Rayleigh was decomposed into its Fourier spectrum and was therefore treated not as monochromatic but rather as polychromatic.

Concerning the problem of white light we again quote a remark² by Lord Rayleigh which refers to the then recent discovery by Röntgen: "The conclusion of Stokes and J. J. Thompson 'that the Röntgen rays are not waves of very short wavelength, but impulses', surprises me. From the fact of their being highly condensed impulses, I should conclude on the contrary that they *are* waves of short wavelength. . . . What then becomes of Fourier's theorem. . . ?"

"Is it contended that previous to resolution (whether merely theoretical, or practically effected by the spectroscope) the vibrations of ordinary (e. g. white) light are regular, and thus distinguished from disturbances made up of impulses? This view. . . has been shown to be untenable by Gouy, Schuster, and the present writer. A curve representative of white light, if it were drawn upon paper, would show no sequences of similar waves."

We wish to add that Emil Wiechert, independently of Stokes, advanced the same hypothesis regarding the nature of Röntgen rays and that the author cooperated with him on several papers at the beginning of this century. In contrast to these theories, Planck, in connection with his discovery of the quantum of action (see p. 9), recognized the necessity of assuming the complete phase independence of "natural light". Laue's discovery subsequently put the individual Fourier components of the continuous Röntgen spectrum into direct evidence and proved the distinction between pulsed and wave radiation to be meaningless, in complete agreement with Rayleigh.

We use the words *white light* and *natural light* synonymously. We perceive the sun's natural light as white, i. e. as lacking all spectral colors, because the eye is *adapted* to the sun; that is to say, because our eye and the associated physiological-psychological vision apparatus has in its evolution-

² Röntgen Rays and Ordinary Light. *Nature* 57, 607, 1898 and *Sci. Papers*, Vol. IV, p. 353.

ary development adapted itself to the spectrum of the sun. If we lived in the vicinity of a red giant, we would presumably perceive its red color as the normal white. As is well known, Goethe abhorred the theory that white light is a mixture of the seven colors of the rainbow (he was certainly correct in regard to the white-sensation which he had primarily in mind). But the rainbow should have convinced him that white light is decomposed into colors by a spectral apparatus (in this case water droplets). In this decomposition the periodicity originates not from the primary sun light but from the frequency-sensitive spectral apparatus.

Gouy (1886) was probably the first to ascertain that a line grating diffracts a single "plane" pulse¹ in the same manner as a plane wave. An obliquely incident primary pulse (or a random sequence of mutually independent pulses) impinges on the various grating lines at rhythmically equal time intervals. Thereupon the grating lines emit a staggered sequence of secondary pulses²; one may refer to fig. 37, bearing in mind, however, that it describes continuous waves, while we are now dealing with a discrete sequence of secondary cylindrical excitations. At a sufficiently large distance from the grating and in any given direction these pulses are spaced a constant distance

$$\lambda = d(\alpha - \alpha_0)$$

apart (α is the direction cosine of the incident pulse with respect to the plane of the grating). This is our former grating formula (32.2) for $h = 1$. However λ is now not the wavelength of a monochromatic sinusoid but is the distance between pulses. The white character of the light has thereby not been lost entirely. Our sequence of secondary pulses is still far from being monochromatic. The periodicity originates in the grating and not in the primary pulse. It differs for different directions of observation. The higher order spectra ($h > 1$) would be obtained if the secondary pulse were decomposed into pure sine waves in the manner of Fourier, that is, into a fundamental oscillation of wavelength λ and harmonics of wavelengths λ/h .

But what is the situation in the case of a *prism* which also produces monochromatic waves from white light without seeming to possess any periodic structure as the grating does? Gouy answers this question in the

¹By "plane pulse" we mean the counterpart of a "plane wave", that is to say an electromagnetic disturbance which has an appreciable intensity only between two parallel infinite planes a small distance apart and which has a constant instantaneous value on every plane parallel to these two.

²By a "secondary pulse" we mean an excitation which has an appreciable intensity only between two coaxial cylinder surfaces centered about a grating line and which propagates outward in this cylindrical shape with the velocity of light.

following manner: a wide spectral line, i. e. a certain portion of the continuous spectrum, represents from a wave-optical viewpoint a *modulated* wave train (with beats between neighboring frequencies). These beats propagate with the *group velocity* and therefore lag behind the wave train which propagates with the *phase velocity*. This results in a periodic variation of the shape of the wave train: the wave train assumes the same shape only when the group has lagged behind the phase by one whole wavelength. The light path inside the prism increases continuously from the edge to the base. Those light paths which satisfy the above condition for any given wavelength define on the surface of emergence a sequence of equidistant parallel lines comparable to the grating structure. The resulting regularity of the emerging waves is thus due to the difference between group and phase velocity or, as we may also say, it is due to dispersion. Again the regularity originates in the prism and not in the incident (more or less white or colored) light.

With this concept of pulses we have extended the older, overly restricted conception of the wave theory in such a way that it comes closer to Einstein's hypothesis of the *light-quantum*. Indeed the mechanics of light quanta has haunted the science of optics from its very beginning.

What else is Fermat's principle of the shortest time of arrival but the principle of the shortest (geodetic) line in the mechanics of a force-free mass point? Both give the same result because in the force-free case the time of travel and the path length are proportional to one another. The same holds for the principle of least action because of the constancy of the kinetic energy, see Vol. I, eq. (37.1) and (37.2). Fermat's principle provides us with a truly popular exercise in the method of maxima and minima: given the starting and end points of the path and given the velocities in the first and second media, the incident particle of light travels along that path by which it reaches the end point in the second medium in the shortest possible time. The same is true if the end point also lies in the first medium and the accessory condition is added that the particle shall touch the surface separating media 1 and 2. Let us here use this principle to calculate the curvature of the trajectory in an inhomogeneous medium which we calculated in the preceding paragraph by means of the theory of the eikonal.

We start with the principle

$$(1) \quad \delta \int_{P_0}^P dt = 0,$$

where P_0 and P are the prescribed starting and end points of the path. Let us consider a stratified medium in which therefore the velocity u of the light particles

(phase velocity of the light) is a given function of the coordinate x alone. We replace u by the velocity ratio $n(x) = c/u(x)$ where c is a standard velocity, the magnitude of which is of no importance here. (1) becomes then

$$(2) \quad \frac{1}{c} \delta \int_{P_0}^P n(x) ds = 0,$$

which we write in the form

$$\delta \int F(x, y') dx = 0, \quad F(x, y') = n(x) \sqrt{1 + y'^2}.$$

The Lagrangian derivative for this variational problem reads

$$\frac{d}{dx} \frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

and hence, since F is independent of y

$$(3) \quad \frac{d}{dx} \frac{\partial F}{\partial y'} = n(x) \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} + \frac{dn}{dx} \frac{y'}{\sqrt{1 + y'^2}} = 0.$$

If α is the angle which the tangent to the curve $y = y(x)$ forms with the x -axis, then

$$(4) \quad \tan \alpha = y', \quad \sin \alpha = \frac{y'}{\sqrt{1 + y'^2}}, \quad \cos \alpha \frac{d\alpha}{dx} = \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}}.$$

Therefore the last term in the double eq. (3) equals $\sin \alpha \, dn/dx$, the next to the last term equals $n(x) \cos \alpha \, d\alpha/dx$, and eq. (3) becomes

$$(5) \quad n(x) \cos \alpha \frac{d\alpha}{dx} + \sin \alpha \frac{dn}{dx} = 0.$$

But now

$$(6) \quad dx = \cos \alpha \, ds, \quad \text{hence} \quad \cos \alpha \frac{d\alpha}{dx} = \frac{d\alpha}{ds} = K,$$

where K is the curvature of the curve $y(x)$. Combining eqs. (5) and (6) we obtain

$$(7) \quad |K| = \frac{1}{n} |\text{grad } n| \sin \alpha.$$

This is our eq. (48.6). As Hamilton had already recognized, geometrical optics is identical with the ordinary mechanics of mass points not only for homogeneous but also for inhomogeneous media.

However to pass from this primitive corpuscular theory of light to the modern "photon theory" required a bold step into quantum theory, a step which was taken by Einstein in 1905: the energy of the photon had to be set equal to $h\nu$ and its momentum to $h\nu/c$. Only in this way does one obtain the energy relations which are so drastically brought into evidence in the

photoelectric effect, the Compton effect, and in the short wave limit of the continuous X-ray spectrum. Only in this way can classical optics be brought into harmony with atomic physics.

It is significant that Einstein made this advance in the quantum theory in the same year in which the theory of relativity was created. L. de Broglie¹ emphasizes that only relativistic mechanics satisfies the requirements of the photon theory. According to classical mechanics we would have for a photon of energy W , momentum g and velocity u

$$W = \frac{m}{2} u^2, \quad g = m u, \quad \text{hence} \quad g = \sqrt{2 m W}.$$

If we substitute for the mass the value obtained from the generally valid relationship $W = m c^2$, then we obtain

$$g = \sqrt{2 \frac{W^2}{c^2}} = \sqrt{2} \frac{W}{c},$$

and not

$$g = \frac{W}{c} = \frac{h\nu}{c},$$

as required by the photon theory. A similar discrepancy by a factor 2 appears in the expression for the light pressure, depending upon whether it is calculated from classical mechanics or, as in Vol. III, Sec. 31, eq. (15), from relativistic electrodynamics.

The photon theory is a corpuscular theory of light such as the one which Newton envisioned. The wave theory of light has equal status with the photon theory. Which of the two will give the correct answer depends on the question that is posed in each particular experiment. Each completes the other — they are *complementary*. At the end of Chapter II we discussed the fact that the two theories do not contradict one another, and we mentioned the resulting far-reaching philosophical consequences. One is taught in school that the eye “sees the light waves”. That is a myth. What our eye “sees” is the photoelectric processes taking place in the retina which, depending on the magnitude $h\nu$ of the incident light quanta, produce the varicolored world of our visual impressions. Certainly as far as our primary sensations go, there is no preference for a wave structure of light (however imbued we may be with it) over a quantum structure. It is fitting that we should conclude our volume on Optics by emphasizing once more that most remarkable and epistemologically most important result, the complementarity of wave and corpuscle.

¹Rev. Mod. Physics Vol. 21, p. 345, 1949, on Einstein's seventieth birthday.

PROBLEMS

CHAPTER I.

I.1. Superposition of two parallel linear oscillations of equal frequency.

Let the two oscillations (in real notation) be

$$(1) \quad x_1 = a_1 \cos(\omega t + \alpha_1), \quad x_2 = a_2 \cos(\omega t + \alpha_2)$$

By forming the vectorial (complex) sum of these oscillations, find the amplitude a and the phase α of the resulting oscillation

$$(2) \quad x = x_1 + x_2 = a \cos(\omega t + \alpha).$$

I.2 The curve described by the electric and magnetic vectors of a plane wave during one period.

In the ideal case (perfectly plane and perfectly monochromatic wave) this curve is an ellipse. Under what conditions does this ellipse degenerate into a circle or a straight line?

I.3. Concerning the surface charge on the boundary between I and II.

Show that the boundary surface must be free of charges in Sec. 3 B as well as in Sec. 3 A.

I.4 A check on figure 4.

Find the equations of the parabolas R_p and R_s as functions of α .

I.5 On the calculation of the reflective power r and the transmissivity d .

Confirm the energy theorem $r + d = 1$ for arbitrary material constants ϵ, μ .

I.6. Elliptic polarization of light through total reflection.

Starting with eq. (5.11), prove eq. (5.12) for the maximum phase difference $\gamma - \delta$ and the associated angle of incidence α_{max} .

I.7. The Perot-Fabry maxima considered as a resonance effect.

Following a suggestion by Kossel¹, investigate the electromagnetic *eigen-vibrations* of the Perot-Fabry air space between the silvered plates of the etalon. Consider only the case where the field depends only on the y -coordinate (perpen-

¹W. Kossel, Ann. d. Phys. (Lpz.) 36, 1939; see remark at the bottom of p. 191 and top of p. 192 of that paper.

dicular to the plates) and where the silvering is so heavy that, for instance, the electric vector \mathbf{E} oscillates everywhere parallel to the plates. Determine the frequency of this *free* oscillation and show that it agrees with the frequency of the maxima of the *forced* oscillation which is excited by a perpendicularly incident *p*-polarized wave.

I.8 Wiener's experiment with obliquely incident light.

Investigate the appearance of interference fringes for arbitrary angles of incidence α and for both cases of polarization.

CHAPTER III.

III.1. The reduced mass in the problem of intra-molecular oscillations.

For a molecule consisting of a positive ion of mass M_1 and a negative ion of mass M_2 , prove the expression

$$\frac{1}{M} = \frac{1}{M_1} + \frac{1}{M_2},$$

which was used in (18.3). The ions are to be thought of as idealized mass points which attract one another with a central force. The same expression appeared in connection with the inelastic collision of two mass points in Vol. I, eq. (3.28 b).

III.2. The deflection angle δ of a prism.

Prove that the deflection angle is a minimum for symmetrical ray paths.

III.3 Direct vision and achromatic prisms.

For small prism angles and small angles of incidence calculate the deflection δ due to a double prism composed of two different glasses (refractive indices n_1, n_2 ; prism angles φ_1, φ_2); prism 1 is upright, prism 2 is upside down so that its edge adjoins the base surface of 1. Determine the ratio φ_2/φ_1 for $\delta = 0$ (direct vision prism) and for $d\delta/d\lambda = 0$ (achromatic prism).

III.4. Zeeman effect and Larmor precession.

Treat the motion of an electron in an arbitrary atomic field describable by a potential $V(r)$ a) when an additional homogeneous magnetic field B is present and b) when, instead, the motion is referred to a coordinate system which rotates with the angular velocity ω about the direction of B as an axis. Show that the motions a) and b) are the same when

$$\omega = \frac{1}{2} \frac{e}{m} B \quad (\text{Formula of the Larmor precession}).$$

Assume that the ordinary centrifugal force is negligible compared to the Coriolis force.

CHAPTER IV

IV.1. Geometrical derivation of the normal surface.

In Vol. II, solution to exercise I.6, the following two theorems were derived from the invariants of a tensor surface represented by an ellipsoid with principal axes a , b , c :

a) The sum of the inverse squares of any three mutually perpendicular semi-diameters is independent of the spatial orientation; hence, in particular, it is equal to

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

b) The volume of any circumscribed parallelepiped of the ellipsoid is independent of its particular position or shape; hence, in particular, it is equal to

$$2a \cdot 2b \cdot 2c.$$

Apply these theorems to the index ellipsoid and to the construction described in Sec. 25, eqs. (12) to (19) and derive in this way the equation of the normal surface.

IV.2 Elementary geometrical derivation of the ray surface.

Apply the theorems a), b) of the previous exercise to Fresnel's ellipsoid and supplement the construction of Sec. 25 by a corresponding construction for the field vector \mathbf{E} . In this way obtain the equation of the ray surface.

IV.3. Proof of the approximate formula (31.9) for the phase difference due to the illumination of a crystal plate with converging light.

A sufficiently good approximation is obtained if in the exact formula for the phase difference between the two rays ABD and AC in fig. 47 one replaces β_1 and β_2 by a suitable mean angle β and if one treats both β_1 and β_2 as small quantities.

SOLUTIONS TO PROBLEMS

I.1. The usual method would be to set

$$x_1 = a_1 \cos \tau, \quad x_2 = a_2 \cos (\tau + \delta), \quad \tau = \omega t + \alpha_1, \quad \delta = \alpha_2 - \alpha_1,$$

$$x_1 + x_2 = (a_1 + a_2 \cos \delta) \cos \tau - a_2 \sin \delta \sin \tau.$$

Comparison with eq. (2) in the exercise yields then

$$a_1 + a_2 \cos \delta = a \cos (\alpha - \alpha_1), \quad a_2 \sin \delta = a \sin (\alpha - \alpha_1)$$

hence

$$(1) \quad a^2 = (a_1 + a_2 \cos \delta)^2 + a_2^2 \sin^2 \delta = a_1^2 + a_2^2 + 2 a_1 a_2 \cos \delta,$$

$$(2) \quad \tan (\alpha - \alpha_1) = \frac{a_2 \sin \delta}{a_1 + a_2 \cos \delta}.$$

It is much simpler, however, to proceed in the following manner: omitting the common factor $e^{i\omega t}$ one writes:

$$x_1 = a_1 e^{i\alpha_1}, \quad x_2 = a_2 e^{i\alpha_2}, \quad x = a e^{i\alpha},$$

where x_1, x_2 represent the vectors OP_1, OP_2 of lengths a_1, a_2 in the complex plane. Their sum is represented in fig. 97 by the diagonal OQ of the parallelogram formed from x_1 and x_2 . By the theorem of Pythagoras the length a of the diagonal is given by

$$(3) \quad a^2 = a_1^2 + a_2^2 + 2 a_1 a_2 \cos \delta$$

as in eq. (1). The angle $\alpha - \alpha_1$ between OQ and OP is calculated from the right triangle OQR to be

$$(4) \quad \tan (\alpha - \alpha_1) = \frac{a_2 \sin \delta}{a_1 + a_2 \cos \delta},$$

as in eq. (2).

For $a_2 = a_1$ and $\delta = \pi - \Delta$ it follows from (3) that

$$(5) \quad a^2 = 2 a_1^2 (1 - \cos \Delta), \quad a = 2 a_1 \sin \frac{\Delta}{2}.$$

We shall meet this formula again in an interference problem in Sec. 31.

I.2. Substituting in (2.1)

$$E_y = \eta, \quad E_z = \zeta, \quad A_y = A e^{-i\alpha}, \quad A_z = B e^{-i\beta}$$

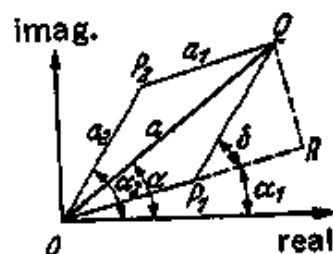


Fig. 97.

Superposition of two parallel oscillations of different phases in the complex plane.

then in real notation;

$$\eta = A \cos(\tau - \alpha), \quad \zeta = B \cos(\tau - \beta), \quad \tau = kx - \omega t.$$

By eliminating $\sin \tau$ or $\cos \tau$ one finds

$$\cos \tau \sin(\beta - \alpha) = \frac{\eta}{A} \sin \beta - \frac{\zeta}{B} \sin \alpha,$$

$$\sin \tau \sin(\beta - \alpha) = -\frac{\eta}{A} \cos \beta + \frac{\zeta}{B} \cos \alpha$$

and by squaring and adding one obtains

$$(1) \quad \left(\frac{\eta}{A}\right)^2 + \left(\frac{\zeta}{B}\right)^2 - 2\frac{\eta}{A}\frac{\zeta}{B}\cos\gamma = \sin^2\gamma, \quad \gamma = \beta - \alpha.$$

This is the polar equation of an ellipse. The two principal axes are in general rotated with respect to the y - and z -axes; they coincide with the latter only when $\gamma = \pm \pi/2$.

If in addition $A = B$, then the ellipse becomes a circle, which corresponds to criterion (2.6) for circular polarization.

The polarization is linear when $\gamma = 0$ or π . For then (1) becomes

$$\left(\frac{\eta}{A} \mp \frac{\zeta}{B}\right)^2 = 0$$

which corresponds to criterion (2.6 a).

In view of (2.5) the same calculation yields for the magnetic vector $H_y = \eta$, $H_z = \zeta$

$$\eta = -B \sqrt{\frac{\epsilon}{\mu}} \cos(\tau - \beta), \quad \zeta = A \sqrt{\frac{\epsilon}{\mu}} \cos(\tau - \alpha)$$

and hence in place of (1)

$$(2) \quad \left(\frac{\eta}{B}\right)^2 + \left(\frac{\zeta}{A}\right)^2 + 2\frac{\eta}{B}\frac{\zeta}{A}\cos\gamma = \frac{\epsilon}{\mu} \sin^2\gamma.$$

Thus the curve described by the magnetic vector becomes a circle or a straight line under the same conditions which hold for the electric vector.

1.3. The general proof of the non-existence of a surface charge rests upon the following considerations: from Maxwell's equations for non-conducting media it follows that $\text{div } \mathbf{D} = \rho$ is independent of t . But since the field is assumed to be purely periodic in time, $\rho = f(x, y, z)$ is excluded and only $\rho = 0$ remains possible. The same statement holds for the surface divergence $\omega = D_n - D_n'$.

This can be formally verified for the case of Sec. 3 B in the following way; at $y = 0$ (see fig. 3 b)

$$E_y = \begin{cases} (A + C) \sin \alpha e^{ik_1 x \sin \alpha} & \text{in medium I.} \\ B \sin \beta e^{ik_2 x \sin \beta} & \text{in medium II.} \end{cases}$$

From this, the law of refraction, and the relation $D = \varepsilon E$ follows

$$D_y = \begin{Bmatrix} \varepsilon_1 (A + C) \sin \alpha \\ \varepsilon_2 B \sin \beta \end{Bmatrix} e^{i k_1 x \sin \alpha}.$$

From eq. (3.14 a), the law of refraction and the definitions of n_{12} and n_{21} it is easily shown that these two values of D_y are equal.

I.4. From the law of refraction it follows that for small α in second approximation:

$$\beta \left(1 - \frac{\beta^2}{6} + \dots \right) = \frac{\alpha}{n} \left(1 - \frac{\alpha^2}{6} + \dots \right),$$

and hence, consistently neglecting higher powers of α ,

$$\begin{aligned} \beta &= \frac{\alpha}{n} \left(1 - \frac{\alpha^2}{6} \right) \left(1 + \frac{\alpha^2}{6 n^2} \right) = \frac{\alpha}{n} \left(1 - \frac{n^2 - 1}{6 n^2} \alpha^2 \right), \\ (1) \quad \alpha \pm \beta &= \alpha \left\{ 1 \pm \frac{1}{n} \left(1 - \frac{n^2 - 1}{6 n^2} \alpha^2 \right) \right\}. \end{aligned}$$

Thus one obtains to the same order of approximation

$$\begin{aligned} (2) \quad \frac{\sin(\alpha - \beta)}{\sin(\alpha + \beta)} &= \frac{n - 1 + \frac{\alpha^2}{6 n^2} (n^2 - 1 - (n - 1)^2)}{n + 1 - \frac{\alpha^2}{6 n^2} (n^2 - 1 + (n + 1)^2)} \\ &= \frac{n - 1}{n + 1} \left(1 + \frac{\alpha^2}{6 n^2} (2n + 4n) \right) = \frac{n - 1}{n + 1} \left(1 + \frac{\alpha^2}{n} \right), \end{aligned}$$

which is given by (4.4) as the expression for R_p .

From (1) one obtains in the same approximation

$$(3) \quad \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = 1 - \frac{2 \alpha^2}{n}$$

and the negative product of (2) and (3) is

$$R_s = -\frac{n - 1}{n + 1} \left(1 - \frac{\alpha^2}{n} \right).$$

as in (4.9).

I.5. In order to give a general proof (not only for the special case $\mu_2 = \mu_1$ considered in Sec. 4) we rely on eqs. (3.9) and (3.15). By (4.18) the equation to be proved becomes

$$\left| \frac{C}{A} \right|^2 + n \frac{\cos \beta}{\cos \alpha} \left| \frac{B}{A} \right|^2 = 1.$$

Dividing by $|B/A|^2$ it can be rewritten

$$\left| \frac{C}{B} \right|^2 + m \frac{\cos \beta}{\cos \alpha} = \left| \frac{A}{B} \right|^2.$$

According to (3.9) this equation becomes for *p-polarization*

$$\left(1 - m \frac{\cos \beta}{\cos \alpha} \right)^2 + 4 m \frac{\cos \beta}{\cos \alpha} = \left(1 + m \frac{\cos \beta}{\cos \alpha} \right)^2$$

and according to (3.15) it becomes for *s-polarization*

$$\left(m - \frac{\cos \beta}{\cos \alpha} \right)^2 + 4 m \frac{\cos \beta}{\cos \alpha} = \left(m + \frac{\cos \beta}{\cos \alpha} \right)^2.$$

Both equations are clearly identities.

I.6. By differentiating (5.11) with respect to α and setting the resulting differential quotient equal to zero one obtains [using the upper signs in the numerator and denominator of (5.11)]

$$(1) \quad 0 = \frac{\cos(\alpha - i\beta')}{\sin(\alpha + i\beta')} \left(1 - i \frac{d\beta'}{d\alpha} \right) - \frac{\sin(\alpha - i\beta')}{\sin^2(\alpha + i\beta')} \cos(\alpha + i\beta') \left(1 + i \frac{d\beta'}{d\alpha} \right).$$

By differentiating the law of refraction $n \sin \alpha = \cos i\beta'$ one finds

$$\frac{d\beta'}{d\alpha} = \frac{i n \cos \alpha}{\sin i\beta'}.$$

Substituting this in (1) one obtains

$$0 = \sin 2i\beta' + n \frac{\sin 2\alpha \cos \alpha}{\sin i\beta'}.$$

A second judicious application of the law of refraction yields

$$0 = 2n \frac{\sin \alpha}{\sin i\beta'} \{ 2 - (n^2 + 1) \sin^2 \alpha \},$$

which contains the second formula (5.12) which was to be derived. Rewriting now (5.11) in the real notation and letting $\Delta = \gamma - \delta$ one writes

$$\frac{e^{i\Delta} - 1}{e^{i\Delta} + 1} = - \frac{\cos \alpha \sin i\beta'}{\sin \alpha \cos i\beta'}$$

or the identical expression

$$i \tan \frac{\Delta}{2} = - \cot \alpha \tan i\beta'.$$

From the above-derived value of $\sin \alpha_{max}$ and the law of refraction the two factors on the right-hand side are found to be equal to

$$\cot \alpha = \sqrt{\frac{n^2 - 1}{2}}, \quad \tan i\beta' = \frac{i}{n} \sqrt{\frac{n^2 - 1}{2}}.$$

Thus also the first formula (5.12) has been proved.

I.7. If in the general trial solution of Sec. 7 A the coefficient A is set equal to zero, that is, if the continuous excitation supplied by the incident wave is omitted, then the solution of the problem of the plane parallel plate represents the free oscillations in the plate instead of the forced oscillations. The wave of amplitude C , which had previously been called the reflected wave, now represents, like the D -wave, the radiation emitted by the free oscillations into outer space. This radiation must clearly be present if the silver layer is not totally reflecting. Let the thickness of the air plate again be $2h$. The formulae of Sec. 7 are based upon the same p -polarization which is assumed in the exercise, but because of the given geometrical characteristics of the free oscillations (independence of x), one must now set $\alpha = \beta = \gamma = 0$. Furthermore, since all three media I, II, and III are now air, $n = n_1 = 1$.

Therefore the four equations (7.30) and (7.31) simplify to

$$\begin{aligned} (1) \quad & -C e^{ikh} - B e^{-ikh} + E e^{ikh} = g C e^{ikh} = g (B e^{-ikh} + E e^{ikh}), \\ (2) \quad & -D e^{ikh} - E e^{-ikh} + B e^{ikh} = g D e^{ikh} = g (E e^{-ikh} + B e^{ikh}). \end{aligned}$$

This way of writing the equations shows that the problem has become symmetrical in C and D as well as in B and E , which is due to the fact that the incident wave has been omitted. Therefore one can set (symmetrical type of solution)

$$(3) \quad D = C, \quad E = B$$

whereby (2) and (1) become identical so that only one double equation remains

$$(4) \quad -C e^{ikh} + 2i B \sin kh = g C e^{ikh} = 2g B \cos kh.$$

By eliminating either B or C one obtains

$$(5) \quad \tan kh = \frac{1 + g}{i}.$$

Alternately, one can set (antisymmetric type of solution)

$$(3a) \quad D = -C, \quad E = -B.$$

Equations (1) and (2) again become identical except for the sign. In place of (4) and (5) one obtains

$$(4a) \quad -C e^{ikh} - 2B \cos kh = g C e^{ikh} = -2ig B \sin kh,$$

$$(5a) \quad \tan kh = \frac{1}{i(1 + g)}.$$

Setting

$$(6) \quad \xi = k h, \quad \tan k h = \frac{e^{i\xi} - e^{-i\xi}}{i(e^{i\xi} + e^{-i\xi})}, \quad \beta = \frac{1}{1+g} \ll 1,$$

(5 a) yields

$$(7) \quad e^{2i\xi} = \frac{1+\beta}{1-\beta} \sim 1 + 2\beta, \quad \xi = \frac{1}{2i} (2\beta + 2m\pi i)$$

and (5) yields

$$(7a) \quad e^{2i\xi} = -\frac{1+\beta}{1-\beta} \sim -(1+2\beta), \quad \xi = \frac{1}{2i} (2\beta + (2m+1)\pi i).$$

From (7) and (7 a) follows, with ξ and β as defined in eq. (6)

$$(8) \quad k h = m\pi - i/g,$$

$$(8a) \quad k h = \left(m + \frac{1}{2}\right)\pi - i/g.$$

The symmetrical and antisymmetrical eigenvalues of kh form an equidistant sequence with the spacing $\pi/2$ between neighboring values. This result is in complete agreement with the *forced* etalon-oscillations as represented in fig. 11.

According to (8) and (8 a) the damping constant $1/g$ is the same for all *free* oscillations. From this one concludes that the half-width is also the same for all *forced* oscillations as indicated in fig. 11. To prove this it is only necessary to compare the results for the simplest mechanical type of damped free oscillations and damped forced oscillations as derived in Vol. I, Sec. 19.

I.8. According to the general expression (3.1), the incident and reflected waves for arbitrary angle of incidence α are represented by

$$(1) \quad \left. \begin{aligned} E_i &= A e^{ik(x \sin \alpha - y \cos \alpha)} \\ E_r &= C e^{ik(x \sin \alpha + y \cos \alpha)} \end{aligned} \right\} e^{-i\omega t}.$$

With *p-polarization* \mathbf{E} is parallel to the z -axis and, because of the boundary condition at $y = 0$, $C = -A$ as in (8.3). Therefore

$$\operatorname{Re}(E_i + E_r) = 2A \cos(\omega t - kx \sin \alpha) \sin(ky \cos \alpha).$$

The locus of points of maximum electric field strength (maximum photographic effectiveness) is the system of parallel planes

$$(2) \quad ky \cos \alpha = \left(m + \frac{1}{2}\right)\pi.$$

The spacing of these planes is larger than the spacing $\lambda/2$ for perpendicular incidence ($\alpha = 0$). In particular, for $\alpha = \pi/4$ it becomes equal to $\lambda/\sqrt{2}$.

With *s-polarization* \mathbf{H} is in the z direction, and \mathbf{E} has components in the x and y directions. From the boundary condition

$$E_{xi} + E_{xr} = 0 \quad \text{for} \quad y = 0$$

llows that $C = -A$ (see fig. 3 b) and according to (1) for $y > 0$:

$$\operatorname{Re} (E_i + E_r)_y = 2A \sin \alpha \sin (\omega t - kx \sin \alpha) \sin (ky \cos \alpha),$$

$$\operatorname{Re} (E_i + E_r)_x = 2A \cos \alpha \cos (\omega t - kx \sin \alpha) \cos (ky \cos \alpha).$$

time average of the sum of the squares of these components gives

$$\begin{aligned} J &= 2A^2 \{ \sin^2 \alpha \sin^2 (ky \cos \alpha) + \cos^2 \alpha \cos^2 (ky \cos \alpha) \} \\ &= 2A^2 \{ \cos^2 \alpha - \cos 2\alpha \sin^2 (ky \cos \alpha) \}. \end{aligned}$$

s, for the angle of incidence $\alpha = \pi/4$ which was used by Wiener one has A^2 ; no fringe system results and the illumination is uniform. For other angles of incidence weak fringes appear superimposed on uniform brightness.

1. If the central force divided by the distance between the two mass points denoted by f , then the equations of motion, written in terms of the cartesian coordinates x, y and x_1, y_1 are

$$\begin{aligned} M_1 \ddot{x}_1 &= f(x_2 - x_1), & M_1 \ddot{y}_1 &= f(y_2 - y_1), \\ M_2 \ddot{x}_2 &= f(x_1 - x_2), & M_2 \ddot{y}_2 &= f(y_1 - y_2). \end{aligned}$$

dition of the equations in each column yields the equations of motion of the center of mass; subtraction yields the equations of the relative motion of the two masses:

$$\ddot{\xi} = -\left(\frac{1}{M_1} + \frac{1}{M_2}\right)f\xi, \quad \ddot{\eta} = -\left(\frac{1}{M_1} + \frac{1}{M_2}\right)f\eta, \quad \begin{cases} \xi = x_1 - x_2, \\ \eta = y_1 - y_2. \end{cases}$$

the definition of M is that given in the exercise, these equations describe the motion of a mass point M with coordinates ξ, η . If the binding force is quasi-elastic as we had generally assumed it to be in our dispersion calculations, then $f = \text{const.}$ and from (2) follows

$$\ddot{\xi} + \omega_0^2 \xi = 0, \quad \ddot{\eta} + \omega_0^2 \eta = 0, \quad \omega_0^2 = f/M.$$

Thus the motion is simply periodic with the frequency ω_0 . The same is true in the case of Coulomb attraction (f proportional to r^{-2}) but not for arbitrary central forces.

I.2. Applied to the front and rear surfaces of the prism, the law of refraction requires that

$$\frac{\sin \alpha}{\sin \beta} = n, \quad \frac{\sin \alpha'}{\sin \beta'} = n'$$

here $n' = 1/n$ if both surfaces border on air. From the sum of the angles in the triangle formed by the prism faces and the interior ray follows

$$\varphi = \beta + \alpha'.$$

At the front face the incident ray is deflected by $\delta_1 = \alpha - \beta$; the emerging ray is deflected by $\delta_2 = \beta' - \alpha'$ at the rear surface. The total deflection is therefore

$$\delta = \delta_1 + \delta_2 = \alpha - \beta + \beta' - \alpha'$$

and because of (2)

$$(3) \quad \delta = \alpha + \beta' - \varphi.$$

Substitution in (1) yields

$$(4) \quad \frac{\sin \alpha}{\sin (\varphi - \alpha')} = n, \quad \frac{\sin \alpha'}{\sin (\delta + \varphi - \alpha)} = \frac{1}{n}.$$

Therefore, by eliminating α' , δ can be represented as a function of α .

Differentiation of (4) with respect to α (before eliminating α') gives the following conditions for the minimum deflection $d\delta = 0$:

$$\begin{aligned} \cos \alpha d\alpha + n \cos (\varphi - \alpha') d\alpha' &= 0, \\ n \cos \alpha' d\alpha' + \cos (\delta + \varphi - \alpha) d\alpha &= 0. \end{aligned}$$

These can be satisfied only if

$$(5) \quad \begin{vmatrix} \cos \alpha & \cos (\varphi - \alpha') \\ \cos (\delta + \varphi - \alpha) & \cos \alpha' \end{vmatrix} = 0.$$

Equating the terms in the first and second column gives $\alpha = (\delta + \varphi)/2$, $\alpha' = \varphi/2$ and applying (2) and (3) yields $\beta = \alpha'$, $\beta' = \alpha$. Hence, if the ray is symmetrical with respect to the bisector plane of the prism angle, one obtains by substitution in one of the eqs. (1) an equation which is much used in the determination of n :

$$(6) \quad n = \sin \frac{1}{2} (\delta + \varphi) / \sin \frac{1}{2} \varphi.$$

III.3. For small angles α , φ , α' it follows from the two eqs. (4) of the preceding solution by eliminating $\alpha + n\alpha'$ that

$$\delta = (n - 1) \varphi.$$

In order to be able to apply this result directly to the twin prisms, it is convenient to imagine prisms 1 and 2 as separated by a narrow air space. Thus, taking into account the opposite positions of the two prism edges, one obtains for the total deflection

$$(1) \quad \delta \approx \delta_1 - \delta_2, \quad \delta_1 = (n_1 - 1) \varphi_1, \quad \delta_2 = (n_2 - 1) \varphi_2.$$

a) For a direct vision prism it is to be required that

$$(2) \quad \delta = 0, \quad \text{i. e.} \quad (n_1 - 1) \varphi_1 - (n_2 - 1) \varphi_2 = 0, \quad \frac{\varphi_2}{\varphi_1} = \frac{n_1 - 1}{n_2 - 1}.$$

Since n_1 and n_2 depend on the wavelength, this condition can only be satisfied for some average wavelength such as $\lambda = 0.590 \mu$.

b) For an achromatic prism one requires that

$$(3) \quad \frac{d\delta}{d\lambda} = 0, \quad \frac{dn_1}{d\lambda} \varphi_1 - \frac{dn_2}{d\lambda} \varphi_2 = 0, \quad \frac{\varphi_2}{\varphi_1} = \frac{dn_1/d\lambda}{dn_2/d\lambda}.$$

Also this condition may in particular be satisfied for $\lambda = 0.590 \mu$. The table below lists the refractive indices n_1 of light boron crown glass and n_2 of heavy flint glass for various wavelengths. For $\lambda = 0.590 \mu$ eqs. (2) and (3) yield

$$(4 a) \quad \frac{\varphi_2}{\varphi_1} = \frac{0.5103}{0.7562}, \quad (4 b) \quad \frac{\varphi_2}{\varphi_1} = \frac{4.18}{13.84}.$$

In the case a) the value of δ is small everywhere in the spectrum but it does depend strongly on color. In the case b) the (also for $\lambda = 0.590 \mu$) non-vanishing deflection is quite independent of color (only at the violet end of the spectrum does the deflection decrease slightly). Once the angle φ_1 is arbitrarily chosen (though it must be small), the angle φ_2 is determined by (4 a, b).

Dispersion of crown glass (n_1) and of flint glass (n_2)

λ	n_1	n_2
0.761	1.5050	1.7390
0.656	1.5076	1.7473
0.590	1.5103	1.7562
0.486	1.5156	1.7792
0.397	1.5245	1.8403

A much more important problem is that of *achromatic lenses*. For these a condition similar to (3) must be satisfied.

III.4. In case a) the inertial force tending to deflect the electron from its orbit must be balanced by the force $-\partial V/\partial r$ of the atomic field and by the Lorentz force $\mathbf{K} = e \mathbf{v} \times \mathbf{B}$. We do not need to go into the shape of the orbit or the velocity variations along it. In case b) the ordinary centrifugal force

$$|\mathbf{Z}| = m \rho \omega^2 \quad (\rho \text{ distance from the axis of rotation})$$

and the Coriolis force

$$\mathbf{C} = 2 m \mathbf{v} \times \vec{\omega} \quad (\mathbf{v} \text{ velocity relative to the rotating system})$$

take the place of \mathbf{K} (see Vol. I, Sec. 29), while the force due to the atom $-\partial V/\partial r$ is the same as in a). It is to be assumed (see wording of problem) that \mathbf{Z} is negligible compared to \mathbf{C} . Then one obtains equilibrium in case b) by setting $\mathbf{C} = \mathbf{K}$. This yields

$$2 m \mathbf{v} \times \vec{\omega} = e \mathbf{v} \times \mathbf{B}; \quad \omega = \frac{1}{2} \frac{e}{m} B$$

which proves *Larmor's theorem*.

In order to be able to neglect \mathbf{Z} we must have

$$m \rho \omega^2 \ll 2 m |\mathbf{v}| \omega.$$

This is equivalent to saying that the velocity $\rho \omega$ imparted to the electron when the magnetic field is switched on is small compared to the velocity $|\mathbf{v}|$ which the electron would have without the magnetic field. For practically attainable fields \mathbf{B} this condition is always fulfilled.

We conclude therefore that the theory of the Zeeman effect developed in Sec. 21 remains valid when the quasi-elastic binding, there assumed, is replaced by a Coulomb field (hydrogen atom) or by an arbitrary atomic field $V(r)$. In particular the theorems on the normal Zeeman effect [$\Delta\omega = 0$ for longitudinal observation, $2\Delta\omega = \pm (e/m)B$ for transverse observation] are preserved because of the general validity of Larmor's theorem.

IV.1. The index ellipsoid [normalized in accordance with (25.12)]

$$(1) \quad u_1^2 x_1^2 + u_2^2 x_2^2 + u_3^2 x_3^2 = C, \quad C = 2W_e/\mu_0$$

and the plane E perpendicular to the wave number vector \mathbf{k}

$$(2) \quad k_1 x_1 + k_2 x_2 + k_3 x_3 = 0$$

intersect in an ellipse. As in (25.19) we denote the reciprocals of the principal axes of this ellipse by u'/\sqrt{C} , u''/\sqrt{C} without, however, presupposing the former definitions of u' , u'' , as wave velocities to hold. We construct a third axis perpendicular to these two and call its length from the origin to the ellipsoid $OP = l$. The coordinates of P are $x_i = l k_i/k$. Substituting these in (1) one obtains

$$(3) \quad \frac{C}{l^2} = \frac{1}{k^2} \sum u_i^2 k_i^2.$$

Then theorem a) gives the following relationship between the three axis lengths \sqrt{C}/u' , \sqrt{C}/u'' , l :

$$\frac{1}{C} (u'^2 + u''^2) + \frac{1}{l^2} = \frac{1}{C} (u_1^2 + u_2^2 + u_3^2),$$

and hence by (3)

$$(4) \quad u'^2 + u''^2 = \sum u_i^2 \left(1 - \frac{k_i^2}{k^2} \right).$$

In order to be able to apply theorem b) we must construct the plane E' which is tangent to the ellipsoid (1) and parallel to E . The equation of any arbitrary tangent plane with the point of tangency $\xi_1 \xi_2 \xi_3$ is

$$(5) \quad \sum u_i^2 \xi_i (x_i - \xi_i) = 0.$$

If this plane is to be perpendicular to \mathbf{K} , then

$$(5a) \quad u_i^2 \xi_i = \rho k_i, \quad (\rho = \text{constant of proportionality}).$$

Since, in particular, the point ξ must lie on the ellipsoid (1):

$$(6) \quad \rho^2 \sum \frac{k_i^2}{u_i^2} = C.$$

Because of (5a) and (1), eq. (5) becomes

$$(7) \quad \rho \sum k_i x_i - C = 0.$$

By the rules of analytic geometry the distance of this plane E' from the center of the ellipsoid is

$$(8) \quad p = \frac{C}{\rho k}, \quad \text{and hence by (6)} \quad p = \frac{1}{k} \sqrt{C \sum k_i^2 / u_i^2}.$$

E' and its diametrically opposite parallel plane E'' , together with the planes which are tangent to the ellipsoid at the end points of the principal axes of the intersectional ellipse, form a circumscribed parallelepiped of the ellipsoid. Its volume is

$$2p \cdot \frac{2\sqrt{C}}{u'} \cdot \frac{2\sqrt{C}}{u''} = \frac{8}{k} \frac{C^{3/2}}{u' u''} \sqrt{C \sum k_i^2 / u_i^2}.$$

According to theorem b) this volume equals the volume of the rectangular parallelepiped formed by the three principal axes of lengths \sqrt{C}/u_i of the ellipsoid; that is, it is equal to $8 C^{3/2} u_1 u_2 u_3$. It follows therefore that

$$(9) \quad u'^2 u''^2 = \frac{u_1^2 u_2^2 u_3^2}{k^2} \sum \frac{k_i^2}{u_i^2}.$$

The two symmetric functions $u'^2 + u''^2$, eq. (4) and $u'^2 u''^2$, eq. (9) yield a quadratic equation in u^2 , the roots of which are u'^2 and u''^2 . It is easy to show that this equation agrees with the eq. (26.19 b) of the *normal surface* when in the latter the expressions (26.19) for the ξ_i are substituted.

IV.2. The equation (24.6 a) of the Fresnel ellipsoid is written in a form analogous to eq. (1) of the preceding problem

$$(1) \quad \frac{x_1^2}{u_1^2} + \frac{x_2^2}{u_2^2} + \frac{x_3^2}{u_3^2} = C, \quad C = 2\mu_0 W_0,$$

where the x_i now denote the components of \mathbf{E} . Since \mathbf{E} is perpendicular to the ray vector \mathbf{S} , hence also to the parallel unit vector $\mathbf{s} = s_1, s_2, s_3$, one must now cut the ellipsoid with the plane

$$(2) \quad s_1 x_1 + s_2 x_2 + s_3 x_3 = 0$$

and calculate the principal axes of the intersectional ellipse so formed. Except for the changed form of the subsidiary conditions (1) and (2), the extremum problem to be solved here is the same as that in Sec. 25.

Let the principal axis lengths of the intersectional ellipse be $\sqrt{C}v'$, $\sqrt{C}v''$; let l be the length from the origin to the ellipsoid of the axis perpendicular to these principal axes. Since its end point has the coordinates $\xi_i = l s_i$, eq. (1) yields

$$(3) \quad \frac{C}{l^2} = \sum \frac{s_i^2}{u_i^2}.$$

Theorem a) gives the following relationship for the three axis lengths $\sqrt{C}v'$, $\sqrt{C}v''$, C :

$$\frac{1}{C} \left(\frac{1}{v'^2} + \frac{1}{v''^2} \right) + \frac{1}{l^2} = \frac{1}{C} \sum \frac{1}{u_i^2}$$

and hence by (3)

$$(4) \quad \frac{1}{v'^2} + \frac{1}{v''^2} = \sum \frac{1-s_i^2}{u_i^2}.$$

Theorem b) concerns the tangential planes E' , E'' of the Fresnel ellipsoid which are parallel to our present intersectional ellipse. The equation of one of these planes is

$$(5) \quad \sum \frac{\xi_i}{u_i^2} (x_i - \xi_i) = 0$$

or, see the preceding problem,

$$(6) \quad \rho \sum s_i x_i - C = 0, \quad \text{where} \quad (7) \quad \rho^2 \sum u_i^2 s_i^2 = C.$$

It follows that the distance of the plane from the origin is

$$(8) \quad \rho = \frac{C}{\rho} = \sqrt{C \sum s_i^2 u_i^2}$$

and the volume of the parallelepiped to be considered here is

$$2\rho \cdot 2\sqrt{C}v' \cdot 2\sqrt{C}v'' = 8C^{3/2}v'v''\sqrt{\sum s_i^2 u_i^2}.$$

From theorem b) one obtains therefore

$$(9) \quad \frac{1}{v'^2 v''^2} = \frac{\sum s_i^2 u_i^2}{u_1^2 u_2^2 u_3^2}.$$

From (4) and (9) it follows that v'^2 , v''^2 are the roots of a quadratic equation in v^2 which we may write

$$0 = \left(\frac{1}{v^2} - \frac{1}{v'^2} \right) \left(\frac{1}{v^2} - \frac{1}{v''^2} \right) = \frac{1}{v^4} - \frac{1}{v^2} \sum \frac{1-s_i^2}{u_i^2} + \frac{1}{u_1^2 u_2^2 u_3^2} \sum s_i^2 u_i^2.$$

It is easy to verify that this equation is identical with the equation (26.13 b) of the ray surface

IV.3. From fig. 47 one finds that

$$(1) \quad AC = d/\cos \beta_2, \quad AB = d/\cos \beta_1, \quad BD = BC \sin \alpha,$$

$$(1a) \quad BC = EC - EB = (\tan \beta_2 - \tan \beta_1) d$$

while from the law of refraction it follows that

$$(2) \quad \sin \alpha = \sin \beta_1 \frac{k_1}{k} = \sin \beta_2 \frac{k_2}{k}.$$

Hence

$$\sin \alpha \tan \beta_1 = \frac{\sin^2 \beta_1}{\cos \beta_1} \frac{k_1}{k}, \quad \sin \alpha \tan \beta_2 = \frac{\sin^2 \beta_2}{\cos \beta_2} \frac{k_2}{k},$$

where the wave number k refers to the surrounding air, and k_1 refers to the more strongly and k_2 to the less strongly refracted ray. Therefore by (1) and (1 a)

$$BD = \frac{d}{k} \left(\frac{\sin^2 \beta_2}{\cos \beta_2} k_2 - \frac{\sin^2 \beta_1}{\cos \beta_1} k_1 \right).$$

The total phase difference Δ is found to be

$$\begin{aligned} \Delta &= k_2 AC - k_1 AB - k BD \\ (3) \quad &= \left(\frac{k_2}{\cos \beta_2} - \frac{k_1}{\cos \beta_1} - \frac{k_2 \sin^2 \beta_2}{\cos \beta_2} + \frac{k_1 \sin^2 \beta_1}{\cos \beta_1} \right) d \\ &= (k_2 \cos \beta_2 - k_1 \cos \beta_1) d. \end{aligned}$$

This result is to be specialized to small angles of incidence α , hence also to small β_1, β_2 . A mean angle of refraction defined by

$$\sin \beta = \sqrt{\sin \beta_1 \sin \beta_2}$$

is to be introduced. By multiplying the two laws of refraction (2) we obtain

$$\sin^2 \alpha = \sin^2 \beta \frac{k_1 k_2}{k^2}$$

and

$$(4) \quad \frac{1}{\cos \beta} = \left(1 - \frac{k^2}{k_1 k_2} \sin^2 \alpha \right)^{-1/2} = 1 + \frac{1}{2} \frac{k^2}{k_1 k_2} \sin^2 \alpha + \dots$$

On the other hand, we have for $i = 1, 2$

$$\begin{aligned} \cos \beta_i &= 1 - \frac{1}{2} \sin^2 \beta_i + \dots = 1 - \frac{1}{2} \sin^2 \alpha \frac{k^2}{k_i^2} + \dots \\ k_2 \cos \beta_2 - k_1 \cos \beta_1 &= k_2 - k_1 - \frac{k^2}{2} \sin^2 \alpha \left(\frac{1}{k_2} - \frac{1}{k_1} \right) + \dots \\ &= (k_2 - k_1) \left\{ 1 + \frac{1}{2} \frac{k^2}{k_1 k_2} \sin^2 \alpha + \dots \right\}. \end{aligned}$$

Substituting this in (3) and applying (4) one obtains

$$(5) \quad \Delta = \frac{(k_2 - k_1) d}{\cos \beta},$$

which agrees with eq. (31.9).

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THERMODYNAMICS AND STATISTICAL MECHANICS

Lectures on Theoretical Physics, Vol. V

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Author's Preface

Thermodynamics is a perfect example of a science which is developed from axioms. In contrast to classical mechanics, thermodynamics has withstood the quantum revolution without having its foundations shaken. In the course of the century of its existence it sprouted only several vigorous offshoots: Nernst's Third Law, Section 12, the theory of dilute solutions, Section 15, the application of the Second Law to electricity and magnetism, Sections 18 and 19. We consider that the thermodynamics of irreversible processes, Section 21, constitutes a promising extension of the classical thermodynamics of equilibria; it is based on Onsager's reciprocal relations and attempts to describe real processes which occur at finite velocities. Even Kirchhoff, as related by Planck in his autobiographical paper (*Naturwissenschaften* 19), restricted the concept of entropy to reversible processes; the firm belief in the general validity of this concept, which Planck stated as early as his doctoral thesis, led him in 1900 to his law of radiation and to quantum theory.

In any case, we do not propose to adhere so rigorously to the axiomatic mode of presentation as to endeavor to develop the science from the smallest possible number of axioms. This was achieved by Carathéodory in his proof of the Second Law which we shall, it is true, duly outline, but to which we shall not accord any preference over the Carnot-Clausius proof. The latter contains so much that is instructive and thoughtful that we consider it indispensable in an introductory course of lectures. The fact that it makes use of concepts derived from engineering is, in our opinion, an advantage rather than a matter for reproach. After all, thermodynamics did originate from the needs of steam engine builders.

Epistemologically there is a certain rivalry between the consideration of *cycles* and the method of *thermodynamic potentials*. The former are preferred in engineering because of their greater appeal to intuitive imagination. However, we shall almost exclusively make use of the latter method. It is much shorter and less arbitrary because it need not rely on artificially invented cycles. Moreover, we shall accord equal prominence to the four Gibbsian potentials, Section 7, although in the applications, the Gibbs function (also known as free enthalpy, or, simply, as the thermodynamic potential) is by far the most important one.

The experimental material which we include is very incomplete. In the case of real gases we restrict ourselves to the consideration of the van der Waals equation; in spite of its very simple form and in spite of the fact that it makes use of only two empirical constants, it reproduces the main outline of the behavior of liquids and their vapors in a very satisfactory manner. In the field of ferromagnetic phenomena Weiss' theory plays a similar part and succeeds, with its single constant of the internal field, to render similar services. A critical re-examination of these two theories must be left to more specialized treatises.

In my original University lectures I used to devote more time to *statistical mechanics* as compared with *classical thermodynamics* because I was personally drawn more to the former on account of its relation to quantum theory. In the present formulation quantum theory must, in principle, be left out and can only occasionally be drawn in as a supplement to *Boltzmann's statistics*. For this reason, the chapters dealing with thermodynamics, Chapters I and II, occupy an increased proportion of space, as compared with Chapters III, IV, and V. *Fermi's statistics* comes into the fore only on the occasion of a short account on metal electrons.

Chapter III contains a preliminary introduction to statistical mechanics, as far, that is, as is possible with elementary methods. The examples quoted in this connection (van der Waals constants, Langevin's theory of paramagnetic phenomena) serve to fill in some of the gaps left open in the sections on thermodynamics. Brownian motion, which is the most important example of statistical fluctuations, is treated together with the theory of the torsional balance. The problems arising in connection with the mean free path are only mentioned and not presented fully because they belong to the most difficult problems in statistical mechanics.

Chapter IV constitutes the summit of our consideration of statistical mechanics. I am of the opinion that Boltzmann's combinatorial method, when it is restricted to stationary processes, surpasses in fruitfulness and boldness its rival, the dynamic method based on Boltzmann's collision equation. In fact, in the first sections of this chapter we shall describe the combinatorial method in the original form given to it by Boltzmann, in which the molecules of a gas are endowed with a physically real existence. We shall free it from the resulting blemishes in Sections 32 to 35 when we shall introduce discrete energy levels of quantum-mechanical origin. However, in this way we are not yet led to quantum statistics proper. Since, in the realm of quantum mechanics, molecules are indistinguishable from each other, the original method due to Boltzmann (distribution of particles over the states) becomes illusory. Moreover, from the point of view of quantum mechanics the states

are given first; the various numerical combinations which govern the way in which the mutually indistinguishable particles are distributed over the states constitute the substance of the new statistics. We reach these points in Sections 36 and 37. Suitable examples are given in Section 38, light quantum gas, and in Section 39, metal electrons.

It is perhaps necessary to apologize for the fact that we have not placed this proper quantum-statistical treatment of states at the very beginning, starting instead with the undoubtedly obsolete method of Boltzmann's statistics of particles. The reason for it is purely didactic. The original method due to Boltzmann achieves so much and is so lucid that it still seems to provide the indispensable foundation for the understanding of the new statistics of states.

Chapter V has been kept very short in comparison with Chapter IV. The assumption of molecular models required here is of a much more specialized nature, the resulting calculations are so much more tedious, than those in the combinatorial method. It is true that in the hands of Hilbert they have led to a consistent theory of such irreversible processes as friction, the conduction of heat, etc. which Maxwell and Boltzmann repeatedly tried to achieve without success. In addition to this, the method due to Chapman and Enskog has been developed numerically to a point where comparison with observation becomes possible. However, such applications by far exceed the scope of a general course of lectures; they illustrate the great difficulties attendant on the exact mathematical development of the problems of the mean free path, which were only cursorily mentioned in Chapter III. Our presentation must necessarily restrict itself to an elucidation of the central problem which Boltzmann posed in his work with statistical methods: to clarify the contradiction between reversible mechanics and the Second Law of thermodynamics.

Arnold Sommerfeld

Editors' Preface

Fate prevented Sommerfeld from completing his treatise on theoretical physics. He died following an accident while he was working on Volume V, the volume on thermodynamics and statistical mechanics. The Editors were entrusted with the task of completing and publishing this volume on the express wish of the Author.

The sections on thermodynamics had then been virtually completed. Unfortunately, the Author could not read Section 21 which had been outlined by one of the Editors. Section 8 existed in two formulations and was completely recast.

The sections on kinetic theory and statistical mechanics existed up to and including Section 35; Section 37 had also been nearly completed. It was, however, clear from the many discussions with the Author that he was not completely satisfied with this portion of the book. We have tried to take this into account by including Section 36 on Gibbs' method, but we realize that the Author might have adopted a different course. The subdivisions and the contents of Sections 38 to 40 had been discussed with the Author, but they could not be written down in time.

Except for the remarks in the Author's Preface there were no clues as to Chapter V. The Author had not yet made up his mind about the contents of this chapter and mentioned casually that it could be included by the editor of future editions. The account of the electron theory of metals is based on the well-known article written by Sommerfeld and Bethe for the *Handbuch der Physik*.

Some of the problems have been taken from the Author's collection. Additional problems have been included following his wishes, which he expressed at one time. Some of them had been brought to his attention, but he could not express his views about them.

Professor G. V. Schubert helped the Author both actively and with advice while Chapters I to III were being written. Professor E. Kappler critically examined the section on Brownian motion. The Author discussed with Professor F. Sauter the contents of Section 26 by correspondence. We are indebted to Messrs. Herbert and Baldus for most of the figures in Chapters I

to III. Dr. Mann assisted with the work of proof reading, as on many previous occasions, and made valuable suggestions and useful criticisms. It is possible that, unknown to the Editors, additional acknowledgements should be made. We wish to thank the Publishers for their willing cooperation.

November 1952.

F. Bopp J. Meixner

Translator's Preface

The present book constitutes as nearly a literal translation of Sommerfeld's Fifth Volume as I could make, without impairing its fluency. Changes, if any, were slight and unimportant.

I would not have been in a position to undertake the translation of this volume, particularly of the chapters on statistical mechanics, were it not for the generous help and assistance which I received from Dr. G. F. Newell of Brown University. He has carefully read and criticized the manuscript and suggested many changes and improvements. I am also indebted to Professor F. Bopp, one of the editors of the German edition, who kindly consented to read the galley proof and to clarify many difficulties. Mr. J. R. Moszynski of Brown University read the page proof and prepared the Index. The Publishers spared no effort to meet my wishes.

The responsibility for any errors, mistakes, and omissions which still remain is, of course, my own.

Providence, R. I. October, 1955.

J. K.

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Lectures on Theoretical Physics

- VOLUME I:** Mechanics. 1952. Translated by Martin O. Stern.
- VOLUME II:** Mechanics of Deformable Bodies. 1950. Translated by G. Kuerti.
- VOLUME III:** Electrodynamics. 1952. Translated by Edward G. Ramberg.
- VOLUME IV:** Optics. 1954. Translated by Otto Laporte and Peter A. Moldauer.
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CHAPTER I

THERMODYNAMICS. GENERAL CONSIDERATIONS

1. Temperature as a property of a system

The science of thermodynamics introduces a new concept, that of *temperature*; it is absent from classical mechanics, as well as from the theory of electricity and magnetism and from atomic physics (with the exception of Joule heat, intensity of spectral lines conceived as interactions between a large number of material particles). Our sense of heat furnishes a qualitative measure, and a quantitative measure, albeit fortuitous to a certain extent, is given by any thermometer. A body which is in thermal equilibrium has the same temperature everywhere. The same is true of two bodies which have remained in thermal contact for a sufficiently long time. *Equality of temperature is a necessary condition of thermodynamic equilibrium.*

Temperature is a *property* or *parameter of state*. It is independent of the previous history of the body and is defined solely by its instantaneous state. It is associated with the behavior of the body at the instant under consideration, or else, it is measured with reference to the instantaneous indication of a thermometer.

The science of thermodynamics, as already stated in the preface, is an *axiomatic science*. In accordance with its spirit we introduce the concept of temperature by stating the following axiom:

There exists a property — temperature. *Equality of temperature is a condition for thermal equilibrium between two systems or between two parts of a single system.*

The preceding statement was purposely formulated in the same way as those which will be used later to state the First and Second Laws of thermodynamics and, following a suggestion by R. H. Fowler,¹ we shall refer to it as to the "Zeroth Law" of thermodynamics.

¹When giving an account of the book on thermodynamics of the great Indian astrophysicist M. N. Saha and his collaborator's, B. N. Srivartava, Allahabad 1931 and 1935.

In order to give a rigorous mathematical definition of the concept of a thermodynamic "property" or "parameter of state" it is necessary to consider its differential. With two independent variables x, y , which must themselves be measurable properties or characteristics of the system (e. g. pressure and volume), we can write it as

$$(1) \quad dT = X dx + Y dy; \quad X = \frac{\partial T}{\partial x}, \quad Y = \frac{\partial T}{\partial y}.$$

Evidently we then have

$$(2) \quad \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x},$$

which is the necessary and sufficient condition for the expression $X dx + Y dy$ to be a *perfect differential*. It is equivalent to the statement that T is a property.

The same condition can be also written in integral form

$$(3) \quad \oint dT = 0$$

for any closed path in the x, y -plane. Denoting the two-dimensional vector which is defined by its components X and Y by the symbol \mathbf{Z} we can apply Stokes' theorem for a two-dimensional field to the expression in eq. (3), obtaining

$$(4) \quad \oint \mathbf{Z} ds = \int \text{curl } \mathbf{Z} dx dy.$$

Since $\text{curl } \mathbf{Z}$ vanishes by eq. (2) it is concluded that statement (3) is, in fact, equivalent to the assertion that T is a property.

The condition for a perfect differential with n independent variables is the vanishing of the n -dimensional curl and can be represented by $n(n-1)/2$ equations of the form (2). Statement (1) generalized in this manner is known as "Pfaff's differential." When there are two independent variables it is always possible to transform the expression $X dx + Y dy$ into a perfect differential, by dividing it by a denominator $N(x, y)$, even if it was not one originally.

With three independent variables x, y, z this is not, generally speaking, possible. The requirement of integrability imposes certain conditions on the components X, Y, Z of a three-dimensional vector \mathbf{Z} which have been investigated in Problem I.7, Vol. II. It was found then that the vector must be normal to its curl:

$$(4 a) \quad \mathbf{Z} \text{ curl } \mathbf{Z} = 0.$$

It was further shown on the example of a field of forces and its potential that this requirement did not uniquely determine the "integrating denominator" ("multiplier" as it was then called) and that any function of one was also such a denominator (multiplier).

These preliminary remarks will help to understand the considerations connected with the Second Law in Sec. 6.

We shall regard the new concept of temperature as a fourth dimension in addition to the mechanical quantities of length, mass, and time in the same way as in the science of electrodynamics when we considered the then new concept of *quantity of electricity* or *charge* as a new fourth dimension. Naturally, in problems of electrochemistry we shall have to deal with five fundamental dimensions, i. e. we shall include the charge. We shall denote the dimension of temperature by the abbreviation "deg" rather than by a new symbol.

In Vol. I we have introduced the concept of a "mechanical system" and understood it to mean a collection of material points or bodies which could be described by specifying geometrically definable links or forces. We shall speak of a "thermodynamic system" when, in order to describe its state, it is necessary to specify in addition the temperatures of its components as well as the details of the quantities of heat transferred between them.

A homogeneous fluid affords the simplest example of a thermodynamic system and we might remark here that this definition will include the special cases of gases and vapors. A fluid possesses only one mechanical degree of freedom, its volume, and only one thermal degree of freedom, its temperature. The volume, V , (extensive property) is associated with its canonical conjugate¹ the pressure, p , (an intensive quantity, also known as tension if its sign is reversed). The temperature T is to be regarded as a thermal intensive quantity; the extensive quantity which constitutes its conjugate will be discussed in Sec. 5 D. Generally speaking p is a function of T and V . The relation $p = f(T, V)$ is known as the equation of state, or the characteristic equation.

The three quantities V , p , and T which have just been introduced can be combined in the *coefficient of thermal expansion*, α , and the *coefficient of*

¹The term originates from Hamiltonian mechanics, Vol. I, Sec. 41. The coordinate q (extensive quantity) and momentum (intensive quantity) were there described as canonically conjugate quantities. The term was, further, extended to include the more general pair of quantities Q , P . This note will suffice to explain the corresponding term in the present text. For more detail see Secs. 7 and 14 of the present volume.

tension, β , the two expressions being referred to the instantaneous values of V or p respectively:

$$(5) \quad \alpha = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_p; \quad \beta = \frac{1}{p} \left(\frac{\partial p}{\partial T} \right)_V.$$

The suffixes denote that in the process of differentiation with respect to T , p is kept constant in the one case, and V is kept constant in the other. Both coefficients have the dimension 1/deg, and their values for gases will be discussed presently. A further derived quantity, the (isothermal) *compressibility coefficient*, κ , is given by the definition

$$(6) \quad \kappa = -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_T.$$

The coefficients α , β , and κ satisfy a remarkable relation (see Problem I.1).

Processes during which T , p , or V remain constant are usually called an *isothermal*, an *isobaric* and an *isochoric* or *isopiestic* process, respectively.

2. Work and heat

Let a fluid occupy a cylindrical vessel of cross-sectional area A and let the vessel be closed by a piston touching the liquid. The piston is acted upon by the fluid with a force $p A$. If the piston is moved by dh , the fluid will perform the work

$$(1) \quad dW = p A dh = p dV.$$

This expression is valid not only for a positive dV , lifting of the piston, but also when it is lowered, i. e. when dV is negative, not only for a cylindrical vessel, but also for any boundary and for any change of shape of the surface of the fluid, when it is only necessary to perform an algebraic summation of all volume changes and to extend it over the boundary.

Equation (1) defines dW . Does it imply that a *property* W exists? Certainly not, as in such a case dW would have to be a "*perfect differential*," and according to (1.3) we should obtain

$$(1 a) \quad \oint dW = 0$$

when the fluid is subjected to a cycle, i. e. when it is made to reach the initial state after having traversed an arbitrary path. Such cycles can be represented graphically in a plane for the system under consideration, i. e. for one with

two degrees of freedom. The system of coordinates will correspond to any of the two properties chosen as independent variables, such as e. g. the pair V, T (one mechanical and one thermal variable), or the pair V, p (two mechanical variables). The latter pair of variables is used in the well-known *indicator diagram* which was introduced by James Watt as early as 150 years ago and which is automatically traced by every reciprocating steam engine, Fig. 1.

The steam cylinder is put into communication with the boiler along the upper horizontal straight line $p = p_1$, whereas along the lower line $p = p_2$ it communicates with the atmosphere or with the low-pressure boiler (condenser). The descending and the ascending branches of the curve correspond to expansion and compression respectively¹. The abscissa is proportional to

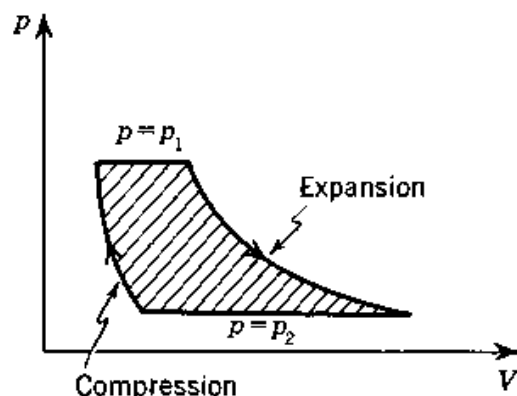


Fig. 1.

Indicator diagram of a steam engine.

the instantaneous distance h between the piston and the dead center and, hence, to the volume of the cylinder which is filled with steam at the moment. The area enclosed by the diagram gives a measure of the quantity

$$(2) \quad \oint p dV = \oint dW$$

and is, evidently, different from zero. In accordance with (1 a) we must say that a property W which would correspond to dW does not exist.

The steam engine performs the work (2) at the expense of the heat introduced. The reverse transformation of work into heat occurs during every process involving friction. The most impressive and the historically most important experiment illustrating this was performed in Munich by Count Rumford (1798): he caused water to boil when a cannon barrel was being bored.

¹Here it is necessary to overlook the fact that the quantity of steam contained in the cylinder is changed as the valves are opened and that it remains constant only during compression and expansion. In all processes which we shall discuss later the mass of the system will remain constant. The mass can be rendered constant in the steam engine example, actually or in the imagination, by condensing the steam as it leaves the cylinder and by returning it to the boiler. In any case the indicator diagram is a classical example of the representation of a cycle in the p, V -plane.

The instantaneous quantity of heat introduced will be denoted by dQ . As far as its measurement is concerned, it is, as is well known, reduced to that of temperature by adopting the definition: The quantity of heat which raises the temperature of one kilogram of water under atmospheric pressure from 14.5 to 15.5 C is called one (large) *calorie* (denoted kcal or, sometimes, Cal). We now recall the definition of specific heat, which we shall also base on the mass of 1 kg. Denoting the quantity of heat added by dq we put:

$$(3\ a) \quad dq = c_v dT \quad c_v = \text{specific heat at constant volume,}$$

$$(3\ b) \quad dq = c_p dT \quad c_p = \text{specific heat at constant pressure.}$$

The distinction between c_p and c_v is essential in the case of gases. It may be neglected in most cases as far as liquids are concerned. Substituting $dq = 1 \text{ kcal/kg}$ and $dT = 1 \text{ deg}$ into eq. (3.b), we find that for water at 15 C

$$(4) \quad c_p = 1 \frac{\text{kcal}}{\text{deg} \cdot \text{kg}}.$$

This statement is, evidently, equivalent to our previous definition of one calorie.

It is found that during all processes involving friction the quantity of work used, dW , bears a definite ratio to the quantity of heat generated, dQ , irrespective of the conditions of the experiment. Joule gave a quantitative proof of this statement by numerous, if at first imprecise, experiments. In particular he performed measurements on the heat generated by an electric current (Joule heat). Somewhat earlier Robert Mayer satisfied himself that water becomes heated on being shaken¹. We write

$$(5) \quad dW = J dQ$$

where J is called the *mechanical equivalent of heat*. Its numerical value is

$$(6) \quad J = 427 \text{ kg m/kcal}$$

if dQ is measured in kcal and dW is expressed in the engineering units of work - kgm. In this context the word "kilogram" denotes, as is known, the kilogram weight, for which it is preferable to use now the designation

¹In a letter dated September 1841 Mayer states that he performed this experiment on many occasions always obtaining a positive result. See Ostwald "Große Männer", Leipzig, 1905, p. 71. In this connection attention may be drawn to a hypothesis expressed by Albrecht von Haller (1708-1777) in accordance with which animal heat was to have been generated by the friction experienced by the blood in the veins. This hypothesis persisted well into the 19-th century.

"kilopond" = kp ¹, reserving the abbreviation kg = kilogram mass for the unit of mass in Giorgi's MKS-system of units. We would thus have

$$1 \text{ kp} = g \times 1 \text{ kg} = 9.81 \text{ MKS}^{-2} = 9.81 \text{ Dyne.}$$

Dyne denotes in this system the unit of force = 10^5 dyne. Following R. W. Pohl (Mechanik p.24) we shall call it the "large dyne". Hence

$$(7) \quad J = 4.19 \times 10^3 \frac{M^2 K S^{-2}}{\text{k cal}} = 4.19 \frac{\text{Erg}}{\text{cal}}.$$

Giorgi's unit of energy Erg = large erg, is equal to 10^7 erg = 1 Joule = = 1 watt sec. The abbreviation cal = small calorie refers to 1 gram of water, in the same way as 1 kcal referred to 1 kilogram of water.

We shall presently revert to the experimental justification of (7). Furthermore, we shall later be in a position to dispense with the use of the special units of heat, kcal or cal, by putting $1 \text{ kcal} = 427 \text{ kgm}$ or $1 \text{ cal} = 4.19 \text{ Erg}$, in accordance with (6) or (7), respectively, implying that $J = 1$.

At this point it is essential to deduce from eq. (5) that: The quantity dQ is not a perfect differential in the same way as dW was not one. There is no property Q , there is no *characteristic heat content which would simply describe the instantaneous state of the system*. It is necessary to point out clearly that our previous definition of a calorie constitutes only a rule for the measurement of the *quantity of heat* dQ (or Q when it is a finite quantity) introduced into the system in some way, but *not* for the quantity of heat *contained within the system*. Equations (3 a) and (3 b) show clearly that the manner of introducing heat is important in this connection.

In many text-books the use of the symbols dQ and dW is avoided and, for example, the symbols δQ , δW or dQ , dW are used instead, in order to warn the reader against erroneously regarding them as perfect differentials. We do not think this necessary because we take the view that the existence of a property and of its subordinated perfect differential constitutes a fundamental peculiarity which we shall always stress explicitly, as we have done in Sec. 1.

¹This suggestion emanated from Germany and has not, so far, gained universal recognition (*Transl.*)

3. The perfect gas

A gas is, so to say, the more perfect, the more difficult it is to liquefy it at a normal pressure of 760 mm Hg = 760 torr, i. e. the lower its boiling point. The degree of perfection is illustrated by the following boiling points in deg C at 760 torr:

He	H ₂	N ₂	O ₂	CO ₂	H ₂ O
- 269	- 259	- 210	- 218	- 78.5	+ 100

Steam, shown at the end of the list, does not, evidently belong to the class of perfect gases. The perfect gas is a *limiting state* to which a real gas will tend as it is expanded indefinitely. The following laws apply to this ideal, limiting state.

A. BOYLE'S LAW (THE LAW OF BOYLE AND MARIOTTE)

$$(1) \quad pV = \text{const},$$

which is valid provided that the temperature is kept constant. The pressure p is usually measured in atmospheres. One atmosphere denotes either the pressure in the surrounding air when the barometer reads 760 mm = 760 torr (physical atmosphere = 1 atm), or, more recently, one engineering atmosphere has been defined as the pressure exerted by 1 kilopond on 1 cm². It is almost exactly equal to the weight of a column of water 10 meters high acting on an area of 1 cm². We can express this engineering atmosphere as follows:

$$1 \text{ at} = 981 \frac{\text{cm}}{\text{sec}^2} \cdot 1000 \text{ g/cm}^2.$$

The first factor denotes the gravitational acceleration, g , the second denotes the mass of the column of water under consideration, the division by cm² denoting that weight has been replaced by pressure. Thus we have

$$(2) \quad 1 \text{ at} = 0.981 \times 10^6 \frac{\text{dyne}}{\text{cm}^2} = 0.981 \text{ bar}.$$

Correspondingly

$$(2 \text{ a}) \quad 1 \text{ atm} = 1.013 \text{ bar}.$$

The unit of 1 bar introduced here denotes

$$1 \text{ bar} = 10^6 \frac{\text{dyne}}{\text{cm}^2}.$$

One thousandth part of this unit constitutes one "millibar," a unit of pressure now often used by meteorologists. In our MKS system, we have

$$(2\text{ b}) \quad 1 \text{ bar} = 10 \frac{\text{Dyne}}{\text{cm}^2} = 10^5 \frac{\text{Dyne}}{\text{M}^2}.$$

If the density $\rho = \text{mass}/V$ is introduced into eq. (1) instead of V , we have

$$(3) \quad p = \rho \times \text{const.}$$

B. CHARLES' LAW (THE LAW OF GAY-LUSSAC)

$$(4) \quad p V = C T.$$

C is a temporary constant which we shall presently express in terms of the universal gas constant, R . We must begin by discussing the temperature scale T defined through eq. (4). According to experience T is the same for all (perfect) gases if C is suitably chosen. To do this we refer to the coefficient of expansion defined in eq. (1.5). Using the same temperature scale as in (4), we find that it can be written as

$$(4\text{ a}) \quad \alpha = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_p = \frac{1}{V} \frac{C}{p} = \frac{1}{T}.$$

According to this equation, α is independent of the nature of the gas, being only a function of temperature. The same is true of the coefficient of tension

$$(4\text{ b}) \quad \beta = \frac{1}{p} \left(\frac{\partial p}{\partial T} \right)_V = \frac{1}{p} \frac{C}{V} = \frac{1}{T}.$$

Equation (4) still leaves the scale unit free. If this is selected so as to obtain the unit on the Celsius (or centigrade¹) scale, then the melting temperature of ice (at 760 torr) becomes

$$(5) \quad T_0 = 273.15 \text{ deg.}$$

Generally we have

$$(5\text{ a}) \quad T = T_0 + t, \quad t = \text{temperature on Celsius scale,}$$

and the coefficients of thermal expansion and tension become

$$(6) \quad \alpha = \beta \approx 1/273 \text{ deg} = 0.00366 \text{ deg}^{-1}.$$

¹According to the 1948 International Temperature Scale the term "centigrade scale" is now obsolete (*Transl.*).

The temperature scale introduced in our eqs. (4), (5) and (5 a) is the *gas thermometer temperature scale*. Equation (5) shows that its zero-point is shifted by 273 (more precisely by 273.15) deg with respect to the Celsius scale.

The air thermometer (or better still, a hydrogen or helium thermometer) can be arranged to measure at constant pressure or at constant volume, the latter arrangement being more convenient. According to eq. (4) T is then proportional to the pressure, p , of the gas. The pressure difference $p - p_0$ is measured with the aid of a suitable barometric arrangement by noting the position of a column of mercury at a temperature T of the gas as compared with that at a temperature T_0 . The definition of a temperature scale with the aid of an air thermometer is sufficient for most practical purposes. The limit of its usefulness is attained at low temperatures when the air ceases to behave like a perfect gas. We shall see later how the temperature scale (Kelvin scale) should then be defined. In the case of a real gas (or in the case of a perfect gas at low temperatures) the Charles-Gay-Lussac equation (4) must be replaced by the already mentioned *general equation of state* of a liquid or gaseous system

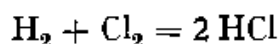
$$(6 a) \quad T = F(p, V).$$

C. AVOGADRO'S LAW AND THE UNIVERSAL GAS CONSTANT

Dalton's *law of multiple proportions* which is valid for all chemical compounds is supplemented with Gay-Lussac's *law of integral volume ratios* when gases are involved.

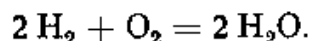
By way of example we take:

1 liter hydrogen + 1 liter chlorine = 2 liters hydrogen chloride, according to the chemical formula



Another example:

2 liters hydrogen + 1 liter oxygen = 2 liters steam, written in chemical form this becomes



The laws of Dalton and Gay-Lussac can be combined into one comprehensive rule due to Avogadro (1811): *Under the same external conditions of pressure and temperature all gases contain equal numbers of molecules in equal volumes* (Avogadro used the term corpuscles instead of the modern term—molecules). This rule remained ignored for a very long time, but since about 1860 it forms

the foundation for all determinations of molecular weights. Nernst gave it prominence when he gave his great text-book the title "Theoretische Chemie vom Standpunkt der Avogadroschen Regel und der Thermodynamik").

The atomistic, microscopic point of view is alien to thermodynamics. Consequently, as suggested by Ostwald, it is better to use *mols* (*moles*) rather than molecules. As is well known one mol (see also vol. II, Sec. 7 footnote¹) represents a mass of as many grams or kilograms (we then differentiate between the gram-mol = mol or gmol, and between the kilogram-mol = kmol) as there are units in the sum of the atomic weights of the constituents of the substance. Thus one gram-mol of O₂ in gaseous form is equal to 32 g, one kilogram-mol of H₂ is approximately equal to 2 kg, one gram-mol of HCl is equal to (1 + 35.5) g in round figures. It is necessary to remember at this point that the fact that hydrogen is diatomic was recognized precisely with the aid of Avogadro's rule; as late as around the year 1850 the chemical formula for water was mostly written HO.

If we introduce the *molar* (or *molal*) volume as a natural unit of volume and if we define it as the volume occupied by exactly one mol of a gas under the given pressure and at the given temperature, then we can put Avogadro's rule in the following simple form: *Under identical external conditions all gases have equal molar volumes.* It is evident that the last statement, just as the preceding ones, is restricted to perfect gases. It may be extended to real gases or even to vapors but only with caution.

We shall calculate the magnitude of this molar volume under a pressure of 1 atm and at a temperature of 0 C. We can utilize the fact that under these conditions the density of H₂ is fairly accurately equal to (see e. g. Vol. IV, eq. (17.14))

$$9.00 \times 10^{-2} \frac{\text{kg}}{\text{m}^3} = 9.00 \times 10^{-2} \frac{\text{g}}{\text{liter}}$$

It follows that 2 g of H₂ occupy a volume of

$$(7) \quad \frac{200}{9.0} = 22.2 \text{ liter.}$$

The accurate value is somewhat higher on account of the atomic weight of H; thus, more precisely

$$V_{\text{mol}} = 22.4 \frac{\text{liter}}{\text{mol}} = 22.4 \frac{\text{m}^3}{\text{kmol}} \quad \text{at 760 torr and 0 C.}$$

According to Avogadro this value of the molar volume applies not only to H₂ but is *universally* valid for all perfect gases.

The equation of state also assumes a universal form if the molar volume is used. We can write it as

$$(8) \quad p V_{mol} = R T$$

where R is called the *universal gas constant*. Substituting the value (7) into eq. (8), we can evaluate it as follows:

$$p = 760 \text{ torr} = 1.03323 \frac{\text{kp}}{\text{cm}^2} = 9.81 \times 1.03323 \frac{\text{Dyne}}{\text{cm}^2}, \text{ see (2) and (2a)}$$

$$T = T_0 = 273.15 \text{ deg} \quad \text{see (6)}$$

$$V_{mol} = 22.4 \text{ liter/mol} = 22.4 \text{ m}^3/\text{kmol}.$$

Consequently

$$(9) \quad R = \frac{9.81 \times 1.03323 \times 224 \text{ Dyne} \times \text{m}}{273.15 \text{ deg} \times \text{mol}} = 8.31 \frac{\text{Erg}}{\text{deg} \times \text{mol}}.$$

Erg is the unit of work in the MKS system and has already been used in (2.7).

Applying (8) to a volume consisting of n mols we obtain, evidently,

$$(10) \quad p V = n R T$$

so that the value of the gas constant C used in (4) becomes

$$(10a) \quad C = n R.$$

If v denotes the so-called specific volume, i. e. the volume of a unit of mass, we have

$$(11) \quad V_{mol} = \mu v$$

where μ is the molecular or, better, molar "weight" (actually the mass of one mol) of the gas under consideration. Thus, for example, it is 32 kg/kmol for O_2 . Substituting (11) into (8), we obtain

$$(11a) \quad p V = \frac{R}{\mu} T$$

4. The First Law. Energy and enthalpy as properties

The so-called "mechanical theory of heat" replaced the theory that regarded heat as a substance, after the latter proved untenable. As the name of the present theory implies, heat is regarded as a manifestation of the random motion of material particles being given by their energy (*vis viva*) or living force. Following these ideas, Helmholtz gave to his book which appeared in 1847 the title: "Über die Erhaltung der Kraft." It is based on the assumption that the whole of the science of physics can be reduced to mechanics and that the interaction between the material particles is due to forces passing through their centers.

The designation "mechanical theory of heat" is evidently too narrow. Solar radiation certainly belongs to the heat balance of the earth and, equally certainly, it is not a mechanical process. For this reason the less vivid designation of "thermodynamics" is preferred nowadays. The ambiguous expression "*vis viva*" has been fortunately replaced by that of "kinetic energy", as suggested by Sir William Thomson. The word *energy* occurs already in Aristotle's writings; it was introduced into the language of science by Rankine (1853); he also used the expression "energetics". Robert Mayer's ("Über die Kräfte der unbelebten Natur," 1842) bold ideas went beyond the framework of classical mechanics and completely corresponded with the modern interpretation of the energy concept, even if he did not yet give them as precise a mathematical formulations as that achieved later by Helmholtz. One of Mayer's outstanding achievements was the emphasis he put on processes involving release (seemingly contradicting the energy principle) which are now so important in the understanding of catalytic phenomena.

We introduce the concept of energy axiomatically and without reference to mechanics and thereby state the *First Law of thermodynamics*: *Every thermodynamic system possesses a characteristic property (parameter of state) — its energy. The energy of the system is increased by the quantity of heat, dQ , absorbed by it and decreased by the external work, dW , performed. In an isolated system, the total amount of energy is preserved.*

A. EQUIVALENCE OF HEAT AND WORK

Introducing Clausius' symbol U to denote energy we can give the following mathematical formulation to the First Law:

$$(1) \qquad dU = dQ - dW.$$

Here dU , unlike dQ or dW , is a *perfect differential*. Hence for any cycle we must have

$$(1\ a) \quad \oint dU = 0.$$

The quantity of heat dQ which appears on the right-hand side of (1) need not be measured in calories; we may assume that it has been converted to mechanical units in accordance with eq. (2.6) or (2.7).

We shall now apply eq. (1) to the simplest possible thermodynamic system, namely to a unit of mass of a homogeneous fluid. The corresponding energy is called specific energy and is denoted by u , in analogy with the symbol v used to denote the specific volume in eq. (3.11 a), or the specific quantity of heat added, dq in eq. (2.3 a, b). Thus

$$(2) \quad du = dq - p\,dv.$$

First of all we shall use this equation to determine the mechanical equivalent of heat, J , and hence to verify eq. (2.7). In order to do this we consider two processes. The first will take place at *constant* v , changing the state of the system from v, T to $v, T + dT$. The second process will take place at *constant* p , the state changing from v, T to $v + dv, T + dT$ with the same T and dT as in the first process. Taking into account the definitions (2.3 a, b) we have:

$$(3) \quad du_1 = c_v\,dT,$$

$$(3\ a) \quad du_2 = c_p\,dT - p\,dv.$$

We now assume that our system is a perfect gas. Then we may apply eq. (3.11 a) to the second process, or

$$p\,dv = \frac{R}{\mu}\,dT$$

so that eq. (3 a) transforms to

$$(3\ b) \quad du_2 = \left(c_p - \frac{R}{\mu}\right)dT.$$

At this stage we supplement the definition of a perfect gas by an additional condition of a caloric nature: *The specific energy u (and, evidently, the total energy U) is a function of temperature T alone*, or, in other words, it is independent of volume or pressure at a given T . Then, according to eq. (3), c_p is

also a function of temperature only ($= du/dT = u'(T)$) and eq. (3) can be written at once in integral form

$$(4) \quad u(T) = \int c_v(T) dT.$$

Since u is a property, the nature of the path, whether at constant or variable volume, is irrelevant.

An experimental and theoretical justification of our additional caloric requirement in the definition of a perfect gas cannot be given here. We shall revert to this point in Secs. 5 C and 7.

Owing to the assumed equality of T and dT for our two processes, we can now write

$$(4 \text{ a}) \quad du_1 = du_2 = u'(T) dT.$$

From (3) and (3 b) we thus obtain

$$(5) \quad c_v = c_p - \frac{R}{\mu}$$

and also

$$(5 \text{ a}) \quad \mu (c_p - c_v) = R.$$

The left-hand side contains the difference of the two molar specific heats which has the numerical value of approximately 2 cal/deg \times mol for all perfect gases; hence

$$(5 \text{ b}) \quad (c_p - c_v)_{\text{mol}} = 2 \text{ cal/deg} \times \text{mol} = 2 \frac{\text{kcal}}{\text{kmol} \times \text{deg}}.$$

Substituting this value into eq. (5 a) and using the value of R from eq. (3.9), we obtain

$$(6) \quad 1 \text{ cal} = 4.16 \text{ Erg}$$

which agrees with our previous statement to within 1%, the discrepancy being due to the inaccuracy in the value of 2 in (5 b).

We can consider the same example and express the unit of work in terms of calories. In this manner, taking into account eq. (5 b), eq. (5 a) gives

$$(7) \quad R = 2 \frac{\text{cal}}{\text{deg} \times \text{mol}}$$

and the equation of state (3.8) assumes the rather odd form

$$(8) \quad p v_{mol} = 2 T \frac{\text{cal}}{\text{deg} \times \text{mol}}$$

where the pressure is measured in cal/unit of volume.

B. THE ENTHALPY AS A PROPERTY

Along with energy, we introduce a new property which is particularly important in engineering applications; it is given the name of *enthalpy* and is defined as

$$(9) \quad H = U + p V.$$

The term enthalpy means "heat function," and the symbol H (originally the Greek letter η was meant to be used) has been introduced in the American standard text-book on thermodynamics by Lewis and Randall; alternative symbols will be listed in Sec. 7 where we shall also deduce the definition in eq. (9) from a general mathematical concept.

From eq. (9), with $dU = dQ - p dV$ we have

$$(10) \quad dH = dQ + V dp.$$

At a constant pressure ($dp = 0$) dH is equal to the quantity of heat introduced to the system from an external source, which explains the name of "heat function" (or "total heat") given to it.

The enthalpy per mol (on occasions also that per unit mass) will be denoted by h , so that

$$(9 \text{ a}) \quad h = u + p v$$

$$(10 \text{ a}) \quad dh = dq + v dp.$$

It follows¹ that the molar specific heat c_p is given by

$$(11) \quad \left(\frac{\partial h}{\partial T} \right)_p = \left. \frac{dq}{dT} \right|_{(p=\text{const})} = c_p.$$

¹In eqs. (11) and (11 a) as well as in succeeding equations we shall avoid writing

$$\left(\frac{dq}{dT} \right)_p \quad \text{or} \quad \left(\frac{dq}{dT} \right)_v$$

because q is not a property.

As a corollary to eq. (4 a), we can write more fully that

$$(11\ a) \quad \left(\frac{\partial u}{\partial T} \right)_v = \frac{dq}{dT} \Big|_{(v=\text{const})} = c_v.$$

In the case of a perfect gas $p v = R T$ and hence h is a function of T alone, in the same way as u ; consequently, we are justified in dropping the indices p and v on the left-hand sides of eqs. (11) and (11 a) respectively. Subtracting (11 a) from (11), we obtain

$$c_p - c_v = \frac{d(h - u)}{dT},$$

which is identical with (5 a) because $h - u = p v = R T$.

The concept of enthalpy is particularly important in engineering applications because it is directly connected with the *flux of energy* during a *steady-state* process involving the performance of work. Imagine a steam turbine which receives high-pressure steam at a constant rate per unit time. The steam is expanded, cooled and rejected by the turbine. We shall now consider, quite generally, the energy balance of an arbitrary machine which functions in a steady manner. We shall assume that all thermal quantities and that all quantities of work have been referred to a unit of mass of the gas (in general — the working fluid) supplied to the machine.

Consider a cross-section 1 (area A_1) through the inlet pipe of the machine, and assume that one unit mass has just crossed it. In this way a quantity u_1 of internal energy (the subscript 1 refers to the state of the gas at cross-section 1 of the pipe) has been transported through A_1 . The body of gas, which follows and whose pressure is p_1 , will have been displaced by a distance v_1/A_1 , because the volume of a unit mass of gas at cross-section 1 occupies a volume v_1 . The external pressure (e. g. the boiler pressure) has thus performed a quantity of work = force \times distance = $(p_1 A_1) (v_1/A_1) = p_1 v_1$. The flux of energy through A_1 , neglecting the kinetic energy, is

$$u_1 + p_1 v_1 = h_1.$$

The same reasoning can be applied to cross-section 2 imagined taken through the exhaust pipe. Assume that the machine performs work (per unit mass of gas) at a rate w (useful power) and that, for the sake of generality, it consumes heat at a rate q . (In particular cases q can be equal to 0.)

The energy equation assumes the simple form

$$(12) \quad h_1 + q = w + h_2.$$

This form of the balance equation has the advantage that any specific processes which may be taking place inside the machine do not come into evidence in it. We shall revert to this example in Sec. 5 C when we shall consider a very important physical process.

C. DIGRESSION ON THE RATIO OF SPECIFIC HEATS c_p AND c_v

At this stage we are compelled to make a digression which falls outside the field of thermodynamics. The science of thermodynamics can supply *relations between properties only*, such as e. g. eq. (5 a), but not their *absolute values*. In order to obtain the latter it is necessary to adopt microscopic models, as is done in the kinetic theory of gases. According to the law of equipartition of energy of the latter theory (see Sec. 31 B, ahead of eq. (9)), the molar specific heat of gases or vapors is given by

$$(13) \quad c_v = \frac{1}{2} f R.$$

Here f denotes the number of degree of freedom and is:

$f = 3$ for monatomic molecules; here only linear translations count, rotations being of no importance;

$f = 5$ for diatomic molecules; they can be regarded as possessing the symmetry of a dumb-bell so that it has two rotational degrees of freedom in addition to the three translational ones; rotation about the link of the atoms is unimportant. At the same time the possibility of the two atoms vibrating with respect to each other, which affects the specific heats only at high temperatures, is here disregarded;

$f = 6$ for more general molecular arrangements, i. e. 3 rotational degrees of freedom + 3 translational degrees of freedom, the possibility of internal motions being again disregarded.

Equation (13) shows that c_v is a characteristic *constant* for each gas, i. e. that it is not only independent of v , but also of T . The corresponding value of c_p , also per mol, is obtained from eq. (7) and is

$$(13 a) \quad c_p = \left(1 + \frac{1}{2} f\right) R.$$

From (13) and (13 a) we find

$$(14) \quad \frac{c_p}{c_v} = \gamma = 1 + \frac{2}{f}.$$

The numerical value of γ , which is the same for the specific heats referred to a unit mass and a mol, is

f	3	5	6
γ	$1 + \frac{2}{3} = 1.66$	$1 + \frac{2}{5} = 1.40$	$1 + \frac{2}{6} = 1.33$

Examples for $f = 3$: mercury vapor and the noble gases He, Ne, A,

Examples for $f = 5$: H_2 , N_2 , O_2 ,, air.

Examples for $f = 6$: all polyatomic gases.

The thermodynamic relation (5) is exact and remains unaffected by quantum corrections. On the other hand the values (13) and (13 a) are more or less accurate approximations and must be refined with the aid of quantum theory. In particular, $\gamma = 1.33$ is only a mean value about which the experimental values for polyatomic gases group themselves more or less closely. It is, however, remarkable that the case $f = 4$, $\gamma = 1.50$ which does not correspond to any geometrical model or to any type of molecular symmetry does not occur in nature.

The purpose of the present digression was to throw some light on the strong and weak points of the science of thermodynamics on the one hand, and of the kinetic theory of gases, on the other.

5. The reversible and the irreversible adiabatic process

We shall begin by emphasizing the difference between *reversible and irreversible processes*.

Reversible processes are not, in fact, processes at all, they are sequences of states of equilibrium. The processes which we encounter in real life are always irreversible processes, processes during which disturbed equilibria are being equalized. Instead of using the term "reversible process" we can also speak of infinitely slow, quasi-static processes during which the system's capacity for performing work is fully utilized and no energy is dissipated. In spite of their not being real, reversible processes are most important in thermodynamics because definite equations can be obtained only by considering reversible changes; irreversible changes can only be described with the aid of inequalities when equilibrium thermodynamics is used.

The actual criterion for a process to be reversible states that during its course there are no lasting changes of any sort in the surroundings if the process is allowed to go forward and then back to the original state.

A. THE REVERSIBLE ADIABATIC PROCESS

The term *adiabatic* implies: exclusion of heat transfer to and from the body; in this connection the thermos flask invented by Dewar may be thought of. The opposite case is that of an isothermal process; in order to maintain the temperature it is necessary to allow heat to be transferred; in this connection one may imagine a water bath in which our quantity of gas is immersed.

Consider a unit mass of a perfect gas and substitute

$$dq = 0 \quad du = c_v dT$$

into (4.2), taking into account (4.4). We then have

$$(1) \quad c_v dT = -p dv.$$

In order to transform this into a relation between v and p we use the equation of state (3.11 a). Instead of (1) we may write

$$\begin{aligned} \frac{\mu}{R} c_v (p dv + v dp) + p dv &= 0 \\ \left(c_v + \frac{R}{\mu} \right) p dv + c_v v dp &= 0 \end{aligned}$$

so that in view of (4.5)

$$c_p p dv + c_v v dp = 0,$$

or, considering (4.14):

$$(2) \quad \frac{dp}{p} + \gamma \frac{dv}{v} = 0$$

We now assume γ to be a constant, see end of Sec. 4, so that we actually exceed the caloric assumption according to which u and hence c_v , c_p and γ depend on T alone. In this case eq. (2) can be integrated directly, so that

$$\log p + \gamma \log v = \text{const.}$$

This is Poisson's equation of a *reversible adiabatic (isentropic)* process. It can be written

$$(3) \quad p v^\gamma = \text{const.}$$

Poisson's equation is very important in meteorology. We may also recall the calculation of the velocity of sound in Vol. II, eq. (13.17 a), with the aid

of Poisson's equation (described as the equation of a polytrope whose exponent $n = \gamma$). Transforming the equation to T, v or T, p coordinates with the aid of the equation of state (3.11 a), we obtain

$$(3 a) \quad T v^{\gamma-1} = \text{const}, \quad \text{or} \quad T p^{(1-\gamma)/\gamma} = \text{const},$$

respectively. The constants in eqs. (3) and (3 a) can be expressed in terms of the initial state p_0, v_0, T_0 , as follows:

$$\text{const} = p_0 v_0^\gamma \quad \text{or} \quad T_0 v_0^{\gamma-1} \quad \text{or} \quad T_0 p_0^{(1-\gamma)/\gamma}$$

According to Boyle's law, isothermals are represented by equilateral hyperbolae in the p, V -plane; on the other hand, according to Poisson's eq. (3), isentropes are steeper downwards (see Fig. 2). In the T, V -plane the isentrope is, evidently, less steep because of the exponent $\gamma - 1$ in eq. (3 a), see Fig. 2 a.

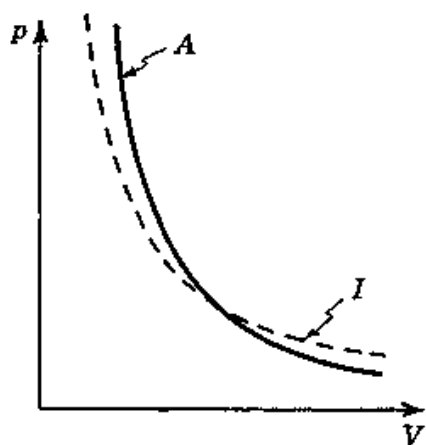


Fig. 2.

Reversible adiabatic (isentropes), A , and isothermal, I , for a perfect gas in the p, V -plane.

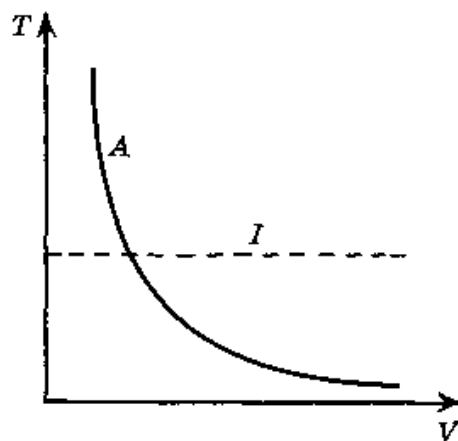


Fig. 2 a.

Reversible adiabatic (isentropes), A , and isothermal, I , for a perfect gas in the T, V -plane.

In order to obtain a clear idea of the *reversible* nature of a process we imagine the gas to be confined in a cylindrical vessel of cross-sectional area A . The vessel is, in turn, enclosed by walls which prevent any exchange of heat, and the gas is contained at the top by a weightless piston. The piston is maintained in equilibrium by a weight $P = p A$ which balances the gas pressure. We imagine P to be subdivided into many small weights δP which will be removed one by one. This causes the piston to rise each time, its pressure p decreasing. Each weight δP is placed outside the vessel at the same level at which it has been removed so that no work is gained or lost in the process. The gas pressure will fall from its initial value p (e. g. 2 kp/cm²)

to a final value p_1 (say 1 kp/cm²), and the volume will increase from an initial value V (e. g. 1 liter) to a final value V_1 (in our example $2^{1/\gamma}$ liter). The center of gravity of each δP has been raised compared with its original level. This work against the forces of gravity stems from the work performed by the gas on the piston. It has not been lost, being found stored in the raised δP 's. If we now replace these weights one by one on the piston, the gas will be re-compressed and heated and will revert to its initial state. The *process is reversible* on condition that it has been carried out in *infinitely small steps* and *sufficiently slowly*,¹ i. e. with a sufficiently fine subdivision of P into elements of δP each.

B. THE IRREVERSIBLE ADIABATIC PROCESS

If the piston (together with the weight P) is raised *suddenly* the gas will first flow into a vacuum performing no external work. The resulting turbulent motion gradually subsides, the gas coming to rest. What is the final state of the gas? Has it become heated owing to internal friction or has it become cooled owing to its having expanded? None of the two: As far as the final state is concerned the process is not only adiabatic but also isothermal, the approximation being as good as that of the gas under consideration is to a perfect gas.

The preceding experiment was first performed by Gay-Lussac in 1807 (flow experiment) and then repeated by Joule with an increased accuracy. Instead of the original cylindrical arrangement two glass jars were used; they were connected through a narrow tube equipped with a cock. One jar was evacuated and the other was filled with the experimental gas. After the cock had been opened and after equilibrium had set in, it was observed that the final temperature, particularly with air or hydrogen, was substantially the same as that at the beginning.

Anticipating this result we shall first consider the *cycle* which Robert Mayer² used for the calculation of the mechanical equivalent of heat and was thus led to the First Law. At the initial state 1, Fig. 3, the gas is under the

¹A reversible process must be carried out infinitely slowly. The reverse is, however, not true, as an infinitely slow process need not be reversible. Example illustrating the latter case: Discharging a condenser through a very large resistance.

²The same calculation, based on specific heats, was found in the papers left by Sadi Carnot, (1796–1832) who died at a young age. He was the son of the geometer and general Lazarus Carnot mentioned in Vol. I in connection with eq. (3.28 b). Hence Sadi Carnot can be regarded as having paved the way not only to the Second Law, but also to the first part of the First Law.

atmospheric pressure p_1 and has the volume V_1 . It is heated at constant volume V_1 until its pressure is changed to p_2 , point 2 in Fig. 3. It now expands to V_3 by being allowed to flow from one vessel to another. Neglecting turbulent deviations, it will reach a state along the isothermal equilateral hyperbola passing through 2. This element of the cycle is shown by a broken line because it is not defined in detail; only the portions of the hyperbola which lie on the other side of points 2 and 3 have been drawn in full. The gas can now be re-compressed at constant pressure p_1 to its initial volume by performing work on it, if V_3 has been so selected as to make the corresponding pressure equal to p_1 .

The change in energy per unit mass of gas along the three paths 12, 23 and 31 is

$$(4) \quad \int_{T_1}^{T_2} c_v dT; \quad 0; \quad \int_{T_2}^{T_1} c_p dT - p_1 (v_1 - v_3).$$

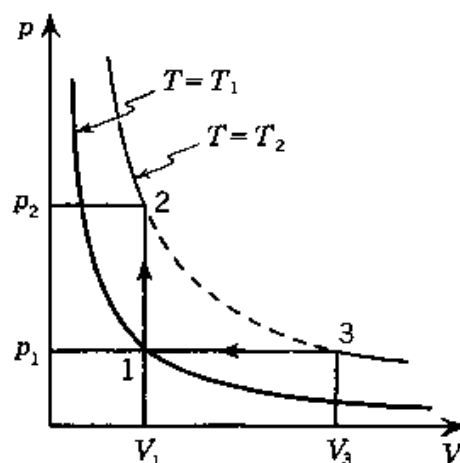


Fig. 3.

Cycle for the determination of the mechanical equivalent of heat.

According to (4.1 a) the sum of these terms must vanish. Thus we are led to (4.5 a) and to the value of the mechanical equivalent of heat in (4.6) if we, in addition, take into account the equation of state of a perfect gas and allow the temperature difference $T_2 - T_1$ to become vanishingly small.

It is evident that the cycle under consideration must give the same result as our differential method in (4.5 a), because both are based on the same assumption, namely on the premiss that the energy of a perfect gas is a pure function of T .

C. THE JOULE-KELVIN POROUS PLUG EXPERIMENT

In order to refine the experiment in which a gas is allowed to flow into an evacuated vessel, William Thomson devised the porous plug experiment and carried it out experimentally in collaboration with Joule. In the experiment the gas is forced through a plug made of cotton wool, the stream being slow and well regulated and proceeding from a higher pressure ahead of to a lower pressure behind the plug. On passing through the cotton wool plug which was accommodated in a pipe made of beechwood, which is, to all intents and purposes, a heat insulator, the gaseous stream became slowed up. After

a steady-state has set in, the temperature in the plug becomes steady, irrespective of how complicated the temperature distribution in its interior is, and the same is true of the temperatures to the left and to the right of the plug.

We shall consider the mass of gas, Fig. 4, contained between an arbitrary section A and the right-hand end of the plug, B , and we shall follow its motion until it reaches the position $A' B'$, when the particles at A have reached the

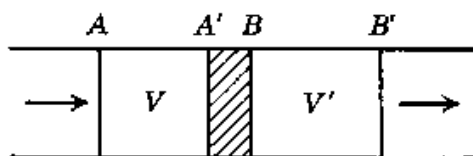


Fig. 4.

The Joule-Kelvin porous plug experiment.

left-hand end of the plug. During the motion the mass of gas is acted upon by a force pA from the left (A = cross-section of the pipe). The opposing force is $p'A$. The path traversed is V/A on the left and $V'A$ on the right, so that the total work performed is

$$(5) \quad \int dW = pV - p'V'.$$

On the other hand there is no transfer of heat either through the gas on the left or on the right, or through the beechwood pipe. Hence

$$(5a) \quad \int dQ = 0.$$

According to the First Law

$$(6) \quad U - U' = -pV + p'V'.$$

So far our reasoning applies to any gas. Thus we note the following general result of the Joule-Kelvin porous plug experiment:

$$(7) \quad U + pV = U' + p'V' \quad \text{or} \quad H = H'.$$

In words: *The Joule-Kelvin experiment is characterized by the fact that the enthalpy of the gas is preserved as it flows through the porous plug.* We recall here the final remark in Sec. 4 B concerning the flow of energy in the inlet and exhaust pipes of a steam engine. It is evident that the quantity of energy calculated in (7) represents the previously considered energy flux (provided that the unit of energy has been suitably chosen), and our present example can be used as a special illustration of the preceding general theorem.

In particular, for a perfect gas the right-hand side of (6) becomes

$$\frac{M}{\mu} R (T' - T)$$

where M denotes the mass of the gas contained in the volume AB .

In actual fact the Joule-Kelvin experiment showed a very small difference between T and T' in the case of air, whereas for hydrogen the difference was hardly measurable. From this result we conclude that: *In the ideal, limiting case we have*

$$(8) \quad U' = U \quad \text{independently of } V,$$

which is the same result as that from the Gay-Lussac experiment, except that it has now been deduced with a much higher degree of accuracy. It is only now that we have based our additional caloric condition in eq. (4.4) on a sure experimental foundation.

D. A CONCLUSION OF GREAT CONSEQUENCE

We shall now consider the First Law and we shall apply it to a reversible process in a perfect gas, e. g. to a unit mass of gas. In view of the now established relations: $u = u(T)$, $c_v = c_v(T)$, $du = c_v(T) dT$ and the equation of state, we write

$$(9) \quad dq = du + p dv = c_v(T) dT + \frac{R}{\mu} \frac{T}{v} dv.$$

Dividing both sides by T we obtain

$$(9 \text{ a}) \quad \frac{dq}{T} = c_v(T) \frac{dT}{T} + \frac{R}{\mu} \frac{dv}{v}.$$

We know that dq is *not* a perfect differential, but eq. (9 a) shows that dq/T is integrable. Putting $ds = dq/T$ we obtain by integrating (9 a) that

$$(10) \quad \int_{T_0, v_0}^{T, v} ds = s - s_0 = c_v \log \frac{T}{T_0} + \frac{R}{\mu} \log \frac{v}{v_0}.$$

We have assumed here that $c_v = \text{const}$, which was convenient but not necessary; s is a *property which is independent of the path between the initial and final state* and depends only on the *instantaneous values of the properties* T, v if the initial properties are fixed at an arbitrary state. With Clausius we shall call this new property *entropy*. The term means "transformability".

In order to recognize, at this stage at least, the formal meaning of entropy, we write the energy equation (9) in the form

$$(11) \quad du = T ds - p dv,$$

since $dq = T ds$. We conclude that s is conjugate to T in the same sense as v is to p : *s is the extensive property which corresponds to the intensive property T* , the problem of finding it having been mentioned already in Sec. 1.

It is evident that the definition of entropy (10) can be extended from a unit of mass to one mol and to any mass M (in which case instead of the lower case symbols we use S, V).

The adiabatic processes which were considered in Section A can be also called "isentropic" because $dq = 0$ implies that they are curves of constant entropy. In fact it is easy to convince oneself that the above eq. (3 a) in the T, v -plane is identical with the equation $s = \text{const}$ from (10).

6. The Second Law

In order to present the most essential considerations in the science of thermodynamics we shall follow the classical path which was initiated by Sadi Carnot in 1824 and then followed by Rudolf Clausius from 1850, and by William Thomson from 1851 onwards. The title of Carnot's paper "Réflexions sur la puissance motrice du feu et les moyens propres à la développer" gives expression to the historical connection between thermodynamics and the development of the reciprocating steam engine.

Carnot based his considerations on the hydraulic analogy: he thought that the heat substance is capable of performing work on passing from a higher to a lower temperature in the same way as water can perform work when it flows from a higher to a lower level. The weakness of this analogy is evidently derived from the fact that no indestructible heat substance exists. In spite of this, however, Carnot's argument proved to be one of permanent value having become essential in the development of the Second Law which was not discovered until 25 years later.

We shall state the Second Law in an axiomatic way, just as we have done with the First Law in Sec. 4 (and with the "zeroth" in Sec. 1):

All thermodynamic systems possess a property which is called entropy. It is calculated by imagining that the state of the system is changed from an arbitrarily selected reference state to the actual state through a sequence of states of equilibrium and by summing up the quotients of the quantities of heat dQ introduced at each step and the "absolute temperature" T ; the latter is to be defined simultaneously in this connection. (First part of Second Law.)

During real (i. e. non-ideal) processes the entropy of an isolated system increases. (Second part of Second Law.)

In what follows we shall provide a "proof" of this proposition, but this can only mean that we shall reduce it to simpler, apparently evident,

assumptions which, by their nature, cannot be proved in turn. The simplest of these seems to be: *Heat cannot pass spontaneously from a lower to a higher temperature level* (Clausius). In this connection it is necessary clearly to define the meaning of the word "spontaneously," and we shall take it to mean that except for the bodies taking part in the exchange of heat there are no permanent changes of any sort caused by the process. The following postulate, due to Kelvin, is equivalent to that due to Clausius: *It is impossible continuously to produce work by cooling only one body down to a temperature below the coldest part of its surroundings*. If that were not so it would be possible to convert the work into heat, for example through friction, and so to bring it to a higher temperature level. Ostwald expressed this principle in a form in which it is now normally quoted: *It is impossible to design a "perpetual motion engine of the second kind,"* i. e. a machine which would work periodically and which would cause no other changes except the lifting of a weight and the cooling of a heat reservoir.¹ (As is well known the First Law expresses the impossibility of building a *perpetual motion engine of the first kind*.)

A. THE CARNOT CYCLE AND ITS EFFICIENCY

We shall use an arbitrary, but homogeneous working fluid. The term "homogeneous" denotes that its state is described by indicating only its two mechanical variables, V and p ; these in turn determine the thermal variable θ with the aid of some general equation of state. The symbol θ instead of T gives expression to the fact that temperature is, at first, measured with the aid of an arbitrary calibrated thermometer (say a thermocouple, etc.).

The path of a Carnot cycle (see Fig. 5) consists of two isotherms 1 2 and 3 4 and of two isentropes 2 3 and 4 1. Along 1 2 it is necessary to add a certain quantity of heat Q from the "boiler" (heat reservoir of temperature θ_1) and along 3 4 it is necessary to reject a quantity of heat Q_2 to a cooler (heat reservoir θ_2). The total amount of heat transferred is

$$\oint dQ = Q_1 - Q_2.$$

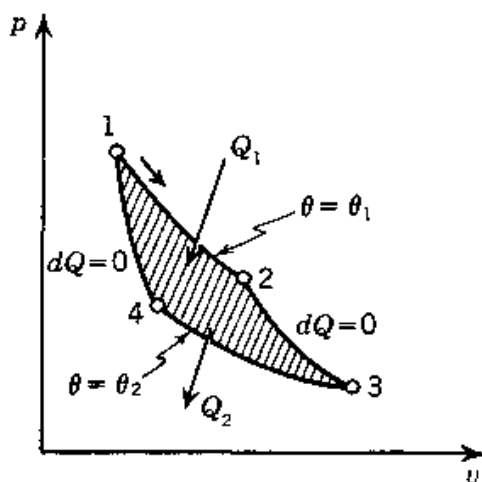


Fig. 5.
The Carnot cycle.

¹Formulation due to Planck, *Thermodynamics*, 8-th German ed. Sec. 116, English ed. "Treatise on Thermodynamics," 3rd. ed. p. 89.

The work performed by the working fluid is equal to

$$\oint dW = \oint p dV = W$$

in the same way as for the indicator diagram in Sec. 2.

According to the First Law

$$(1) \quad W = Q_1 - Q_2$$

since the internal energy U returns to its initial value at point 1. The efficiency of the cycle is defined as

$$(2) \quad \eta = \frac{\text{work performed}}{\text{heat added}} = \frac{W}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

i. e. in the same way as for a steam engine.

Carnot considers an engine **E** which realizes the process 1 2 3 4 infinitely slowly (without frictional or radiation losses) so that the working fluid is always in thermal equilibrium. (In such a case the isentrope must be qualitatively the same as that described in Sec. 5 A in connection with the special case of a perfect gas.) Such an engine is called *reversible*: It can equally well traverse the sequence of states of equilibrium in the direction 1 4 3 2 in which case it does not operate as a *prime mover* but as a refrigerator ($W < 0, Q_2 > Q_1$; it is now necessary to add the amount of work $|W|$ in order to depress still further the lower temperature level of the cooler).

Carnot shows that the efficiency of such an engine is independent of the properties of the working fluid. In order to do this he considers two engines **E** and **E'** which operate on different working fluids but between the same heat reservoirs θ_1 and θ_2 developing equal power W . The quantities of heat processed by **E'** are denoted by Q_1' and Q_2' respectively. Let us assume that

$$(3) \quad \eta' > \eta.$$

In this case let us arrange **E** and **E'** in such a way that **E** operates as a refrigerator, i. e. in the direction 1 4 3 2, being driven by **E'**. From (2) and (3) we have

$$\frac{|W|}{Q_1'} > \frac{|W|}{Q_1}, \quad \text{i. e.} \quad Q_1 > Q_1'.$$

The hotter reservoir receives more heat from **E** than it loses to **E'**. Owing to the simultaneous operation of **E** and **E'**, this difference $\Delta Q = Q_1 - Q_1'$ is taken from the lower level θ_2 . The total effect is to transfer the quantity of

heat ΔQ from the lower level θ_1 without the performance of work and without making any permanent changes in \mathbf{E} , \mathbf{E}' , or in the surroundings. According to the preceding postulate this is impossible. Thus assumption (3) is untenable.

The assumption $\eta > \eta'$ is equally untenable: It suffices to interchange the roles of \mathbf{E} and \mathbf{E}' in order to arrive once more at a contradiction with our postulate. Consequently we must have

$$(4) \quad \eta = \eta'.$$

All reversible engines which exchange heat only at two temperatures θ_1 and θ_2 have equal efficiencies. In view of (2), eq. (4) can be replaced by

$$(5) \quad \frac{Q_1}{Q_2} = f(\theta_1, \theta_2),$$

where f denotes a universal function which is independent of the working fluid and of the design of the heat engine.

B. THE FIRST PART OF THE SECOND LAW

In order to split the function of two variables $f(\theta_1, \theta_2)$ into two functions of one variable each it is necessary to span two reversible Carnot cycles between the two temperature levels θ_1 , θ_2 and a heat reservoir of an arbitrary but constant intermediate temperature θ_0 so that the heat reservoir θ_0 acting as a cooler for one cycle absorbs the same quantity of heat, Q_0 , as it is forced to reject when serving as a heater for the other. In this manner the reservoir θ_0 will not enter the heat balance equation and the simple cycle (θ_1, θ_2) is seen to operate with the same quantities of heat as the compound cycle $(\theta_1, \theta_0) + (\theta_0, \theta_2)$. In addition to eq. (5) we can write the equations

$$(6) \quad \frac{Q_1}{Q_0} = f(\theta_1, \theta_0); \quad \frac{Q_0}{Q_2} = f(\theta_0, \theta_2)$$

in which the same quantities of heat Q_1 and Q_2 appear. On multiplying, we have

$$(6 \text{ a}) \quad \frac{Q_1}{Q_2} = f(\theta_1, \theta_0) \times f(\theta_0, \theta_2).$$

Comparing with (5), we find

$$(6 \text{ b}) \quad f(\theta_1, \theta_2) = f(\theta_1, \theta_0) \times f(\theta_0, \theta_2).$$

Inserting $\theta_1 = \theta_2^1$ as a special case, so that according to (5) we also have $f(\theta_1, \theta_1) = 1$, we have

$$f(\theta_0, \theta_2) = 1/f(\theta_2, \theta_0).$$

Consequently, eq. (6 b) can also be written as

$$(6\ c) \quad f(\theta_1, \theta_2) = \frac{f(\theta_1, \theta_0)}{f(\theta_2, \theta_0)}.$$

Since θ_0 cancels out, eqs. (5) and (6 c) lead to

$$(7) \quad \frac{Q_1}{Q_2} = \frac{\phi(\theta_1)}{\phi(\theta_2)}.$$

With the intrinsically arbitrary temperature scale θ we can now associate an *absolute scale* in such a way that to each mark on θ there corresponds the mark

$$(7\ a) \quad T = \phi(\theta)$$

on the latter scale. We shall see in Sec. 10 how this can be done in practice. At the moment we shall only remark that this *absolute temperature* T coincides with the temperature measured on a gas thermometer over a range in which the thermometric substance behaves like a perfect gas if a suitable value is chosen for the still arbitrary constant factor in $\phi(\theta)$. The proof of this proposition will be advanced in Problem 1.

Equations (7) and (7 a) can be combined into the Carnot ratio:

$$(8) \quad Q_1 : Q_2 = T_1 : T_2.$$

From this we deduce the formula for efficiency, *viz.*

$$(8\ a) \quad \eta = \frac{T_1 - T_2}{T_1},$$

and applying it to an infinitely narrow Carnot diagram (finite temperature difference, but infinitely small quantities of heat added and rejected, dQ_1 and dQ_2), we obtain

$$(8\ b) \quad \frac{dQ_1}{T_1} = \frac{dQ_2}{T_2}.$$

¹It will be noticed that we now abandon the stipulation that θ_0 is intermediate between θ_1 and θ_2 , but this point has no bearing on the result.

We shall consider an arbitrary, but still reversible cycle. We shall represent it with the aid of the continuous contour in the p, V -diagram of Fig. 6, and select two arbitrary points A and B on it. We now replace the process by infinitely narrow Carnot cycles. The fact that the continuous contour is now replaced by a sequence of small saw-teeth, as shown in Fig. 6 at A and B , makes no difference for the integration. If we consider that the rejected heat dQ_2 is negative, which is entirely consistent, we obtain at once from (8 a) that

$$(9) \quad \oint \frac{dQ_{rev}}{T} = 0$$

where integration is carried out over the whole contour. The subscript of dQ explicitly stresses the reversible nature of the cycle under consideration. According to Sec. 1 eq. (9) is the necessary and sufficient condition for

$$(10) \quad dS = \frac{dQ}{T}$$

to be a *perfect differential*, provided that dQ is added reversibly (utilizing the available work in full). Reversibility is assured if we put $dQ = dU + dW$ according to the First Law, i. e. if instead of (10) we write (10 a)

$$(10 a) \quad dS = \frac{dU + p dV}{T}$$

for the simple working fluid now being considered. The absolute temperature, see eq. (7 a), defined in the above sense is seen to be the *integrating denominator* of the incomplete differential which appears in the numerator of (10 a).

Equation (9) has been shown to be true for a general path of integration but it still applies only to a very special thermodynamic system (homogeneous fluid). It is, however, true for any system, composed of different substances, appearing in different phases, and possessing any number of degrees of freedom (e. g. electrical or magnetic), *provided that the system does not perform any irreversible processes, such as friction, Joule heat, etc.*

If we first consider a single, say the i -th homogeneous component of the system possessing two degrees of freedom, we find according to (10) that

$$(10 b) \quad dS_i = \frac{dQ_i}{T_i}$$

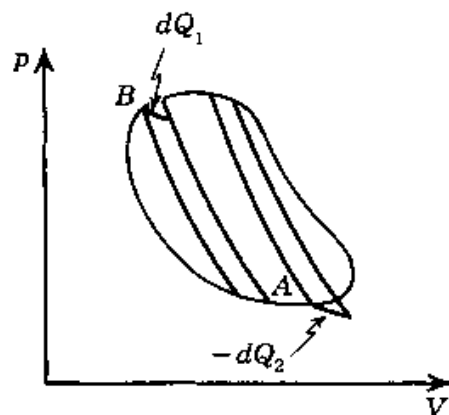


Fig. 6.

Representation of an arbitrary reversible process as a sum of infinitely narrow Carnot cycles.

is a perfect differential; here T_i denotes the absolute temperature of this i -th component and dQ_i denotes the total quantity of heat added reversibly to it, whether externally or by the other components of the system.¹

We now form the sum

$$(10\ c) \qquad dS' = \sum_i dS_i = \sum_i \frac{dQ_i}{T_i},$$

and find that it is also a total differential independently of the choice of the variables of the system to describe the processes. This sum is simpler than the sum of the individual expressions in (10 b), because in (10 c) the quantities of heat transferred between the individual components need not be included. Since such transfers of heat have been assumed reversible they must take place between the components at equal temperatures (ordinary conduction of heat must be excluded!). Denoting two such sub-systems by i and i' we have $T_i = T_{i'}$, and $dQ_i = -dQ_{i'}$, (the heat added to i is rejected by i'). *It is thus seen that the terms which are due to such transfers cancel each other.* The same is true of the quantities of heat transferred at phase equilibrium which, as will be stressed in Sec. 8 B, stipulate equal temperatures for the two phases. *Consequently we may restrict the meaning of dQ_i to denote only the heat added externally to the i -th component.*

In principle an inequality between T_i and $T_{i'}$ is not excluded if the two sub-systems are separated from each other by an adiabatic wall. This would, however, necessitate fairly artificial combinations of the sub-systems. Normally we shall find that there is thermal equilibrium not only throughout a single component (T_i constant within i) but throughout the whole system, ($T_i = T$). In such cases (10 c) reduces to

$$(10\ d) \qquad dS = \frac{1}{T} \sum_i dQ_i = \frac{dQ}{T}$$

which is identical with (10); dQ denotes here, as in eq. (10), the quantity of heat added reversibly to the whole system from the external surroundings. Equations (10), (10 a), as well as the slightly more general eq. (10 c), directly demonstrate *the existence of entropy S as a property of the system* and thus gives the proof of the first part of the Second Law.

¹When more than two degrees of freedom are present it is possible to fix all degrees of freedom except two, using different combinations, and to apply eq. (10) to each partial process.

The difference in the entropy between two arbitrary states A and B is calculated with the aid of the equation

$$(11) \quad S_B - S_A = \int_A^B \frac{dQ_{rev}}{T}.$$

We wish to emphasize the fact that the path of integration bears no relation to the way in which the system reaches B from A in actual fact. Real processes are always at least partly irreversible. Our rule (11) demands, however, the selection of an imaginary *reversible* path. The actual path selected is immaterial, because S is a property, and as such it is independent of the path.

The simplest example of such a calculation is afforded by the porous plug experiment described in Sec. 5 C, Fig. 4, where the points 2 and 3 in the sketch correspond to states A and B in eq. (11). Since the process is adiabatic, we have $dQ = 0$, so that for the *real* process

$$\int_2^3 \frac{dQ}{T} = 0$$

independently of how much the instantaneous temperature which prevails during the turbulent transition departs from the isotherm (shown dotted in the sketch, Fig. 3). On the other hand for the imaginary *reversible* process, which we may select along the isotherm, we have $dU = 0$, $dQ_{rev} = dU + p dV = p dV$, so that per mol of gas flowing:

$$\Delta S = S_3 - S_2 = \int_2^3 \frac{p dV}{T} = R \int_2^3 \frac{dv}{v} = R \log \frac{v_3}{v_2}.$$

We shall, obviously, find the same value if we integrate along 23 + 31 in Fig. 3 instead of the isotherm, as can easily be verified. Attention may be drawn to the fact that the preliminary evaluation of entropy in eq. (3.10) made use of the heat added reversibly in the sense of the preceding eq. (10 a), and the same is true of the van der Waals gas in Sec. 9 B.

Our example shows clearly that the existence and the value of entropy at the final state depend only on the state itself and not on whether it has been reached reversibly or irreversibly. Its value, denoted here by S_3 , is determined except for a constant, denoted here by S_2 .

In connection with the transition from eq. (10 b) to (10 c) we wish to remark that it implies that *partial entropies are additive*. This is usually assumed in classical thermodynamics, but from the higher point of view of statistical mechanics it is not necessarily so, see Sec. 31 A.

A system will be termed *isolated* when it does not interact with the surroundings, i. e. when no heat is transferred and no work is performed. The *energy* of such a system is constant, because $dQ = 0$ and $dW = 0$. According to (11) the *entropy* of such a system would also be constant:

$$(12) \quad S_B = S_A.$$

This is a paradox and it seems to contradict the second part of the Second Law. The reason for it lies in the too narrow interpretation which we gave to the concept of a "thermodynamic system" in eq. (10 b) and following, because we have excluded all irreversible interactions between the components of the system thus implying thermodynamic equilibrium. It was in fact necessary to do so to calculate the entropy difference in (11). The proposition contained in (12), *namely that the entropy of an isolated system is constant, the system being in thermal equilibrium*, is true only under the above restrictive assumption.

C. THE SECOND PART OF THE SECOND LAW

We now assume that of the two engines **E** and **E'** considered in Section A one, say **E'**, is *not reversible*. In this case we can still achieve the mode of operation assumed in (3) when **E** driven by **E'** worked as a refrigerator between the same temperatures as **E'**, and can thus prove the impossibility of $\eta' > \eta$, but the reverse operation is not possible. Instead of eq. (4) we now have, therefore,

$$(13) \quad \eta > \eta'$$

because $\eta = \eta'$ is also excluded by the assumption that **E'** is irreversible. The reversible Carnot cycle has a larger efficiency than an irreversible Carnot cycle which operates between the same temperatures and produces the same power. The latter is less economical than the former; it requires a greater expenditure of fuel for the same power: $Q_1' > Q_1$.

Retaining the definition of absolute temperature given in (8) and in accordance with (8 a) and (2) we conclude from $1 - \eta < 1 - \eta'$ that

$$T_2/T_1 = Q_2/Q_1 < Q_2'/Q_1'$$

and consequently

$$(13\ a) \quad \frac{Q_1'}{Q_2'} < \frac{T_1}{T_2}.$$

For an infinitely narrow Carnot diagram we have

$$\frac{dQ_1'}{T_1} < \frac{dQ_2'}{T_2}$$

instead of (8 b). Following the same reasoning as before (namely by subdividing into infinitely many cycles) we can prove that for an arbitrary cycle which is partly or wholly irreversible we must have

$$(14) \quad \oint \frac{dQ'}{T} < 0,$$

if rejected heat quantities, dQ_2' etc., are considered negative. We now split this cycle into two segments, $A \rightarrow B$ and $B \rightarrow A$ and we assume that segment $B \rightarrow A$ consists of infinitesimal reversible processes only, whereas all irreversible processes take place along $A \rightarrow B$. Applying eq. (11) to the segment $B \rightarrow A$ we can rewrite eq. (14) to read

$$\int_A^B \frac{dQ'}{T} + S_A - S_B < 0$$

or

$$(15) \quad S_B - S_A > \int_A^B \frac{dQ'}{T}.$$

This inequality applies to any kind of system. To be consistent we can now lift the restriction placed in connection with (10 c) and allow irreversible processes to take place within it. Consequently dQ' can be represented as the sum of dQ_e (heat introduced from the outside) and dQ_i (irreversible transfer of heat within the system). For an *isolated system* ($dQ_e = 0$) we can write now

$$(15\ a) \quad S_B - S_A > \sum \int_A^B \frac{dQ_i}{T}.$$

For each individual process dQ_i the integral in (15 a) is positive because dQ_i appears twice, once as a positive quantity, and once as a negative quantity, the denominator being smaller in the former case (an example is afforded

by the porous plug experiment or by the conduction of heat under a finite temperature difference). Thus *a fortiori*

$$(16) \quad S_B > S_A.$$

The entropy of an isolated system can only increase. The Second Law ascribes a definite direction to natural phenomena which was absent from the mechanistic point of view.

In order to clarify the conflicting statements in eqs. (16) and (12) we shall introduce the concept of *retarded equilibrium*. We shall assume that state *A* is one of equilibrium both in (16) and in (12), but we shall suppose that it contains different components which can be prevented from interacting with each other by the application of artificial devices. In this connection we may think of an impermeable wall which separates two gaseous phases and prevents their mixing. If such a wall is removed, it being possible to do so with an arbitrarily small expenditure of work (e. g. the opening of a valve, the closing of an electric contact), an irreversible process will set in and it will continue until a new state of equilibrium, *B*, has been reached. We may also think of two substances which cannot combine chemically under the conditions prevailing at *A* but which can be prompted to react in the presence of a catalyzer. The latter takes no part in the transfer of energy but it makes an irreversible chemical process possible so that transition to a new state of equilibrium, *B*, takes place. The Second Law makes no statements about the details of such processes but it enables us to calculate the change in entropy on transition from equilibrium state *A* to equilibrium state *B*. In order to do this it suffices to think of any *reversible* process which leads from *A* to *B* and to form the integral

$$S_B - S_A = \int_A^B \frac{dQ_{rev}}{T};$$

its value is independent of the particular choice of the reversible path between *A* and *B*.

The conditions of equilibrium at state *B* in eq. (16) are now different from those at state *A*. In this manner an irreversible retarded process at *A* becomes possible and a new state of equilibrium *B* with $S_B > S_A$ can be reached.

D. SIMPLEST NUMERICAL EXAMPLES

According to (8 a), the efficiency of a perfect steam engine would be

$$(17) \quad \eta = \frac{T_1 - T_2}{T_1} = \frac{80}{373} = 22\%,$$

if it operated on a Carnot cycle. We have assumed here that $T_1 = 100^\circ\text{C}$ and that $T_2 = 20^\circ\text{C}$. The real indicator diagram of a steam engine is not identical with the Carnot cycle (Fig. 1 compared with Fig. 5); nevertheless, the high-pressure line in Fig. 1 is identical with the isotherm of boiling water and the low-pressure line is approximately that of atmospheric temperature. The indicated limit of 22% is reached approximately in modern designs but it can never be exceeded.

If we assume that T_2 is kept constant in (17), we find that η increases with an increasing temperature T_1 .

Superheated steam (locomotive) is more effective than steam at a normal boiling temperature. However, there are certain limits to the magnitude of pressure used in engineering practice. For this reason the work of developing mercury vapor turbines was initiated in the USA many years ago. By coupling a mercury vapor turbine with a steam turbine it is possible to obtain a unit which operates between 535°C and 35°C . This corresponds to an ideal efficiency of

$$\eta = 62\%.^1$$

The diesel engine (ignition temperature 400°C) operates at a larger temperature difference than the steam plant and has a considerably larger ideal efficiency. We remark here that the efficiency of a diesel cycle cannot be inferred directly from a Carnot cycle because its indicator diagram differs too much from the latter.

It is possible to state generally: Heat at a higher temperature is more precious than at a lower temperature. Work can be regarded as being equivalent to heat at an infinitely high temperature.

An efficiency of $\eta = 100\%$ could be attained in a prime mover if it were possible to maintain absolute zero in its cooler. We shall discover in Sec. 12 that, strictly speaking, such an efficiency cannot be achieved.

¹These remarks are not quite relevant at the present time, as modern engineering practice has developed means of handling steam at very high (near-critical and super-critical) pressures and temperatures exceeding the 535°C quoted. (*Transl.*).

At this stage we shall interpose a remark of a historical nature. Written in terms of our original θ temperature scale, and assuming an infinitely small temperature difference $\theta_1 = \theta$ and $\theta_2 = \theta - d\theta$, the efficiency from eqs. (2) and (7) is given by

$$(17\ a) \quad \eta = 1 - \frac{\phi(\theta - d\theta)}{\phi(\theta)} = \frac{\phi'(\theta)}{\phi(\theta)} d\theta = C(\theta) d\theta.$$

The function $C(\theta) = \phi'(\theta)/\phi(\theta)$ is designated as "Carnot's function" in older references. The same designation in the absolute temperature scale belongs to the function

$$(17\ b) \quad C(T) = \frac{1}{T}.$$

A slightly unexpected application of the Second Law to the derivation of algebraic inequalities is contained in Problem 4.

We shall refrain here from discussing the application to the universe which was already given by Clausius and which predicts its "thermal death." The increase in entropy is supposed to equalize all temperature differences so that the performance of work will supposedly become impossible. We think that the nature of the universe, i. e. whether it is open or closed, expanding (even pulsating!) or stationary is still too much in doubt to enable us to discuss this problem.

Planck¹ opposes (and rightly so) the view of certain physicists that the essence of the Second Law consists in the statement that energy tends to degrade. Evidently an increase in entropy causes in many cases a decrease in the available temperature difference and hence also in the availability of work. Planck quotes the obvious example in which heat is transformed into work completely, namely the example of an isothermal expansion of a perfect gas with heat transfer from a reservoir of higher temperature and with complete utilization of the pressure of the gas to perform work. In this process energy is not degraded but quite to the contrary, it is ennobled (heat completely transformed into work).

In our and in Planck's opinion, the essence of the Second Law consists in the existence of entropy and in the impossibility of its decreasing under well defined conditions.

¹"Theorie der Wärme", Vol. V of Einführung in die theoretische Physik, Sec. 36, or "Thermodynamik", Sec. 108. See also engl. transl. "Theory of Heat", Vol. V of "Introduction to Theoretical Physics", p. 50, or "Treatise on Thermodynamics", 3rd. ed. p. 81.

E. REMARKS ON THE LITERATURE OF THE SECOND LAW

Our proof of the Second Law was based on that due to Clausius.¹ The proof given by Planck (*l. c.*) is perhaps simpler and certainly more precise; it is, however, more abstract and less instructive than ours. The proof given by Carathéodory² is even more abstract and, at the same time, simpler if the simplicity of a proof is judged by the small number of assumptions required. In fact, using a system of two fluids which can be separated by a heat conducting or a heat insulating wall, as required, Carathéodory needs only the assumption: *In the neighborhood of every state which can be reached reversibly there exist states which cannot be reached along a reversible adiabatic path, or, in other words, which can only be reached irreversibly or which cannot be reached at all.*

This exceedingly economical postulate suffices to provide a mathematical proof of the existence of the property known as entropy.

We shall now quote the point of view which was adopted by Carathéodory in his inaugural address to the Prussian Academy:³ "It is possible to ask the question as to how to construct the phenomenological science of thermodynamics when it is desired to include only directly measurable quantities, that is volumes, pressures, and the chemical composition of systems. The resulting theory is logically unassailable and satisfactory for the mathematician because, starting solely with observed facts, it succeeds with a minimum of hypotheses. And yet, precisely these merits impede its usefulness to the student of nature, because, on the one hand, temperature appears as a derived quantity, and on the other, and above all, it is impossible to establish a connection between the world of visible and tangible matter and the world of atoms through the smooth walls of the all too artificial structure."

In connection with the last question Planck⁴ makes the casual remark: It is true that the First Law applies to 10 molecules enclosed in a fixed volume, but with the aid of such a system it is impossible to build a heat engine owing to excessive fluctuations. Applied to such a system the Second Law loses its sense. Carathéodory's proof does not exclude such systems in advance; it requires additional restrictive assumptions in order to adapt itself to reality.

¹R. Clausius: *Mechanische Wärmetheorie*, 1876. 2nd. ed. of "Abhandlung über mechanische Wärmetheorie".

²C. Carathéodory, *Math. Ann.* 67, 1909 and Prussian Academy, Jan. 1925. Reference should also be made to M. Born: *Natural Philosophy of Cause and Chance*, Oxford 1949, who, in Carathéodory's own judgement, has given a particularly clear presentation of his method.

³*Sitzungsberichte* of 3rd. July 1919, No. XXXIII.

⁴*Ibid.* 1921, p. 453.

In order to obtain at least an approximate idea of Carathéodory's method let us consider the two fluid systems already mentioned, namely Σ_1 and Σ_2 , whose states will be described by pressure, volume and an additional parameter θ . These properties are related through an equation of state each of which we can write in the form:

$$\theta_1 = F_1(p_1, V_1); \quad \theta_2 = F_2(p_2, V_2).$$

Let the two systems be brought into thermal contact which may be defined by stipulating $\theta_1 = \theta_2$. The thermally coupled system $\Sigma = \Sigma_1 + \Sigma_2$ satisfies the equation

$$F_1(p_1, V_1) = F_2(p_2, V_2),$$

so that of the four variables only three are independent. We can choose them arbitrarily and we may denote them by x , y , and z . The First Law states then that the quantity of heat added to the system during a reversible process is given by Pfaff's differential expression (see Sec. 1):

$$(18) \quad dQ = X dx + Y dy + Z dz,$$

where X , Y , and Z denote functions of x , y , z . Generally speaking dQ is not a perfect differential, as seen from eq. (4 a) in Sec. 1. The corresponding Pfaff differentials for sub-systems Σ_1 , Σ_2 containing two variables each can always be transformed into perfect differentials by the adoption of an integrating denominator, as also mentioned in Sec. 1. From this, together with Carathéodory's postulate, it is possible to conclude that the expression (18) also possesses an integrating denominator (in fact a family of them). This proves the existence of absolute temperature and of entropy and the fact that they are properties.

F. ON THE RELATIVE RANK OF ENERGY AND ENTROPY

We quote here a note¹ by Robert Emden whose deep understanding of thermodynamics has withstood the test of time in fundamental papers on astrophysics (gaseous spheres!) and meteorology (grey atmosphere): "Why do we have Winter Heating? The layman will answer: 'To make the room warmer.' The student of thermodynamics will perhaps so express it: 'To impart the lacking (inner, thermal) energy.' If so, then the layman's answer is right, the scientist's wrong."

¹Nature, Vol. 141, May 1938, p. 908 entitled: Why do we have Winter Heating?

"We suppose, to correspond to the actual state of affairs, that the pressure of the air in a room always equals that of the external air. In the usual notation, the (inner, thermal) energy is, per unit mass,

$$u = c_v T.$$

(An additive constant may be neglected). Then the energy content is, per unit of volume,

$$(19) \quad u_1 = c_v \rho T$$

or, taking into account the equation of state, we have

$$(20) \quad u_1 = c_v \mu p / R.$$

The energy content of the room is thus independent of the temperature, solely determined by the state of the barometer. The whole of the energy imparted by the heating escapes through the pores of the walls of the room to the outside air."

"I fetch a bottle of claret from the cold cellar and put it to be tempered in the warm room. It becomes warmer, but the increased energy content is not borrowed from the air of the room but is brought in from outside."

"Then why do we have heating? For the same reason that life on the earth needs the radiation of the sun. But this does not exist on the incident energy, for the latter, apart from a negligible amount, is re-radiated, just as a man, in spite of continual absorption of nourishment, maintains a constant body-weight. Our conditions of existence require a determinate degree of temperature, and for the maintenance of this there is needed not addition of energy but addition of entropy."

"As a student, I read with advantage a small book by F. Wald entitled 'The Mistress of the World and her Shadow'. These meant energy and entropy. In the course of advancing knowledge the two seem to me to have exchanged places. In the huge manufactory of natural processes, the principle of entropy occupies the position of manager, for it dictates the manner and method of the whole business, whilst the principle of energy merely does the book-keeping, balancing credits and debits."

Numerical examples and critical remarks are given in Problem 2.

7. The thermodynamic potentials and the reciprocity relations

We have at our disposal two pairs of variables

$$p, v \quad \text{and} \quad T, s$$

for each simple, homogeneous system (possessing one mechanical and one thermal degree of freedom, e. g. a gas, a vapor, or a liquid). The First Law expressed in terms of them, whether per mol or per unit mass, see Sec. 5 D, eq. (11), has the form

$$(1) \quad du = T ds - p dv.$$

The two "extensive" quantities, s, v , are the independent variables, the two "intensive" quantities T, p being conjugate to them. The internal energy, u , is to be regarded as a function of the variables s, v :

$$u = u(s, v).$$

According to (1) the remaining two variables are given by

$$(2) \quad T = \left(\frac{\partial u}{\partial s} \right)_v, \quad -p = \left(\frac{\partial u}{\partial v} \right)_s.$$

However the selection of independent variables is largely a matter of free choice. There are four possibilities of making such a choice with one mechanical and one thermal variable in the pair:

$$(3) \quad s, v; \quad s, p; \quad T, v; \quad T, p.$$

At this point we recall the Legendre transformation whose great importance for analysis, for mechanics and for thermodynamics has already been stressed in Vol. I, Sec. 42. It gives us the rule: *If it is desired to replace one of the independent variables (e. g. s) in a Pfaff differential of the form (1) by its conjugate, it is necessary to subtract from the dependent variable (in our case u) the product of the two conjugate independent variables (in our case $T s$).*

A corresponding rule applies when it is desired to substitute both initial variables and when there are more variables than two. In this manner there are four expressions associated with the four possibilities (3), namely

$$(4) \quad \begin{array}{cccc} u(s, v); & h(s, p) = u + p v; & f(T, v) = u - T s; & g(T, p) = u - T s + p v. \\ \text{energy} & \text{enthalpy} & \text{free energy} & \text{free enthalpy} \end{array}$$

The expression for h , taken per mol or per unit mass, corresponds to the quantity H introduced previously in eq. (4.9). The remaining symbols and definitions are summarized in Table (6) below.

We shall now clarify the usefulness of the preceding definitions in a purely formal way by forming the corresponding differentials and by substituting du from (1):

$$(5) \quad \begin{cases} dh = du + p dv + v dp = T ds + v dp, \\ df = du - T ds - s dT = -p dv - s dT, \\ dg = du - T ds - s dT + p dv + v dp = -s dT + v dp. \end{cases}$$

The last terms show that the differentials dh , df , dg have the same simple form when expressed in terms of the independent variables associated with them as du has when it is expressed in terms of the variables s and v . The expressions (4) are known as the *thermodynamic potentials* because the variables can be deduced from them by differentiation with respect to the independent variable associated with them in the same way as the components of a force are derived from a force potential. The same designation can be properly applied to energy as seen from eq. (2). The corresponding equations for the potentials h , f , and g will be found in the Table. We should like to stress here that the choice of a potential determines the choice of the associated variables. For example the free energy f possesses the properties of a potential with respect to the variables v , T ; it loses this property with a different choice of variables.

From the representation in (2) and from the analogous expressions given in the Table there follow the most important and significant thermodynamic relations summarized in the fourth column of our Table.

We shall begin by considering the third of these relations:

$$(7) \quad \left(\frac{\partial p}{\partial T} \right)_v = \left(\frac{\partial s}{\partial v} \right)_T.$$

Substituting $ds = dq/T$, we obtain

$$(7a) \quad \left(\frac{\partial p}{\partial T} \right)_v = \frac{1}{T} \frac{dq}{dv} \bigg|_{T = \text{const}}.$$

The left-hand side is the coefficient of tension, β , from eq. (1.5), except for the p in the denominator. The second factor on the right-hand side is the "isothermal heat of expansion"¹ which must be introduced during the expansion to maintain a constant temperature.

¹The symbol M was used to denote it in older papers and it was measured in calories. Introducing the Carnot function $C(T)$ for $1/T$, eq. (6.17 a) and denoting the mechanical equivalent of heat by J , we find that the right-hand side reads $J.C.M$. This is the reason for which James Clerk Maxwell used dp/dt as his pen name. (Mnemonic rule for examinees who were asked for Clapeyron's equation.)

(6)

Table

Potential	Independent variables	Conjugate variables	Thermodynamic relations	Definitions and notation
U, u	v, s $du = T ds - p dv$	$T = \left(\frac{\partial u}{\partial s}\right)_v$ $p = -\left(\frac{\partial u}{\partial v}\right)_s$	$\left(\frac{\partial T}{\partial v}\right)_s = -\left(\frac{\partial p}{\partial s}\right)_v$ $= \frac{\partial^2 u}{\partial v \partial s}$	Energy u (Clausius) ϵ (Gibbs)
H, h $h = u + p v$	p, s $dh = T ds + v dp$	$T = \left(\frac{\partial h}{\partial s}\right)_p$ $v = \left(\frac{\partial h}{\partial p}\right)_s$	$\left(\frac{\partial T}{\partial p}\right)_s = \left(\frac{\partial v}{\partial s}\right)_p$ $= \frac{\partial^2 h}{\partial p \partial s}$	Enthalpy H (Lewis and Randall) X (Gibbs) J (Heat engineering in certain countries)
F, f $f = u - T s$	v, T $df = -s dT - p dv$	$s = -\left(\frac{\partial f}{\partial T}\right)_v$ $p = -\left(\frac{\partial f}{\partial v}\right)_T$	$\left(\frac{\partial s}{\partial v}\right)_T = \left(\frac{\partial p}{\partial T}\right)_v$ $= -\frac{\partial^2 f}{\partial v \partial T}$	Free energy F (Helmholtz) ψ (Gibbs)
G, g $g = h - T s$ $= f + p v$ $= u - T s + p v$	p, T $dg = -s dT + v dp$	$s = -\left(\frac{\partial g}{\partial T}\right)_p$ $v = \left(\frac{\partial g}{\partial p}\right)_T$	$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p$ $= -\frac{\partial^2 g}{\partial p \partial T}$	Free enthalpy ζ (Gibbs), also called thermodynamic potential

It is remarkable that the relation (7 a) which we have justified for homogeneous systems contains a statement which is true for the transition between *two* homogeneous systems, namely two different phases. Equilibrium between water and steam is of particular interest. Equation (7 a) becomes identical with the famous Clapeyron equation which played such an important role in the development of the steam engine (see Sec. 16) if we interpret p as the vapor pressure at temperature T and replace $(dq/dv)_T$ by $\Delta q/\Delta v$, where now Δq denotes the heat of evaporation per mol (or per unit mass).

At present eq. (7) is used to obtain a remarkable formula for $c_p - c_v$ of general validity. Assuming that T is kept constant we can obtain from the First Law written in form (1) the expression

$$\left(\frac{\partial u}{\partial v}\right)_T = T \left(\frac{\partial s}{\partial v}\right)_T - p.$$

In view of (7) we have

$$(8) \quad \left(\frac{\partial u}{\partial v}\right)_T + p = T \left(\frac{\partial p}{\partial T}\right)_v.$$

On the other hand the First Law can be written as:

$$dq = du + p dv = \left\{ \left(\frac{\partial u}{\partial v}\right)_T + p \right\} dv + \left(\frac{\partial u}{\partial T}\right)_v dT,$$

and at constant v we have

$$(8 a) \quad c_v = \left. \frac{dq}{dT} \right|_{v=\text{const}} = \left(\frac{\partial u}{\partial T}\right)_v$$

whereas at constant p we obtain

$$(8 b) \quad c_p = \left. \frac{dq}{dT} \right|_{p=\text{const}} = \left\{ \left(\frac{\partial u}{\partial v}\right)_T + p \right\} \left(\frac{\partial v}{\partial T}\right)_p + \left(\frac{\partial u}{\partial T}\right)_v.$$

Subtracting (8 b) and (8 a) we find

$$(8 c) \quad c_p - c_v = \left\{ \left(\frac{\partial u}{\partial v}\right)_T + p \right\} \left(\frac{\partial v}{\partial T}\right)_p$$

and from (8)

$$(9) \quad c_p - c_v = T \left(\frac{\partial p}{\partial T}\right)_v \left(\frac{\partial v}{\partial T}\right)_p.$$

The last two factors represent the "coefficient of tension β " and the "coefficient of thermal expansion α ," respectively, provided that the factors p and v appearing in the denominator of eq. (1.) are taken into account. Thus eq. (9) can also be written in the form

$$(9 a) \quad c_p - c_v = \alpha \beta v p T$$

For a perfect gas we had $\alpha = \beta = 1/T$ and eq. (9 a) takes on the form

$$(9 b) \quad c_p - c_v = \frac{pv}{T} = \begin{cases} R & \text{for one mol} \\ R/\mu & \text{for a unit of mass,} \end{cases}$$

as it should, in accordance with Sec. 3.

It will be noted that in deducing eq. (9 b) for the perfect gas there was no need to make use of the additional caloric condition of Sec. 4 A which stated that the internal energy of a perfect gas depended only on its temperature. This is due to the fact that the caloric condition does not really constitute a new requirement imposed on the gas but represents a property of the perfect gas which is a consequence of the Second Law. In order to see this it is sufficient to express the pressure on the right-hand side of eq. (8) in terms of the equation of state of a perfect gas; we then have $(\partial p/\partial T)_v = p/T$; hence eq. (8) leads to $(\partial u/\partial v)_T = 0$ which means that the internal energy is independent of the volume, being a function of temperature alone.

We now propose to examine more closely the last thermodynamic relation in our Table. Replacing once more ds by dq/T we obtain

$$(10) \quad \left(\frac{\partial v}{\partial T} \right)_p = - \frac{1}{T} \frac{\partial q}{\partial p} \bigg|_{T=\text{const}}$$

The left-hand side is the product of the coefficient of thermal expansion and volume. The last term on the right-hand side is known as the "isothermal heat of compression". It is, generally speaking, *negative*, which means that heat must be rejected if the system is to be maintained at the same temperature at a higher pressure. If this were not so it would become heated on compression. Correspondingly α is, generally speaking, *positive* (two negative signs in (10) cancel each other). There are, however, exceptions. The best known exception is water between 0 C and 4 C. Equation (10) shows that in this interval the heat of compression is *positive*: it is necessary to add heat in order to prevent the water from cooling on compression. (See also Problem I.6) The same is true of *raw rubber* and *silver iodide* in certain temperature intervals. The anomaly of water has led Roentgen to suppose that water tends to polymerize in the neighborhood of its freezing point, the supposition having been confirmed later by others. Thus the process of crystallization which takes place at 0 C occurs to a certain extent before it.

All four thermodynamic relations were deduced by Maxwell in his "Theory of Heat," London, 1883, from an elementary geometrical figure; he also stated them in words. It is evident that he felt himself that in this case the differential representation is much simpler than the elementary treatment given in his text-book; for this reason he appended the analytical formulation contained in our Table in a remark to Chap. IX. An intuitive understanding of the signs in these reciprocity relations is contained in the *principle due to Braun and Le Chatelier* somewhat in the manner of Lenz's rule in electrodynamics;

it does not, however, attain the same degree of precision as that possessed by the statements in our Table.¹

The impressive regularity of our Table is due to the great student of thermodynamics and statistical mechanics – Willard Gibbs. His papers, which were at first buried in the Transactions of the Connecticut Academy of 1876 and 1878, became generally known only after Ostwald published them in German in 1902 under the title “*Thermodynamische Studien*.” Adopting Gibbs’ point of view we consider that the “four potentials” u, h, f, g , or U, H, F, G , are equivalent, the choice between them depending on the choice of independent variables, eq. (3). We have already stressed in eq. (5.7) that the simplest formulation of the theory of the Joule-Kelvin *porous plug experiment* is obtained in terms of *enthalpy*, H , which is equal on both sides. In relation to phase equilibria the same simplification is achieved by the use of the *free enthalpy*, G . The *free energy*, F , is the principal potential in physical chemistry and in electrochemistry. It furnishes a measure of *chemical affinity*. Planck prefers, as a rule, to use the “*potential function*”

$$\Phi = -\frac{G}{T} = S - \frac{U + pV}{T},$$

which is in fact convenient in problems involving statistical questions; it does not, however, fit Gibbs’ beautiful system.

8. Thermodynamic equilibria

A. UNCONSTRAINED THERMODYNAMIC EQUILIBRIUM AND MAXIMUM OF ENTROPY

We have found in Sec. 6 C that the entropy of an isolated system cannot decrease. A system was called isolated when it absorbed no heat and performed no work. These conditions are equivalent to stating that the internal energy U and the volume V are kept constant ($dU = 0, dV = 0$). An isolated system will tend to a final state at which the entropy has a maximum if all constraints within the system are removed. We shall call this a state of unconstrained thermodynamic equilibrium.

¹In this connection see: P. Ehrenfest, Z. Phys. Chem. 77, 1911; Planck, Ann. d. Phys. 19, 1934 with an Appendix *ibid* 20, 1935; further Mrs. Tatiana Ehrenfest and Mrs. de Haas-Lorentz, Physica 2, 1935 with reply from Planck, *ibid*. The main problem during this discussion was the distinction between intensive and extensive properties which is of paramount importance for an unambiguous formulation of the principle of Braun-Le-Chatelier.

A process which starts spontaneously from a state of unconstrained equilibrium is impossible; if this were not so the entropy would have to increase again in contradiction to our assumption that the entropy already has a maximum value. We can, however, consider virtual processes δ which are compatible with the restrictions $dU = 0, dV = 0$ and which cannot, evidently, occur spontaneously. (Example: let a vessel be filled with a gas whose pressure and temperature are constant; we now let half of the gas be heated to a temperature $T + \delta T$, the other half being cooled to $T - \delta T$.) Such virtual changes of state from unconstrained thermodynamic equilibrium satisfy the relations

$$(1) \quad \delta S \leq 0 \quad \text{when} \quad \delta U = 0; \quad \delta V = 0,$$

or, in another form

$$(1 \text{ a}) \quad S = S_{\max} \quad \text{when} \quad U = \text{const}, \quad V = \text{const}.$$

If it were possible to indicate a process with $\delta U = 0, \delta V = 0$ for which $\delta S > 0$, we would conclude that the initial state was not one of unconstrained equilibrium. We would conclude that there existed constraints whose removal caused the entropy to increase further. Equation (1), or eq. (1 a), constitutes one of the two conditions of equilibrium established by Gibbs. The second, which is less important to us, has the form

$$(2) \quad \delta U \geq 0 \quad \text{when} \quad \delta S = 0, \quad \delta V = 0.$$

The condition

$$(2 \text{ a}) \quad U = U_{\min} \quad \text{when} \quad S = \text{const}, \quad V = \text{const}.$$

is equivalent to it. *At a state of equilibrium the internal energy assumes a minimum value.* This last proposition is reminiscent of the criterion for equilibrium in general mechanics which requires the potential energy to assume a minimum value (see e. g. Vol VI Sec. 25).

B. AN ISOTHERMAL AND ISOBARIC SYSTEM IN UNCONSTRAINED THERMODYNAMIC EQUILIBRIUM

It follows from the characteristics of the state of an unconstrained equilibrium in eq. (1) that the pressure and temperature throughout the system are independent of space coordinates. If this were not so we could select two elements of space in which the temperatures were T_1 and T_2 , say, the corresponding pressures being p_1 and p_2 . We could now assume a virtual process during which the energy of the first element changed by δU_1 its

volume changing by δV_1 . The corresponding variations for the second element of space would then be $\delta U_2 = -\delta U_1$, $\delta V_2 = -\delta V_1$ if the conditions in (1) were to be satisfied. Changes in concentration or of the masses contained in the individual phases present will now be excluded. According to (1) we must then have

$$\begin{aligned} 0 \geq \delta S &= \delta S_1 + \delta S_2 = \frac{1}{T_1} (\delta U_1 + p_1 \delta V_1) + \frac{1}{T_2} (\delta U_2 + p_2 \delta V_2) = \\ &= \left(\frac{1}{T_1} - \frac{1}{T_2} \right) \delta U_1 + \left(\frac{p_1}{T_1} - \frac{p_2}{T_2} \right) \delta V_1. \end{aligned}$$

The virtual changes δU_1 and δV_1 are arbitrary and independent of each other. The above inequality cannot, therefore, be satisfied for $T_1 \neq T_2$, $p_1 \neq p_2$, if δU_1 , δV_1 are to have any values, contrary to our assumption.

C. ADDITIONAL DEGREES OF FREEDOM IN RETARDED EQUILIBRIUM

The state of a system which is in unconstrained thermodynamic equilibrium is often specified by indicating the internal energy, U , the volume, V , and the masses of the independent components (cf. Sec. 14). We shall now consider a system Σ which is not yet in equilibrium. Its state can only be determined if in addition to U , V and the masses of the independent components we specify further quantities x_i ; these may denote, for example, the distribution of the independent components over the phases and the concentrations of the individual components which can interact chemically; they may, furthermore, describe local differences if the system is subdivided into sufficiently small elements of volume, and if the preceding quantities are specified for each element. We shall consider only states of disequilibrium of a kind, and this is an essential assumption for Sec. 21, which can be interpreted as states of constrained equilibrium, in which the x_i are kept constant, so that the *entropy of the system* may be taken to be *the sum of the entropies of all volume elements for such a state of constrained equilibrium*. We shall restrict ourselves to the consideration of isothermal and isobaric systems. On transition from constrained equilibrium U, V, x_i to the constrained equilibrium $U + dU, V + dV, x_i + dx_i$ the change in the entropy must be calculated from

$$(3) \quad T dS = dU + p dV + \sum_i X_i dx_i,$$

which is a generalization of eq. (7.1). The kind of process taking place, whether reversible or irreversible, is here of no importance, because dS denotes

the difference between the entropy of the final and initial state, both (only infinitesimally different) being constrained equilibria. The coefficients X_i will be called forces associated with the additional degrees of freedom x_i .

D. EXTREMUM PROPERTIES OF THE THERMODYNAMIC POTENTIALS

We shall now add to Σ a "surroundings" which we may imagine in the form of a very large heat reservoir, Σ_0 . All quantities referring to Σ_0 will be denoted by the subscript 0 ; the combined system consisting of Σ and Σ_0 will be assumed to be thermally insulated; under these assumptions the total entropy cannot decrease:

$$(4) \quad dS + dS_0 > 0.$$

The sign of equality will be excluded by assuming that the *changes* of state which take place in Σ are irreversible like all real processes. As already mentioned, we assume that the system Σ is isobaric and isothermal which means that it is in mechanical and thermal equilibrium but not necessarily in chemical or phase equilibrium. The transfer of heat between Σ and Σ_0 will be allowed only on condition that Σ has the temperature T_0 of the heat reservoir. We assume that the system Σ_0 is so large that it can exchange *reversibly*, i. e. absorb or reject, a quantity of heat with Σ without markedly changing its own temperature. Thus

$$(5) \quad dS_0 = \frac{dQ_0}{T_0}.$$

The changes in volume in Σ are assumed to take place at a pressure p which is always equal to that outside. Applying the First Law to Σ and to the quantity of heat $dQ = -dQ_0$ transferred to Σ from Σ_0 , we obtain

$$(6) \quad dU + p dV = -dQ_0.$$

Consequently

$$(7) \quad dS > \frac{1}{T} (dU + p dV)$$

because, according to our assumption, $T = T_0$ when there is a flow of heat, and eq. (7) follows from (4), (5) and (6). If, however, $T \neq T_0$ then there is no exchange of heat, $dU + p dV$ vanishes according to the First Law, and (7) states simply that $dS > 0$, as already derived in (6.16) for an isolated system. Hence eq. (7) is true for any process involving the transfer of heat and the performance of work by our system.

From (3) and (7) we infer that for any process dU , dV , dx_i which can take place spontaneously we must have

$$(8) \quad \sum_i X_i dx_i > 0.$$

In the case of *reversible* processes of the combined system $\Sigma + \Sigma_0$ and of the system Σ alone we must have a sign of equality in eq. (4), which leads to a sign of equality in (8). We can now make the following statement:

The necessary and sufficient conditions for a process in an isothermal and isobaric system Σ to be reversible, are: heat must be exchanged with the surroundings in a reversible way (i. e. at a temperature T of Σ equal to that of the surroundings), the internal pressure p must be equal to the external pressure and, in addition, we must have

$$(9) \quad \sum_i X_i dx_i = 0$$

during the whole process.

The latter condition is satisfied, for example, when all x_i are kept constant. It is also satisfied when non-vanishing x_i 's are associated with vanishing X_i 's. Any transition from state 1 with U_1 , V_1 , x_{i1} to a state 2 with U_2 , V_2 , x_{i2} can be conducted in many different ways and always so that condition (9) is satisfied during the whole process. An application of this rule to an actual example is given in Section E.

Let us now consider an isothermal system of fixed temperature and volume (e. g. a system immersed in a bath of constant temperature T); the differential of free energy is then $dF = dU - T dS$. If the changes in F , U , S refer to a spontaneous process we can apply eq. (7) with $dV = 0$ and we obtain

$$(10) \quad dF \leq 0.$$

*The free energy decreases.*¹ There exists a minimum of free energy beyond which no spontaneous changes of state are possible. The condition of equilibrium of the present system is

$$(11) \quad F = F_{min} \quad \text{when} \quad T = \text{const}, \quad V = \text{const}.$$

¹The condition that the pressure is constant throughout the system (see C *infra*) is not required for the validity of this result because the existence of any differences in pressure between different parts of the system has no influence on the result in (7) owing to $dV = 0$.

The condition that for any virtual process

$$(11\ a) \quad \delta F \geq 0 \quad \text{when} \quad \delta T = 0, \quad \delta V = 0$$

is equivalent to that in (11).

In the case of an isothermal-isobaric system (for example one immersed in a heat bath of temperature T and pressure p in a way that ensures thermal and static equilibrium with the system) we make use of the free enthalpy $G = U - TS + pV$. Its differential is $dG = dU - TdS + pdV$ because $dT = 0$, $dp = 0$. If the changes in G , U , S , V , refer to a spontaneous process we can apply eq. (7) once more, and we obtain

$$(12) \quad dG \leq 0.$$

The free enthalpy decreases. There is a minimum of *free enthalpy* beyond which spontaneous changes are no longer possible. The present condition of equilibrium is

$$(13) \quad G = G_{min} \quad \text{when} \quad T = \text{const}, \quad p = \text{const.}$$

The condition

$$(13\ a) \quad \delta G \geq 0 \quad \text{when} \quad \delta T = 0, \quad \delta p = 0$$

is equivalent to it.

This last statement is most important in the theory of processes which involve transformation of substances into different forms. It will be seen later that of the four potentials introduced in Sec. 7, the free enthalpy will prove, generally speaking, to be first in importance.

The proof of eqs. (2) and (2 a), as well as the derivation of an extremum property for the enthalpy H of an isobaric system under the conditions $\delta S = 0$, $\delta p = 0$ are left to the reader.

E. THE THEOREM ON MAXIMUM WORK

We now propose to calculate the quantity of work which a system Σ can perform on its surroundings when it passes reversibly from a state 1 of given temperature T_0 to another state 2 of equal temperature. We assume that the transition need not be isothermal but we postulate that Σ may exchange heat with its surroundings only at that temperature T_0 ; in other words, in ranges where $T \neq T_0$ the processes must be adiabatic and reversible.

Making use of the definition of free energy $F = U - TS$ we find that according to (3) we may write

$$(14) \quad dF = dU - T dS - S dT = -S dT - p dV - \sum_i X_i dx_i$$

for an elementary process. The requirement of reversibility means that the sum $\sum X_i dx_i = 0$ during the whole process of transition $1 \rightarrow 2$. Thus the quantity of work performed by the system becomes

$$\int_1^2 p dV = F_1 - F_2 - \int_1^2 S dT.$$

We now assert that

$$(15) \quad \int_1^2 p dV = F_1 - F_2$$

i. e. that the integral of $S dT$ vanishes under our assumptions. This results from the following argument: The transition $1 \rightarrow 2$ may contain isothermal components at temperature T_0 ; along these we have $dT = 0$. In addition it contains one or more adiabatic and reversible components each of which corresponds to a constant entropy and to a temperature T_0 which is the same at the beginning as at the end. Hence such an adiabatic segment satisfies the relation

$$\int_{T_0}^{T_0} S dT = S \int_{T_0}^{T_0} dT = 0.$$

If the process is irreversible the quantity of work performed is smaller than $F_1 - F_2$. If this were not the case we could bring the system to its initial state in a reversible way performing a quantity of work which is ≥ 0 at the expense of an equivalent quantity of heat from the surroundings at T_0 . This, however, contradicts the Second Law. For this reason the change in the *free energy* is also described as the *maximum work* which the system Σ can perform on its surroundings when it undergoes a process during which its initial and final temperature is equal to that of the surroundings, T_0 , on condition that it exchanges heat with the surroundings only at that temperature.

In many text-books on thermodynamics the above proposition on maximum work is stated on the additional restrictive assumption that the transition $1 \rightarrow 2$ is isothermal. Our formulation goes further because for some systems it is impossible to find a *reversible isothermal* process $1 \rightarrow 2$. In spite of that the maximum of work can be produced. Helmholtz's term "free energy" derives from eq. (15); its difference between two states 1 and 2 of equal temperature T_0 represents that portion of the change in energy, $U_1 - U_2$, which can be converted into external work (which is "available") during a reversible process $1 \rightarrow 2$, or that which it is necessary to perform externally on the system during the reverse process on condition that heat is exchanged with the surroundings only reversibly at a temperature T_0 . Consistently we can call $U - F = TS$ the "bounded", or "unavailable" energy.

We shall now show with the aid of a simple example how it is possible to perform the process $1 \rightarrow 2$ in a way which will ensure that $\sum_i X_i dx_i = 0$ along the whole path; the specific example has been chosen so that it is representative of the general case. For this purpose we shall consider a dissociating gas whose velocity of dissociation we can impede at will. Let x denote its degree of dissociation; its value at the state of unrestrained equilibrium will be denoted by $\bar{x}(T, V)$. Instead of (3) we can write $T dS = dU + p dV + X dx$. The initial state is given by T_0, V_1, x_1 , and the final state is denoted by T_0, V_2, x_2 . Of the two degrees of dissociation x_1 and x_2 at least one will be assumed to correspond to a deviation from equilibrium; otherwise the succeeding argument would become trivial. The first process is assumed to be adiabatic and reversible from T_0, V_1, x_1 at constant x_1 (i. e. $dx_1 = 0$) to such values T', V' for which $x_1 = \bar{x}(T', V')$. In other words we are asking for those values of T and V for which the prescribed value x_1 denotes the degree of dissociation in unconstrained equilibrium. We can now remove the restraint which kept x_1 constant without causing any further changes in the system. Next we perform an adiabatic reversible change from T', V' to such values T'', V'' as to effect a change from x_1 to $x_2 = \bar{x}(T'', V'')$. All along the path of change the equilibria are unconstrained so that according to (9) $X dx$ vanishes at all points (it is seen that $X = 0$ is the condition of dissociation equilibrium owing to $dx \neq 0$). We now maintain $x_2 = \text{const}$ and perform a further adiabatic and reversible change from T'', V'' until the initial temperature T_0 and a volume V''' are reached. Finally we perform an isothermal reversible process at constant x_2 during which V''' is changed to the desired volume V_2 . In the preceding example the whole process could have been performed isothermally because dissociation equilibrium depends on pressure. In order to achieve this we can perform a reversible isothermal

process changing the pressure p_0 at constant x_1 until $x_1 = \bar{x}(T_0, p')$ or, in other words, until dissociation equilibrium is reached at the prescribed value of x_1 . At this stage we again remove the constraint and change p' to p'' , where p'' is so chosen that $x_2 = \bar{x}(T_0, p'')$ and $X = 0$ during the process. Finally we again keep $x_2 = \text{const}$ and vary the pressure until the prescribed final volume V_2 has been reached.

In conclusion we shall make the following remark. The *maximum work* which can be made available during an elementary process with equal initial and final temperature T_0 is $\oint p dV + \sum_i X_i dx_i$. For this reason the preceding

expression is known as the generalized differential of work. It is a perfect differential and there exists a property, our free energy F at constant temperature T_0 , whose difference is equal to its integral.

9. The van der Waals equation

We shall now consider real gases, having so far devoted our attention to perfect gases, and we shall base our description on a dissertation entitled "The continuity of the gaseous and liquid states" published by van der Waals in Leiden in the year 1873.

In fact van der Waals succeeded in establishing an equation of state which reproduces qualitatively the process of liquefaction (condensation) of a gas and which introduces a quantitative correction into the equation of state of a perfect gas. Boltzmann¹ described van der Waals as the Newton of real gases.

Written for one mol, the equation established by van der Waals has the form:

$$(1) \quad p = \frac{RT}{v-b} - \frac{a}{v^2}.$$

The constant b introduced here is due to the volume of the molecules; the constant a is a measure for the forces of cohesion between the gaseous molecules and is connected with the capillarity of the free liquid surface. (The atomic significance of a and b is discussed in Sec. 26.) For $a = b = 0$, or, which amounts to the same, for sufficiently large v eq. (1) transforms into the perfect-gas equation, as it should. Instead of (1) we can also write

$$(1a) \quad (p + p_a)(v-b) = RT, \quad p_a = a/v^2.$$

¹Enzykl. der Mathem. Wiss., Vol. V. 1, p. 550.

The quantity p_a denotes the "cohesion pressure" which must be added to the "kinetic pressure" p . We shall begin by calculating the coefficient of thermal expansion from (1). Putting $dp = 0$ and differentiating (1), we obtain

$$0 = \frac{dT}{v-b} - \left(\frac{T}{(v-b)^2} - \frac{2a}{Rv^3} \right) dv,$$

and consequently

$$(2) \quad \alpha = \frac{1}{v} \left(\frac{\partial v}{\partial T} \right)_p = \frac{v-b}{v T - \frac{2a}{R} \left(\frac{v-b}{v} \right)^2}$$

which is a generalization of the value $\alpha = 1/T$ for a perfect gas and which can be obtained from (2) by substituting $a = b = 0$.

We shall also calculate the difference

$$(2a) \quad \alpha - \frac{1}{T} = \left\{ \frac{2a}{RT} \left(\frac{v-b}{v} \right)^2 - b \right\} / \left\{ v T - \frac{2a}{R} \left(\frac{v-b}{v} \right)^2 \right\}.$$

This can be simplified by retaining only the first powers of the parameters a, b implying $v \gg b, RTv \gg a$:

$$(2b) \quad \alpha - \frac{1}{T} = \left(\frac{2a}{RT} - b \right) / v T.$$

For most gases, e. g. O_2, N_2 , the right-hand side of (2) is positive in the range of ordinary temperatures, meaning that the coefficient of expansion is larger than that for the perfect gas. Hydrogen H_2 and the noble gases are the only exceptions. The forces of cohesion of these gases, as defined by a , are so small that the right-hand side of (2a) becomes negative at ordinary temperatures. For this reason H_2 was designated in the past as a "*gaz plus que parfait*."

A. COURSE OF ISOTHERMS

Figure 7 shows the course of the van der Waals isotherms. The v -axis and the straight line $v = b$ parallel to the p -axis are their asymptotes. Equation (1) has no physical meaning for $v < b$. According to (1) the points of intersection of an isobar $b = \text{const}$ with an isotherm $T = \text{const}$ are determined by a cubic equation. This has either one or three real roots. The limit between these two cases lies along the critical isotherm $T = T_{cr}$ on which the three points of intersection coalesce into one point of inflection with a horizontal tangent - the *critical point* $v = v_{cr}, p = p_{cr}$.

In order to determine v_{cr} and T_{cr} we calculate from (2):

$$\frac{\partial p}{\partial v} = 0 \quad \text{i. e.} \quad \frac{RT}{(v-b)^2} = \frac{2a}{v^3}, \quad \frac{1}{2} \frac{\partial^2 p}{\partial v^2} = 0 \quad \text{i. e.} \quad \frac{RT}{(v-b)^3} = \frac{3a}{v^4}.$$

It follows that

$$(3) \quad \begin{cases} v = v_{cr} = 3b, \\ RT = RT_{cr} = \frac{8}{27} \frac{a}{b}. \end{cases}$$

The corresponding value of p is found from (1)

$$(4) \quad p = p_{cr} = \frac{1}{27} \frac{a}{b^2}.$$

Since the constants a, b can be expressed in terms of the critical parameters eq. (1) can be rewritten to contain only the ratios

$$v = \frac{v}{v_{cr}}; \quad p = \frac{p}{p_{cr}}; \quad t = \frac{T}{T_{cr}}.$$

We then obtain

$$(5) \quad \left(p + \frac{3}{v^2}\right)(3v - 1) = 8t.$$

The preceding equation expresses the *law of corresponding states* due to van der Waals and establishes a universal law of similarity; its accuracy is the same as that of eq. (1). Incidentally, we may note that an analogous law of similarity can be established for any equation which contains only three individual constants.¹ It suffices to eliminate these three constants by introducing new dimensionless properties v, p, t .

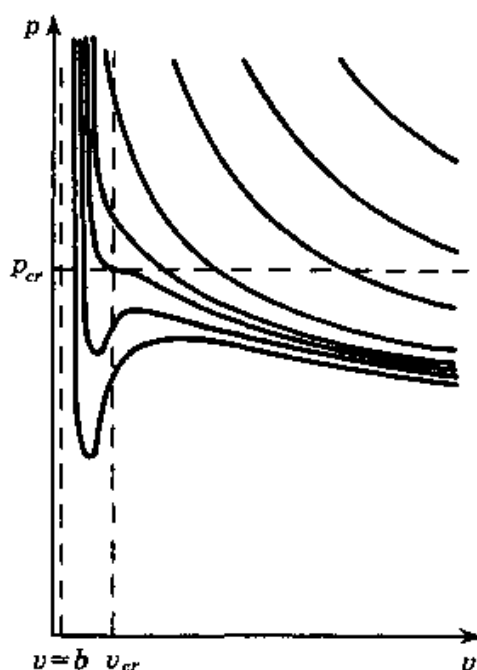


Fig. 7.

The van der Waals isotherms in the p, v plane

B. ENTROPY AND THE CALORIC BEHAVIOR OF THE VAN DER WAALS GAS

In discussing the behavior of a perfect gas in Sec. 5 D we have proved the existence of entropy from our additional caloric condition, but in the case of the van der Waals gas we shall proceed in a reverse manner in that

¹ J. de Boer and collaborators, *Physica* 14, 139, 149, 320 (1948).

we shall deduce its caloric behavior from the proposition on the existence of entropy. In the case of a perfect gas this was defined by eq. (5.7 a), i. e. by $\partial u / \partial v = 0$; in the present case we shall be led to the more general condition

$$(6) \quad \left(\frac{\partial u}{\partial v} \right)_T = \frac{a}{v^2}.$$

This condition is physically revealing: the internal energy of a gas is now seen to consist not only of the kinetic energy of the molecules but also of the potential energy of their forces of cohesion which is associated with the constant a . As is the case with a gravitating system of mass points, this energy is negative and approaches zero with increasing expansion. Consequently the energy u contained in the gas must increase with v , as borne out by our eq. (6).

In point of fact when van der Waals established his equation he was already in full possession of the fundamental propositions of thermodynamics. He was thus able to adapt the form of his equation to the entropy principle. We shall demonstrate that eq. (6) can be deduced from that principle.

According to the definition in eq. (6.10 a), we write

$$(7) \quad ds = \frac{du + p dv}{T}.$$

Inserting the value for p from (1) and considering u to be a function of T and v , we obtain at first

$$(8) \quad \begin{aligned} ds &= \frac{1}{T} \left(\frac{\partial u}{\partial T} dT + \frac{\partial u}{\partial v} dv \right) + \left(\frac{R}{v-b} - \frac{a}{v^2 T} \right) dv \\ &= \frac{1}{T} \frac{\partial u}{\partial T} dT + \frac{1}{T} \left(\frac{\partial u}{\partial v} + \frac{R}{v-b} - \frac{a}{v^2 T} \right) dv. \end{aligned}$$

The necessary and sufficient condition for this expression to be a perfect differential is:

$$(9) \quad \frac{1}{T} \frac{\partial^2 u}{\partial v \partial T} = \frac{\partial}{\partial T} \left(\frac{1}{T} \frac{\partial u}{\partial v} + \frac{R}{v-b} - \frac{a}{v^2 T} \right).$$

In carrying out the differentiation indicated on the right-hand side it is noticed that the middle term vanishes and that the last term gives $a/v^2 T^2$; the first term gives rise to two, one which cancels the left-hand side, the other being equal to $-\frac{\partial u}{\partial v} / T^2$. Thus eq. (9) becomes

$$(9 \text{ a}) \quad 0 = -\frac{1}{T^2} \frac{\partial u}{\partial v} + \frac{1}{T^2} \frac{a}{v^2}.$$

This is identical with eq. (6) which we seek to prove.

By partial differentiation we can deduce from (6) that

$$(10) \quad \frac{\partial c_v}{\partial v} = \frac{\partial^2 u}{\partial T \partial v} = \frac{\partial}{\partial T} \frac{a}{v^2} = 0,$$

so that c_v is a pure function of T , as for a perfect gas.

We shall now proceed to calculate the difference between the molar heats $c_p - c_v$ which for a perfect gas was equal to the characteristic value R . With reference to the general eq. (7.8 c), we obtain

$$(11) \quad c_p - c_v = \left\{ \left(\frac{\partial u}{\partial v} \right)_T + p \right\} \left(\frac{\partial v}{\partial T} \right)_p.$$

The value of $(\partial v / \partial T)_p$ can be taken from (2), that of $(\partial u / \partial v)_T$ follows from (6). Hence for a van der Waals gas we find that

$$(12) \quad c_p - c_v = R \left/ \left(1 - \frac{2a}{R T} \frac{(v-b)^2}{v^3} \right) \right.$$

In this equation a may be considered small and the product of the small quantities a , b may be omitted. Equation (12) now becomes

$$(13) \quad c_p - c_v = R \left/ \left(1 - \frac{2a}{R T v} \right) \right.$$

Reverting to the expression for entropy in (8) we can simplify it to

$$(14) \quad ds = \frac{c_v dT}{T} + R \frac{dv}{v-b},$$

in view of (6) and (10). Integrating on the assumption that the molar specific heat c_v is almost a constant, as for a perfect gas, we obtain

$$(15) \quad \int_{T_0, v_0}^{T, v} ds = s - s_0 = c_v \log \frac{T}{T_0} + R \log \frac{v-b}{v_0-b}.$$

10. Remarks on the liquefaction of gases according to van der Waals

A. THE INTEGRAL AND THE DIFFERENTIAL JOULE - THOMSON EFFECT

The condition that the *enthalpy* H is *constant* is valid for any equation of state and not only for perfect gases; this has already been stressed in eq. (5.7). From the Table in Sec. 7 we obtain

$$(1) \quad \Delta h = T \Delta s + v \Delta p$$

if the infinitesimal quantities ds, dp are extrapolated to small finite differences $\Delta s, \Delta p$ by the way of an approximation. Changing from the variables s, p to the variables T, p we may put

$$\Delta s = \left(\frac{\partial s}{\partial T} \right)_p \Delta T + \left(\frac{\partial s}{\partial p} \right)_T \Delta p.$$

Recalling the significance of c_p we have

$$\left(\frac{\partial s}{\partial T} \right)_p = \frac{1}{T} \left(\frac{\partial q}{\partial T} \right)_p = \frac{c_p}{T}.$$

Making use of the relation

$$\left(\frac{\partial s}{\partial p} \right)_T = - \left(\frac{\partial v}{\partial T} \right)_p$$

from our Table in Sec. 7 we can transform eq. (1) to read:

$$(2) \quad \Delta h = c_p \Delta T + \left[v - T \left(\frac{\partial v}{\partial T} \right)_p \right] \Delta p.$$

Thus the fact that the enthalpy is constant gives

$$(3) \quad \frac{\Delta T}{\Delta p} = \frac{1}{c_p} \left[T \left(\frac{\partial v}{\partial T} \right)_p - v \right] = \frac{v}{c_p} \left(\alpha - \frac{1}{T} \right),$$

where α denotes the coefficient of thermal expansion. We restrict our attention to the van der Waals gas in that we substitute the special value for α from eq. (9.2). Taking into account eq. (9.2 b) we obtain

$$(4) \quad \frac{\Delta T}{\Delta p} = \left(\frac{2a}{RT} - b \right) / c_p,$$

from which we conclude: When the gas is expanded, $\Delta p < 0$, there will be cooling, $\Delta T < 0$, if

$$(4\ a) \quad \frac{2a}{RT} > b.$$

This is the case with air and with most other gases. *Air can be cooled at will by repeated expansion and can finally be liquefied.*

The industrial production of liquid air (and its separation from, Ne, A, . . .) in Linde-type plants need not, of course, be restricted to an exact realization of the Joule-Thomson effect in all its details. Instead of "porous plugs" throttling valves are used and the performance is improved by Linde's regenerative counter-flow heat exchanger.

We shall encounter eq. (3) once more in Sec. 11. It represents the *finite Joule-Thomson effect*. It originated by extrapolation from the *differential Joule-Thomson effect* which has been rigorously defined earlier. From eq. (3) we find that for the latter

$$(5) \quad \left(\frac{\partial T}{\partial p} \right)_h = \frac{v}{c_p} T \left(\alpha - \frac{1}{T} \right).$$

B. THE INVERSION CURVE AND ITS PRACTICAL UTILIZATION

We shall now try to determine quite generally the region in the p, T plane in which expansion ($\Delta p < 0$) is associated with a decrease in temperature ($\Delta T < 0$), as in eq. (4 a) or, in other words, where $(\partial T / \partial p)_h > 0$. This, from the point of view of practical applications, desirable region will be called positive. It is bounded by the *inversion curve* on which $(\partial T / \partial p)_h = 0$, so that from (5) we see that it is given by

$$(5\ a) \quad \alpha(p, T) = \frac{1}{T}.$$

It separates the desirable from the undesirable, *negative* region. As mentioned previously the states of air and of most other gases which correspond to ordinary conditions of pressure and temperature always lie within the inversion curve. This is confirmed by Fig. 8.

Making use of the accurate van der Waals equation we obtain the following expression for the inversion curve from eq. (9.2 a):

$$(5\ b) \quad \frac{2a}{RT} \left(\frac{v-b}{v} \right)^2 = b,$$

where it is only necessary to express v in terms of p and T . Introducing at the same time the reduced coordinates p and t we obtain after some rearrangement:

$$(5c) \quad p = 24 \sqrt{3} t - 12 t - 27.$$

Figure 8 contains an inversion curve determined experimentally for H_2 (W. Meissner) in addition to the inversion curve from eq. (5c). The conversion data for hydrogen and air are given in the caption to the figure. It is seen that $(\partial T / \partial p)_h$ for air is positive at room temperature up to

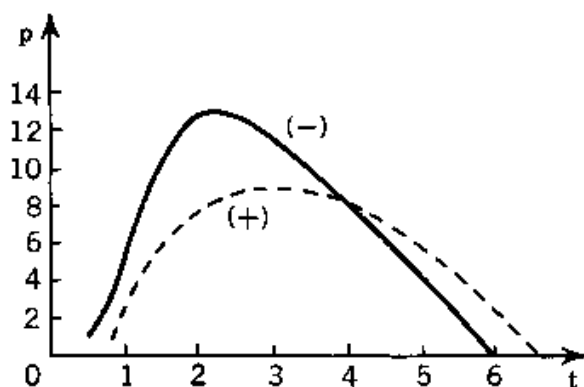


Fig. 8.

Inversion curve for the differential Joule-Thomson effect in reduced coordinates:

— determined experimentally for H_2 by W. Meissner

----- calculated for a van der Waals gas from eq. (10.5c)

For air: $T = 132.5 \times t$ K; $p = 34.5 \times p$ at.

For H_2 : $T = 33.2 \times t$ K; $p = 13.2 \times p$ at.

450 at; on the other hand for H_2 at room temperature it is always negative. This circumstance was responsible for many accidents due to the fact that highly compressed hydrogen ignited spontaneously when leaking from damaged pipes. Hydrogen can be cooled on sudden expansion ("throttling") only after its temperature had been reduced below -80°C .

Returning to the performance of a machine operating on the Joule-Thomson process we find that it depends on the *integral Joule-Thomson effect*

$$(5d) \quad (T_2 - T_1) = \int_{p_1}^{p_2} \left(\frac{\partial T}{\partial p} \right)_h dp.$$

In practical applications the pressure p_2 after expansion is, in most cases, approximately equal to atmospheric. The temperature T_1 is determined essentially by the choice of the fluid for pre-cooling (for the liquefaction of air cooling water is used, whereas liquid nitrogen is used for the liquefaction of H_2). The pressure p_1 at which the gas enters the liquefaction chamber is the only variable which may be adjusted freely within certain limits. We shall now try to determine that value for p_1 which leads to a maximum cooling effect. Differentiating the integral in (5d) with respect to the lower limit and equating to zero we have

$$\left(\frac{\partial T}{\partial p} \right)_h = 0 \quad \text{for} \quad p = p_1, \quad T = T_1.$$

This, however, is exactly the condition for the state p_1, T_1 to lie on the inversion curve of the differential Joule-Thomson effect. In designing a liquefaction plant it is necessary to satisfy this condition. For example, it is found that the most favorable temperature of pre-cooling for the liquefaction of H_2 is 64.5 K (it is generated by evaporating liquid nitrogen under reduced pressure). The corresponding favorable pressure is found from the inversion curve to be 160 at. In practice the values of 72 K and 140 at are often selected for operation. In the case of helium a temperature of pre-cooling of 14 K and a pressure $p_1 = 29$ at is used. The temperature of 14 K is produced either with liquid H_2 which is allowed to boil under a reduced pressure (Kammerlingh-Onnes), or, more recently, with helium gas which is cooled with the aid of a reversible adiabatic process (performance of work in an expansion engine). (Kapitza and Meissner; the latter gave a complete analysis of the required thermodynamic conditions.)

C. THE BOUNDARY OF THE REGION OF CO-EXISTING LIQUID-VAPOR PHASES IN THE p, v PLANE

The word *phase* has several meanings; generally speaking it denotes the "form of a phenomenon." In this connection we may think of the phase of an optical vibration, of the phases of the moon, of the various phases in a political development; later we shall speak of the multi-dimensional "phase space" in statistical mechanics. In thermodynamics the term "phase" is used to denote the different states of aggregation of a single substance including the various structural forms of the solid (crystal structures, amorphous structures). When applied to several substances the term includes the different chemical groupings of which they are capable.

In the theory evolved by van der Waals we shall study the equilibrium between the gaseous phase 2 and the liquid phase 1, i. e. that between a saturated vapor over its liquid. *Equality of pressure* constitutes the mechanical condition for such equilibrium and *equality of temperature* is the thermal condition of equilibrium. Of the four potentials listed in the Table in Sec. 7 the *free enthalpy* G is the one most suitable for use when p and T are constant. According to eq. (8.13 a) the equality of p and T carries with it the equality of g in both phases: $g_1 = g_2$. If the masses of substance contained in the two phases, 1 and 2, are denoted by m_1 and m_2 then according to eq. (8.13 a) we may write

$$\delta G = \delta(m_1 g_1 + m_2 g_2) = (g_1 - g_2) \cdot \delta m_1 = 0$$

because $\delta m_1 = -\delta m_2$; the total mass $m_1 + m_2$ remains constant during the variation.

With reference to Fig. 7 we consider an isotherm $T < T_{cr}$ and make it intersect with an isobar $p < p_{cr}$. Of the three points of intersection we shall denote the two external ones by A and B , Fig. 9, the corresponding values of g being denoted by g_A and g_B . These are also equal because they belong to the same p and T . The equality of g_A and g_B leads to

$$(6) \quad u_B - u_A - T(s_B - s_A) + p(v_B - v_A) = 0,$$

because $g = u - Ts + pv$. The difference $s_B - s_A$ can be taken from eq. (9.15). Taking into account the equality of T at both points of intersection A and B we find that

$$(7) \quad s_B - s_A = R \log \frac{v_B - b}{v_A - b}.$$

We now calculate $u_B - u_A$ integrating du along the isobar $p = \text{const}$ (any other path of integration would lead to the same result):

$$u_B - u_A = \int_A^B du = \int_A^B \left\{ \left(\frac{\partial u}{\partial v} \right)_T dv + \left(\frac{\partial u}{\partial T} \right)_v dT \right\}.$$

Making use of eqs. (9.6), (9.10) we have:

$$u_B - u_A = \int_A^B \frac{a}{v^2} dv + \int_A^B c_v(T) dT,$$

where the second term on the right-hand side vanishes as $T_A = T_B$. The first term gives:

$$(8) \quad u_B - u_A = -\frac{a}{v_B} + \frac{a}{v_A}.$$

Substituting (7) and (8) into (6) we find:

$$(9) \quad -\frac{a}{v_B} - RT \log(v_B - b) + \frac{a}{v_A} + RT \log(v_A - b) + p(v_B - v_A) = 0.$$

We now calculate the area in the p, v plane bounded by the isotherm $T = \text{const}$, the axis of abscissae and the two straight lines $v = v_A$, $v = v_B$. According to the van der Waals equation this area is equal to

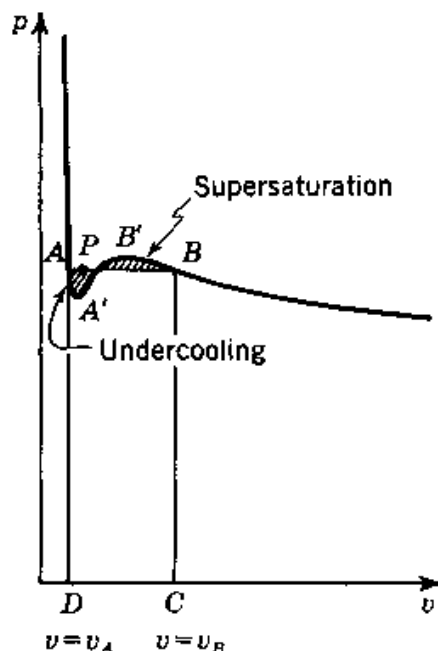


Fig. 9.

Definition of Maxwell's line.

$$(10) \quad \int_B^A p \, dv = R T \int_A^B \frac{dv}{v-b} - a \int_A^B \frac{dv}{v^2} = \left[R T \log (v-b) + \frac{a}{v} \right]_{v=v_A}^{v=v_B}.$$

The expression on the right-hand side is equal to the first four terms in eq. (9), except for the sign. Substituting (9) into (10) we have

$$(11) \quad \int_A^B p \, dv = p (v_B - v_A).$$

Geometrically $p(v_B - v_A)$ represents the rectangle $ABCD$ in Fig. 9. According to (9) its area is equal to that between the isotherm and the axis of abscissae considered in (10). Hence the two crescents shaded in the figure must have equal areas. This rule gives a convenient graphical method of determining the boundary points A and B for phase equilibrium. It was first given by Maxwell (Nature 1875); the line AB is known as "Maxwell's line."

Carrying out the above construction for every isotherm below the critical in Fig. 7 we obtain the boundary enclosing that region in the p, v plane within which the two phases, *liquid* and *gaseous*, co-exist in equilibrium. The boundary is shown sketched in Fig. 10. To the right of this line only gas can exist, whereas to the left of it only the liquid will be present. The vertex of this region coincides with the critical point v_{cr}, p_{cr} . The two branches of the boundary curve meet at that point. The locus of points A is the water (or liquid) line, and that of points B is the steam (or vapor) line. Proceeding from the gaseous phase we find that the first droplets of liquid appear on the vapor line; proceeding from the liquid phase we shall notice the first bubbles of vapor on crossing the liquid line.

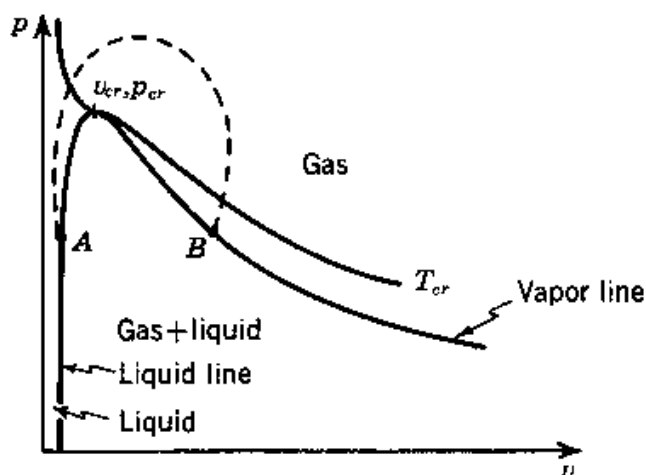


Fig. 10.

Boundary of the region of co-existing phases gas + liquid in the p, v plane.

Let us return for a moment to Fig. 9 and let us inquire into the physical meaning of the points along Maxwell's line AB . They are points of varying volume but equal pressure and temperature. The variation in volume is produced by the varying proportions of the liquid and gaseous phase present.

At B we have pure saturated vapor of molar volume v_B , whereas at A we have pure liquid of molar volume v_A . Denoting the relative mass of the liquid by x and that of the vapor by $1 - x$, we find that at A $x = 1$ and at B $x = 0$. At an intermediate point P we have

$$v_P = x v_A + (1 - x) v_B.$$

It follows that

$$(11\ a) \quad x = \frac{v_B - v_P}{v_B - v_A} = \frac{PB}{AB}; \quad 1 - x = \frac{v_P - v_A}{v_B - v_A} = \frac{AP}{AB}.$$

The parts x and $1 - x$ can be inferred from the diagram as they are equal to the ratios PB/AB and AP/AB respectively.

We shall now explain the meaning of the captions "supersaturation" and "undercooling" in Fig. 9. They denote *unstable* states of equilibrium, which are *not* states of *true equilibrium* like those corresponding to the points on Maxwell's line. Under favorable conditions (for example steam in dust-free atmosphere) it is possible to obtain supersaturated steam when at constant temperature the pressure assumes a higher value than that at point B on the vapor line. Similarly the liquid state can be maintained at a temperature which exceeds slightly the boiling point if heated in a vessel which is completely free from vibration. At a temperature equal to that at point A the pressure decreases. It must be realized, however, that these unstable extensions over the limits on the boundary lines are very small and the states between A' and B' cannot be attained at all because they correspond to an isothermal increase in pressure upon expansion. It is very remarkable that the theory due to van der Waals is in a position to predict, at least qualitatively, the existence of the unstable states along the branches AA' or BB' .

We now return to Fig. 10 and directly deduce from it the following facts:

1. At temperatures $T > T_c$ it is impossible to liquefy a gas however large the compression.
2. In order to liquefy a gas it is not enough to reduce the temperature to $T < T_i$; according to eqs. (5 b) and (9.3) the inversion temperature T_i is about seven times larger than T_c . It is, however, possible to liquefy a gas, as was the case with air, by a succession of Joule-Thomson expansions and to reach the range of partial and, finally, total liquefaction, provided that $T < T_i$ has been attained.
3. It is possible to reach the liquid region at the lower left-hand corner of the diagram in Fig. 10 directly from the rarefied gaseous state at the lower

right-hand region without crossing the shaded two-phase region by following a path which traverses the region above the critical point. According to the van der Waals equation the states along such a path form a continuum of states of stable equilibrium. There is no discontinuity even on crossing the critical isotherm $T = T_{cr}$, either on its lower branch where $p < p_{cr}$ or on its upper branch for which $p > p_{cr}$. This behavior explains the title of van der Waals' paper "The continuity of the gaseous and liquid states."

4. In particular we can traverse from one end, B , of a Maxwell line to the other at A along the path shown by a broken line in Fig. 10. The analytic expression for g_B will vary continuously until the value g_A is attained. This has been anticipated in eq. (6). Strictly speaking we could have added to this equation a linear function of T owing to the fact that the zero levels of energy and entropy are undetermined. Our considerations of continuity do, however, demonstrate that this should, in fact, have been set equal to zero.

In the preceding description the concept of the critical point, as well as that of Maxwell's line, have been explained with the aid of the idealized model of a van der Waals gas.

In actual fact the course of the isotherms of a real gas shows qualitative agreement with the van der Waals model. In particular the existence of a critical point is always observed being determined by the condition from Sec. 9, namely

$$\left(\frac{\partial p}{\partial v}\right)_T = 0; \quad \left(\frac{\partial^2 p}{\partial v^2}\right)_T = 0$$

together with the equation of state $p = p(v, T)$. Moreover, for analytically formulated equations of state the relation (11) for Maxwell's line remains true in general, as it is easy to prove.

In order to do this let us consider an isotherm below the critical whose shape is qualitatively reproduced by that shown in Fig. 9. From the equality of free enthalpy at A and B we have that

$$(12) \quad f_A + p v_A = f_B + p v_B$$

since $g = f + p v$. On the other hand

$$(13) \quad f_A - f_B = - \int_A^B \left(\frac{\partial f}{\partial v}\right)_T dv.$$

According to our Table in Sec. 7 we find that $(\partial/\partial v)_T = -p$ so that eqs. (12) and (13) yield

$$p(v_B - v_A) = \int_A^B p \, dv$$

where the integral extends over the extrapolated isotherm $AA'B'B$. This proves that (11) is true for any analytic form of the equation of state.

11. The Kelvin temperature scale

The absolute temperature has been defined in Sec. 6 as that function $\phi(\theta)$ of an arbitrary, conventionally measured temperature θ which satisfies Carnot's ratio

$$(1) \quad \frac{Q_1}{Q_2} = \frac{\phi(\theta_1)}{\phi(\theta_2)} = \frac{T_1}{T_2}.$$

This definition was proposed by Thomson as early as 1848. It has also been pointed out that a temperature measured with the aid of a gas thermometer would satisfy condition (1) with a degree of precision which is directly related to the deviation of the thermometric substance within the range under consideration from that of a *perfect gas*. We now proceed to show how the temperature θ measured with the aid of a *real* gas thermometer can be reduced to the absolute temperature T .

Evidently eq. (1) is not very suitable for this purpose because it is not possible to measure Q calorimetrically with a high enough degree of precision. In its place, however, we may use any relation deduced from the Second Law which contains the absolute temperature. Lord Kelvin recognized that his analytical formulation of the Joule-Thomson effect given in Sec. 8 was particularly suitable. Clapeyron's equation given in Sec. 14 constitutes another practical starting point.

Equation (10.3) had the form:

$$(2) \quad \frac{\Delta T}{\Delta p} = \frac{v T}{c_p} \left(\alpha - \frac{1}{T} \right).$$

The quantities c_p and α , which have been defined with the aid of derivatives with respect to T can be rewritten in terms of derivatives with respect to θ , because $T = \phi(\theta)$ is a pure function of θ :

$$\alpha = \frac{1}{v} \left(\frac{\partial v}{\partial T} \right)_p = \frac{1}{v} \left(\frac{\partial v}{\partial \theta} \right)_p \frac{d\theta}{dT} = \alpha' \frac{d\theta}{dT};$$

$$c_p = \left(\frac{\partial q}{\partial T} \right)_{p=\text{const}} = \left(\frac{\partial q}{\partial \theta} \right)_{p=\text{const}} \frac{d\theta}{dT} = c_p' \frac{d\theta}{dT}.$$

The newly defined quantities α' and c_p' are to be regarded as being empirically determined functions of θ . Dividing the numerator and denominator on the right-hand side of eq. (2) by $d\theta/dT$ we find that it is equal to

$$(2 \text{ a}) \quad \frac{v T}{c_p'} \left(\alpha' - \frac{1}{T} \frac{dT}{d\theta} \right).$$

The left-hand side can be rewritten as

$$(2 \text{ b}) \quad \frac{\Delta \theta}{\Delta p} \frac{dT}{d\theta}$$

where the quantity $\Delta \theta / \Delta p$ is an empirically given function of θ , namely that measured with the aid of the Joule-Thomson effect. Equating (2 a) and (2 b) we have

$$c_p' \frac{\Delta \theta}{\Delta p} \frac{1}{T} \frac{dT}{d\theta} = v \left(\alpha' - \frac{1}{T} \frac{dT}{d\theta} \right)$$

so that

$$(3) \quad \frac{1}{T} \frac{dT}{d\theta} = \frac{v \alpha'}{v + c_p' \frac{\Delta \theta}{\Delta p}},$$

or

$$(4) \quad \log \frac{T}{T_0} = \int_{\theta_0}^{\theta} \frac{v \alpha'}{v + c_p' \frac{\Delta \theta}{\Delta p}} d\theta.$$

In general the integration must, of course, be carried out numerically because it involves only empirically determined functions of θ (v also belongs to this group). If θ is measured in deg Celsius and if the unit on the scale of T is suitably chosen then we have the correspondence

$$\theta = 0 \quad \text{and} \quad T = T_0,$$

$$\theta = 100 \text{ C} \quad \text{and} \quad T = T_0 + 100 \text{ C}.$$

In order to determine T_0 we can use eq. (4), or

$$(4a) \quad \log \left(1 + \frac{100 \text{ C}}{T_0} \right) = \int_0^{100 \text{ C}} \frac{v \alpha'}{v + c_p' \frac{\Delta \theta}{\Delta p}} d\theta.$$

The scale of temperature defined above is now universally known as the *Kelvin temperature scale*, and we write $T = \dots$ K. In the interval 0 to 100 C the differences between T and θ are very small for most thermometric substances. In accordance with the determinations carried out by the German Physikalisch-Technische Reichsanstalt the maximum deviation when θ is determined by an air thermometer is -0.0026 C. It is smaller for the more perfect hydrogen for which it reaches 0.0007 C.

These differences naturally increase as the temperature approaches the point of liquefaction of the respective substance, and the Joule-Thomson process fails below it. The most effective method for the generation of the lowest temperatures near the (unattainable) absolute zero is that based on the *magneto-caloric effect* (P. Debye 1926, W. J. Giauque 1937); the two authors suggested (independently of each other) to use the paramagnetic salt-gadolinium sulphate as the most suitable substance for the purpose. The process is as follows: The substance is first cooled in a liquid helium bath of very low temperature (~ 1.3 K) and placed in a powerful magnetic field until thermal equilibrium has been reached. In this way all magnetic dipoles become practically unidirectional, provided that the field is sufficiently powerful. The entropy at this ordered state is smaller than at the state of disorder which prevails in the absence of a field (*cf.* Sec. 19), and the result is an appreciable reduction in entropy. At this moment the paramagnetic salt is insulated thermally and the field is switched off. The total entropy must remain constant in the process, but as the external field is decreased a certain amount of disorder will set in, the magnetic contribution to entropy increasing from zero to an appreciable value. Owing to the presence of the thermal insulation the total entropy remains constant which causes the temperature to decrease in consequence. The specific heat c_p is very small at these very low temperatures and the magnetic contribution to entropy constitutes the leading term so that on adiabatic demagnetization the temperature can be made to decrease to a value of the order of several thousandths of one degree K (de Haas). We shall see in Sec. 12 that the point of absolute zero cannot be reached in this way. We shall only note here that the thermodynamic theory of the process of adiabatic demagnetization (*cf.* Sec. 19) supplies us with a method

of extending the definition of the Kelvin temperature scale down to temperatures in the neighborhood of absolute zero.

The skeptical Mach¹ expressed the view that the zero point on the absolute temperature scale has a meaning only in relation to the special case of a gas. This view is flatly contradicted by the creation of the absolute Kelvin temperature scale. The whole structure of the science of thermodynamics would collapse without the existence of this (fixed, but unattainable) lower limit of temperature.

12. Nernst's Third Law of thermodynamics

Nernst's proposition is quite rightly termed the Third Law of Thermodynamics. It does not lead to a new property, like the First and Second Laws of thermodynamics but it makes the properties S , F , G , . . . numerically determinate and hence usable.

The entropy is defined in terms of its differential dS but S itself is only defined short of a constant S_0 . In itself this does not constitute a drawback because in applications we almost always deal with differences in entropy. The same is true of the energy U in whose expression a constant of integration remains undefined. However, in the expressions of the potentials F and G a linear function of the form $S_0T + \text{const}$ remains undetermined in view of the term TS . Thus their usefulness for states at different temperatures and in the equations of chemical equilibrium becomes illusory.

Thus there arises the question of the absolute value of entropy: As is the case with all fundamental questions, nature provides the simplest conceivable and the mathematically most satisfactory answer: *As the temperature of a system tends to absolute zero its entropy tends to a constant value S_0 which is independent of pressure, state of aggregation, etc.* We may put it equal to zero so that the entropy of every substance becomes normalized in an absolute way. This also removes the indeterminateness in the potentials F , G because in the integral formulae for entropy (cf. (6.11)) we may assume the lower limit to be at $T = 0$.

This is the formulation which was given by Planck, but it is interesting to quote here Nernst's *own* first statement of the same fact. In connection with his researches on electrochemical phenomena Nernst was able to handle the Second Law of thermodynamics with greater success than anybody else but, remarkably enough, he disliked the concept of entropy and preferred to

¹ "Prinzipien der Wärmelehre", Leipzig 1896, p. 341. Gay Lussac's inaccessible account of his free-expansion experiment has been published in an Appendix to this book.

use that of "maximum (available) work" which he also used as a measure of chemical affinity. Introducing Nernst's symbol A for it we find that according to eq. (8.15) we have

$$(1) \quad A = \Delta F \quad \text{with} \quad \Delta F = F_1 - F_2.$$

From $F = U - TS$ together with the relation

$$S = - \left(\frac{\partial F}{\partial T} \right)_V, \text{ see Table in Sec. 7}$$

we have

$$F = U + T \left(\frac{\partial F}{\partial T} \right)_V$$

so that from (1) we find that

$$(2) \quad A - \Delta U = T \left(\frac{\partial A}{\partial T} \right)_V \quad \text{with} \quad \Delta U = U_1 - U_2.$$

Nernst regarded (2) as the *fundamental equation of thermodynamics*¹ because it combines the First Law with the Second.

According to Berthelot and Nernst, ΔU denotes the heat tone of the process at constant V .² Berthelot thought that the left-hand side of (2) should be equal to zero which, however, is not the case. Nernst (*l. c.*) made the following comment: "It occurred to me that we have here a limiting law since often the difference between A and ΔU is very small. It seems that at absolute zero A and ΔU are not only equal but also asymptotically tangent to each other. Thus we should have

$$(3) \quad \lim_{T \rightarrow 0} \frac{dA}{dT} = \lim_{T \rightarrow 0} \frac{d\Delta U}{dT} = 0 \quad \text{for} \quad T = 0."$$

This is the historical origin of the most far-reaching generalization of classical thermodynamics of our century.

Returning from A to

$$\Delta F = \Delta U - T \Delta S$$

we obtain at once from (3) that

$$\lim_{T \rightarrow 0} \left\{ \frac{d\Delta U}{dT} - \Delta S - T \frac{d\Delta S}{dT} \right\} = \lim_{T \rightarrow 0} \frac{d\Delta U}{dT}$$

¹ "Theoretische Chemie", ed. 1926, p. 795. We have written ΔU instead of Nernst's U and we have added the subscript V in the partial derivative of A which was omitted by Nernst.

² In most cases it is more useful to consider the heat tone at constant pressure and to define it as the difference of enthalpies ΔH .

and consequently

$$(4) \quad \Delta S \rightarrow 0 \quad \text{i. e.} \quad S_2 \rightarrow S_1 \quad \text{for} \quad T \rightarrow 0.$$

Thus we have deduced Planck's formulation $S \rightarrow S_0$ and the possibility of normalizing $S_0 = 0$ for any substance from Nernst's statement (3). Reciprocally, it is evident that eq. (3) follows from (4).

Originally Nernst limited the validity of his theorem to pure condensed substances (solids or liquids) and excluded the gaseous state. However, in a subsequent publication¹ he considered the question of degenerate gases which was beginning to occupy the attention of physicists at the time. This problem was formulated shortly afterwards in the Bose-Einstein and the Fermi-Dirac statistics. The two statistics show that gases degenerate at the lowest temperatures. In particular in his sketch of a theory of metals (1927) the present author demonstrated that, with respect to all its properties, the free electron gas obeys Nernst's Third Law. It is not at all astonishing that the expressions (5.10) and (9.15) for the entropy of a perfect gas and for that of a van der Waals gas respectively do not lead to $S \rightarrow 0$ as $T \rightarrow 0$ because they do not claim any validity in this region. Apparent contradictions to the Third Law observed in connection with deuterium compounds have been resolved by Clausius who showed that they represented frozen, i. e. metastable states of equilibrium.

The science of thermodynamics can give no information about the duration of such exceptional states; this problem must be left to be dealt with by the methods of quantum mechanics. In any case we did not find it necessary to restrict the validity of Nernst's Third Law to condensed substances and we take the view that its validity is universal.

We shall now summarize a series of the most simple consequences of this law as regards homogeneous substances. Since we are now referring to mols or units of mass we shall use lower case symbols s, v instead of the capital letters used so far.

1. *The coefficient of thermal expansion tends to zero as $T \rightarrow 0$.* According to the Table in Sec. 7, line G, we have

$$\left(\frac{\partial s}{\partial p} \right)_T = - \left(\frac{\partial v}{\partial T} \right)_p.$$

The left-hand side vanishes because the limit s_0 of s is independent of pressure. Hence the change in volume which appears on the right-hand side must also vanish. The same is true of the coefficient of thermal expansion, α .

¹"Die theoretischen und experimentellen Grundlagen des neuen Wärmethorems," Knapp, Halle, 1918.

2. *The coefficient of tension tends to zero as $T \rightarrow 0$.* According to the table in Sec. 7, line *F*, we have

$$\left(\frac{\partial s}{\partial v}\right)_T = \left(\frac{\partial p}{\partial T}\right)_v.$$

Hence, again, the left-hand side vanishes because the limit s_0 is independent of v as well. The same is true of the right-hand side and of the coefficient of tension β , provided $p \neq 0$.

3. *The specific (or molar) heats c_p and c_v tend to zero as $T \rightarrow 0$.* From the definitions

$$c_v = T \left(\frac{\partial s}{\partial T}\right)_v, \quad c_p = T \left(\frac{\partial s}{\partial T}\right)_p$$

we obtain by integration

$$(5) \quad s(v, T) = \int_0^T \frac{c_v}{T} dT; \quad s(p, T) = \int_0^T \frac{c_p}{T} dT.$$

The constants of integration would be functions of v only in the first, and of p only in the second integral. However, for sufficiently small values of T their existence would contradict Nernst's Third Law which stipulates that the limiting value s_0 is asymptotically independent of both v and p . We have normalized the value of s_0 in both eqs. (5) at $s_0 = 0$ by assuming the lower limit of integration to be equal to zero. Now these integrals show that c_v and c_p must vanish for $T = 0$; if this were not so the integrals would diverge in view of the lower limit of integration $T = 0$. The German Reichsanstalt carried out extensive measurements under Nernst's direction providing a full confirmation of the fact that the specific heats, and that c_p in particular, decrease as absolute zero is approached. The rate at which these quantities tend to zero is left undetermined by eqs. (5). According to Debye (*cf.* Vol. II, Sec. 44) the rate of decrease for an elastic body is proportional to T^3 , for the electron gas to T (*cf.* this volume, Sec. 39). These theoretical results have also been fully confirmed by experiment.

4. *The absolute zero temperature cannot be reached by any finite process; it may only be reached asymptotically.* We have made use of the process of adiabatic expansion, see Sec. 5, to cool real gases. In the range of lowest temperatures its place, as the most efficient process, is taken by adiabatic demagnetization, see Sec. 11.

The T, S diagram in Fig. 11 represents the curve of constant field intensity, $H = \text{const.}$ ¹ for a salt in a field of strength H . The pressure is also considered to be constant (e. g. $p = 0$, vacuum; in the case of a salt in the solid state the volume is unimportant). This curve passes through the origin 0 and its slope increases with T because the ordering of dipoles forced on the salt by the field is increasingly disturbed as T increases. We now supplement the diagram with the curve $H = 0$ for an unmagnetized salt. It lies above the curve $H = \text{const}$ because in this case no magnetic ordering exists. According to Nernst's Third Law this curve must also pass through the origin at $T = 0$. If we now start with the magnetized state at the initial temperature T_1 and proceed along an adiabatic (isentropic) curve, i. e. along a horizontal line $S = \text{const}$ until the demagnetized state on the curve $H = 0$ ² has been reached we attain a temperature T_2 , which is, as is seen from the drawing, considerably lower than T_1 ; but absolute zero is certainly not attained.

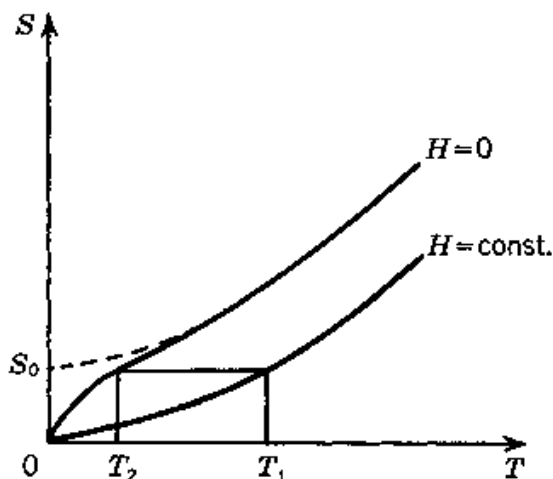


Fig. 11.

Illustrating the impossibility of reaching absolute zero by adiabatic demagnetization

This would not be so if the Third Law were not true. Suppose that the curve $H = 0$ had a limit $S_0 \neq 0$ for $T = 0$, as shown by the broken line in Fig. 11. With the present choice of the initial temperature T_1 we would attain absolute zero starting from $H = \text{const}$ in *one step*, namely we would reach point $S = S_0$ at $H = 0$.

It is quite natural to try to come nearer to absolute zero by magnetization at T_2 and repeated demagnetization. However, such a scheme would encounter practical difficulties. The only effective way of going one step further would be to demagnetize the nuclei, but this is still in the future.

In any case the following conclusion can be drawn from the Third Law: *The point of absolute zero temperature can only be attained asymptotically.* Consequently, the realization of a Carnot process with heat rejection at $T = 0$ and an efficiency of $\eta = 1$, cf. Sec. 6, is impossible.

¹In the present Section and in Secs. 19 and 25 the letter H is used to denote the magnetic field strength and not enthalpy.

²The characteristic bulge in the curve is due to the fact at very low temperatures paramagnetic materials exhibit a kind of spontaneous magnetization (ordering of magnetic moments), similar to that exhibited by ferromagnetic substances.

CHAPTER II

THE APPLICATION OF THERMODYNAMICS TO SPECIAL SYSTEMS

13. Gaseous mixtures, Gibbs' paradox. The law due to Guldberg and Waage

The air which surrounds us is a mixture of

78% N_2 , 21% O_2 and almost 1% A.

The surprisingly high argon content escaped the notice of earlier investigators because of its nobility; it was only discovered by Lord Rayleigh and W. Ramsay as late as the beginning of the century. The preceding figures represent molar percentages and are proportional to the number of molecules of the respective gas in the mixture. The ratio of percentages by weight is obtained by multiplying the above numbers with the molecular weights 28, 32, 40 respectively.

On the assumption that the gases are perfect, which is permissible at ordinary temperatures (point-molecules with zero cohesion; the van der Waals constants $b = 0$, $a = 0$), each component of the mixture behaves as if it alone occupied the volume V . It is, therefore, possible to define *partial pressures* p_i and to add them in order to obtain the total pressure p .

$$(1) \quad p = \sum_i p_i, \quad \text{Dalton's law.}$$

Since each component obeys the perfect-gas law due to Gay-Lussac in the form (3.10), we also have

$$(1 \text{ a}) \quad V p_i = n_i R T.$$

Adding eqs. (1 a) for all components we obtain

$$(2) \quad V p = n R T; \quad n = \sum_i n_i.$$

The energy of the mixture is also the sum of the molar energies of the components:

$$(3) \quad U = \sum_i n_i u_i; \quad u_i = c_{vi} T.$$

It follows that the specific heats C_v and C_p of the mixture are given by

$$(3\ a) \quad C_v = \sum_i n_i c_{vi}; \quad C_p = \sum_i n_i c_{pi}; \quad C_p - C_v = n R.$$

Equation (3) can be understood by imagining that each component is compressed isothermally to the smaller volume

$$(3\ b) \quad V_i = V \frac{p_i}{p},$$

that all these volumes are placed side by side in compartments of equal cross-section A (similar to those in Fig. 12 below), and that they are separated by partitions, so that the sum of the volumes $V = \sum_i V_i$. If the partitions are

withdrawn the i -th component will spread from the smaller volume V_i over the larger volume V and during this process its temperature T , its molar energy u_i , and its total energy $n_i u_i$ will remain constant. According to our preceding representation the presence of the remaining components may be disregarded. The physical nature of this process is better described by the term *diffusion*.

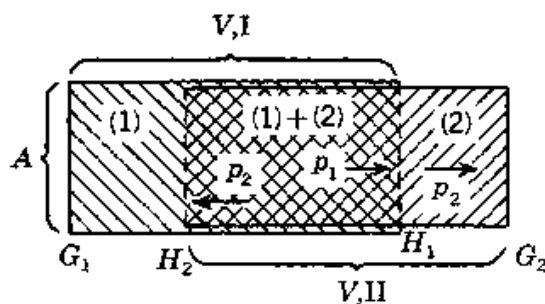


Fig. 12.

Reversible separation of two gases.

A. REVERSIBLE SEPARATION OF GASES

In the following argument we may restrict ourselves to the consideration of two gases: 1 and 2. In order to achieve separation we shall need *semi-permeable membranes*. These occur in nature as walls of organic cells and can also be realized approximately with the aid of chemical means (copper ferrocyanide membrane). Their practical and theoretical application is due to the botanist Pfeffer, *cf.* Sec. 15. Figure 12 shows two cylinders, I and II of equal volumes V , which can slide into one another without friction. The side surfaces and the ends (G_1 of I and G_2 of II) are assumed *impermeable* to both gases 1 and 2. However, the end H_2 of II is assumed impermeable only to 2, allowing 1 to pass through it. The reverse is assumed about the end H_1 of I.

The two cylinders are assumed to have been pushed into each other initially, both gases being mixed. Now cylinder II is imagined drawn out infinitely slowly by a distance dx to the right. The compartment $G_1 H_2$ can only be entered by gas 1 which, so to say, ignores the presence of membrane H_2 .

The end H_2 is acted upon only by the pressure p_2 which opposes the displacement dx . The work performed by it during the process is

$$(4) \quad dW = -A p_2 dx.$$

The membrane H_1 , being at rest, does not perform any work and the same is true of wall G_1 . However, there is work associated with wall G_2 which is moved by dx . Since compartment $H_1 G_2$ contains only gas 2 at a pressure equal to that in $H_2 H_1$ (H_1 is, so to say, ignored by the gas) the work done by p_2 on G_2 is

$$(4 a) \quad dW = +A p_2 dx,$$

i. e. equal and opposite to (4). The process of separation requires no work, and occurs without the exchange of heat, as we shall assume from now on; this means that the energy and the temperature of the two components remain constant. The final outcome of this experiment is that both gases 1 and 2 become separated, either of them in a volume V and at a pressure p_1 or p_2 . Their entropies will be denoted by S_1 and S_2 . Since this imaginary process of separation proved to be reversible, requiring neither work nor heat, we can say that the entropy S of the mixture:

$$(5) \quad S = S_1 + S_2.$$

It is evident that the process represented in Fig. 12 can be reversed by sliding the cylinders slowly into each other. In this way the *mixing* of the gases also becomes *reversible*.

Equation (5) provides a sure basis for the calculation of the entropy of a mixture. Written out more fully it is

$$(5 a) \quad S(T, V) = S_1(T, V) + S_2(T, V),$$

or (by changing the mathematical definition of the symbol S for the function)

$$(5 b) \quad S(T, p) = S_1(T, p_1) + S_2(T, p_2).$$

In words: *In order to calculate the entropy of a mixture as the sum of the entropies of its components it is necessary to imagine each component occupying the same volume V as the mixture so that the pressure p is reduced to p_1 and p_2 respectively.*

Generalizing to more than two components we have, evidently:

$$(6) \quad S(T, p, n_1, n_2, \dots) = \sum_i n_i s_i(T, p_i),$$

where n_i is the number of mols and s_i is the molar entropy of the i -th component.

B. THE INCREASE IN ENTROPY DURING DIFFUSION AND GIBBS' PARADOX

During the process of diffusion considered in the preceding Section the initial volumes V_i of the components, imagined arranged side by side, have been assumed to be different from each other, each being smaller than V ; their pressure was brought to the common value p by isothermal compression. Let S_0 denote the sum of the entropies of the components before diffusion. In analogy with (6) we can write

$$(6a) \quad S_0 = \sum_i n_i s_i(T, p).$$

From (6) and (6a) we can calculate the change in entropy due to diffusion:

$$(6b) \quad S - S_0 = \sum_i n_i \{s_i(T, p_i) - s_i(T, p)\}.$$

Using our original definition of entropy for a perfect gas as given in (5.10) we can calculate this difference per mol and at constant temperature¹ and we obtain:

$$s_i(T, p_i) - s_i(T, p) = R \log \frac{v}{v_i};$$

owing to (3b) we can replace the right-hand side by $R \log p/p_i$. Thus eq. (6b) transforms into

$$(7) \quad S - S_0 = R \sum_i n_i \log \frac{p}{p_i}.$$

Diffusion is an irreversible process in the same way as the conduction of heat. The increase in entropy found in (7) can be reversed only by the performance of work.

Taking into account the preceding equations of state (2) and (1a) we have $p/p_i = n/n_i$. The increase in entropy becomes a pure function of the numbers of mols n_i and of their sum $n = \sum_i n_i$:

$$(8) \quad S - S_0 = R \sum_i n_i \log \frac{n}{n_i} = R \left[n \log n - \sum_i n_i \log n_i \right].$$

¹It is true that Clausius and Waldmann have shown that diffusion is accompanied by measurable temperature differences. However, we are here only interested in the final state which sets in after the temperatures have become equalized.

The right-hand side of (8) may be called the *mixing term*. It depends solely on the number of molecules and not on their nature. This leads to the *paradox enunciated by Gibbs*: On going over to the limit of identical molecules eq. (8) would, apparently, remain unchanged. This is absurd, because when the partitions are removed from compartments enclosing completely identical molecules there is no diffusion. *The process of going over to the limit is inadmissible*. It contradicts the *atomistic nature of matter* and it is inconsistent with the fact that there is no continuous transition between different kinds of molecules (e. g. the atoms H and He).

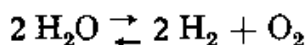
In order to explain it in greater detail we consider the case of very similar molecules, e. g. the isotopes of a noble gas. Equation (8) applies to this case without any correction; the same is true of a mixture of ortho- and para-hydrogen whose components differ only by their spin as well as of a mixture of molecules some of which are in the ground state, the remainder being in an excited state of energy. On the other hand eq. (8) fails when the molecules of the components are completely indistinguishable.

C. THE LAW OF MASS ACTION DUE TO GULDBERG AND WAAGE

So far we have assumed that the components of the mixture under consideration are chemically inert with respect to each other. When chemical reactions are possible in the system a state of equilibrium between the reactants and products will be reached eventually so that we now propose to study the precise nature of such an equilibrium. We shall assume that the chemical reaction takes place at constant pressure, p , (e. g. at constant atmospheric pressure) and at constant temperature, T . Such conditions can almost always be achieved if the experiment is suitably arranged. In accordance with Sec. 8 we can write down the conditions of equilibrium as

$$(9) \quad \delta G = 0 \quad \text{with} \quad G = U - TS + pV = H - TS.$$

The dissociation of steam into hydrogen and oxygen affords a simple example of such a process during which a stoichiometric mixture of oxy-hydrogen gas remains in equilibrium with water vapor. (The fact that steam cannot be regarded as a perfect gas at ordinary temperatures need not concern us here because no appreciable dissociation occurs unless the temperature has become high.) The chemical formula of the reaction, namely



asserts that 2 mols of H_2O must disappear for every two mols of H_2 and every mol of O_2 appearing in the system. If we denote the number of mols of H_2 ,

O₂ and H₂O present in the system by n_1 , n_2 and n_3 , respectively, and if we denote the integers associated with the chemical equation by ν_1 , ν_2 and ν_3 (reckoned positive for the substances on the right-hand side of the chemical equation, and negative for the other side) then we have for the present example

$$(9 \text{ a}) \quad \begin{cases} \nu_1 = 2, & \nu_2 = 1, & \nu_3 = -2 \\ \text{for H}_2, & \text{O}_2, & \text{and H}_2\text{O respectively.} \end{cases}$$

Further, considering a real or virtual change in the number of mols in the system we can establish the following proportions

$$(10) \quad \delta n_1 : \delta n_2 : \delta n_3 = \nu_1 : \nu_2 : \nu_3.$$

We shall now proceed to express the condition of equilibrium in eq. (9) in terms of the variables p , T , n_i and the parameters ν_i . The molar energies, as shown previously, are functions of T only. Hence

$$U = \sum_i n_i u_i(T).$$

The same is true of enthalpy, as seen from eqs. (1) and (1 a), or, of the molar enthalpy which will be denoted by h_i . Thus we have

$$(11) \quad H = \sum_i n_i h_i(T).$$

The rule concerning entropy is, however, different. In this connection it is necessary to recall that the molar entropies, $s_i(T, p)$, must be augmented by the term due to mixing and given in eq. (8), i. e. .

$$(12) \quad S = \sum_i n_i \{s_i(T, p) + R \log n/n_i\}.$$

From eqs. (9), (11) and (12) we can deduce that

$$G = \sum_i n_i \{h_i(T) - T s_i(T, p) - RT \log n/n_i\}$$

which can also be written as

$$(13) \quad G = \sum_i n_i \{g_i(T, p) - RT \log n/n_i\}$$

where g_i denotes the free enthalpy (Gibbs' function) of one mol of each individual component.

Considering that T and p are constant, we can obtain from eq. (13) that

$$\delta G = \sum_i \delta n_i \{g_i(T, p) - RT \log n/n_i\} - R T \sum_i n_i \delta \log n/n_i.$$

Since, however, $n = \sum_i n_i$, the last term in the above equation vanishes.

The condition of equilibrium $\delta G = 0$ can thus be rewritten as

$$(14) \quad \sum_i \nu_i \{g_i(T, p) - RT \log n/n_i\} = 0$$

where ν_i has been substituted for δn_i from the condition of proportionality, eq. (10), suitably modified. Taking anti-logarithms we obtain the *law of mass action* in the form

$$(15) \quad \prod_i (n_i/n)^{\nu_i} = K, \text{ where } \log K = -\frac{1}{RT} \sum_i \nu_i g_i(p, T).$$

This law was first discovered in 1867 by the Norwegian scientists Guldberg and Waage who used a line of argument based on statistical mechanics (probability of molecular encounters). Shortly afterwards, Gibbs demonstrated the validity of this law for perfect gases with the aid of purely thermodynamic considerations. He further extended the scope of this law by actually computing the value of the constant K . With certain limitations, the law of mass action can also be applied to vapors¹ and this constitutes one of the foundations of physical chemistry whose early development took place at the period under consideration.

If we now introduce molar concentrations, i. e. if we introduce the molar fractions $c_i = n_i/n$, we obtain from eq. (15)

$$(15 \text{ a}) \quad \prod_i c_i^{\nu_i} = K.$$

The quantity K is known as the "constant" of the equation of mass action or the *equilibrium constant*. In the example of the dissociation of water vapor we see from eq. (15 a) that

$$(16) \quad \frac{c_1^2 c_2}{c_3^2} = K.$$

¹The processes of polymerization and dissociation which may occur with vapors can also be included in the preceding argument if they are described with the aid of additional equations, and if the law is applied by setting up the conditions of equilibrium for several simultaneous reactions.

In order to determine the individual values of the three unknowns c_1 , c_2 and c_3 , we can utilize the additional condition:

$$(16 a) \quad c_1 + c_2 + c_3 = 1$$

which follows from $\sum n_i = n$, as well as the known (directly or by measurement) ratio of the number of atoms of hydrogen to that of oxygen which exist in the system:

$$(16 b) \quad \frac{N_H}{N_O} = \frac{2 c_1 + 2 c_3}{2 c_2 + c_3}.$$

In the more general case when more than three components take part in the reaction and when more than two atomic species are involved, the number of available equations is still sufficient to compute the individual values of c_i .

It is possible to eliminate the molar fractions, c_i , from eq. (15 a) and to use partial pressures, p_i , instead. We then have

$$(17) \quad \prod_i p_i^{v_i} = K_p \quad \text{where} \quad K_p = p^{\sum_i v_i} \times K.$$

This form of the law of mass action is useful because K_p is independent of p being a function of T alone. This fact can be proved as follows. Differentiating K in eq. (15) with respect to pressure we obtain

$$(18) \quad \frac{\partial \log K}{\partial p} = -\frac{1}{R T} \sum_i \left\{ v_i \frac{\partial g_i(p, T)}{\partial p} \right\}.$$

From the Table in Sec. 7 we find that

$$(18 a) \quad \frac{\partial g_i}{\partial p} = v_i(p, T) \quad \text{and} \quad \left(\frac{\partial g_i}{\partial T} \right) = s_i(p, T).$$

Here v_i denotes the molar volume of the i -th component under the pressure p and is equal to $R T/p$. Hence eq. (18) may be transformed into

$$\frac{\partial \log K}{\partial p} = -\frac{\sum_i v_i}{p} = \frac{\partial}{\partial p} \log p^{-\sum_i v_i}.$$

Furthermore, integrating with respect to p , we have

$$(18 b) \quad K = C \times p^{\sum_i v_i},$$

where C is independent of p . However, as seen from eq. (17), C is identical with K_p which proves our proposition.

We can obtain the relation between K and T in a similar manner. Differentiating eq. (15) with respect to T , we obtain

$$(19) \quad \frac{\partial \log K}{\partial T} = \frac{1}{R T^2} \sum_i \nu_i g_i - \frac{1}{R T} \sum_i \nu_i \frac{\partial g_i}{\partial T}$$

In view of eq. (18 a) we can write

$$(19 \text{ a}) \quad \frac{1}{R T^2} \sum_i \nu_i (g_i + T s_i) = \frac{1}{R T^2} \sum_i \nu_i h_i$$

for the right-hand side of eq. (19).

Having found the derivatives of $\log K$ with respect to p and T we can calculate the value of K itself and we find that it is determined except for a constant factor which depends only on the nature of the system under consideration. We shall obtain the latter in Sec. 14 C.

A discussion of the numerous applications of the law of mass action to problems of chemistry and engineering is beyond the scope of these lectures. It is only possible to sketch some general consequences of the equations just deduced. We shall repeat the most important ones here in a somewhat more complete form for the sake of future reference:

$$(15 \text{ a}) \quad \prod c_i^{\nu_i} = K(p, T)$$

$$(18) \text{ with } (18 \text{ a}) \quad \frac{\partial \log K}{\partial p} = - \frac{\sum_i \nu_i v_i}{R T} = - \frac{\Delta v}{R T}$$

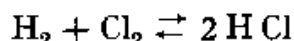
$$(19 \text{ b}) \text{ with } (19 \text{ a}) \quad \frac{\partial \log K}{\partial T} = \frac{\sum_i \nu_i h_i}{R T^2} = \frac{\Delta h}{R T^2}$$

Here the symbol Δv denotes the change in the molar volume of the whole system and Δh denotes the change in the total molar enthalpy during a process which involves the completion of the chemical reaction from left to right on the assumption that both the pressure and the temperature remain constant. It is seen from the first equation that an increase in K causes an increase in that c_i which is associated with a positive value of ν_i , i. e. with the substances which appear on the right-hand side of the chemical equation in accordance with the previously adopted convention (cf. eq. (9 a)).

The second of the above equations shows that an increase in pressure at constant temperature shifts the equilibrium in favor of that side of the chemical equation which corresponds to the *smaller volume*. Since it has been assumed

that only perfect gases take part in the reaction the change in volume, Δv , follows from Gay-Lussac's law of integral volume ratios discussed in Sec. 3 C and can be evaluated directly from the chemical equation. Thus in the case of the dissociation of steam considered previously in eq. (10) steam has the smaller volume, 2, compared with that of the sum of the volumes of the products of dissociation 2 H_2 and O_2 which is equal $2 + 1 = 3$. Hence an increase in pressure causes the concentration of steam to increase.

In the case of the explosive mixture of chlorine and hydrogen



$\Delta v = 0$ and equilibrium depends on temperature alone.

The last of the foregoing equations demonstrates that an increase in temperature at constant pressure shifts the position of equilibrium in the direction of that side in the equation which is associated with the higher enthalpy. At lower temperatures the gases which correspond to higher enthalpy are, practically, non-existent. Thus in the case of steam the degree of dissociation at the boiling point at 100 C is negligible.

In this connection it should be borne in mind that the science of thermodynamic equilibrium is only concerned with final stable states and makes no statements about the speed of reaction with which a certain state of equilibrium is reached. These speeds may be so low that "metastable" states become possible, but the latter cannot be dealt with on the ground of classical thermodynamics. This explains why a mixture composed of two parts hydrogen and one part oxygen by volume can exist for any length of time even though it is not in a thermodynamically stable state of equilibrium. Such metastable states can be included under the heading of "constrained equilibria" as defined in Sec. 6 C.

The *synthesis of ammonia* from hydrogen and atmospheric nitrogen is of great practical importance in industry. It occurs in accordance with the equation



which shows that the molar volume decreases from 4 to 2. In accordance with the second of the foregoing equations the concentration of ammonia increases with pressure. The extraordinary success with which this synthesis is now carried out in industry is due to the complete understanding of the conditions for thermodynamic equilibrium (Haber), to the mastery of the engineering problems connected with high pressure (Bosch) and, finally, to the successful selection of catalyzers which promote high reaction rates (Mittasch).

14. Chemical potentials and chemical constants

In the preceding Sections we encountered the parameters n_i , i. e. the numbers of mols of the individual components, in addition to the properties p , T or v , T . They are properties, being typical *extensive quantities*.

In eq. (8.3) we have already admitted the possibility of having any number of properties (always, naturally, a finite number of them). Starting with this equation we can combine the First Law with the Second and we may write

$$(1) \quad T dS = dU + p dV + \sum_i X_i dx_i.$$

We shall now identify the quantities x_i with our n_i 's. Following Gibbs, the *intensive quantities* which are canonically conjugate with them will be denoted by $-\mu_i$, so that eq. (1) now becomes

$$(2) \quad T dS = dU + p dV - \sum_i \mu_i dn_i,$$

$$(2 a) \quad dU = T dS - p dV + \sum_i \mu_i dn_i.$$

Instead of (2) we can also write

$$(2 b) \quad dH = dU + d(pV) = T dS + V dp + \sum_i \mu_i dn_i,$$

$$(2 c) \quad dF = dU - d(TS) = -S dT - p dV + \sum_i \mu_i dn_i,$$

$$(2 d) \quad dG = dH - d(TS) = -S dT + V dp + \sum_i \mu_i dn_i.$$

A. THE CHEMICAL POTENTIALS μ_i

First we notice that no changes have to be introduced to our Table in Sec. 7, which was originally limited to two independent variables, provided that the new additional variables n_i are kept constant. If, however, their variation is permitted it is necessary to add the following to the preceding differential relations, depending on whether we use (2 a, b, c, d)¹:

¹The subscript n_j denotes that all n_j 's with respect to which we do not differentiate are kept constant.

$$(3\ a) \quad \mu_i = \left(\frac{\partial U}{\partial n_i} \right)_{S, V, n_j},$$

$$(3\ b) \quad \mu_i = \left(\frac{\partial H}{\partial n_i} \right)_{S, p, n_j},$$

$$(3\ c) \quad \mu_i = \left(\frac{\partial F}{\partial n_i} \right)_{T, V, n_j},$$

$$(3\ d) \quad \mu_i = \left(\frac{\partial G}{\partial n_i} \right)_{T, p, n_j}.$$

It is evident that it is possible to deduce in the same way relations which correspond to Maxwell's equations in Sec. 7. For example:

$$(4) \quad \frac{\partial V}{\partial n_i} = \frac{\partial \mu_i}{\partial p}$$

(it is implied that on the right-hand and on the left-hand sides $S, p, n_j \neq n_i$ and S, n_i , respectively, are kept constant). Of the eqs. (3 a, b, c, d) the last one is the most important relation. In order to examine it more closely let us imagine that our system has been increased by a given factor, say γ . All extensive quantities, i. e. the numbers of mols n_i , the volume V , and the entropy S , will become multiplied by γ , whereas all intensive quantities, pressure, temperature, and the potentials μ_i , will remain unchanged. We conclude from (3 d) that G becomes multiplied by γ in the same way as the n_i 's. Thus we see that G must be a *homogeneous function of the first degree in the n_i 's*:

$$(5) \quad G(p, T, \gamma n_i) = \gamma G(p, T, n_i).$$

We can make the same assertion with regard to the quantities U, H, F , when we consider them to be functions of T, p, n_i ; but, for example, the free energy, regarded as a potential and expressed with the aid of the variables T, V, n_i which correspond to it, must satisfy the relation (c/. Sec. 7) $F(T, \gamma V, \gamma n_i) = \gamma F(T, V, n_i)$.

Applying Euler's rule for homogeneous functions (differentiation with respect to γ , with $\gamma = 1$) we obtain

$$(6) \quad G = \sum_i n_i \left(\frac{\partial G}{\partial n_i} \right)_{T, p, n_j}.$$

Combining this with (3 d), we have

$$(7) \quad G = \sum_i \mu_i n_i.$$

The μ_i' s depend on p , T , and the n_i' s. The dependence on the latter must be such as to yield a homogeneous function of order zero, i. e. pure functions of the ratios of numbers of mols, or of the so-called "molar concentrations." It follows, further, from eq. (3 d) that the condition of equilibrium $\delta G = 0$ can be written

$$(8) \quad \delta G = \sum_i \left(\frac{\partial G}{\partial n_i} \right)_{T, p, n_j} \delta n_i = \sum_i \mu_i \delta n_i = 0, \quad \delta p = 0, \quad \delta T = 0.$$

This must be supplemented with the auxiliary conditions which express the fact that the number of chemical atoms is preserved. Applying this to a single equation for the chemical reaction in the same way as in the case of the law of mass action in eq. (13.10) we have:

$$(9) \quad \delta n_1 : \delta n_2 : \delta n_3 : \dots = \nu_1 : \nu_2 : \nu_3 : \dots$$

Combining this with eq. (8), we obtain

$$(10) \quad \sum_i \mu_i \nu_i = 0.$$

In this equation we have not stipulated, as was the case with the law of mass action, that the reactants are perfect gases. Thus we may regard eq. (10) as a *universally valid* formulation of the *law of mass action*.

The real difficulty consists now in the determination of the μ_i' s; it constitutes the main problem in the science of physical chemistry and can be solved only on the basis of measurements. Fortunately the extent to which this is required is reduced by the existence of certain identities, and we shall now proceed to deduce them. From (7) we have quite generally:

$$(11) \quad dG = \sum_i n_i d\mu_i + \sum_i \mu_i dn_i.$$

Since G is a property we may write:

$$(11 a) \quad dG = \left(\frac{\partial G}{\partial p} \right)_{n_j, T} dp + \left(\frac{\partial G}{\partial T} \right)_{n_j, p} dT + \sum_i \left(\frac{\partial G}{\partial n_i} \right)_{T, p, n_j} dn_i.$$

According to (2 d) the factors of dp and dT are equal to V and $-S$ respectively, and it is seen from eq. (3 d) that the factors of dn_i are exactly equal to our μ_i' s. On comparing (11) with (11 a) we conclude that

$$(11 b) \quad \sum_i n_i d\mu_i = V dp - S dT.$$

We shall now apply this general equation to the case when, of the independent variables p, T, n_i in eq. (11 a), p, T are kept constant and only the n_i 's are varied. Taking into account that the μ_i 's are properties, we obtain

$$(11\ c) \quad \sum_i \sum_k n_i \left(\frac{\partial \mu_i}{\partial n_k} \right)_{T, p, n_j} dn_k = 0.$$

Now in this equation the n_i 's are independent variables (the auxiliary conditions which had to be taken into consideration, for example in connection with the law of mass action, are unimportant, because (11 b) is true not only at chemical equilibrium, being also valid in the case when the numbers of atoms involved vary), and eq. (11 c) must be satisfied for each dn_k . Hence

$$(12) \quad \sum_i n_i \left(\frac{\partial \mu_i}{\partial n_k} \right)_{T, p, n_j} = 0 \quad (k = 1, 2, \dots, K).$$

Since the μ_i 's depend only on the ratios of the n_k , i. e. on the concentrations c_k , the system of equations (12) can be written as

$$(12\ a) \quad \sum_i c_i \left(\frac{\partial \mu_i}{\partial c_k} \right)_{T, p, c_j} = 0 \quad (k = 1, 2, \dots, K-1),$$

assuming K substances, or $K-1$ concentrations. The μ_i 's are to be regarded as functions of the $K-1$ variables c_1, c_2, \dots, c_{K-1} . Equations (12 a) occur already in Gibbs' writings; they are, however, usually described as the *Duhem-Margule conditions*.

B. RELATION BETWEEN THE μ_i 's AND THE g_i 's FOR IDEAL MIXTURES

We have expressed in Sec. 13 the free enthalpy for the special case when perfect gases are involved in terms of all the variables p, T, n_i in giving eq. (13.13). This equation has the form of our present eq. (7) so that on comparing the latter with (13.13) we can at once write down an equation for the chemical potentials:

$$(13) \quad \mu_i = g_i(T, p) - R T \log \frac{n}{n_i}; \quad n = \sum_i n_i.$$

It has already been stressed in Sec. 13 that the symbol g_i denotes the molar free enthalpy of the pure component i . As we can see now, the chemical potential differs from it. What is the origin of the term $R T \log n/n_i$? The answer is: The increase in entropy on mixing (cf. (13.8)). Mixtures which

obey eq. (13) will be called *ideal mixtures* even when no perfect gases are involved. It is easy to verify that the chemical potentials given in (13) satisfy the Duhem-Margule conditions (see Problems to II). In the case of non-ideal mixtures the interaction of components may give rise to heat tones, changes in volume, etc.

C. THE CHEMICAL CONSTANT OF A PERFECT GAS

We now revert to the question of how far we can predict the equilibrium constant $K(p, T)$ in the law of mass action, already posed at the end of Sec. 13. According to (13.15) it is necessary to know completely the quantities g_i which we shall now write in the form

$$(14) \quad g_i = h_i - T s_i.$$

We know that for a perfect gas and for temperatures which are not too low we have

$$(15) \quad h_i = c_{pi} T + h_{io};$$

$$(16) \quad s_i = c_{pi} \log T - R \log p + s_{io}.$$

Here h_{io} and s_{io} denote integration constants whose values cannot be determined with the aid of thermodynamics alone. The Third Law does not help here directly because the laws under consideration must not be extrapolated to $T \rightarrow 0$. We shall stress here that quantum mechanics confirms eq. (16) for sufficiently large temperatures, T , and thus leads to a definite value of s_0 . The exact value of c_p is also given by quantum mechanics so that the quantity

$$(17) \quad i_j = \frac{s_{jo} - c_{pj}}{R}$$

can be calculated for each component j . The quantities i_j are called *chemical constants* and their interpretation will become clear on substituting our present eqs. (14), (15), (16) into eq. (13.15). We thus obtain

$$\log K = \sum_j v_j \left[\frac{c_{pj}}{R} \log T - \log p - \frac{h_{jo}}{RT} + \frac{s_{jo} - c_{pj}}{R} \right],$$

where the last term in the square bracket is identical with i_j . Hence the law of mass action can be written

$$(18) \quad \prod_i c_i^{v_i} = p^{-\sum_j v_j} T^{\sum_j v_j c_{pj}/R} \times e^{\sum_j v_j i_j - \tau_0/RT}.$$

The only quantity which must be determined experimentally, and which we may formally denote as "the heat of reaction at absolute zero" is

$$(19) \quad r_0 = \sum_j v_j h_{j0}.$$

This is the constant which was mentioned in connection with eq. (13.19 a).

15. Dilute solutions

A solution is called dilute when the quantity of the solvent (e. g. water) is much larger than that of the solute (e. g. sugar). Dilute solutions differ from concentrated solutions by the simplicity of their behavior in a way similar to the simplicity of the behavior of perfect gases, as compared with real gases, with the exception of strong electrolytes.

A. GENERAL AND HISTORICAL REMARKS

When the solute is introduced into the solvent it will diffuse *uniformly* over the solvent irrespective of the initial state, in the same way as a gas will diffuse over the volume at its disposal. This behavior, in the same way as with a gas, is ascribed to the action of a pressure acting on the solute. It is called *osmotic pressure* and it is denoted by P . Its presence can be shown by the application of a semipermeable membrane which is permeable to the solvent but not to the solute, *cf.* Sec. 13 B. This membrane experiences only the osmotic pressure P being insensitive to the pressure in the solvent.

If the membrane is placed between the solution and the solvent and if it is free to move then work will have to be performed in order to move it towards the side on which there is the solution. The solvent will cross the membrane and the solution will become more concentrated. Conversely, work can be obtained by moving the membrane in the direction of the solvent thus allowing it to penetrate into the solution so that the concentration decreases. There is, consequently, a bias towards dilution which manifests itself by the availability of this positive work. We can say that the membrane exerts a suction on the solvent which is opposed to the solute's tendency to spread out and which is proportional to the latter's osmotic pressure.

This corresponds to the arrangement which was first used by Pfeffer (investigations into osmosis, Leipzig 1877) to measure osmotic pressures. A long tube is inserted into a beaker filled with water, the tube being closed at the bottom with the aid of a semipermeable membrane (copper ferrocyanide, *cf.* Sec. 13 A). Since the membrane is permeable to water the levels in the

tube and beaker will at first be equal. If now sugar is added to the water in the tube, the water will begin to rise in it in proportion to the quantity of sugar added. Equilibrium is reached when the hydrostatic pressure at the lower end of the tube becomes equal to the osmotic pressure P of the solution. When the solutions are concentrated the differences in level may be equal to several meters of water and Pfeffer was forced to use a closed mercury manometer in his later experiments.

Osmotic pressure and semipermeable membranes play a most important part in nature's economy. The riddle of how the juices can penetrate to the tops of tall trees can only be solved by the recognition of the existence of osmotic pressure. The walls of organic cells both in animals and plants are all semipermeable. The protoplasm in the walls of cells must have the same osmotic pressure as the external fluid in which the cell is immersed: both must be "isotonic" (iso-osmotic). If the external osmotic pressure is larger, the cell will contract, and it will burst if the reverse is true. In the field of medicine, for example in illnesses involving blood corpuscles, both possibilities play a remarkable role.

B. VAN 'T HOFF'S EQUATION OF STATE FOR DILUTE SOLUTIONS

If we now wish to obtain quantitative results in addition to the preceding general statements we must consider the special case of reversible processes in dilute solutions. Van 't Hoff followed this line of thought in 1885 and discovered a certain similarity between dilute solutions and perfect gases.

In order to prove the analogy it is customary to consider *cycles* involving a moving piston which is assumed to be semipermeable during one stroke and impervious during the next and to compare the amounts of work performed by or on the system. We shall, however, base the argument on the consideration of our general *equilibrium conditions*, because in this way we shall achieve our goals much faster.

The system will be assumed to consist of two parts — the pure solvent and the solution, both interacting across a semipermeable wall. All quantities relating to the substance of the "solvent" will be denoted by the subscript 1, and those relating to the "solute" will obtain the subscript 2. The sub-system "solution" will be denoted by the superscript 1 and the sub-system "pure solvent" on the other side will be denoted by the superscript 2. Thus we shall distinguish between the mol numbers n_1^1 , n_1^2 and n_2^1 . Since the sub-system 2 contains no solute, we have

$$(1) \quad n_2^2 = 0.$$

We stipulate that the semipermeable wall shall not prevent an exchange of heat so that the temperature is assumed uniform throughout the system. We must, however, admit the possibility of the pressures p^1 and p^2 on the two sides of the wall being different.

When deducing the equilibrium condition in Sec. 8 we could have considered the case $T = \text{const}$, $p^1 = \text{const}$ and not equal to $p^2 = \text{const}$. It is, however, easy to see that we would have obtained nothing new and that the old condition

$$(2) \quad \delta G = 0$$

would turn out to apply to the system $p^1 \neq p^2$.

The auxiliary conditions are

$$(3) \quad \delta n_1^1 + \delta n_1^2 = 0,$$

$$(4) \quad \delta n_2^1 = \delta n_2^2 = 0.$$

Equation (3) expresses the conservation of mass for the solvent, and eq. (4) is a mathematical expression of the properties of the semipermeable wall. Introducing the chemical potentials from Sec. 14 A into (2) and taking into account (4), we obtain

$$(5) \quad \mu_1^1 \delta n_1^1 + \mu_1^2 \delta n_1^2 = 0.$$

In view of (3) we have further

$$(6) \quad \mu_1^2(p^2, T) = \mu_1^1\left(p^1, T, \frac{n_2^1}{n_1^1}\right).$$

The respective variables have been shown in the brackets in order to make the relations clearer. We have already established in Sec. 14 that the μ 's depend only on the ratios of the numbers of mols — the "concentrations." The argument n_2^2/n_1^2 could be omitted from μ_1^2 because it refers to the homogeneous sub-system consisting of pure solvent in which, according to (1), the concentration n_2^2/n_1^2 is equal to zero. We can now see quite clearly that a non-zero value of n_2^1 must necessarily imply a difference $p^2 - p^1 \neq 0$ because otherwise the condition of equilibrium (6) could not be satisfied. The reason lies in the relation $\delta n_1^1 + \delta n_1^2 = 0$ which is so characteristic in the consideration of equilibrium and which expresses a conservation law. The difference

$$(7) \quad P = p^1 - p^2$$

is called *osmotic pressure*.

Equation (6) is the exact equation of state for any solution; in order to be in a position to apply it, it would be necessary to know the chemical potentials. Since, generally speaking, this is not the case, we are forced to use experimental results and semi-empirical formulae, just as was the case with real gases. In analogy with gases, where the perfect gas constitutes a limiting case, we can consider a limiting case in this connection too; it is, namely, possible to treat the case of highly dilute solutions with purely theoretical means. Hence we now imply that

$$n_1^1 \gg n_2^1.$$

When the substance 2 penetrates into the solvent 1 the only change *with respect to substance 1* is an increase in molecular disorder, i. e. the generation of entropy due to mixing. The assumption implied here is borne out by experiment to a high degree of accuracy; it can be further reinforced by considerations of a thermodynamic or of a statistical nature.

Denoting the molar free enthalpy of the solvent by g_1 we have to assume that

$$(8) \quad \begin{cases} \mu_1^1 = g_1(p^1, T) - R T \log \frac{n_1^1 + n_2^1}{n_1^1} \\ \mu_1^2 = g_1(p^2, T) \end{cases}$$

in accordance with Sec. 14 B. In view of the assumption that $n_2^1 \ll n_1^1$ we have $\log(1 + n_2^1/n_1^1) \approx n_2^1/n_1^1$. Hence it follows from (6) and (8) that

$$(9) \quad g_1(p^2, T) = g_1(p^1, T) - R T \frac{n_2^1}{n_1^1}.$$

We now expand the left-hand side into a Taylor series and retain the first term only:

$$(10) \quad g_1(p^2, T) = g_1(p^1, T) + (p^2 - p^1) \left[\frac{\partial g_1(p^1, T)}{\partial p} \right]_T + \dots$$

According to our fundamental Table in Sec. 7 we have

$$(11) \quad \frac{\partial g_1}{\partial p} = v_1,$$

where v_1 is the molar volume of the pure solvent. Substituting (10), (11) and (7) into (9), we obtain

$$(12) \quad P v_1 = R T \frac{n_2^1}{n_1^1}.$$

Since the partial volume of the solute is small compared with the volume V of the solution we may put

$$V = n_1^1 v_1,$$

with the same degree of approximation as that in all preceding formulae. From now on we shall use the symbol n to denote the number of mols n_2^1 of the solute. Hence

$$(13) \quad P V = n R T.$$

This is van 't Hoff's proposition: *The osmotic pressure of a dilute solution of n mols of solute is equal to the pressure of a perfect gas which would be measured if n mols of the gas exerted pressure on the walls of a vessel of total volume V equal to that of solvent and solute.*

When more solutes exist in the solution, their quantities being n_1, n_2, \dots , we would find in the same way that

$$(14) \quad P V = (n_1 + n_2 + \dots) R T,$$

which is the equation of state of a mixture of perfect gases. It is, therefore, possible to define the partial pressure P_i of a single solvent

$$(14 \text{ a}) \quad P_i V = n_i R T$$

and the total osmotic pressure

$$P = \sum_i P_i, \quad .$$

in the same way as for perfect gases. The unexpected form of eq. (13) was difficult to grasp when first discovered. Any doubts have now been removed by the results of an enormous volume of experimental material and by considerations based on kinetic theory (due to H. A. Lorentz among others).

16. The different phases of water. Remarks on the theory of the steam engine

In the present Section we shall study the equilibrium between the different phases of water. On this occasion we wish to make some remarks on the origins of thermodynamics and, in particular, on its connection with the development of the steam engine; the reader is also reminded of the paragraph on Carnot in the introduction to Sec. 6.

A. THE VAPOR-PRESSURE CURVE AND CLAPEYRON'S EQUATION

We begin by considering the following well-known facts: Imagine a cylindrical vessel filled with water and a piston which fits snugly over its upper surface. There is no air between the water and the piston. We now pull the piston out of the cylinder, the temperature being kept constant all the time, and notice that some water evaporates. The quantity of steam formed between the water level and the piston is just sufficient to maintain a constant

pressure which is independent of that volume and is a function of temperature alone. The steam is said to be "saturated." On reversing the piston the steam is not compressed but water is formed, its quantity being again just sufficient to insure that the steam remains saturated. It is thus seen that there exists an equation:

$$(1) \quad \phi(p, T) = 0$$

which is independent of volume, and which connects the prescribed temperature T with the pressure p of saturated steam. The plot of eq. (1) in p, T coordinates yields

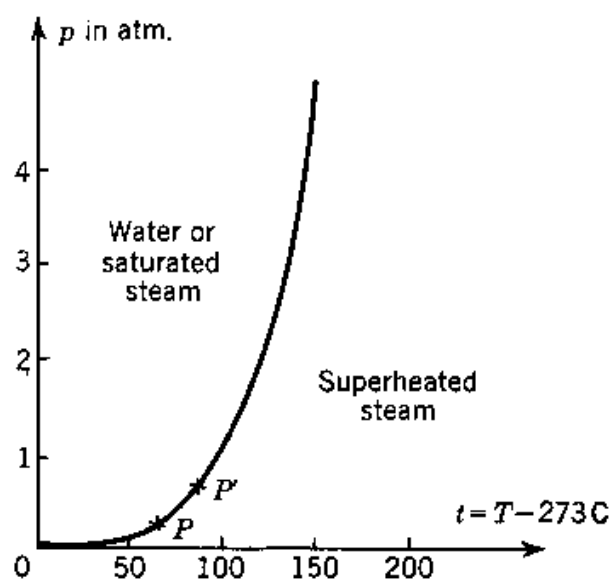


Fig. 13.

Phase equilibrium between water and steam.

the vapor-pressure curve, as shown by the curve in Fig. 13. Some pairs of values are given as follows:

$t = 0^\circ \text{C}$	50°C	100°C	120°C	200°C
$p = 6 \times 10^{-3}$	0.126	1.03	2.02	15.9
kp/cm ² .				

This independence of volume is consistent with the van der Waals *model of the process of liquefaction*. Reverting to Fig. 10 we can imagine that the temperature is the third coordinate and that it is plotted at right angles to the p, V plane, and we can fix our attention on the resulting "state surface." Looking at the surface from the direction of the V -axis, i. e. projecting it onto the p, T plane we shall notice that each Maxwell line AB will yield one point in the p, T plane. The fact that along every such straight line the volume varies according to the ratio of the mass of liquid to that of vapor, as given by eq. (10.12), cannot be deduced from Fig. 13. In particular, the two curves denoted as "liquid line" and "vapor line" in the p, V plane give one

single projection in the p, T plane, namely our vapor-pressure curve (1). This curve ends abruptly at the point which corresponds to the critical point on the state surface. It is seen that the van der Waals equation takes into account the fact expressed in eq. (1) (not only in the case of steam, but for all condensing gases).

We shall now endeavor to find an analytic expression for eq. (1). Since the variables p, T are being considered, use will be made of the free enthalpy G and of the equilibrium condition $\delta G = 0$, in accordance with Sec. 8. We put

$$(2) \quad G = n g_2 + (N - n) g_1,$$

where the subscript 2 refers to the "higher" phase (steam) and the subscript 1 to the "lower" phase (water). A phase will be called calorically higher than another if the transition from the latter to the former is connected with an *addition* of heat. Later we shall use the subscript 0 to denote the solid phase. The symbols g_1 and g_2 denote the free enthalpies per mol of liquid and vapor respectively, n denotes the number of mols of steam, N that for vapor + liquid; g_1 and g_2 are pure functions of p and T .

The condition $\delta G = 0$ applied to (2) at constant p, T and N leads to $(g_2 - g_1) \delta n = 0$, so that

$$(3) \quad g_2 = g_1.$$

This is the analytical expression for eq. (1). We shall now proceed to evaluate it, but before doing so we shall deduce from it a useful differential relationship.

Consider two neighboring points P and P' , Fig. 13, whose coordinates are p, T and $p + dp, T + dT$. With reference to the Table in Sec. 7 we calculate

$$(4) \quad \begin{aligned} g(p + dp, T + dT) &= g(p, T) + dp \left(\frac{\partial g}{\partial p} \right)_T + dT \left(\frac{\partial g}{\partial T} \right)_p = \\ &= g(p, T) + dp \times v - dT \times s. \end{aligned}$$

We now form the difference for phases 1 and 2, denoting it by Δ , for example

$$\Delta v = v_2 - v_1; \quad \Delta s = s_2 - s_1; \quad \Delta g = g_2 - g_1,$$

and conclude from (4) that

$$(4a) \quad \Delta g(p + dp, T + dT) = \Delta g(p, T) + dp \times \Delta v + dT \times \Delta s.$$

The left-hand side must vanish as well as the first term on the right-hand side in view of the fact that eq. (3) applies to both points P and P' . Hence we have

$$(5) \quad \frac{dp}{dT} = \frac{\Delta s}{\Delta v}.$$

It is convenient to express Δs in terms of enthalpy h . Remembering the relation $g = h - T s$, cf. Sec. 7, and noting that $\Delta g = 0$ and $\Delta T = 0$, we obtain

$$(5 a) \quad \Delta h = T \Delta s.$$

$\Delta h = h_2 - h_1$ represents the quantity of heat per mol required for the phase transition $1 \rightarrow 2$ at constant pressure. It is called the "latent heat of evaporation" and it will be denoted by r . Substituting (5 a) into (5), we obtain

$$(6) \quad \frac{dp}{dT} = \frac{r}{T \Delta v}.$$

This is the famous equation due to Clapeyron; it was proved thermodynamically for the first time by Clausius. If Δv denotes the difference in the specific volume as is usual, and not that in the molar volumes, the meaning of the symbol r must be adjusted accordingly (specific and not molar latent heat of evaporation).

On comparing (6) with the preceding formula (7.7) we notice that the total derivative dp/dT and the ratio $\Delta s/\Delta v$ replace the partial derivatives $(\partial p/\partial T)_v$ and $(\partial s/\partial v)_T$ in the expression for a single-phase system.

Clapeyron's equation provides the means for the point by point calculation of the vapor-pressure curve from its tangents if r and v are known from measurement. Instead of performing such a step-by-step process of integration we revert to eq. (3) which must, evidently, contain the result of such an integration. We shall assume that the pressure is so low that the vapor can be treated like a perfect gas. According to Sec. 14, eqs. (14) to (17), we obtain

$$(7) \quad g_2 = R T \left\{ \log p - \frac{c_{p2}}{R} \log T + \frac{h_{20}}{R T} - i_2 \right\}.$$

In order to establish the corresponding expression for g_1 , we neglect the changes in volume, as is usual for liquids and solids. It is now superfluous to differentiate between c_p and c_v and it suffices to consider one specific heat, c_{liq} . Hence we obtain the following expressions for entropy and enthalpy:

$$(8) \quad h_1 = h_{1m} + \int_{T_m}^T c_{liq} dT,$$

$$(9) \quad s_1 = s_{1m} + \int_{T_m}^T \frac{c_{liq}}{T} dT.$$

The symbols h_{1m} and s_{1m} denote the values of enthalpy and entropy at a provisionally arbitrary temperature T_m which may be chosen as that at the melting point. The constants h_{1m} and s_{1m} can thus be determined with the aid of the Third Law and the caloric properties of the solid phase.

It follows that the free enthalpy of the liquid is given by

$$(10) \quad g_1 = \int_{T_m}^T c_{liq} dT - T \int_{T_m}^T \frac{c_{liq}}{T} dT + h_{1m} - T s_{1m}$$

and our conditions of equilibrium (3) together with (7) and (10) yield:

$$(11) \quad \log p = \frac{c_{p2}}{R} \log T + \frac{1}{R T} \int_{T_m}^T c_{liq} dT - \frac{1}{R} \int_{T_m}^T \frac{c_{liq}}{T} dT - \frac{h_{20} - h_{1m}}{R T} + i_2 - \frac{s_{1m}}{R}.$$

In principle all quantities in the above equation, with the exception of the chemical constant i_2 , must be obtained from measurements on the solid phase. If it is considered that the value of i_2 given by quantum mechanics is not reliable enough it is possible to check it with the aid of a single measurement of vapor pressure. The vapor-pressure curve for water obtained in the above way, and fully confirmed by experiment, is shown plotted in Fig. 13. It ascends monotonically as could have been already inferred from the Clapeyron equation. In fact r is always positive (this is in agreement with our previous definition of a "higher phase"); in addition, we must naturally have $\Delta v > 0$ (since $v_2 \gg v_1$).

The highest temperature which need be considered (see above) is the critical temperature. This is the physically natural end point of the curve (it is not shown in Fig. 13 because it lies outside its range). The curve also has a natural initial point which lies near the origin of our diagram in the case of water, *c/* Sec. 17 A; it can be used to make a direct check on the value of i_2 .

B. PHASE EQUILIBRIUM BETWEEN ICE AND WATER

Ice is the lower phase with respect to water because the melting of, for example, one gram of ice requires the introduction of the *latent heat of fusion* (melting), r . Since we have agreed to denote the solid phase by the subscript 0, the symbol Δv will now denote the difference $v_{water} - v_{ice}$. Applying the proper interpretation of the symbol Δ to g , h , s we again obtain formally Clapeyron's

equation (6) from eqs. (4) and (5), with the important difference that Δv is now negative:

$$\Delta v = v_1 - v_0 = (1.00 - 1.091) \text{ cm}^3/\text{g} = -0.091 \text{ cm}^3/\text{g}.$$

This fact is fundamental for the existence of life on earth. *Ice floats on water.* If this were not so, all fishes would die in winter and no life could develop at our latitudes. (As is well known, land animals have evolved from water animals.) *Water expands on freezing.* The erosion of mountains which allows fertile soil to reach the valleys is a consequence of this fact (bursting of rock when water freezes in fissures).

Clapeyron's equation shows that: The *melting curve* descends as T increases, unlike the *vapor-pressure curve*. In the neighborhood of 0 C the numerical value of the latent heat is

$$r = 80 \frac{\text{cal}}{\text{g}} = 80 \times 42.7 \frac{\text{at cm}^3}{\text{g}}.$$

The last value follows from (4.6), (3.2) and (3.2 a); hence at $T \sim 273 \text{ K}$ and $t = T - 273 \text{ C} = \text{Celsius temperature}$, we have

$$(12) \quad \frac{dp}{dt} = - \frac{80 \times 42.7}{273 \times 0.091} \frac{\text{at}}{\text{deg}} = -138 \frac{\text{at}}{\text{deg}}.$$

Accordingly the melting curve $\phi(p, t) = 0$ starts at $p \approx 0$ at $t \approx 0 \text{ C}$ and passes through the second quadrant of Fig. 13; it is a very steep, nearly straight, line, it being necessary to reach $p = 138$ at in order to depress the melting temperature to $t = -1 \text{ C}$. This "depression of the melting point" plays an important part in the motion of glaciers, notwithstanding the fact that it is so small. The deeper parts of a glacier begin to move under the pressure of the masses above them but freeze again when the pressure decreases (regelation). The ease with which a skater moves on ice also depends on this fact; the ice which melts under the pressure of the skate acts as a lubricant.

C. THE SPECIFIC HEAT OF SATURATED STEAM

So far we have only discussed the specific heats c_p and c_v . It is, however, possible to define a specific heat for any process, i. e. for any path in the p, v plane. It is immediately clear that on progressing along an isentrope ($dq = 0$) we have $c_s = 0$; on the other hand we can assert that: $c_T \approx \infty$ because on progressing along an isotherm there is no change in temperature no matter

how large dq is. We can also define a specific heat along any path in the p, T plane.

We are particularly interested in the specific heat of steam, c_ϕ , on progressing along the vapor-pressure curve $\phi(p, T) = 0$. Applying the definitions of the latent heat of evaporation $r = \Delta h$ from eq. (5 a.) and taking into account that $h = u + p v$, we obtain

$$(13) \quad \frac{dr}{dT} = \frac{d\Delta u}{dT} + p \frac{d\Delta v}{dT} + \frac{dp}{dT} \cdot \Delta v.$$

According to the First Law

$$(13 a) \quad \frac{dq}{dT} = \frac{du}{dT} + p \frac{dv}{dT}$$

along any path, so that in the case of steam, along the vapor-pressure curve we must have in particular:

$$(13 b) \quad c_\phi = \frac{dq_\phi}{dT} = \frac{du_2}{dT} + p \frac{dv_2}{dT}.$$

The corresponding equation for the liquid phase is

$$(13 c) \quad c_{liq} = \frac{du_1}{dT} + p \frac{dv_1}{dT}$$

because the difference between c_p and c_v for the liquid phase can be neglected, as already remarked in connection with eq. (2.4). Hence we may put $c_p \approx c_v = c_{liq}$ and it follows from (13 b, c) that

$$c_\phi - c_{liq} = \frac{d\Delta u}{dT} + p \frac{d\Delta v}{dT}.$$

Substituting this into (13), we find:

$$(13 d) \quad \frac{dr}{dT} = c_\phi - c_{liq} + \frac{dp}{dT} \cdot \Delta v.$$

Taking, finally, into account the Clapeyron equation (6), we have:

$$(14) \quad c_\phi = \frac{dr}{dT} + c_{liq} - \frac{r}{T}.$$

According to the very precise measurements carried out by engineers¹ we have at $T \sim 373 \text{ K}$:

$$(14 a) \quad \frac{dr}{dT} = -0.64 \frac{\text{cal}}{\text{deg g}},$$

¹In this connection cf. Problem II. 2.

so that with the value $r = 539$ cal/g for $T = 373$ K and $c_{liq} = 1$ cal/deg \times g

$$(14\ b) \quad c_\phi = (1 - 0.64 - 1.44) \frac{\text{cal}}{\text{deg g}} = -1.07 \frac{\text{cal}}{\text{deg g}}.$$

Saturated steam requires no heat when its state is changed along the vapor-pressure curve; it is in a position to reject heat. On the other hand, if saturated steam is expanded without the addition of heat it will, as seen from Fig. 13, enter the region below the vapor-pressure curve, marked "water or super-saturated steam." It tends to condense in this region.

We shall now quote two examples, one trivial, the other fundamental for modern physics: In a bottle containing mineral water the atmosphere between the free surface of the liquid and the plug is one of saturated steam. When the bottle is opened suddenly the speed of the process insures its being adiabatic. The steam condenses into drops. This phenomenon finds a beautiful application in C. T. R. Wilson's (1912) *cloud chamber*.

The chamber contains saturated steam and is expanded very suddenly. If just prior to the expansion ionizing material particles have been allowed to penetrate into it, the resulting ions will act as nuclei of condensation and thus the paths of the particles will be made visible. The importance of this method of research (cosmic rays, discovery of positrons, mesons, etc.) is very well known.

The fact that c_ϕ is negative is of some importance in steam engineering. When steam is expanded isentropically, the corresponding curve is less steep than the isentrope $p v^\gamma = \text{const}$ for a perfect gas. Owing to this fact the indicator diagram, Fig. 14, increases by the area shown shaded in the sketch. This has some advantages from the point of view of the design of a reciprocating steam engine.

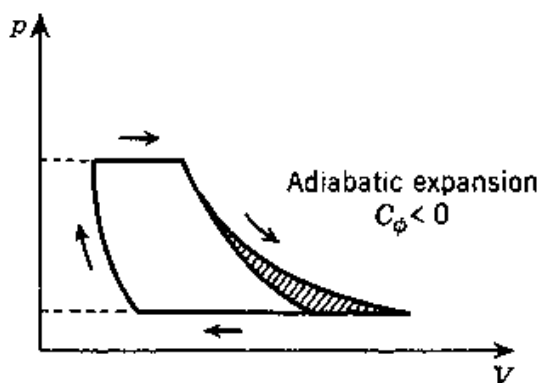


Fig. 14.

The indicator diagram, corrected for negative specific heat along the vapor-pressure line.

17. General remarks on the theory of phase equilibria

The study of the different phases of water is only a fortuitous example which belongs to the much more general problem of the co-existence of the phases of substances of arbitrary chemical composition. Even in the case of water the preceding analysis was incomplete. In the solid state, in addition

to the ordinary hexagonal ice whose structure is so beautifully exhibited by microscopic pictures of snow flakes, there exist, according to Tammann, numerous allotropic modifications; these show a preference for other regions in the p, T plane. Furthermore, the complete study of the phases of water would have to include its dissociation into oxy-hydrogen gas, *cf.* Sec. 13, when the system ceases to be *homogeneous*, (H_2O) and becomes *heterogeneous* (H_2, O_2).

A. THE TRIPLE POINT OF WATER

We now revert to the state diagram for water in the p, T plane. With reference to Fig. 13 we now draw a diagram in which the p -axis is horizontal and the t -axis is vertical (t denotes the Celsius temperature), both to a very much enlarged scale. The vapor-pressure curve has been drawn schematically and it is seen that it now is convex upwards. Further, the melting line is also seen sketched; according to Sec. 16 B it is nearly a straight line inclined downwards at a very small angle of $1/138$ with respect to the p -axis. At this stage it is necessary to settle the question as to how to draw the boundary between the ice and steam regions in the p, t plane. It is known that ice can be transformed directly into steam and not only through the intermediate stage of being melted first. This process is called *sublimation*. (The term did not originate with water but with mercury.) The process can be observed in the spring during a light frost accompanied by brilliant sunshine when the snow seems to disappear quickly, without melting. In actual fact the resulting water vapor escapes into the atmosphere. This condition is again described by the equation $\phi(p, T) = 0$ which defines the *sublimation curve* in the p, t plane. We now propose to prove that it passes through the point of intersection of the melting curve with the vapor-pressure curve. In order to do this consider the analytic representation of the three curves

	melting curve	$\phi_{01} = 0$	$g_0 = g_1,$
(1)	vapor-pressure curve	$\phi_{12} = 0$	$g_1 = g_2,$
	sublimation curve	$\phi_{02} = 0$	$g_0 = g_2.$

It is seen that the equation $g_0 = g_2$ is satisfied when the two equations, $g_0 = g_1$ and $g_1 = g_2$, are satisfied simultaneously. The common point of intersection of all three curves is termed the *triple point*. It is defined by

$$(1 \text{ a}) \quad g_0 = g_1 = g_2.$$

The latent heats which correspond to the three transitions will be denoted by r_{01} , r_{12} , r_{02} so that r_{02} is the *latent heat of sublimation*. According to the First Law they must satisfy the relation

$$(2) \quad r_{01} + r_{12} + r_{20} = 0.$$

(A path around the triple point along which the energy reverts to its initial value.) It follows from (2) that

$$(2 \text{ a}) \quad r_{02} = -r_{20} = r_{01} + r_{12}.$$

Making use of the preceding values $r_{01} = 80$ cal/g from Sec. 16 B and¹ $r_{12} = 603$ cal/g, we find that

$$(2 \text{ b}) \quad r_{02} = 683 \frac{\text{cal}}{\text{g}}.$$

Since the analytic expressions for g_0 and g_1 are not known (that for g_2 is known only approximately from the perfect gas equation), we are not in a position to solve eq. (1) and we must resort to experiment to find the *thermodynamic coordinates of the triple point*. The respective experimental values, Fig. 15, are

$$(3) \quad t = 0.0074 \text{ C}, \quad p = 0.0061 \text{ atm.}$$

The Clapeyron equation is again satisfied along the sublimation curve:

$$(4) \quad \frac{dp}{dT} = \frac{r_{02}}{T \Delta v},$$

where Δv at the triple point can be found as follows:

$$(4 \text{ a}) \quad \Delta v = \Delta v_{20} = v_2 - v_0 = v_2 - v_1 + v_1 - v_0 = \Delta v_{21} + \Delta v_{10}.$$

The three equilibrium curves have been sketched in Fig. 15. They have been extended beyond the triple point with the aid of broken lines in order to stress the fact that they correspond to states of unstable equilibrium in the respective regions of the p, t plane.

Summing up we can state: The *three* phases of water under consideration can coexist only at a single *point* in the p, t plane; the coexistence of two of the three phases is possible only along one specific *curve*; each phase taken *singly* can exist only in a well-defined *area*.

¹Extrapolated from the values for $t = 100 \text{ C}$: $r = 539$ cal/g and $dr/dt = -0.64$ cal/g deg, eq. (16.14 a).

Figure 16 shows an isometric projection of a three-dimensional model for water. The temperature axis, T , is drawn upwards, the axis of volumes, v , is drawn horizontally inwards and the pressure axis, p , is drawn to the left and inwards. The hyperboloidal surface rising steeply corresponds to the gaseous phase, its equation being $T = p v/R$; the isobars and isochores which are straight lines are seen marked on the model. Its lower edge is the *vapor line*. The *solid phase* appears in the form of a bulge at the bottom on the

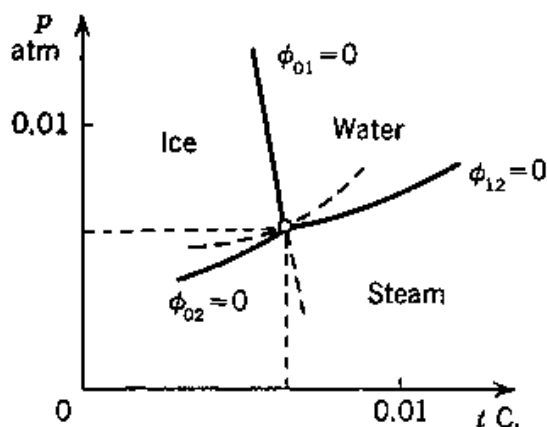


Fig. 15.

The neighborhood of the triple point in the p, t plane.

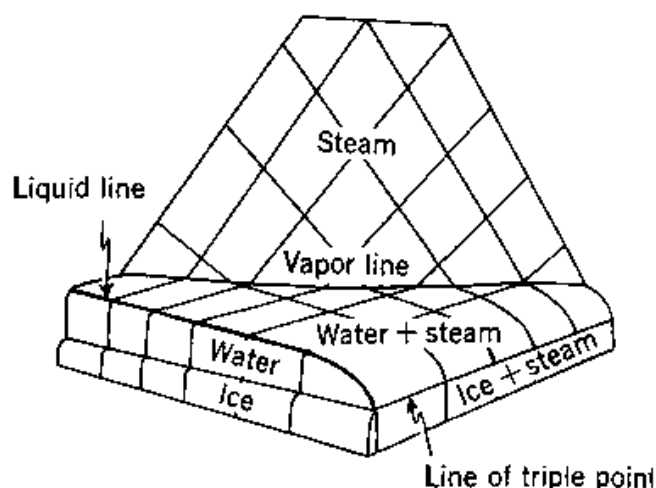


Fig. 16.

Model illustrating the three phases of water. In order to facilitate comparison with Fig. 15 it is necessary to imagine that the p -axis has been drawn to the left.

left. It borders on the *liquid phase* along the *melting curve*. The upper boundary of the latter, in turn, lies along the *liquid line*. The developable surface between the liquid line and the vapor line corresponds to the region of *water + steam*. The projection of this surface onto the p, T plane in the direction of v from left inwards lies along the vapor-pressure curve as already described in connection with eq. (16.1). The bulge at the right-hand bottom edge represents the coexistence of *ice + steam*; it is also developable. It borders on the region *water + steam* along the *triple - point edge*. The most forwards point on the *triple - point edge* is the *triple point* itself. The *sublimation curve* appears on projecting this part of the model onto the p, T plane.

B. GIBBS' PHASE RULE

In the preceding Sections we have restricted ourselves to the consideration of a *single component*, H_2O , and we now propose to extend our description to *any number* of substances (molecular or atomic species). We shall number them as follows:

$$1, 2, \dots, k, \dots, K.$$

Furthermore we shall now admit the existence of *any number* of physical states and chemical groupings of the K components, instead of the *two* or *three* phases considered so far. We shall again denote them as phases and we shall number them consecutively

$$1, 2, \dots, i, \dots, J.$$

The symbol J denotes the largest number of phases which can coexist, and K denotes the total number of substances which react with one another.

The number of mols of the k -th component present in the i -th phase will be denoted by n_{ik} . The free enthalpy, G , of the whole system is the sum of the contributions from the individual phases and components. The condition $\delta G = 0$ leads to a system of simultaneous equations of the form

$$(5) \quad \sum_{i=1}^J \sum_{k=1}^K g_{ik} \delta n_{ik} = 0.$$

The number of conditions included in (5) would equal $J \times K$ if the δn_{ik} were independent of each other. They are, however, linked by the condition that the total mass of each component (the sum of the numbers of mols over all the phases) must be preserved. Consequently, the following additional conditions must be satisfied:

$$(5 \text{ a}) \quad \sum_{i=1}^J \delta n_{ik} = 0 \quad \text{for} \quad k = 1, 2, \dots, K.$$

The number of independent equations does not equal $J \times K$; it is only

$$(6) \quad J \times K - K.$$

We now inquire into the number of variables which we have at our disposal and which must satisfy these equations. These are given by the numbers of mols n_{ik} whose number is, again, $J \times K$. Since, however, G is homogeneous in the n_{ik} 's, *cf.* Sec. 14 A, the conditions of equilibrium (5), (5 a) written for each phase will contain only ratios of the n_{ik} 's. The number of such ratios per phase is equal to $K - 1$ so that in the sum for all phases we shall have

$$J(K - 1) = J \times K - J$$

ratios. In addition there are the two variables p, T which determine a point in the p, T plane. *The total number of independent variables at our disposal is thus*

$$(7) \quad J K - J + 2.$$

When this number is smaller than the number of equations to be satisfied, eq. (6), the system cannot, generally speaking, be solved. It is, therefore, necessary to stipulate

$$JK - J + 2 \geq JK - K.$$

It follows that

$$(8) \quad J \leq K + 2.$$

This is the famous *phase rule* discovered by Willard Gibbs. Instead of (8) we can also write

$$J + F = K + 2,$$

where F denotes the *number of degrees of freedom* possessed by the system of K components forming J coexisting phases.

When $F = 0$ the corresponding state of the system is called *invariant* which is the case, for example, for water, $K = 1$, at the triple point: we have here $J = 3$, ice, water, and steam coexisting.

When $F = 1$ the corresponding system is called *monovariant*. In the particular case when $K = 1$ we have $J = 2$, i. e. the coexistence of two phases along the vapor-pressure, melting, or sublimation curve.

When $F = 2$ the system is *bivariant*. For $K = 1$ we have also $J = 1$, no phases coexisting; instead, there is the possibility of the existence of a single phase in a two-dimensional region of the p, T plane.

If we now consider one of Tammann's allotropic modifications of ice in addition to the ordinary (hexagonal) phase we find that the phase rule allows their coexistence with only *one* more phase, *either* the liquid, *or* the vapor. There can be no "quadruple point" in cases when $K = 1$. Such a point can exist in at least a "binary" system composed of two substances: $K = 2$ gives $J = 4$ for $F = 0$.

The study of the phase rule in application to ternary or to multi-component systems leads to the consideration of multi-dimensional spaces describing phase equilibria. We must, however, leave this topic to physical chemistry.

C. RAOULT'S LAWS FOR DILUTE SOLUTIONS

A particularly simple case of phase equilibrium occurs when dilute solutions are evaporated, on condition that the solute is not volatile. We shall refer here to our derivation of van 't Hoff's law in Sec. 15 B, excluding, from our considerations the case of electrolytes, as before. We wish to make the following changes in notation: Instead of denoting phases by subscripts as

heretofore, we shall now denote the liquid and gaseous phase of the solvent by superscripts 1 and 2 respectively; these have been used in Sec. 15 to distinguish between the two subsystems "solution" and "pure solvent." The subscripts 1 and 2 are now required to denote the substances "solvent" and "solute," as was done in Sec. 14. The place of the "pure solvent" of Sec. 15 is now taken by the gaseous phase (number of mols n_1^2). Instead of the semipermeable wall we now have to consider the free surface of the fluid which is impermeable to the solute. The numbers of mols in the dilute solution are now, as before, denoted by n_1^1 and n_2^1 .

The difference between the two cases consists in the fact that the pressures of the two phases are now equal. The former condition of equilibrium (15.5)

$$(9) \quad \mu_1^1 \delta n_1^1 + \mu_1^2 \delta n_1^2 = 0$$

and the former auxiliary condition (15.3)

$$(10) \quad \delta n_1^1 + \delta n_1^2 = 0$$

remain unaltered. Consequently, the equation for our present "vapor-pressure line" is determined by the preceding eq. (15.6)

$$(11) \quad \mu_1^2(p, T) = \mu_1^1\left(p, T, \frac{n_2^1}{n_1^1}\right)$$

but with the proviso that $p^1 = p^2 = p$. In analogy with the previous eq. (15.8) we must assume now

$$(12) \quad \begin{aligned} \mu_1^1 &= g_1^1(p, T) - R T \log \frac{n_1^1 + n_2^1}{n_1^1} \\ \mu_1^2 &= g_1^2(p, T), \end{aligned}$$

where, unlike in Sec. 15, the symbol g_1^2 denotes the free enthalpy of the pure vapor, g_1^1 denotes that for the pure solvent. Thus the condition of equilibrium becomes:

$$(13) \quad g_1^2(p, T) = g_1^1(p, T) - R T \log \frac{n_1^1 + n_2^1}{n_1^1}.$$

If the solution is dilute to a sufficient degree we have $n_2^1 \ll n_1^1$ and we may replace the logarithm by n_2^1/n_1^1 or by n_2/n_1 if we omit the superscript which has now become superfluous. Hence eq. (13) changes to

$$(14) \quad g_1^2(p, T) = g_1^1(p, T) + R T \frac{n_2}{n_1}.$$

This is the equation for the *vapor-pressure curve of the solvent above the solution* in the p, T plane.

For the purpose of comparison we now write the equation for the *vapor-pressure curve for the pure solvent*. In order to make a distinction we shall denote its coordinates by p^*, T^* :

$$(14\ a) \quad g_1^2(p^*, T^*) = g_1^1(p^*, T^*).$$

Subtracting (14) from (14 a) we have

$$(15) \quad g_1^2(p^*, T^*) - g_1^2(p, T) = g_1^1(p^*, T^*) - g_1^1(p, T) + R T \frac{n_2}{n_1}.$$

We now expand both sides into Taylor series and retain the first term of each only. Hence

$$(16) \quad \begin{aligned} (p^* - p) \frac{\partial g_1^2}{\partial p} + (T - T^*) \left[-\frac{\partial g_1^2}{\partial T} \right] = \\ = (p^* - p) \frac{\partial g_1^1}{\partial p} + (T - T^*) \left[-\frac{\partial g_1^1}{\partial T} \right] + R T \frac{n_2}{n_1}. \end{aligned}$$

The derivatives can be taken from the Table in Sec. 7. Thus eq. (16) becomes

$$(17) \quad (p^* - p) [v_1^2(p, T) - v_1^1(p, T)] + (T - T^*) [s_1^2(p, T) - s_1^1(p, T)] = R T \frac{n_2}{n_1}.$$

This equation can be analyzed from two points of view. (a) First we shall inquire into the change in the vapor pressure at constant T on adding n_2 mols of substance 2 to the solvent. This means that we have to put $T = T^*$ in (17) and so we determine the *decrease in vapor pressure* $\Delta p = p^* - p$:

$$(18) \quad \Delta p = \frac{R T}{v_1^2 - v_1^1} \cdot \frac{n_2}{n_1}.$$

Equation (18) applies to any dilute solution, that is, whenever $n_2 \ll n_1$. In particular if the vapor may be treated as a perfect gas

$$v_1^2 = \frac{R T}{p}$$

and if the molar volume v_1^1 of the liquid may be neglected with respect to that of the vapor, we can simplify (18) to read

$$(19) \quad \frac{\Delta p}{p} = \frac{n_2}{n_1}.$$

This is the *law of vapor pressures*. It is independent of the solvent as well as of the nature of the solute and is given directly by the ratio of the number of mols of the two. This surprisingly simple law was discovered empirically by Raoult in 1886. It was proved thermodynamically shortly afterwards by van 't Hoff.

(b) Secondly we shall inquire into the change in the boiling point $T - T^* = \Delta T$ at constant pressure on adding substance 2 to the solvent 1. We now put $p = p^*$ in (17) and take into account that

$$(s_1^2 - s_1^1) T = r \quad (r = \text{latent heat of evaporation}).$$

It follows from eq. (17) for $\Delta T = T - T^*$ that

$$(20) \quad \Delta T = \frac{R T^2}{r} \cdot \frac{n_2}{n_1}.$$

Raoult's law of *decrease in vapor pressure* is seen to be associated with the law of *rise in boiling point*. Concerning the validity and the historical origin of this law the same remarks can be made as in connection with the law of decrease in vapor pressure.

To the *rise in boiling point* there corresponds a *freezing point depression* on transition from phase 1 to 0 (freezing). Equation (20) remains valid for this case as well except that $r = r_{12}$ must be replaced by the heat of reaction $r_{10} = -r_{01}$ which is equal to the negative of the heat of solidification.

In all these considerations the assumption was made that all molar masses are preserved during the phase transition which implies the exclusion of polymerization and dissociation. There is no difficulty in including such phenomena. It is only necessary to include the mass ratio m_2/m_1 associated with the respective process where necessary, e. g. on the right-hand side of eq. (11). Planck¹ stressed the importance of this circumstance for the determination of molecular weights.

D. HENRY'S LAW OF ABSORPTION (1803)

The condition of equilibrium $\delta G = 0$ finds a simple application in the study of the solubility of a gas in a liquid whose vapor pressure is negligible compared with that in the gas chamber. In such cases only the chemical potential of the gas in the gaseous phase, μ^2 , and in the solution, μ^1 , are

¹ Thermodynamics, Sec. 269 and 270.

important (the subscript 1 denoting "gas" can be dropped here and subsequently). The auxiliary condition

$$\delta n^1 + \delta n^2 = 0$$

leads to

$$(21) \quad \mu^1 = \mu^2$$

in the same way as in (11). We shall again assume that the gaseous phase may be regarded as a perfect gas, so that

$$(22) \quad \mu^2 = g^2 = R T \log p + \psi(T).$$

The function $\psi(T)$ which includes the chemical constant of the gas need not be described in more detail. Regarding the solution we make the assumption

$$(23) \quad \mu^1 = g^1(p, T) + R T \log c,$$

which is analogous to eq. (12). Here c denotes the molar concentration of the dissolved gas (represented as n_1^1/n with $n = n_1^1 + n_2^1$ in the above mentioned eq. (12)). With certain reservations the free enthalpy of a liquid of fixed chemical nature is practically independent of pressure, so that (23) may be written

$$\mu^1 = R T \log c + \chi(T).$$

It follows from (21), (22) and (23) that

$$\log c = \log p - \frac{\chi(T) - \psi(T)}{R T}.$$

and also

$$(24) \quad c = p \times f(T) \quad \text{with} \quad \log f(T) = \frac{\psi(T) - \chi(T)}{R T}.$$

This is the very well known law due to Henry: *The quantity of gas absorbed by a liquid is proportional to the partial pressure of the gas remaining above it.* The coefficient of proportionality depends only on the temperature (equal for the gas and the liquid); it is unaffected, for example, by the presence of any additional gases in the chamber.

18. The electromotive force of galvanic cells

We have until now considered only thermodynamic systems which were composed of electrically neutral particles (atoms, molecules). We now propose to investigate the changes to be introduced into our equations when charged particles (electrons, ions) are included. This field includes problems in the thermal and caloric equations of state for electrons in metals, electrolytes, ionized gases, etc. We shall continue to restrict ourselves to problems of equilibrium thermodynamics which implies the exclusion of such problems as the flow of an electric current through metals or electrolytes, etc.

From among the rich collection of remaining problems we shall now fix our attention on the question of the electromotive force (emf for short) of open galvanic cells because we can formulate several general statements without undue labor. We have purposely added the qualification "open" in the preceding sentence in order to emphasize the fact that we are interested in the static cases only. It is assumed that equilibrium has been attained and that it is not disturbed by any irreversible processes, such as the generation of Joule heat during the passage of a current. It is, therefore, necessary to imagine that the emf of the cell is measured with the aid of an electrostatic voltmeter.

When two phases which are capable of exchanging charged particles are permanently separated by a boundary there will appear between them a difference in potential in the same way as the existence of equilibrium between two solutions of different concentrations, separated by a semipermeable wall, implies that an osmotic pressure difference is permanently maintained between them.

A. ELECTROCHEMICAL POTENTIALS

We shall now consider a thermodynamic system Σ and its "surroundings" Σ' as was done in Sec. 8. The system Σ will be described by specifying the properties V, T, n_i . According to (14.1) we can write for Σ :

$$(1) \quad T dS = dU + p dV - \sum_i \mu_i dn_i.$$

Let us fix our attention on an infinitesimal process

$$dV = 0, \quad dT = 0, \quad dn_i = 0 \quad \text{for} \quad i \neq j \\ dn_j \neq 0;$$

the product $\mu_j dn_j$ represents that quantity of work which must be performed on system Σ in order to change the number of mols n_j by an increment dn_j , as already mentioned.

We shall suppose that the component j is no longer electrically neutral as was the case before, but that it is charged positively, in other words, that it consists of molecules or atoms which have lost a certain number of electrons each, that number being denoted by z . Consequently, one mol of that j -th component carries a charge zF where F denotes Faraday's equivalent charge of 96494 coulombs and is equal to the product of the elementary charge \times Avogadro's number per mol. The system Σ is assumed to have an electrical potential Φ with respect to Σ' . Without loss of generality it is possible to assume that Σ' is grounded so that its potential is zero. Since we are now performing only an imaginary experiment we need not be concerned with the manner in which the potential Φ is maintained. If we now introduce into the system dn_j mols of charged particles we shall have to perform the work

$$\Phi z_j F dn_j$$

in addition to the "chemical" work $\mu_j dn_j$. The total amount of work to be performed on Σ is

$$+ (\mu_j + z_j F \Phi) dn_j.$$

The transfer of the charge must occur infinitely slowly, and must not generate Joule heat if the process is to be reversible.

Accordingly, eq. (1) must be replaced by

$$(2) \quad T dS = dU + p dV - \sum \eta_i dn_i$$

where

$$(3) \quad \eta_i = \mu_i + z_i F \Phi.$$

The η_i 's are known as the *electrochemical potentials* as distinct from the chemical potentials μ_i . For negatively charged particles z_i is to be taken to be negative.

Generally speaking, however, a galvanic cell will be described by more than one potential. It consists of a whole *chain* of phases and each phase has its own electrical potential and is in equilibrium only with its two neighboring phases. We can imagine, at least during our imaginary experiment, that the individual phases are separated from each other by semipermeable membranes which allow only certain ions to pass. These remarks will be made clear on the example of the Daniell cell.

B. THE DANIELL CELL, 1836

This is represented schematically in Fig. 17. The individual phases have been marked by Roman numerals. The wall M (made of clay) allows only the SO_4^{--} ions to pass. We imagine that a copper terminal V is connected to the zinc electrode IV in order to insure that the emf $\Phi_I - \Phi_V$ of the open cell is measured between identical metals (copper) thus excluding any contact emf's when measured electrostatically. The electrochemical potentials, η , and the numbers of mols, n , will be distinguished from each other respectively by writing the phase in the upper left superscript and the kind of particle in the lower right subscript, e. g. ${}^{\text{II}}\eta_{\text{Cu}^{++}}$. The symbol \ominus will be used to denote electrons. The condition of equilibrium $\delta G = 0$, e. g. between the phases I and II and the auxiliary condition which expresses the preservation of molar masses are

$$(4) \quad {}^{\text{I}}\eta_{\text{Cu}^{++}} + \delta^{\text{I}}n_{\text{Cu}^{++}} + {}^{\text{II}}\eta_{\text{Cu}^{++}} + \delta^{\text{II}}n_{\text{Cu}^{++}} = 0,$$

$$(5) \quad \delta^{\text{I}}n_{\text{Cu}^{++}} + \delta^{\text{II}}n_{\text{Cu}^{++}} = 0.$$

These equations are similar to those in Sec. 17, eqs. (9) and (10) because the virtual changes are performed at constant p and T and the chemical potentials must be replaced by the electro-chemical potentials. From (4) and (5) we obtain

$$(6) \quad {}^{\text{I}}\eta_{\text{Cu}^{++}} = {}^{\text{II}}\eta_{\text{Cu}^{++}}.$$

According to (3) with $z = +2$ we have:

$$(7) \quad \text{Phases I/II:} \quad {}^{\text{I}}\mu_{\text{Cu}^{++}} + 2F\Phi_{\text{I}} = {}^{\text{II}}\mu_{\text{Cu}^{++}} + 2F\Phi_{\text{II}}.$$

The conditions of equilibrium between the remaining pairs of phases are analogous and can be written

$$(8) \quad \text{Phases II/III:} \quad {}^{\text{II}}\mu_{\text{SO}_4^{--}} - 2F\Phi_{\text{II}} = {}^{\text{III}}\mu_{\text{SO}_4^{--}} - 2F\Phi_{\text{III}},$$

$$(9) \quad \text{Phases III/IV:} \quad {}^{\text{III}}\mu_{\text{Zn}^{++}} + 2F\Phi_{\text{III}} = {}^{\text{IV}}\mu_{\text{Zn}^{++}} + 2F\Phi_{\text{IV}},$$

$$(10) \quad \text{Phases IV/V:} \quad {}^{\text{IV}}\mu_{\ominus} - F\Phi_{\text{IV}} = {}^{\text{V}}\mu_{\ominus} - F\Phi_{\text{V}}.$$

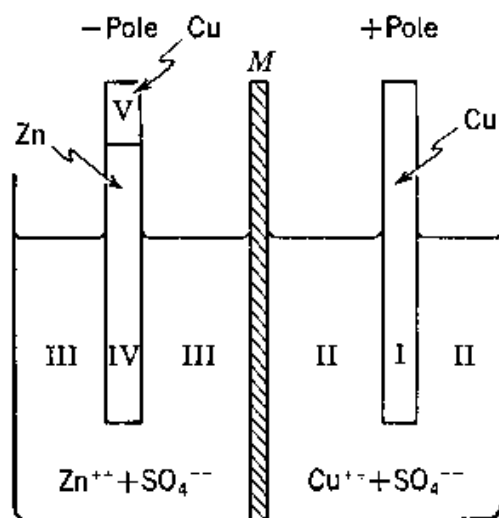
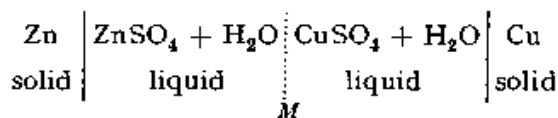


Fig. 17.

Schematic diagram of Daniell cell.



In addition there are the equations describing the chemical reactions at the copper and the zinc electrode



This means the neutralization of the Cu^{++} ions on deposition on the copper electrode and the charging of the atoms of zinc with two positive elementary charges on crossing the boundary $\text{Zn}-\text{ZnSO}_4$. This occurs according to the two additional conditions:

$$(11) \quad {}^I\mu_{\text{Cu}} - 2 {}^I\mu_{\ominus} = {}^I\mu_{\text{Cu}^{++}},$$

$$(12) \quad {}^{IV}\mu_{\text{Zn}} - 2 {}^{IV}\mu_{\ominus} = {}^{IV}\mu_{\text{Zn}^{++}}.$$

Finally, the condition

$$(13) \quad {}^V\mu_{\ominus} = {}^I\mu_{\ominus}$$

must be added to the scheme owing to the fact that the material (Cu) of the two electrodes is the same. This completes the description of the reactions taking place in our chain at equilibrium in all its details.

C. CONTRACTION OF INDIVIDUAL REACTIONS INTO A SIMPLIFIED OVERALL REACTION

It is possible successively to eliminate the potentials Φ_{II} , Φ_{III} , Φ_{IV} from eqs. (7), (8), (9), (10). In this way we find that

$$(14) \quad 2F(\Phi_I - \Phi_V) = ({}^{II}\mu - {}^I\mu)_{\text{Cu}^{++}} + ({}^{IV}\mu - {}^{III}\mu)_{\text{Zn}^{++}} + ({}^{II}\mu - {}^{III}\mu)_{\text{SO}_4^{--}} + 2({}^{IV}\mu - {}^V\mu)_{\ominus}.$$

In view of (11) the first term on the right-hand side can be transformed to

$$(15) \quad {}^{II}\mu_{\text{Cu}^{++}} - {}^I\mu_{\text{Cu}} + 2 {}^I\mu_{\ominus},$$

and, similarly, in view of (12) the second term can be transformed to

$$(16) \quad {}^{IV}\mu_{\text{Zn}} - {}^{III}\mu_{\text{Zn}^{++}} + 2 {}^{IV}\mu_{\ominus}.$$

Taking into account condition (13) we see that the sum of the terms in the last two equations, together with the fourth term on the right-hand side, is equal to zero. Hence it follows from (14) that

$$(17) \quad \Phi_I - \Phi_V = \frac{1}{2F} \left\{ {}^{IV}\mu_{\text{Zn}} - {}^{III}\mu_{\text{Zn}^{++}} + {}^{II}\mu_{\text{Cu}^{++}} - {}^I\mu_{\text{Cu}} + ({}^{II}\mu - {}^{III}\mu)_{\text{SO}_4^{--}} \right\}.$$

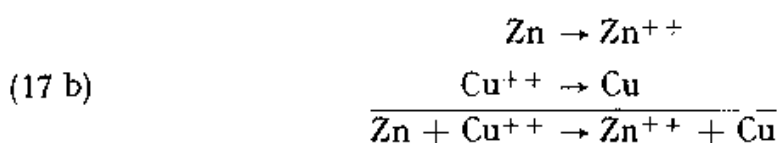
$\Phi_I - \Phi_V$ is the emf of the open cell, and we shall denote it by E .

It is permissible to assume that ${}^{\text{II}}\mu_{\text{SO}_4^{--}} = {}^{\text{III}}\mu_{\text{SO}_4^{--}}$ which implies $\Phi_{\text{II}} = \Phi_{\text{III}}$. This can be justified with the aid of certain artifices. Instead of (17) we then have

$$(17 \text{ a}) \quad E \cdot 2F = {}^{\text{IV}}\mu_{\text{Zn}} + {}^{\text{II}}\mu_{\text{Cu}^{++}} - {}^{\text{III}}\mu_{\text{Zn}^{++}} - {}^{\text{I}}\mu_{\text{Cu}}.$$

The physico-chemical interpretation of this relation is as follows:

Imagine that a charge $2F$ is transferred in a reversible (electrostatic) manner from the positive pole to the negative pole outside the cell so that an equal charge is transferred from the negative to the positive pole within the Daniell cell causing the following reaction to take place:



for one mol at constant pressure and temperature. The product $2F \times E$ represents the (electrical) work performed by the cell. Hence the right-hand side of (17 a) can be interpreted as a decrease ΔG in the free enthalpy during the occurrence of the process described by (17 b). This can be inferred from the representation

$$(18) \quad \Delta G = \sum_i \mu_i (-\Delta n_i)$$

which follows from (14.7) for $\Delta n_{\text{Zn}} = -1$, $\Delta n_{\text{Zn}^{++}} = +1$, $\Delta n_{\text{Cu}} = +1$ and $\Delta n_{\text{Cu}^{++}} = -1$ and for constant chemical potentials. Furthermore, to be precise it would be necessary to conduct the preceding reaction with an infinitesimally small fraction of one mol if the latter condition is to be satisfied.

The reaction (17 b) represents the *overall* reaction in the cell. It results from the individual reactions (7) – (13). Equations (7) – (13) result from *virtual* changes, (18) corresponds to a *real* transfer of charge. It will, however, be seen from eqs. (17 a, b) that the individual reactions are unimportant. Their function was to make it clear that the emf is determined by the *equilibrium of the individual sub-systems*.

During the transfer of the charge $2F$ as described by eq. (17 b), i. e. when one mol undergoes the overall reaction, the system will liberate a certain heat of reaction which can be measured directly. We now propose to deduce a relation between this heat and the emf.

D. THE GIBBS-HELMHOLTZ FUNDAMENTAL EQUATION

We shall begin by generalizing from the special case of a Daniell cell to any galvanic cell. It is natural to suppose that eqs. (17 a) and (18) have general validity:

$$(19) \quad E = \frac{1}{zF} \Delta G.$$

Here ΔG denotes the decrease in free enthalpy per one mol at $p = \text{const}$, $T = \text{const}$ and z is the valency of the anions or kations or their least common multiple.

By forming differences we find from our Table in Sec. 7 that (e. g. $G = H - TS$; $S = -(\partial G/\partial T)_p$)

$$(20) \quad \Delta G - T \left(\frac{\partial \Delta G}{\partial T} \right)_p = \Delta H.$$

If the reaction took place at $V = \text{const}$, $T = \text{const}$ eqs. (19) and (20) would have been replaced¹ by

$$(21) \quad E = \frac{1}{zF} \Delta F,$$

$$(22) \quad \Delta F - T \left(\frac{\partial \Delta F}{\partial T} \right)_v = \Delta U.$$

Here ΔF , ΔH , and ΔU denote the decrease in F , H , and U per one mol undergoing the overall reaction and ΔH and ΔU denote the heats of reaction at $p = \text{const}$ and $V = \text{const}$ respectively. It follows at once from (19) – (22) that

$$(23) \quad E - T \left(\frac{\partial E}{\partial T} \right)_p = \frac{\Delta H}{zF} \quad (\text{Gibbs}),$$

$$(24) \quad E - T \left(\frac{\partial E}{\partial T} \right)_v = \frac{\Delta U}{zF} \quad (\text{Helmholtz}).$$

These two equations played a very important part when Nernst formulated the Third Law, because experimental results showed that even at moderately low temperatures the "naive" formulae

$$(25) \quad E \sim \frac{\Delta H}{zF} \quad \text{or} \quad E \sim \frac{\Delta U}{zF}$$

appear to be correct. From this fact Nernst concluded that the curves for U and F must osculate, and not only meet, at absolute zero (*cf.* Sec. 12).

¹In the following we shall provisionally use the symbol F to denote free energy to avoid confusion with the Faraday constant F .

E. NUMERICAL EXAMPLE

We now return to the Daniell cell. Measurement of the emf gives

$$E = 1.0999 \text{ volt at the ice point}$$

$$\frac{dE}{dT} = -4.3 \times 10^{-4} \frac{\text{volt}}{\text{deg}} \text{ at the ice point.}$$

It is seen from the way the Daniell cell is made that no difference between $(\partial E/\partial T)_p$ and $(\partial E/\partial T)_v$ is to be expected and, in fact, none is observed. Consequently, the two heats of reaction ΔH and ΔU given in (23) and (24) respectively do not differ from each other. Denoting their common value by q we obtain from (23) or (24), with $z = 2$:

$$\begin{aligned} q &= 2 \times 96494 \text{ coulomb} \times (1.0999 + 273 \times 4.3 \times 10^{-4}) \text{ volt} = \\ &= 192988 \text{ coulomb} \times (1.0999 + 0.1173) \text{ volt} = 2.35 \times 10^5 \text{ coulomb} \times \text{volt.} \end{aligned}$$

The unit coulomb \times volt = Joule = Erg. Since 1 Erg = 0.239 cal, we have

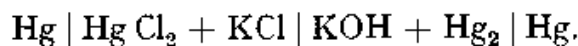
$$q = 56100 \text{ cal}$$

compared with the measured value $q = 55200$ cal.

If we calculated "naively" from (25) we would obtain

$$E = \frac{55200}{2 \times 96494 \times 0.239} \text{ volt} = 1.195 \text{ volt}$$

instead of 1.0999 volt. The primitive equation is seen to be fairly good and this can easily be understood if it is noticed that $T \cdot dE/dT = 0.1173$ is relatively small compared with $E = 1.0999$. The same is not true for the cell



Here we have

$$E = 0.1483 \text{ volt at the ice point}$$

$$\frac{dE}{dT} = 8.37 \times 10^{-4} \frac{\text{volt}}{\text{deg}} \text{ at the ice point}$$

The observed heat of reaction is

$$-3280 \text{ cal}$$

so that on applying the naive formula we would obtain a negative emf. In actual fact at 0 C the term $T \cdot dE/dT = 0.227$ volt exceeds E itself in value which is consistent with the negative value of the heat of reaction.

It is sometimes stated in text-books that the emf of a galvanic cell, particularly that of a Daniell cell, can be deduced from the First Law and that, generally speaking, the Second Law introduces only a correction term. The last example shows that the statement is erroneous. It is necessary, in principle, to base the calculation on the Gibbs-Helmholtz equations, which have been deduced with the aid of the First *and* the Second Law.

F. REMARKS ON THE INTEGRATION OF THE FUNDAMENTAL EQUATION

In view of the identity

$$(26) \quad E - T \frac{dE}{dT} = -T^2 \frac{d}{dT} \left(\frac{E}{T} \right)$$

eq. (23) can also be written

$$(27) \quad \frac{d}{dT} \left(\frac{E}{T} \right) = - \frac{\Delta H}{zF \cdot T^2}.$$

We can split ΔH into a term ΔH_0 which is independent of temperature and into a term ΔH_T which depends on it. We shall prove that the latter vanishes very rapidly as $T \rightarrow 0$. We can write that

$$(28) \quad \Delta C_p = \frac{\partial \Delta H}{\partial T} = \frac{\partial \Delta H_T}{\partial T}.$$

According to the Third Law ΔC_p vanishes for $T \rightarrow 0$, consequently ΔH_T must tend to zero *more rapidly* than T (like T^4 in most cases; no constant term in ΔH_T which would vanish on differentiating (28) need be considered because it would have had to be included in ΔH_0). On integrating it follows at once from (27) that

$$(29) \quad E = \frac{\Delta H_0}{zF} - \frac{T}{zF} \int_0^T \frac{\Delta H_T}{T^2} dT.$$

By choosing the lower limit of integration at $T = 0$ we have given the right value to the constant of integration, namely, the value is such that it

corresponds to our original definition of E in eq. (19). In fact, because $TS \rightarrow 0$ as $T \rightarrow 0$, as required by the Third Law, we have

$$\Delta G_0 = \Delta H_0, \quad \text{that is} \quad E = \frac{\Delta H_0}{zF}.$$

This "limiting condition" which must be satisfied at absolute zero is seen to be satisfied by (29) which justifies our choice of the lower limit of integration.

Equation (29) enables us to predict the emf together with its variation with temperature from measured values of the variation of the heat of reaction with temperature. The emf of many cells was determined by this method, excellent agreement with direct measurement having been found. Fundamentally this agreement is equivalent to an additional verification of the Second and Third Laws.

19. Ferro- and paramagnetism

Diamagnetic phenomena are independent of temperature but para- and ferromagnetism depends very strongly on it. Both increase with decreasing temperature. Above a certain limit, known as the Curie point, ferromagnetic substances behave like paramagnetic solids. It is now our purpose to apply the principles of thermodynamics to such phenomena. As usual we can only expect to obtain a general framework within which such phenomena take place. Their details must be obtained with the aid of statistical methods and supplemented with statements from the field of atomic physics (magnetic moment of an electron, *c.f.* Vol. III Sec. 14 B). Diamagnetic processes belong entirely to the realm of atomic physics.

A. WORK OF MAGNETIZATION AND MAGNETIC EQUATION OF STATE

According to Vol. III., eqs. (5.66) and (12.2) the differential of the magnetic energy density is given by $(H, d\mathbf{B})$ with $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ where \mathbf{M} is the magnetization (magnetic moment per unit volume) and μ_0 is a constant for the vacuum which must be added on dimensional grounds. We are not interested in the term

$$\mu_0(\mathbf{H}, d\mathbf{H})$$

of this differential because it is present even in the absence of magnetism. Disregarding the vectorial nature of the process of magnetization (inadmissible in the case of single crystals) we find that the contribution to the energy

density which is due to the second term and which is the only important one in the present considerations can be written as (*cf.* first footnote on p. 75).

$$(1) \quad \mu_0 H dM.$$

It is, furthermore, convenient to interpret M as the magnetization per mol (rather than per unit volume); thus eq. (1) represents the work performed by the external field when changing the magnetization by dM per mol, i. e. a quantity of energy added to the system.

We now have to consider two magnetic variables, H, M ¹ in addition to the thermal variables T, s . We can also disregard the two mechanical variables p, V because they are unimportant here (they would come into play only if the phenomena of magnetostriction were considered). Combining the First and the Second Laws we have

$$(2) \quad du = T ds + \mu_0 H dM$$

or

$$(2 a) \quad ds = \frac{du}{T} - \frac{\mu_0 H}{T} dM.$$

In his first paper on this subject (1905) Paul Langevin made the tentative assumption that both terms in ds are perfect differentials. Hence $\mu_0 H/T$ must be independent of T , and so it must be a function of M alone. This is equivalent to saying that M is a function of H/T only:

$$(3) \quad M = f\left(C \cdot \frac{H}{T}\right).$$

At the same time u in eq. (2 a) must be independent of M and a function of T alone:

$$(3 a) \quad \left(\frac{\partial u}{\partial M}\right)_T = 0.$$

The argument of the function f in eq. (3) contains a constant C which is characteristic of the material and which can be assumed to include the coefficient μ_0 from eq. (2 a); furthermore, this constant must be such as to

¹It would be more consistent to choose $\mu_0 H$ instead of H itself as the first magnetic variable. Being an intensive quantity it corresponds to the variables T and p in the two other pairs of variables. However, in order not to obscure the text with the unimportant coefficient μ_0 we shall use H alone.

render the argument of the function f dimensionless. In connection with eq. (3 a) it is necessary to remark that u is here considered to be a function of T and M and not of s and M as in eq. (2) (disregarding its dependence on p, v which we continue to neglect).

The preceding equations are reminiscent of those for a perfect gas if the magnetic properties H, M, C are imagined replaced by the mechanical quantities $1/v, p, 1/R$ respectively. Equation (3 a) is seen to transform into the fundamental caloric equation for perfect gases $(\partial u / \partial v)_T = 0$ and eq. (3) becomes the equation of state if $f(x)$ is replaced by x . Thus.

$$\frac{1}{v} = \frac{p}{R T}.$$

This analogy between the perfect gas and the type of magnetic substance which we are now considering suggests that eq. (3) may be termed "the equation of state of a perfect magnetic substance." In any case it should be borne in mind that this analogy applies to *paramagnetic* substances only. Putting again $f(x) = x$, we obtain

$$(4) \quad M = C \cdot \frac{H}{T}; \quad \frac{M}{H} = \chi = \frac{C}{T}.$$

This equation is known as *Curie's law for paramagnetic solids*, already mentioned in Vol. III, eq. (13.10); C is *Curie's constant* and χ is the *magnetic susceptibility per mol*. It is seen from (4) that C has the dimension of temperature.

The equation of state for diamagnetic substances does not conform to the scheme of eq. (3), because M is proportional to H and independent of temperature. However (2) is valid for diamagnetic substances. The equation for *ferromagnetic substances* which is of great interest to us here also differs from the scheme in (3). Consequently, eq. (3 a) does not apply to ferromagnetic substances as will be explained in Section D.

B. LANGEVIN'S EQUATION FOR PARAMAGNETIC SUBSTANCES

A pivoted elementary magnet of moment m placed in an external magnetic field will point in the latter's direction. If all n elementary magnets contained in one mol were so directed, the magnetization would assume its saturation value

$$(5) \quad M_{\infty} = m n.$$

This state is resisted by thermal agitation. It is now our task to determine that state of compromise between saturation $M = M_{\infty}$ and complete disorder

$M = 0$ which corresponds to a given temperature T . Langevin deduced the result under consideration by a comparatively simple application of Boltzmann's statistics, cf. Sec. 25. It will be shown there that his result has the form

$$(5\ a) \quad \frac{M}{M_\infty} = \frac{\cosh \alpha}{\sinh \alpha} - \frac{1}{\alpha}; \quad \alpha = \frac{\mu_0 M_\infty H}{R T},$$

where α is dimensionless, because the numerator as well as the denominator in eq. (5 a) defining α have the dimensions of energy.

The expression

$$(5\ b) \quad L(\alpha) = \frac{\cosh \alpha}{\sinh \alpha} - \frac{1}{\alpha}$$

is known as *Langevin's function*. It is represented graphically by the monotonically increasing curve OBA in Fig. 18 and corresponding to the following approximations:

$$(6) \quad \alpha \rightarrow 0, \quad L(\alpha) = \frac{1 + \frac{1}{2}\alpha^2 + \cdots}{\alpha + \frac{1}{6}\alpha^3 + \cdots} - \frac{1}{\alpha} = \frac{1}{\alpha} - \frac{\frac{1}{3}\alpha^2}{1 + \frac{1}{6}\alpha^2} \rightarrow \frac{1}{3}\alpha$$

$$(6\ a) \quad \alpha \rightarrow \infty; \quad L(\alpha) = \frac{e^\alpha + e^{-\alpha}}{e^\alpha - e^{-\alpha}} - \frac{1}{\alpha} \approx 1 - \frac{1}{\alpha} \rightarrow 1.$$

From (6) and (5 a) we obtain the following expression which is valid for almost all values of α which can be attained in practice:

$$(7) \quad \chi = \frac{M}{H} = \frac{M_\infty \alpha}{H \cdot 3} = \frac{\mu_0 M_\infty^2}{3 R T}.$$

This is Curie's law (4) where the Curie constant C has the value

$$(7\ a) \quad C = \frac{\mu_0 M_\infty^2}{3 R}.$$

It breaks down only for $T \rightarrow 0$, when eq. (4) leads to $\chi \rightarrow \infty$ instead of the correct finite value

$$\chi = M_\infty / H$$

which follows from (7) and (6 a).

The approximations (6) and (6 a) are seen plotted in Fig. 18. They are represented, respectively, by the dash-dot lines near the origin 0 and by the asymptote. Generally speaking (cf. *infra*), all paramagnetic states which are attainable in practice lie near the lowest end of the tangent curve.

The statistical theory due to Langevin neglects mutual interactions between the elementary magnets. It assumes that they are only subjected to the influence of the external field, which, obviously, represents a far-reaching idealization. This idealization is equivalent to the assumption in (3 a) that the energy u is independent of M ; this would not be true if interactions between elementary magnets were taken into account. It is now clear that Langevin's equation of state (5) is compatible with the scheme (3) because the latter was thermodynamically linked with the condition (3 a). In order to justify this drastic simplification we may mention that saturation effects are, generally speaking, unobservable in paramagnetic substances, and can only be expected to occur at the lowest temperatures. (This is confirmed by the observations on gadolinium sulphate made at temperatures down to 1.3 K by Woltjer and Kammerlingh Onnes at the Cryogenic Laboratory in Leiden, *cf.* Section E.) Debye¹ demonstrated that Langevin's function ceases to be applicable at such low temperatures because it contradicts Nernst's Third Law as T is made to tend to zero.

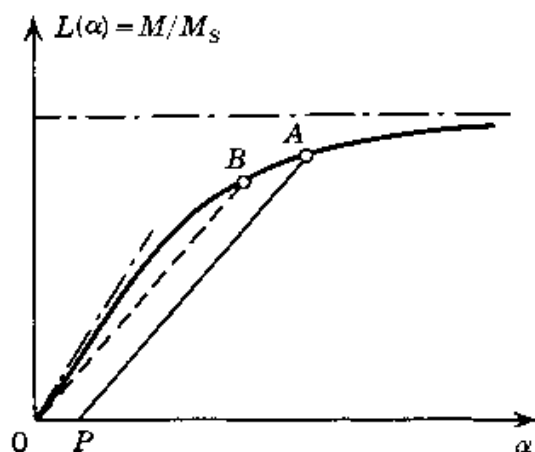


Fig. 18.

Langevin's curve for paramagnetic substances and its application in Weiss' theory of ferromagnetic phenomena.

C. THE THEORY OF FERROMAGNETIC PHENOMENA DUE TO WEISS

Pierre Weiss stated the extremely fruitful supposition that in ferromagnetic bodies it is possible to discern small regions or domains in which the elementary magnets cause each other to become parallel, thus giving rise to an internal, molecular field H_m whose strength exceeds observed, external fields by many orders of magnitude, *cf.* Vol. II, Sec. 14 A. Weiss assumes that this field is proportional to the magnetization, M , present in it, the coefficient of proportionality, N , being very large and depending on the material under consideration:

$$(8) \quad H_m = N \cdot M.$$

¹Ann. d. Phys. 81, 1154 (1926). The required simplification of Langevin's function is furnished by quantum mechanics, see *infra*, Section D.

These "Weiss domains" are aligned individually but the direction of a molecular field H_m varies from domain to domain. Consequently the body appears to be non-magnetic in the absence of an external field H . The moment exerted by such an external field H on the domains is quite different from that exerted on individual elementary magnets; the external magnetic influence of a ferromagnetic order of magnitude is due to the alignment of these domains with the external field.

The change in direction, particularly in the case of weak fields, is not due to a rotation in the directions of the Weiss domains; the principal effect is due to irreversible, abrupt turns performed by the elementary magnets at the boundaries of Weiss' domains and to the wall displacements connected with them (Vol. III Sec. 14 C).

We shall refrain from analyzing the interactions which occur between the Weiss domains and we shall restrict ourselves to the consideration of the influence of the external magnet on the magnetization in a single domain.

Quantitatively Weiss superimposes the inner field H_m on the external field H in Langevin's expression for α :

$$(9) \quad \alpha = \frac{\mu_0 M_\infty}{R T} (H + H_m) = \frac{\mu_0 M_\infty}{R T} (H + N M).$$

It is seen that the value of α becomes much larger than in the paramagnetic case because $H_m \approx H$. Furthermore, on substituting (9), Langevin's assumption (5 a) becomes an implicit equation in M , because M appears not only explicitly on the left-hand side of (5 a) but also on the right-hand side as given in (9).

Figure 18 shows a graphical method of solving this equation. On the one hand, the point which is determined by the two unknowns M/M_∞ and α must lie on Langevin's curve and on the straight line defined by eq. (9), on the other. It must, therefore, satisfy the two equations

$$(9 a) \quad \frac{M}{M_\infty} = L(\alpha),$$

$$(9 b) \quad \alpha = \alpha_0 + \beta \frac{M}{M_\infty}; \quad \alpha_0 = \frac{\mu_0 M_\infty H}{R T}; \quad \beta = N \frac{\mu_0 M_\infty^2}{R T}.$$

The straight line (9 b) intersects the axis of abscissae at the point $\alpha = \alpha_0$ (denoted by P in Fig. 18); according to the definition of α in eq. (5 a) this is also the abscissa in the paramagnetic case and lies very near the origin.

According to (9 b) the tangent of the angle which it includes with the axis of abscissae is given by

$$(9\ c) \quad \frac{1}{\beta} = \frac{1}{N \mu_0} \frac{R T}{M_\infty^2}.$$

This slope depends on temperature and decreases with it. The point of intersection A of this straight line with Langevin's curve moves to the right as T is decreased; in this way M approaches to the saturation value M_∞ which corresponds to perfectly aligned elementary magnets.

The position of the point of intersection changes very little when the external field is removed, i. e. when we make $H = 0$. The straight line PA is then translated parallel to itself until it passes through the origin 0 (since $\alpha_0 = 0$) and the new point of intersection is at B. The field of a single Weiss domain remains almost unchanged. Thus there exists the possibility of a residual *spontaneous magnetization* of Weiss' domains leading to *permanent magnetization*. The preceding argument has shown that eq. (9) reproduces the essential features of ferromagnetic behavior at *low temperatures*, such as *the existence of spontaneous magnetization which increases as T is decreased*. The fact that this spontaneous magnetization cannot always be observed follows from the interaction between the individual fields whose directions may differ from domain to domain causing them to cancel each other.

Until now we have implied that the straight line PA is less steep than e. g. the tangent to the Langevin's curve at 0. When the opposite is true the point of intersection will lie near the origin 0 provided that the external field is sufficiently weak. In this case the approximation (6) for $L(\alpha)$ may be used and eqs. (9 a, b) yield

$$\frac{M}{M_\infty} = \frac{1}{3} \cdot \frac{\mu_0 M_\infty}{R T} (H + NM).$$

It follows that

$$(10) \quad M \left(T - \frac{\mu_0 M_\infty^2 N}{3 R} \right) = \frac{\mu_0 M_\infty^2}{3 R} H.$$

The coefficient of H on the right-hand side is equal to *Curie's constant* C from eq. (7 a). The left-hand side contains its multiple NC which we shall denote by Θ , or

$$(11) \quad \Theta = \frac{\mu_0 M_\infty^2 N}{3 R}.$$

Θ is known as the *Curie point*.

Since the factor $1/3$ in the preceding equations derives from the first term in the series expansion for $L(\alpha)$ and is identical with $L'(0)$, we can replace (11) by

$$(11\ a) \quad \Theta = \mu_0 L'(0) \frac{M_\infty^2 N}{R}.$$

It is convenient to retain this form rather than that in (11) because some of the succeeding calculations can thus be made independent of the particular choice of Langevin's function $L(\alpha)$. This will make the results more suitable for the introduction of generalizations which are suggested by quantum mechanics.

Substituting the abbreviation Θ from (11) or (11 a) we can transform eq. (10) into

$$(12) \quad M = \frac{CH}{T - \Theta}.$$

It is seen that above the Curie point the body behaves like a paramagnetic substance and obeys the Curie-Weiss law (12). Its graphical representation is given by a straight line on plotting H/M in terms of T . Strictly speaking experiments suggest the existence of two slightly different Curie points depending on whether Θ is defined with the aid of this straight line or on the basis of the disappearance of spontaneous magnetization. A more thorough consideration of these and other details of the extensive field of ferromagnetic experimental data would exceed the scope of these lectures.

We must, however, examine a little more closely the neighborhood of the Curie point. It is evident that eq. (12) remains valid for $T \rightarrow \Theta$ only on condition that H tends to zero sufficiently strongly at the same time. Thus we may apply eq. (12) above the Curie point when $H = 0$ so that we obtain $M = 0$ denoting *no* spontaneous magnetization.

We shall now consider the magnetization when $H = 0$, i. e. we shall investigate the spontaneous magnetization close to but below the Curie point; the corresponding value of α will be denoted by α_{sp} . It is no longer permissible to approximate the Langevin function by a straight line even though $H = 0$ if the point of intersection B from Fig. 18 is not to be lost. Moreover, it is necessary to take into account the higher derivatives of $L(\alpha)$ at the origin. All even derivatives vanish at the origin since $L(\alpha)$ is an odd function of α . Neglecting the derivatives of the 5-th, 7-th, etc. order, we obtain from (9 a) that

$$(13) \quad \frac{M_{sp}}{M_\infty} = \alpha_{sp} L'(0) + \frac{\alpha_{sp}^3}{6} L'''(0).$$

On the other hand it follows from (9) with $H = 0$ and in view of (11 a) that:

$$(14) \quad \frac{M_{sp}}{M_{\infty}} = \alpha_{sp} \frac{R T}{\mu_0 N M_{\infty}^2} = \alpha_{sp} L'(0) \frac{T}{\Theta}$$

On comparing (13) with (14) we obtain an equation whose non-vanishing solution can be written

$$(15) \quad \alpha_{sp} = \sqrt{\frac{6 L'(0)}{-L'''(0)} \left(1 - \frac{T}{\Theta}\right)}.$$

The value of $L'''(0)$ calculated from (5 a) is equal to $-2/15$. Substituting (15) into (13) we obtain, after a short calculation, that the spontaneous magnetization just below the Curie point is given by

$$(16) \quad M_{sp} = M_{\infty} \sqrt{\frac{6 [L'(0)]^3}{-L'''(0)}} \cdot \frac{T}{\Theta} \cdot \sqrt{\left(1 - \frac{T}{\Theta}\right)}.$$

Figure 19 shows that the plot of M_{sp} in terms of decreasing T has a vertical tangent at the Curie point, in agreement with eq. (16). The curve increases up to $T = 0$ where $M_{sp} = M_{\infty}$. It must, however, be noted that the diagram is only qualitatively correct. In actual fact its shape must be changed due to quantum effects (directional quantization of spin moments; the curve obtains a horizontal tangent at $M_{sp} = M_{\infty}$ and its slope is not like that shown in Fig. 19).

It is clear that the diagram is true for a single Weiss domain. The degree of permanent magnetization which will be discernible in a macroscopic aggregate depends on the structure of the material and cannot be described with the aid of the present theory.

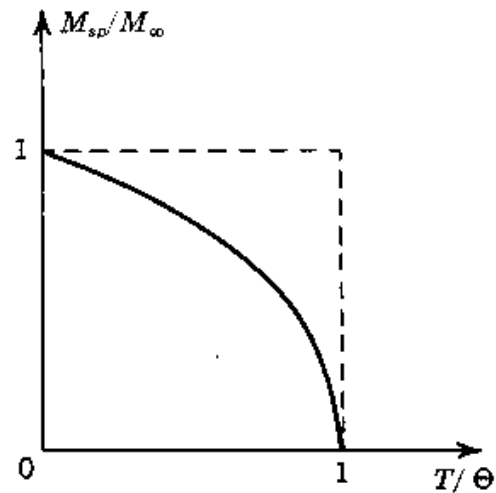


Fig. 19.

Spontaneous magnetization below the Curie point according to Weiss' theory (subject to quantum corrections).

D. THE SPECIFIC HEATS c_H AND c_M

The magnetic equation of state is no longer of the simple type (3), owing to the presence of the term NM in (9). Consequently the caloric equation (3 a) ceases to apply. The equation which takes its place can be deduced from the expression for entropy (2 a) by writing it in terms of T and M as the independent variables:

$$(17) \quad ds = \frac{1}{T} \left(\frac{\partial u}{\partial T} \right)_M dT + \frac{1}{T} \left[\left(\frac{\partial u}{\partial M} \right)_T - \mu_0 H \right] dM.$$

Taking the partial derivative of the factor before dT with respect to M , and of that before dM with respect to T we have

$$\frac{1}{T} \frac{\partial^2 u}{\partial M \partial T}$$

and

$$\frac{1}{T} \left[\frac{\partial^2 u}{\partial T \partial M} - \mu_0 \left(\frac{\partial H}{\partial T} \right)_M \right] - \frac{1}{T^2} \left[\left(\frac{\partial u}{\partial M} \right)_T - \mu_0 H \right].$$

Since both derivatives must be equal to each other, we obtain

$$(18) \quad \left(\frac{\partial u}{\partial M} \right)_T = \mu_0 \left\{ H - T \left(\frac{\partial H}{\partial T} \right)_M \right\}.$$

The addition of the subscripts T and M in the last formula is required for the sake of clarity. First we note that the preceding derivation is completely analogous to that of $\partial u / \partial v$ in (9.6) from van der Waals' equation. Substituting (18) into (17) we find that the heat added reversibly is given by

$$(19) \quad T ds = \left(\frac{\partial u}{\partial T} \right)_M dT - T \mu_0 \left(\frac{\partial H}{\partial T} \right)_M dM.$$

If T and H are chosen as independent variables instead of T and M it is only necessary to substitute

$$dM = \left(\frac{\partial M}{\partial T} \right)_H dT + \left(\frac{\partial M}{\partial H} \right)_T dH.$$

It follows from (15) that

$$(20) \quad T ds = \left\{ \left(\frac{\partial u}{\partial T} \right)_M - T \mu_0 \left(\frac{\partial H}{\partial T} \right)_M \left(\frac{\partial M}{\partial T} \right)_H \right\} dT - T \mu_0 \left(\frac{\partial H}{\partial T} \right)_M \left(\frac{\partial M}{\partial H} \right)_T dH.$$

In strict analogy with the molar specific heats, c_v , and, c_p , we now define the molar specific heats at constant magnetization, c_M , and that at constant field intensity, c_H . According to (19) and (20), we obtain

$$(21) \quad c_M = T \left(\frac{\partial s}{\partial T} \right)_M = \left(\frac{\partial u}{\partial T} \right)_M,$$

$$(21 \text{ a}) \quad c_H = T \left(\frac{\partial s}{\partial T} \right)_H = \left(\frac{\partial u}{\partial T} \right)_M - T \mu_0 \left(\frac{\partial H}{\partial T} \right)_M \left(\frac{\partial M}{\partial T} \right)_H,$$

whence, by subtraction

$$(22) \quad c_H - c_M = -T \mu_0 \left(\frac{\partial H}{\partial T} \right)_M \left(\frac{\partial M}{\partial T} \right)_H.$$

This, again, is an exact analogue of the already familiar general expression for $c_p - c_v$ (eq. (7.9) with $-p, v$ replaced by $\mu_0 H, M$).

The two derivatives on the right-hand side of (22) can be found from the parametric representation (9 a), (9 b); on differentiating with respect to T at constant M , we obtain

$$(23) \quad 0 = L'(\alpha) \left\{ \frac{\mu_0 M_\infty}{R T} \left(\frac{\partial H}{\partial T} \right)_M - \frac{\alpha}{T} \right\}$$

and at constant H we have

$$(24) \quad \left(\frac{\partial M}{\partial T} \right)_H = M_\infty L'(\alpha) \left\{ \frac{\mu_0 N M_\infty}{R T} \left(\frac{\partial M}{\partial T} \right)_H - \frac{\alpha}{T} \right\}.$$

It follows at once from (23) that

$$(25) \quad \left(\frac{\partial H}{\partial T} \right)_M = \frac{R}{\mu_0 M_\infty} \alpha.$$

Taking into account the definition (11 a) of Θ , we calculate from (24) that

$$(26) \quad \left(\frac{\partial M}{\partial T} \right)_H = \frac{M_\infty L'(0) L'(\alpha) \alpha}{\Theta L'(\alpha) - T L'(0)}.$$

Hence, according to (22) the difference of the specific heats becomes:

$$(27) \quad c_H - c_M = \frac{R L'(0) L'(\alpha) \times \alpha^2}{L'(0) - (\Theta/T) L'(\alpha)}.$$

We shall now discuss this result for the special case when $H = 0$ (removal of external field) putting $M = M_{sp}$ accordingly (spontaneous magnetization), see Fig. 20. As we already know at $T > \Theta$ we have $M_{sp} = 0$ (paramagnetic behavior). Consequently it follows from (9) that $H = 0$ implies $\alpha = 0$. Hence eq. (27) yields

$$(28) \quad c_H = c_M$$

for $T > \Theta$ and $H = 0$.

Of the region $T \leq \Theta$ we shall first consider case a) $T \ll \Theta$ which, according to (9), implies $\alpha \gg 1$. It then follows from (6 a) that

$$L'(\alpha) \approx \frac{1}{\alpha^2}$$

and from (27) that

$$(29) \quad c_H - c_M = \frac{R L'(0)}{L'(0) - \frac{\Theta}{T} \cdot \frac{1}{\alpha^2}}.$$

From (9) with $H = 0$ and $M = M_{sp} = M_\infty$ (c/. Fig. 17), we have:

$$\alpha = \frac{\mu_0 M_\infty^2 N}{R T}$$

which can be replaced by

$$\alpha = \frac{\Theta}{T} / L'(0)$$

in view of (11 a). Substituting this into (29), we have

$$(29a) \quad c_H - c_M \approx \frac{R}{1 - \frac{T}{\Theta} L'(0)} \approx R.$$

Considering the neighborhood of the Curie point we assume

$$(29b) \quad \Theta - T \ll \Theta.$$

It is now necessary to proceed in the same way as in connection with eq. (13) and to take into account the third derivative of the Langevin function in the denominator of (27), replacing $L'(\alpha)$ in the numerator by $L'(0)$ which is permissible because $\alpha = \alpha_{sp} \ll 1$: Thus we obtain

$$(30) \quad c_H - c_M = \frac{R \alpha^2 [L'(0)]^2}{L'(0) - \frac{\Theta}{T} \left[L'(0) + \frac{1}{2} \alpha^2 L'''(0) \right]}.$$

Substituting α from (15) and cancelling the common factor $\Theta - T$ in the numerator and in the denominator, we find that

$$(31) \quad c_H - c_M = 3 R \frac{[L'(0)]^2}{-L'''(0)} \cdot \frac{T}{\Theta}.$$

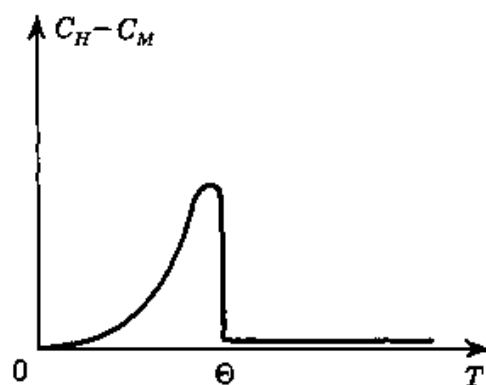


Fig. 20.

Qualitative representation of the maximum in $c_H - c_M$ at the Curie point. The behavior near $T = 0$ is sketched in accordance with the results of quantum mechanics and unlike eq. (29).

For $T > \Theta$ we have $c_H - c_M = 0$.

Thus there is a jump in the specific heats at $T = \Theta$, because, as already mentioned, $c_H = c_M$ for $T > \Theta$.

Making use of the preceding numerical values $L'(0) = 1/3$, $L'''(0) = -2/15$ we find that it is

$$(31 a) \quad c_H - c_M = \frac{5}{2} R.$$

In actual fact this sudden jump is smoothed out into a maximum (owing to the small latitude in the value of the Curie point, *cf. supra*) which decreases steeply on the side $T > \Theta$ and which is much more gradual on the side $T < \Theta$. Such a maximum persists also in the case when $H \neq 0$, i. e. in the case when the magnetization is influenced by the external field and is not spontaneous.

It will be recalled that in Section C it was found necessary to stress the fact that the results contained in it were restricted in their application to a single Weiss domain and that they were less pronounced in the case of a complete macroscopic system and depended on the particular material. This restriction is unnecessary as far as the specific heats are concerned. The specific heats superimpose themselves one on the other like *scalars* and not like the fields whose summation obeys the laws of *vectors*. Consequently our present formulae remain valid for the macroscopic system.

However, it is necessary to remember that our present results must be corrected in the light of *quantum mechanics*. This may be inferred at once from the fact that eq. (29 a) implied $c_M - c_H \rightarrow R$ for $T \rightarrow 0$ whereas Nernst's Third Law requires that $c_M - c_H \rightarrow 0$; *cf.* clause 3 in Sec. 12.

The quantum theory leads to much lower values than the value $5 R/2$ in eq. (31 a); for example, the value $3 R/2$ may be obtained depending on the kind of quantization of direction which must be assumed on atomistic grounds for a given choice of the Langevin function L .

A comprehensive and critical presentation of ferromagnetic phenomena is given in a book by Becker and Doering¹ which has already been quoted in Vol. III Sec. 14 D. We have assumed in the preceding argument that Weiss' model provides a sufficiently accurate approximation to reality; the book by Becker and Doering does, on the other hand, contain a detailed comparison of this model with the existing pertinent experimental material.

¹R. Becker and W. Doering, *Ferromagnetismus*, Berlin 1939. The book also discusses the atomistic aspects of the problem which had to be omitted from this course of lectures.

E. THE MAGNETO-CALORIC EFFECT

Isentropic demagnetization causes a drop in temperature in the case of ferro- and paramagnetic substances. According to (20) and (21) we can calculate it from

$$(32) \quad c_H \left(\frac{\partial T}{\partial H} \right)_s = T \mu_0 \left(\frac{\partial H}{\partial T} \right)_M \left(\frac{\partial M}{\partial H} \right)_T = - T \mu_0 \left(\frac{\partial M}{\partial T} \right)_H.$$

It is known as the *magneto-caloric effect*. (Conversely, a sudden, and therefore adiabatic, magnetization involves a corresponding increase in temperature.) We now proceed to calculate this effect, having described it qualitatively in terms of the disorder associated with demagnetization at the end of Sec. 11 and Sec. 12.

We shall restrict our considerations to the particularly interesting case of a paramagnetic salt (e. g. gadolinium sulphate) and assume that it obeys Curie's law down to the lowest temperatures. We thus have

$$M = \frac{C}{T} H, \quad \left(\frac{\partial H}{\partial T} \right)_M = \frac{M}{C}, \quad \left(\frac{\partial M}{\partial H} \right)_T = \frac{C}{T},$$

and from (32), we find that

$$c_H \left(\frac{\partial T}{\partial H} \right)_s = T \mu_0 \frac{M}{C} \cdot \frac{C}{T} = \mu_0 \frac{C}{T} H.$$

It follows that the isentropic process under consideration is governed by the differential equation

$$(33) \quad T dT = \frac{\mu_0 C}{c_H} H \cdot dH.$$

The process is described by

$$H \rightarrow 0, \quad T \rightarrow T_0.$$

Assuming that the coefficients C and c_H are constant we can integrate (33) to obtain

$$(34) \quad T_0^2 - T^2 = - \frac{\mu_0 C}{c_H} H^2, \quad T_0 = T \sqrt{1 - \frac{\mu_0 C}{c_H} \cdot \frac{H^2}{T^2}}.$$

It is seen that the temperature does, in fact, decrease, the drop in temperature being larger for a stronger original field H and for a lower initial temperature T .

The preceding calculation is also superficial to a certain extent because of the extrapolation of Curie's law to the lowest temperatures. This implies that interactions between elementary magnets have been neglected and this is no longer permissible. Nevertheless, eq. (34) does give an idea of the very effective step taken by Debye, Giauque and Kammerlingh Onnes in order to come nearer to the absolute zero of temperature.

20. Black body radiation

All hot bodies emit electromagnetic radiation. As the temperature increases the body changes from a red through a yellow to a white glow. It must be realized, however, that bodies emit radiation even at ordinary or low temperatures, except that the wavelength then lies in the infrared region. All thermal radiations are wave-like in their character but within the field it is possible to analyze them exactly according to the laws of geometrical optics which means that they may be resolved into pencils of rays.

Let us now imagine a hollow box whose walls are maintained at a constant temperature. The radiation present inside it is in thermal equilibrium with its walls. Consequently we must ascribe to it the same temperature T as that possessed by the walls. This is true for every element of volume in the cavity and specifies *homogeneous* radiation throughout, i. e. one which is independent of the space coordinates. The cavity constitutes a thermodynamic system (proof in Section A) which is independent of the particular physical and chemical processes of emission and absorption taking place in the walls.

It is found that the internal equilibrium is not disturbed appreciably if a small hole is made in the box so that the radiation can leave the cavity and can thus be made accessible to observation. Radiation from outside which may fall on the opening will not be reflected; it is completely absorbed by the walls after having reflected from them a large number of times and after having been partly absorbed on each reflection. Since a surface which absorbs radiation completely is usually called "black" it is natural to call the radiation emitted through an opening in the box "black body radiation."

The introduction of a "speck of soot," i. e. of a perfectly absorbing body of very small heat capacity into the cavity does not disturb the state of equilibrium. On the other hand when the inner walls of the cavity are made of a perfectly reflecting material and cannot, therefore, influence the rays falling on them, the radiation filling the cavity may become one which is not in equilibrium. The introduction of a speck of soot into the cavity will turn the radiation into black body radiation. (The speck of dust performs the role of a catalyzer.)

The view which presents itself to an observer inside the box is not very interesting: he perceives the same luminosity at every point and in all directions. He cannot see the shape of the cavity and is not aware of the differences in the distances from the walls in varying directions. Using a Nicol prism he could verify that the radiation is not polarized. On changing the temperature he will only notice a change in the intensity and in the color of the radiation.

A. KIRCHHOFF'S LAW

We recall from Electrodynamics that electromagnetic radiation carries energy and momentum, *cf.* Vol. III. Sec. 31. The energy density was denoted there by W . In a monochromatic field of radiation its average with respect to time depends on the space coordinate, on the frequency, ν , and on the amplitude, or, more precisely, on its square, the intensity. We now consider all radiation within a small spectrum interval, $d\nu$, as distinct from monochromatic radiation. The energy density contained within this interval will be denoted by $u\,d\nu$, and that of the whole spectrum will be denoted by u . We then have

$$(1) \quad u(T) = \int_0^{\infty} u(\nu, T) d\nu.$$

The argument, T , has been added here to emphasize the fact that the amplitude (or intensity) of black body radiation depends only on temperature if equilibrium prevails and is the same at every point in the cavity. The symbol u is here used in a slightly different sense, because u denotes now energy per unit volume and not per unit mass (or per mol):

$$(1\ a) \quad [u] = \frac{\text{erg}}{\text{cm}^3}.$$

It follows from (1) that the dimension of u is

$$(1\ b) \quad [u] = \frac{\text{erg sec}}{\text{cm}^3}.$$

Kirchhoff (1859) proved that u is a function of the arguments ν and T and that it is *independent of the nature of the walls of the cavity*. This proposition is known as Kirchhoff's law. In order to indicate the method of proving it let us consider two hollow boxes A and B whose walls are different. Let us assume that u in A is larger than in B , in a certain spectral region ($\nu, d\nu$). We now

connect A to B through a small tube which is opaque to all wavelengths except ν (color filter). In such an arrangement more heat would flow from A to B than in the reverse direction, thus upsetting the state of equilibrium; the temperature of B would increase and that of A would decrease until the two values of u would have become equal. In this way a temperature difference would be created "spontaneously" (without work being done on the system) and such a result is inconsistent with the Second Law. *We conclude that u must be a universal function of ν and T ; it follows from (1) that u is a universal function of T .*

We now consider the *flux* of energy (denoted by S in electrodynamics) as distinct from its density. It is defined as the amount of energy radiated per unit area and time. The vectorial character of S corresponds to the direction of the normal to the unit of area under consideration. Since black body radiation is *isotropic* (uniform in all directions) it loses its vectorial character and we are justified in speaking of a scalar radiation intensity. We shall not associate it with a discrete direction (the flux of energy in any given single direction is zero) but with a small cone of radiation $d\Omega$. We now imagine that the direction is enclosed in such a cone and denote the energy radiated through $d\Omega$ by $K d\Omega$. Consequently, the energy radiated through an elementary cone which forms an angle θ with the normal is given by

$$(2) \quad K \cos \theta d\Omega \quad \text{where} \quad d\Omega = \sin \theta \cdot d\theta \cdot d\phi.$$

The amount of radiation passing through an elementary area da during a time dt in a "forward" (or "rearward") direction is then

$$(2a) \quad K da dt \int_0^{\pi/2} \cos \theta \sin \theta d\theta \int_0^{2\pi} d\phi = \pi K da dt.$$

If K is analyzed spectrally and if the two directions of polarization are distinguished by a dash we may write

$$(3) \quad K(T) = \int_0^{\infty} [K(\nu, T) + K'(\nu, T)] d\nu = 2 \int_0^{\infty} K(\nu, T) d\nu,$$

the last equality being a consequence of the absence of polarization in black body radiation. The dimension of K is the same as that of S ; that of K follows from (3) and is

$$(3a) \quad [K] = \frac{\text{erg}}{\text{cm}^2 \text{sec}}; \quad [K] = \frac{\text{erg}}{\text{cm}^2}.$$

The quantities u and K satisfy the relation

$$(4) \quad u = 4\pi K/c.$$

We shall refrain here from giving the proof because it can be deduced from a simple premiss of geometrical optics, assuming the cavity to be evacuated. If this were not so it would be necessary to replace c by c/n . In view of (3) and (1) we have from (4) that

$$(4 \text{ a}) \quad u = 8\pi K/c.$$

We now proceed to obtain an *extension of Kirchhoff's law* by applying the equilibrium principle to the walls of the cavity.

The *absorptive power* of a wall element da will be denoted by A ; in other words A denotes that fraction of the impinging radiation $K(\nu, T)$ (assumed spectrally decomposed) which is converted into heat as it penetrates into the wall. Thus the amount of energy (per unit area and time and per solid angle $d\Omega$) which is deducted from the system at equilibrium is

$$(5) \quad A K(\nu, T).$$

This energy must be replaced in the cavity by the *emissivity* E of the same element of wall. In the case of a blackened surface ($A = 1$) we have

$$(5 \text{ a}) \quad E = K(\nu, T).$$

The emissivity of a perfectly white perfectly reflecting surface ($A = 0$) must be $E = 0$. In such a case, as already stated previously, the wall cannot contribute to the establishment of thermodynamic equilibrium. In an average surface E must replace the amount (5) withdrawn from the cavity. Thus we must have

$$(6) \quad \frac{E}{A} = K(\nu, T).$$

For pure thermal radiation the ratio of the emissivity to the absorptive power is a universal function of the frequency and temperature.

Kirchhoff's law and the present extension thereof have now become very important not only in problems of black body radiation but also in illumination engineering. It contributed to the discovery of spectral analysis which was made by Kirchhoff and Bunsen at about that time.

B. THE STEFAN-BOLTZMANN LAW

It has already been stated at the beginning of Section A that radiation carries with it momentum in addition to energy. This is the cause of the pressure of light discovered by Maxwell. According to the last equations in Sec. 31 of Vol. III, the pressure exerted by a wave forming an angle θ with the normal to an element of area da is $u \cos^2 \theta$; it follows that for radiation coming from all sides the pressure is

$$(7) \quad p = u \int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\theta = u/3.$$

The preceding equation is valid for a partly reflecting surface as well as for a black one because the thrust due to the reflected radiation is added to that due to emitted radiation.

Let us now imagine an evacuated cylindrical vessel fitted with a sliding piston and filled with black body radiation at a temperature T . The volume V can be changed at will by moving the piston (infinitely slowly). The preceding constitutes a thermodynamic system with two variables and its energy is

$$U = V u(T),$$

whereas, according to (7), the work on the piston is given by

$$dW = p \, dV = \frac{1}{3} u(T) \, dV.$$

The change in entropy is

$$(8) \quad dS = \frac{dU + dW}{T} = \frac{V \, du}{T \, dT} dT + \frac{4}{3} \frac{u}{T} dV.$$

Since dS is a perfect differential we must have

$$\frac{1}{T} \frac{du}{dT} = \frac{4}{3} \frac{d}{dT} \left(\frac{u}{T} \right);$$

after a short calculation we find that

$$(9) \quad \frac{du}{u} = 4 \frac{dT}{T}; \quad \log u = 4 \log T + \text{const, or} \\ u = a T^4.$$

In order to determine the constant of integration a we substitute K for u from eq. (4) and we obtain

$$(10) \quad K = \frac{c a}{4\pi} T^4.$$

According to (2 a) the left-hand side represents the total radiation of a black surface (e. g. the hole in the wall of any black body cavity) per unit area and time. The constant $c a/4\pi$ which appears on the right-hand side is usually denoted by σ ; it can be determined from observation. Equation (10) contains a statement of the law of radiation discovered empirically by Stefan. The preceding thermodynamic derivation was first given by Boltzmann in 1884. In his memorial address devoted to Boltzmann, H. A. Lorentz called it "a veritable pearl of theoretical physics."¹

Substituting (9) into (8) we obtain

$$(11) \quad dS = 4 a \left(V T^2 dT + \frac{1}{3} T^3 dV \right) = \frac{4}{3} a d(T^3 V).$$

The integration of (11) does not lead to a new constant because according to the Third Law we must have $S = 0$ for $T = 0$. Thus we obtain

$$(12) \quad S = \frac{4}{3} a T^3 V.$$

The equation of an isentrope in the T, V plane is represented by

$$(12 a) \quad T^3 V = \text{const.}$$

It describes the change in temperature which accompanies an adiabatic and reversible change in volume (and hence, according to (9), also the change in the energy density u). Equation (12 a) is seen to be identical with the isentropic equation for a perfect gas whose ratio of specific heats $K = 4/3$.

C. WIEN'S LAW

The most significant idea which W. Wien used to determine the relation between frequency and temperature for black-body radiation consisted in his inquiring into the change in the spectrum of radiation on reflection from a moving mirror. It will be recalled from Vol. IV, Sec. 13, that the frequency of reflected light differs from that of incident light when the mirror moves

¹Verh. d. Deutsch. Physik. Gesellschaft, 1907.

in the direction of its normal. The same is true of the intensity of incident and reflected radiation. Making use of the premiss that the modified spectrum must retain the properties of equilibrium radiation if the process is conducted in a suitable way it is possible to deduce the *shift* in the maximum of intensity and hence the color of the radiation which accompanies a change in temperature.

We shall refrain from proving Wien's law on the basis of a suitable model¹ and will concentrate on the widely discussed problem of whether it can be made plausible with the aid of dimensional analysis, that is on the basis of considerations of similarity.² As always we shall assume four fundamental units, one of them being temperature (symbol θ). The remaining three are the mechanical units, it being convenient to replace the unit of mass by the unit of energy (erg, symbol e).³ Time and length will be denoted by t and l respectively.

According to (1 b) the dimension of u is $e l/l^3$. It is now necessary to express u in terms of ν and T (dimension t^{-1} or θ , respectively) and of certain universal constants. The latter include the speed of light c (dimension $l t^{-1}$) and the universal constant R , whose dimension is $e \theta^{-1}$ because $R T$ denotes an energy, as seen from the equation of state of a perfect gas. R is usually referred to one mol of some substance. In what follows, however, it is more convenient to refer it to a single molecule which can be effected by dividing it by the number of molecules per mol. It is known as Boltzmann's constant, k , and its dimension $e \theta^{-1}$ is the same as that of R .

The five quantities in question are shown listed together with their dimensions in the following (we shall refer to the last column presently):

$$(13) \quad \frac{u}{e l^{-3} t} \left| \frac{\nu}{t^{-1}} \right| \left| \frac{T}{\theta} \right| \left| \frac{c}{l t^{-1}} \right| \left| \frac{k}{e \theta^{-1}} \right| \left| \frac{h = k \alpha}{e t} \right|$$

We now try to form a product of these five quantities, each raised to a certain (positive or negative) power satisfying the condition that it has the dimension zero in all four units

$$(14) \quad e, l, t, \theta.$$

¹The simplest proof of this kind was given by von Laue, Ann. d. Phys. (5) 48, 220, (1943). The model consists of a single pencil of monochromatic rays and the proof is based on its invariance with respect to Lorentz transformations. Our argument only assumes invariance with respect to change of scale.

²Cf. a note by Glaser, Sitzungsber. d. Akad., Wien, Vol. 156, p. 87; our considerations are partly based on this note.

³It is assumed that the fourth unit of our electrodynamical system, the unit of electricity Q , does not occur in the argument.

We can assume that one of the exponents has a prescribed value, say unity, without loss of generality. In this manner the four remaining exponents are seen to be uniquely determined by the four equations which result from equating to zero the sum of the exponents for each of the units (14). There exists only *one* such product. Assuming that the exponent of u is equal to 1 we can deduce from Table (13) that $u c^3$ and $\nu^2 k T$ have the dimension $e t^{-2}$, so that the product in question becomes

$$(15) \quad \Pi = \frac{u c^3}{\nu^2 k T},$$

and Π denotes an unknown universal number. The spectral distribution function becomes

$$(16) \quad u = \frac{\nu^2 k T}{c^3} \Pi.$$

This is the *unique* (except for an undetermined factor) answer which is supplied by *classical* physics to the problem of the spectrum of black body radiation. The adjective "classical" means here that the argument is confined to the application of the two universal constants c and k which have been in use in physics for a very long time.

Equation (16) was first deduced by Lord Rayleigh in 1900 who obtained it from classical statistics, finding at the same time that the numerical constant Π was equal to 8π . The equation was further developed by J. H. Jeans (the Rayleigh-Jeans radiation formula). It is, however, clear that the equation gives absurd results for large values of ν , because it leads to an infinite value of u for $\nu \rightarrow \infty$, and because the integral for total radiation, $u = \int u d\nu$ is divergent.

In order to reach agreement with experiment we are forced to give up the limitation of using only two universal constants. There must be a third such constant, because it follows from Kirchhoff's law that apart from u , ν , and T no other *variables* enter into the problem.

The third constant will lead to an additional dimensionless group Π' which is independent of eq. (15) and which may be assumed independent of u and depending on the first power of ν without any loss in generality¹. Thus we find that

$$(17) \quad \Pi' = \alpha \nu T^n.$$

¹If this were not the case it would suffice to multiply the number Π' by a suitable power of Π in order to eliminate u and to raise the result to such a power as to render the exponent of ν equal to unity. The last operation is always possible because experiments show that Π' cannot be independent of ν .

The constant α in this equation is a combination of c , k together with the new universal constant. Consequently

$$(17\ a) \quad \Pi = f(\Pi')$$

or

$$(17\ b) \quad u(\nu, T) = \frac{\nu^2}{c^3} k T \cdot f(\alpha \nu T^n).$$

The exponent n must be so selected as to yield eq. (9) on integration over all frequencies. From

$$u = \frac{k T}{c^3} \int_0^\infty f(\alpha \nu T^n) \nu^2 d\nu$$

with the abbreviation $\alpha \nu T^n = x$, we have

$$u = \frac{k T^{1-3n}}{\alpha^3 c^3} \int_0^\infty f(x) x^2 dx.$$

The result will be proportional to T^4 only if we put $n = -1$. In this way eq. (17 b) leads to Wien's law:

$$(18) \quad u(\nu, T) = \frac{\nu^2 k T}{c^3} f\left(\frac{\alpha \nu}{T}\right).$$

The unknown function of two variables, $u(\nu, T)$, has thus been reduced to the unknown function, f , of a single variable, $\alpha \nu/T$. This is the great achievement of Wien's law.

It is convenient to include Boltzmann's constant, k , in the argument of f and to put $k \alpha = h$. This gives the more familiar form

$$(18\ a) \quad u(\nu, T) = \frac{\nu^2 k T}{c^3} f\left(\frac{h \nu}{k T}\right).$$

The quantity h represents a new constant and has the dimension of "action" i. e. ϵt . It completes our Table (13). We add here parenthetically that h is Planck's quantum of action which has now become a familiar fundamental constant and which has been anticipated, at least as far as its dimension is concerned, by Wien's law. Multiplying and dividing the coefficient of f in eq. (18 a) by $h \nu$, we obtain

$$(18\ b) \quad u(\nu, T) = \frac{h \nu^3}{c^3} \cdot \frac{f(x)}{x} = \frac{h \nu^3}{c^3} f_1(x); \quad x = \frac{h \nu}{k T}.$$

Consequently, the Stefan-Boltzmann constant a from eq. (9) becomes

$$(19) \quad a = k \left(\frac{k}{h c} \right)^3 \times F, \quad F = \int_0^{\infty} x^3 f_1(x) dx.$$

To conclude we shall give reasons for describing the preceding law as "Wien's displacement law." We now ask for that value of ν which corresponds to the maximum in the intensity u for a given temperature, that is that value of ν for which $\partial u / \partial \nu = 0$. From eq. (18 a), we find that it is given by

$$(20) \quad 2 f(x) + x f'(x) = 0, \quad x = \alpha \nu / T.$$

We shall denote the real positive root of this equation by $x = x_m$ corresponding to $\nu = \nu_m$. Thus

$$(20 a) \quad \nu_m = x_m T / \alpha.$$

As T increases the point of maximum intensity is "displaced" towards larger values of ν . Since the value of ν_m determines the general coloring perceived on observing the whole spectrum, eq. (20 a) is seen to supply an explanation for the transition from a red to a white glow at increasing temperature.

It has become customary to associate the values of λ rather than those of ν with our color perception. Since

$$\nu = \frac{c}{\lambda}; \quad |d\nu| = \frac{c}{\lambda^2} |d\lambda|; \quad u |d\nu| = u_\lambda |d\lambda|$$

the variation of the intensity u_λ on the scale of λ becomes

$$(21) \quad u_\lambda = \frac{k T}{c \lambda^2} / \left(\frac{\alpha c}{\lambda T} \right) \frac{|d\nu|}{|d\lambda|} = \frac{k T}{\lambda^4} / \left(\frac{\alpha c}{\lambda T} \right),$$

as seen from eq. (18). Introducing a new variable y and a new function $g(y)$ by

$$(21 a) \quad y = \frac{\lambda T}{\alpha c}; \quad g(y) = y / \left(\frac{1}{y} \right)$$

we obtain

$$(22) \quad u_\lambda = \frac{\alpha k c}{\lambda^5} g(y)$$

as seen from (21). Hence

$$(22 a) \quad \frac{\partial u_\lambda}{\partial \lambda} = - \frac{\alpha k c}{\lambda^6} [5 g(y) - y g'(y)].$$

and the position of maximum intensity is given by the equation

$$(23) \quad 5g(y) - yg'(y) = 0.$$

Making use of the real positive value y_m of the root of this equation we find from eq. (21) that

$$(23a) \quad \lambda_m T = \alpha c y_m.$$

The root $y = y_m$ differs from the root x_m in eq. (20) because y and x have different meanings. Qualitatively the conclusion regarding the displacement in color is, evidently, the same as before: As the temperature is increased the value of λ_m is shifted towards shorter wavelengths (higher frequencies ν).

D. PLANCK'S LAW OF RADIATION

Planck inserts into the field of radiation a linear oscillator which reacts with it to a certain extent: it is a Hertz dipole of a definite natural frequency ω_0 whose dimensions are small compared with the relevant wavelengths. If the oscillator were free it would perform damped oscillations because of the electromagnetic radiation and with small damping it would react sharply on the frequencies ω of incident radiation which lie in the neighborhood of ω_0 . Assuming that the incident and the excited oscillation are given by

$$(24) \quad C \sin \omega t \quad \text{and} \quad D \sin(\omega t + \delta),$$

respectively, and applying the result in Vol. I, eq. (19.10) we find that

$$(24a) \quad D = \frac{C}{M} \{(\omega^2 - \omega_0^2)^2 + 4\rho^2 \omega^2\}^{-\frac{1}{2}}.$$

The oscillation equation must be assumed to be of the form

$$(25) \quad m(\ddot{x} + 2\rho \dot{x} + \omega_0^2 x) = e E_x$$

in accordance with eq. (19.9) of Vol. I. E_x denotes here the component of the electrical field of radiation, E , which coincides with the direction of motion x ; e and m denote the charge and mass of the oscillating electron. According to (25) the opposing damping force is equal to

$$(26) \quad R = -2\rho m \dot{x}.$$

Comparing it with the damping force ("reaction force") of radiation from Vol. III, eq. (36.4), we have

$$R = \frac{e^2}{6\pi\epsilon_0 c^3} \ddot{\ddot{x}},$$

which can also be written

$$(26\ a) \quad R = -\frac{e^2}{6\pi\epsilon_0 c^3} \omega^2 \dot{x}$$

because of the dependence of $x = D \sin(\omega t + \delta)$ on time. It follows from eqs. (26) and (26 a) that

$$(26\ b) \quad \rho = \frac{1}{12\pi} \frac{e^2}{m\epsilon_0 c^3} \omega^2.$$

We now proceed to calculate the energy of the oscillator. Its kinetic energy is

$$\frac{m}{2} \dot{x}^2 = \frac{m}{2} D^2 \omega^2 \cos^2(\omega t + \delta),$$

and its potential energy can be found from (25):

$$\frac{m}{2} \omega_0^2 x^2 = \frac{m}{2} D^2 \omega_0^2 \sin^2(\omega t + \delta).$$

Taking into account eq. (24 a) we find that their sum averaged over time is equal to

$$(27) \quad U_\omega = \frac{m}{4} D_\omega^2 (\omega^2 + \omega_0^2) = \frac{C_\omega^2}{4m} \frac{\omega^2 + \omega_0^2}{(\omega^2 - \omega_0^2)^2 + 4\rho^2 \omega^2}.$$

We have added the subscript ω to the energy U and to the amplitudes C , D in order to emphasize that, so far, we have been considering a single monochromatic oscillation. However, an oscillator placed in a field of radiation is excited by a continuous spectrum of mutually incoherent oscillations C_ω . The requirement of incoherence is as essential for our black body radiation as it was for natural "white" light in Vol. IV, Sec. 49.

It follows that in this case the squares of the amplitudes (intensities) are added, and not the amplitudes themselves as for coherent light. Hence (27) yields

$$(28) \quad U = \int U_\omega d\omega = \frac{1}{4m} \int C_\omega^2 \frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\rho^2 \omega^2} d\omega.$$

The fraction in the integrand on the right-hand side varies strongly with ω and possesses a sharp maximum in the neighborhood of $\omega = \omega_0$ (the maximum is sharp owing to the smallness of the term $\rho^2 \omega^2$). On the other hand C^2

varies slowly and may be replaced by its value C^2 at $\omega = \omega_0$. Thus instead of (28) we may write

$$(28 \text{ a}) \quad U = \frac{C^2}{4m} \int_0^\infty \frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\rho^2 \omega^2} d\omega.$$

This integral can be further simplified because the numerator and the term $4\rho^2 \omega^2$ vary slowly. They may be replaced by

$$(28 \text{ b}) \quad 2\omega_0^2 \text{ and } 4(\sigma\omega_0^2)^2 \omega_0^2 \text{ respectively with}$$

$$(28 \text{ c}) \quad \sigma = \frac{1}{12\pi} \frac{e^2}{m\epsilon_0 c^3} \quad \text{from eq. (26 b).}$$

We may further write

$$(\omega_0^2 - \omega^2)^2 = (\omega - \omega_0)^2 4\omega_0^2.$$

In this way the integral under consideration becomes

$$\frac{1}{2} \int_0^\infty \frac{d\omega}{(\omega - \omega_0)^2 + (\sigma\omega_0^2)^2} = \frac{1}{2\sigma\omega_0^2} \int_{-1/\sigma\omega_0}^\infty \frac{d\xi}{\xi^2 + 1}; \quad \xi = \frac{\omega - \omega_0}{\sigma\omega_0^2}.$$

Since $\sigma\omega_0 \ll 1$ we obtain

$$\frac{1}{2\sigma\omega_0^2} \tan^{-1} \xi \Big|_{-\infty}^{+\infty} = \frac{\pi}{2\sigma\omega_0^2}.$$

and (28 a) transforms into

$$(29) \quad U = \frac{\pi}{8m\sigma\omega_0^2} C_0^2.$$

It remains now to express C_0 in terms of the energy density, u , of black body radiation. The energy density is equal to twice its electrical contribution, i. e. to

$$(\mathbf{E}, \mathbf{D}) = \epsilon_0 \mathbf{E}^2.$$

Taking a time average for black, isotropic radiation, we have

$$(30) \quad \epsilon_0 \overline{\mathbf{E}^2} = 3\epsilon_0 \overline{\mathbf{E}_x^2} = u_\omega.$$

The last term on the right-hand side denotes the energy density on the ω -scale as distinct from that on the ν -scale used before. It will be recalled from (24) and (25) that C is equal to the amplitude of $e\mathbf{E}_x$; averaging over time, we find that at $\omega = \omega_0$ we have

$$\frac{1}{2} C_0^2 = e^2 \overline{\mathbf{E}_x^2}.$$

Substituting $\overline{\mathbf{E}_x^2}$ from (30), we obtain

$$(30\ a) \quad C_0^2 = \frac{2}{3} \frac{e^2}{\epsilon_0} u_\omega$$

where u_ω denotes the energy density over the interval $d\omega$ at $\omega = \omega_0$. Consequently

$$u_\omega d\omega = u_\nu d\nu; \quad u_\omega = \frac{1}{2\pi} u_\nu.$$

Hence (30 a) can be replaced by

$$(30\ b) \quad C_0^2 = \frac{1}{3\pi} \frac{e^2}{\epsilon_0} u_\nu.$$

Substituting this expression into (29) and taking into account (28 c) we find that the quantities e and m which refer to the specific model of the oscillator are cancelled (it is seen that the constant ϵ_0 also vanishes as might have been expected on dimensional grounds), and we obtain simply:

$$(31) \quad U = \frac{\pi c^3}{2 \omega_0^2} u_\nu = \frac{c^3}{8\pi \nu^2} u_\nu.$$

The preceding argument shows that the energy of the *oscillator* is just as universal as the energy density of *black body radiation* so that in the succeeding reasoning Planck could use the former instead of the latter. He associated with the oscillator the entropy S , in addition to the temperature T , the former being given by

$$(32) \quad dS = \frac{dU}{T},$$

at constant radiation volume ($dV = 0$).

In his Nobel Prize inaugural address delivered in 1920 Planck gave a fine example of objectivity; his law of radiation is described at first as "an interpolation formula which resulted from a lucky guess."

The experimental results obtained before 1900 (Paschen, Lummer, and Pringsheim) for the case of short wavelengths appeared to confirm an empirical hypothesis advanced by W. Wien. It follows from (18 b) by putting

$$f_1(x) = A e^{-x}, \quad x = \alpha \nu / T.$$

Thus

$$(33) \quad u(\nu, T) = \frac{\alpha k A}{c^3} \nu^3 e^{-\alpha \nu / T}.$$

Correspondingly we obtain from (31)

$$(33 \text{ a}) \quad U = A_1 e^{-\alpha \nu / T}; \quad A_1 = \frac{\alpha k}{8} \nu A.$$

Evaluating $1/T$ and taking into account (32), we have

$$(33 \text{ b}) \quad \frac{dS}{dU} = -\frac{1}{\alpha \nu} \log \left(\frac{U}{A_1} \right)$$

and

$$(33 \text{ c}) \quad \frac{d^2S}{dU^2} = -\frac{1}{\alpha \nu U}.$$

Later measurements performed at long wavelengths (infrared region) by Rubens and Kurlbaum revealed a completely different pattern of behavior which appeared to confirm the Rayleigh-Jeans equation (16). According to (31) the corresponding oscillator energy becomes

$$(34) \quad U = k T,$$

if the numerical factor H in (16) is given its value of 8π calculated by Rayleigh. Hence, according to (32), we find

$$(34 \text{ a}) \quad \frac{dS}{dU} = \frac{1}{T} = \frac{k}{U},$$

$$(34 \text{ b}) \quad \frac{d^2S}{dU^2} = -\frac{k}{U^2}.$$

Planck now uses the following formula

$$(35) \quad \frac{d^2S}{dU^2} = -\frac{1}{\alpha \nu U + U^2/k}$$

to interpolate between (33 c) and (34 b). The right-hand side can be written

$$(35 \text{ a}) \quad -\frac{1}{\alpha \nu} \cdot \frac{1}{U + \beta U^2} = -\frac{1}{\alpha \nu} \left(\frac{1}{U} - \frac{\beta}{1 + \beta U} \right); \quad \beta = \frac{1}{\alpha \nu k},$$

and it is seen that (35) can be integrated. The constant of integration will be determined from the condition that for $U = \infty$ we must have $T = \infty$ so that $dS/dU = 0$. Hence

$$(36) \quad \frac{dS}{dU} = -\frac{1}{\alpha \nu} \log \frac{\beta U}{1 + \beta U}.$$

As seen from (32), the derivative dS/dU can be replaced by $1/T$ so that

$$(36 \text{ a}) \quad \log \frac{\beta U}{1 + \beta U} = -\frac{\alpha \nu}{T}; \quad \beta U = \frac{1}{e^{\alpha \nu/T} - 1}.$$

Substituting from (35 a) we obtain

$$(36 \text{ b}) \quad U = \frac{\alpha \nu k}{e^{\alpha \nu/T} - 1}.$$

According to Table (13) αk has the dimension of energy \times time = action. It is seen that the new universal constant, the *quantum of action*

$$(37) \quad h = \alpha k$$

which has already been mentioned previously, now makes its appearance. The energy of the oscillator becomes

$$(38) \quad U = \frac{h \nu}{e^{h \nu/k T} - 1}$$

and (31) leads to Planck's law of radiation

$$(39) \quad u_\nu = \frac{8 \pi \nu^2}{c^3} \frac{h \nu}{e^{h \nu/k T} - 1}.$$

The statistical derivation of the same law, *c/f.* Sec. 33, goes much deeper than this somewhat cumbersome argument and places the revolutionary character of the constant h in its proper light. The preceding argument outlined Planck's original train of thought and the reason for describing it here lies not only in its very great historical importance; it has been quoted also in order to demonstrate that the application of the concept of entropy to the oscillator plays a very important part in it.

Figure 21 shows a three-dimensional model of Planck's law of radiation in which u_ν has been plotted vertically upwards, ν to the right in the horizontal plane, and T has been plotted rearwards. The model consists of six plane profiles placed one behind the other. The profiles represent the dependence of u_ν on ν for $T = 100, 200, \dots, 600$ K. The vertical profile which passes through the maxima ν_m as given by eq. (20 a) is developable owing to the linearity of the equation linking ν_m with T .

We now proceed to show how the limiting cases represented by the equations of Rayleigh-Jeans and Wien, respectively, can be deduced from (39):

For small values of ν and a fixed value of T we can expand the denominator of (39) into a series and obtain

$$(40) \quad u = \frac{8\pi k}{c^3} \nu^2 T.$$

For large values of ν and a fixed value of T we can neglect 1 in the denominator of (39), and we have

$$(40 a) \quad u = \frac{8\pi h}{c^3} \nu^3 e^{-h\nu/kT}.$$

Equation (40) is identical with (16) if we put $H = 8\pi$, as already stated; eq. (40 a) transforms into eq. (33) if the previous constant A is also replaced by 8π . Finally the constant a in the Stefan-Boltzmann law obtains a definite theoretical justification. On comparing eq. (18.6) with (39) we obtain the following expression for the function f_1 :

$$f_1(x) = \frac{8\pi}{e^x - 1}.$$

Hence the integral F in eq. (19) becomes:

$$(41) \quad F = 8\pi \int_0^\infty \frac{x^3}{e^x - 1} dx.$$

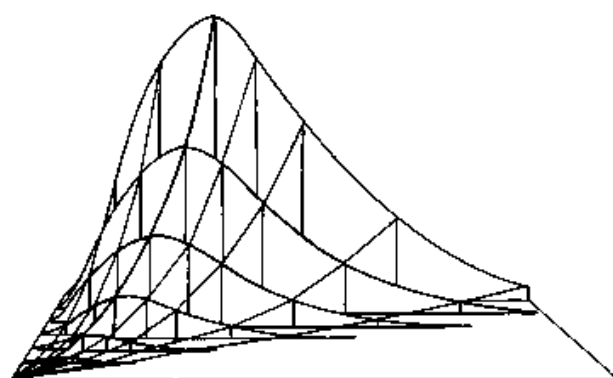


Fig. 21.

Cardboard model of Planck's law of radiation $u_\nu = f(\nu, T)$; ν is measured to the right, T is measured rearwards. The graded shading in the photograph is due to the light falling on the model. At $T = 600$ K the maximum is at $\nu_m = 4 \times 10^{13} \text{ sec}^{-1}$, at $T = 200$ K the much lower maximum is at $\nu_m = 12 \times 10^{12} \text{ sec}^{-1}$. The profile corresponding to $T = 100$ K protrudes so little that it is hardly visible.

Since for all values of $x > 0$ e^{-x} is less than unity, we can rewrite eq. (41) to give

$$(41\ a) \quad \frac{F}{8\pi} = \int_0^{\infty} \frac{e^{-x}}{1-e^{-x}} x^3 dx = \int_0^{\infty} (e^{-x} + e^{-2x} + e^{-3x} + \dots) x^3 dx.$$

Denoting $2x, 3x, \dots$ in the 2nd, 3rd, \dots term of the series respectively by ξ , we have

$$(41\ b) \quad \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right) \int_0^{\infty} \xi^3 e^{-\xi} d\xi.$$

The integral is equal to $\Gamma(4) = 3!$, and the value of the series in the brackets in front of the integral can be taken from Vol. VI, eq. (2.18), where it has been shown to be equal to $\pi^4/90$. Hence eq. (41 b) gives $\pi^4/15$ and eq. (41 a) yields

$$(41\ c) \quad F = 8\pi^5/15.$$

Substituting this value into (19) we obtain the following theoretical value of the Stefan-Boltzmann constant a :

$$(42) \quad a = \frac{8\pi^5}{15} \frac{k^4}{(hc)^3}.$$

Since a and the constant $\alpha = h/k$ from Wien's displacement law are known from measurements, eqs. (42) and (23 a) with (37) can be, in turn, used to evaluate h and k . At the present time the following are regarded as their most accurate values

$$(43) \quad h = 6.624 \times 10^{-27} \text{ erg sec}; \quad k = 1.380 \times 10^{-16} \text{ erg/deg.}$$

21. Irreversible processes. Thermodynamics of near-equilibrium processes

A. CONDUCTION OF HEAT AND LOCAL ENTROPY GENERATION

So far we have considered, essentially, only states of thermodynamic equilibrium. Concerning irreversible processes we were able only to establish that they are associated with an increase in entropy, provided that they take place in a closed system within an adiabatic boundary. We now propose to determine in greater detail where that entropy increase is located and how it depends on the parameters of the system.

We shall begin by considering a particularly simple example, namely the conduction of heat through a homogeneous and isotropic solid body disregarding its thermal expansion. If the temperature varies from point to point then, generally speaking, the internal energy per unit mass $u(x, y, z, t)$ will depend on the space coordinates and on time. The same is true of the flux of heat \mathbf{W} . The principle of the conservation of energy (see Vol. VI, eq. (7.11); it should be noted that the symbol u in that equation denoted temperature) can be written as

$$(1) \quad \rho \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{W} = 0$$

where ρ denotes density. To complete the description of the process of heat conduction it is necessary to write down the relation between internal energy and temperature, e. g. in differential form

$$(2) \quad du = c dT,$$

where c denotes the specific heat, and Fourier's hypothesis for the relation between the heat flux and temperature gradient (Secs. 44, 45 and Vol. VI, eq. (7.12)):

$$(3) \quad \mathbf{W} = -\kappa \operatorname{grad} T,$$

where κ is the thermal conductivity.

For the time being we shall disregard eqs. (2) and (3) and we shall concentrate our attention on eq. (1). The internal energy and entropy are connected through the relation

$$(4) \quad du = T ds$$

because changes in volume have been neglected. From eqs. (1) and (4) we obtain

$$(5) \quad \rho \frac{\partial s}{\partial t} = -\frac{1}{T} \operatorname{div} \mathbf{W},$$

or rearranged:

$$(6) \quad \rho \frac{\partial s}{\partial t} + \operatorname{div} \frac{\mathbf{W}}{T} = -\frac{1}{T^2} (\mathbf{W} \cdot \operatorname{grad} T).$$

Equation (1) is an equation of continuity which means that it expresses a conservation principle; in this case, that of energy. Equation (6) would also express a principle of conservation if its right-hand side vanished. Now

it is known that entropy satisfies no conservation principle, moreover, it increases in an isolated system during an irreversible process, such as the conduction of heat, within it. This increase in entropy must now be related to the right-hand side of eq. (6), and to achieve it we shall integrate eq. (6) over the heat conducting body. Using Gauss' theorem (see Vol. II, eq. (3.1)) we obtain

$$(7) \quad \rho \frac{\partial}{\partial t} \int s dV + \oint \frac{\mathbf{W}_n}{T} dA = - \int \frac{1}{T^2} (\mathbf{W} \cdot \text{grad } T) dV.$$

Assuming, at first, that the surface of the heat conducting body is adiabatically insulated we shall find that \mathbf{W}_n vanishes. Then the term on the left-hand side of eq. (7) is the change in the entropy of the body per unit time. It is expressed on the right-hand side in terms of temperature, temperature gradient and heat flux. Since by Clausius' principle heat cannot flow spontaneously from a lower to a higher temperature ($\mathbf{W} \cdot \text{grad } T$) must be negative if $\mathbf{W} \neq 0$ and $\text{grad } T \neq 0$. Hence the right-hand side of eq. (6) is positive as required by the Second Law.

The change in entropy per unit time is thus given by a volume integral; it is natural to define the integrand as the change in entropy per unit time and per unit volume. We shall regard this quantity as the *entropy generated locally*. Hence the local entropy increase is defined as

$$(8) \quad \theta = - \frac{1}{T^2} (\mathbf{W} \cdot \text{grad } T)$$

and depends only on the state which prevails at the given instant and at the given point. In this context it is necessary to interpret the concept of state more widely than hitherto. We have, namely, stipulated constant temperature in eq. (8) and it is noted that in order to specify the state it is necessary to specify in addition the temperature gradient and hence (see eq. (3)) the heat flux.

If we now drop the restriction regarding the adiabatic nature of the boundary and imagine that it is in contact with heat sources, then \mathbf{W}_n denotes the heat transferred to the respective source of heat per unit area of the boundary and per unit time (= energy, since work = 0), and \mathbf{W}_n/T represents the entropy transferred from the body to the source. It is, therefore, natural to define \mathbf{W}_n/T as the *entropy flux*

$$(9) \quad \mathbf{S} = \frac{\mathbf{W}}{T}.$$

With these conventions eq. (6) becomes

$$(10) \quad \rho \frac{\partial s}{\partial t} + \operatorname{div} \mathbf{S} = \theta$$

and integration over any portion of the body gives

$$(11) \quad \rho \frac{\partial}{\partial t} \int s dV + \oint \mathbf{S}_n dA = \int \theta dV.$$

Reading from right to left we can now interpret the physical meaning of this equation: The quantity of entropy generated inside the volume of integration is partly conducted away from it and partly contributes to the change in the entropy of the volume. Evidently the latter contribution can also be negative.

The preceding argument allowed us to determine the sources of entropy and their output for the case of the conduction of heat. In accordance with the Second Law we can, further, establish that this output can never be negative. Equation (10) together with the inequality $\theta \geq 0$ can be regarded as the *differential formulation of the Second Law of Thermodynamics*. The statement in integral form, namely that the entropy in an isolated system cannot decrease, can be replaced by its corollary in differential form which asserts that the quantity of entropy generated locally cannot be negative irrespective of whether the system is isolated or not, and irrespective of whether the process under consideration is irreversible or not.

We shall now compare Fourier's hypothesis, eq. (3), with the expression for the quantity of entropy generated locally in eq. (8). We find that the latter contains as factors precisely the quantities \mathbf{W} and $\operatorname{grad} T$ which enter into Fourier's equation. This fact, as will be seen later, has a more general significance. Moreover, from

$$\theta = \frac{\kappa}{T^2} (\operatorname{grad} T)^2 \geq 0$$

we deduce that

$$\kappa \geq 0.$$

B. THE CONDUCTION OF HEAT IN AN ANISOTROPIC BODY AND ONSAGER'S RECIPROCAL RELATIONS

We now proceed to consider the more general case of the conduction of heat in an anisotropic body such as a crystal of arbitrary constitution. The preceding argument remains unchanged except for eq. (3) which must now be replaced by a tensor relation between the components of the temperature

gradient and of the heat flux (we now denote the coordinates by x_1, x_2 , and x_3).

$$(12) \quad \begin{cases} W_1 = -\kappa_{11} \frac{\partial T}{\partial x_1} - \kappa_{12} \frac{\partial T}{\partial x_2} - \kappa_{13} \frac{\partial T}{\partial x_3} \\ W_2 = -\kappa_{21} \frac{\partial T}{\partial x_1} - \kappa_{22} \frac{\partial T}{\partial x_2} - \kappa_{23} \frac{\partial T}{\partial x_3} \\ W_3 = -\kappa_{31} \frac{\partial T}{\partial x_1} - \kappa_{32} \frac{\partial T}{\partial x_2} - \kappa_{33} \frac{\partial T}{\partial x_3} \end{cases}$$

It expresses the fact that in a crystal the temperature gradient and heat flux are not, generally speaking, parallel (more precisely anti-parallel).

If, as before, we now compare the assumption in eq. (12) (such an assumption, as well as similar ones made in connection with the other irreversible processes, is known as a phenomenological hypothesis) with the expression for local entropy generation in eq. (8), rewriting the latter as

$$(13) \quad \theta = -\frac{1}{T^2} \left(W_1 \frac{\partial T}{\partial x_1} + W_2 \frac{\partial T}{\partial x_2} + W_3 \frac{\partial T}{\partial x_3} \right),$$

we notice that the phenomenological hypothesis, eq. (12), expresses the first factors W_1, W_2, W_3 in eq. (13) in terms of the second factors, $\partial T/\partial x_1, \partial T/\partial x_2$ and $\partial T/\partial x_3$, as linear homogeneous functions. In any case the phenomenological hypothesis in eq. (12) is not arbitrary. It must, first, satisfy the condition that $\theta \geq 0$ for any temperature gradient, i. e.

$$(14) \quad \sum_i \sum_k \kappa_{ik} \frac{\partial T}{\partial x_i} \frac{\partial T}{\partial x_k} \geq 0$$

which shows that the tensor κ_{ik} turns out to be non-negative definite. Secondly, we must have

$$(15) \quad \kappa_{ik} = \kappa_{ki} \quad (i, k = 1, 2, 3)$$

which shows that the tensor is symmetrical.

The last relation is confirmed by experiment¹ and follows from kinetic theories of heat conduction. It is, finally, a particular case of quite general symmetrical relations which were postulated by Onsager². We shall revert to a more general formulation of these reciprocal relations later.

¹M. Voigt, Nachr. Ges. Wiss. Göttingen, Math. Phys. Class, p. 87 (1903); Ch. Soret, Arch. de Genève, Vol. 29, p. 355 (1893), Vol. 32, p. 611 (1894).

²L. Onsager, Phys. Rev. Vol. 37, p. 405, Vol. 38, p. 2265 (1931).

In accordance with the language of the general thermodynamic theory of irreversible processes we consider that in the preceding example there are three elementary irreversible processes which are superimposed on one another. Each of them corresponds to an elementary irreversible process, associated with one coordinate direction. Furthermore, in accordance with the hypothesis in eq. (12) and from eq. (13) it is seen that when several elementary irreversible processes interact with each other the quantity of entropy generated locally can be split into three terms each of which is due to one irreversible process only. On the other hand the phenomenological hypothesis shows that such elementary processes may be coupled, meaning that one temperature gradient, e. g. that in the direction x_1 , can give rise to a heat flux in another direction, such as x_2 and x_3 .

This feature is quite general and may appear during an interaction of completely dissimilar irreversible processes such as heat conduction and diffusion, the conduction of heat and electricity, etc. In these cases the coupling of irreversible processes through the respective phenomenological hypotheses leads to thermal diffusion (known as the Soret effect when condensed phases are concerned) and the Dufour effect or thermal effusion (temperature gradient evoked during diffusion), or to thermoelectric phenomena.

C. THERMOELECTRIC PHENOMENA

In the case of thermoelectric effects we are dealing, on the one hand, with the flux of energy and electricity and, on the other, with temperature gradients and electric field intensities as their causes. We consider a metal which carries an electric current and throughout which there exists a temperature gradient. We can write down the principle of the conservation of energy from which, in turn, we can deduce the entropy equation which corresponds to eq. (6). We assume here that the specific internal energy, u , and the specific entropy, s , are independent of the current density I . This hypothesis is of the same nature as the implicit assumption in Section A that the internal energy depends on temperature but not on the heat flux or temperature gradient. The electron theory of metals furnishes further justification for such an assumption (see also Sec. 45).

We shall now postulate the existence of only one kind of mobile carriers of electricity. These can be regarded as being endowed with a negative charge $-e$ ($e > 0$) without introducing any essential limitation into our argument. The same idea forms also the basis of the electron theory of metals.

In formulating the energy equation it is necessary to take into account that the metal receives a quantity $(I \cdot E)$ of electrical energy per unit time

and volume and that due to charging with $-\text{div } \mathbf{l}$ per unit time there is an increase in potential energy by an amount $-\Phi \text{ div } \mathbf{l}$, where Φ denotes the electrical potential and $\mathbf{E} = -\text{grad } \Phi$ is the electric field strength. It is hereby implied that the electric currents vary at a slow rate. The principle of the conservation of energy, in analogy with eq. (1), now becomes

$$(16) \quad \rho \frac{\partial u}{\partial t} = -\text{div } \mathbf{W} + (\mathbf{l} \cdot \mathbf{E}) - \Phi \text{ div } \mathbf{l}.$$

The differential of specific entropy from eqs. (18.2) and (18.3) assumes the form

$$(17) \quad T ds = du - (\mu - F\Phi) dn$$

on the assumption that changes in volume are negligible and taking into account that $z = -1$. Here Ln is the number of carriers of electricity per gram ($L = \text{Avogadro's number}$), μ denotes the chemical and $\mu - F\Phi$ the electrochemical potential of the carriers. Since $F = Le$ and since $-\rho L n e$ represents the charge per unit volume, we have

$$\rho F \frac{\partial n}{\partial t} = -\rho \frac{\partial(-L n e)}{\partial t} = \text{div } \mathbf{l}.$$

Hence

$$(17 a) \quad T \rho \frac{\partial s}{\partial t} = \rho \frac{\partial u}{\partial t} + \left(\Phi - \frac{\zeta}{e} \right) \text{div } \mathbf{l},$$

where we have put $\mu/F = \zeta/e$ i. e. $\zeta = \mu/L$. In the electron theory of metals the quantity ζ is referred to as the chemical potential per electron. Eliminating $\partial u/\partial t$ from eqs. (16) and (17 a) and rearranging slightly, we obtain

$$(17 b) \quad \begin{aligned} \rho \frac{\partial s}{\partial t} + \text{div } \frac{1}{T} \left(\mathbf{W} + \frac{\zeta}{e} \mathbf{l} \right) = \\ = \frac{1}{T} \left(\mathbf{W} \cdot -\frac{1}{T} \text{grad } T \right) + \frac{1}{T} \left(\mathbf{l} \cdot \mathbf{E} + T \text{grad } \frac{\zeta}{e T} \right). \end{aligned}$$

This equation is the counterpart of eq. (6). Here again we shall regard the quantity $\frac{1}{T} \left(\mathbf{W} + \frac{\zeta}{e} \mathbf{l} \right)$ as the flux of entropy, whereas the right-hand side of the equation represents the quantity of entropy generated locally, θ . The results derived in Sections A and B show how to obtain the assumptions for the fluxes of heat and current of electricity from the local entropy increase.

First we find from eq. (17 b) that θ is a linear function of the fluxes \mathbf{W} and \mathbf{l} and express them again as linear functions of their coefficients $-1/T \text{ grad } T$ and $\mathbf{E} + T \text{ grad } (\zeta/e T)$, namely

$$(18) \quad \begin{cases} \mathbf{W} = -\frac{\alpha}{T} \text{ grad } T + \beta \left(\mathbf{E} + T \text{ grad } \frac{\zeta}{e T} \right) \\ \mathbf{l} = -\frac{\gamma}{T} \text{ grad } T + \delta \left(\mathbf{E} + T \text{ grad } \frac{\zeta}{e T} \right). \end{cases}$$

Solving for \mathbf{E} and \mathbf{W} , we have, with the usual notation

$$(18 \text{ a}) \quad \mathbf{W} = -\kappa \text{ grad } T - \left(\Pi + \frac{\zeta}{e} \right) \mathbf{l}$$

$$(18 \text{ b}) \quad \mathbf{E} = \frac{1}{\sigma} \mathbf{l} - \varepsilon \text{ grad } T - \text{grad } \frac{\zeta}{e}$$

with the coefficients κ , Π , ε , $1/\sigma$ whose significance will be further investigated later. Their relation to β and γ is noted for further reference:

$$(18 \text{ c}) \quad \beta = -\sigma \left(\Pi + \frac{\zeta}{e} \right); \quad \gamma = -\sigma \left(\varepsilon T + \frac{\zeta}{e} \right).$$

Equation (18 b) connects the jump in potential across a boundary between two metals with the jump in ζ . Namely, if eq. (18 b) is integrated along a very small path crossing the boundary between metals I and II, we obtain

$$(18 \text{ d}) \quad \Phi_{\text{II}} - \Phi_{\text{I}} = \frac{1}{e} (\zeta_{\text{II}} - \zeta_{\text{I}})$$

because the contributions of the first two terms on the right-hand side of the equation can be made as small as we please. The difference $\Phi_{\text{II}} - \Phi_{\text{I}}$ is the contact potential between the two metals. At equilibrium $\mathbf{l} = 0$, $\mathbf{W} = 0$, $\text{grad } T = 0$ and hence $\mathbf{E} = -\text{grad } \zeta/e$. It is found from eq. (18 b) that at equilibrium electrical field strengths are only present in regions where ζ varies at constant temperature. In other words they exist only in regions where the material is non-homogeneous, i. e. in particular across the boundary between two different homogeneous materials.

If no electric current is present eq. (18 a) reduces to Fourier's law of heat conduction and κ is the thermal conductivity as seen upon comparing with eq. (3). Equation (18 b) shows that in such a case there exists everywhere a field of strength $\mathbf{E} = -\varepsilon \text{ grad } T - \text{grad } \zeta/e$. The coefficient ε is known as the absolute *thermal force* (see also eq. (25) in Sec. 45, which contains an explicit expression for ε).

If the temperature is uniform everywhere and if the material is homogeneous so that $\text{grad } \zeta = 0$, eq. (18 b) will reduce to Ohm's law with σ denoting the electrical conductivity. In general, i. e. when the temperature is not constant throughout the space, eq. (18 b) can be rewritten as

$$(19) \quad \mathbf{l} = \sigma(\mathbf{E} + \mathbf{E}^e).$$

The quantity denoted by \mathbf{E}^e is the *impressed electric field strength*. In this connection eq. (18 a) shows that an energy flux may be present even when there are no temperature differences, provided that $\mathbf{l} \neq 0$. In other words, the transport of electricity is coupled with a transport of energy. This is contingent on the fact that the transport of electricity and that of energy have a common cause in the motion of electrons in the metal and the electron theory of metals confirms this assumption in all respects. In accordance with the present notation the energy transported per Coulomb is equal to $-(\Pi + \zeta/e)$.

We shall now proceed to consider thermoelectric phenomena and we shall begin by discussing the *Thomson effect*. This effect occurs in an electric conductor which carries a current and along which a temperature gradient is maintained and consists in the fact that so-called *Thomson heat* appears in addition to Joule heat, as is seen by substituting (18 b) into (17). The amount of Thomson heat per unit volume and time is equal to $\mu (\mathbf{l} \cdot \text{grad } T)$ where μ (not to be confused with the μ from eq. (17)) is the *Thomson coefficient*. This additional quantity of heat can be positive or negative depending on the relative direction of \mathbf{l} and $\text{grad } T$. It is customary to refer to this effect as being reversible because it changes sign with a change in the direction of \mathbf{l} or $\text{grad } T$. This term is, however, a misnomer, because the Thomson effect constitutes only one aspect of the whole process, and, moreover, it is intimately interlocked with heat conduction and with the generation of Joule heat, both of them typically irreversible processes.

The existence of Thomson heat is implied in the preceding fundamental equations and this is readily seen when considering the accumulation of heat per unit volume and time, or $\rho \partial u / \partial t = (\mathbf{l} \cdot \mathbf{E}) - \text{div } \mathbf{W} - \Phi \text{div } \mathbf{l}$ with the substitution of \mathbf{W} and \mathbf{E} from eqs. (18 a) and (18 b) respectively. Since $\text{div } \mathbf{l} = 0$ (which is always true for direct current, and approximately so if the current varies but slowly) we obtain

$$\rho \frac{\partial u}{\partial t} = \text{div} (\kappa \text{grad } T) + \left(\frac{\partial \Pi}{\partial T} - \epsilon \right) (\mathbf{l} \cdot \text{grad } T) + \frac{1}{\sigma} \mathbf{l}^2.$$

The first term on the right-hand side gives the accumulation of heat for heat conduction alone, the last term being the Joule heat. The second term has a

form to be expected of Thomson heat. From the definition of the Thomson coefficient *the first Thomson relation* is deduced, i. e.

$$(20) \quad \mu = \frac{\partial \Pi}{\partial T} - \varepsilon.$$

It was first obtained by Thomson; speaking more precisely Thomson inferred the existence of an additional heat term from the fact that, generally speaking, $\partial \Pi / \partial T - \varepsilon \neq 0$ and that, otherwise the energies would not balance.

The coefficient Π is known as the *Peltier coefficient*. It is connected with the *Peltier effect*, i. e. with the positive or negative flow of heat at a boundary between two different metals. Considering the arrangement in Fig. 22 and assuming that the temperature remains constant throughout we can calculate that the flow of energy from left to right in metal I is given by: $-[\Pi_I + (1/e)\zeta_I]l$, whereas that in metal II is given by: $-[\Pi_{II} + (1/e)\zeta_{II}]l$, assuming a unit cross-sectional area of both conductors at the boundary.

Hence we obtain an accumulation of energy of a magnitude

$$(21) \quad \left[\Pi_{II} - \Pi_I + \frac{1}{e} (\zeta_{II} - \zeta_I) \right] l.$$

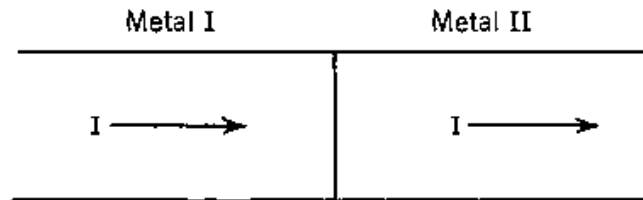


Fig. 22.
Notation for Peltier effect.

The quantity $1/e(\zeta_{II} - \zeta_I)l$ is used up in lifting the carriers of electricity through a potential difference from Φ_I to Φ_{II} across the boundary, so that only the quantity $(\Pi_{II} - \Pi_I)l$ is left over.

We can, finally, calculate the emf of a circuit which is composed of two different metals in which the metals are not kept at a constant temperature. Thus

$$\oint (\mathbf{E}^e \cdot d\mathbf{r}) = \oint \varepsilon (\text{grad } T \cdot d\mathbf{r}) + \oint \left(\text{grad } \frac{\zeta}{e} \cdot d\mathbf{r} \right) = \oint \varepsilon dT.$$

It is convenient to express the integral with the aid of two part integrals, each of which is taken over one metal. If T_1 and T_2 denote the temperatures of the junctions and ε_I and ε_{II} denote the absolute values of the emf's of the two metals respectively, we obtain

$$(22) \quad \oint (\mathbf{E}^e \cdot d\mathbf{r}) = \int_{T_1}^{T_2} \varepsilon_{II} dT + \int_{T_2}^{T_1} \varepsilon_I dT = \int_{T_1}^{T_2} (\varepsilon_{II} - \varepsilon_I) dT.$$

This shows that the emf of a closed circuit depends only on the temperatures of the junctions and that it vanishes when both metals are identical (as in such a case $\varepsilon_I = \varepsilon_{II}$) and, finally, that for sufficiently small temperature differences it is approximately equal to $(\varepsilon_{II} - \varepsilon_I) (T_2 - T_1)$. It is necessary to remark here that from measurements of the Peltier heat as well as from measurements of the emf's it is possible to obtain only differences of Peltier coefficients or emf's for *pairs* of metals. Equations (20) and (23) can, therefore, be verified only with respect to two metals. On the other hand, the electron theory of metals is capable of defining Π and ε for a single metal.

Equation (20) shows that the three thermoelectric effects are coupled with each other. Onsager's reciprocal relations lead to another important equation connecting these quantities. According to it the matrix of coefficients in eq. (18) must be symmetrical which leads to *Thomson's second relation* $\beta = \gamma$ or, from eq. (18 c)

$$(23) \quad \Pi = T \varepsilon.$$

Thomson obtained eq. (23) by a different line of reasoning. He separated the thermoelectric effects from heat conduction and Joule heat, which are coupled with them in reality, and considered that the thermocouple constituted a Carnot engine when operated in steady-state. In such a case only the Peltier heat at the hot junctions is considered as the heat absorbed by the system. Hence the efficiency of a cycle with a temperature difference dT between the source and the sink and with a quantity of work equal to the electrical energy

$\oint (\mathbf{E} \cdot d\mathbf{r})$ becomes

$$\frac{\Delta T}{T} = \frac{(\varepsilon_{II} - \varepsilon_I) \Delta T}{\Pi_{II} - \Pi_I}.$$

which is a consequence of eq. (23).

There is little justification for such a separation of the so-called reversible and irreversible effects, in spite of the fact that it leads to a result which can be verified experimentally.¹ It is, therefore, very gratifying that the electron

¹ Thomson himself explicitly stated that such a separation involves a new hypothesis and that an experimental proof is required, because "Not only are the conditions prescribed in the second Law of the Dynamical Theory not completely fulfilled, but the part of the agency which does fulfil them is in all known circumstances of thermo-electric currents excessively small in proportion to agency inseparably accompanying it and essentially violating those conditions" (Trans. Roy. Soc. of Edinburgh, Vol. XXI, p. 128, 1 May 1854). In his careful analysis, Boltzmann showed that this hypothesis is untenable, thus fully confirming Thomson's misgivings (Sitzungsber. d. Akad. d. Wiss., Vienna, Math. Naturw. Klasse, II Div. 96, 1258 (1888)).

theory of metals was able to confirm Thomson's second relation without the need for such additional 'ad hoc' hypotheses.¹ It provided, moreover, one of the first examples for the direct proof of an Onsager reciprocal relation and articulated the basic principle which underlies it, namely the principle of microscopic reversibility (i. e. the property of invariance of fundamental laws with respect to a change in the direction of time). This principle was later so imaginatively generalized by Onsager.

In this connection it is necessary to point to certain generalizations which are important when considering thermoelectric phenomena in anisotropic bodies and when an external magnetic field \mathbf{B} is superimposed. It is particularly noteworthy that in the case of magnetic fields Onsager's principle must be modified. In the simplest case for the tensor of thermal conductivity we obtain

$$\kappa_{ik}(\mathbf{B}) = \kappa_{ki}(-\mathbf{B})$$

instead of eq. (15).

This change of sign is connected with the fact that the principle of microscopic reversibility applies only when the magnetic field is reversed simultaneously. Considering, for example, the equation of motion of an electron in a magnetic field, i. e. $m \dot{\mathbf{v}} = -e(\mathbf{v} \times \mathbf{B})$ it is noticed that it remains unaltered by the substitution $t, \mathbf{B} \rightarrow -t, -\mathbf{B}$.

The process of *thermal diffusion*, and that of *thermal effusion* which was discovered by Dufour² and very convincingly demonstrated by Clausius and Waldmann³ can be analysed by similar methods. An apparent complication occurs due to the fact that changes in volume and flow phenomena must be taken into account. The application of Onsager's principle leads to a relation connecting the two effects and that relation is confirmed experimentally. To-day use is made of this relation to obtain very accurate values of coefficients of thermal diffusion from measurements on thermal effusion.⁴

D. INTERNAL TRANSFORMATIONS

In the preceding sections we have considered transport phenomena (the transport of energy and electricity) and now we propose to discuss irreversible processes in non-homogeneous matter, i. e. so-called internal transformations or relaxation phenomena which are not accompanied by transport phenomena.

¹A. Sommerfeld, *Zeitschr. f. Physik* Vol. 47, pp. 1 and 43 (1928).

²Dufour, *Ann. Physik*, Vol. 28, p. 490 (1873).

³Ki. Clausius and L. Waldmann, *Naturw.* Vol. 30, p. 711 (1942).

⁴L. Waldmann, *Z. f. Physik*, Vol. 124, (1944) p. 30.

(If the internal transformations consist of chemical reactions in the usual sense they are also termed homogeneous reactions.) In this case the volume must be kept constant and the introduction of heat must be excluded, so that we put $dV = 0$ and $dU = 0$. We shall consider 1 gram of matter consisting of three components A_0 , A_1 and A_2 which may take part in two reactions $A_0 \rightleftharpoons A_1$ and $A_0 \rightleftharpoons A_2$. All other possible cases involving an arbitrary number of components and arbitrary types of chemical reactions can be treated in the same way so that we may restrict ourselves to this simple scheme. With $dV = 0$ and $dU = 0$ eq. (2) of Sec. 14 becomes

$$T ds = -\mu_0 dn_0 - \mu_1 dn_1 - \mu_2 dn_2.$$

The quantities μ_i represent the chemical potentials of the single components and they will be referred here to 1 gram instead of to 1 mol of substance, whereas the quantities n_0 , n_1 and n_2 denote the quantity of each component respectively, all per 1 gram of mixture. Thus $n_0 + n_1 + n_2 = 1$ and only n_1 and n_2 are independent. Hence

$$(24) \quad T ds = (\mu_0 - \mu_1) dn_1 + (\mu_0 - \mu_2) dn_2.$$

The rate of change of entropy with time or, since transport phenomena are excluded, the rate at which entropy is created becomes

$$(25) \quad \rho \frac{ds}{dt} = \frac{\rho}{T} (\mu_0 - \mu_1) \frac{dn_1}{dt} + \frac{\rho}{T} (\mu_0 - \mu_2) \frac{dn_2}{dt}.$$

This expression is also a linear function of dn_1/dt and dn_2/dt and their coefficients describe the deviation from equilibrium. In equilibrium $ds \leq 0$, as shown by eq. (8.1), for every virtual change of state dn_1 , dn_2 which implies $\mu_0 - \mu_1 = \mu_0 - \mu_2 = 0$.

For small deviations from thermodynamic equilibrium we may, again, express dn_1/dt and dn_2/dt in eq. (25) as linear functions of their coefficients:

$$(26) \quad \begin{cases} \rho \frac{dn_1}{dt} = a_{11} (\mu_0 - \mu_1) + a_{12} (\mu_0 - \mu_2) \\ \rho \frac{dn_2}{dt} = a_{21} (\mu_0 - \mu_1) + a_{22} (\mu_0 - \mu_2) \end{cases}$$

where $a_{ik} = a_{ik}(T, v)$ and in accordance with Onsager's principle we have $a_{12} = a_{21}$. The connection with the principle of detailed equilibrium is discussed in the solution to Problem II 7.

To conclude this section we shall add some general remarks. It was necessary to impose the condition $dU = 0$ in order to exclude transport

phenomena. A change in internal energy can take place only through the performance of work, i. e. through a change in volume or through an exchange of heat. On the other hand the assumption that only the two reactions $A_0 \rightleftharpoons A_1$ and $A_0 \rightleftharpoons A_2$ took place was superfluous; the argument will not change if the reaction $A_1 \rightleftharpoons A_2$ is also admitted. This circumstance is due to the fact that only *independent* reactions are important and their number determines the number of independent n_i 's. In the case under consideration the reaction $A_1 \rightleftharpoons A_2$ would not be independent being the difference between the reactions $A_0 \rightleftharpoons A_1$ and $A_0 \rightleftharpoons A_2$. Evidently any two reactions can be taken as the independent ones. Furthermore, the reactions as written down need not faithfully represent their molecular mechanism i. e. they are so-called elementary reactions. It is even permissible to choose arbitrary linear combinations of such elementary reactions with arbitrary numerical coefficients without in any way affecting the argument. This is a consequence of the fact that the thermodynamic theory of irreversible processes concerns itself only with the phenomenological aspects of the processes and does not consider their molecular mechanism.

At this point it is useful to recall the statement in eq. (8.9) which asserted that for a *reversible* process $\sum X_i dx_i = 0$. Equation (25) shows that the rate of entropy creation, θ , per time dt for a process involving the changes dx_i is given by $\rho \sum_i X_i dx_i$ and that for an *irreversible* process we always have $\sum_i X_i dx_i > 0$.

E. GENERAL RELATIONS

The discussion of more general irreversible processes involving the coupling of internal changes with transport phenomena exceeds the scope of these lectures. We shall confine ourselves to the remark that in each particular case the change in entropy of an isolated unit of mass considered along its path in the field of motion can be calculated with the aid of the conservation laws and Gibb's equation (14.2) and that it can always be written in the form¹

$$(27) \quad \rho \frac{ds}{dt} + \text{div } \mathbf{S} = \theta$$

where \mathbf{S} denotes the *entropy flux*. The entropy flux is a linear function of the energy flux and of the diffusion fluxes and electrical current density if

¹This fact was first clearly formulated by G. Jaumann; his name has already been mentioned in another connection in Vol. III.

they exist in the system. It does not contain the convective entropy flux, i. e. the entropy transported by matter in motion because eq. (27) is valid for an observer who travels with the element of matter. The quantity θ is the *entropy created locally* and can always be written as

$$(28) \quad \theta = \frac{1}{T} \sum_i K_i X_i.$$

The quantities X_i are generalized fluxes such as the energy flux (or its three components), the flux of momentum due to internal friction, i. e. its six components p_{ik} considered in Vol. II eq. (10.18), the quantities dn_i/dt considered in the preceding section etc. The coefficients K_i will be referred to as thermodynamic forces. They are, e. g. $-1/T \text{ grad } T$, $T \text{ grad } \mu_i/T$, $\partial v_i/\partial x_k$, the chemical potentials themselves or their differences (as in eq. (25)) etc. The energy dissipation term $T\theta$ is thus always a sum of products of fluxes and forces.

When the deviations from thermodynamic equilibrium are small the fluxes X_i are linear functions of the forces K_k , thus

$$(29) \quad X_i = \sum_{k=1}^n a_{ik} K_k.$$

From this point onwards we are in a position to formulate *Onsager's reciprocal relations*. However, contrary to what might be expected from previous sections their general formulation cannot be taken to be

$$(30) \quad a_{ik} = a_{ki}.$$

Casimir¹ was the first to point out that these relations are true only for cases when the forces K_i and K_k which are associated with the pair of subscripts i, k are both even or both odd functions of the flow velocities or of the molecular velocities. If, however, one of the forces is even and the other odd, we have to write with Casimir, that

$$(31) \quad a_{ik} = -a_{ki}.$$

We cannot give here the general proof of eqs. (30) and (31). Following Onsager and Casimir it is possible to obtain these relations from the *principle*

¹H. B. G. Casimir, Rev. Mod. Physics, Vol. 17, p. 343 (1945).

of *microscopic reversibility*, or, in special cases, e. g. with the aid of the kinetic theory of gases as applied to gaseous flows. In the latter case it would be necessary at least to consider gaseous mixtures if non-trivial Onsager-type relations are to be obtained. For a homogeneous gas the energy dissipation, as will also be shown in Chap. V, is given by

$$(32) \quad T\theta = \frac{1}{2} \sum_i \sum_k p_{ik} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) + \left(\mathbf{W} \cdot -\frac{1}{T} \text{grad } T \right)$$

(for an explanation of the second term see Section B, and for the explanation of the first term see Vol. II, Sec. 10.18). The phenomenological relations in eq. (29) would represent the quantities W_x , W_y , and W_z as linear functions of the terms $-(1/T)(\partial T/\partial x_i)$ and of the six terms $(\partial v_i/\partial x_k + \partial v_k/\partial x_i)$. The three phenomenological relations must remain covariant with respect to a rotation of the system of coordinates, since the gas is isotropic at all points, or, in other words, its form must be the same, with the same coefficients, when expressed with the aid of the components in the new system of coordinates.

It follows that the coefficients of the terms $(\partial v_i/\partial x_k + \partial v_k/\partial x_i)$ must vanish (a vector cannot depend linearly on a tensor if the continuum is isotropic). Further, it also follows from the condition of isotropy that the heat conduction tensor must reduce itself to a multiple of the unit tensor δ_{ik} . A similar argument involving the phenomenological assumptions for the p_{ik} leads to the conclusion that there can be no coupling with the temperature gradient. Thus almost all mixed phenomenological coefficients vanish, only some mixed coefficients in the relation between p_{ik} and $(\partial v_i/\partial x_k + \partial v_k/\partial x_i)$ remain. Thus, as already shown in Vol. II, Sec. 10.21, and as we shall show again in Chap. V of the present volume, we can write

$$(33) \quad p_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) + \left(\zeta - \frac{2}{3} \eta \right) \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \delta_{ik}.$$

In this equation η denotes the ordinary viscosity and ζ denotes the volume, or bulk viscosity. If we consider the coefficient of $(\partial v_2/\partial x_2)$ in p_{11} and the coefficient of $(\partial v_1/\partial x_1)$ in p_{22} we find that they must be equal. In this manner it can be seen that all reciprocal relations follow from considerations of symmetry in the present example. However, insofar as mixtures of two gases are concerned, *one* non-trivial reciprocal relation can be deduced. It expresses the relation between thermal diffusion and thermal effusion.

F. LIMITATIONS OF THE THERMODYNAMIC THEORY OF IRREVERSIBLE PROCESSES

All previous considerations were restricted to small deviations from thermodynamic equilibrium and it is now necessary to specify more precisely the degree of deviation from thermodynamic equilibrium which may be regarded as small. It is natural to postulate that the thermodynamic concepts of temperature as well as of thermodynamic functions still retain their meaning. However, a rigorous specification of the limits of applicability becomes possible only when a more general theory can be formulated. Such a theory would then contain the theory of irreversible processes as a special limiting case. The kinetic theory of gases satisfies these requirements for gases. It does, in fact, as we shall see in Chap. V, confirm the conservation laws of fluid dynamics, it leads to the expression for the dissipation function in eq. (32) and also yields the phenomenological hypotheses concerning the flow of heat and the tensor of viscous stresses. If it were desired to establish the range of validity of these equations it would be necessary to carry the expansions in Chap. V further by adding the next term in order of magnitude. This was done e. g. by Enskog.¹ This method leads to clear statements about the conditions under which the next term in order of magnitude can be neglected. In this manner it is found that the temperature variations along a mean free path must be small compared with the absolute temperature and that changes in velocity must be small compared with the velocity of sound, or compared with the mean molecular velocity of random thermal motion. Considering that under normal conditions the mean free path is of the order of 10^{-4} cm it must be concluded that these limitations still leave an extremely wide margin which is seldom transgressed, excepting shock wave phenomena. It is true that in the case of numerous other irreversible phenomena it is not now possible to indicate quantitatively and in a simple manner the field of validity of the respective formulations but the results with gases allow us to expect with confidence that in many other cases there exists a field of validity which is sufficient for many purposes.

¹D. Enskog, *Zeitschr. f. Physik*, Vol. 54, p. 498 (1929).

CHAPTER III

THE ELEMENTARY KINETIC THEORY OF GASES

The beginnings of the kinetic theory of gases can be traced to Daniel Bernoulli. A derivation of the expression for the pressure of a gas from the change in momentum of the molecules impinging on its walls can be found in his book "Hydrodynamica." Strassbourg, 1738 (*cf.* Vol. II, Sec. 11). The further development of this theory was resumed in the middle of the 19th century: Krönig 1856, Clausius 1857, Maxwell 1860. Ludwig Boltzmann's papers in which Maxwell's law of velocity distribution was given its most general form stand at the peak of this development.

22. The equation of state of a perfect gas

Let us now center our attention on the collisions which a solid, flat (or continuously curved), smooth (and hence frictionless) wall experiences from the impact of the gas molecules. We shall notice that the graph of these collisions taken for any element of surface $d\sigma$ is represented by a curve $f(t)$ which possesses a very large number of sharp dents corresponding to the enormous number of collisions. The pressure (force per unit area) on the element is defined as the smoothed time average of this curve. The contribution of a single collision is equal to the change in the momentum of one molecule resulting from the impact and from the subsequent reflection. When impact occurs at a velocity c and in a direction forming the angle θ with the normal to the wall, the change in momentum is equal to

$$(1) \qquad 2 m c \times \cos \theta.$$

The factor 2 stems from the recoil experienced by the wall during the appearance of the reflected impulse (angle of reflection = angle of incidence, the magnitude of velocity $|\mathbf{v}| = c$ remains preserved owing to the absence of friction). Taking the three Cartesian components ξ, η, ζ of \mathbf{v} and placing the ξ -axis normal to the element with its positive direction outwards, we can replace (1) by

$$(1 a) \qquad 2 m \xi, \qquad \xi > 0.$$

In order to account for all collisions we construct an oblique cylinder over the element of area $d\sigma$, Fig. 23, inclining its axis at an angle θ with respect to the normal and making its length equal to c . In the interior of this cylinder we now mark all molecules whose velocities point towards the wall and have

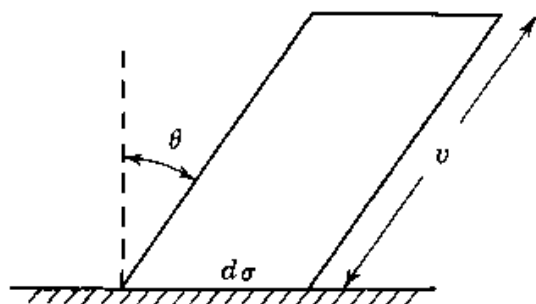


Fig. 23.

Illustrating the calculation of kinetic pressure.

values which lie between c and $c + dc$, and directions confined between θ and $\theta + d\theta$ and ϕ and $\phi + d\phi$. The angles θ and ϕ are measured with respect to the normal to the elementary area and around it. All velocity vectors drawn from each such molecule intersect our elementary area; during a unit of time all these, and only these molecules will impinge on the element $d\sigma$ under consideration.

The volume of the cylinder is equal to cross-sectional area \times height $= d\sigma \times \xi$. Denoting the number of molecules per unit volume by n we find that the number contained within the cylinder is

$$(2) \quad n \times \xi d\sigma.$$

Of these only such molecules count as will meet the wall and only such whose velocity lies in a given region $d\omega$ of the "velocity space." This region has, so far, been described by the quantities $dc, d\theta, d\phi$. The range of velocities has thus been described in a polar system of coordinates which is natural from the point of view of the element $d\omega$ in the physical space. It will, however, be more convenient from the point of view of the succeeding argument to express $d\omega$ in terms of the rectangular coordinates ξ, η, ζ and to put $d\omega = d\xi d\eta d\zeta$.

In any case, and independently of the choice of the system of coordinates, the number of colliding molecules is given by

$$(2 a) \quad dv = v \xi d\sigma d\omega.$$

The symbol v which must be carefully distinguished from n , denotes the density of molecules per unit volume and per unit velocity space $d\omega$. Evidently

$$n = \int v d\omega.$$

The corresponding change in momentum is found from (2 a) by multiplication by (1), or

$$(2 b) \quad 2 v m \xi^2 d\sigma d\omega.$$

The contribution of this group of collisions to the pressure is thus (division by $d\sigma$)

$$(2\ c) \quad dp = 2\ v\ m\ \xi^2\ d\omega, \quad \xi > 0.$$

The total pressure follows when we construct our cylinder for all possible velocity ranges:

$$(3) \quad p = n\ m\ \bar{\xi^2}, \quad \bar{\xi^2} = \frac{2}{n} \int v\ \xi^2\ d\omega, \quad \xi > 0;$$

the value $\bar{\xi^2}$ is obtained by averaging over half the velocity space $\xi > 0$. Since, however, to each molecule which travels towards the wall ($\xi > 0$) there corresponds one which travels away from it ($\xi < 0$) with equal probability we can extend the integration indicated in (3) over the whole of the velocity space and write

$$(3\ a) \quad p = n\ m\ \bar{\xi^2}, \quad \bar{\xi^2} = \frac{1}{n} \int v\ \xi^2\ d\omega, \quad \text{with} \quad \xi \geq 0,$$

instead of (3). The symbol $\bar{\xi^2}$ denotes the mean value of ξ^2 at a given point in space. A more rational representation, and one independent of the choice of the system of coordinates, is obtained by taking into account the fact that all directions are equivalent (isotropy of the velocity space). We then have

$$(3\ b) \quad \bar{\xi^2} = \bar{\eta^2} = \bar{\zeta^2} = \frac{1}{3} \bar{c^2}$$

because $v^2 = \xi^2 + \eta^2 + \zeta^2 = c^2$. Hence (3) yields

$$(4) \quad p = \frac{n\ m}{3} \bar{c^2} = \frac{2}{3} n\ \bar{E}_tr, \quad \bar{E}_tr = \frac{1}{2} m\ \bar{c^2},$$

where \bar{E}_tr denotes the kinetic energy of *translation*; rotational energy or that associated with internal motions need not be considered in the calculation of pressure.

Equation (4) contains more than the kinetic explanation of *pressure*; it also contains a kinetic definition of *temperature*. In order to see this we put

$$(4\ a) \quad n = \frac{N}{V},$$

where N denotes the total number of molecules in a volume V . Thus eq. (4) leads to

$$(4\ b) \quad pV = \frac{2}{3} N\ \bar{E}_tr.$$

Applying eq. (4 b) to one mol, we have $V = V_{mol}$ and $N = L =$ number of molecules per mol = Loschmidt-Avogadro number. Hence eq. (4 b) transforms into

$$(4\ c) \quad p V_{mol} = \frac{2}{3} L \bar{E}_{tr}.$$

Comparing this result with the equation of state of a perfect gas in its forms (3.11) and (3.11 a) it is seen that its right-hand side must be equal to $R T$. In this way

$$(5) \quad \bar{E}_{tr} = \frac{3}{2} \frac{R}{L} T = \frac{3}{2} k T,$$

where k denotes the Boltzmann constant, already defined in Sec. 20, i. e. the gas constant reduced to a single molecule, R/L . In a three-dimensional space translation has three degrees of freedom. Consequently the contents of (5) (effecting simultaneously a change from subscript tr to subscript f) can be expressed as follows: *The mean kinetic energy per degree of freedom is given by*

$$(6) \quad E_f = \frac{1}{2} k T.$$

This statement contains our (provisional) *kinetic definition of temperature*.

It should be noted here that eq. (5) contains the theory of the specific heats of a monatomic gas. Since its energy is wholly due to translation we can find u and c_v per mol of such a gas from eq. (5). Thus

$$(6\ a) \quad u = L \bar{E}_{tr} = \frac{3}{2} R T, \quad c_v = \frac{du}{dT} = \frac{3}{2} R \approx 3 \text{ kcal/kmol}^1$$

As $c_p - c_v = R$ eq. (6 a) determines c_p as well, and we have

$$(6\ b) \quad \frac{c_p}{c_v} = 1 + \frac{2}{3} = 1.66.$$

We now reiterate what we have already said in Sec. 4 C: The kinetic theory of gases is capable of filling the general framework of thermodynamics with actual numerical values which agree with experiment. In Chap. IV we shall revert to the values concerning polyatomic gases as discussed in Sec. 4 C.

In the preceding argument we have encountered the mean values of the squares $\bar{\xi}^2$, \bar{c}^2 etc. The linear mean $\bar{\xi}$ is, evidently, equal to zero, because the velocity space is isotropic and the positive semi-axis of ξ is statistically

¹ 1 kmol = 1000 mol = molar weight in kg.

indistinguishable from its negative semi-axis, or from any other semi-axis. The mean value of c which, obviously, differs from zero will be calculated in Sec. 23 B. The quantity

$$(7) \quad (\overline{c^2})^{1/2}$$

is given directly from experimental results and constitutes a measure of the velocity of the gas. By way of example we shall now find its numerical value for the gases, H_2 , He, O_2 , N_2 . As long as the gases remain perfect the mean velocity depends on temperature only, and not on pressure.

Denoting the mass of a molecule by m we can find from (5) that

$$\frac{m}{2} \overline{c^2} = \frac{3}{2} k T$$

or, multiplying by the Loschmidt-Avogadro number, that

$$\frac{L m}{2} \overline{c^2} = \frac{3}{2} R T.$$

The product $L m$ is the mass μ of a mol and it is seen that the mean of the square of the molecular velocity $\overline{c^2} = 3 R T / \mu$ is found from macroscopic measurements alone. For example for hydrogen we have $\mu = 2 \text{ kg/kmol}$. With $T = 273 \text{ K}$ and the value (3.9) of R we obtain

$$\overline{c^2} = \frac{3}{2} \times 8.31 \times 273 \times 10^2 \text{ m}^2 \text{sec}^{-2}.$$

Hence

$$(8) \quad \sqrt{\overline{c^2}} = 1.85 \text{ km/sec.}$$

Correspondingly for helium (monatomic, atomic weight 4, i. e. double that of the molar weight 2 of H_2) we find

$$\sqrt{\overline{c^2}} = \frac{1.85}{\sqrt{2}} \text{ km/sec} = 1.30 \text{ km/sec.}$$

The molar weight of oxygen is 16 times larger than that of hydrogen. Consequently it is necessary to divide the velocity (8) by $\sqrt{16} = 4$. In the case of nitrogen it is necessary to divide by $\sqrt{14} = 3.74$.

Even at 0 C the velocities of molecules appear to be extraordinarily high; with increasing temperatures their values increase somewhat, namely in proportion to $(1 + t/273)^{1/2}$, where t is the Celsius temperature. We can gain

some insight into these relations if we consider that the velocity of sound cannot exceed the velocity of the molecules which propagate it. Hence it must be of the same order of magnitude, namely $c^2 = \frac{f+2}{f} \cdot \frac{RT}{\mu}$ [cf. Vol. II, eq. (13.17 a)]. The same is true about the velocity of compressed gases at the exit of an expansion nozzle.

The mean velocities of O_2 , N_2 , . . . in atmospheric air *differ from each other* as is the case with any gaseous mixture. In accordance with one definition of temperature the *mean energies of translation are equal* to each other at *thermal equilibrium*. Consequently, on the average, the total energy of translation is distributed equally among the different components of a gaseous mixture and in proportion to their masses. The above constitutes the simplest example of the more general *law of the equipartition of energy*.

In the preceding argument we have considered the pressure on the wall only. However, all relevant statements can be seen to apply to the pressure in the interior of the gas if it is imagined that a small membrane is introduced there to measure the pressure. Hence the pressure inside the gas and including that at the walls appears to be independent of the point at which it is measured (cf. however, Sec. 26). This is due to the fact that external forces, e. g. gravitation, have been neglected. This influence on the statistical considerations concerning gases will be considered in Sec. 23 C.

23. The Maxwellian velocity distribution

In the preceding Section we have made a distinction between the physical space and the velocity space. We considered, e. g. in eq. (4 a), that the *physical space* is uniformly filled with molecules, in apparent agreement with macroscopic observation, it being implied that no external forces are acting on the molecules. We now turn our attention to the *velocity space*. It will be noted that in the preceding argument we only needed to know the mean values $\overline{\xi^2}$, . . . , $\overline{c^2}$.

A. THE MAXWELLIAN DISTRIBUTION FOR A MONATOMIC GAS. PROOF OF 1860

If we select an arbitrary molecule of the gas we find that its velocity components have some arbitrary values, say ξ, η, ζ . We shall now consider the *probability* that the first component has a value which lies between ξ and $\xi + d\xi$ (a lamina confined between two planes at right angles to the ξ -axis), and denote it by

$$f(\xi) d\xi.$$

The same can be said about the components η and ζ , the probability function being the same as before because there is no preferred direction in the space. It is not, however, evident, that the probable value for η is unaffected by a value which has been found for, say, ξ . As is well known this is the case with lotteries: Having won the great prize in one year we still have exactly the same chance of winning it next year. At first Maxwell assumed that this independence of probabilities was true in the theory of gases, but proved it later explicitly (*cf.* Sec. C and Chap. V).

The velocity vector \mathbf{v} resulting from ξ, η, ζ belongs to the volume element $d\xi, d\eta, d\zeta$ of the velocity space (intersection of the three laminae ξ, η, ζ), and the probability that the tip of \mathbf{v} will be found in this volume element is equal to

$$(1) \quad f(\xi) f(\eta) f(\zeta) d\xi d\eta d\zeta,$$

because of the lottery assumption. Taking into account the isotropy of all velocity directions and hence the fact that they are all equally probable we can introduce a new unknown function F which depends only on the velocity. Thus

$$(1 a) \quad F(c) d\omega, \quad d\omega = d\xi d\eta d\zeta.$$

Comparing (1) with (1 a), we find that

$$(2) \quad F(\sqrt{\xi^2 + \eta^2 + \zeta^2}) = f(\xi) f(\eta) f(\zeta).$$

In order to determine the functions F and f from the preceding functional equation we can (purely formally) proceed as follows:

a) Logarithmic differentiation of (2) with respect to ξ :

$$(3) \quad \frac{\xi F'(c)}{c F(c)} = \frac{f'(\xi)}{f(\xi)}.$$

b) Introduction of the abbreviations

$$(3 a) \quad \Phi(c) = \frac{1}{c} \frac{F'(c)}{F(c)}, \quad \phi(\xi) = \frac{1}{\xi} \frac{f'(\xi)}{f(\xi)},$$

whence (3) becomes

$$(3 b) \quad \Phi(c) = \phi(\xi).$$

c) Differentiation with respect to η or ζ leads to

$$(3 c) \quad \Phi'(c) = 0, \quad \Phi(c) = \text{const.}$$

Assuming¹ that the const = -2γ we find that in view of (3 b) we also have

$$\phi(\xi) = -2\gamma,$$

so that according to (3 a)

$$(3\ d) \quad \frac{d \log f(\xi)}{d\xi} = -2\gamma \xi, \quad \log f(\xi) = \alpha - \gamma \xi^2.$$

Putting $e^{-\alpha} = a$, we have

$$(4) \quad f(\xi) = a e^{-\gamma \xi^2}.$$

It is remarkable as well as gratifying to find that the probability is given by the standard form of all statistical laws, namely by Gauss' error distribution function. The most probable value of the velocity component ξ is $\xi = 0$; the deviations from it distribute themselves symmetrically on either side and trace Gauss' probability curve (see Fig. 24a).

The constant a in (4) can be determined from the condition that it is absolutely certain that ξ has some value between $-\infty$ and $+\infty$. Hence

$$(5) \quad \int_{-\infty}^{+\infty} f(\xi) d\xi = 1.$$

Making use of Laplace's integral, we have

$$(5\ a) \quad a \int_{-\infty}^{+\infty} e^{-\gamma \xi^2} d\xi = a \left(\frac{\pi}{\gamma} \right)^{\frac{1}{2}} = 1 \quad \text{hence} \quad a = \left(\frac{\gamma}{\pi} \right)^{\frac{1}{2}}.$$

In order to determine the value of γ we shall calculate the mean kinetic energy of the degree of freedom of the component ξ :

$$\begin{aligned} \frac{m}{2} \overline{\xi^2} &= \frac{m}{2} \left(\frac{\gamma}{\pi} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \xi^2 e^{-\xi^2} d\xi = -\frac{m}{2} \left(\frac{\gamma}{\pi} \right)^{\frac{1}{2}} \frac{d}{d\gamma} \int_{-\infty}^{+\infty} e^{-\gamma \xi^2} d\xi = \\ &= -\frac{m}{2} \left(\frac{\gamma}{\pi} \right)^{\frac{1}{2}} \frac{d}{d\gamma} \left(\frac{\pi}{\gamma} \right)^{\frac{1}{2}} = \frac{m}{4\gamma}. \end{aligned}$$

¹The negative sign is required in order to satisfy, for example, the following eq. (5); the factor 2 is introduced for convenience.

Equating this with $kT/2$ in accordance with eq. (22.6), we find

$$(6) \quad \gamma = m/2kT.$$

Taking into account (5) and (6) we obtain the final form of eq. (4)

$$(7) \quad f(\xi) = \left(\frac{m}{2\pi kT} \right)^{1/2} \times e^{-E_1/kT}; \quad E_1 = \frac{m}{2} \xi^2.$$

The symbol E_1 denotes here the kinetic energy of the component ξ which corresponds to the point of the velocity space under consideration. Analogous equations apply to the remaining components η , ζ . The new distribution function $F(c)$ can be obtained at once with the aid of eq. (2). Thus

$$(8) \quad F(c) = \left(\frac{m}{2\pi kT} \right)^{3/2} \times e^{-E/kT}, \quad E = \frac{1}{2} m (\xi^2 + \eta^2 + \zeta^2).$$

The symbol E denotes now the kinetic energy of the translation which results from the components ξ , η , ζ .

B. NUMERICAL VALUES AND EXPERIMENTAL RESULTS

If instead of the velocity \mathbf{v} we are interested in the distribution of the absolute value of the velocity, denoted by c = celeritas, we consider the spherical shell described about the origin by the radii c and $c + dc$. Its probability will be denoted by

$$\phi(c) dc$$

and it is equal to the volume $4\pi c^2 dc$ within the spherical shell times the value (8) of $F(c)$. Thus we obtain

$$(9) \quad \phi(c) = 4\pi c^2 \left(\frac{m}{2\pi kT} \right)^{3/2} \times e^{-E/kT}, \quad E = \frac{1}{2} m c^2.$$

In this way we are led to a distribution function which is no longer Gaussian, as shown in Fig. 24, and which ceases to be symmetrical with respect to the most probable value. For large values of c it decreases to zero exponentially, in the same way as the previous curve, but for small values of c it tends to zero only quadratically; for $c < 0$, obviously, ϕ remains undefined.

The maximum of the curve is found from $\phi'(c) = 0$ and according to (9) we have

$$(10) \quad c_w = \left(\frac{2kT}{m} \right)^{1/2} = \text{most probable velocity.}$$

It is different from the root mean square, $\sqrt{\overline{c^2}}$ as well as from the linear mean

$$(10 a) \quad \bar{c} = \int_0^{\infty} c \phi(c) dc.$$

These three velocities satisfy the ratios:

$$(11) \quad c_w : \bar{c} : \sqrt{\overline{c^2}} = 1 : 1.13 : 1.22,$$

as shown in Problem III.2.

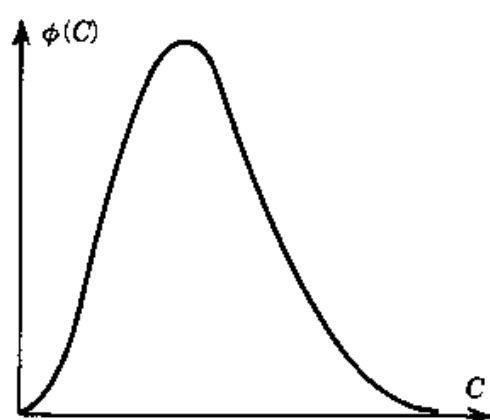


Fig. 24.
Maxwellian distribution.

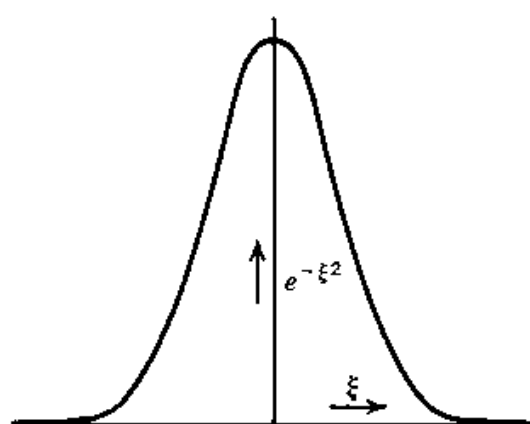


Fig. 24 a.
Gaussian distribution.

A qualitative confirmation of Maxwell's distribution is obtained by observing the *broadening of spectral lines* of a luminous gas with increasing temperature. This is due to the *Doppler effect*. If ν_0 denotes the natural frequency of the luminous particle (atom or molecule) and $\lambda_0 = c_L/\nu_0$ (we shall provisionally denote the velocity of light by c_L in order to distinguish it from the preceding c) then the observer who is looking, say, in the x -direction, perceives the wavelength λ_0 only of such particles whose velocity is almost zero in the ξ -direction. Generally speaking, he will observe a wavelength $\lambda_0 + \Delta\lambda$. According to Vol. IV, Sec. 11 (and neglecting relativistic corrections, we have $\Delta\lambda/\lambda_0 = \xi/c_L$, so that

$$(12) \quad \Delta\lambda = \xi/\nu_0.$$

All particles with equal ξ 's contribute towards the specific intensity, J , of the spectrogram in equal degrees at a positive or negative distance, $\Delta\lambda$, from its center. Their number is determined by the distribution function, $f(\xi)$, from eq. (4). We are justified in assuming that all particles are excited with equal

strength and that their intensities (not amplitudes) are superimposed owing to the lack of coherence in the emission. Consequently, the observed intensity becomes directly proportional to $f(\xi)$ where ξ is to be taken equal to $v_0 \Delta\lambda$ in accordance with (12). In this way we obtain from (4) that

$$(13) \quad J = J_0 \exp [-\gamma(v_0 \Delta\lambda)^2].$$

J_0 denotes the specific intensity in the center of the spectrogram; in accordance with eq. (6) γ is inversely proportional to T and determines the width of the spectral line. Its width at half intensity is given by $J = J_0/2$, or

$$\Delta\lambda_H = \frac{1}{v_0} \left(\frac{\log 2}{\gamma} \right)^{\frac{1}{2}} = \frac{1}{v_0} \left(\frac{kT}{m} 2 \log 2 \right)^{\frac{1}{2}}.$$

It is seen that the form of the spectral line given by (13) constitutes a direct image of Gauss' error distribution curve and hence also of Maxwell's distribution function.

The first measurements on the shape and half-width of *emission spectra* at varying temperature and atomic weight m are due to Michelson.¹ The shape of Fraunhofer's absorption spectra is of fundamental importance in astrophysics. In this connection, in addition to the Doppler effect, it is necessary to take into account the broadening due to pressure (damping due to collisions), whereas the natural breadth of the lines (cf. Vol. III, Sec. 36) becomes insignificant by comparison. Otto Stern² was the first to succeed in directly verifying the Maxwellian distribution when he made use of his method of atomic beams.

C. GENERAL REMARKS ON THE ENERGY DISTRIBUTION. THE BOLTZMANN FACTOR

In the preceding Section we have made use of the original, and somewhat primitive, first proof advanced by Maxwell. We shall justify the lottery assumption in Chap. IV when we shall make use of a much more general and essentially simpler "combinatorial method," based on classical mechanics. It will be shown that the more precise method leads to Maxwell's distribution function for the case of monatomic gases in the absence of external forces. The restriction to monatomic gases was stressed in the title of Sec. A: the restriction to gases with no external forces acting on them was implied when, e. g. in eq. (4 a), we assumed that the density was uniform throughout the physical space. This would not be true, for example, of a gas in a gravitational field.

¹Phil. Mag. **34**, 280 (1892).

²Zeitschr. f. Phys. **3**, 417 (1920).

Anticipating the results given in Chap. IV we now proceed to generalize our argument to include a polyatomic gas in an external field of force possessing a potential Φ . We shall proceed from eq. (8) but we must now replace the translational energy E by the total energy of a particle

$$(14) \quad \epsilon = E_{tr} + E_{rot} + \dots + \Phi.$$

In addition to the translational energy, which is the only form applicable to monatomic gases, we have to consider the rotational energy E_{rot} as well as the energy of internal motions of particles (vibrational energy etc.) indicated as \dots in the above equation, as well as the potential energy Φ in the field of forces. Furthermore, we shall extend the scope of an element $d\omega = d\xi d\eta d\zeta$ of the velocity space and we shall introduce the element $d\Omega$ of the "phase space" to be defined later. Thus instead of (8) we obtain

$$(15) \quad F d\Omega = A e^{-\epsilon/kT} d\Omega.$$

The constant A introduced here is determined by the normalizing condition

$$(15 a) \quad \int F d\Omega = 1$$

in the same way as the constant a was normalized in eq. (4). Concerning the meaning of $d\Omega$ we shall remark here only that in the case of a monatomic gas $d\Omega = d\tau \times m^3 d\omega$; $d\tau$ denotes the element of volume $dx dy dz$ in the physical space; the factor m^3 is due to the fact that in the phase space we shall introduce the momenta $m\xi, m\eta, m\zeta$ instead of the velocities ξ, η, ζ themselves. We now center our attention on the factor

$$(16) \quad e^{-\Phi/kT}$$

contained in (15). Since we assume that only Φ depends on the space coordinates x, y, z , the value of this factor indicates the probability that a gas molecule may be found in a cell at x, y, z . Thus it permits us to calculate the spatial density distribution ρ of the particles. If ρ_0 denotes the density at a reference level of the potential, we can put generally that

$$(17) \quad \frac{\rho}{\rho_0} = e^{-\Phi/kT}.$$

In the simplest case of a gravitational field, for which $\Phi = mgz$, eq. (17) transforms into the *barometric formula* cf. Vol. II, Sec. 7, eq. (15 a) or (15 c); J. Perrin's experiments on models of the atmosphere, see *ibid*, can be regarded as a macroscopic confirmation of the Boltzmann factor.

24. Brownian motion

The oscillatory motion performed by the smallest particles (specks of dust, colloidal particles), suspended in a liquid or a gas and observable with the aid of a microscope, was described in 1826 by the botanist Robert Brown. Its nature remained a puzzle for a long time. A final clarification was given in a paper by Einstein published in the memorable year 1905. Even in 1906 the critical Röntgen endeavored to counter the assertion that Brownian motion might after all be due to the energy of the lighting system of the microscope by a series of suitable control experiments.

Brownian motion comes under the general heading of *fluctuations*, i. e. departures from *thermodynamic equilibrium*. In line with the elementary character of this volume of our Lectures we shall confine ourselves here to a special derivation due originally to Langevin¹ because it leads to Einstein's result in a very simple way.

In Sec. 33 of Vol. IV we have proved the following theorem: If we place a very large number N of unit vectors of entirely arbitrary directions in a plane, then the resultant vector is equal to \sqrt{N} . In the present case we are interested in the collisions undergone by the colloidal particle owing to the thermal agitation of its surroundings. Complete directional isotropy of collisions is statistically assured and the number of collisions is proportional to the observation time t . The distances traversed by particles between two collisions (in actual fact reference is made to the projections of the paths \mathbf{r} of a particle onto the focal plane of the microscope of which the observed zig-zag is composed) do not constitute unit vectors; they are small distances which fluctuate about a mean value whose magnitude, in turn, depends on the properties of the surrounding fluid as well as on the particle whose motion (so-called "random walk") is being observed. The resultant translation \mathbf{r} of a particle can be calculated from eq. (33.4) in Vol. IV by interchanging S with \mathbf{r} and by replacing the unit vector by \mathbf{r}_i . The mean value is then

$$(1) \quad \overline{\mathbf{r}^2} = \sum \overline{\mathbf{r}_i^2} = \overline{r_i^2} \times N = P t.$$

In order to determine the factor of proportionality in this equation we shall make reference to the equation of motion of the particle:

$$(2) \quad M \ddot{\mathbf{r}} = \mathbf{K}(t) - C \dot{\mathbf{r}}.$$

M denotes the mass of the colloidal particle, \mathbf{r} is the vectorial displacement of its center of mass from a fixed initial point 0; $\mathbf{K}(t)$ is the force which varies

¹Comptes rendus 1908, p. 530.

in jumps both as regards magnitude and direction, and which transfers the collisions to M ; the last term in the equation represents the frictional resistance assumed proportional to the velocity $\dot{\mathbf{r}}$ as in Stokes' hypothesis; this assumption implies that the surrounding medium is regarded as a continuum which, obviously, is permissible only if the particle is many times larger than the molecular structure of the fluid; this is a reasonable assumption in the circumstances. Assuming that the particle is a sphere of radius a and that the viscosity of the fluid is η , we can assume that

$$(2a) \quad C = 6\pi\eta a$$

as given in eq. (35.20) of Vol. II. Taking the scalar product with \mathbf{r} , we find from eq. (2) that

$$(3) \quad M(\mathbf{r} \cdot \ddot{\mathbf{r}}) = (\mathbf{r} \cdot \mathbf{K}) - C(\mathbf{r} \cdot \dot{\mathbf{r}}),$$

where

$$\mathbf{r} \cdot \dot{\mathbf{r}} = \frac{1}{2} \frac{d}{dt} (r^2).$$

The product $(\mathbf{r} \cdot \mathbf{K})$ is known as the "virial of force \mathbf{K} ," the term being used in the study of the mechanics of material points; its usefulness in the kinetic theory of gases was first recognized by Clausius. We now apply the elementary transformation used in connection with the virial theorem:

$$\mathbf{r} \cdot \ddot{\mathbf{r}} = \frac{d}{dt} (\mathbf{r} \cdot \dot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = \frac{1}{2} \frac{d^2}{dt^2} (r^2) - v^2;$$

thus eq. (3) transforms to

$$(4) \quad \left(\frac{1}{2} M \frac{d^2}{dt^2} + \frac{1}{2} C \frac{d}{dt} \right) r^2 - M v^2 = (\mathbf{r} \cdot \mathbf{K}).$$

We now integrate this equation with respect to time from 0 to t and divide all terms by t . During the time interval t the product $(\mathbf{r} \cdot \mathbf{K})$, i. e. the projection of the rapidly varying force on the direction of \mathbf{r} , changes sign many times. Dividing by the large value of t (large compared with the interval which corresponds to a change in sign of $\mathbf{r} \cdot \mathbf{K}$) we may expect that

$$(4a) \quad \frac{1}{t} \int_0^t (\mathbf{r} \cdot \mathbf{K}) dt = 0.$$

With $\mathbf{r}_0 = 0$ (for the initial position of the particle) the integrated eq. (4) becomes

$$(5) \quad \frac{M}{2t} \frac{d\mathbf{r}^2}{dt} + \frac{C}{2t} \mathbf{r}^2 = \frac{1}{t} \int_0^t M \mathbf{v}^2 dt.$$

The right-hand side contains the temporal mean of double the kinetic energy of the particle. According to the law of equipartition the mean value of the kinetic energy averaged over a large number of particles is equal to kT (2 degrees of freedom in the two-dimensional motion under consideration; in the case of linear motion it would be necessary to write $\frac{1}{2} kT$ instead). We can now make use of our result in Sec. 23 B concerning gaseous mixtures: The mean velocities of our colloidal particles are much smaller than those of the molecules in the surroundings owing to the much larger mass of the former but the mean kinetic energy is equal for both and can, therefore, be expressed in terms of the absolute temperature of the surrounding fluid, as indicated. Denoting the mean for the larger aggregate by a bar over the symbol (transition from a single particle to a certain aggregate of particles), we obtain

$$(5 \text{ a}) \quad \frac{1}{t} \int_0^t \frac{1}{2} M \bar{\mathbf{v}}^2 dt = kT.$$

We shall verify presently that the first term on the left-hand side of (5) decreases exponentially with t so that we are able provisionally to neglect it. Performing the transition to an aggregate of particles on the left-hand side of (5) and taking into account eqs. (2 a) and (5 a), we conclude that

$$(6) \quad \bar{\mathbf{r}}^2 = \frac{2kT}{3\pi\eta a} t.$$

This contains our rough estimate in eq. (1) together with the evaluation of the factor P which appeared in it.

When observations are made only on one-dimensional translations of the particle, e. g. on those in the x -direction, eq. (6) is replaced by

$$(6 \text{ a}) \quad \bar{x}^2 = \frac{kT}{3\pi\eta a} t$$

in accordance with the principle of equipartition. This is *Einstein's equation* which has been confirmed experimentally in numerous ways and which was

used, for example, to determine the Boltzmann constant, k , or the Loschmidt-Avogadro number $L = R/k$.

We shall now supplement our derivation with a more accurate integration of eq. (5), performing the transition to the aggregate before the integration. Putting $\bar{r}^2 = u$ in eq. (4) and integrating once we obtain

$$(7) \quad \dot{u} + \frac{C}{M} u = \frac{4 k T}{M} t.$$

The integral of the associated homogeneous equation is

$$(7 a) \quad u_1 = A e^{-Ct/M}$$

and it is easy to find that

$$(7 b) \quad u_2 = \frac{4 k T}{C} \left(t - \frac{M}{C} \right)$$

constitutes a particular solution of the non-homogeneous equation. Now $M = \frac{4\pi}{3} \rho a^3$ where ρ denotes the density of the particle and hence, according to (2 a), we have

$$\frac{M}{C} = \frac{2}{9} \cdot \frac{\rho a^2}{\eta}.$$

Assuming $a = 10^{-4}$ cm (limit of visibility), $\eta \approx 10^{-2} \frac{\text{g}}{\text{cm sec}}$ (water) and $\rho \approx 1 \text{ g} \times \text{cm}^{-3}$ (the particle floats in water), we find that

$$(7 c) \quad \frac{M}{C} = \frac{2}{9} \times 10^{-6} \text{ sec.}$$

The presence of M/C in the brackets in (7 b) denotes an unmeasurably small shift in the zero of the time scale. Its presence in the exponent of (7 a) represents a very fast decay of any initial disturbance A that may have been present. Thus our assumption that $u = u_1 + u_2$ simplifies to

$$u \approx u_2 \approx \frac{4 k T}{C} t$$

which is identical with (6).

It is evident that observations performed on a single particle will deviate considerably from the mean values in (6) or (6 a); furthermore, as is easily demonstrated, the scatter will follow Gauss' error distribution curve so that,

consequently, it is necessary to take mean values averaged over large numbers of single observations in order to obtain an experimental verification.

The behavior of a torsional microbalance constitutes an extremely instructive variant of Brownian motion. The investigation of the fluctuations of a microbalance was first suggested by M. von Smoluchowski who also gave the relevant theory. The method was improved by E. Kappler¹ to such an extent that it could be used to determine the Loschmidt-Avogadro number to within 1 per cent.

In the case of Brownian motion the observation is concerned with the quadratic mean ($\overline{r^2}$ or $\overline{x^2}$) of a displacement; in the case of a microbalance the relevant quantity is given by the quadratic mean $\overline{\phi^2}$ of the angular displacement.

The following remarks may suffice to give a description of circumstances encountered during an experiment: A thin mirror of about 1 mm² in area is suspended from a quartz strand several μ in diameter. The torsional fluctuations caused by the impacts from air molecules are registered on a photographic film with the aid of reflected light. It is, of course, necessary to maintain a constant temperature and to insure freedom from vibrations. The "directional force," i. e. the elastic constant D of the quartz strand, is determined in the usual way by observing the free oscillations of the system when provided with an additional mass. In order to exclude radiometric influences it is desirable to keep the pressure either very low (e. g. 1/100 mm Hg), or comparatively high (e. g. 1 atm). The duration of one film recording was about 10 hrs.

When making a theoretical analysis of these fluctuations it is necessary to note that the mirror possesses not only the kinetic energy

$$(8\ a) \quad \frac{1}{2} I \dot{\phi}^2, \quad I = \text{moment of inertia}^2$$

but also the potential energy

$$(8\ b) \quad \frac{1}{2} D \phi^2, \quad D = \text{directional force (elastic constant)}.$$

Since the time-averaged values of kinetic and potential energy are equal, they have each to be ascribed the statistical mean energy $\frac{1}{2} k T$, corresponding to one degree of freedom. On taking mean values we find from eqs. (8 a) and (8 b) respectively that

$$(9\ a) \quad \overline{\dot{\phi}^2} = \frac{k T}{I},$$

$$(9\ b) \quad \overline{\phi^2} = \frac{k T}{D}.$$

¹Ann. d. Phys. 11, 233 (1931); cf. Naturw. 649 and 666 (1939).

²Cf. Sec. 31, remark following eq. (7).

The statistical deviations from these two mean values have been registered by Kappler; both plot exactly along a Gaussian error distribution curve. The distribution of $\dot{\phi}^2$ around the mean value (9 a) can be directly denoted as a "Maxwellian distribution" for the angular velocity.

It is realized that Kappler's method furnishes an elementary and reasonably accurate way of determining the Loschmidt-Avogadro number L , because the equation $R = L k$ permits us to calculate the Loschmidt-Avogadro number when k is known.

It is evident that no analogy to eq. (9 b) occurs in the consideration of the Brownian motion of a freely floating particle, because the particle is not restricted to a definite position of equilibrium. Equation (9 a) is replaced by our preceding eq. (5 a). Moreover the angular displacement ϕ (measured with respect to a fixed axis) of a Brownian particle satisfies a relation which is analogous to Einstein's equation (6 a), as already shown by Perrin, in which it suffices to make a suitable change in Stokes' frictional constant (2 a) (*cf.* Vol. II, eq. (35.21)).

In order to complete the dynamical analysis of the torsional balance experiment we again follow Langevin's method. In analogy with eq. (2) we can write the equation of motion as

$$(10) \quad I \ddot{\phi} = \mathbf{M}(t) - C \dot{\phi} - D \phi,$$

where $\mathbf{M}(t)$ denotes the torque due to the molecular collisions with the mirror; $C \dot{\phi}$ denotes Stokes' frictional moment in the surrounding air (or at low pressure); $D \phi$ is the elastic couple of the quartz strand (this term was absent in eq. (2)). Multiplying by ϕ and applying the virial theorem we have

$$(11) \quad \left(\frac{1}{2} I \frac{d^2}{dt^2} + \frac{1}{2} C \frac{d}{dt} \right) \phi^2 - I \dot{\phi}^2 + D \phi^2 = \phi \mathbf{M}(t),$$

instead of (4). Integrating with respect to t and dividing by t we find that the right-hand side vanishes. The last two terms on the left-hand side cancel each other in accordance with eqs. (9 a, b), when the mean over the aggregate is taken. Hence eq. (11) reduces to

$$I \frac{du}{dt} + C u = 0, \quad u = \overline{\phi^2}$$

because the constant of integration is zero in view of the fact that the mean value of the moment is equal to zero. Repeated integration gives

$$(11 a) \quad u = u_0 e^{-C t/I}.$$

The decay of an initial angular displacement (or velocity) indicated by eq. (11 a) was investigated by Kappler. By reducing the pressure of the air we can cause the value of the frictional constant C to decrease to an arbitrarily small value and thus it is possible to increase the "time of decay" I/C to an order of magnitude of several seconds. In this way it was possible to obtain an experimental verification of the relation in eq. (9 a) for the torsional balance.

The same could not be achieved in the case of a freely floating particle, eq. (5 a), because of the order of magnitude of M/C calculated in (7 c). The latter also indicates the time interval during which the motion can be regarded as being linear for all intents and purposes (i. e. without significant changes in direction). Because of this order of magnitude the motion of a Brownian particle observed by eye is in fact seen to be a mean of a very large number of displacements.

This is the reason why the very careful measurements performed by Franz Exner (1900) led to values of *velocity* which were smaller by many orders of magnitude than those implied in eq. (5 a). The existence of these measurements proved to be an obstacle to the acceptance of the view that Brownian motion is essentially of a molecular nature. The latter view gained universal acceptance only after Einstein deduced theoretically in 1905 the expression for a quantity which could be measured directly, namely one for the *mean square of displacement*.

Reverting to the torsional balance we conclude with a remark of general interest: *Thermal fluctuations which can be studied quantitatively on the example of a torsional balance set an insurmountable limit to the sensitivity of all indicating instruments*; this principle was first stated in relation to ultra-sensitive galvanometers by Ising, 1926.

25. Statistical considerations on paramagnetic substances

In order to give a statistical derivation of Langevin's function which has been used in Sec. 10 in anticipation of this proof, it is sufficient to make a simple application of the Boltzmann factor from eq. (23.15).

We suppose that the paramagnetic body is made of single, independent elementary magnets which are free to rotate and which are arranged in a disorderly manner. We shall denote their magnetic moments by $\mathbf{m} = p l \mathbf{e}$ (p = magnetic pole strength, l = distance between poles, \mathbf{e} = unit vector). This model is sufficient to deduce not only the properties of paramagnetic gases (O_2 , NO , ...) and liquids, but also of those of solid salts, both when classical considerations are applied as in Section A, and when quantum

mechanics is used, as in Section B. From the classical point of view all directions of the magnetic axes are permissible and equally probable. From the point of view of quantum mechanics only several discrete orientations of the magnetic axis with respect to the external magnetic field are admitted. In any case the probability of a given orientation is determined by the interaction between the directed force of the external field and thermal agitation, i. e. the temperature of the system.

A. THE CLASSICAL LANGEVIN FUNCTION

Let θ denote the angle between the axis of the elementary magnet and the direction of the external field. The latter will be denoted by B (denoted by $\mu_0 H$ in Sec. 19). The moment acting between them is

$$D = m B \sin \theta$$

and tends to *decrease* θ , that is to render \mathbf{m} parallel to B . The work of *increasing* θ against the field is

$$D d\theta = m B \sin \theta d\theta = -m B d \cos \theta.$$

On turning by $d\theta$ the change in the potential energy Φ of our elementary magnet is of equal magnitude. Thus we have

$$(1) \quad d\Phi = -m B d \cos \theta, \quad \Phi = -m B \cos \theta.$$

Φ increases as θ is increased in the same way as the gravitational potential $\Phi = m g z$ increases as the distance z from the surface of the earth is increased. Our present Φ is so normalized as to place its minimum $\Phi = -m B$ at the stable position $\theta = 0$ and its maximum at the unstable position $\theta = \pi$.

From the classical point of view the *a priori* probability dW of a given orientation of our elementary magnet is *the same* for equal ranges of angles $d\omega = \sin \theta d\theta d\phi$. It becomes *different* for different temperatures when multiplied by the Boltzmann factor, namely

$$(2) \quad dW = A e^{-\Phi/kT} d\omega.$$

This exhibits the opposing influences of temperature and field. At high temperatures all directions have roughly the same probability, as it was *a priori*, because then $\exp(-\Phi/kT) \approx 1$. At low temperatures the stable orientation $\theta = 0$, $\Phi = \Phi_{min}$ outweighs all other possibilities. The coefficient A

introduced in (2) can best be evaluated from the condition that the probability of finding the magnet at *any* direction is equal to 1. It follows that

$$\int dW = 1 = A \int_0^{2\pi} d\phi \int_0^\pi e^{-\phi/kT} \sin \theta d\theta$$

so that

$$(2a) \quad 1/A = 2\pi \int_0^\pi e^{-\phi/kT} \sin \theta d\theta.$$

Substituting (2) and integrating with respect to ϕ , we have

$$(3) \quad dW = \frac{e^{-\phi/kT} \sin \theta d\theta}{\int e^{-\phi/kT} \sin \theta d\theta}.$$

We now proceed to calculate the mean value of the component of \mathbf{m} in the direction of the field and we shall denote it by \bar{m} . With dW from (3) we have

$$(4) \quad \bar{m} = \int_0^\pi m \cos \theta dW.$$

When the number of elementary magnets is very large only this component and only its probable value enter into the calculation because the components at right angles to the field lie in different azimuthal planes ϕ in an irregular manner and cancel each other. Multiplying the mean value \bar{m} by the number n of elementary magnets per mol we obtain a quantity M which in Sec. 19 was called the magnetization:

$$M = n \bar{m}.$$

On the other hand $M_\infty = n m$ (uniform orientation of all elementary magnets in a sufficiently strong field), eq. (19.7), denotes the *magnetization at saturation*. Introducing the abbreviations

$$(5) \quad x = \cos \theta, \quad \alpha = \frac{m B}{k T}$$

we find that eq. (4) yields

$$(6) \quad \frac{M}{M_\infty} = \frac{\int_{-1}^{+1} e^{\alpha x} x dx}{\int_{-1}^{+1} e^{\alpha x} dx}.$$

Performing the integration indicated in the denominator, we have

$$\frac{1}{\alpha} (e^{\alpha} - e^{-\alpha}) = \frac{2}{\alpha} \sinh \alpha.$$

The numerator is the derivative of the denominator with respect to α , i. e. equal to

$$\frac{2}{\alpha} \cosh \alpha - \frac{2}{\alpha^2} \sinh \alpha.$$

Hence it follows from (6) that

$$(7) \quad \frac{M}{M_{\infty}} = \frac{\cosh \alpha}{\sinh \alpha} - \frac{1}{\alpha}.$$

The right-hand side represents the Langevin function from eq. (19.5 b); it is easy to see that our definition of α in (5) agrees with that given in (19.5 a). Our present eq. (7) fills the gap left open in Sec. 19 B and completes the classical theory of paramagnetic substances as well as Weiss' theory of ferromagnetic substances which followed in Sec. 19 C.

B. MODIFICATION OF LANGEVIN'S FUNCTION WITH THE AID OF QUANTUM MECHANICS

According to the point of view of quantum mechanics the assemblage of all possible orientations of the elementary magnets does *not* constitute a *continuum*, $-1 \leq \cos \theta \leq +1$, but is restricted to certain *discrete* values of $\cos \theta$ ("angular quantization"). Their number is determined by the spectroscopic character of the atom (or molecule) in its ground state; it is equal to 2, 3, 4, ..., depending on whether the ground state corresponds to a doublet, triplet, quadruplet, (The ground state is *not* associated with a magnetic moment in the case of a singlet system; the atom does *not* then constitute an elementary magnet.)

The principles of quantum mechanics lead to the following rule, which we cannot, obviously, justify here: Let r denote the multiplicity of the term system ($r = 2$ doublet, $r = 3$ triplet, ...); put $r = 2j + 1$ ($j =$ azimuthal quantum number) and $\cos \theta = s/j$ (so that $|s| \leq j$ because $|\cos \theta| \leq 1$). The rule states that only such values of s are admitted as differ from each other by unity. They are seen tabulated below.

$r =$	2	3	4	5
$j =$	1/2	1	3/2	2
$s =$	$\pm 1/2$	$\pm 1; 0$	$\pm 3/2; \pm 1/2$	$\pm 2; \pm 1; 0$

In the case of a doublet system, for example, only the two orientations which correspond to $\cos \theta = \pm 1$, i. e. parallel and antiparallel to the field, are permissible. In a system composed of triplets the orientation at right angles to the field, corresponding to $\cos \theta = 0$, is also permitted; there are four possible orientations in a system composed of quadruplets, namely those given by $\cos \theta = \pm 1$, $\cos \theta = \pm 1/3$ etc. It is assumed *a priori* that such orientations are associated with equal probabilities. When calculating the mean value \overline{m} of the component in the field direction it is necessary again to take into account the Boltzmann factor. Generally speaking the integral in (4) is now replaced by the sum of r terms. Introducing the abbreviations in (5), we have for $r = 2$

$$(8) \quad \frac{\overline{m}}{m} = (e^{\alpha} - e^{-\alpha}) / (e^{\alpha} + e^{-\alpha}) = \tanh \alpha,$$

and for $r = 3$

$$(8a) \quad \frac{\overline{m}}{m} = (e^{\alpha} - e^{-\alpha}) / (e^{\alpha} + 1 + e^{-\alpha}) = \frac{2 \sinh \alpha}{1 + 2 \cosh \alpha}.$$

Similarly for $r = 4$

$$(8b) \quad \frac{\overline{m}}{m} = \left(e + \frac{1}{3} e^{\alpha/3} - \frac{1}{3} e^{-\alpha/3} - e^{-\alpha} \right) / (e^{\alpha} + e^{\alpha/3} + e^{-\alpha/3} + e^{-\alpha}).$$

Depending on the spectroscopic nature of the state of the atom it is necessary to replace Langevin's function by the right-hand side of one of the preceding equations. The classical Langevin function appears as a limit for $r \rightarrow \infty$. We shall denote the latter by $L_{\infty}(\alpha)$, denoting the preceding modified forms by $L_2(\alpha)$, $L_3(\alpha)$, The function L_2 was introduced by W. Lenz¹ in 1920. We shall now compare it with Langevin's function

$$(9) \quad L_2 = \frac{\sinh \alpha}{\cosh \alpha}, \quad L_{\infty} = \frac{\cosh \alpha}{\sinh \alpha} - \frac{1}{\alpha}.$$

On a graph the curves L_3, L_4, \dots will be seen to fall between the limiting curves (9).

This is exhibited, in particular, by the slope of the tangent at the origin. According to (9) we have (see also eq. (19.6))

$$L_2'(0) = 1, \quad L_{\infty}'(0) = \frac{1}{3}$$

¹Phys. Zeitschr. 21, 613 (1920).

whereas (8 a, b) give

$$L_3'(0) = \frac{2}{3}, \quad L_4'(0) = \frac{5}{9}, \dots$$

The slope is steepest for L_2 and decreases in steps until it reaches $1/3$ for L_∞ .

Since, according to (19.11 a), the *Curie temperature* Θ from eq. (19.11 a) depends on $L'(0)$, its value is also modified by quantum mechanics as compared with the classical value. The same applies to the *Curie constant* whose classical value (19.7 a) should be replaced by, for example,

$$C = \frac{\mu_0 M_\infty^2}{R}$$

in the case of a doublet.

Additional modifications introduced by quantum mechanics have already been described at the end of Sec. 19 C and in connection with Fig. 19. We shall not pursue this point here but we wish to emphasize that the theory of para- and ferromagnetic phenomena, being of statistical origin, often illustrates the difference between classical statistical mechanics and quantum mechanics.

26. The statistical significance of the constants in van der Waals' equation

In Sec. 9 we introduced the constants a and b of the van der Waals equations in a purely phenomenological way giving only a brief account of their physical significance. In his dissertation, van der Waals fully established their statistical origin and his derivation was subsequently simplified by Boltzmann in his "Vorlesungen über Gastheorie," 1898. In our description we shall follow that given in Chapter A of F. Sauter's paper¹ which contains a modern account of the essential points of Boltzmann's method. The circumstance that only the simplest statements of the elementary theory of gases are required for the purpose, as stressed by Sauter, allows us to fill the gap left open in Sec. 9 already at this stage and without having to draw on the general methods described in Chap. IV.

A. THE VOLUME OF A MOLECULE AND THE CONSTANT b

When considering perfect gases it is assumed that the volume of a molecule is equal to zero. This assumption is justified under certain conditions (not too low a temperature, and sufficiently high rarefaction). Generally speaking, and in connection with real gases in particular, it is necessary to assume that

¹Ann. d. Physik (6) 6, 59 (1949).

the presence of a molecule whose (very small) volume is equal to v_0 prevents other molecules, whose volumes are also equal to v_0 , from penetrating into its sphere of influence. It suffices here to assume that v_0 corresponds to a rigid sphere. The sphere of influence can thus be defined as that sphere whose surface can be reached by the *centers* of the other, impinging molecules, without causing the molecular volumes to overlap. It is evident that this sphere of influence has double the radius of a molecule and hence its volume is $8 v_0$.

Let us now consider a unit of volume within the gas and let n_i denote the number of molecules present in it (molecular density). The portion of space barred to an additional molecule is then $8 n_i v_0$, so that

$$(1) \quad \frac{v_i}{v_1} = 1 - 8 n_i v_0,$$

where v_1 denotes the unit volume; it has been included in the denominator for reasons of dimensional consistency.

Let us now consider an element of area, $d\sigma$, inside the gas. It is seen that eq. (1) is satisfied on both sides of it. If, however, $d\sigma$ denotes an *element* of a *solid wall* then only half of the molecules present will constitute a bar to penetration and it will be due to the molecules present on the side of the element exposed to the gas. Thus we must make a distinction between n_i and n_w (molecular density near a wall). The volume open to the penetration of an additional molecule is larger than (1) and is equal to

$$(2) \quad \frac{v_w}{v_1} = 1 - 4 n_w v_0.$$

The ratio v_w/v_i represents simultaneously the ratio of the probabilities of finding an additional molecule near the wall as compared with the interior of the gas (the probability of being able to place a molecule at one point as compared with the other). This ratio is equal to n_w/n_i , so that comparing (2) with (1), we have

$$(3) \quad \frac{v_w}{v_i} = \frac{n_w}{n_i} = \frac{1 - 4 n_w v_0}{1 - 8 n_i v_0}.$$

Solving for n_w we find that

$$(3 a) \quad n_w = \frac{n_i}{1 - 4 n_i v_0}.$$

The number of molecules, n , and hence the density of a gas is slightly higher near a wall than in the bulk. This is the real reason for the correction term b in the van der Waals equation, as shown by Sauter who followed Boltzmann in this respect.

In order to show this we refer once more to the calculation of pressure given in Sec. 22. We have considered then a cylinder on base $d\sigma$ and finite height ξ . We can, however, decrease the height ξ as much as we please, and so we can confine the whole cylinder to the neighborhood of the wall, if we simultaneously allow the base to increase in proportion. Thus eq. (22.4) remains valid, except that n must be replaced by n_w . Taking into account the definition of temperature in eq. (22.5) as well as eq. (3 a), we have

$$(4) \quad p = n_w \times k T = \frac{n_i k T}{1 - 4 n_i v_0}.$$

In order to reduce this equation to the form due to van der Waals and given in Sec. 9 we first apply it to one mol of a gas. The number of all molecules is then equal to the Loschmidt-Avogadro number L and the number of molecules in a unit of volume $n_i = L/v$, where v now denotes the molar volume. It follows from eq. (4) that

$$(4 a) \quad p = \frac{L k T/v}{1 - 4 L v_0/v} = \frac{R T}{v - 4 L v_0}.$$

On comparing with eq. (9.1) in which we must put $a = 0$, because cohesion forces have so far been left out of account, we find

$$(5) \quad b = 4 L v_0;$$

the constant b is equal to four times the total volume of all molecules present in a mol. This is the physical significance of the constant b ; it was deduced already by van der Waals.

B. THE VAN DER WAALS COHESION FORCES AND THE CONSTANT a

In the present argument we shall neglect the volume of each molecule but we shall take into account the short-range cohesion forces acting between neighboring molecules and given by $f(r)$; for example in the case of noble gases these forces decrease approximately in proportion to r^{-7} as r increases. We may now put $n_i = n$ as if no cohesion forces were present, because in the *bulk of the gas* all forces acting on one molecule are evenly distributed and cancel each other. However, near a *boundary* conditions are different because the forces of cohesion exert a pull towards the interior of the gas ("upwards")

because the molecules which would exert a pull in the opposite direction ("downwards") do not exist. There is thus a kind of sedimentation in the upward direction, i. e. away from the wall, which leads to the normal value n outside the boundary zone but generates a value $n_w < n$ in the neighborhood of the wall. Qualitatively speaking, the influence of the forces of cohesion consists in creating a density distribution near the boundary which differs from the normal, in a way similar to the influence of molecular volumes.

In order to obtain a quantitative expression it is necessary to recall the barometric formula (23.16). In deducing it, account was taken of the gravitational potential i. e. of the force of gravity which was independent of position and which was directed downwards. Its value was

$$(6) \quad \Phi = m g z \quad \text{or} \quad -\frac{\partial \Phi}{\partial z} = -m g.$$

In the present case, however, the force is not constant; it is equal to zero above the boundary zone (mutual compensation of cohesion forces), and it is directed upwards within it. We imagine that the sphere of influence of cohesion forces is represented by a spherical surface centered at the point under consideration and we use the symbol $\gamma(z)$ to denote the influence from the segment of the sphere intercepted by the wall. The "upward pull" will be proportional to this well-defined quantity $\gamma(z)$. Furthermore, in the present case we are not considering a molecule in a given field of forces, as was the case with the gravitational field, but we wish to enquire into the interaction between the molecule which experiences the pull upwards and the surrounding neighboring molecules. This interaction depends on the density with which the neighboring molecules would fill the segment $\gamma(z)$ in the absence of the wall. It is thus seen to be proportional to the number of molecules n ; since our present considerations are directed towards the establishing of a correction term, the variation of n with space coordinates may be neglected. Hence, as distinct from (6), we put

$$(7) \quad -\frac{\partial \Phi}{\partial z} = n \gamma(z); \quad \Phi = n \int_z^{\infty} \gamma(z) dz,$$

where the coefficient of proportionality has been included in $\gamma(z)$. In this equation Φ is normalized to give $\Phi = 0$ for $z = \infty$, which is different from the condition $\Phi = 0$ at $z = 0$ in the case of gravity. Hence at the wall

$$\Phi(0) = n \bar{\gamma}, \quad \bar{\gamma} = \int_0^{\infty} \gamma(z) dz,$$

and the preceding barometric formula

$$\frac{n_z}{n_0} = e^{-\Phi/kT}$$

transforms into

$$(8) \quad \frac{n_w}{n} = e^{-\Phi(0)/kT} = e^{-n\bar{\gamma}/kT}$$

at $z = 0$.

Applying eq. (4), or Bernoulli's construction, we find that the pressure is given by

$$(9) \quad p = n_w k T = n k T e^{-n\bar{\gamma}/kT}.$$

Expanding into a series for large values of kT , and neglecting higher-order terms we have

$$(9 \text{ a}) \quad p \approx n k T - n^2 \bar{\gamma}.$$

We apply this equation once more to one mol putting $n = L/v$ with $v = v_{mol}$. Hence¹

$$(10) \quad p = \frac{R T}{v} - \frac{L^2 \bar{\gamma}}{v^2}.$$

¹At this point one might ask whether eq. (9) does not lead to a pressure correction term which is more precise than that in the van der Waals' equation. When the full expression is retained the van der Waals equation is replaced by that due to Dieterici

$$p = \frac{R T}{v - b} e^{-a/vRT}.$$

The critical values (*cf.* Sec. 9 A) implied are:

$$v_{cr} = 2b, \quad R T_{cr} = \frac{a}{4b}, \quad p_{cr} = \frac{a}{4b^2} e^{-2},$$

and the critical constant is

$$K = \frac{R T_{cr}}{p_{cr} v_{cr}} = \frac{1}{2} e^2 = 3.69$$

(instead of $K = 8/3 = 2.67$ given by the van der Waals equation). Empirical values for He, A and Xe are $K = 3.31$; 3.42, and 3.57, respectively.

The first term is of the same form as for a perfect gas, as expected, because the volume of the molecules has been neglected. On comparing with the van der Waals equation the second term explains the physical significance of a in eq. (9.1):

$$(11) \quad a = L^2 \bar{\gamma}.$$

The constant a in the van der Waals equation is equal to L^2 times a certain cohesive action at the boundary which vanishes in the bulk of the gas. It may be worth mentioning that an artificial wall is created when a manometer is introduced into the interior of the gas and so it measures the "cohesion pressure" $p_a = a/v^2$ in addition to the normal pressure.

Sauter (*l. c.*) deduced an expression for the cohesion pressure from the statistical behavior of the whole collection of molecules and not, as we have done it in the preceding argument, from the behavior of a single isolated molecule, which experiences the pull of the rest. Consequently, Sauter's derivation is more satisfactory but also more tedious. In addition Sauter proved that the nature of the wall and the supposed existence of long-range cohesion forces between wall and gas have no influence on the pressure being measured.

27. The problem of the mean free path

The concept of the mean free path was introduced by Clausius as early as 1858. It represents the expected length l which is traversed by any molecule under consideration and moving within the conglomeration of molecules which constitute the gas, during a time interval between an arbitrary initial position and the first collision with another molecule. In the case of perfect gases whose molecules have been assumed to be shrunk to points we would have $l = \infty$. Consequently, it is necessary to take into account the finite volume of a molecule, v_0 , as in Sec. 26. We have found previously that this volume entered into the van der Waals equation in the form of the constant b which was in turn proportional to $4 v_0$. It will be shown that in the present case the final equation will contain the cross-section of v_0 multiplied by the factor 4. As in the preceding argument, the present considerations are restricted to rigid, spherical molecules.

We shall, further, restrict the calculation to the case when the velocity of the impinging molecule is large with respect to that of the molecule suffering the impact. The "mean free path" which was calculated already by Maxwell on the assumption that the molecules satisfy the Maxwellian distribution is

of the same order of magnitude (coefficient $1/\sqrt{2}$ in the denominator). It appears to us that the present restriction is justified because in the exact theory, *cf.* Sec. 44, it is not necessary to know the mean free path as it can be replaced by an argument in a series expansion which must then be properly defined depending on the particular problem under consideration.

It will be remarked at this point that the problems to be considered in the following Sections B and C are concerned with *irreversible processes* and hence transgress the scope of ordinary thermodynamics and statistical mechanics (*cf.* here Sec. 21 and Chap. V).

A. CALCULATION OF THE MEAN FREE PATH IN ONE SPECIAL CASE

When the impinging particles move very fast it is possible to assume that the slow particles suffering the impact are at rest. We now attach the volume of the latter to the former, as was done in Sec. 26 A, and construct an "effective cross-section." The radius of the effective cross-section (the sum of the radii r_1 and r_2 of the two molecules under consideration) will be denoted by s (in the present case $r_1 = r_2 = r$ and $s = 2r$). The volume swept by the sphere of influence per unit time has the form of a circular cylinder of cross-section πs^2 and height c . When the number of molecules is n , the interior of the circular cylinder will contain on the average

$$n\pi s^2 c$$

molecules suffering impact, for these can now be regarded as points, as they have been deprived of their volume. The preceding expression represents also the number of collisions per unit time, denoted by ν . Since ν collisions are associated with the path traversed per unit time we find that the mean free path for a single collision is

$$(1) \quad l = \frac{c}{\nu} = \frac{1}{n\pi s^2}.$$

In the case of rigid spherical molecules of effective cross-section πs^2 , l denotes precisely the smallest thickness of layer for which the effective cross-section of the molecules would cover the whole area if there were no intersections.

We now proceed to calculate the order of magnitude of l . In order to do this we introduce a reference length l_1 defined by the condition that a cube of side l_1 contains on the average just *one* molecule. Thus $n = 1/l_1^3$ and according to (1), we have

$$(2) \quad \frac{l}{l_1} = \frac{1}{\pi} \left(\frac{l_1}{s} \right)^2.$$

The numerical value of the radius of a hydrogen atom (first Bohr orbit) is about 0.5×10^{-8} cm. Estimating that the radius of a hydrogen molecule is equal to the double of that for an atom, we find that the radius s of the sphere of influence is equal to 2×10^{-8} cm. The number of molecules n should be taken per unit of volume and can be found from the Loschmidt-Avogadro number. At 0 C and 1 at pressure it is $2.8 \times 10^{19}/\text{cm}^3$. Hence

$$(2\ a) \quad l_1^3 = \frac{1}{28} \times 10^{-18} \text{ cm}^3, \quad \frac{l_1}{s} = \frac{1/3 \times 10^{-6}}{2 \times 10^{-8}} = \frac{50}{3},$$

and according to (2)

$$(2\ b) \quad \frac{l}{l_1} = \frac{10^4}{36\pi} \approx 100.$$

Consequently the following proportion can be set up:

$$(3) \quad l : l_1 : s = 10^4 : 10^2 : 6.$$

Under the present assumptions the mean free path l is 100 times larger than the mean distance l_1 between molecules; at atmospheric pressure it is, in turn, much larger than the radius s of the cross-section, or, as we may also say, than the molecular diameter.

According to Avogadro's law the length l_1 is the same for all perfect gases; furthermore the diameters of molecules do not differ materially from each other. Consequently the proportion (3) is also approximately valid for O_2 and N_2 .

We can conclude, further, that the molecules of a perfect gas suffer enormously large numbers of collisions on moving over a distance of several cm from an initial position. This shows that the unexpectedly large values of $\sqrt{\overline{c^2}}$ found in Sec. 22 are confined to ultra-microscopic distances. We can also see from eq. (1) (l being inversely proportional to pressure) that the concept of a mean free path becomes illusory when applied to very high vacua: The molecules are in a position to reach the walls of the vessel without first colliding with other molecules. This limiting case of the kinetic theory of gases was clarified both experimentally and theoretically in detailed papers due to M. Knudsen. In the opposite limiting case (liquefaction, touching of molecules) we have $l \approx s$ and the concept of the mean free path loses its sense once more.

B. VISCOSITY

We now propose to consider the case when the molecules of a gas possess a *molar* (macroscopic) velocity in addition to their molecular velocity. We assume that the former is directed along the x -axis and that it increases linearly in the direction of y . In the phenomenological science of fluid dynamics, *cf.* Vol. II, Sec. 10, it is assumed that a small element of area at right angles to y is acted upon by a shearing stress in the x -direction given by

$$(4) \quad \sigma_{yz} = \pm \eta \frac{\partial u}{\partial y}.$$

It decelerates the velocity of the upper (faster) layers and accelerates that of the lower (slower) layers (see the respective sketch in Sec. 10 of Vol. II). It is now our task to deduce the same result with the aid of kinetic theory and to calculate the value of the coefficient η (viscosity). The proof consists in setting up a balance equation for the *transfer of momentum*.

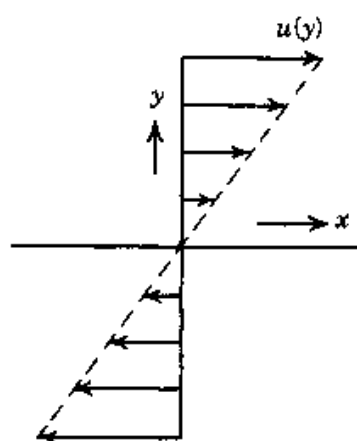


Fig. 25.
Mean velocity in Couette
flow.

Let m denote the mass of a molecule, u —the molar, and ξ —the molecular velocity component in the direction of x . The total momentum of the molecules in the direction of x is thus:

$$(5) \quad g_x = m(u + \xi),$$

where u depends on y , Fig. 25. The probability distribution of the values of ξ is independent of y if, as we shall assume, the temperature does not vary from point to point. The cross-section at $y = 0$ under consideration is continually traversed by slower molecules coming from the lower half space and entering the upper region; faster molecules traverse it in the opposite direction.

We shall now inquire into the point from which the faster and slower molecules, respectively, arrive. This depends on the last preceding collision in the half space from which they come. In order to determine the probable distance at which this collision took place we produce the mean free path l starting at $y = 0$ and direct it along the path of the molecule. Denoting the angle between this direction and the axis of ordinates by θ we find that the abscissa of the position of the last collision is equal to $-l \cos \theta$ for a molecule coming from below. When it arrives from above, the abscissa is equal to $+l \cos \theta$, *cf.* Figs. 26 and 26 a.

We now proceed to calculate the molar momentum $g = m u$ which is lost by the upper half-space $y > 0$ owing to its having lost the molecules which have passed our cross-section at $y = 0$ per unit area. It is not necessary to take into account the molecular momentum $m \xi$ because ξ is independent of y and its change is, therefore, cancelled by the molecules arriving from below.

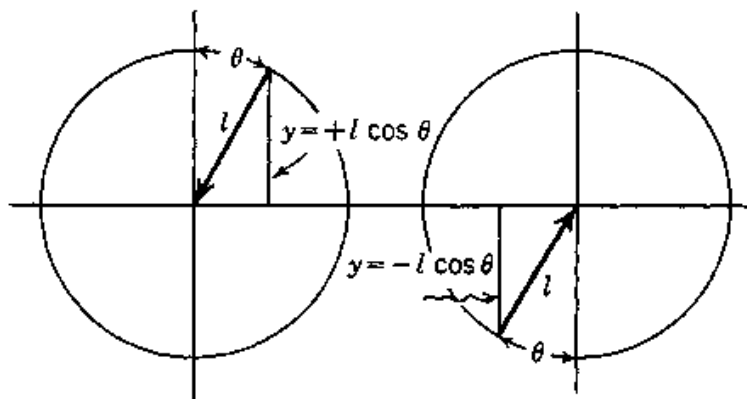


Fig. 26.

Fig. 26 a.

Particles from above Particles from below
arriving from a distance l ($=$ radius of circle).

We begin by considering the group of molecules which move upwards with a velocity between c and $c + dc$ and whose directions lie between θ and $\theta + d\theta$. Since they come from below, Fig. 26 a, their contribution is

$$(5 a) \quad g \uparrow = m u(-l \cos \theta) = m u(0) - m l \cos \theta \left(\frac{\partial u}{\partial y} \right)_0.$$

(Since $u(y)$ is linear there are no higher-order terms in the Taylor expansion for u .) Correspondingly the group of molecules travelling upwards ($\theta =$ angle with the negative semi-axis of ordinates), Fig. 26 a:

$$(5 b) \quad g \downarrow = m u(+l \cos \theta) = m u(0) + m l \cos \theta \left(\frac{\partial u}{\partial y} \right)_0.$$

The difference of the two is

$$(6) \quad g \uparrow - g \downarrow = -2 m l \cos \theta \left(\frac{\partial u}{\partial y} \right)_0.$$

This value must now be multiplied by the number of molecules dv , equal for the two groups, which cross our section at $y = 0$ per unit time and area. According to eqs. (22.9 a) and (23.9) the number of molecules which arrive from the volume in Fig. 23 per unit time and area is equal to

$$(7) \quad dv = \frac{n}{4\pi c^2} \phi(c) \times c \cos \theta \times \sin \theta d\theta d\phi \times c^2 dc,$$

where the normal to the surface is taken in the ζ -direction (also the polar axis). The total change in momentum due to the molecules in the volume shown in Fig. 23 is obtained by multiplying eq. (6) by this expression. Integrating with respect to the azimuth ϕ and rearranging, we have

$$-n m \times l c \phi(c) dc \times \cos^2 \theta \sin \theta d\theta \times \left(\frac{\partial u}{\partial y} \right)_0.$$

The values of $g \uparrow$ and $g \downarrow$ have been multiplied by the same factor because the Gaussian distribution is symmetrical. In order to evaluate the total momentum transferred, it is necessary to integrate over all velocities c and over all directions θ from which molecules may come. Integration over θ is thus extended from 0 to $\frac{1}{2}\pi$ and then total change in momentum becomes

$$\begin{aligned} \bar{g} &= \int (g \uparrow - g \downarrow) dv = -m n \left(\frac{\partial u}{\partial y} \right)_0 \int_0^\infty l c \phi(c) dc \cdot \int_0^{\pi/2} \cos^2 \theta \sin \theta d\theta \\ (8) \qquad &= -\frac{m n}{3} \left(\frac{\partial u}{\partial y} \right)_0 \int_0^\infty l c \phi(c) dc. \end{aligned}$$

This equation takes into account the fact that, strictly speaking, l depends on c . It would be possible to determine this relation, the calculation being somewhat complicated, on the assumption that the molecules obey Maxwell's distribution law. It is, however, necessary to note that the problems under consideration do not involve states of equilibrium so that, strictly speaking, the distribution law is not Maxwellian, *cf.* Sec. D. We shall, therefore, abstain from further investigating the integral in (8) replacing l by its mean value, for example by $l/\sqrt{2}$ (see here the introduction to the present Section). This allows us to place it in front of the integral, so that (8) yields

$$(8 a) \qquad \bar{g} = -\frac{m n l}{3} \left(\frac{\partial u}{\partial y} \right)_0 \int_0^\infty c \phi(c) dc.$$

Making use of eq. (23 10 a) and replacing the integral in (8 a) by \bar{c} , we find

$$(9) \qquad \bar{g} = -\frac{m n l \bar{c}}{3} \left(\frac{\partial u}{\partial y} \right)_0.$$

Evaluating the change in momentum for the negative half space we obtain the same formula, except that the sign is reversed. Thus

$$(9\ a) \quad \bar{g} = + \frac{m n l \bar{c}}{3} \left(\frac{\partial u}{\partial y} \right)_0.$$

Comparing (9) and (9 a) with (4) we see that the viscosity is given by

$$(10) \quad \eta = \frac{m n l \bar{c}}{3}.$$

It will be noticed that $m n$ represents the mass per unit volume, i. e. the density ρ of the gas. The preceding equation (10) was first deduced by Maxwell in 1860. It only seemingly implies a dependence on pressure, because the product of

$$(11) \quad \rho = m n \quad \text{and} \quad l \approx \frac{1}{\sqrt{2} n \pi s^2}$$

is independent of the number of molecules n , which cancels in the numerator and denominator. The remaining quantity is independent of the mass and volume of the particles. This *lack of dependence* of viscosity on *pressure* was at first considered paradoxical; it has, however, been verified by Kundt and Warburg in 1875 down to pressures of the order of 1/60 at. The equation is also limited in its application at the high pressure range because the gas then approaches its point of liquefaction when the concept of a mean free path ceases to have a meaning, as is also the case for very low pressures.

The *dependence on temperature* defined by the factor \bar{c} in eq. (10) is also of great interest. In actual fact \bar{c} (as well as $(\bar{c}^2)^{\frac{1}{2}}$) increases in proportion to the square root of absolute temperature. Whereas the viscosity of liquids (e. g. of oils) decreases fast with temperature, *the viscosity of a gas increases in proportion to \sqrt{T} .*

C. THERMAL CONDUCTIVITY

The preceding considerations, concerning the transfer of momentum g , apply to the transfer of any quantity G which may be transported by the molecules. In particular, we are interested in the transport of energy, E , when we consider a gas which is not in thermal equilibrium, and whose temperature increases linearly along the axis of ordinates. We shall once more use Figs. 26 and 26 a to illustrate the argument.

The energy E is given by eq. (23.14) in which the potential Φ may now be omitted. Denoting the number of degrees of freedom by f (translation, rotation, and internal vibrations) and making use of the law of equipartition

of energy we find that each degree of freedom is associated with the energy $\frac{1}{2} k T$, so that the total energy becomes

$$E = \frac{1}{2} f k T$$

or, in terms of y

$$E = \frac{1}{2} f k \left[T(0) + y \left(\frac{\partial T}{\partial y} \right)_0 \right].$$

Thus the quantity G being transferred is given by

$$G \uparrow - G \downarrow = -2 \cdot \frac{1}{2} \cdot f k l \cos \theta \left(\frac{\partial T}{\partial y} \right)_0$$

instead of (6) ($m u$ has been replaced by $\frac{1}{2} f k T$). Instead of (8 a), we obtain

$$\overline{G} = -\frac{1}{3} \frac{n f k l}{2} \left(\frac{\partial T}{\partial y} \right)_0 \cdot \int_0^\infty c \phi(c) dc = -\frac{n l}{3} \cdot \frac{f k \bar{c}}{2} \left(\frac{\partial T}{\partial y} \right)_0.$$

Comparing this expression with Fourier's phenomenological assumption about the conduction of heat along a corresponding temperature gradient (Q = quantity of heat transferred, κ = thermal conductivity)

$$Q = -\kappa \left(\frac{\partial T}{\partial y} \right)_0,$$

we obtain the following interpretation of κ from kinetic theory

$$(12) \quad \kappa = \frac{n l f k \bar{c}}{3 \cdot 2}.$$

It is seen that κ is *independent of pressure* in the same way as η from eq. (10) but *varies with temperature* (owing to the factor \bar{c}). The lack of dependence of thermal conductivity on pressure was demonstrated experimentally by Kundt and Warburg (when the temperature is not too low) at the same time when they performed experiments on viscosity.

The value of the ratio κ/η which results from the preceding considerations is also of great interest, because it does not contain the questionable quantities \bar{c} and l . Combining (12) with (10) we obtain

$$\frac{\kappa}{\eta} = \frac{f k}{2 m}.$$

Multiplying the numerator and the denominator by the Loschmidt-Avogadro number L per mol, we obtain

$$\frac{\kappa}{\eta} = \frac{\frac{1}{2} f R}{\mu}$$

(because $k L = R =$ universal gas constant and $m L = \mu =$ molecular weight). It is seen from eq. (4.13) that the numerator contains the molar specific heat at constant volume, c_v , of the gas. This quantity can be *measured directly* as distinct from the original quantities \bar{c} and l which appear in the expressions for κ and η . The resulting equation

$$(13) \quad \frac{\kappa}{\eta} = \frac{c_v}{\mu}$$

in which c_v/μ represents the specific heat. In a more accurate derivation it would still contain a numerical factor on the right-hand side. The numerical value of the constant for rigid, spherical molecules (Enskog) is 2.52, becoming rather smaller for polyatomic molecules. It may be worth noting that eq. (13) is to a certain extent reminiscent of the *Wiedemann-Franz law* in the theory of metals (*cf.* Sec. 45).

$$(13 \text{ a}) \quad \frac{\kappa}{\sigma} = \frac{2}{3} \left(\frac{k}{e} \right)^2 T,$$

($\sigma =$ electrical conductivity, $e =$ electron charge). The analogy consists in the fact that the questionable quantity l (as well as the number of free electrons, n) has been canceled in the quotient (13 a). The two equations differ in that the temperature dependence has been suppressed in (13) as distinct from (13 a).

D. SOME GENERAL REMARKS ON THE PROBLEMS ASSOCIATED WITH THE CONCEPT OF THE MEAN FREE PATH

It has been mentioned on several occasions that the assumption of a Maxwellian distribution in connection with non-equilibrium states constitutes a rough approximation. The essential task in problems of this kind is to *determine the velocity distribution in terms of space and time*. The importance of this problem was recognized quite early by Boltzmann; it was subsequently formulated by Hilbert. When solving it, it is found that a certain length l , which can be interpreted as the mean free path, enters naturally into the equations and that it can be defined only in terms of the velocity distribution.

It is by no means evident that it will be equal for problems of internal friction and heat conduction, as assumed hitherto. We shall revert to this point in Sec. 44.

The relations become particularly difficult in problems involving diffusion, but we do not propose to pursue this matter here. We feel, however, forced to consider a difficulty which is contained right in the foundations of probability and which becomes particularly acute in problems involving the mean free path. We shall explain its nature with reference to the throwing of dice by inquiring into the number of throws which are required in order to make it probable for a prescribed numeral, for example 1, to show up for the first time. This very imprecisely formulated question is traditionally answered by introducing the concept of "mathematical expectation" defined as

$$(14) \quad \bar{x} = \sum_{x=1}^{\infty} x w_x$$

where w_x denotes the probability that the prescribed number will first occur at the x 'th throw. In the case of dice w_x is known uniquely so that the sum in (4) can be calculated without difficulty. The computation is shown carried out in Problem III. 4 and it is found that the result is 6. This number can be said to be equal to the "mean free path of the numeral one" in a series of throws.

However, it is possible to imagine that this series has been extended rearwards and we can inquire as to when the numeral one most probably showed up for the last time before the present series of throws had been instituted. The mathematical expectation for this event is again given by eq. (14) and is, consequently, also equal to 6. Thus it appears that there is a contradiction between the "mean free path 6" and the interval of 12 throws between the last appearance of unity in the series extended rearwards and its first appearance in the series of throws following. It must be realized that such a contradiction is to be expected owing to the lack of regularity in chance and is conditioned by the arbitrary cut which was introduced into the sequence of events by selecting a definite start for the series of throws.

This cut finds its corollary in the kinetic theory of gases in the arbitrary cut which we introduced at $y = 0$ in Figs. 26 and 26 a; in a similar manner it is possible to argue that the collisions in the upper and lower half-spaces respectively correspond to the occurrence of the unity before and after the cut. Consequently it should not be surprising that the path covered by a molecule between two such collisions is equal to twice the distance which, in accordance with (14), we would define as the mathematical probability for the occurrence of a single collision; its approximate value was given in eq. (1).

CHAPTER IV

GENERAL STATISTICAL MECHANICS: COMBINATORIAL METHOD.

It has been shown in Chap. III that statistical considerations concerning states of equilibrium lead to simple general laws, the Maxwell-Boltzmann velocity distribution law, the equation of state of a perfect gas, and the values of its specific heats. On the other hand, towards the end of Chap. III we were forced to conclude that the considerations of irreversible processes and of any arguments involving the use of the mean free path required the introduction of fairly arbitrary assumptions and cumbersome modes of reasoning. Surveying the problem from a higher vantage point we are compelled to suppose that states of equilibrium can be treated in a simple and comprehensive manner, whereas it is necessary to employ more complex methods in order to obtain satisfactory solutions of problems associated with irreversible processes. A sketch of the latter will be given in Chap. V. The present Chap. IV rests on the work of Ludwig Boltzmann (1847—1906) and is astonishingly simple. In any case, the simplicity of the theory must not be judged from the point of view of elementary and trivial intuition but must be assessed from the point of view of higher mathematical lucidity. The concept of *entropy*, this fundamental pillar of thermodynamics which was conspicuously absent in the elementary Chap. III will only now be given its due place and in the form of *Boltzmann's principle* it will turn out to be the arithmetical consequence of a simple procedure of a combinatorial character.

28. Liouville's theorem, Γ -space and μ -space

Before applying any calculations involving probability it is necessary to make a decision about "events of equal probability". In the case of dice the equal probability attached to the numbers 1 to 6 is warranted by their geometrical shape and by the homogeneity of the material of which they are made. In card games an attempt is made to give an equal chance to each player by careful shuffling.

A. THE MULTIDIMENSIONAL Γ -SPACE (PHASE SPACE)

In the kinetic theory of gases it is expected that any initial regularities in the physical or velocity space will be quickly smoothed out owing to the enormous number of particles and to their intense motion. It is expected that a state of complete "molecular chaos" prevails so that it is possible to apply the calculus of probability to it. The gas is taken to be a mechanical system of F degrees of freedom. Let N be the number of molecules present, all being assumed identical and each possessing f degrees of freedom, so that $F = fN$. The system will be described by Hamilton's canonical equations given in Vol. I, eq. (41.4),

$$(1) \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad \dot{q}_k = +\frac{\partial H}{\partial p_k}, \quad H = H(p_1, \dots, p_F, q_1, \dots, q_F)$$

where the q_k 's denote the space coordinates, and the p_k 's represent the associated momentum coordinates of the molecules. It is assumed that the p 's and q 's are numbered (*cf.* e. g. Sec. 12 of Vol. I), so that a given molecule is associated with many values of k . The Hamiltonian function H represents the total energy of the system and is represented in terms of p and q ; the molecular interactions during collisions and the repulsion caused by the walls is assumed included in H . It is derived from Lagrange's function

$$L = L(q_1, \dots, q_F, \dot{q}_1, \dots, \dot{q}_F)$$

and is

$$H = \sum_k p_k \dot{q}_k - L, \quad p_k = \frac{\partial L}{\partial \dot{q}_k},$$

(see e.g. Vol. I, Sec. 41, eq. (1)).

The space of $2F$ dimensions defined by the p, q will be called the Γ -space or the phase space.¹ The instantaneous state of a system is represented by a point in that space, the point describing a path with the course of time. We do not, however, consider one single such Γ point but a large number of them, in particular, such as lie within a small spatial element

$$(2) \quad \Delta \Omega = \Delta p \Delta q \quad \text{with} \quad \begin{cases} \Delta p = \Delta p_1 \dots \Delta p_F \\ \Delta q = \Delta q_1 \dots \Delta q_F. \end{cases}$$

¹Following P. and T. Ehrenfest in their now standard Sec. 32 of the *Mathem. Enzykl.* Vol. IV, 4, which deals with the axiomatic foundations of the theory of gases. The letter Γ stands for "gas".

We further imagine that the curves emanating from it are marked in some way. They represent gases of identical molecular species and identical numbers of molecules, differing only somewhat as regards their micro-states; they cannot be distinguished macroscopically when they possess equal (or almost equal) energies.¹ The spatial element $\Delta \Omega$ changes its shape with time in a multiplicity of ways. We do assert, however, that it retains its volume in the process.

B. LIOUVILLE'S THEOREM

In order to arrange the proof in the simplest way possible we shall recall the results of ordinary kinematics given in Sec. 1 of Vol. II. The relative change of volume (volumetric dilatation) was shown to be given by

$$(3) \quad \Theta = \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z},$$

where x, y, z denoted arbitrary Cartesian coordinates fixed in space, ξ denoted the translation of a point of a body in the direction of x ; η, ζ denote translations in the directions y and z respectively. Let it now be imagined that this formula is extended to many dimensions, x and y corresponding to our p_1 and q_1 . The symbols ξ and η now denote the translations $\dot{p}_1 dt$ and $\dot{q}_1 dt$ in the Γ -space and $\partial \xi / \partial x + \partial \eta / \partial y$ transforms into

$$(4) \quad \left(\frac{\partial \dot{p}_1}{\partial p_1} + \frac{\partial \dot{q}_1}{\partial q_1} \right) dt \quad \text{or, in view of (1) into} \quad \left(-\frac{\partial}{\partial p_1} \frac{\partial H}{\partial q_1} + \frac{\partial}{\partial q_1} \frac{\partial H}{\partial p_1} \right) dt = 0.$$

In a similar manner pairs of succeeding terms of the sum in (3) imagined extended to the many dimensions under consideration can be combined to give zero. Such partial sums have no independent physical meaning because they represent projections on a coordinate plane placed in an arbitrary manner. However, the sum of them all, our dilatation Θ , is an invariant of the Γ -space.

It is clear that its geometrical meaning is given by $\frac{d\Delta \Omega}{dt} / \Delta \Omega$. It follows that

$$(5) \quad \frac{d\Delta \Omega}{dt} = 0, \quad \Delta \Omega = \text{const}$$

which is the statement of Liouville's theorem.

¹When it is prescribed that the energies are exactly equal, the number of dimensions of the Γ -space is reduced from $2F$ to $2F - 1$ owing to the existence of the energy equation $H(p, q) = \text{const}$; when the energies are approximately equal, $H_1 < H < H_2$, attention is centered on a shell-like portion of the space of $2F$ dimensions. The latter case occurs, for example, when the system is in thermal equilibrium with its surroundings.

Equation (5) can also be expressed as follows: If we imagine that the phase points of different micro-states are distributed over the Γ -space with uniform density, then their motion is like that of an *incompressible fluid*. Interpreting the density as a measure of the probability of finding a phase point in an element of the phase space we can conclude that the assumption that equal elements $\Delta\Omega$ enclose regions of equal probability is true at all times if it was true at any instant.

Consequently, it is possible to replace the initial conditions of mechanics by the statistical assumption that equal phase elements in the Γ -space are associated with equal probabilities. It corresponds to the assumption that the faces of dice are associated with equal probabilities. In the last resort the question as to whether or not this is true, i. e. whether the sides of dice and equal phase elements are, in fact, associated with equal probabilities, is a matter to be decided in the light of experience. The preceding argument could only demonstrate that the fundamental hypothesis of statistical mechanics is compatible with the equations of motion. Generally speaking, such statistical rules replace the initial conditions of the mechanics of systems whereas the equations of motion remain unaltered.

In conclusion we wish to stress once more that considerations involving the phase space are not related to a single gas but to a very large group of states of the gas under consideration or of an arbitrary system, which we imagine placed one beside the other and whose behavior is surveyed at the same time. Their states differ only microscopically; macroscopically they are alike.

C. EQUALITY OF PROBABILITY FOR THE PERFECT GAS

We now proceed to introduce the simple representation of the *phase space which is associated with a single molecule, or the " μ -space"*, according to the terminology introduced by P. and T. Ehrenfest. This has only $2f$ dimensions instead of the $2Nf$ of the Γ -space. It is thus six-dimensional in the case of a monatomic gas ($f = 3$), and ten-dimensional in the case of a diatomic gas composed of two rigid molecules ($f = 5$ owing to the two additional rotational degrees of freedom) etc.

It should be realized that the transition from the Γ - to the μ -space is possible only in cases when the mechanical system described by $H(p, q)$ possesses very special properties. So far we have assumed it to be quite general in its properties (arbitrary interactions between the molecules were admitted, e. g. repulsive forces at impact or cohesive forces at larger separations), but now we shall stipulate that the system corresponds to a *perfect gas* (volume of molecule $v_0 = 0$, hence no collisions, infinite mean free path). Conse-

quently the molecules are seen to move independently of each other; their phase spaces are separated and identical. H is the sum of the Hamiltonian functions corresponding to the individual molecules; it contains no terms which involve the coordinates of several molecules simultaneously so that during the motion the contributions of the individual molecules remain separate. From the fact that the volume elements in the Γ -space are constant and from the fact that the $\Delta \Omega_r$'s have equal probabilities (statistical hypothesis and Liouville's theorem) it is possible to conclude that, under the restrictions under consideration, the $\Delta \Omega_\mu$'s also have *equal probabilities*. This equality of probability forms the basis of *Boltzmann's combinatorial method* which will occupy our attention in this Section. The method *a priori* attaches equal probability to the elements

$$(6) \quad \Delta \Omega = \Delta p \Delta q \quad \text{with} \quad \begin{cases} \Delta p = \Delta p_1 \dots \Delta p_f \\ \Delta q = \Delta q_1 \dots \Delta q_f \end{cases}$$

The simplification of this formula compared with (2) should be noted: The subscripts 1, 2, ..., f in (6) refer to the same molecule, whereas in (2) the indices 1, 2, ..., F were enumerated through all the molecules.

When deriving eq. (2) we have assumed that the element $\Delta \Omega$ in the Γ -space was "small" and in (4) we applied the rules of differential calculus to the differences Δp , Δq . We have not, however, decided *how small* these elements should be. When describing the method, Boltzmann stresses repeatedly that it is necessary to stipulate finite elements Δp , Δq to make sure that the phase elements enclose numerous molecules; he does, however, always pass to the limits $\Delta p \rightarrow 0$, $\Delta q \rightarrow 0$ in the final results. In fact the answer to the question of "how small" these are to be was only given by the *quantum theory*.

In anticipation we wish to make the following remark: The product $p_k q_k$ (and hence the product $\Delta p_k \Delta q_k$) has the dimension of *an action*. This is quite clear as far as Cartesian coordinates are concerned ($[q] = \text{cm}$; $[p] = [m \dot{q}] = g \text{ cm sec}^{-1}$); it is, however, true of coordinates q with arbitrary dimensions, as seen from the relation $p = \partial L / \partial \dot{q}$ already quoted: Since L has the dimension *erg* it follows, in fact, that

$$\text{Dimension of } p_k q_k = \text{erg sec} = \text{action.}$$

Quantum theory teaches that the action is composed of elementary quanta in a discontinuous way. We shall see that Planck's constant h ("quantum of action") puts a *lower limit* to the size of $\Delta p_k \Delta q_k$ for every pair of coordinates p_k, q_k :

$$(7) \quad \Delta p_k \cdot \Delta q_k = h.$$

We shall make use of this result here. According to (6) the magnitude of an elementary cell of the phase space for a molecule of f degrees of freedom is

$$(8) \quad \Delta \Omega = h^f.$$

Two states p, q of a molecule whose representations lie within the same (properly bounded) elementary cell cannot be distinguished statistically. The cells $\Delta \Omega$ defined in this way constitute the *units* with which it is necessary to work when applying the combinatorial method. They correspond to the six events of equal probability in the game of dice.

To conclude, we must pose one more difficult question: Are these elementary cells large enough to contain, as required by Boltzmann, a large number of molecules? Normally this is not the case. Earlier reasoning was based on sufficiently large phase cells so that, generally speaking, there was no difficulty associated with too small numbers. This is, however, not *a priori* possible when the size of an elementary cell is determined by quantum mechanics. However, with the aid of a simple transformation it is possible to show that, nevertheless, a large number of elementary cells can be combined into a higher entity (Sec. 29 C) thus creating conditions in which the assumptions which are essential for the application of Boltzmann's method are satisfied. Thus the small size of the elementary cells constitutes no serious obstacle.

Darwin and Fowler have developed a different method which allows us to calculate the mean values even in cases when the numbers of molecules per cell are small. The results agree with those due to Boltzmann, but their field of applicability is larger because variations in energy from cell to cell impose limitations on the process of combining a large number of cells into one big cell. We shall refrain here from describing the derivation due to Darwin and Fowler,¹ but we shall use this method to advantage in another connection (*c.f.* Sec. 37).

The succeeding argument will be concerned mostly with the μ -space. However, the inability to discern between equal particles, demanded by quantum mechanics, will force us to revert to the Γ -space. It will be found that the formulae which are applicable in this case are quite similar to those which we are about to derive for the μ -space. They can, moreover, be derived in a similar manner, but we shall justify them in Sec. 36 with the aid of methods which can be traced to W. Gibbs, to whom, together with L. Boltzmann, we owe the existence of statistical mechanics. It will turn out that Gibbs' method is quite independent of the problems connected with the smallness of cells.

¹Reference should be made to: R. H. Fowler, 'Statistical Mechanics', Cambridge 1929 or to the small, but eminently readable booklet by M. Born: "Natural Philosophy of Cause and Chance", Oxford, 1949.

29. Boltzmann's principle

Boltzmann's principle interprets entropy in terms of the probability of states and expresses it in the terse formula:

$$(1) \quad S = k \log W.$$

So it stands carved out on Boltzmann's memorial in the Central Cemetery in Vienna, floating in the clouds over his majestic bust.

It is immaterial that Boltzmann himself never wrote down the equation in this form. This was first done by Planck, e. g. in the First Edition of his "Vorlesungen über die Theorie der Wärmestrahlung," 1906. The constant, k , was also introduced by Planck and not by Boltzmann. Boltzmann only referred to the proportionality between S and the logarithm of the probability of a state. The designation of "Boltzmann's principle" was advocated by Einstein for the reverse of (1), namely

$$(1 a) \quad W = e^{S/k}$$

in which S was considered to be known empirically, the quantity W being the unknown for which an expression was sought. According to it the "second part of the Second Law" signifies, as recognized already by Helmholtz, a transition from an artificial state of order to a more probable state of disorder.

The right-hand side of (1), particularly in the case of non-homogeneous systems, must be augmented by a constant, i. e. by a quantity which is independent of the parameters of state but is related only to the numbers of mols of the components. In general, eq. (1) has the form

$$(1 b) \quad S = k \log W + \text{const.}$$

In 1877 in connection with eq. (1) (which he had not written down) Boltzmann remarked,¹ still quite vaguely, that: "It might even be possible to calculate the probability of the different states from the ratio of the number of the various distributions, and this might lead to an interesting calculation of thermal equilibrium". Shortly afterwards² he added: "I do not think that one is justified in accepting this result without reservations as something evident, at least not until an exact definition of what we are to understand by the term 'most probable distribution' has been given." In the same paper Boltzmann intimates that the existence of Liouville's theorem constitutes a necessary limitation for the parameters of state which are to be chosen.

¹Vienna Academy, No. 39 in "Gesammelte Werke", p. 121.

²*Ibid*, No. 42, p. 193.

A. PERMUTABILITY AS A MEASURE OF THE PROBABILITY OF A STATE

Let us now consider a perfect gas; we assume that it consists of N molecules, that it is enclosed in a volume V , and that its total energy is U . If the number of degrees of freedom of a single molecule is f , then the phase space has $2f$ dimensions. We subdivide it into M cells,

$$1, 2, \dots, i, \dots, M.$$

In view of the fact that the cells are finite, as given in eq. (28.8), and in view of the prescribed finite values of U and V , M is a finite, if extremely large, number. At first, we distribute the molecules over the cells in a completely arbitrary fashion, and denote the numbers of molecules in the cells by

$$(2) \quad n_1, n_2, \dots, n_i, \dots, n_M,$$

where, naturally,

$$(2a) \quad \sum_i n_i = N.$$

Any distribution n_i determines a definite microstate of the gas. We now propose to inquire into the number of ways in which one specific microstate can be realized with N molecules distributed over M cells, and denote this number as the thermodynamic probability (or weight), W , of the state.¹ It is given by the permutability²

$$(3) \quad W = \frac{N!}{n_1! n_2! \dots n_M!}.$$

In order to clarify the concept we shall first consider a case when M and N are small.

¹It is essential to note that the thermodynamic probability, being an integer, has not been normalized to 1. Thus W depends on the size of the elementary cells.

²The term permutability (Germ. Permutabilität) was suggested by Boltzmann on p. 191 of the paper last cited. It is more graphic than the normally employed German term "Komplexion."

(It denotes the number of combinations of N elements taken n_1, n_2, \dots at a time—*Transl.*)

Example: $N = 2, \quad M = 2;$

$$\text{a)} \quad n_1 = 1, \quad n_2 = 1, \quad W = \frac{2!}{1!1!} = 2,$$

$$\text{b)} \quad n_1 = 2, \quad n_2 = 0, \quad W = \frac{2!}{2!0!} = 1,$$

$$\text{c)} \quad n_1 = 0, \quad n_2 = 2, \quad W = \frac{2!}{0!2!} = 1.$$

In connection with a) we can put the first molecule either in cell 1 or in cell 2; the placement of the second molecule is then determined; hence $W = 2$. In connection with b) and c) there is no freedom of choice; hence $W = 1$.

Although eq. (3) is known from the elementary theory of combinatorials to represent the polynomial coefficient, we shall, for completeness, prove it by induction from $N - 1$ to N . Thus we assume that eq. (3) has been proved to be true up to $N - 1$ molecules. In order to obtain W_N from the known expression for W_{N-1} it is necessary to consider one of the following W_{N-1} arrangements:

$$\begin{array}{llll} n_1 - 1, & n_2, \dots & n_M, & W_{N-1}^{(1)} = \frac{(N-1)!}{(n_1-1)!n_2!\dots} = \frac{(N-1)!n_1}{n_1!\dots n_M!}, \\ n_1, & n_2 - 1, \dots & n_M, & W_{N-1}^{(2)} = \frac{(N-1)!}{n_1!(n_2-1)!\dots} = \frac{(N-1)!n_2}{n_1!\dots n_M!}, \\ \dots & \dots & \dots & \dots \\ n_1, & n_2, \dots & n_M - 1, & W_{N-1}^{(N)} = \frac{(N-1)!}{n_1!\dots(n_M-1)!} = \frac{(N-1)!n_M}{n_1!\dots n_M!}. \end{array}$$

Each of these arrangements leads to the required arrangement (2) if the N th molecule is properly placed. Thus we obtain W_N if we add up the values of $W_{N-1}^{(i)}$ on the right-hand sides. In this way we obtain

$$W_N = \sum_{i=1}^M W_{N-1}^{(i)} = \frac{(N-1)!}{n_1!\dots n_M!} (n_1 + n_2 + \dots + n_M) = \frac{N!}{n_1!\dots n_M!},$$

which is identical with (3) in view of (2 a).

The use of Stirling's formula suggests itself in order to simplify the expression for W . In our case (since N is very large) it is sufficient to use the approximation

$$(4) \quad N! = \left(\frac{N}{e}\right)^N.$$

A slightly more accurate estimate would give

$$(4 \text{ a}) \quad N! = (2\pi N)^{\frac{1}{2}} \cdot \left(\frac{N}{e}\right)^N.$$

We may note the elementary derivation of eq. (4) (logarithmic curve replacing an inscribed step-like series):

$$\begin{aligned} \log N! &= \log 1 + \log 2 + \dots + \log N \approx \int_1^N \log x \, dx \\ &= [x(\log x - 1)]_{x=N} - [x(\log x - 1)]_{x=1} = N(\log N - 1) + 1 \approx N \log \frac{N}{e}. \end{aligned}$$

Taking the *numerus logarithmi* we are led to eq. (4).

We shall use the approximation in (4) to represent the numbers n_i as well as N which implies that they are assumed to be large too. This is undoubtedly permissible if the series of n_i 's includes any large numbers, because the small numbers are then negligible and can be replaced by unities. However, the method fails when all n_i 's are small numbers. Unfortunately this is exactly the case where Boltzmann's method is applicable and if the argument is associated with the phase elements h' . It is, for example, possible to show that in the case of a perfect gas under normal conditions out of 30,000 quantum cells, in round figures, at most only one contains a molecule at all (*cf.* Sec. 37). It is, in this case, most unlikely that two molecules will be contained in one cell. Fortunately, as mentioned at the end of Sec. 28, this presents no serious obstacle. We shall, therefore, disregard this difficulty at first, and perform the calculation as if the n_i 's included large numbers. At the conclusion of the argument we shall investigate the changes which must be effected when a large number of elementary cells are combined into one big cell, so that the n_i 's really include large numbers.

Following Boltzmann we substitute for $n_i!$ the approximations

$$(4 \text{ b}) \quad n_i! = \left(\frac{n_i}{e}\right)^{n_i}, \quad \log n_i! = n_i (\log n_i - 1)$$

into eq. (3), and obtain

$$(5) \quad \log W = \text{const} - \sum_{i=1}^M n_i \log n_i.$$

The constant in (5) includes all terms which are independent of the n_i 's: hence we have

$$(5\ a) \quad \text{const} = N (\log N - 1) + \sum_{i=1}^M n_i = N \log N.$$

B. THE MAXIMUM OF PROBABILITY AS A MEASURE OF ENTROPY

We now ask for the "most frequent arrangement" of molecules, or, in other words, we shall try to calculate the values of the n_i 's for which W becomes a *maximum*. In order to do this we shall apply a virtual variation δn_i to n_i , taking into account that, according to the condition in eq. (2 a), we must have

$$(6) \quad \sum \delta n_i = 0.$$

Hence, we obtain from eq. (5) that

$$\delta \log W = - \sum \delta n_i (\log n_i + 1).$$

In view of condition (6) the unity in the brackets can be omitted. As long as the n_i 's only have to satisfy condition (6), the criterion of maximum probability becomes

$$(7) \quad \delta \log W = - \sum \delta n_i \log n_i = 0.$$

This means, in agreement with our statistical hypothesis, that all the n_i 's must be equal, because in view of (6), eq. (7) can be satisfied only if $n_1 = n_2 = \dots$

It is, however, necessary to subject the n_i 's to another condition. Since the total energy is prescribed (see the beginning of this section), it follows that

$$(8) \quad U = \sum n_i \varepsilon_i,$$

where U denotes the sum taken over all the cells, and ε_i is the total energy of the molecules in a cell whose coordinates in the phase space are p_i, q_i . The value of ε_i changes from cell to cell, but it must be considered fixed within a cell by quantum theory. When n_i is varied in (8) at constant values of U and ε_i , we obtain

$$(8\ a) \quad \sum \varepsilon_i \delta n_i = 0.$$

In order to satisfy the two conditions (6) and (8 a) simultaneously it is best to make use of the elegant method of Lagrange's multipliers, as explained in Vol. I, eq. (12.5). Thus (7) is replaced by

$$(9) \quad \delta \log W = - \sum \delta n_i (\log n_i + \alpha + \beta \epsilon_i) = 0.$$

Since the multipliers α and β are yet to be fixed, we may regard the n_i 's as independent. In this manner eq. (9) leads to:

$$(10) \quad \log n_i = -\alpha - \beta \epsilon_i, \quad n_i = e^{-\alpha} \cdot e^{-\beta \epsilon_i}.$$

Substituting this value of $\log n_i$ into (5), we obtain the maximum

$$(11) \quad \log W_{max} = \text{const} + \alpha \sum n_i + \beta \sum n_i \epsilon_i.$$

Here α and β refer to the state of the whole system, and not to that of a single cell so that we were justified in placing them in front of the summation sign which is extended over i .

Making use of (2 a) and (8) and inserting the value of the constant from (5 a), we obtain simply

$$(12) \quad \log W_{max} = N \log N + \alpha N + \beta U.$$

It will be shown later that the maximum of W which we have just calculated is extremely large as compared with all states associated with only a slightly different value of n_i . For this reason we are justified in identifying the state of maximum probability with the "real state" which we would expect to find experimentally. If this is so, then, according to Boltzmann's principle (1), eq. (12) represents the value of S/k . It follows at once from (11) that for a process during which N remains constant but during which the energy U is changed by external interaction, we must have

$$(13) \quad dS/k = N d\alpha + U d\beta + \beta dU.$$

The changes in α and β are related to each other by the condition

$$(14) \quad N = \sum n_i = e^{-\alpha} \sum e^{-\beta \epsilon_i} = \text{const.}$$

The sum in eq. (14) is the so-called *partition function* (in German *Zustandssumme*):

$$(15) \quad Z_0 = \sum e^{-\beta \epsilon_i}.$$

More precisely, the sum in eq. (14) refers to the partition function in the μ -space. Its importance stems from the circumstance that all thermodynamic properties can be derived from it (cf. also eq. (33.14)). According to (14), we have $\alpha = \log (Z_0/N)$. Thus eq. (12) becomes:

$$(12') \quad \frac{1}{k} S = \log W_{max} = N \log Z_0 + \beta U$$

and the arrangement of molecules and the energy follow from eqs. (8) and (10), respectively, and are

$$(16) \quad n_i = e^{-\alpha} e^{-\beta \epsilon_i} = -\frac{N}{\beta} \frac{\partial \log Z_0}{\partial \epsilon_i}$$

$$U = e^{-\alpha} \sum \epsilon_i e^{-\beta \epsilon_i} = -N \frac{\partial \log Z_0}{\partial \beta}.$$

C. THE COMBINING OF ELEMENTARY CELLS

There is no doubt about the fact that the approximation in Section B is inadmissible. The results, however, are correct. It is easy to show that they apply at least to cases when the values of energy differ little from cell to cell. It is then possible to regard them as being constant over a large region and to combine a large number of cells, say κ , into a higher entity, a *macro-cell*, as we shall call it.

Let N_1, N_2, \dots, N_m denote the numbers of molecules per macro-cell. In analogy with eqs. (2 a) and (8), we may write

$$(17) \quad \sum_i N_i = N, \quad \sum_i N_i \bar{\epsilon}_i = U.$$

The summation \sum_i extends over all macro-cells and $\bar{\epsilon}_i$ denotes the mean energies in them. The thermodynamic probability of a given distribution of molecules over the macro-cells is obtained by summing up all elementary probabilities associated with the arrangements of molecules which lead to the same distribution over the macro-cells. Hence

$$(18) \quad W' = \frac{N!}{N_1! \dots N_m!} \sum_{(n)} \frac{N_1!}{n_{11}! \dots n_{1\kappa}!} \cdot \frac{N_2!}{n_{21}! \dots n_{2\kappa}!} \dots$$

Mathematically this means that the sum must be taken over all arrangements of N_1, N_2 etc. Consequently the sum changes into a product of sums

$$W' = \frac{N!}{N_1! \dots N_m!} \prod_j \sum \frac{N_j!}{n_{ji}! \dots n_{j\kappa}!}$$

which can all be calculated with the aid of the binomial theorem. The factors are: $\kappa^{N_1}, \kappa^{N_2}, \dots$, and we have:

$$(19) \quad W' = \frac{N!}{N_1! \dots N_m!} \kappa^N.$$

This equation differs from (3) only by the term κ^N . It is now possible to apply Boltzmann's methods to W' because we can choose κ large enough for the N_j 's to become large numbers. In analogy with eq. (10), we obtain

$$(20) \quad N_j = e^{-\alpha - \beta \bar{e}_j},$$

Furthermore, eq. (12) is replaced by

$$(21) \quad \log W'_{\max} = N \log \kappa N + \alpha N + \beta U$$

and (14) is replaced by

$$(22) \quad N = e^{-\alpha} \sum_j e^{-\beta \bar{e}_j}.$$

We can now write down an expression for the partition function (15) in the μ -space

$$(23) \quad Z_0 = \kappa \sum_j e^{-\beta \bar{e}_j},$$

and it follows from (22) that

$$e^{\alpha} = \frac{Z_0}{\kappa N}, \quad \alpha = \log Z_0 - \log \kappa N.$$

Accordingly, we are once more led to eq. (12')

$$(21') \quad \log W'_{\max} = N \log Z_0 + \beta U = \log W_{\max},$$

i. e. a value which is independent of κ . The same can be said about the remaining quantities, and the results are seen to be the same as for $\kappa = 1$. This time, however, the justification for them is more sound.

30. Comparison with thermodynamics

A. CONSTANT VOLUME PROCESS

If $V = \text{const}$, the division of the phase space into elementary cells remains unaltered and it follows from (29.14) by logarithmic differentiation that

$$(1) \quad 0 = -d\alpha - d\beta \frac{\sum \varepsilon_i e^{-\beta \varepsilon_i}}{\sum e^{-\beta \varepsilon_i}} = -d\alpha - d\beta \frac{U}{N}$$

Thus eq. (29.13) reduces to

$$(2) \quad dS = k \beta dU.$$

Now, in accordance with the Second Law (cf. Sec. 6), we have

$$(2a) \quad dS = dQ_{\text{rev}}/T = dU/T.$$

On comparing (2) with (2a), we obtain

$$(3) \quad \beta = \frac{1}{kT}.$$

We shall see later that this fundamental relation remains true also in cases B and C.

B. GENERAL PROCESS PERFORMED BY A GAS IN THE ABSENCE OF EXTERNAL FORCES

We assume that the process consists not only of a change in energy, dU , but also of a change in volume, dV . Thus in addition to the phase cells 1 to M we shall have cells $M+1$ to M' (i. e. if dV is positive; if dV is negative cells $M'+1$ to M will disappear). The corresponding change of $\sum_{i=1}^M$ will be denoted by

$$(4) \quad d\Sigma = \sum_{i=M+1}^{M'} \left(\text{or } = - \sum_{i=M'+1}^M \right)$$

where, in the additional terms in the sum, the α 's and β 's which are associated with the whole system, and not with the single phase elements, retain the same values in the same way as in the original terms. Differentiating eq. (29.14) logarithmically we obtain, instead of (1), that

$$(4a) \quad -d\alpha - d\beta \frac{U}{N} + \frac{d\Sigma}{\Sigma} = 0.$$

In the absence of external forces we shall have

$$(4\ b) \quad \frac{d\Sigma}{\Sigma} = \frac{dV}{V}$$

as will be proved shortly. Substitution of (4 a, b) into (29.13) yields

$$dS = k \beta dU + k N \frac{dV}{V}.$$

On comparing with the statement of the Second Law

$$dS = \frac{dQ_{rev}}{T} = \frac{dU + p dV}{T}$$

which now differs from (2 a) we obtain, in addition to (3), the *equation of state of a perfect gas*

$$(5) \quad \frac{p}{T} = \frac{k N}{V} \quad \text{as well as} \quad p v = R T.$$

The latter is obtained when the number of molecules, N , is chosen equal to the Loschmidt-Avogadro number, i. e. when V is made equal to the molar volume, v .

In order to prove eq. (4 b) we are obliged to examine more closely the structure of the phase element $\Delta \Omega$. If x, y, z denote the coordinates of a molecule (e. g. of its center of gravity) in the physical space, we may put

$$(5\ a) \quad \Delta \Omega = \Delta \tau \Delta \Omega', \quad \Delta \tau = \Delta x \Delta y \Delta z$$

where $\Delta \Omega'$ denotes the volume element of all momentum coordinates of the molecule, as well as of the space coordinates of its internal degrees of freedom, if they exist. Our statement in Sec. 28 that the $\Delta \Omega$'s are equal to each other, being all equal to h^f , can now be supplemented by the statement that the $\Delta \tau$'s can also be chosen equal, because

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = \frac{\partial^2 H}{\partial x \partial p_x} + + = \left(\frac{\partial K_x}{\partial p_x} + + \right) = 0$$

since the external force K has been assumed to be equal to zero.

In this case the values of energy ϵ_i are independent of x, y, z . Thus in the summation with respect to i in eq. (29.14) for each $\Delta \Omega'$ there are as many equal terms as there are space cells, $\Delta \tau$. This number is $V/\Delta \tau$. Consequently, eq. (29.14) may be replaced by

$$(6) \quad N = \sum_i n_i = e^{-\alpha} \frac{V}{\Delta \tau} \sum_i e^{-\beta \epsilon_i}$$

where the subscript j denotes now the summation over the phase elements $\Delta \Omega_j'$ only. By logarithmic differentiation of (6), subject to the requirement that N is constant, and for varying V , α , and β (the summation over j remains unaffected thereby), it follows that

$$(6\ a) \quad -d\alpha - d\beta \frac{U}{N} + \frac{dV}{V} = 0.$$

The additional term dV/V in this equation corresponds to the term $d\Sigma/\Sigma$ in eq. (4 a) and hence eq. (4 b) is seen to have been proved.

C. A GAS IN A FIELD OF FORCES; THE BOLTZMANN FACTOR

It was found in (5) that the pressure was uniform throughout the gas. This, however, is true only when the energy ϵ is independent of the space coordinates. When external forces are present (we assume that they possess a single-valued potential $\Phi(x, y, z)$, since otherwise no state of equilibrium could exist, *cf.* Vol. II, end of Sec. 7), we assume

$$(7) \quad \epsilon = \Phi(x, y, z) + \epsilon',$$

where ϵ' denotes that portion of the energy of a molecule (inclusive of rotational energy, etc.) which is independent of x, y, z . The problem is now seen to depend on the change in volume dV (i. e. on how we insert the volume elements $\Delta \tau$).

In the present case we shall restrict ourselves to an isochoric process as in Sec. A. Since the external forces are independent of the process, and since $V = \text{const}$, we obtain

$$(8) \quad \beta = \frac{1}{kT}$$

in exactly the same way as in eqs. (1) to (3); here k is a constant of the system, and independent of the space coordinates, in spite of the fact that Φ does depend on them. *Hence T is independent of the space coordinates.* In particular this is also true of the atmosphere in the gravitational field, on condition that it is in thermal equilibrium. (Some meteorologists have doubted this in the past.)

On the other hand, however, the pressure and density depend on the space coordinates. The latter follows directly from eq. (29.10). The number of particles contained in a cell $\Delta \tau$ becomes

$$(9) \quad n = e^{-\alpha - \Phi/kT} \cdot \sum e^{-\epsilon_j/kT}$$

if the value of ϵ_j is taken from (7) and if the summation is extended over the phase space excluding the cells in the physical space. Multiplying by the mass of the molecule, m , and dividing by $\Delta \tau$, we obtain the *density* ρ . Comparing with the density ρ_0 at the reference level of Φ , we obtain

$$(10) \quad \frac{\rho}{\rho_0} = e^{-\Phi/kT}.$$

This is the Boltzmann factor from eqs. (23.16) and (23.17) of which use has already been made on several occasions.

D. THE MAXWELL-BOLTZMANN VELOCITY DISTRIBUTION LAW

Maxwell's velocity distribution law for a monatomic gas with zero external forces can be deduced in an equally simple manner. We consider a definite cell $\Delta \omega_j$ of the momentum space and an element $\Delta \tau$ of the physical space (which is arbitrary when no forces are acting). The momentum cell is specified by the coordinates

$$p_x = m \xi, \quad p_y = m \eta, \quad p_z = m \zeta,$$

where ξ, η, ζ denote the velocity components, as before.

In the case of a monatomic gas, when the inner degrees of freedom are absent, the cell is associated with the energy

$$\epsilon = \frac{m}{2} (\xi^2 + \eta^2 + \zeta^2).$$

Substituting the universal relation $\beta = 1/k T$ into eq. (29.10), we obtain

$$(11) \quad n_i = e^{-\alpha} \exp \left\{ -\frac{m}{2} (\xi^2 + \eta^2 + \zeta^2)/k T \right\}.$$

Summation over all elements of space, yields

$$(11 \text{ a}) \quad n_j = \sum_i n_i = \frac{V}{\Delta \tau} e^{-\alpha} \exp \left\{ -\frac{m}{2} (\xi^2 + \eta^2 + \zeta^2)/k T \right\}.$$

Putting

$$(11 \text{ b}) \quad \frac{n_j}{N} = F_i \Delta \omega_i,$$

we see that $F_i \Delta \omega_i$ denotes the probability that an atom selected arbitrarily from among the N present will belong to the momentum cell $\Delta \omega_i$ or, in other words, that its velocity will be ξ, η, ζ .

Since $N = \sum n_j$ eqs. (11 a) and (11 b) give

$$(12) \quad F = \frac{\exp \{ \quad \}_j}{\sum_i \exp \{ \quad \}_j \Delta \omega_j}.$$

The factor $\Delta \omega_j$ with which F_j was multiplied in eq. (11 a) has been here taken into the denominator and each individual term of the sum has been multiplied by it. (Since the phase cells $\Delta \Omega_j$ are equal, and since the $\Delta \tau$'s from eq. (5 a) have been assumed equal, it follows that the momentum cells $\Delta \omega_j$ are equal.)

In order to evaluate the denominator in (12), which, as it is easy to see, is closely related to our partition function Z_0 in (29.15), we go over to the limit $\Delta \omega_j \rightarrow 0$. The denominator then becomes equal to:

$$(12 \text{ a}) \quad m^3 \int_{-\infty}^{+\infty} e^{-m \xi^2 / 2 k T} d\xi \cdot \int_{-\infty}^{+\infty} e^{-m \eta^2 / 2 k T} d\eta \cdot \int_{-\infty}^{+\infty} e^{-m \zeta^2 / 2 k T} d\zeta.$$

Performing the integrations indicated in (23.5 a), we find that

$$(12 \text{ b}) \quad m^3 \left(\frac{2 \pi k T}{m} \right)^{3/2} = (2 \pi m k T)^{3/2}.$$

Consequently, it follows from (12) that

$$(13) \quad F = (2 \pi m k T)^{-3/2} \exp \left\{ -\frac{m}{2} (\xi^2 + \eta^2 + \zeta^2) / k T \right\}.$$

This expression is identical with that in (23.8) with the only formal difference that our present F refers to an element of momentum space, whereas the previous one referred to one in the velocity space; this explains the occurrence of the factor $m^{3/2}$ in the numerator of (23.8) as against its presence in the denominator of (13).

We can now state the following conclusions: It is seen that the path followed in the derivation of Maxwell's velocity distribution law proves to be a *royal* one. It leads, moreover, directly to Boltzmann's generalization for polyatomic molecules which was formulated at the end of Sec. 23. In fact the preceding derivation can be applied without change to a polyatomic gas possessing internal degrees of freedom as well as to a gas in a potential field because the respective additional factors will cancel each other in the numerator and in the denominator of eq. (12).

E. GASEOUS MIXTURES

We may restrict ourselves to the case of two gases accommodated in the same volume V and possessing a total energy U ; let the masses of their molecules be m_1, m_2 , the numbers of molecules being N_1, N_2 . We introduce a phase space for each of the gases; the numbers of molecules per cell in the phase space will be denoted by n_{i1} and n_{j2} respectively.

According to (29.3) the permutabilities are

$$W_1 = \frac{N_1!}{\prod n_{i1}!}, \quad W_2 = \frac{N_2!}{\prod n_{j2}!}.$$

Since both distributions are independent of each other, the thermodynamic probability for the mixture is

$$W = W_1 \times W_2.$$

Applying Stirling's formula, eq. (29.5) becomes

$$(14) \quad \log W = \text{const} - \left(\sum_i n_{i1} \log n_{i1} + \sum_j n_{j2} \log n_{j2} \right).$$

The calculation of the maximum of W is subject to the following three conditions:

$$(15) \quad \sum n_{i1} = N_1, \quad \sum n_{j2} = N_2, \quad \sum n_{i1} \varepsilon_{i1} + \sum n_{j2} \varepsilon_{j2} = U.$$

In order to satisfy them it is necessary to introduce three Lagrangian multipliers $\alpha_1, \alpha_2, \beta$. From (14), we have

$$\delta \log W = - \sum_i \delta n_{i1} (\log n_{i1} + \alpha_1 + \beta \varepsilon_{i1}) - \sum_j \delta n_{j2} (\log n_{j2} + \alpha_2 + \beta \varepsilon_{j2})$$

and consequently:

$$\log n_{i1} = -\alpha_1 - \beta \varepsilon_{i1}, \quad \log n_{j2} = -\alpha_2 - \beta \varepsilon_{j2}.$$

Substitution into (14) yields

$$\log W_{\max} = \text{const} - \alpha_1 \sum_i n_{i1} - \alpha_2 \sum_j n_{j2} - \beta \left(\sum_i n_{i1} \varepsilon_{i1} + \sum_j n_{j2} \varepsilon_{j2} \right),$$

and in view of (15), we have

$$(16) \quad \log W_{\max} = \text{const} - \alpha_1 N_1 - \alpha_2 N_2 - \beta U.$$

This is Boltzmann's principle for the representation of the entropy of a mixture. It follows once more that

$$(17) \quad \beta = \frac{1}{k T}$$

as before. This means that the *temperature of the mixture is uniform* and changes of volume satisfy the *equation of state of a mixture* in the form:

$$(18) \quad p = p_1 + p_2, \quad p_1 = \frac{k N_1 T}{V}, \quad p_2 = \frac{k N_2 T}{V},$$

where p_1 and p_2 denote the partial pressures exerted by each of the components, on the assumption that it alone fills the volume V .

The velocity distributions are also superimposed on each other, as are the pressures, and each separately retains the Maxwellian form.

31. Specific heat and energy of rigid molecules

In spite of the fact that the concept of a rigid molecule, just as the concept of a rigid body in mechanics, is undoubtedly unacceptable from the physical point of view, it is useful to perform a careful study of the thermal behavior of a gas consisting of rigid molecules because such a study will, in particular, determine for us the limits of validity of classical statistical mechanics.

The difficulties attendant on such a task were mentioned by *Lord Kelvin* in his Baltimore Lectures in 1884, Appendix B, when he referred to them as the *Nineteenth Century Clouds over the Dynamical Theory of Heat*; they led him to the conclusion, most revolutionary at the time, that the principle of equipartition would have to be abandoned. We shall see in Secs. 33 to 35 that the physics of the Twentieth Century, the quantum theory in particular, was able to throw brilliant light on all hitherto dark regions of statistical mechanics.

A. THE MONATOMIC GAS

In view of its lack of a definite structure, the assumption that a monatomic molecule is rigid presents, as yet, no difficulties. We have found in Sec. 22, eqs. (6 a) and (6 b), that the molar energy and the molar specific heats of a monatomic gas are given by

$$(1) \quad u = \frac{3}{2} R T, \quad c_v = \frac{3}{2} R, \quad \frac{c_p}{c_v} = \frac{5}{3}$$

and it only now remains to show how its entropy, already known to us from thermodynamics, fits into the statistical method of expression.

According to Boltzmann's principle, eq. (29.12) becomes

$$(2) \quad S = k N \log N + \alpha k N + \beta k U.$$

The last term on the right-hand side of this equation is a constant and equals $3/2 k N$ (because $k = 1/T$, and $U = 3/2 N k T$); it can, therefore, be combined with the first term. Thus eq. (2) simplifies to

$$(2 a) \quad S = k N \left(\frac{3}{2} + \alpha + \log N \right).$$

The value of α for a monatomic gas can be deduced from the argument in Sec. 30 D. We begin with

$$(3) \quad N = \sum n_i = e^{-\alpha} \frac{V}{\Delta \tau} \sum_j \exp \{ \quad \}_j.$$

The term in the brackets $\{ \}$ has the same meaning as in (30.12) and, as before, the summation over j extends only over the momentum cells. The factor $V/\Delta \tau$ denotes the number of space cells contained in the volume V , and hence the multiplicity with which each term j must be counted in order to supplement the original summation over i with that over j . Multiplying both sides of (3) by

$$\Delta \Omega = \Delta \tau \cdot \Delta \omega$$

and putting $\Delta \omega$ as a factor in the summation sign \sum_j , in accordance with the remark to (30.12), we obtain

$$(4) \quad e^{\alpha} = V \sum_j \exp \{ \quad \}_j \Delta \omega_j / N \Delta \Omega.$$

The sum in the above equation is identical with the denominator in (30.12); hence, according to (30.12 a/12 b) it is equal to

$$(2\pi m k T)^{3/2},$$

and we conclude from (4) that

$$(4 a) \quad \alpha = \log V + \frac{3}{2} \log T + \frac{3}{2} \log (2\pi m k) - \log (N \Delta \Omega).$$

Substituting this value of α into (2 a), we obtain

$$(5) \quad S = k N \left(\log V + \frac{3}{2} \log T \right) + C.$$

This is the thermodynamic entropy equation, known to us from Sec. 5, eq. (10), except that it refers here to N particles of a monatomic gas, rather than to one gram.

It must, however, be realized that our present result by far transcends that obtained in thermodynamics because the constant C now has a definite numerical value. It is, namely, equal to the product of $k N$ and the constant from eq. (4 a), augmented by the constant in eq. (2 a):

$$(5 a) \quad C = \frac{3}{2} k N \left(1 + \log (2 \pi m k) - \frac{2}{3} \log \Delta \Omega \right).$$

According to Boltzmann's theory, $\Delta \Omega$ is undefined, but according to quantum theory we have $\Delta \Omega = h^3$. Thus for a monatomic gas we put $\Delta \Omega = h^3$, and we obtain from eq. (5 a) that

$$(5 b) \quad C = k N \log \frac{(2 \pi m k)^{3/2} e^{3/2}}{h^3}.$$

Substituting this value into (5) and simplifying, we have

$$(5 c) \quad S_{transl} = k N \log \left\{ V [(2 \pi m k T)^{3/2} e^{3/2} / h^3] \right\}$$

or per mol,

$$(6) \quad s_{transl} = R \log \left\{ v [(2 \pi m k T)^{3/2} e^{3/2} / h^3] \right\}.$$

The subscript attached to s indicates that the validity of this expression is not restricted to monatomic gases; it also represents the contribution of the translatory motion of a polyatomic molecule to the entropy of the gas. It is obvious that an extrapolation to $T = 0$ is not permissible because the ideal gas state ceases to exist at $T = 0$. Thus there is no contradiction between eq. (6) and Nernst's Third Law.

Equation (6) was first deduced by Sackur.¹ At about the same time, and independently, Tetrode² established an equation which formally differed only slightly from it. We shall return to this famous Sackur-Tetrode equation in Sec. 37 A.

¹O. Sackur, Ann. d. Phys. **36**, 958 (1911); **40**, 67 (1913).

²H. Tetrode, Ann. d. Phys. **38**, 434 (1912).

At this stage we must draw attention to two points. First, the constant in the expression for entropy contains the finite volume of the phase cells. We are not allowed arbitrarily to choose the size of a phase element, as was still done by Boltzmann. Moreover, the constant in the expression for entropy determines its size. Unfortunately, this also implies that the assumption that single cells contain large numbers of molecules, n , can never be satisfied, because of the small value of a quantum of action (*c/.* here Sec. 29 C).

Secondly, the entropy must be proportional to the number of molecules N . According to (5 c) this will be the case when we subtract from it the value $N \log N$. Since the volume contained in the logarithm in eq. (5 c) is proportional to the number of molecules at constant temperature, the entropy would increase more than proportionately, if the term $N \log N$ had not been subtracted from it. However, this difficulty can only be removed with the aid of quantum theory. At this stage reference should be made to the derivation of Tetrode's equation in Sec. 37 A.

B. GAS COMPOSED OF DIATOMIC MOLECULES

We imagine a rigid diatomic molecule in the form of a dumb-bell: The two atoms are assumed to be point-masses which are connected by a massless link of length l .¹ In addition to the two atomic masses, m_1, m_2 , the model is assumed to possess two equal moments of inertia about axes at right angles to the line connecting m_1 and m_2 ; the moment of inertia about the axis joining m_1 to m_2 is assumed to be equal to zero. The same is true about any linear arrangement of atoms, such as occurs, for example, in the case of CO_2 .

As is well known, the model possesses 5 degrees of freedom corresponding to the coordinate of the center of gravity x, y, z and the two angular coordinates θ, ψ which describe the position of the axis of the dumb-bell with respect to an arbitrary reference position. The third angle, ϕ , which measures the angular deflection about the axis of the dumb-bell does not count because it corresponds to a zero angular momentum. According to eq. (35.12) in Vol. I, the kinetic energy of rotation is

$$(7) \quad \epsilon_{rot} = \frac{1}{2} I (\dot{\theta}^2 + \sin^2 \theta \times \dot{\psi}^2),$$

because here $C = 0$ and $A = I$. The symbol I denotes the moment of inertia of the body and it should be noted that the symbol Θ was used in Vol. I.

¹ Quantum theory demonstrates that we may here neglect the finite extent of atomic nuclei and of the electrons.

This change in notation is made necessary because we shall need the symbol Θ to denote the characteristic temperature of the rotator in Sec. 33.

The momentum coordinates are thus given by

$$(7 \text{ a}) \quad p_\theta = \frac{\partial \epsilon_{rot}}{\partial \dot{\theta}} = I \dot{\theta}, \quad p_\psi = \frac{\partial \epsilon_{rot}}{\partial \dot{\psi}} = I \sin^2 \theta \times \dot{\psi}.$$

The phase space has 10 dimensions. The phase element can be written as

$$(7 \text{ b}) \quad \Delta \Omega = \Delta x \Delta y \Delta z \cdot m^3 \Delta \xi \Delta \eta \Delta \zeta \cdot \Delta \theta \Delta \psi \cdot \Delta p_\theta \Delta p_\psi.$$

For the following argument it is far more convenient to transform ϵ_{rot} into a sum of squares with constant coefficients. As is well known, this can be achieved by introducing the angular velocities about two mutually perpendicular axes, both being at right angles to the axis of the dumb-bell:

$$(7 \text{ c}) \quad \omega_1 = \dot{\theta} = p_\theta/I, \quad \omega_2 = \sin \theta \times \dot{\psi} = p_\psi/I \sin \theta.$$

Here ω_1, ω_2 represent "non-holonomic" velocities, see Vol. I, Sec. 35.4; the quantities $I \omega_1, I \omega_2$ were designated as "momentoids" by Boltzmann, but we prefer to call them "impulsoids." The phase element of the impulsoid space of the $I \omega_1, I \omega_2$ differs from that of the momentum space of the p_θ, p_ψ by the functional determinant:

$$(7 \text{ d}) \quad \frac{\Delta p_\theta \Delta p_\psi}{\Delta(I \omega_1) \Delta(I \omega_2)} = \begin{vmatrix} \frac{\partial p_\theta}{\partial(I \omega_1)} & \frac{\partial p_\theta}{\partial(I \omega_2)} \\ \frac{\partial p_\psi}{\partial(I \omega_1)} & \frac{\partial p_\psi}{\partial(I \omega_2)} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & \sin \theta \end{vmatrix} = \sin \theta.$$

The transformation (7 c) changes (7 b) into

$$(8) \quad \Delta \Omega = \Delta x \Delta y \Delta z \cdot m^3 \Delta \xi \Delta \eta \Delta \zeta \cdot \sin \theta \Delta \theta \Delta \psi \cdot \Delta \omega_1 \Delta \omega_2,$$

and (7) assumes the following form, commonly used in mechanics:

$$(8 \text{ a}) \quad \epsilon_{rot} = \frac{1}{2} I (\omega_1^2 + \omega_2^2).$$

The extension of the phase space from 6 to 10 dimensions does not affect the universally valid results derived in Sec. 29, i. e. neither the meaning of $\beta = 1/k T$, or the equation of state, or the Boltzmann factor. However, the additional degrees of freedom demand their share of energy, each of a value of $\frac{1}{2} R T$, at least in accordance with the classical calculation, to be given here. Hence eqs. (1) must be changed to

$$(9) \quad u = \frac{5}{2} R T, \quad c_v = \frac{5}{2} R, \quad \frac{c_p}{c_v} = 1 + \frac{2}{5} = \frac{7}{5}.$$

In order to prove it we express the molar energy in the form

$$(10) \quad \frac{u}{L} = - \frac{\partial}{\partial \beta} \log \sum e^{-\beta \epsilon_i}$$

in accordance with the universal formula (29.16). We change from the summation over i , which extends over the physical and the momentum spaces, to the summation over j which is only concerned with the momentum space. In analogy with (3) every term must be multiplied by the number of cells in the physical space. This is now equal to:

$$\frac{v \cdot 4\pi}{\Delta \tau \cdot \Delta \sigma} \text{ with } \begin{cases} \Delta \tau = \Delta x \Delta y \Delta z, & \Delta \sigma = \sin \theta \Delta \theta \Delta \psi, \\ 4\pi = \int \sin \theta \Delta \theta \Delta \psi = \text{surface of unit sphere.} \end{cases}$$

We can now extend the numerator and the denominator of the preceding fraction with the aid of the five-dimensional $\Delta \omega$ of the momentum space, so that in the denominator we obtain $\Delta \Omega = \Delta \tau \Delta \sigma \Delta \omega = h^5$, and $\Delta \omega$ in the numerator can be taken into the Σ as $\Delta \omega_j$. Thus the partition function Z_0 for (10) becomes

$$(11) \quad Z_0 = \frac{4\pi v}{\Delta \Omega} \sum e^{-\beta \epsilon_j} \Delta \omega_j.$$

The contribution of the rotation to the distribution function can be factored out because the energies are additive. With $Z_0 = Z_{\text{transl}} \times Z_{\text{rot}}$ we have

$$\begin{aligned} Z_{\text{rot}} &= \frac{4\pi}{h^2} \sum_j \exp \left(-\frac{1}{2} \beta I (\omega_1^2 + \omega_2^2) \right) \Delta \omega_j \text{ (rot)} \\ &= \frac{4\pi I^2}{h^2} \int \exp \left(-\frac{1}{2} \beta I (\omega_1^2 + \omega_2^2) \right) d\omega_1 d\omega_2. \end{aligned}$$

These are two Laplace integrals, each of which is equal to $\sqrt{\frac{2\pi}{\beta I}}$. Hence

$$(12) \quad Z_{\text{rot}} = \frac{8\pi^2 I}{h^2 \beta}.$$

According to (29.16), the contribution to the energy is

$$(13) \quad U_{\text{rot}} = -N \frac{\partial \log Z_{\text{rot}}}{\partial \beta} = \frac{N}{\beta} = N k T.$$

Thus the energy per mol, including the contribution from translation is

$$(13 \text{ a}) \quad u = \frac{5}{2} R T, \quad c_v = \frac{5}{2} R, \quad \text{Q. E. D.}$$

The entropy of a diatomic gas must be changed accordingly. It is seen from eq. (29.12') that the contribution from rotation per mol is equal to

$$(14) \quad s_{rot} = R \log T + R \log \frac{8\pi^2 e k I}{h^2}.$$

The total entropy of one mol becomes:

$$(15) \quad s = R \log v + \frac{5}{2} R \log T + \text{const}$$

in complete agreement with the thermodynamic equation (5.10)

C. THE POLYATOMIC GAS AND KELVIN'S CLOUDS

Assuming a rigid molecule of a general structure in which the atoms are not arranged along a single line, we have to consider an additional degree of freedom: There are now three angular coordinates, θ, ψ, ϕ and three angular velocities, $\omega_1, \omega_2, \omega_3$ about the principal axes of the ellipsoid of inertia; we shall assume that the principal moments of inertia are I_1, I_2, I_3 . The calculation which led us to eq. (12), will now give

$$\left(\frac{2\pi}{\beta m}\right)^{3/2} \left(\frac{2\pi}{\beta I_1}\right)^{1/2} \left(\frac{2\pi}{\beta I_2}\right)^{1/2} \left(\frac{2\pi}{\beta I_3}\right)^{1/2}$$

and instead of (9), we now have

$$(16) \quad u = \frac{6}{2} R T, \quad c_v = 3 R \approx 6 \frac{\text{cal}}{\text{deg mol}}, \quad \frac{c_p}{c_v} = \frac{4}{3}.$$

It is very satisfactory to note that simple rule from Sec. 4 C, namely that $c_v = 5/2 R$, or that $c_p = 3 R$, now becomes comprehensible. However, on closer inspection the above rule turns out to be too simple. Consider, for example, the angular model of a molecule of water. According to spectroscopic results, the valency bonds which connect O with the two atoms of H form an angle $\gamma > \frac{1}{2}\pi$ (band spectrum of water vapor). The three moments of inertia are different from each other and from zero; if we consider steam to be an approximately perfect gas, we find that c_v is equal to 6 cal/deg mol independently of the moments of inertia and of temperature. The same would be true about molecules with more obtuse angles, but in the limiting case $\gamma = \pi$ the arrangement would become linear and c_v would jump to the diatomic value $5/2 R \approx 5$ cal/deg mol. The discontinuity of values 3, 5, and 6 cal/deg mol

which reflects the 3, 5, and 6 degrees of freedom of mono- dia- and polyatomic molecules, signifies one of the clouds which obscure the kinetic theory of gases.

However, there exists an even darker cloud. Induced by Nernst's representation of gas degeneration, Eucken performed measurements on the molar specific heat of H_2 at decreasing temperature. He discovered that it continuously decreases from 5 cal/deg mol and becomes equal to 3 cal/deg mol at 80 K. *The rotational degrees of freedom have died out, they have become, as we sometimes say, frozen, and H_2 has become monatomic.* Paraphrasing a quotation from Schiller,¹ the author stated in 1911, during a scientific congress in Karlsruhe, that: "Degrees of freedom should be weighted, not counted." Quantum theory shows how this is to be achieved.

32. The specific heat of vibrating molecules and of solid bodies

We now drop the physically untenable hypothesis of rigid molecules and take into account the fact that atoms in a molecule are capable of performing small vibrations about their position of equilibrium when, in addition to their kinetic energy, they also possess potential energy. The same is true about the atoms in the giant molecule of a solid body.

A. THE DIATOMIC MOLECULE

The force with which the two atoms of such a molecule act on each other coincides with the axis of the arrangement, irrespective of the origin of the force, i. e. irrespective of whether it is of an electrical, polar or homeopolar nature. Let r_1, r_2 denote the two amplitudes of the atoms, measured outwards with respect to the position of equilibrium. In view of the equality of the *actio* and *reactio*, the two amplitudes are seen to be coupled. For the sake of simplicity we shall assume a quasi-elastic bond and hence a harmonic vibration. We put $r = r_1 + r_2$ and calculate the potential and kinetic energies of the coupled system. Evidently

$$(1) \quad E_{pot} = \frac{C}{2} r^2, \quad E_{kin} = \frac{M}{2} \dot{r}^2.$$

The "reduced" mass of the two atoms (cf. e. g. Vol. IV, Problem III. 1) of mass m_1, m_2 respectively, is given by

$$(1a) \quad M = \frac{m_1 m_2}{m_1 + m_2}.$$

¹Demetrius, 1st Act, end of Scene 1.

The elastic constant of the link has been denoted here by C . The rotary motion is superimposed on the vibration along the link. Strictly speaking, the rotation is not completely independent of the vibration, because the moment of inertia varies with the varying distance between the atoms. However, this influence may be neglected in a first approximation because the amplitude is small compared with the distance between the atoms (at $T = 2000$ K and for H_2 molecules it is about 10 per cent).

The phase space of a vibrating diatomic molecule must, thus, be enlarged, as compared with that of a rigid molecule, namely from 10 to 12 dimensions. The phase element $\Delta\Omega$ in (31.8) is to be supplemented with the factors Δr and $M \Delta \dot{r}$. This, however, constitutes no additional difficulty, since the additional energy terms (1) are quadratic in form. The previous integral (31.12) only becomes multiplied by the factors:

$$(2\ a) \quad \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\beta C r^2\right) dr = \sqrt{\frac{2\pi}{\beta C}},$$

$$(2\ b) \quad \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}M \dot{r}^2\right) d\dot{r} = \sqrt{\frac{2\pi}{\beta M}},$$

if the changes in the moment of inertia are neglected. Since we are only interested in the dependence on β , the thus extended eq. (31.13) gives at once

$$(3) \quad u = \frac{7}{2} R T, \quad c_v = \frac{7}{2} R \approx 7 \frac{\text{cal}}{\text{deg mol}}, \quad \frac{c_p}{c_v} = 1 + \frac{2}{7} = 1.29.$$

The preceding argument leads to an important remark and to an even more important question:

1. The potential energy in (2a) appears on an equal footing with the kinetic energy in (2b); according to (3) each contributes $\frac{1}{2} R T$ per mol to the energy distribution.

2. Why does the vibration remain unexcited in the case of air and other diatomic gases under ordinary conditions? If this were not so we should observe the smaller value 1.29 for c_p/c_v instead of 1.4.

B. POLYATOMIC GASES

We shall recall here a general proposition from the science of vibrations (*cf.* e. g. Vol. VI, Sec. 25): The number of free vibrations which any mechanical system can perform about a stable position of equilibrium is equal to the number of degrees of freedom of the system less the number of degrees of freedom for translations and rotations. Regarding each of these vibrations we can make the same statement as we have made about the linear vibration of a diatomic gas; each of them would have to increase the value of c_v by $R T$ so that for increasingly complex systems of molecules c_v would increase without limit. Consequently, the value of c_p/c_v would approach unity. Why then, do we observe in the case of organic molecules an average value of c_p/c_v in the neighborhood of 1.33 (*cf.* Sec. 4 C) instead of this value, that is a value which corresponds only to active translational and rotational degrees of freedom?

C. THE SOLID BODY AND THE DULONG-PETIT RULE

The analysis of the structure of crystals reveals that every crystal is composed of atoms arranged in a lattice. Because of the mutual links between atoms, and disregarding translations and rotations of the whole body, we see that the atoms only vibrate about their positions of equilibrium. Since every atom possesses 3 degrees of freedom, a lattice composed of N atoms possesses $3 N$ degrees of freedom, and deducting the 6 degrees of freedom of the rigid body, we are left with $3 N - 6$ degrees of freedom for vibrational modes and an equal number of independent oscillators. In this connection it is necessary to remember that it is always possible to regard the motion of a number of *coupled* oscillators, such as, for example, that of the atoms, as the sum of an equal number of independent normal vibrations,¹ (*cf.* Vol. VI, Sec. 25), whose potential energies must be taken into account in the same way as in A.

Thus at thermodynamic equilibrium there is associated with each oscillator an average energy $k T$. Hence the molar specific heat of a solid body (in the small, every solid body is a crystal) becomes $c_v = 3 R \approx 6$ cal/deg mol independently of the temperature. This is the very well known *rule* due to *Dulong and Petit*. However, this rule contradicts Nernst's Third Law (*cf.* Sec. 12.3), according to which $c_v \rightarrow 0$ for $T \rightarrow 0$, and it does not agree with experiment at lower temperatures. In fact, in the case of hard substances (diamond, carborundum) a decrease in c_p (and in c_v) is observed even at room temperatures.

¹A normal mode occurs when all atoms vibrate simultaneously in a characteristic manner. Two sympathetic pendulums oscillating in parallel or in counter-stroke, or the natural vibrations of a string, may be recalled as examples.

33. The quantization of vibrational energy

Planck's discovery of the quantum of action, h , has induced us from the beginning to regard the phase elements $\Delta \Omega$ as being constant and to define the corresponding values of energy, ϵ_i , as a discontinuous series. It turned out that this was very important for the constant in the entropy equation. This series can be treated like a continuum only in cases when the difference between two succeeding levels of energy is vanishingly small in comparison with the equipartition energy $k T$, i. e. when

$$(1) \quad \epsilon_{i+1} - \epsilon_i \ll k T.$$

The transition from the partition function expressed as a sum

$$(2) \quad Z_0 = \sum_i e^{-\epsilon_i/kT}$$

to the integral (cf. e. g. eq. (30.12 a))

$$(2 a) \quad \frac{1}{\Delta \omega} \int e^{-\epsilon/kT} d\omega$$

is justified only in such cases.

A. THE LINEAR OSCILLATOR

In what follows we shall replace the subscript i by the more common subscript n and we shall denote the natural frequency of the oscillator by ν . According to the original assumption made by Planck in 1900, we then have

$$(3) \quad \epsilon_n = n \cdot h \nu.$$

According to his suggestion made in 1911 we put

$$(3 a) \quad \epsilon_n = \left(n + \frac{1}{2} \right) h \nu,$$

which agrees with the final result in quantum mechanics. In both assumptions condition (1) implies

$$(4) \quad h \nu \ll k T.$$

Introducing the *characteristic temperature*

$$(4 a) \quad \Theta = \frac{h \nu}{k},$$

we find that (4) reduces to

$$(5) \quad \Theta \ll T.$$

The preceding calculations with the aid of the integral partition function instead of the sum, is justified only if the above condition is satisfied. We shall prove that the specific heat vanishes in the reverse case, i. e. when $T \ll \Theta$.

The value of ν , and thus the value of Θ can be obtained from spectroscopical data (infrared rotational spectra), and so, for example, for HCl, for which the spectra have been investigated particularly thoroughly, we have $\Theta \approx 4000$ K.

We now proceed to perform the calculation for arbitrary values of $T \leq \Theta$ making, at first, use of assumption (3). From eq. (29.16) we find the molar energy is given by

$$(6) \quad u = -L \frac{\partial}{\partial \beta} \log \sum_{n=0}^{\infty} e^{-\beta n h \nu}.$$

The geometrical series which appears in the above equation is convergent for any values of $\beta > 0$ and is equal to

$$1/(1 - e^{-\beta h \nu}).$$

Hence according to (6) we have

$$(6 \text{ a}) \quad u = L h \nu \frac{e^{-\beta h \nu}}{1 - e^{-\beta h \nu}}.$$

Substituting $\beta = 1/kT$ and $\Theta = h\nu/k$, we obtain

$$(7) \quad u = \frac{R\Theta}{e^{\Theta/T} - 1}$$

$$(8) \quad c_v = \frac{du}{dT} = \frac{R\Theta^2/T^2}{(e^{\Theta/T} - 1)^2} e^{\Theta/T}.$$

Thus in the two limiting cases we have

$$(9) \quad \begin{array}{c|c} T \gg \Theta & T \ll \Theta \\ \hline u = RT & u = R\Theta e^{-\Theta/T} \rightarrow 0 \\ c_v = R & c_v = \frac{R\Theta^2}{T^2} e^{-\Theta/T} \rightarrow 0 \end{array}$$

In the first limiting case we have complete participation of the vibrational degree of freedom in equipartition, in the same way as in Sec. 32 A. In the

second case we have no appreciable excitation of vibrations. The transition between these two limiting cases is shown in Figs. 27 and 27a; for example, for $T = \Theta$, we have according to (6) and (7) that

$$u = \frac{R\Theta}{e-1} = 0.58 R\Theta, \quad c_v = \frac{R e}{(e-1)^2} = 0.92 R.$$

The above two diagrams settle the question which we have posed at the end of Sec. 32 A. The vibrational degree of freedom is now "weighted" according to temperature and not simply "counted" as in the classical theory. It now becomes clear that at normal temperatures and with values of Θ of the order of several thousand degrees, *the vibrational degree of freedom does not take part in equipartition.*

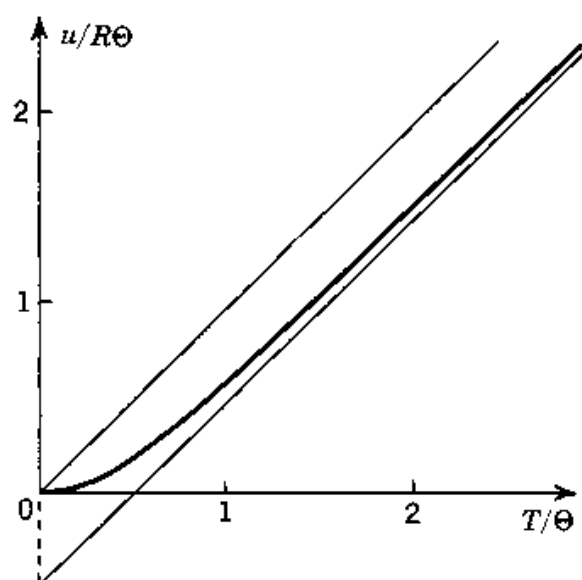


Fig. 27.

Molar energy of quantum mechanical oscillators as a function of temperature; (units: $R\Theta, \Theta$).

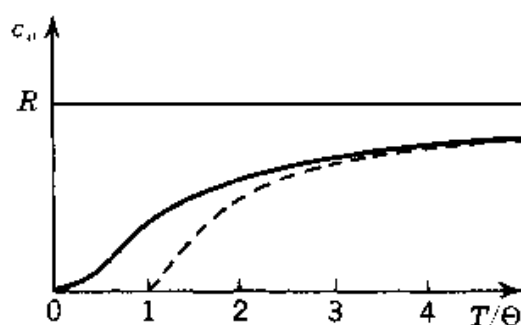


Fig. 27 a.

Molar specific heat of quantum mechanical oscillators as a function of temperature; (units: R, Θ).

It is, however, necessary to include two additional remarks:

1. If the correct quantum equation (3a) is substituted for the older equation (3) we find that (7) is replaced by

$$(10) \quad u = \frac{1}{2} R\Theta + \frac{R\Theta}{e^{\Theta/T} - 1}$$

which is obtained easily from (6) by substituting $n + \frac{1}{2}$ for n . It is seen that (7) is increased by the "zero-point energy" $\frac{1}{2} R\Theta$; its existence has been confirmed by numerous experiments at very low temperatures. The curve

representing u in Fig. 27 becomes thus shifted upwards by a certain distance (equal for all values of T).

2. We are, naturally, most interested in the relations for symmetrical molecules, such as H_2 , N_2 , O_2 , Unfortunately, the preceding formulae do not rigorously apply to such molecules but we are forced to refrain from considering the reasons for it, because this would lead us too deeply into the minutiae of wave mechanics. We shall content ourselves with stating that the "freezing" of the vibrational degrees of freedom at $T \ll \Theta$ is unaffected by these additional considerations.

For molecules possessing more vibrational modes it is, of course, necessary to consider a whole series of natural frequencies, ν_1, ν_2, \dots whose influences are superimposed on one another. Instead of Fig. 27 a we obtain a curve for c_v which exhibits several steps, the vertical distance from step to step increasing by R each time. It is clear that at ordinary temperatures only the first step need be taken into account.

B. THE SOLID BODY

The inconsistency of the theory with Nernst's Third Law which we were forced to record in Sec. 32 C resolves itself if we consider that every atom of the solid body is an independent oscillator and that each must be "quantized" separately. Thus for $T \rightarrow 0$ the specific heat per mol is given by the right-hand columns of (9): *The molar specific heat tends to zero exponentially.*

This application of the theory of quanta was suggested by Einstein (Ann. der Physik, Vol. 22) as early as 1907. However, the experiments carried out in Nernst's institute showed a weaker than exponential decrease towards zero. The reason for it is easy to perceive: The atoms do not vibrate independently but in larger or smaller groups, depending on the temperature (cf. Sec. 32 C and also Sec. 35).

C. GENERALIZATION TO ARBITRARY QUANTUM STATES

We define the general molecular partition function:

$$(11) \quad Z_0 = \sum_{n=0}^{\infty} g_n e^{-\epsilon_n k/T}.$$

The equation (11) differs from our previous definition by the "weighting factor", g_n . It serves to collect different quantum states of equal energy into one term. In the case of two unequal atoms vibrating we had $g_n = 1$, but already

in the case of two equal atoms g_n varies from term to term (cf. remark 2 in Sec. A).

In the same way as in eq. (6) we now derive an expression for the total energy U from Z_0 but instead of one mol we consider a system composed of an arbitrary number of molecules, N . At the same time we shall replace the differentiation with respect to β by one with respect to T in that we write

$$-\frac{d}{d\beta} = -\frac{dT}{d\beta} \frac{d}{dT} = k T^2 \frac{d}{dT}.$$

Equation (6) now transforms into

$$(12) \quad U = N k T^2 \frac{d \log Z_0}{dT}.$$

An expression for S is obtained from (29.12')

$$(13) \quad S = N k \log Z_0 + \frac{U}{T}.$$

In order to prove it by the *methods of thermodynamics* we now vary T keeping V , as well as the remaining parameters, constant. Thus we obtain from (13) that

$$(13 a) \quad dS = N k \frac{d Z_0}{Z_0} - \frac{U}{T^2} dT + \frac{dU}{T}.$$

The second term on the right-hand side can be calculated from (12):

$$-N k \frac{d \log Z_0}{dT} dT = -N k \frac{dZ_0}{Z_0},$$

and it is seen that it cancels the first term on the right-hand side of (13a). Hence we have

$$dS = \frac{dU}{T}$$

as we must have for an isochoric process. Thus the validity of (13) has been proved, except for an additive constant whose value will be adjusted in accordance with Nernst's Third Law. Combining (12) with (13), we obtain

$$(13 b) \quad S = N k \frac{d(T \log Z_0)}{dT}.$$

The expression for free energy $F = U - T S$ which follows directly from (13) turns out to be particularly simple:

$$(14) \quad F = -N k T \log Z_0.$$

34. The quantization of rotational energy

The levels of rotational energy which are admissible by quantum theory also form a discontinuous, step-like series. For the simplest atomic model of a rotator (diatomic molecule), Planck's energy levels of a linear oscillator given in eq. (33.3) are now replaced by the expression

$$(1) \quad \epsilon_n = n(n+1) \frac{\hbar^2}{2I}, \quad n = 0, 1, 2, \dots, \quad \hbar = \frac{h}{2\pi}.$$

The proof of (1) given in wave mechanics is based on the theory of spherical harmonics, but its detailed consideration would exceed the scope of these lectures.

We shall employ here the partition function from eq. (33.11) formally. In our case the weighting factor is given by

$$(2) \quad g_n = 2n + 1.$$

This also is a consequence of the theory of spherical harmonics, according to which, apart from zonal harmonics (dependence on θ alone), there are in addition $2n$ associated surface harmonics (dependence on θ and ϕ). These rotational states which differ from each other in the physical space, correspond to the same energy (1). Every energy level in the series (1) is, as we say, $(2n+1)$ -fold degenerated. Thus the weighting factor corresponds to a summation over the physical space, which in this case must be carried out together with that over the momentum space.

Consequently, and in accordance with (33.11), we obtain the partition function

$$(3) \quad Z_{\text{rot}} = \sum_{n=0}^{\infty} (2n+1) e^{-n(n+1)q} = 1 + 3e^{-2q} + 5e^{-6q} + \dots$$

where q denotes the contraction

$$(3a) \quad q = \frac{\hbar^2}{2IkT} = \frac{\Theta}{T} \quad \text{with} \quad \Theta = \frac{\hbar^2}{2Ik}.$$

We now proceed to calculate Θ for H_2 as this constitutes a particularly interesting case. From spectroscopic data (multiple line spectra) we have¹

$$I = 0.46 \times 10^{-40} \text{ g cm}^2.$$

¹Using the equation $I = 2m_H(a/2)^2$, we can calculate the distance between the two atoms of hydrogen, as it is known that $m_H = 1.67 \times 10^{-24} \text{ g}$. Thus $a = 0.74 \times 10^{-8} \text{ cm} = 0.74 \text{ \AA}$, i. e. the same order of magnitude as the cross-section of a hydrogen atom. This remark serves to render the above value for I plausible from the point of view of its order of magnitude.

Making use of the values which were given at the end of Sec. 20 (eq. (20.43)):

$$\hbar = 1.06 \times 10^{-27} \text{ erg sec}, \quad k = 1.38 \times 10^{-16} \text{ erg/deg}$$

and of the definition of Θ in (3a), we calculate that

$$(4) \quad \Theta \approx 80 \text{ K.}$$

This result leads to the following conclusions: For $T \ll 80 \text{ K}$ the second term in the series (3) is already exponentially small compared with the first. Thus

$$Z_{\text{rot}} \approx 1, \quad \log Z_{\text{rot}} \approx 0.$$

According to eq. (33.12) this expresses the fact that the contributions from rotational energy to the energy, U , and to the specific heat, c_v , vanish. The specific heat is reduced to the contribution from translation

$$(5) \quad c_v = \frac{3}{2} R.$$

Hydrogen has become, so to say, "monatomic". This agrees with Eucken's discovery which was described at the end of Sec. 31 as the darkest cloud threatening the development of classical statistical mechanics,¹ particularly with regard to the numerical value of Θ .

We now proceed to consider the second limiting case when $T \gg \Theta$, i. e. when $q \ll 1$. The series (3) converges very slowly because its terms decrease to zero only for extremely large values of n . According to its analytical character, it belongs to the group of theta functions; the latter occur in the theory of elliptic functions and in heat conduction problems, see Vol. VI Sec. 15. The use of an analogue to the "transformation formula of the theta function", Vol. VI, eq. (15.8) would in the case of our eq. (3) also lead to a more exact and a more convenient formula for numerical computation. However, at this stage it suffices to perform an approximate estimation of the limit for $T \gg \Theta$, as the more accurate calculation is considered in Problem IV. 6.

Since q is small, we may write

$$(6) \quad p = n(n+1)q$$

¹The rotational energy of the electron shell and that due to the atomic nucleus remain unexcited for the same reason. The value of Θ becomes extremely large (100,000 to 10 million deg in round numbers), owing to the small mass of an electron, in the first case, and owing to the small radius in the second.

which is almost continuous and which varies from 0 to ∞ as n increases. The difference Δp between p_n and p_{n-1}

$$(6a) \quad \Delta p = p_n - p_{n-1} = n(n+1)q - n(n-1)q = 2nq$$

is a small number for all finite values of n for which the terms of (3) give a significant contribution. Thus in the following argument we shall denote it by dp . Consequently, without committing an appreciable error,¹ we can write $2n+1 = dp/q$. Thus,

$$(7) \quad Z_{rot} = \frac{1}{q} \int_0^{\infty} e^{-p} dp = \frac{1}{q} = \frac{T}{\Theta}.$$

and it follows from eq. (33.12) that

$$(7a) \quad U = NkT^2 \frac{d}{dT} \log \frac{T}{\Theta} = NkT \quad \text{and hence} \quad \frac{dU}{dT} = Nk.$$

The last result demonstrates that the contribution of rotation to the molar heat for $T \gg \Theta$ is exactly equal to R . This contribution is represented graphically in Figs. 28 and 28a. They demonstrate clearly, just as was the case with the vibrational energy, that the rotational degree of freedom dies out gradually as we pass to $T < \Theta$; the full amount of $R = 2$ cal/deg mol is attained asymptotically only above $T = \Theta$.

In the preceding argument we have purposely considered "hydrogen" because of Eucken's discovery. Strictly speaking we should have referred to the semi-heavy hydrogen HD ($D = \text{deuterium}$), because in the case of the ordinary hydrogen H_2 (as well as in the case of the heavy hydrogen D_2) there are complications, as mentioned in remark 2 on p. 240. However, even in the case of atoms composed of two identical molecules, the general qualitative features of the behavior of the rotational heat c_v remain unchanged in essence.

We shall refrain here from discussing the problems in quantum mechanics which are connected with the rotational energy of a general *polyatomic* molecule. Even in the special case of a symmetrical molecule, such as NH_3 , CH_4 , . . . , there appears a new degree of freedom, namely rotation about the axis of symmetry. The scale of energy (1) must now be extended by a term which contains the moment of inertia about this axis and a new index of summation. The single summation in the partition function is now replaced by a double summation.

¹The magnitude of the error can be estimated with the aid of Euler's summation formula.

At sufficiently high temperatures the new degree of freedom causes the molar heat to increase by $\frac{1}{2} R$, but decays at lower temperatures depending on the magnitude of the temperature Θ' which is characteristic for this degree of freedom. In this way the polyatomic molar heat $c_v = 3 R$ changes continuously until it reaches the diatomic value $c_v = 5/2 R$. For large values of Θ' (small moment of inertia about the axis of symmetry) the polyatomic molecule behaves like a *diatomic* one at ordinary temperatures.

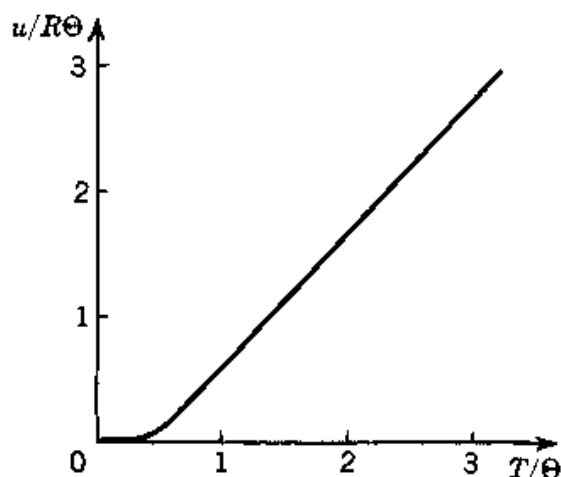


Fig. 28.

Molar energy of quantum-mechanical rotators in terms of temperature (units: $R \Theta$, Θ).

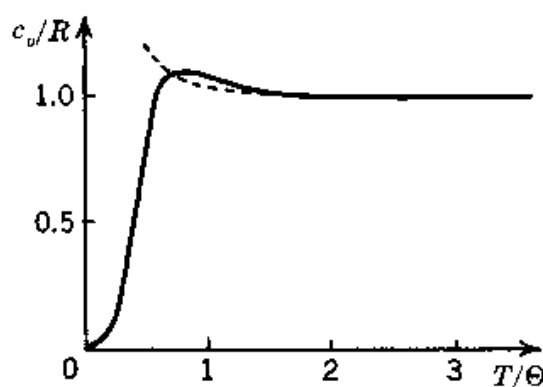


Fig. 28 a.

Molar specific heat of quantum-mechanical rotators in terms of temperature (units: R , Θ).

35. Supplement to the theory of radiation and to that of solid bodies

In Sec. 20, eq. (38), we have deduced the following expression for the mean energy of a linear oscillator:

$$(1) \quad U = \frac{h \nu}{e^{h\nu/kT} - 1}.$$

This is the energy assumed by the oscillator in a radiation cavity of temperature T when it is in equilibrium with the surrounding black-body radiation. We pointed out at the time that this equation can be derived in a much simpler manner by use of statistical methods.

In order to obtain the proof it suffices to recall the linear oscillator which was considered in Sec. 33 A. It is true that the latter was not placed in a radiation cavity, but it was in equilibrium with its surroundings, it being immaterial to consider the way in which equilibrium had been achieved, i. e. whether equilibrium had been brought about by radiation or by the coupling

of the molecules of a gas. According to Planck's quantum hypothesis (33.3), we can use the expression in (33.6a) directly for the energy at equilibrium where, naturally, the factor L (Avogadro-Loschmidt number) should be omitted, because we are now considering a single oscillator and not one mol of substance. The value obtained in this way agrees with (1) precisely.

A. METHOD OF NATURAL VIBRATIONS

In using statistical methods it is irrelevant whether we are concerned with material objects or with states, with economic data or experimental errors, etc. The application of statistical methods to the natural electromagnetic vibrations of a cavity (parallelepiped or cube) has proved to be particularly fruitful. In Vol. II, Sec. 44, we have calculated the number and arrangement of natural vibrations of an elastic slab and we have discussed the simplifications which may be introduced in the case of a cavity containing electromagnetic radiation but entirely devoid of matter. In the latter case it is possible to satisfy the boundary conditions exactly ($E_{tang} = 0$) with the aid of the elementary sine-law, whereas in the former this is so only when the so-called "mixed" boundary conditions are used.

In addition, the following should be noted: Equation (44.16a) in Vol. II was applicable to the number Z of elastic vibrational modes whose frequency is smaller than ν . In order to adapt this equation to the electromagnetic case it is necessary to put:

$$c_{trans} = c = \text{velocity of light}; \quad c_{long} = \infty,$$

as already mentioned on p. 325 of Vol. II. In this manner we obtain the equation

$$(2) \quad Z = \frac{8\pi V \nu^3}{3c^3}$$

which was deduced by Lord Rayleigh as early as 1900. Here V denotes the volume of the parallelepiped (or of the cube). The number of vibrational modes per interval $d\nu$ becomes

$$(2a) \quad dZ = \frac{8\pi V}{c^3} \nu^2 d\nu.$$

If we now equip each of these with the energy kT in accordance with the law of equipartition (not $\frac{1}{2}kT$ because it is necessary to take into account the potential energy as well), we find that the energy density per frequency interval becomes

$$(3) \quad u_\nu = \frac{kT}{V} \frac{dZ}{d\nu} = \frac{8\pi}{c^3} \nu^2 kT.$$

This is the Rayleigh-Jeans law, already quoted in eq. (20.16). As we know, it is grossly inconsistent with experiment because it predicts that $u \rightarrow \infty$ as $\nu \rightarrow \infty$.

If, however, we adopt Debye's suggestions (Ann. d. Phys. Vol. 33), and regard each frequency mode as a quantized oscillator (c/. Sec. 33 C) ascribing to it an energy given by (1), we find from (1) and (2a) that

$$(4) \quad u_\nu = \frac{U}{V} \frac{dZ}{d\nu} = \frac{8\pi \nu^2}{c^3} \cdot \frac{h\nu}{e^{h\nu/kT} - 1}$$

instead of (3), i. e. directly Planck's radiation law, as given in eq. (20.39). Planck adopted this method of derivation in the fourth edition of his "Strahlungstheorie" published in 1921, and described it as an "exceedingly simple derivation" of his law radiation.

B. DEBYE'S THEORY OF THE SPECIFIC HEAT OF A SOLID

The main difference between the elastic body and the radiation cavity consists in the fact that the degrees of freedom of the former are restricted by its lattice structure, whereas (as far as we know to-day), that of the latter is unlimited. Since the number of normal modes of a body is equal to the number of degrees of freedom, and since every atom in the lattice possesses 3 degrees of freedom, the number Z of normal modes is found to be equal to its upper limit Z_g . Thus for a solid body of a mass equal to one mol we have

$$(5) \quad Z_g = 3L \quad (L = \text{Loschmidt-Avogadro number}).$$

To this upper limit there corresponds an upper limit ν_g of the admissible natural frequencies. As explained in detail in Vol. II, Debye cuts off the spectrum of natural vibrations at $\nu = \nu_g$. The energy of the solid body becomes

$$(6) \quad u = \int_{\nu=0}^{\nu_g} U dZ$$

where U is the quantized value of energy from eq. (1). The limiting frequency ν_g defines the *characteristic temperature* Θ of the solid body,

$$(7) \quad \Theta = \frac{h\nu_g}{k} = \text{Debye temperature}.$$

Performing the integration indicated in (6) we find

$$(8a) \quad u = \frac{3\pi^4 R T^4}{5 \Theta^3}, \quad c_v = \frac{12\pi^4}{5} R \left(\frac{T}{\Theta}\right)^3 \quad \text{for } T \ll \Theta,$$

$$(8b) \quad u = 3RT, \quad c_v = 3R \approx 6 \frac{\text{cal}}{\text{deg mol}} \quad \text{for } T \gg \Theta,$$

as explained in detail in Vol. II, Sec. 44. The limiting case (8b) is seen to contain the *rule* due to *Dulong and Petit*; in the limiting case (8a) u has the same form as in *Stefan-Boltzmann's law of radiation*; c_v is given by *Debye's third power law* (T^3 - law) for the molar specific heat of a solid. It has proved itself as an excellent qualitative rule (disregarding details of the crystal structure) and corrects Einstein's assumption from Sec. 33 B. The T^3 -law, just as Einstein's law, satisfies Nernst's Third Law, but it does not converge exponentially. The convergence towards absolute zero is that of a parabola of the third degree. The reason for it is clear: Einstein treated the atoms of a solid body as independent oscillators, whereas Debye collects the molecules which simultaneously perform the natural¹ vibrations into groups. The wavelength of these groups increases as the temperature becomes lower; the correlation between molecules increases in like measure which renders Einstein's assumption of independence quite illusory. The sketch in Fig. 73 of Vol. II gives a qualitative description of the variation of u and c_v between the two limiting cases, (8 a) and (8 b).

This argument dispels the last of Kelvin's clouds mentioned in Sec. 32.

36. Partition function in the Γ -space

We have seen in Sec. 29 that Boltzmann's combinatorial method is not directly applicable to real bodies. It is true that the collecting of numerous elementary cells into a higher unit leads to the right answers and to an approximation which is entirely satisfactory from the practical point of view but, since the thermodynamic relations enjoy general validity, a rigorous justification seems desirable. We shall combine this problem with that of representing the fundamental equations of statistical mechanics in the Γ -space, as we shall require the results in Sec. 37.

A. THE GIBBS CONDITION

The succeeding argument which throws new light on Boltzmann's hypotheses of the equality of the probability of equal phase cells is due to J. W. Gibbs.² On comparing two phase elements at two different points in the phase space, it is found that the probabilities of finding the phase point

¹The fact that these (as distinct from the atoms in the lattice) may be regarded as being independent of each other can be inferred from the circumstance that the velocity of sound is almost the same for all frequencies.

²J. W. Gibbs, "Elementary Principles in Statistical Mechanics", Yale University Press, New Haven, 1902.

of an actual system in one place or in the other can be different and can vary with time. Quite generally, we can write that such probabilities are given by

$$(1) \quad \Delta w = f(p_1 \dots p_F, q_1 \dots q_F; t) \frac{\Delta \Omega}{h^F}.$$

Here $\Delta \Omega$ denotes the phase element, as before, h^F (with $F = Nf$) is the size of an elementary cell, and the function $f(p, q; t)$ indicates the manner in which this probability changes from point to point and with the course of time. It constitutes the probability per elementary cell h^F . This general form of Δw is important, for example, when we inquire into the manner in which errors in our knowledge of the initial conditions of a mechanical system propagate themselves in time.

In studying thermodynamical statistics we concentrate our attention on states of equilibrium. Consequently $f(p, q)$ need not depend on time explicitly and the question now arises as to what are the consequences of such a statement.

We consider two phase elements $\Delta \Omega$ and $\Delta \Omega'$ which result one from the other because of the motion. The time interval between them will be denoted by $\Delta t = t' - t$. According to eq. (1) the two probabilities can be written as

$$\Delta w = f(p, q, t) \frac{\Delta \Omega}{h^F} \quad \text{and} \quad \Delta w' = f(p', q', t + \Delta t) \frac{\Delta \Omega'}{h^F}.$$

Both are equal because the moving phase element encloses the same phase points at any instant. Thus $\Delta w = \Delta w'$.

Since, according to Liouville's theorem, we also have $\Delta \Omega = \Delta \Omega'$, and since at equilibrium f does not depend on time, we obtain the equation

$$(2) \quad f(p, q) = f(p', q'),$$

on condition that (p', q') results from (p, q) in the course of the motion. Thus we have proved the following proposition: *In the case of thermodynamic equilibrium the probability per elementary cell is an integral of the motion.*

In the case of canonical systems there always exists at least one integral of the motion: the energy $E = H(p, q)$. Every function of energy is also an integral. Hence we may put

$$(3) \quad f = f(H).$$

Apart from energy the argument may contain additional integrals: *partial energies*, if the system is composed of independent sub-systems; *angular momentum* if the system is free to rotate, etc. However, all integrals except

H are constant only under certain restrictive conditions. Small modifications of the system which often do not at all disturb the total energy of the system, but which are important for the establishing of equilibrium, remove accidental integrals. Here one may think of the example of perfect gases in which, for all intents and purposes, the individual molecules exist in isolation from each other; they do, however, exchange energy on collision. Another example is afforded by cosmic matter whose rotation is not confined within any "walls of a vessel" so that on applying statistical considerations it is found that the angular momentum must be taken into account in addition to energy. However, generally speaking, thermodynamical statistics is determined by a single function of energy.

What is the form of the function in eq. (3)? At this point we shall make use of Gibbs' assumption which replaces Boltzmann's hypothesis: *Two coupled mechanical systems in statistical equilibrium remain in equilibrium in the limiting case of vanishingly small coupling even if they are separated.* For example, when two systems equalize their temperatures on being brought in contact, the equilibrium temperature which they both have attained will not change if they are subsequently separated. More precisely, the temperature will not change markedly, if the work performed against the forces of cohesion can be neglected in the overall energy balance; for example, when the bodies are sufficiently large in all directions.

Let H_1 and H_2 denote the Hamiltonians of the two sub-systems and let $H = H_1 + H_2 + \delta H$ ($\delta H \rightarrow 0$) be the integral of the coupled system, δH denoting the energy of coupling. According to eq. (3) we have to expect the following distribution functions:

$$f_1(H_1) \cdot f_2(H_2) \quad \text{and} \quad f(H) \approx f(H_1 + H_2).$$

Gibbs' hypothesis leads to the equation

$$(4) \quad f_1(H_1) \cdot f_2(H_2) = f(H_1 + H_2)$$

because the probability of finding each system at a given state must be the same before and after separation.

B. CONNECTION WITH BOLTZMANN'S METHOD

First, it should be recognized that in the limit when $\delta H \rightarrow 0$ the left- and right-hand sides of eq. (4) become integrals of the motion so that Gibbs' hypothesis leads to a statement about the initial conditions only, in the same way as Boltzmann's hypothesis did before. In addition, taking the derivative of (4) with respect to H_1 and H_2 we obtain

$$f_1'(H_1 + H_2) = f_1'(H_1) \cdot f_2(H_2) = f_1(H_1) f_2'(H_2),$$

so that the ratios

$$\frac{f_1'(H_1)}{f_1(H_1)} = \frac{f_2'(H_2)}{f_2(H_2)} = -\beta$$

must be constant, because if this were not so, two functions with different variable arguments could not be identical in form. Integration leads at once to Boltzmann's distribution function

$$(5) \quad f(H) = e^{-\alpha - \beta H}.$$

The constant factor has been written in the form $e^{-\alpha}$ in analogy with eq. (29.10). Its value is obtained from the condition that $\Sigma \Delta W = 1$, so that

$$(6) \quad e^{\alpha} = \int e^{-\beta H} \frac{d\Omega}{h^F} = Z;$$

Z is now the partition function in the Γ -space and appears still in the form of an integral. The factor $1/h^F$ hints at quantum corrections to be introduced in Sec. C.

All quantities which are characteristic of the state of equilibrium can be deduced from Z . The variational derivate is, in particular,

$$(7) \quad -\frac{1}{\beta} \frac{\delta \log Z}{\delta H(p, q)} = \frac{e^{-\beta H}}{\int e^{-\beta H} d\Omega} = f(p, q).$$

In analogy with eq. (29.16), we obtain an expression for *mean energy*

$$(8) \quad U = -\frac{\partial \log Z}{\partial \beta}.$$

The *temperature* and *entropy* are given by the Second Law:

$$T \delta S = \delta U - \int \delta H \cdot f(p, q) \frac{d\Omega}{h^F}.$$

The second term represents the change in energy as required by the Second Law because on varying H , while keeping the distribution of molecules in the phase space constant, the integral represents the amount of work added to the system. Applying eq. (7), we obtain

$$T \delta S = \delta U + \frac{1}{\beta} \int \delta H \frac{\delta \log Z}{\delta H(p, q)} d\Omega = \delta U + \frac{1}{\beta} \delta \log Z - \frac{1}{\beta} \frac{\partial \log Z}{\partial \beta} \delta \beta.$$

The term $\delta \log Z$ includes the variation of β in addition to that of H , and the contribution of the former is subtracted in the third term. Applying eq. (8), we have

$$T \delta S = \frac{1}{\beta} \delta(\beta U + \log Z).$$

Comparing the factors in front of and behind the sign of variation and integrating we can show that

$$(9) \quad T = \frac{1}{k\beta}, \quad S = k \left(\log Z + \frac{U}{kT} \right).$$

The factor k which is at present undetermined is identical with Boltzmann's constant.

That this is so can be seen on comparing eq. (9) with Boltzmann's equation in (29.1). First, from eqs. (6) and (9) we have

$$S = k(\alpha + \beta U) = k \overline{(\alpha + \beta H)} = -k \overline{\log f(p, q)}.$$

The bar over a symbol in the above equation denotes a mean value, which is explicitly given as

$$(10) \quad S = -k \int f \cdot \log f \cdot \frac{d\Omega}{h^F}$$

and can be directly compared with eqs. (29.5) and (29.5 a), according to which

$$\log W = -N \sum_i \frac{n_i}{N} \log \frac{n_i}{N}.$$

This gives the equation

$$(11) \quad S = k \log W,$$

which is equivalent to (29.1). It will be realized that eq. (5) gives that distribution function, from among all the possible distribution functions f leading to the same energy U , which renders $\log W$ in eq. (11) a maximum. The relevant calculations are identical with those in Sec. 29. This shows that Gibbs' and Boltzmann's hypotheses are equivalent.

If the system consists of N equal, independent parts of f degrees of freedom each, according to $p_1' \dots q_f', p_1'' \dots q_f'', \dots, p_1^{(N)} \dots q_f^{(N)}$, then its energy is given by

$$H = H_0(p', q') + H_0(p'', q'') + \dots + H_0(p^{(N)}, q^{(N)}).$$

It follows at once that the partition function can be split into N factors:

$$Z = \int e^{-\beta H_0(p', q')} \frac{d\Omega'}{h^f} \times \dots \times \int e^{-\beta H_0(p^{(N)}, q^{(N)})} \frac{d\Omega^{(N)}}{h^f},$$

which differ only in the symbols employed for the variables under the sign of integration. They are, thus, all equal. Consequently

$$(12) \quad Z = Z_0^N$$

and

$$(13) \quad Z_0 = \int e^{-\beta H_0(p, q)} \frac{d\Omega_0}{h^f}$$

is the partition function in the μ -space. This gives us a direct connection with our preceding arguments.

C. CORRECTION FOR QUANTUM EFFECTS

It must be realized that, generally speaking, it is necessary to base the argument on the partition function in the Γ -space. The splitting of Z into factors Z_0 implies very special assumptions which cease to be valid for the case of a perfect gas with quantum effects included. We shall prove this statement in Sec. 37. In order to introduce the quantum corrections into our statistical considerations, we find it necessary to take several consecutive steps. The first two have already been taken, namely in the μ -space. The third step which makes it necessary to go over to the Γ -space will be described in Sec. 37. The fourth step, the quantum-mechanical derivation of the partition function requires so much knowledge of quantum mechanics that we shall have to be satisfied with a simple transposition of the classical partition function, referring the reader to specialized papers.¹

The first step consists in replacing the integral form of the partition function in (6) by the sum

$$(14) \quad Z = \sum_{(n)} e^{-\beta E(n)}$$

¹M. Delbrück, G. Molière: Proc. Prussian Ac. of Sci., Phys-Math. Class, 1936, No. 1.

in which the summation is extended over all phase cells of size h^F in the Γ -space. The results are, to all intents and purposes, identical with those which followed from (6), because within one cell the variation suffered by the integrand is insignificant. The size of the phase cell is the only parameter of importance. Its value influences the value of the constant in the expression for entropy (*cf.* Sec. 31, eq. (5 c)). It will be noted, however, that it has already been taken into account in (6) by the assumption of $d\Omega/h^F$ for the phase element. In fact, eq. (6) can be regarded as a continuous approximation to the quantum sum (14), in the sense of Euler's sum equation.

We now have to show that the sum (14) can be split according to eq. (12) in the same way as the partition function (6). Dividing the phase space of a single molecule into cells of size h^f we find that the energy $E(n)$ in eq. (14) is given by

$$(15) \quad E(n) = \sum_i n_i \varepsilon_i.$$

Here n_i denotes the number of molecules in a given cell in the μ -space and ε_i is the energy of a molecule in such a cell. The sign \sum denotes summation over all cells in the μ -space and (n) in eq. (14) denotes that the sum should be taken over all decompositions (partitions) per arrangement of the number of molecules

$$(16) \quad N = \sum_i n_i.$$

The weighting factor g which appeared in eq. (33.11) has been omitted in (14). It is implied in the convention that every value of energy which occurs several times must be written out as many times. The equality of $E(n)$ in various phase cells in the Γ -space can result from the fact that the values of ε_i in different phase cells of the μ -space are equal (degenerate molecular states). Such degeneration leads to the weighting factor in eq. (33.11).

As long as it is assumed that it is possible to distinguish individual molecules of one kind one from another, as is done in classical statistical mechanics, it is possible to obtain definite values of energy $E(n)$ in many ways because the individual molecules can be distributed over the phase cells in a variety of ways. The distribution is, evidently, such that we always have n_1 molecules in the first cell, n_2 molecules in the second, etc. The number of possibilities is given by the permutability in eq. (29.3), so that the partition function (14) must be provided with a weighting factor:

$$(17) \quad Z = \sum_{(n)}' \frac{N!}{n_1! n_2! \dots} e^{-\beta \sum_i n_i \varepsilon_i}$$

if every partition of N is written down only once, as indicated by the prime added to the summation sign. Putting

$$(18) \quad e^{-\beta \epsilon_i} = z_i,$$

we see that eq. (17) becomes

$$(17 \text{ a}) \quad Z = \sum'_{(n)} \frac{N!}{n_1! n_2! \dots} z_1^{n_1} z_2^{n_2} \dots$$

The permutabilities are seen to be equal to the binomial coefficients. Every power product

$$z_1^{n_1} z_2^{n_2} \dots$$

occurs in the last sum exactly as often as in the calculation of the N -th power of the sum of all z_i . Hence

$$Z = (z_1 + z_2 + \dots)^N$$

or

$$(19) \quad Z = Z_0^N, \quad Z_0 = \sum_i z_i = \sum_i e^{-\beta \epsilon_i}$$

which agrees with eqs. (12) and (13). Z_0 is the partition function in the μ -space; this time, however, it is represented as a sum over all phase cells h' .

The second step towards quantum mechanical statistics is obtained when we no longer consider that the ϵ_i are the energy values of the different phase cells, but as the energy values of the different quantum states. Instead of the sum over different phase cells of the μ -space we obtain the sum over all quantum states of a molecule. Boltzmann's hypothesis about phase cells having equal probabilities associated with them is now replaced by the hypothesis that quantum states have equal probabilities ascribed to them. The consequences of this change are studied in Secs. 33 to 35. It was seen that the first step led us to the value of the entropy constant; the second step corrects the principle of equipartition. It will be recalled that we still have to root out a sensitive error which we have encountered when deriving Sackur's formula (*cf.* Sec. 31, eq. (5 c)). We shall achieve this in Sec. 37, when taking the third step; the same step will lead us to gas degeneration. The results contained in Sec. 37 will turn out to be consistent with Nernst's Third Law.

D. ANALYSIS OF GIBBS' HYPOTHESIS

We now revert to eq. (5) once more and inquire into the significance of the dependence on H in the Γ -space. When we consider an actual system which is completely isolated from the surroundings, we find that its energy has a definite value. Thus there is no distribution $f(H)$. It follows that by assuming (5) we are not, at first, considering a completely isolated system, and the derivation of (5) from (4) shows that, in fact, the system under consideration has been separated from a larger system. Thus the *canonical* distribution in (5) is seen at first to apply to a thermodynamic system in a thermal bath.

In order to consider an isolated system it is necessary to refer to eq. (3) directly. Here we have

$$(20) \quad f(H) = \begin{cases} \text{const in a narrow interval } U - \delta U < H < U + \delta U \\ 0 \text{ outside of this interval.} \end{cases}$$

This is the so-called *microcanonical* distribution. It is obtained when Gibbs' assumption of equilibrium is replaced by the requirement that $H = U = \text{const.}$ The calculation based on the microcanonical distribution and carried out in the Γ -space is less simple, but the result agrees, to all intents and purposes, with the consequences of the assumption of a canonical distribution because the fluctuations about the mean value of energy (provided that the systems are not too small) are, in most cases, exceedingly small. The disturbances introduced by the bath can normally be only of the order of the energy of interaction, and this we could neglect (see, however, Problem IV. 9).

Thus there is no difference in the μ -space. Since in making the transition to the μ -space it is implied that every molecule interacts only slightly with its surroundings, it may be said that every molecule remains in contact with a thermal bath formed by all the others. On the other hand the mean behavior of a single molecule is determined by the canonical distribution in the μ -space, which is the Boltzmann distribution (29.10), irrespective of whether we begin with the canonical or microcanonical distribution.

37. Fundamentals of quantum statistics.¹

A. QUANTUM STATISTICS OF IDENTICAL PARTICLES

In the view adopted in quantum mechanics, identical particles cannot be distinguished from each other. This is true not only in relation to electrons in the shell of an atom or in a metal, about which it has long been recognized that they possess no individuality, not only in relation to light quanta and elementary particles in general, but also in relation to the atoms and molecules of our gas. They differ from each other only by special features (ionization, excitation, spin moments). For this reason we must not isolate single *particles* from the total number N and distribute them over the cells $\Delta\Omega_i$ of the phase space, as we have done in Sec. 29. We can only distinguish between the *states of the system*, meaning the states of the *gas* as a whole, and not the states ε_i of single particles. The former is given by

$$(1) \quad E(n) = \sum n_i \varepsilon_i.$$

Consequently, from the point of view of quantum statistics, it makes no sense to start with the partition function for a single molecule, eq. (29.15),

$$(2) \quad Z_\mu = Z_0 = \sum e^{-\beta \varepsilon_i},$$

but it is necessary to use the "partition function for the gas" from eq. (36.14)

$$(2a) \quad Z_T = Z = \sum'_{(n)} e^{-\beta E(n)}.$$

Since particles cannot be distinguished from each other, an interchange of two particles between two different cells does not lead to a new case. Every distribution given by eq. (36.16) must thus be counted only once. Instead of the permutability in eq. (36.17) we must put a factor of unity. In other words we must use the summation \sum' as we have done in eq. (36.7).

The summation becomes more difficult to perform after dropping the factor due to permutability. First of all it can no longer be calculated directly from the partition function for the μ -space. The property of being reducible

¹This Section is based on the short, but significant presentation due to Schrödinger, *Statistical Thermodynamics*, Cambridge University Press, 1948.

to the μ -space applies only to the classical perfect gas. Before proceeding to compute the sum (2 a) in a systematic way, we propose to consider a limiting case which will allow us to perceive why the use of the partition function (36.17) is almost sufficient in many cases.

We now revert to the remark (cf. Sec. 29, p. 216) that in the case of a perfect gas under normal conditions at most one of about 30,000 cells contains just one molecule.¹ Thus in this case we only have to consider occupation numbers 0 and 1, so that the terms $n_i!$ in the denominator of the expression for permutability are, practically, all equal to unity. This signifies that the sums (36.17) and (2 a) differ only by a factor $N!$ which is independent of n_i and which may be taken outside the summation sign \sum_n . It follows that for perfect gases under normal conditions we may write:

$$(3) \quad Z_{\text{class}} = Z_0^N \approx N! Z_{\text{quant}} = N! Z$$

and that the new partition function is given by

$$Z \approx \frac{Z_0^N}{N!},$$

and

$$(3 a) \quad \log Z \approx N \log Z_0 - N (\log N - 1)$$

in which $\log N!$ has been expanded with the aid of Stirling's formula.

The argument in the preceding Section remains unchanged except for an additive constant. It is, however, seen that this constant suffices to correct Sackur's equation. In fact, if we add the term $-N(\log N - 1)$ to the entropy in eq. (31.5 c), we obtain precisely the formula due to *Sackur and Tetrode*, which is

$$(4) \quad S = k N \log \frac{V}{N} (2\pi m k T)^{3/2} e^{5/2} / h^3.$$

As required *a priori*, this expression is proportional to N . Nevertheless the equation does not yet satisfy Nernst's theorem. In order to correct for this deficiency it is necessary to perform a more accurate evaluation of the sum and this becomes possible by:

¹Mean number of molecules per phase cell $e^{-\alpha - \beta \epsilon_i}$, hence the maximum is $e^{-\alpha}$; mean number of cells per particle according to eqs. (29.14 and 15): $e^\alpha = Z_0/N$ so that for monatomic gases $(V/N) (2\pi m k T)^{3/2} / h^3 \approx 30\,000$ for $m = 1.67 \times 10^{-24}$ g.

B. THE METHOD DUE TO DARWIN AND FOWLER¹

The method allows us to take into account the distribution condition contained in eq. (36.16). In mentioning the names of Darwin and Fowler in the title of this Section we are compelled to make one reservation. The two authors applied their method to classical statistical considerations, as witnessed by the date (1922) of the publication just mentioned; it preceded the formulation of quantum statistics. If they had employed the partition function Z , they could have made reference only to its form (36.17). We shall see in greater detail that such a scheme leads to the classical relation between Z and Z_0 , that is to no new discovery. In fact, as seen from eqs. (36.17) to (36.19), it is possible to take into account the distribution condition (36.16) in an elementary way.

The object which the two authors have set themselves consisted in demonstrating the way in which it is possible to take into account the energy condition

$$(5) \quad U = \sum n_i \varepsilon_i$$

in Boltzmann's statistics, when the n_i 's are small numbers. Since their method relies on the fact that the terms in the sum (5) are integers, they were forced to measure the energies in "such small units" that all ε_i 's, and hence the total energy U , could be approximated by integers. We shall denote this unit by ε_0 ; (it was not introduced by them explicitly). We are forced to introduce it in order to be able to expand the function in the ζ -plane which corresponds to our $Y(\zeta)$ (see eq. (8) below) in terms of integral powers of ζ so as to be able to use Cauchy's theorem. In using this method we cannot avoid passing to the limit $\varepsilon_0 \rightarrow 0$ which implies that the energy scale has been divided into infinitely small elements which is inconsistent with the finite dimensions of phase elements. Pascual Jordan mentions this point in his "Statistical Mechanics,"² but "passes over these disturbing circumstances without a more detailed discussion." The energy condition (5) has already been taken into account by the form of the partition function. The latter has been deduced in a different way in Sec. 36, and we no longer need use the Darwin-Fowler method in connection with (5). At this stage, however, we have to introduce the distribution condition (36.16) which is not trivial in the realm of quantum

¹C. G. Darwin and R. H. Fowler, Phil. Mag. Vol. 44, 450, 823, (1922); see also "Statistical Mechanics" by R. H. Fowler, Cambridge University Press, 1929.

²Vol. 87 of the series "Wissenschaft", 2. ed. footnote 2, p. 33, Vieweg, 1944.

statistics. We can apply to it the Darwin-Fowler method (*cf.* Schrödinger, *l. c.*, Chap. VII //). In fact its application is less problematical because the terms in (36.17), unlike those in (5), are, by their nature, integral numbers.

In order to calculate the partition function (2 a) we refer to eq. (36.18). Substituting the energy from eq. (1), we obtain

$$(6) \quad Z = \sum_{(n)} z_1^{n_1} z_2^{n_2} \dots$$

If we now extended the summation over *all* n_i 's we would introduce a large number of terms which do not belong in the partition function (6). It is, however, possible to reject them by the use of a stratagem: The quantities z_i from eq. (36.18) are replaced by ζz_i , so that

$$(7) \quad \prod_i z_i^{n_i} \text{ is replaced by } \zeta^{\sum_i n_i} \prod_i z_i^{n_i},$$

where

$$\sum = \sum_i n_i.$$

Let the symbol $Y(\zeta)$ denote the sum resulting from (6) with unrestricted values of n_i . We expand it in powers of ζ and concentrate our attention on that group of terms which is multiplied by ζ^N . According to (7) this is our partition function (5 a). Thus we may write

$$(8) \quad Y(\zeta) = \dots + \zeta^N Z + \dots$$

A second stratagem isolates the terms with ζ^N from all others. Following Darwin and Fowler we make use of Cauchy's theorem on residues, and we obtain

$$(9) \quad Z = \frac{1}{2\pi i} \oint Y(\zeta) \zeta^{-N-1} d\zeta$$

where ζ is regarded as a complex variable; the integration is to be performed along a closed path in the ζ -plane encircling the origin but no other singularity. In this way all terms in the series in (8) denoted by \dots are excluded and we retain only that residue (the term with ζ^{-1}) which yields Z directly, in accordance with (9).

C. BOSE-EINSTEIN AND FERMI-DIRAC STATISTICS

We now proceed to analyze the auxiliary function $Y(\zeta)$ a little more closely. Having substituted ζz for z in (6) and having expressly lifted the distribution condition (36.16), we obtain the general expression:

$$(10) \quad Y(\zeta) = \sum_{n_1=0}^{\infty} (\zeta z_1)^{n_1} \cdot \sum_{n_2=0}^{\infty} (\zeta z_2)^{n_2} \cdot \dots$$

The summations are easy to perform if the n_i 's can assume all values

$$n_i = 0, 1, 2, \dots$$

as already indicated in (9). We then have simply

$$(11) \quad Y(\zeta) = \prod_i \frac{1}{1 - \zeta z_i}.$$

From the point of view of wave mechanics the above result means that the eigen-functions of the system are *symmetrical* functions of the coordinates of its components. This case was developed in 1924 by S. N. Bose for the light-quantum gas and extended shortly afterwards by Einstein to include material gases.¹

There exists another case which is realized in nature and which corresponds to the *antisymmetrical* eigen-functions of wave mechanics. In such a case we have

$$n_i = 1 \quad \text{or} \quad 0.$$

The latter case was introduced into wave mechanics in 1926 by Fermi, who made use of *Pauli's exclusion principle*, and independently by Dirac. The most important application of this statistics occurs in relation to metal electrons.

In this case eq. (10) leads directly to

$$(11 \text{ a}) \quad Y(\zeta) = \prod_i (1 + \zeta z_i).$$

The two cases can be represented by the single equation

$$(12) \quad Y(\zeta) = \prod_i (1 \mp \zeta z_i)^{\mp 1}$$

(the upper sign gives the Bose-Einstein statistics, and the lower sign leads to the Fermi-Dirac statistics).

¹A detailed discussion of the relation of this result to symmetrical and antisymmetrical wave functions in wave mechanics would exceed the scope of this book.

In both cases we have $Y(0) = 1$ and $Y(\zeta)$ can be expanded into a series of integral powers of ζ in the neighborhood of $\zeta = 0$, as already assumed in (8). In the Fermi-Dirac case $Y(\zeta)$ is a holomorphic function which increases monotonically along the real positive axis. In the Bose-Einstein case $Y(\zeta)$ is a meromorphic function which possesses poles at all points where

$$\zeta = \zeta_i = 1/z_i.$$

In accordance with the definition of z_i in eq. (36.18) all such points lie on the positive real axis of ζ . If we normalize the energy so that all $\varepsilon_i \geq 0$, the poles are all on the other side of $\zeta = 1$ and for ζ increasing they cluster at infinity. For values $\zeta < (\zeta_i)_{min}$, $Y(\zeta)$ behaves monotonically in the Bose-Einstein case as well.

We then consider the logarithm of the integrand in (9), denoting

$$(13) \quad F(\zeta) = \log Y(\zeta) - (N + 1) \log \zeta.$$

It is equal to $+\infty$ at $\zeta = 0$ (because $\log \zeta = -\infty$) and decreases very fast as long as the second term on the right-hand side is predominant. However, the first term begins to predominate before ζ reaches the value of unity; this term increases monotonically like $Y(\zeta)$. Thus on the positive real axis there exists a point ζ_0 at which

$$(13a) \quad F'(\zeta_0) = 0.$$

The corresponding value $F''(\zeta_0)$ is very large and positive, because the transition from $F(\zeta)$ decreasing to $F(\zeta)$ increasing takes place rapidly; in fact, the change is faster for larger values of N , as we shall show later.

The last remarks serve to prepare the ground for the following. To conclude this section we wish to make a remark regarding the introduction of the two "new" kinds of statistics: The statistics themselves are not new, but the objects to which we apply them are. The new objects consist of indistinguishable particles and their quantum states which are of a symmetrical or an antisymmetrical nature.

D. THE SADDLE-POINT METHOD

We now proceed to evaluate the integral (9). Making use of the logarithm of the integrand defined in (13) we may replace (9) by

$$(14) \quad Z = \frac{1}{2\pi i} \oint e^{F(\zeta)} d\zeta.$$

We now expand $F(\zeta)$ into a power series at the point $\zeta = \zeta_0$, its linear term vanishing in accordance with (13 a):

$$(15) \quad F(\zeta) = F(\zeta_0) + \frac{1}{2} F''(\zeta_0) (\zeta - \zeta_0)^2 + \dots$$

Two-dimensional potential functions cannot possess a real maximum or minimum. Since $\nabla^2 \Phi = 0$, the second derivatives $\partial^2 \Phi / \partial x^2$ and $\partial^2 \Phi / \partial y^2$ must always be of opposite signs which means that the surfaces $u = \text{const}$ are always convex upwards in one direction, and convex downwards at right angles to it. Thus points where $\partial \Phi / \partial x = \partial \Phi / \partial y = 0$ are saddle points. The same is true about the real and imaginary parts u, v , of any complex function $F(\zeta)$ at points $\zeta = \zeta_0$ where $F'(\zeta_0) = 0$.

When discussing the behavior of our function $F(\zeta)$ along the positive real axis it was shown that it possesses a sharply marked minimum. Considering the topography of a saddle-like surface of the type of a potential function we see that it must possess a sharply marked maximum at the same point and along a line passing through ζ_0 parallel to the imaginary axis.

In evaluating our integral it is necessary to proceed along a path encircling the origin, e. g. along a circle. Drawing the circle through ζ_0 we see that during the process of integration we pass through a steep path ("steepest descent" on the one side, and "steepest ascent" on the other). The only important contribution to the integral comes from the neighborhood of the saddle point; in its neighborhood the circle can be replaced by a segment of the tangent to it and the remainder of the circle may be neglected, see Fig. 29. Along this segment of the tangent we have

$$(16) \quad \zeta = \zeta_0 + i y; \quad -y_0 < y < +y_0.$$

Neglecting the higher terms we find from (14) and (15) that

$$(17) \quad Z = \frac{1}{2\pi} e^{F(\zeta_0)} \int_{-y_0}^{+y_0} \exp \left\{ -\frac{1}{2} F''(\zeta_0) y^2 \right\} dy.$$

Introducing the new variable

$$\eta = y \sqrt{\frac{1}{2} F''(\zeta_0)}$$

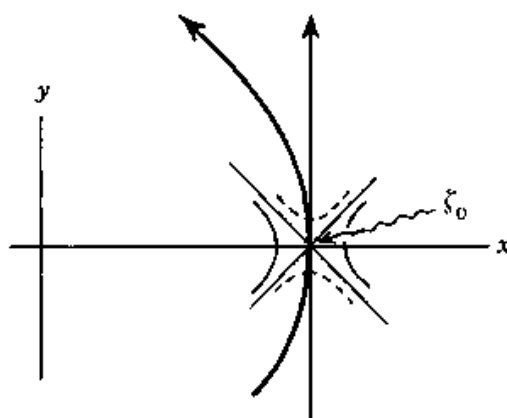


Fig. 29.

The ζ -plane ($\zeta = x + i y$) in the neighborhood of the saddle point ζ_0 with a qualitative representation of lines of constant elevation.

and assuming that $F''(\zeta_0)$ is sufficiently large, we obtain

$$Z = \frac{1}{2\pi} \sqrt{\frac{2}{F''(\zeta_0)}} e^{F(\zeta_0)} \int_{-\infty}^{+\infty} e^{-\eta^2} d\eta = \frac{e^{F(\zeta_0)}}{\sqrt{2\pi F''(\zeta_0)}},$$

and hence

$$(18) \quad Z = \frac{Y(\zeta_0)}{\zeta_0^{N+1} \sqrt{2\pi F''(\zeta_0)}}$$

in view of (13).

It is very instructive to prove this result in the classical case first, even though in this case the sum can be calculated exactly without difficulty, and without the use of the stratagem due to Darwin and Fowler. Instead of starting with the definition (6) of the partition function, we would now start with its definition in (36.17 a). If we now drop the partition condition (36.16), as we have done in (6), and if we write the denominators $n_1!$, $n_2!$, ... under the respective signs of summation over n_1 , n_2 , ..., we obtain

$$(19) \quad Y_{class} = N! \prod_i \left(\sum_{n_i} \frac{(\zeta z_i)^{n_i}}{n_i!} \right) = N! e^{\zeta(z_1 + z_2 + \dots)} = N! e^{\zeta Z_0}$$

which replaces the preceding function $Y(\zeta)$. This case shows again that the partition function Z_0 has a legitimate meaning in classical mechanics.

From eqs. (13) and (19) we can deduce the relations

$$(20 \text{ a}) \quad F(\zeta) = \zeta Z_0 + \log N! - (N+1) \log \zeta,$$

$$(20 \text{ b}) \quad F'(\zeta) = Z_0 - \frac{N+1}{\zeta}$$

$$(20 \text{ c}) \quad F''(\zeta) = (N+1)/\zeta^2,$$

so that according to (13), we have

$$(20 \text{ d}) \quad \zeta_0 = \frac{N+1}{Z_0}, \quad \sqrt{2\pi F''(\zeta_0)} = \left(\frac{2\pi}{N+1} \right)^{\frac{1}{2}} Z_0,$$

and eq. (18) transforms into

$$(21) \quad Z = \frac{N! e^{N+1} Z_0^N}{(N+1)^N \sqrt{2\pi(N+1)}} \rightarrow Z_0^N.$$

According to Stirling's formula (29.4 a), the numerical factors are equal to unity; for $N \gg 1$ we have, namely:

$$(21 \text{ a}) \quad \frac{N! e^{N+1}}{(N+1)^N \sqrt{2\pi(N+1)}} = \frac{e}{(1+1/N)^N} \cdot \frac{1}{\sqrt{1+1/N}} \approx 1.$$

Thus our eq. (21) leads to nothing new, but reproduces the already familiar eq. (39.16), as mentioned in Sec. B. The return to the classical case in the preceding argument can thus be regarded as a check on the not very simple analytical methods used in the approximation. Furthermore, we may consider that it offers a substitute for the proof that $F''(\xi_0)$ is very large for large values of N . We have, in fact, assumed in the transition from (17) to (18) that this was necessary hypothesis for the application of the method of steepest descent (saddle-point method). We can now infer from eq. (20 c) that this assumption is satisfied in the classical case: We know that ξ_0 is finite and that $\xi_0 < 1$ and conclude from (20 c) that $F''(\xi_0)$ increases to infinity proportionately with N . We shall assume that the same occurs in quantum statistics.

Reverting to the latter, we form the logarithm of the partition function given in (18), making use of the representation of $Y(\xi)$ from (12):

$$\log Z = \mp \sum_i \log (1 \mp \xi_0 z_i) - (N+1) \log \xi_0 - \frac{1}{2} \log [2\pi F''(\xi_0)].$$

In this equation we may neglect 1 compared with N . The last term is of order $\log N$ which in the limit $N \rightarrow \infty$ is negligible compared with N . It can be seen that in the limit of $N \rightarrow \infty$ it can also be neglected in comparison with the others. In this manner we obtain

$$(22) \quad \log Z = \mp \sum_i \log (1 \mp \xi_0 z_i) - N \log \xi_0,$$

where ξ_0 is determined by eqs. (13) and (13 a). Thus

$$(22') \quad \frac{d}{d\xi_0} [\log Y(\xi_0)] = \frac{N}{\xi_0}$$

when we neglect 1 against N once more. Consequently

$$(23) \quad \sum_i \frac{\xi_0 z_i}{1 \mp \xi_0 z_i} = N,$$

in accordance with eq. (12).

Substituting the expressions (36.18) into eqs. (22) and (23) and putting $\zeta_0 = e^{-\alpha}$, we have

$$(22 \text{ a}) \quad \log Z = \mp \sum_i \log (1 \mp e^{-\alpha - \beta \epsilon_i}) + N \alpha$$

and

$$(23 \text{ a}) \quad \sum_i \frac{1}{e^{\alpha + \beta \epsilon_i} \mp 1} = N.$$

It follows from (23 a) that for large values of α

$$e^{-\alpha} Z_0 = N, \quad \alpha = \log Z_0 - \log N$$

and from (22 a) that

$$\log Z = N \log Z_0 - N (\log N - 1)$$

in agreement with eq. (3 a). It is seen that $\alpha \gg 1$ corresponds to the limiting case of ordinary gases. As α becomes small, or even negative, we obtain degenerate gaseous states. The most important example of a degenerate gas is afforded by the conduction electrons in metals which we propose to consider in the following Sections (38 and 39).

38. Degenerate gases

A. BOSE-EINSTEIN AND FERMI-DIRAC DISTRIBUTION

We have based our considerations in Sec. 37 on the partition function in the I -space because identical molecules are indistinguishable, but the results, as given in eqs. (37.22 a) and (37.23 a), contain only sums over the μ -space. The sum in eq. (37.22 a)

$$(1) \quad \log Y = \mp \sum_i \log (1 \mp e^{-\alpha - \beta \epsilon_i}) = \Phi(\alpha, \beta)$$

plays a similar part to that played by the partition function itself. It is a thermodynamic potential. It follows from (37.22') that:

$$(2) \quad N = - \frac{\partial \Phi}{\partial \alpha}.$$

The internal energy follows from (36.8) and is

$$(3) \quad U = -\frac{\partial \Phi}{\partial \beta}$$

and the number of particles follows from (36.7):

$$(4) \quad \bar{n}_i = -\frac{1}{\beta} \frac{\partial \Phi}{\partial \epsilon_i} = \frac{1}{e^{\alpha + \beta \epsilon_i} \mp 1}.$$

Applying eq. (4) we can deduce from (2) and (3) that

$$(5) \quad N = \sum \bar{n}_i, \quad U = \sum \bar{n}_i \epsilon_i.$$

Applying (36.9), we obtain an expression for entropy

$$(6) \quad S = k(\Phi + \alpha N + \beta U).$$

The differential form of Φ has the form:

$$(7) \quad d\Phi = -N d\alpha - U d\beta - \beta \sum \bar{n}_i d\epsilon_i.$$

Using the language of thermodynamics we can state it represents the potential in terms of α , β , and ϵ_i as independent variables.

It is seen from eq. (7) that Φ defined in (1) represents a kind of thermodynamic potential. It differs from previous potentials in that it contains the number of particles as an independent variable, in addition to energy (Problem II. 1). In the case of a completely isolated system the energy remains constant ($U = \text{const}$) and all phase points lie on one surface in the phase space. Every phase element of equal volume between this surface and the energy surface $U + \Delta U$ is of equal probability. We speak of a microcanonical distribution and of a *microcanonical ensemble*.

So far we have carried out our consideration in relation to *canonical ensembles*. The latter occur when the system remains in thermal contact with the surroundings. For every temperature we obtain a canonical distribution function which admits fluctuations of the mean energy; these are due to the energy fluctuations between system and bath. In the case of systems with very many degrees of freedom the differences between the results of calculations for microcanonical systems, as against canonical systems, are, generally speaking, not significant, because the energy fluctuations are exceedingly small.

The function of state, Φ , refers to so-called "*grand canonical ensembles*". When α is constant and is not eliminated, unlike in the following argument, the number of molecules N is also subject to fluctuations. Now there is more

than only thermal contact with the surroundings, and an exchange of particles becomes possible. This causes fluctuations in the total number of particles. Again, in the case of large systems, the fluctuations in the number of particles are small, and no appreciable differences result. We shall restrict ourselves to these remarks and will now revert to the canonical distributions, eliminating α with the aid of eq. (2) (see also Sec. 40).

In the limiting case of large values of α , we obtain from (1) that

$$(1') \quad \log Y = \Phi = e^{-\alpha} \sum e^{-\beta \epsilon_i} = e^{-\alpha} Z_0.$$

With the aid of eqs. (2) to (4) we can deduce from it that

$$N = e^{-\alpha} Z_0, \quad U = -e^{-\alpha} \frac{\partial Z_0}{\partial \beta}, \quad \bar{n}_i = -\frac{e^{-\alpha}}{\beta} \frac{\partial Z_0}{\partial \epsilon_i},$$

or, eliminating α , we have

$$(8) \quad U = -N \frac{\partial \log Z_0}{\partial \beta}, \quad \bar{n}_i = -\frac{N}{\beta} \frac{\partial \log Z_0}{\partial \epsilon_i}$$

in agreement with (29.6). Here \bar{n}_i represents the ordinary Boltzmann distribution function.

The same calculation can be performed without assuming that α is very large, but the elimination of α ceases to be elementary. In any case, we can now perform our calculations in the μ -space. It will be noticed that in the Γ -space we always have the distribution function $\exp\{-\alpha - \beta E(n)\}$ and that the only change occurs in the number of energy steps in $E(n)$. On the other hand, according to (4), in the μ -space, the Boltzmann distribution function $\exp(-\alpha \beta \epsilon_i)$ is now replaced by new distribution functions. In the *Bose-Einstein case*:

$$(4a) \quad \bar{n}_i = \frac{1}{e^{\alpha + \beta \epsilon_i} - 1},$$

and in the *Fermi-Dirac case*:

$$(4b) \quad \bar{n}_i = \frac{1}{e^{\alpha + \beta \epsilon_i} + 1}.$$

Accordingly quantum mechanical gases can also be described in the μ -space if we, as it is said, adopt new statistics. It is usual to derive eqs. (4a) and (4b) with the aid of combinatorial methods which take into account the fact that particles cannot be distinguished one from another, as well as Pauli's

principle, right in the μ -space. We shall, however, refrain from pursuing this point.

The entropy equation (6) leads to a number of important conclusions. It follows from eq. (4) that

$$e^{\alpha + \beta \epsilon_i} = \frac{1 \pm \bar{n}_i}{\bar{n}_i}.$$

Substituting this expression into eq. (6), we obtain

$$\frac{1}{k} S = \sum_i \left\{ \mp \log \left(1 \mp \frac{\bar{n}_i}{1 \pm \bar{n}_i} \right) + \bar{n}_i \log \frac{1 \pm \bar{n}_i}{\bar{n}_i} \right\}$$

or, after an elementary rearrangement:

$$(9) \quad \frac{1}{k} S = - \sum_i \{ \bar{n}_i \log \bar{n}_i \mp (1 \pm \bar{n}_i) \log (1 \pm \bar{n}_i) \}.$$

The first term is identical with Boltzmann's $\log W$ in eq. (36.10). The whole equation represents the quantum mechanical expression for $\log W$, because the distribution functions (4 a) and (4 b) yield, precisely,

$$(10) \quad \log W^\pm = - \sum_i \{ f_i \log f_i \mp (1 \pm f_i) \log (1 \pm f_i) \}$$

subject to the additional conditions that $N = \sum f_i = \text{const}$, $U = \sum f_i \epsilon_i = \text{const}$ for the maximum.

These changes in the thermodynamic probability can be easily interpreted in the Fermi-Dirac case. Taking into account that

$$(11) \quad f_i^0 = 1 - \bar{n}_i, \quad f_i' = \bar{n}_i$$

represent the probabilities of finding either *no* molecule or *one* molecule in the i -th quantum state, we can infer from (10) that

$$(12) \quad \log W^- = - \sum_i (f_i^0 \log f_i^0 + f_i' \log f_i').$$

This expression differs from that due to Boltzmann by one term which represents Boltzmann's thermodynamic probability of the empty places. Fermi's distribution can now be obtained by determining the maximum of W^- subject to the conditions

$$(13) \quad f_i^0 + f_i' = 1, \quad \sum_i f_i' = N, \quad \sum_i \epsilon_i f_i' = U.$$

The same is true of the Bose-Einstein case. In this case also it is necessary to take into account the Boltzmann thermodynamic probabilities of the empty spaces. Let $f_i^{(n)}$ denote the probability of finding n molecules at the i -th quantum state; the thermodynamic probability is then equal to the sum of expressions of Boltzmann's type:

$$(14) \quad \log W^+ = - \sum_i (f_i^0 \log f_i^0 + f_i' \log f_i' + f_i'' \log f_i'' + \dots).$$

The Bose-Einstein distribution (4 a) follows from this expression and $\log W^+$ follows from eq. (9) on calculating the maximum of (14), subject to the additional conditions that

$$(15) \quad \sum_{n=0}^{\infty} f_i^{(n)} = 1, \quad \sum_{i,n} n f_i^{(n)} = N, \quad \sum_{i,n} n f_i^{(n)} \epsilon_i = U.$$

A proof is given in Problem IV. 7.

B. DEGREE OF GAS DEGENERATION

In the present Section we shall consider monatomic quantum mechanical perfect gases. In this case the sum in eq. (1) can be written as

$$(16) \quad \Phi = \mp \frac{4\pi V}{h^3} \int_0^{\infty} \log (1 \mp e^{-\alpha - \beta p^2/2m}) p^2 dp$$

if a continuous approximation with the aid of an integral is used. Alternatively

$$(16 a) \quad \Phi = \frac{V}{h^3} \left(\frac{2\pi m}{\beta} \right)^{3/2} \cdot \chi(\alpha),$$

where

$$(16 b) \quad \chi(\alpha) = \mp \frac{2}{\sqrt{\pi}} \int_0^{\infty} \log (1 \mp e^{-\alpha - t}) \sqrt{t} dt$$

with $\beta p^2 = 2 m t$. A correction to this equation is given later in (26). The number of particles can be calculated from (16 a) with the aid of (2), (3) and (4), when we find

$$(17) \quad N = - \frac{V}{h^3} (2\pi m k T)^{3/2} \chi'(\alpha).$$

Similarly the *energy* is:

$$(17\text{ a}) \quad U = \frac{3}{2} \frac{V}{h^3} (\sqrt{2\pi m k T})^3 \cdot k T \cdot \chi(\alpha) = \frac{3}{2} N k T \cdot [-\chi(\alpha)/\chi'(\alpha)],$$

and the *pressure* is:

$$(17\text{ b}) \quad p = \frac{1}{\beta} \frac{\partial \Phi}{\partial V} = \frac{1}{h^3} (\sqrt{2\pi m k T})^3 \cdot k T \cdot \chi(\alpha),$$

(taking into account that $\Phi + N\alpha = -\beta F$ and that $dF = -S dT - p dV$). From the last two equations we can deduce the relation

$$(18) \quad p V = \frac{2}{3} U$$

which is independent of $\chi(\alpha)$ and which thus remains valid in quantum mechanics.

According to eq. (17) α is a function of the ratio

$$(19) \quad \rho = \frac{N}{V} \cdot \frac{h^3}{(\sqrt{2\pi m k T})^3}$$

which is a measure of the deviation of eqs. (17), (17 a), and (17 b) from the ideal gas state. The quantity ρ is defined as the *degree of gas degeneration*. From eqs. (16 b), (17), and (19), we find that

$$(20) \quad \rho = -\chi'(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{t} dt}{e^{\alpha+t} \mp 1}$$

showing that ρ is small for large values of α . A small value of ρ also means that the behavior of the gas is classical. We have estimated in Sec. 37 that for a gas under normal conditions $\rho \approx 1/30,000$. The influence of ρ on the energy of a Bose-Einstein and of a Fermi-Dirac gas is shown graphically in Fig. 30. It serves to eliminate α from eqs. (17) and (17 a).

An example for large values of ρ and large degeneration is afforded by the conduction electrons in a metal. We shall see in Sec. 39 that to a first approximation, the conduction electrons behave like free particles. In other words they move through the metal like gaseous molecules in a vessel, but the electron gas is highly degenerate. On the assumption that one electron per atom in copper is a conduction electron we find $\rho \approx 5000$.

The electron gas is of the Fermi-Dirac type because electrons behave in accordance with Pauli's principle. For small values of ρ both types of gases (i. e. Fermi-Dirac and Bose-Einstein) behave identically, but in the limiting case of large values of ρ the differences are considerable, and it is necessary to discuss them separately.

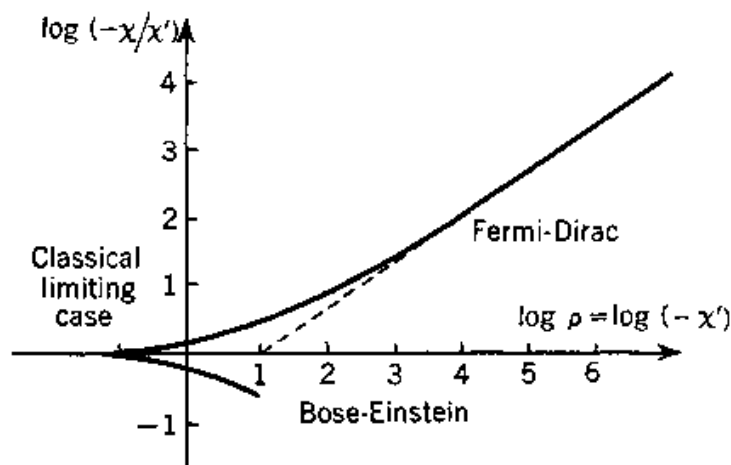


Fig. 30.

The logarithm of the energy factor $U/U_{Boltzm.} = -\chi/\chi'$ from eq. (38.17 a) plotted in terms of the logarithm of the degree of degeneration $\rho = -\chi'$ as seen from eq. (38.20). (after F. L. Bauer).

C. HIGHLY DEGENERATE BOSE-EINSTEIN GAS

Until recently, apart from light quanta, no instances of appreciable degeneration in the case of Bose-Einstein statistics were known. It appears, however, that superfluidity¹ in helium at very low temperatures is linked with gas degeneration. In any case it is a fact that superfluidity occurs only in the isotope He^4 which satisfies Bose-Einstein statistics. It is absent in He^3 which behaves in accordance with Fermi-Dirac statistics.

According to eqs. (17) and (20) the number of particles of a Bose-Einstein gas is proportional to

$$(21) \quad -\chi'(\alpha) = \frac{2}{V\pi} \int_0^\infty \frac{\sqrt{t} dt}{e^{\alpha+\beta t} - 1}.$$

This integral reaches its maximum $\alpha = 0$. Values of $\alpha < 0$ must be excluded because $-\chi'(\alpha)$ then diverges. The fact that the above integral just converges for $\alpha = 0$ and assumes the value $\zeta(3/2) = 2.612$ ($\zeta = \text{Zeta function}$)

¹F. London, "Superfluids", Vol. I, Introduction, Sec. 4; Structure of matter series, New York, London, 1950.

could be interpreted to mean, in accordance with eq. (17), that only a finite number of Bose-particles can be accommodated in a finite volume, and that their number decreases like $T^{3/2}$ as the temperature decreases and becomes equal to zero, at absolute zero. Such a consequence does not appear plausible, and contradicts the assumption that in the case of Bose-Einstein statistics it is possible to have an arbitrary number of particles in each quantum state.

In fact, the conclusion is false. It is a consequence of the continuous approximation used for the sum (1). At low temperatures when $\beta \epsilon_i \gg 1$ it is not permissible to replace the sum by an integral. If the energy is so normalized that its lowest value is equal to zero, and if we arrange the values of energy in their order of magnitude

$$0 = \epsilon_0 < \epsilon_1 \leq \epsilon_2 \leq \dots,$$

we find that the mean numbers of molecules for the different energy states are given by eq. (4). They are

$$(22) \quad \bar{n}_0 = \frac{1}{e^{\alpha} - 1}, \quad \bar{n}_i = \frac{g_i}{e^{\alpha + \beta \epsilon_i} - 1} \quad (i = 1, 2, \dots).$$

The factors g_i which appear here indicate the multiplicity of the energy values ϵ_i , in the same way as in eq. (33.11). The ground state is assumed to be simple. Since in the limit $\alpha \rightarrow 0$, \bar{n}_0 increases without limit, the number of particles ceases to be bounded.

The fact that (21) remains finite at $\alpha = 0$ means only that the number of particles $\bar{n} = \sum_{i=1}^{\infty} \bar{n}_i$ present in excited states cannot exceed an upper limit so that an increase in the number of particles beyond that limit contributes only to the ground state. The particles condense to a certain extent into the ground state (Einstein condensation). The number \bar{n} is determined by the integral (21). Defining the degree of degeneration in the same way as before, in eq. (19), we see that eq. (17) yields¹:

$$(23) \quad \bar{n} = -\frac{N}{\rho} \chi'(\alpha) \leq \frac{N}{\rho} \zeta\left(\frac{3}{2}\right) = N_0.$$

because the disappearance of the lowest energy level in the definition of \bar{n} makes practically no difference. It is possible to estimate that the error is of the order $\delta \bar{n} / \bar{n} = (\rho / N)^{1/2}$, which is negligible even in cases of appreciable degeneration.

¹It is evident that eq. (20) ceases to apply because it implies $\bar{n} = N$.

We now direct our attention to the consideration of high degrees of degeneration. According to (23) we then have $\bar{n} \ll N$, and practically all particles are at the ground state. Thus eq. (20) shows that α is very small. We have $N_0 \ll N$ and

$$(24) \quad \bar{n} = N_0, \quad \bar{n}_0 = N - N_0, \quad \alpha = \frac{1}{N - N_0}.$$

As a first approximation the numbers of particles in *excited* states is given by

$$(25) \quad \bar{n}_i \approx n_i^0 = \frac{g_i}{e^{\beta \epsilon_i} - 1}$$

where $\alpha \approx 0$ has been substituted into (22). We now inquire into when we may substitute the n_i^0 's, defined in eq. (25), for the \bar{n}_i 's, or, in other words we ask for the conditions under which we have

$$\frac{n_i^0 - \bar{n}_i}{\bar{n}_i} = \frac{(e^\alpha - 1) e^{\beta \epsilon_i}}{e^{\beta \epsilon_i} - 1} = \frac{e^{\beta \epsilon_i / \bar{n}_0}}{e^{\beta \epsilon_i} - 1} \ll 1.$$

First it follows that this expression is small for large values of $\beta \epsilon_i$. Furthermore, we may have $\beta \epsilon_i \ll 1$ on condition that for all excited states ϵ_i we also have:

$$\beta \epsilon_1 \gg \frac{1}{\bar{n}_0} \approx \frac{1}{N}.$$

The lowest excited quantum state ϵ_1 is of the order

$$\epsilon_1 \approx h^2 / 2 m V^{2/3},$$

which corresponds to a de Broglie wavelength of the order of magnitude of the linear dimensions of the volume. Thus the above inequality is satisfied if

$$\frac{h^2}{2 m k T} \cdot V^{-2/3} \gg \frac{1}{\bar{n}_0}.$$

In view of (19), the following condition may be written down

$$(26) \quad \rho \pi^{3/2} \gg \frac{N}{\bar{n}_0^{3/2}}$$

which is satisfied in the limiting case of large values of ρ , because we then have $\bar{n}_0 \approx N$.

The numbers of particles of a highly degenerate Bose-Einstein gas occupying excited states are given by eq. (25) and depend on temperature alone. By way

of a first approximation they can be calculated from the distribution function which we have already encountered in connection with light quanta (Sec. 20). The more exact values in eq. (22) can be deduced from the modified potential:

$$(16') \quad \Phi = -\log(1 - e^{-\alpha}) - \frac{V}{h^3} (2\pi m k T)^{3/2} \cdot \frac{2}{\sqrt{\pi}} \int_0^{\infty} \log(1 - e^{-\alpha-t}) \sqrt{t} dt.$$

The first term constitutes a correction to eq. (16) for the lowest quantum state. The pressure is independent of the correction term. According to (17 b) we have

$$(27) \quad p = \chi(\alpha) \left(\frac{2\pi m}{h^2} \right)^{3/2} (k T)^{5/2} \approx \chi(0) \left(\frac{2\pi m}{h^2} \right)^{3/2} (k T)^{5/2}.$$

Thus a kind of vapor-pressure curve has been obtained.

In conclusion we wish to show that the degeneration of a Bose-Einstein gas is compatible with Nernst's Third Law. Taking into account (1) to (3) we can deduce from (6) that

$$\frac{1}{k} S = \sum_i \left\{ -\log(1 - e^{-\alpha - \beta \epsilon_i}) + \frac{\alpha + \beta \epsilon_i}{e^{\alpha + \beta \epsilon_i} - 1} \right\}.$$

The value of α in this equation can be derived from (2):

$$N = \sum \frac{1}{e^{\alpha + \beta \epsilon_i} - 1}.$$

However, in the neighborhood of absolute zero all excited states satisfy the condition $\beta \epsilon_i \gg 1$, so that all terms in the sum for which $i \neq 0$ vanish at least in proportion to $e^{-\beta \epsilon_i}$. Only the terms which are independent of temperature remain:

$$\frac{1}{k} S_0 = -\log(1 - e^{-\alpha}) + \frac{\alpha}{e^{\alpha} - 1}, \quad N = \frac{1}{e^{\alpha} - 1}.$$

This equation shows that S tends exponentially to the constant value

$$S_0 = k(\log N + 1).$$

The *specific entropy* at absolute zero is given by

$$(28) \quad s_0 = \frac{S_0}{N} = k \frac{\log N + 1}{N} \approx 0$$

except for terms of an order which has been neglected anyway.

So far we have assumed that the lowest energy state is $\epsilon_0 = 0$, and it remains to enquire into the changes in the argument which would have to be introduced if we had $\epsilon_0 \neq 0$. Since, according to eq. (2), α depends on the temperature, we may replace α by $\alpha' - \beta \epsilon_0$ and thus we are led to eq. (1) in the same form as before. Equations (2) and (4) also remain unaltered. Equation (3) is replaced by

$$U = - \left(\frac{\partial \Phi}{\partial \beta} \right)_{\alpha = \text{const}} = - \left(\frac{\partial \Phi}{\partial \beta} \right)_{\alpha' = \text{const}} - \left(\frac{\partial \Phi}{\partial \alpha'} \right) \frac{\partial \alpha'}{\partial \beta} = U' + N \epsilon_0,$$

from which it follows that

$$U' = U - N \epsilon_0.$$

Consequently if α' is kept constant instead of α in finding the derivative with respect to β , the energy will decrease by its zero-point value $N \epsilon_0$.

39. Electron gas in metals

A. INTRODUCTORY REMARK TO DRUDE'S METHOD

Since the discovery of the electron there could be no doubt that an electric current is carried by electrons. The suggestion that the electrons in a metal behave like molecules of a gas and participate in thermal equilibrium is due to P. Drude. The greatest success achieved by Drude's theory consists in the derivation of the Wiedemann-Franz law which states that the ratio of thermal conductivity, κ , to electrical conductivity, σ , is given by the equation

$$(1) \quad \frac{\kappa}{\sigma} = 3 \frac{k^2}{e^2} T.$$

Here $-e$ denotes the elementary charge of an electron. Drude was also able to derive an expression for electrical conductivity which is of some importance even to-day:

$$(2) \quad \sigma = \frac{e^2 l n}{m \bar{v}}$$

(l = "mean free path" of an electron, n = number of *free* electrons per m^3 , \bar{v} = mean velocity, e = elementary charge, m = mass of an electron). Numerous thermoelectrical and thermomagnetic phenomena, such as thermal emf, voltaic emf, thermal emission of electrons from metals etc., can be

explained with the aid of Drude's supposition, at least qualitatively, and their inner connection can be recognized.

However, the fundamental supposition is inconsistent with experimental facts in the matter of specific heats. According to Drude's hypothesis every free electron should possess an energy of $\frac{1}{2} k T$ per degree of freedom at a state of equilibrium, and the molar energy of electrons should have the value of $3 R/2 = 3 \text{ cal/mol deg}$ which is in flat contradiction to the Dulong-Petit rule. When taking into account Maxwell's velocity distribution in a more precise way, it turns out that the numerical factor in eq. (1) should have a value of 2, instead of 3, as shown, which destroys its agreement with the measurements due to Jäger and Diesselhorst. The above and other additional difficulties disappear only if it is assumed *ad hoc* that the number of free electrons is considerably smaller than that of atoms.

The preceding difficulties completely destroyed the faith in Drude's hypothesis of the existence of an "electron gas". The lack of success appeared, essentially, comprehensible, because the electrons in a metal do not move in a *zero field of forces* but in a periodical potential field created by the ions of the metal. In addition the interaction between electrons must also play a part.

However, the existence of metal electrons which are not subjected to any forces can, nevertheless, be justified to a certain extent on the ground of wave mechanics. Moreover, Sommerfeld successfully reverted in 1928 to Drude's assumption of the existence of free electrons and was able to show that the difficulties just discussed disappear if it is taken into account that the electron gas possesses the properties of a highly degenerate Fermi-Dirac gas (*cf.* Sec. 38, eq. (20)). We now proceed to discuss some of the consequences of this theory basing our considerations on a paper by A. Sommerfeld and H. Bethe.¹

B. THE COMPLETELY DEGENERATE FERMI-DIRAC GAS

We now wish to recall eq. (38.1) remembering that the lower sign applies to the Fermi-Dirac case. For α we substitute $\alpha = -\beta \zeta$ because it can be shown that in the limiting case of $T \rightarrow 0$, ζ and not α remains finite.¹ In this way we obtain

$$(3) \quad \Phi = \sum_i \log (1 - e^{-\beta(e_i - \zeta)}).$$

¹"Elektronentheorie der Metalle", in "Handbuch der Physik" edited by H. Geiger and K. Scheel, Vol. XIV 2, Chap. 3, I, pp. 333-368.

²It is no longer possible to confuse the present ζ with that in Sec. 37.

The quantity ζ in the above equation has a simple thermodynamical interpretation. The free enthalpy in eq. (7.4), namely

$$G = U - TS + pV$$

can now be evaluated with the aid of eqs. (38.6) and (38.17 b). We thus obtain

$$G = U - \frac{1}{\beta} \Phi + \zeta N - U + \frac{V}{\beta} \frac{\partial \Phi}{\partial V}.$$

The terms in this equation which contain U and Φ cancel each other in pairs. In the latter case it should be noted that Φ is proportional to V so that $V \partial \Phi / \partial V = \Phi$. Hence we obtain

$$(4) \quad \zeta = -\frac{\alpha}{\beta} = \frac{G}{N},$$

ζ is seen to represent the *free enthalpy per electron*. Corresponding equations can also be derived for the Bose-Einstein case.

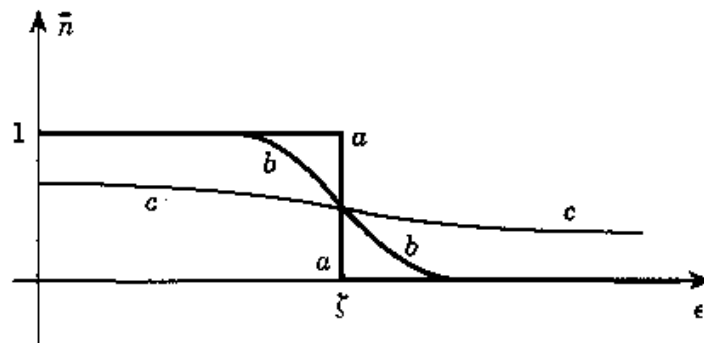


Fig. 31.

Mean number of particles per phase cell

a) at absolute zero b) for $\beta = 10/\zeta$ c) for $\beta = 1/\zeta$ ($\zeta = \text{const}$).

As distinct from the latter, α can assume negative values so that ζ becomes positive. According to eq. (38.4) the particle density is given by

$$(5) \quad \bar{n}_i = \frac{1}{e^{\beta(\epsilon_i - \zeta)} + 1}.$$

In the limiting case of $T = 0$ ($\beta \rightarrow \infty$) we obtain a value of either 0 or 1, depending on whether $\epsilon_i > \zeta$ or $\epsilon_i < \zeta$. Thus in the case of *complete degeneracy* we obtain

$$(6) \quad \bar{n}_i = \begin{cases} 1 & \text{for } \epsilon_i < \zeta_0 \\ 0 & \text{for } \epsilon_i > \zeta_0 \end{cases}$$

where the quantities at absolute zero have been denoted by the subscript 0 (see curve a in Fig. 31), and where ζ_0 plays the part of a limiting energy. All energy levels below ζ_0 are occupied, all levels above it are empty. Thus it is seen that at absolute zero the lowest levels are occupied; according to Pauli's principle every level is associated with two electrons, one each for each possible orientation of the spin of the electrons.

The total number of electrons determines the *limiting energy*. Defining the limiting momentum P_0 by

$$(7) \quad \zeta_0 = \frac{P_0^2}{2m}, \quad P_0 = \sqrt{2m\zeta_0},$$

we see that the number of particles is

$$(7a) \quad N = 2 \frac{V}{h^3} \cdot \frac{4\pi}{3} P_0^3,$$

and that the *energy* is

$$(7b) \quad U = 2 \frac{V}{h^3} \cdot \frac{4\pi}{5} \frac{P_0^5}{2m}.$$

The numerical factor 2 is due to the fact that according to quantum mechanics every phase cell includes two quantum states, owing to the two spins and hence also two electrons. The evaluation of P_0 gives

$$(8) \quad P_0 = h \left(\frac{3}{8\pi} \cdot \frac{N}{V} \right)^{1/3}.$$

Thus the *energy* and the *limiting energy* from (7b) and (7) at absolute zero are given by

$$(9) \quad U_0 = \frac{3N h^2}{10m} \left(\frac{3}{8\pi} \cdot \frac{N}{V} \right)^{2/3}, \quad \zeta_0 = \frac{h^2}{2m} \left(\frac{3}{8\pi} \frac{N}{V} \right)^{2/3},$$

so that the pressure, eq. (38.18), becomes

$$(10) \quad p_0 = \frac{8\pi h^2}{15m} \left(\frac{3}{8\pi} \cdot \frac{N}{V} \right)^{5/3}.$$

In the case of copper this pressure has a value of $p_0 \approx 3.8 \times 10^5$ atm. This enormous value corresponds to the electrical attraction between electrons and ions. The limiting energy for copper is $\zeta_{\text{Cu}} \approx 11.3 \times 10^{-12}$ erg ≈ 7 eV. It is comparable with the ionization energy of hydrogen atoms (13.54 eV). The

total energy per mol is $U_0 = 3/5 L \zeta_0$ which is roughly equal to the heat of combustion of carbon. Furthermore, the force acting on the cross-section of an ion is equal to the electrostatic attraction: $p_0 r^2 \approx e^2/4\pi\epsilon_0 r^2$ for $r \approx (V/N)$ which is of the order of several Ångström units. Leaving r provisionally undetermined we can see that, generally speaking, there will be no equilibrium. Furthermore

$$p_0 r^2 = \gamma e^2/4\pi\epsilon_0 r^2,$$

Substituting p_0 from eq. (10), it follows that

$$\gamma = \frac{4\pi\epsilon_0 r^4}{e^2} \cdot \frac{1}{5} \left(\frac{3}{8\pi} \right)^{2/3} \frac{h^2}{m r^5}$$

or, with $A = h/mc$ (Compton wavelength) and $\alpha = e^2/4\pi\epsilon_0 \hbar c$ (fine structure constant):

$$\gamma = \frac{2\pi}{5} \left(\frac{3}{8\pi} \right)^{2/3} \frac{A}{\alpha r} \approx \frac{1\text{Å}}{r}.$$

When $\gamma < 1$ (i. e. when $r > 1\text{Å}$) the pressure is too small to balance the forces of attraction. The electrons, and with them the ions, draw closer to each other. When $\gamma > 1$ ($r < 1\text{Å}$) the pressure becomes too large, and the metal will expand. It is seen that the conduction electrons are mainly responsible for the cohesion of the metal.

The preceding numerical values refer to absolute zero. In actual fact, however, the pressure, energy, and the limiting energy depend on temperature. Nevertheless, the degree of degeneration is so high that their dependence on temperature is very weak. It is seen that the degree of degeneration, eq. (38.19), becomes equal to unity at the *characteristic temperature of degeneration*.

$$\Theta = \frac{h^2}{2\pi m k} \left(\frac{N}{V} \right)^{2/3} = \frac{\zeta_0}{\pi k} \left(\frac{8\pi}{3} \right)^{2/3} \approx 100,000 \text{ K.}$$

C. ALMOST COMPLETE DEGENERACY

With increasing temperature the step-function from eq. (6) becomes smoothed out (see Fig. 31, curves *b* and *c*), but the transition between $\bar{n}_i = 1$ and $\bar{n}_i = 0$ takes place very rapidly at absolute zero, i. e. as long as $T \ll \Theta \approx 100,000 \text{ K}$, or $\hbar T \ll \zeta_0$, i. e. $\beta \zeta_0 \gg 1$. This enables us to evaluate the integral (38.16) and the integrals derived from it with a good degree of accuracy.

With reference to eq. (38.17) we put $t + \alpha = t - \beta \zeta = x$, and add the spin factor 2. Thus we obtain for the *number of particles*

$$(11) \quad N = \frac{4\pi V}{h^3} \left(\frac{2m}{\beta} \right)^{3/2} \int_{-\beta\zeta}^{\infty} \frac{(x + \beta\zeta)^{1/2} dx}{e^x + 1}.$$

In a similar way by partial integration we obtain from eq. (38.17 a) the following expression for *energy*:

$$(12) \quad U = \frac{4\pi V}{2m h^3} \left(\frac{2m}{\beta} \right)^{5/2} \int_{-\beta\zeta}^{\infty} \frac{(x + \beta\zeta)^{3/2} dx}{e^x + 1}.$$

Both integrals can now be further transformed by partial integration. It follows from (11) that

$$N = \frac{8\pi V}{3h^3} \left(\frac{2m}{\beta} \right)^{3/2} \int_{-\beta\zeta}^{\infty} \frac{e^x}{(e^x + 1)^2} (x + \beta\zeta)^{3/2} dx.$$

The first factor in the integrand is a symmetrical function of x . For large values of $\beta\zeta$ the second factor does not vary considerably in the interval in which $e^x/(e^x + 1)^2$ differs appreciably from zero. This enables us to expand the root in powers of x . Since, moreover, the integral decreases exponentially on both sides we can replace the lower limit of integration by $-\infty$. In this manner we obtain finally

$$(11a) \quad N = \frac{8\pi V}{3h^3} (2m\zeta)^{3/2} \int_{-\infty}^{+\infty} \frac{e^x}{(e^x + 1)^2} \left(1 + \frac{3x}{2\beta\zeta} + \frac{3x^2}{8\beta^2\zeta^2} + \dots \right) dx.$$

The integral in the middle vanishes because the integrand is odd. The first integral can be evaluated without difficulty, its value being unity. The last integral is proportional to

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{x^2 e^x}{(e^x + 1)^2} dx &= 2 \int_0^{\infty} x^2 (e^{-x} - 2e^{-2x} + 3e^{-3x} - \dots) dx = \\ &= 2 \times 2! \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right) = \frac{\pi^2}{3}. \end{aligned}$$

Substituting the above values of the integrals into eq. (11 a), we have

$$(11\ b) \quad N = \frac{8\pi V}{3h^3} (\sqrt{2m\zeta})^3 \left(1 + \frac{\pi^2}{8} \frac{k^2 T^2}{\zeta^2} + \dots \right).$$

The deviation from the limiting case of complete degeneracy is of the order $(T/\Theta)^2$. A corresponding evaluation of (12) yields

$$(12\ a) \quad U = \frac{4\pi V}{5m h^3} (\sqrt{2m\zeta})^5 \left(1 + \frac{5\pi^2}{8} \frac{k^2 T^2}{\zeta^2} + \dots \right).$$

The first terms of (11 b) and (11 a) are, naturally, identical with (7 a) and (7 b), respectively.

Denoting the value of ζ at absolute zero by ζ_0 , as before, (Fermi's limiting energy), and equating the values of N for $T = 0$ and $T \neq 0$, we obtain from eq. (11 b) that

$$\left(\sqrt{\frac{\zeta}{\zeta_0}} \right)^3 \cdot \left(1 + \frac{\pi^2}{8} \frac{k^2 T^2}{\zeta^2} + \dots \right) = 1$$

or

$$(13) \quad \zeta = \zeta_0 \left(1 - \frac{\pi^2}{12} \frac{k^2 T^2}{\zeta_0^2} + \dots \right)$$

and by (12 a)

$$(14) \quad U = U_0 \left(1 + \frac{5\pi^2}{12} \frac{k^2 T^2}{\zeta_0^2} + \dots \right)$$

so that according to eq. (38.18), we now have

$$(15) \quad p = p_0 \left(1 + \frac{5\pi^2}{12} \frac{k^2 T^2}{\zeta_0^2} + \dots \right).$$

D. SPECIAL PROBLEMS

The dependence on temperature is due to the fact that some electrons now exceed the Fermi limit, instead of all of them being below it, as at absolute zero. According to (11) and (7 a) their number is given by

$$\delta N/N = \frac{3}{2} \left(\sqrt{\frac{\zeta}{\zeta_0}} \right)^3 \frac{1}{\beta \zeta} \int_{\beta(\zeta_0 - \zeta)}^{\infty} \frac{(1 + x/\beta \zeta)^{1/2} dx}{e^x + 1}$$

which is equal to the expression for $\zeta = \zeta_0$:

$$(16) \quad \frac{\delta N}{N} = \frac{3}{2} \frac{1}{\beta \zeta_0} \int_0^\infty \frac{dx}{e^x + 1} = \frac{3}{2} \log 2 \cdot \frac{k T}{\zeta_0}$$

except for correction terms of the order T/Θ . If we take into account that only electrons which have exceeded Fermi's limit can make themselves felt thermally as well as the free places given by eq. (38.12), we shall come to the conclusion that only the small fraction δN of electrons is significant for Drude's hypothesis, and not the total number, N , of free electrons. Thus the theory leads us to a decrease in the number of effective electrons in a natural way.

The *specific heat* of electrons is seen to be so small as not to affect the Dulong-Petit rule. The heat capacity can be calculated from eq. (14), and we have

$$C = \frac{dU}{dT} = U_0 \cdot \frac{5\pi^2 k^2 T}{6 \zeta_0^2} = \frac{4\pi^3}{3} \frac{V}{h^3} (\sqrt{2m\zeta_0})^3 \frac{kT}{\zeta_0} \cdot k.$$

In the transformation use is made of equations (7 b) and (9). By (7 a) we have

$$C = \frac{\pi^2}{2} \frac{k T}{\zeta_0} \cdot N k.$$

Evaluating per mol ($N = L =$ Loschmidt-Avogadro number):

$$(17) \quad c_{\text{electr}} = \frac{\pi^2}{2} \frac{k T}{\zeta_0} \cdot R = \frac{\pi^2}{6} \frac{k T}{\zeta_0} \cdot c_{\text{Dulong-Petit}}.$$

The fraction $c_{\text{electr}}/c_{\text{Dulong-Petit}}$ is of the order of $\delta N/N$ from eq. (16). In particular, for copper we obtain $T/54,000$, so that for $T \approx 300$ K it is equal to $1/180$. The contribution due to the electrons is seen to be less than 1%.

We now propose to apply eqs. (5) or (38.8 b) to the calculation of *Richardson's effect*, i. e. to the emission of electrons from incandescent cathodes. Our assumption that the electrons move as if no forces acted between them implies a constant potential in the interior of the metal, Fig. 32. On the other hand at the boundary strong forces balance the electron pressure, so that the potential must increase rapidly from negative values, reaching the value of zero in the exterior. The quantity $W - \zeta_0$ denotes the gain in energy which occurs when an external electron assumes the Fermi limiting energy in the interior; W denotes the full difference between the potential energy of an electron in the interior and one in the exterior. The real variation

in the potential is, naturally, different from that shown in Fig. 32. In the interior there are the periodic variations, already mentioned, and at the boundary the transition does not occur in a step, but continuously, even if it is very rapid. The simplification introduced in Fig. 32 does not, however, destroy the characteristic features of Richardson's effect.

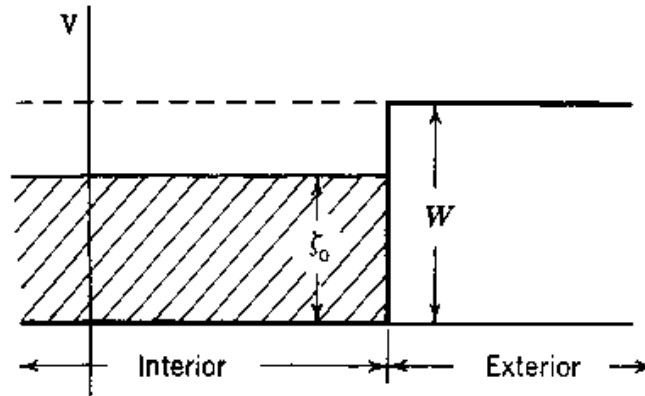


Fig. 32.

Illustrating Richardson's effect.

When the energy of the electrons in the metal becomes sufficiently large, the electrons become capable of overcoming the threshold, and can escape from the metal. The condition for this to happen is that the portion of the kinetic energy of the electron which is due to its motion at right angles to the wall exceeds W . Assuming that the z -axis is normal to the boundary, we can write

$$p_z^2/2m > W.$$

The density of the current due to the electrons leaving the metal is given by the integral

$$I_z = 2 \frac{e}{m} \cdot \frac{1}{h^3} \int_0^\infty \frac{p_z dp_x dp_y dp_z}{1 + \exp[(\beta p^2/2m) - \beta \zeta]}.$$

The lower limits of integration are: $p_x, p_y = -\infty$, and $p_z = \sqrt{2mW}$. Putting $\beta/2m(p_x^2 + p_y^2) = t$, $(\beta p_z^2/2m) - \beta W = s$, and $\zeta \approx \zeta_0$, we have

$$I_z = \frac{\pi e}{m h^3} \left(\frac{2m}{\beta} \right)^2 \int_0^\infty \frac{ds dt}{1 + \exp[\beta(W - \zeta_0) + s + t]}.$$

Since $W > \zeta_0$ and, in general, also $\beta (W - \zeta_0) \gg 1$, we may neglect the unity in the denominator as compared with the exponential function, and we can obtain Richardson's formula:

$$(18) \quad I_z = \frac{4\pi e m}{h^3} k^2 T^2 \exp\left(-\frac{W - \zeta_0}{k T}\right).$$

If we had based our calculation on the Boltzmann distribution instead of that due to Fermi, we would have obtained

$$(18') \quad I_z = \frac{e N_0/V}{\sqrt{2\pi m}} \sqrt{k T} \exp(-W/k T).$$

It is at once clear that the quantum value of the exponent should be $(W - \zeta_0)/k T$ instead of $W/k T$, because it must be equal to the difference in energy required to overcome the Fermi threshold. In the classical formula N_0 denotes the number of free electrons. Equating the factors in (18) and (18') we have

$$(19) \quad N_0 = \frac{2 V}{h^3} (\sqrt{2\pi m k T})^3 = \frac{3}{4} \sqrt{\pi} \cdot N \cdot \left(\sqrt{\frac{k T}{\zeta_0}}\right)^3.$$

This is, again, much smaller than the number of free electrons, but the ratio differs from that given in eqs. (16) and (17). Hence it can be clearly seen that it is not possible to correct Drude's theory simply by introducing a reduced number of free electrons, and thus to obtain the modern electron theory.

We are not yet in a position to discuss conductivity, or to derive the Franz-Wiedemann law, and we propose to defer this topic until we reach Chap. V. In conclusion we shall show that the electron gas satisfies Nernst's Third Law. According to eq. (38.16), the equation for the potential of spin electrons is

$$\Phi = \frac{4\pi V}{h^3} \left(\sqrt{\frac{2m}{\beta}}\right)^3 \int_0^\infty \log(1 + e^{\beta \zeta - t}) \sqrt{t} dt.$$

Integration by parts gives

$$\Phi = \frac{8\pi V}{3 h^3} \left(\frac{2m}{\beta}\right)^{3/2} \int_0^\infty \frac{t^{3/2} dt}{e^{t - \beta \zeta_0} + 1}$$

or, in view of (12):

$$(20) \quad \Phi = \frac{2}{3} \beta U.$$

In accordance with eq. (38.6), the entropy is given by

$$(21) \quad S = k \beta \left(\frac{5}{3} U - \zeta N \right).$$

Further, according to eqs. (7 a), (7 b) and (14), we have

$$(21 a) \quad U_0 = \frac{3}{5} \zeta_0 N, \quad U = \frac{3}{5} \zeta_0 N \left(1 + \frac{5\pi^2}{12} \frac{1}{\beta^2 \zeta_0^2} \right).$$

Substituting these values together with the value of ζ into eq. (21), we obtain

$$S = N k \beta \zeta_0 \left(1 + \frac{5\pi^2}{12} \frac{1}{\beta^2 \zeta_0^2} - 1 + \frac{\pi^2}{12} \frac{1}{\beta^2 \zeta_0^2} \right)$$

or

$$(22) \quad S = \frac{\pi^2}{2} N k \cdot \frac{k T}{\zeta_0} (= C_{electr} \text{ for } k T \ll \zeta_0).$$

The entropy is seen to vanish in the limiting case of $T \rightarrow 0$; it increases in proportion to T . It is seen from (16) that it is of the order $k \delta N$.

40. The mean square of fluctuations

So far we have dealt with mean values or even with magnitudes associated with the maximum of probability implying that they are identical with those observed on a macroscopic scale. Such an attitude is justified by the fact that laws involving mean values are identical with the laws of thermodynamics and that the properties of substances required in thermodynamics can be calculated with the aid of suitable molecular models.

It is by no means evident that this must be so. The concept of a mean value includes the possibility of larger or smaller deviations, and single measurements can yield values which *fluctuate* to a larger or lesser extent about this mean value. The good agreement between the statistical mean values and the macrophysical experimental data may be interpreted to signify that the fluctuations encountered in statistical considerations of a thermodynamical nature are, generally speaking, very small; this is a conclusion which is a consequence of the *law of large numbers*.

In order to prove this proposition we shall require a measure for the fluctuations. The mean value of the fluctuations is certainly equal to zero, because the mean value of a quantity is so defined as to render the deviations

in both directions equally probable. A possible measure is given by the mean value of the squares of the fluctuations: *the mean square*, for short.

Denoting the mean values by a bar, as we have already done on occasions, we find from eq. (36.14) that the mean value of the energy of a system in the Γ -space is given by:

$$(1) \quad \bar{E} = \frac{\sum_n E(n) e^{-\beta E(n)}}{\sum_n e^{-\beta E(n)}} = - \frac{\partial \log Z}{\partial \beta}.$$

The fluctuations are equal to the differences between special measured values $E(n)$ and the mean value \bar{E} , or

$$\Delta E(n) = E(n) - \bar{E}.$$

Thus the mean value of the square of the fluctuation in energy is given by

$$(2) \quad (\Delta E)^2 = \overline{(\Delta E(n))^2} = \overline{(E(n) - \bar{E})^2}.$$

Here ΔE is the root mean square of the fluctuation, and is usually implied when referring to the mean fluctuation for short. Written explicitly eq. (2) becomes

$$(3) \quad (\Delta E)^2 = \frac{\sum_n [E(n) - \bar{E}]^2 e^{-\beta E(n)}}{\sum_n e^{-\beta E(n)}}.$$

Since the formation of a mean is a linear process (and since any mean value is a constant with respect to further operations of taking a mean: $\overline{\bar{E}} = \bar{E}$), we infer from (2) or (3) that

$$(4) \quad (\Delta E)^2 = \overline{E^2} - 2 \bar{E} \bar{E} + \bar{E}^2 = \overline{E^2} - \bar{E}^2.$$

The mean square of a fluctuating quantity is equal to the difference between the mean value of the square of the quantity and the square of its mean value. It is, incidentally, clear that this difference must always be positive.

We shall now make use of eq. (4) to calculate the mean square of the fluctuation in energy. Since

$$\overline{E^2} = \frac{\sum_n E(n)^2 e^{-\beta E(n)}}{\sum_n e^{-\beta E(n)}}$$

we can see that the numerator can be obtained by differentiating the denominator twice with respect to β . Hence

$$(5) \quad \overline{E^2} = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}.$$

Substituting the mean values (1) and (5) into (4), we have

$$(\Delta E)^2 = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \frac{1}{Z^2} \left(\frac{\partial Z}{\partial \beta} \right)^2.$$

This is exactly the derivative of the quotient Z'/Z , and we can write finally that:

$$(6) \quad (\Delta E)^2 = \overline{(E - \bar{E})^2} = \frac{\partial^2 \log Z}{\partial \beta^2}.$$

In view of (1) and of the definition of β we may also write

$$(7) \quad (\Delta E)^2 = -\frac{\partial \bar{E}}{\partial \beta} = k T^2 \frac{\partial U}{\partial T} = k T^2 C.$$

The mean square of the fluctuation in energy is seen to be determined by thermodynamic quantities only. It is proportional to the heat capacity, C .

For monatomic perfect gases we have $U = 3/2 N k T$ see eq. (22.6 a), and

$$(8) \quad (\Delta E)^2 = \frac{3}{2} N k^2 T^2, \quad \frac{\Delta E}{U} = \sqrt{\frac{2}{3N}}.$$

For one mol of gas ($N = L$) the mean fluctuation is equal to the one $\sim 10^{12}$ th part of the mean energy, which is utterly unobservable. For $N = 150$ we should have $\Delta E/U = 6.7\%$.

The preceding examples illustrate the effect of very large numbers, and demonstrate that fluctuations are unimportant in relation to large masses, but may play a significant part in small regions. Fluctuations in energy of the order of 6.7% at room temperature correspond to temperature fluctuations of ± 20 C. The importance of fluctuations in relation to small regions has already been discussed in connection with the study of Brownian motion (*cf.* Sec. 24).

Equation (7) is valid universally. In the case of a system of quantum-mechanical oscillators eq. (33.8) leads to

$$(9) \quad (\Delta E)^2 = \frac{N (k \Theta)^2}{(e^{\Theta/T} - 1)^2} e^{\Theta/T},$$

or, in analogy with eq. (8), to

$$(9 \text{ a}) \quad \Delta E/U = 1/\sqrt{N}.$$

In the case of a solid body eqs. (35.8 a) and (35.8 b) lead to

$$(10) \quad \frac{\Delta E}{U} = \begin{cases} \frac{2}{\pi^2} \sqrt{\frac{5}{3}} (\Theta/T)^{3/2} \frac{1}{\sqrt{N}} & \text{if } T \ll \Theta \\ \frac{1}{\sqrt{3} N} & \text{if } T \gg \Theta. \end{cases}$$

It is seen that at absolute zero the fluctuations themselves die out, but that their ratio to the thermal energy U need not do so. The fact that ΔE tends to zero as $T \rightarrow 0$ is also a consequence of Nernst's Third Law (*cf.* Sec. 12) because $C = \partial U/\partial T$ in eq. (7) is proportional to the specific heat. In the case of an electron gas, we have from eq. (39.21 a) that

$$(11) \quad \frac{\Delta E}{U} = \frac{5\pi}{3\sqrt{2}} \left(\frac{kT}{\zeta_0} \right)^{3/2} \frac{1}{\sqrt{N}}$$

which is analogous to the first eq. (10).

The mean squares of fluctuation can be easily calculated for the occupation numbers n_i . In analogy with eq. (38.4), we have

$$(12) \quad (\Delta n_i)^2 = \overline{(n_i - \bar{n}_i)^2} = \frac{1}{\beta^2} \left(\frac{\partial^2 \log Z}{\partial \epsilon_i^2} \right)_{\alpha, \beta} = -\frac{1}{\beta} \left(\frac{\partial \bar{n}_i}{\partial \epsilon_i} \right)_{\alpha, \beta}$$

that is, in the case of a Bose-Einstein, or a Fermi-Dirac gas, respectively

$$(13) \quad (\Delta n_i)^2 = \frac{e^{\alpha + \beta \epsilon_i}}{(e^{\alpha + \beta \epsilon_i} \mp 1)^2}$$

and

$$(14) \quad \Delta n_i = \sqrt{\bar{n}_i (1 \pm \bar{n}_i)}.$$

In the Fermi-Dirac case the above expression markedly differs from zero only at the Fermi threshold. Directly at $\epsilon_i = \zeta > \zeta_0$ (*cf.* eq. (39.5)) Δn_i assumes its maximum value $\Delta n_i = \frac{1}{2}$. For the excited states of a highly degenerate Bose-Einstein gas we have $\bar{n}_i \ll 1$, and hence $\Delta n_i = \sqrt{\bar{n}_i}$. The same equation is valid for the classical limiting case. The ground state of a Bose-Einstein gas is of particular interest; since $1 \ll \bar{n}_0 (\approx N)$, we have

$$\Delta n_0 \approx \bar{n}_0.$$

and the fluctuation is seen to be of the order of the mean value, and hence numerically very large.

In this connection it should be noted that eq. (14) is valid only for grand canonical ensembles because in the derivation of eq. (12) we have kept α , and not the number of molecules N , constant. Moreover, the sub-systems of the grand canonical ensemble continually exchange particles which explains the manner in which large fluctuations may originate.

From the practical point of view the fluctuations which occur in groups of phase cells are much more important in view of the small size of a single phase cell. Let \sum'_i denote the sum taken over a well-defined group of cells; it is then seen from eq. (38.4) that the mean number of particles within it is given by

$$\bar{n}_i = \sum'_i \bar{n}_i = -\frac{1}{\beta} \sum'_i \frac{\partial \log Z}{\partial \varepsilon_i}$$

and that the mean square of fluctuation becomes

$$(15) \quad (\Delta n_j)^2 = \overline{(n_j - \bar{n}_j)^2} = \frac{1}{\beta^2} \sum_{i_1, i_2} \frac{\partial^2 \log Z}{\partial \varepsilon_{i_1} \partial \varepsilon_{i_2}}$$

Since in eq. (38.4) the \bar{n}_i depend only on the ε_i , the mixed terms in the double sum disappear. Consequently we have

$$(16) \quad (\Delta n_j)^2 = -\frac{1}{\beta} \sum'_i \frac{\partial \bar{n}_i}{\partial \varepsilon_i} = \sum'_i \bar{n}_i (1 \pm \bar{n}_i).$$

For example, the fluctuations in a volume element ΔV , as given by eq. (16) become

$$(17) \quad (\Delta n)^2 = \frac{4\pi \Delta V}{h^3} \int_0^\infty \frac{e^{\alpha + \beta p^2/2m}}{(e^{\alpha + \beta p^2/2m} \mp 1)^2} p^2 dp,$$

(where the spin factor 2 has been omitted). Thus in the limiting case of Boltzmann's statistics, we have

$$(\Delta n)^2 = \frac{4\pi \Delta V}{h^3} e^{-\alpha} \left(\sqrt{\frac{2m}{\beta}} \right)^3 \frac{\sqrt{\pi}}{4}.$$

Substituting the expression for e^α from eq. (31.4), we obtain

$$(18) \quad (\Delta n)^2 = \frac{N \Delta V}{V},$$

or, introducing the mean number of molecules in ΔV from $\bar{n} = N \Delta V / V$:

$$(18a) \quad \frac{\Delta n}{\bar{n}} = \sqrt{\frac{V}{N \Delta V}}.$$

Such fluctuations in density can be observed because they cause the scattering of light. The blue color of the sky is due to the scattering of solar light and is caused by the fluctuations in the density of the atmospheric air.

It is, naturally, possible to deduce eq. (18 a) from the fact that the mean density is constant, if direct combinatorial methods are used, and the method would, certainly, be much simpler. On the other hand the preceding argument allows us to show its relation to the fundamental equations of statistical mechanics which would otherwise be lost. The direct method of derivation is discussed in Problem IV. 8.

In the case of an electron gas, eq. (17) yields

$$(\Delta n)^2 = 8\pi \frac{\Delta V}{h^3} \left(\sqrt{\frac{2m}{\beta}} \right)^3 \int_{-\beta\zeta_0}^{\infty} \frac{e^x}{(e^x + 1)^2} \sqrt{x + \beta\zeta} \frac{dx}{2}$$

(now with the spin factor 2 included). The evaluation is identical with that in Sec. 39, and in the case of complete degeneration the integration gives:

$$(\Delta n)^2 = 8\pi \frac{\Delta V}{h^3} (\sqrt{2m\zeta_0})^3 \frac{1}{2\beta\zeta_0}$$

or, in accordance with eqs. (39.7) and (39.7 a),

$$(\Delta n)^2 = \frac{3}{2} \frac{N \Delta V}{V} \frac{1}{\beta\zeta_0}.$$

Hence, with $\bar{n} = N \Delta V / V$, we have

$$(19) \quad \frac{\Delta n}{\bar{n}} = \sqrt{\frac{V}{N \Delta V}} \cdot \sqrt{\frac{3kT}{2\zeta_0}}.$$

The density fluctuations of the Fermi gas are seen to disappear at absolute zero.

In conclusion we wish to remark that the higher powers of fluctuation can also be deduced from the partition function. Thus, for example, we have

$$(20) \quad \overline{(E(n) - \bar{E})^3} = -\frac{\partial^3 \log Z}{\partial \beta^3},$$

since

$$\begin{aligned}
 -\frac{\partial^3 \log Z}{\partial \beta^3} &= -\left(\frac{Z'}{Z}\right)'' = \left(\frac{Z''}{Z} - \frac{Z'^2}{Z^2}\right)' = -\left(\frac{Z'''}{Z} - \frac{3Z''Z'}{Z^2} + \frac{2Z'^3}{Z^3}\right) = \\
 &= \overline{E^3} - 3\overline{E^2}\overline{E} + 2\overline{E^3} = \overline{(E - \overline{E})^3}.
 \end{aligned}$$

In the case of a perfect gas, we have $\log Z = -3/2 N \log \beta + \dots$ and it follows that

$$\overline{(E - \overline{E})^3} / \overline{E^3} = \frac{8}{9} \frac{1}{N^2}, \quad \text{or} \quad \left\{ \frac{\overline{(E - \overline{E})^3}}{\overline{E^3}} \right\}^{1/3} = \sqrt[3]{\frac{8}{9}} N^{-2/3}.$$

CHAPTER V

OUTLINE OF AN EXACT KINETIC THEORY OF GASES.

In the study of statistical mechanics given in Chap. IV we have succeeded in providing an atomistic justification for thermodynamics, i.e. for the science of thermal equilibrium. It has been rendered more complete by demonstrating that the thermodynamic potentials, for example the free energy in eq. (29.12'), and in particular the partition function in eq. (29.15), can be deduced from atomistic data.

An atomistic description of non-equilibrium processes is much less simple. The phenomenological propositions have already been given in Sec. 21. In the present Chapter we shall restrict ourselves to the consideration of the behavior of molecules in perfect gases thus following the course of historical development. In this manner the object of the present chapter may be stated as an attempt to provide an exact formulation of the kinetic theory of gases given in Chap. III. The kinetic theory of condensed matter has been considerably advanced in recent times, but to describe it here we would have to exceed the scope of this text-book. In the present Chapter we shall be forced more often to refer the reader to specialized papers. This is true, in particular, in relation to the methods of solving the collision equation,¹ about to be derived.

The lack of completeness in the present Chapter is not only due to the limitation of its objects. It follows also from the fact that explicit calculations are only possible for very crude molecular models, and this is particularly true when it is desirable to take into account the quantum mechanical properties of molecules. An important step forward in this field was taken with the development of the theory of conduction electrons in metals, particularly when Sommerfeld succeeded in deriving the Wiedemann-Franz law, whose description will be given at the end of this Chapter.

41. The Maxwell-Boltzmann collision equation

A. DESCRIPTION OF A STATE IN THE KINETIC THEORY OF GASES

The perfect gas is characterized by the fact that the state of any of its molecules is independent of that of all the others, except for the instant of collision. We can describe it completely by specifying the position and velocity

¹See M. Born, "Cause and Chance", footnote on p. 203.

of every molecule at a given instant. We shall restrict our considerations to monatomic molecules which we shall assume to be rigid spheres. We shall denote position by specifying the *space coordinates* $\mathbf{r} = (x_1, x_2, x_3)$ and we shall also specify the velocity $\mathbf{v} = (\xi_1, \xi_2, \xi_3)$.¹

The volume elements in the physical and velocity spaces are

$$(1) \quad dx = dx_1 dx_2 dx_3, \quad d\xi = d\xi_1 d\xi_2 d\xi_3.$$

In the present Chapter, unlike Chap. IV, we shall disregard the quantum nature of these volume elements, except in Sec. 45. We assume them to be so large that they can contain a large number of molecules, and yet small enough to allow us to disregard variations in density within one element. Mathematically this means that dx and $d\xi$ in eq. (1) may be treated like differentials in spite of their large size. This change in definition may serve to justify the difference between the notation in eq. (1) and that employed earlier.

We now proceed to determine the number of molecules, dv , at a place (\mathbf{r}, \mathbf{v}) in the phase element of the μ -space, but replace momentum by velocity, so that the latter becomes equal to $dx d\xi$, and

$$(2) \quad dv = f(\mathbf{r}, \mathbf{v}, t) dx d\xi.$$

Thus the total number of particles is given by the integral

$$(3) \quad N = \int f(\mathbf{r}, \mathbf{v}, t) dx d\xi.$$

The integration is extended over the whole space (or over the volume of a vessel) and over all velocities. Hence the *total* mean value of a function $g(\mathbf{r}, \mathbf{v})$ is given by

$$(4) \quad \bar{g} = \frac{1}{N} \int g(\mathbf{r}, \mathbf{v}) f(\mathbf{r}, \mathbf{v}, t) dx d\xi.$$

However, in the kinetic theory of gases importance is attached only to *local* mean values. They determine mean values in the velocity space which, generally speaking, vary from point to point. If $\phi(\mathbf{v})$ denotes any function of velocity, then the local mean of ϕ is given by the integral

$$(5) \quad \bar{\phi}(\mathbf{r}) = \frac{1}{n} \int \phi(\mathbf{v}) f(\mathbf{r}, \mathbf{v}) d\xi.$$

¹The fact that we may regard the molecules as being rigid spheres and that we may restrict ourselves to the consideration of translational motion finds its justification in quantum mechanics. Cf. here the footnote on p. 242, Sec. 34, regarding the rotational energy of an electron.

Here n denotes the local particle density

$$(5\ a) \quad n = \int f(\mathbf{r}, \mathbf{v}) d\xi.$$

Thus $n dx$ is the number of particles in a volume element dx irrespective of its velocity. For example, the mean velocity is given by

$$(6) \quad \mathbf{u} = \frac{1}{n} \int \mathbf{v} f(\mathbf{r}, \mathbf{v}) d\xi,$$

or, written in terms of components, the triple integral

$$(6\ a) \quad u_i(x_1, x_2, x_3) = \frac{1}{n} \int \int \int \xi_i f(x_1, x_2, x_3; \xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3.$$

We assume that in addition to intermolecular forces, about which we shall have more to say later, there acts an external force, given as a function of space coordinates:

$$(7) \quad \mathbf{F} = \mathbf{F}(\mathbf{r}) = [X_1(\mathbf{r}), X_2(\mathbf{r}), X_3(\mathbf{r})].$$

We shall disregard forces which depend on velocities, such as the forces acting on a charged particle as it moves through a magnetic field.

These assumptions are sufficient to provide a complete justification for the thermodynamics of gases in motion. A beautiful and non-trivial example of the theory under consideration is afforded by the discovery of the effect known as thermal effusion made by Clausius and Waldmann. The effect is obtained in the process of finding a higher-order approximation to the solution of the collision equation for several molecular species. We know from thermodynamics that perfect gases do not change their temperature on mixing. However, the process of mixing itself is accompanied by thermal effects. They are implied in the calculations due to Chapman¹ and Enskog,² but their importance in experimental science was first recognized by Clausius and Waldmann, who were also the first ones to observe it. Concerning the relation with thermal diffusion reference should be made to Sec. 21 C (reciprocal relations).

¹Chapman, Phil. Trans. **211** (1911) 433, **216** (1916) 279, **217** (1916) 115.

²D. Enskog, Kinetic energy of processes in moderately dense gases, Inaugural dissertation (Uppsala 1917), Ark. for Matem. **16** (1921) No. 16, Kungl. Svenska Akad. **63** (1922) 4.

B. THE VARIATION OF f WITH TIME

The Maxwell-Boltzmann collision equation is obtained by inquiring into the variation of f with time. We assume that f is continuous and sufficiently differentiable, this being possible owing to our definition of the volume elements dx and $d\xi$. The phase density $f(\mathbf{r}, \mathbf{v}, t)$ in the μ -space changes owing to the motion of the particles and to their collisions. We now consider a time interval Δt which is, on the one hand, large compared with the duration of a collision τ_s , so that most collisions which have begun during Δt are also completed within it. On the other hand, we shall stipulate that Δt is small compared with the mean collision time τ , i. e. with the interval between two collisions. Thus, generally speaking, *one* molecule will suffer at most *one* collision with another molecule during the interval Δt . This implies that the radius of action of intermolecular forces is sufficiently small compared with the distance between atoms, and, *a fortiori*, small compared with the mean free path (Sec. 27).

If no collisions occurred during Δt we could make the transformations

$$\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r} + \mathbf{v} \Delta t \quad \text{and} \quad \mathbf{v} \rightarrow \mathbf{v}' = \mathbf{v} + \frac{1}{m} \mathbf{F} \Delta t$$

so that

$$(8) \quad f(\mathbf{r}, \mathbf{v}, t) dx d\xi \rightarrow f(\mathbf{r} + \mathbf{v} \Delta t, \mathbf{v} + \frac{1}{m} \mathbf{F} \Delta t, t + \Delta t) dx' d\xi' = \\ = \left[f(\mathbf{r}, \mathbf{v}, t) + \Delta t \left\{ \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \frac{1}{m} \mathbf{F} \frac{\partial f}{\partial \mathbf{v}} + \frac{\partial f}{\partial t} \right\} + \dots \right] dx' d\xi'.$$

The last equation applies in cases when we may neglect higher-order terms, i. e. when f does not change appreciably during the interval Δt . It may be noted that such an assumption is compatible with considerable changes within one mean free path because $\Delta t \ll \tau$. Furthermore, according to Liouville's theorem (Sec. 28), we can write

$$(8a) \quad dx' d\xi' = dx d\xi$$

for the preceding motion which is not impeded by collisions. Consequently the differential factors can be cancelled on both sides of eq. (8).

The collisions between molecules cause some molecules to leave $dx d\xi$, and some pass from $dx_1 d\xi_1$ to $dx d\xi$. They are equivalent to a loss or gain in particles in $dx d\xi$ owing to collisions. Thus the balance equation for the particles consists in stating that the change in the number of particles,

according to (8) and due to flow, must be equal to the difference between the numbers gained (J_{gain}) and lost (J_{loss}) as a consequence of collisions. Hence per unit time and phase space element we may write

$$(9) \quad \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{1}{m} \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{v}} = J_{gain} - J_{loss}.$$

This is the Maxwell-Boltzmann collision equation. The quantities J_{gain} and J_{loss} must be calculated in accordance with the laws of elastic collision. In order to shorten the equations, i. e. eqs. (8) and (9), we have denoted the spatial gradient of f by $\partial f / \partial \mathbf{r}$ and that in the velocity space by $\partial f / \partial \mathbf{v}$.

C. THE LAWS OF ELASTIC COLLISION

These depend on the nature of the forces acting between molecules. In the preceding Sections we only stipulated that the radius of action of the forces was small. Thus, e. g. a force law $F \sim 1/r^n$ would be compatible with the assumption, provided that n were large enough. This case is the one normally discussed. It leads to particularly simple results for $n = 5$, and the limiting case of $n = \infty$ corresponds to rigid molecules. We shall restrict ourselves here to the latter assumption. The diameter of such a sphere will be denoted by s . It indicates the smallest distance between the centers of two atoms imagined to be spherical in shape. It is evident that real molecules differ greatly from what we have assumed here. Monatomic molecules at moderate temperatures seem to be nearest to this model.¹

When two molecules collide, the total energy and total momentum must preserve their values. Denoting the velocities of the two particles before collision by \mathbf{v}, \mathbf{v}_1 and by $\mathbf{v}', \mathbf{v}_1'$, after collision, we may write

$$(10) \quad \begin{aligned} \mathbf{v} + \mathbf{v}_1 &= \mathbf{v}' + \mathbf{v}_1', \\ v^2 + v_1^2 &= v'^2 + v_1'^2. \end{aligned}$$

If, further, \mathbf{V} denotes the relative velocity before collision, i. e.

$$(11) \quad \mathbf{V} = \mathbf{v}_1 - \mathbf{v},$$

we can write down the solution of (10) in the form

$$(12) \quad \mathbf{v}' = \mathbf{v} + (\mathbf{V} \cdot \mathbf{e}) \mathbf{e}, \quad \mathbf{v}_1' = \mathbf{v}_1 - (\mathbf{V} \cdot \mathbf{e}) \mathbf{e}.$$

¹The higher energy levels of atoms do not become excited at normal temperatures so that the law of attraction is determined by the polarization of the atoms.

Here \mathbf{e} denotes an arbitrary unit vector. Hence the relative velocity after collision is given by

$$(12\ a) \quad \mathbf{V}' = \mathbf{v}_1' - \mathbf{v}' = \mathbf{V} - 2(\mathbf{V} \cdot \mathbf{e})\mathbf{e},$$

so that

$$(12\ b) \quad (\mathbf{V}' \cdot \mathbf{e}) = -(\mathbf{V} \cdot \mathbf{e}).$$

Solving eq. (11) for \mathbf{v} and \mathbf{v}_1 yields

$$(13) \quad \mathbf{v} = \mathbf{v}' + (\mathbf{V}' \cdot \mathbf{e})\mathbf{e}, \quad \mathbf{v}_1 = \mathbf{v}_1' - (\mathbf{V}' \cdot \mathbf{e})\mathbf{e}.$$

This equation is identical in form with eq. (12) and it is seen that the transformations for the velocities are mutually reciprocal.

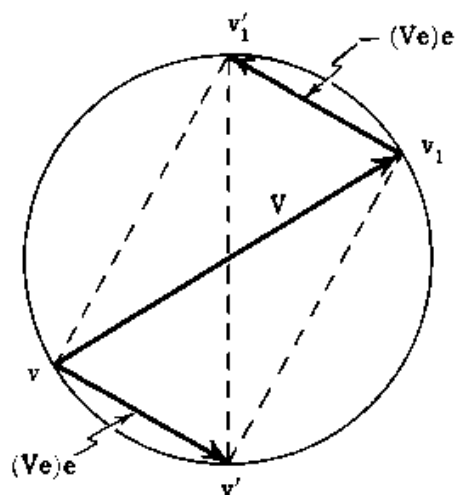


Fig. 33.

Vector diagram for the velocities associated with the elastic collision of two equal spheres.

The collision equation (11) for identical particles can be represented graphically in a simple way; we shall employ the same geometrical construction later. The points \mathbf{v} and \mathbf{v}_1 in Fig. 33 represent the end points of the vectors \mathbf{v} and \mathbf{v}_1 . The vector drawn from \mathbf{v} to \mathbf{v}_1 represents the relative velocity \mathbf{V} . We now draw a ray through \mathbf{v} in the direction $+\mathbf{e}$ and one through \mathbf{v}_1 in the direction $-\mathbf{e}$ and obtain the vectors $\pm(\mathbf{V} \cdot \mathbf{e})\mathbf{e}$ from eq. (11) by projecting \mathbf{V} onto the two parallel rays. These projections determine the points \mathbf{v}' and \mathbf{v}_1' , as shown. The four points $(\mathbf{v}, \mathbf{v}_1, \mathbf{v}', \mathbf{v}_1')$ are seen to lie on a rectangle irrespective of the direction of \mathbf{e} . They all lie on a sphere whose diameter is \mathbf{V} and whose center is at the mid-point of

the vector $|\mathbf{V}|$; furthermore, the pairs $(\mathbf{v}, \mathbf{v}_1)$ and $(\mathbf{v}', \mathbf{v}_1')$ are each diametrically opposed, Fig. 33.

It is now necessary to indicate the meaning of \mathbf{e} for the collision of rigid spheres. According to eq. (12) we may write

$$(14) \quad \mathbf{V} - \mathbf{V}' = 2(\mathbf{V} \cdot \mathbf{e})\mathbf{e}.$$

During a collision there is a transfer of momentum. *On the one hand* this must be normal to the plane tangent to the two spheres at the point of impact, i. e. in the direction of the line through the two centers, the so-called *central*

axis. On the other hand, the amount of momentum transferred is equal to the change in the momentum of either sphere, or to

$$(14 a) \quad \mathbf{v}' - \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_1' = (V \mathbf{e}) \mathbf{e}.$$

This shows that the vector \mathbf{e} lies along the central axis. Moreover, since V and V' are equal in magnitude, as seen from eq. (12 a) and Fig. 33, the central axis bisects the angle formed V and V' . Fig. 34 represents the directions of the vectors and of the central axis as seen by an observer moving with either of the two spheres, say with that whose velocity is \mathbf{v} before collision.

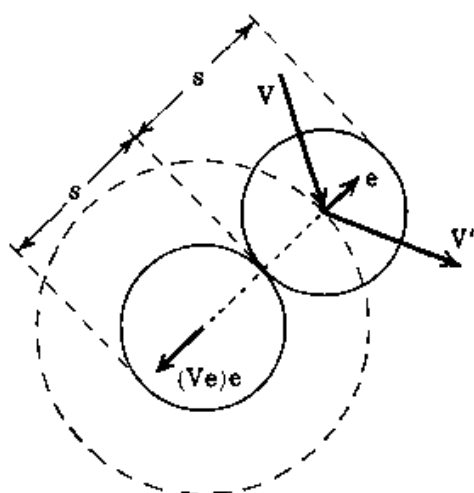


Fig. 34.

The kinematics of an elastic collision.

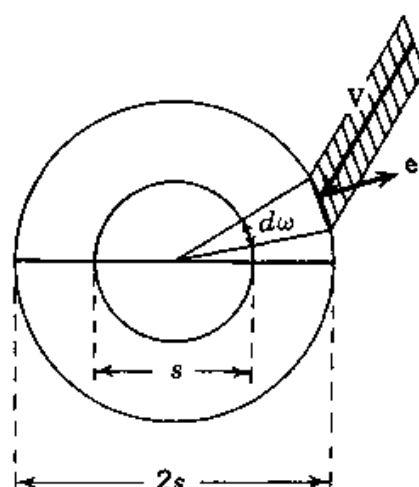


Fig. 35.

Sphere of influence and rate of collision.

D. BOLTZMANN'S COLLISION INTEGRAL

The preceding geometrical picture provides a starting point for the calculation of the right-hand side of eq. (9). It is perhaps somewhat more convenient to represent the motion of the center of the impinging sphere relative to the sphere of influence of radius s , drawn dotted in Fig. 35. The number of molecules which impinge on an area $s^2 d\omega$, where $d\omega$ is an elementary solid angle, during an interval of time Δt is

$$s^2 d\omega |\mathbf{V} \mathbf{e}| \Delta t \cdot f(\mathbf{r}, \mathbf{v}_1, t) d\xi_1.$$

The first term, $s^2 d\omega |\mathbf{V} \mathbf{e}| \Delta t$, represents the volume of the oblique cylinder from which particles of a given direction and a given velocity arrive during time Δt , as shown in Fig. 35. The second term denotes the density of these special particles.

Now, in the gas there are $f(\mathbf{r}, \mathbf{v}, t) dx d\xi$ molecules present whose velocity is \mathbf{v} . Thus the total number of collisions between molecules whose velocities are \mathbf{v} and \mathbf{v}_1 and whose central axis vector is \mathbf{e} ¹ is

$$(15) \quad s^2 d\omega |\mathbf{V} \mathbf{e}| \Delta t \cdot f(\mathbf{r}, \mathbf{v}_1, t) \cdot f(\mathbf{r}, \mathbf{v}, t) dx d\xi d\xi_1$$

Integration over all velocities \mathbf{v}_1 and over all directions \mathbf{e} gives the total number of collisions which deflect the paths of the particles whose velocity is \mathbf{v} . This number, computed per unit time and per unit $dx d\xi$, is the one entering eq. (9). Thus

$$(16) \quad J_{loss} = \frac{s^2}{2} \int |\mathbf{V} \mathbf{e}| f(\mathbf{r}, \mathbf{v}_1, t) f(\mathbf{r}, \mathbf{v}, t) d\omega d\xi_1.$$

The factor $\frac{1}{2}$ is due to the fact that the integration is performed over the whole sphere, whereas the physical argument applies only to half of it, as it is easy to see by considering the variation of \mathbf{e} on shifting \mathbf{v} parallel to itself.

The calculation of J_{gain} is quite analogous. We must now arrange it in such a way as to make the velocities after collision equal to \mathbf{v} and \mathbf{v}_1 , because J_{gain} corresponds to the increase in the number of particles moving with a velocity \mathbf{v} due to collisions. Let the corresponding velocities before the impact be \mathbf{v}' and \mathbf{v}_1' . The number of impacts on an element of area $s^2 d\omega$ of the sphere of influence is analogous to (15):

$$(15 a) \quad s^2 d\omega |\mathbf{V}' \mathbf{e}| \Delta t \cdot f(\mathbf{r}, \mathbf{v}', t) f(\mathbf{r}, \mathbf{v}_1', t) dx d\xi' d\xi_1'.$$

According to (12 b) we have $|\mathbf{V}' \mathbf{e}| = |\mathbf{V} \mathbf{e}|$, and according to (13) (and also to Liouville's theorem given in Sec. 28) we have, furthermore, $d\xi' d\xi_1' = d\xi d\xi_1$. Moreover the same result follows from the Jacobian of (13)

$$(16 a) \quad \frac{\partial(\mathbf{v}, \mathbf{v}_1)}{\partial(\mathbf{v}', \mathbf{v}_1')} = \begin{vmatrix} 1 & . & . & 0 & . & . \\ . & 1 & . & . & 0 & . \\ . & . & 0 & . & . & 1 \\ 0 & . & . & 1 & . & . \\ . & 0 & . & . & 1 & . \\ . & . & 1 & . & . & 0 \end{vmatrix} = -1,$$

¹A short expression, like the one used here, implies that the vectors have the given values within the elements $d\omega$ and $dx d\xi$.

where we may assume that $\mathbf{e} = (0, 0, 1)$ without loss of generality.¹ Thus the gain in momentum (again for $\Delta t = 1$ and $dx d\xi = 1$) is:

$$(16\ b) \quad J_{\text{gain}} = \frac{s^2}{2} \int |\mathbf{V} \mathbf{e}| f(\mathbf{r}, \mathbf{v} + (\mathbf{V} \mathbf{e}) \mathbf{e}, t) f(\mathbf{r}, \mathbf{v}_1 - (\mathbf{V} \mathbf{e}) \mathbf{e}, t) \cdot d\omega d\xi_1.$$

Equations (16) and (16 b) are known as *Boltzmann's collision integrals*. Substituting these into eq. (9) and introducing the usual abbreviations

$$(17) \quad \begin{aligned} f &= f(\mathbf{r}, \mathbf{v}, t), & f_1 &= f(\mathbf{r}, \mathbf{v}_1, t), \\ f' &= f(\mathbf{r}, \mathbf{v}', t) = f(\mathbf{r}, \mathbf{v} + (\mathbf{V} \mathbf{e}) \mathbf{e}, t), \\ f'_1 &= f(\mathbf{r}, \mathbf{v}'_1, t) = f(\mathbf{r}, \mathbf{v}_1 - (\mathbf{V} \mathbf{e}) \mathbf{e}, t), \end{aligned}$$

we obtain the *Maxwell-Boltzmann collision equation*:

$$(18) \quad \frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \frac{1}{m} \mathbf{F} \frac{\partial f}{\partial \mathbf{v}} = \frac{s^2}{2} \int |\mathbf{V} \mathbf{e}| (f' f'_1 - f f_1) d\omega d\xi_1.$$

The mathematical problem of an exact kinetic theory of gases consists in the solution of the preceding non-linear, integro-differential equation.

E. BOLTZMANN'S HYPOTHESIS ABOUT MOLECULAR CHAOS

Before proceeding to study the properties of this equation it is necessary to discuss an important assumption implied in the preceding argument. In deriving the expression for the collision integrals we need the probability $W(\mathbf{r}, \mathbf{v}; \mathbf{r}_1, \mathbf{v}_1)$ of finding a molecule at phase point (\mathbf{r}, \mathbf{v}) in collision with another at phase point $(\mathbf{r}_1, \mathbf{v}_1)$. However, we only know at first the probabilities $W(\mathbf{r}, \mathbf{v})$ and $W(\mathbf{r}_1, \mathbf{v}_1)$ of finding a molecule at (\mathbf{r}, \mathbf{v}) , or $(\mathbf{r}_1, \mathbf{v}_1)$, respectively. The latter can be easily calculated from $f(\mathbf{r}, \mathbf{v})$, because according to (3) we can write per unit volume in the phase space:

$$W(\mathbf{r}, \mathbf{v}) = \frac{1}{N} f(\mathbf{r}, \mathbf{v}).$$

The occurrence of products of two functions f in eqs. (15) and (15 a) means that $W(\mathbf{r}, \mathbf{v}; \mathbf{r}_1, \mathbf{v}_1) = 0$ for $\mathbf{r} \neq \mathbf{r}_1$, and that for $\mathbf{r} = \mathbf{r}_1$ we have assumed that

$$W(\mathbf{r}, \mathbf{v}; \mathbf{r}, \mathbf{v}_1) = W(\mathbf{r}, \mathbf{v}) \cdot W(\mathbf{r}, \mathbf{v}_1)$$

¹The change in sign in the Jacobian is compensated by that in the factor $\mathbf{V}' \mathbf{e}$. It is necessary to remember that $\frac{1}{2} |\mathbf{V}'' \mathbf{e}| d\xi' d\xi'_1$ originated from $(\mathbf{V}' \mathbf{e}) d\xi d\xi'_1$ with $(\mathbf{V}' \mathbf{e}) > 0$.

i. e. that the first function is simply a product of the other two. This implies that the probabilities for the possible *velocities* of one particle are independent of the velocity of the other.

That this constitutes an additional assumption can be seen by considering the function $W(\mathbf{v}, \mathbf{v}_1)$. The function $W(\mathbf{v}, \mathbf{v}_1)$ must be symmetrical in \mathbf{v} and \mathbf{v}_1 , since no molecule is preferred compared with another. On integration we find that

$$\int W(\mathbf{v}, \mathbf{v}_1) d\xi_1 = \int W(\mathbf{v}_1, \mathbf{v}) d\xi_1 = W(\mathbf{v}).$$

Evidently, owing to symmetry, the same function $W(\mathbf{v})$ is obtained in either case. It is, however, by no means permissible to conclude that $W(\mathbf{v}, \mathbf{v}_1) = W(\mathbf{v}) \times W(\mathbf{v}_1)$. In general this will not be the case at all.

The separation of W into a product is a consequence of Boltzmann's hypothesis of *complete molecular chaos*. It corresponds to Maxwell's assumption in Sec. 23, to the assumption of equal probabilities in Sec. 28, or to Gibbs' hypothesis in Sec. 36. This assumption, essentially, justifies the validity of the entropy theorem which we shall derive from eq. (18) in Sec. 42. Born¹ and his co-workers have recently given a more detailed analysis of this hypothesis.

42. The H -theorem and Maxwellian distribution

A. THE H -THEOREM

We now turn our attention to the entropy theorem. The expression for entropy is given by eq. (36.10), namely:²

$$(1) \quad \bar{H} = -k \int \log f \cdot f dx d\xi.$$

It should, however, be noted that this equation refers to the μ -space, whereas originally it has been written for the Γ -space, and that we do not now attribute quantum properties to the phase cells. The expressions in eq. (29.5) for a Boltzmann gas, in eq. (38.12) for a Fermi-Dirac gas, and in eq. (38.14) for an Einstein-Bose gas, referred to the μ -space. In the preceding Chapter, when dealing with thermodynamic equilibrium, we have restricted ourselves to

¹Cf. "Cause and Chance", *l. c.* p. 223, footnote 1.

²With Boltzmann, we now use the symbol H instead of S .

the consideration of the conditions under which the integral (1) attained its maximum. In the present Section we shall investigate the variation of H with time. It might be expected that \bar{H} will never decrease under the conditions known from thermodynamics.

When studying non-equilibrium processes it is found that the *local* entropy is more revealing than the total entropy. Hence, instead of using eq. (1) we shall investigate the variation of

$$(2) \quad H = -k \int \log f \cdot f d\xi$$

with time. We obtain

$$\frac{\partial H}{\partial t} = -k \int (1 + \log f) \frac{\partial f}{\partial t} d\xi,$$

or according to eq. (41.18)

$$(3) \quad \frac{\partial H}{\partial t} = k \int (1 + \log f) \left(\mathbf{v} \frac{\partial f}{\partial r} + \frac{1}{m} \mathbf{F} \frac{\partial f}{\partial \mathbf{v}} - J_f \right) d\xi,$$

where J_f denotes the collision integral appearing on the right-hand side of eq. (41.18), namely

$$(4) \quad J_f = \frac{s^2}{2} \int |\mathbf{V} \mathbf{e}| (f' f_1' - f f_1) d\omega d\xi_1.$$

The first term on the right-hand side of eq. (3) can be transformed as follows:

$$\int (1 + \log f) \mathbf{v} \frac{\partial f}{\partial r} d\xi = \operatorname{div} \int \mathbf{v} \log f \cdot f d\xi.$$

The integral

$$(5) \quad \mathbf{S} = -k \int \mathbf{v} \log f \cdot f d\xi$$

denotes the flux vector associated with H defined in eq. (2). Thus eq. (3) assumes the following form

$$(6) \quad \frac{\partial H}{\partial t} + \operatorname{div} \mathbf{S} = -k \int (1 + \log f) J_f d\xi.$$

This has taken into account that the second term on the right-hand side of (3) vanishes. In fact, since F depends only on the space coordinates, the second term can be written

$$\frac{k}{m} F \int (1 + \log f) \frac{\partial f}{\partial \mathbf{v}} d\xi = \frac{k}{m} F \int \frac{\partial}{\partial \mathbf{v}} (f \log f) d\xi.$$

This integral can be represented in the form of a surface integral over the sphere at infinity in the velocity space. Since the energy is finite, f vanishes, and we have

$$(7) \quad \frac{k}{m} F \int (1 + \log f) \frac{\partial f}{\partial \mathbf{v}} d\xi = 0.$$

In this manner only the integral on the right-hand side of eq. (6) remains and we can substitute J_1 from eq. (4), when we obtain the form

$$(8) \quad -\frac{k s^2}{2} \int |\mathbf{V} \mathbf{e}| (1 + \log f) (f' f_1' - f f_1) d\omega d\xi d\xi_1.$$

Interchanging the two triples of variables of integration, \mathbf{v} and \mathbf{v}_1 , does not affect the value of the integral, so that we may also write

$$-\frac{k s^2}{2} \int |\mathbf{V} \mathbf{e}| (1 + \log f_1) (f' f_1' - f f_1) d\omega d\xi d\xi_1.$$

It should be noted that the following transformation may be performed. First $\mathbf{V} \rightarrow -\mathbf{V}$ and

$$\mathbf{v}' = \mathbf{v} + (\mathbf{V} \mathbf{e}) \mathbf{e} \rightarrow \mathbf{v}_1 - (\mathbf{V} \mathbf{e}) \mathbf{e} = \mathbf{v}_1'$$

$$\mathbf{v}_1' = \mathbf{v}_1 - (\mathbf{V} \mathbf{e}) \mathbf{e} \rightarrow \mathbf{v} + (\mathbf{V} \mathbf{e}) \mathbf{e} = \mathbf{v}'.$$

Consequently, we can write the right-hand side of eq. (6) in the following, more symmetrical form

$$(9) \quad -\frac{k s^2}{4} \int |\mathbf{V} \mathbf{e}| (2 + \log f + \log f_1) (f' f_1' - f f_1) d\omega d\xi d\xi_1.$$

Instead of integrating with respect to \mathbf{v} and \mathbf{v}_1 , we may also integrate with respect to \mathbf{v}' and \mathbf{v}_1' . Thus, according to eq. (41.8 a), eq. (9) becomes

$$-\frac{k s^2}{4} \int |\mathbf{V} \mathbf{e}| (2 + \log f + \log f_1) (f' f_1' - f f_1) d\omega d\xi' d\xi_1'.$$

Now it is necessary to assume that \mathbf{v} and \mathbf{v}_1 have been eliminated with the aid of eq. (41.13) rather than eliminating \mathbf{v}' and \mathbf{v}_1' with the aid of eq. (41.11). Having done this we can change our notation and write \mathbf{v} and \mathbf{v}_1 respectively for \mathbf{v}' and \mathbf{v}_1' . This will cause no confusion because \mathbf{v} and \mathbf{v}_1 do not appear in the equation. Having performed the change of variables it will, nevertheless, be found convenient to define new variables \mathbf{v}' and \mathbf{v}_1' with the aid of eq. (41.11), so that the integral on the right-hand side of (8), now denoted by G , becomes

$$(10) \quad G(\mathbf{r}, t) = -\frac{k s^2}{4} \int |\mathbf{V} \mathbf{e}| (2 + \log f' + \log f_1') (f f_1 - f' f_1') d\omega d\xi d\xi_1.$$

It is easy to see that the factor $|\mathbf{V} \mathbf{e}|$ in the integrand remains unchanged. It will turn out that $G \equiv \theta$, cf. Sec. 21, eq. (8).

The integral can be made even more symmetrical if it is replaced by half the sum of the two equal expressions in eqs. (8) and (10). In doing so it is necessary to note the change in the sign of the term in the last bracket in the integrand. Thus we obtain

$$(11) \quad \begin{aligned} G(\mathbf{r}, t) &= -k \int (1 + \log f) \cdot J_f d\xi = \\ &= -\frac{k s^2}{8} \int |\mathbf{V} \mathbf{e}| (\log f + \log f_1 - \log f' - \log f_1') (f' f_1' - f f_1) d\omega d\xi d\xi_1. \end{aligned}$$

or, after a simple rearrangement:

$$(12) \quad G(\mathbf{r}, t) = \frac{k s^2}{8} \int |\mathbf{V} \mathbf{e}| \left(\log \frac{f' f_1'}{f f_1} \right) (f' f_1' - f f_1) d\omega d\xi d\xi_1.$$

At this stage it might be remarked that we shall encounter an identical transformation of an integral of the type (8), except that an arbitrary function $\psi(\mathbf{v})$ will occur instead of $1 + \log f$. We would then obtain

$$(13) \quad \begin{aligned} &\int |\mathbf{V} \mathbf{e}| \psi(\mathbf{v}) (f' f_1' - f f_1) d\omega d\xi d\xi_1 = \\ &= \frac{1}{4} \int |\mathbf{V} \mathbf{e}| (\psi + \psi_1 - \psi' - \psi_1') (f' f_1' - f f_1) d\omega d\xi d\xi_1 \end{aligned}$$

in complete analogy with the preceding case. The different ψ -functions in eq. (13), are defined in the same way as the f 's in eq. (41.17).

First we note that the integrand in eq. (12) cannot be negative, because $\log (f' f_1' / f f_1)$ and $f' f_1' - f f_1$ always have the same signs. Hence

$$(14) \quad \dot{H} + \operatorname{div} \mathbf{S} = G \geq 0.$$

The relation between this equation and eq. (21.10) will be discussed later. Integration over a finite volume yields

$$(15) \quad \frac{d}{dt} \int H dx + \int S_n d\sigma = \int G dx \geq 0,$$

where the volume integral in the second term has been transformed with the aid of Gauss' theorem into a surface integral.¹ The integral $\int H dx$ is seen to change owing to two causes; first, there is a flow of entropy through the surface and, secondly, there exists within the volume a distribution of sources which are either zero or positive. When the system is isolated from the surroundings there is no flow of entropy across the surface and we must have

$$(16) \quad \frac{d}{dt} \int H dx = \int G dx \geq 0.$$

The entropy of an isolated system cannot decrease. It should be realized that the scope of eq. (14) exceeds that of the entropy principle in thermodynamics. It determines the magnitude of the irreversible change in H . Furthermore, eq. (5) defines the entropy flux.

B. MAXWELLIAN DISTRIBUTION

When $G = 0$ the change in entropy is determined solely by the flow of entropy. Since the integrand in eq. (12) cannot be negative, this can occur only if

$$(17) \quad f' f_1' = f f_1.$$

Putting

$$(17 \text{ a}) \quad \log f = \psi$$

we find that (17) is equivalent to

$$(17 \text{ b}) \quad \psi' + \psi_1' = \psi + \psi_1.$$

The sum $\psi + \psi_1$ is seen to remain constant during a collision; it is an *additive invariant of the collision*.

¹ $d\sigma$ denotes a surface element on the surface and S_n is the component of the vector \mathbf{S} in the direction of the normal outwards.

We can at once write down five functions which satisfy eq. (17 b), namely a constant, and the expressions for momentum and energy:

$$(18) \quad \psi_0 = 1, \quad \psi_1 = \xi_1, \quad \psi_2 = \xi_2, \quad \psi_3 = \xi_3, \quad \psi_4 = \frac{1}{2} v^2.$$

In fact, these are the only additive invariants for a collision.

In order to prove this proposition we revert to the representation in Fig. 33, Sec. 41. We shall call $\psi(\mathbf{v})$ an antipodal function if for antipodal points \mathbf{v} and \mathbf{v}_1 on an *arbitrary* sphere in the velocity space we have

$$\psi(\mathbf{v}) + \psi(\mathbf{v}_1) = \text{const.}$$

The constant may, evidently, vary from sphere to sphere. Since on collision the points \mathbf{v} and \mathbf{v}_1 change to antipodal points on the same sphere, eq. (17 b) is seen to be satisfied. Antipodal functions are thus equivalent to the additive invariants for a collision.

It is easy to show¹ that a continuous antipodal function vanishes identically if it vanishes at the following five points:

$$(19) \quad \mathbf{v} = (0, 0, 0); (1, 0, 0); (0, 1, 0); (0, 0, 1); (-1, 0, 0).$$

Using the five functions (18) it is always possible to construct a function, by the use of linear superposition, which would assume arbitrarily prescribed values at the characteristic points (19), i. e. one that would assume the same values at those points as an arbitrary antipodal function. Since the difference between the prescribed and the so constructed antipodal function is also an antipodal function, namely one which vanishes at the five points (19), it must vanish identically. In other words the only antipodal functions, i. e. the only additive collision invariants are

$$(20) \quad \psi = a_0 + \mathbf{a} \cdot \mathbf{v} + a_4 v^2.$$

According to eq. (17 a) we may also write

$$(20 \text{ a}) \quad \log f = \alpha - \gamma(\mathbf{v} - \mathbf{u})^2,$$

with a different set of constants. Putting $a = e^\alpha$ we obtain Maxwell's distribution law

$$(21) \quad f = f_0(\mathbf{v}) = a e^{-\gamma(\mathbf{v} - \mathbf{u})^2}$$

with the difference that a , γ , and \mathbf{u} may still be functions of \mathbf{r} and t . We refer to it as to the *local* Maxwellian distribution.

¹The proof was given by Harold Grad, Comm. pure appl. Maths., 2 (1949) 311.

It now remains to prove Grad's lemma. We begin by considering at first only the ξ_x, ξ_y -plane, as shown in Fig. 36a. Of the first points in (19) the ones denoted by \times lie in this plane; they are A, B, C , and D . At those points we have $\psi = 0$ by definition. The same can be said about all nodal points of the quadratic lattice in Fig. 36a, because we can always find pairs of antipodal points of which we know that $\psi = 0$ for three points and hence must be so at the fourth. For example $(A, D; B, 1)$, $(C, D; B, 2)$, $(A, C; D, 7)$ etc.

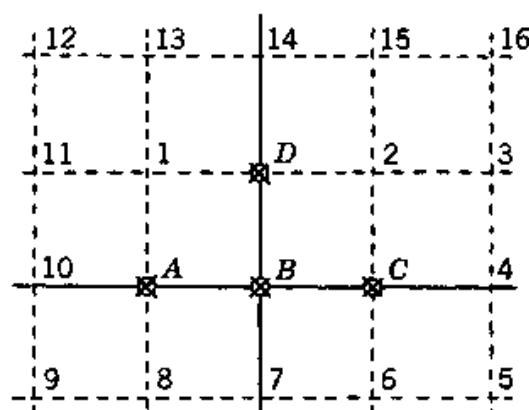


Fig. 36 a,

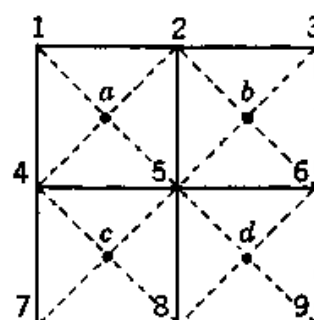


Fig. 36 b.

Illustrating the derivation of local Maxwellian distribution.

Using the same construction we can find additional points at which ψ vanishes. The nodal points from Fig. 36a have been denoted by \times in Fig. 36b. It is seen from the diagram that ψ must vanish also at the mid-points denoted by \cdot , as it is easy to infer from the following antipodal pairs: $(a, b; 2, 5)$, $(b, d; 5, 6)$, $(c, d; 5, 8)$, $(a, c; 4, 5)$; $(a, d; b, c)$. Since

$$\psi(a) + \psi(b) = \psi(2) + \psi(5) = 0, \quad \psi(b) + \psi(d) = \psi(5) + \psi(6) = 0,$$

$$\psi(c) + \psi(d) = \psi(5) + \psi(8) = 0, \quad \psi(a) + \psi(c) = \psi(4) + \psi(5) = 0,$$

$$\psi(a) + \psi(d) = \psi(b) + \psi(c)$$

we also have

$$\psi(a) = \psi(b) = \psi(c) = \psi(d) = 0.$$

This procedure leads to another quadratic lattice which is now smaller and oblique, and can, therefore, be continued. In this manner we can obtain a lattice of points which is as dense as we please and at whose nodes ψ vanishes. Assuming continuity we have $\psi(\mathbf{v}) = 0$, q.e.d.

Making use of all points in eq. (19) we can easily extend the construction and the proof to three dimensions.

C. EQUILIBRIUM DISTRIBUTIONS

Equation (21) contains all distributions which are compatible with an entropy whose value changes only owing to a flow. It must, however, be realized that α , γ and \mathbf{u} considered as functions of \mathbf{r} and t in eq. (21) cannot be arbitrary if they are to be compatible with the Maxwell-Boltzmann equation. Moreover, it follows from eq. (41.18) that

$$(22) \quad \frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \frac{1}{m} \mathbf{F} \frac{\partial f}{\partial \mathbf{v}} = 0.$$

Introducing eq. (20 a) we can deduce the following equations from (22):

$$(23) \quad \begin{aligned} \text{grad } \gamma &= 0, & \frac{\partial u_i}{\partial k} + \frac{\partial u_k}{\partial i} &= \frac{\dot{\gamma}}{\gamma} \delta_{ik}; \\ \dot{\alpha} &= -(\mathbf{u} \nabla) (\alpha - \gamma u^2) - \dot{\gamma} u^2 \\ \mathbf{F} &= m \left\{ \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2\gamma} \text{grad } (\alpha - \gamma u^2) + \frac{\dot{\gamma}}{\gamma} \mathbf{u} \right\} \end{aligned}$$

in which i and k represent x , y , or z .

The first two lead to the following form:

$$(24) \quad \gamma = \gamma(t), \quad \mathbf{u} = \frac{\dot{\gamma}}{2\gamma} \mathbf{r} + \mathbf{a}(t) \times \mathbf{r} + \mathbf{b}(t)$$

Since \mathbf{u} denotes the mean local velocity, eq. (24) is seen to represent a special superposition of a *translation*, a *rotation*, and a radial *expansion*. The whole motion remains *isotropic*, because during time δt , \mathbf{r} transforms to $\mathbf{r} + \mathbf{u} \delta t$ and $d\mathbf{r} \rightarrow d\mathbf{r} + d\mathbf{u} \delta t$, so that

$$d\mathbf{r}^2 \rightarrow d\mathbf{r}^2 + 2(d\mathbf{r} d\mathbf{u}) \delta t,$$

if the term with δt^2 is dropped. According to eq. (24), we have

$$d\mathbf{u} = \frac{\dot{\gamma}}{2\gamma} d\mathbf{r} + \mathbf{a} \times d\mathbf{r},$$

and

$$(25) \quad d\mathbf{r}^2 \rightarrow \left(1 + \frac{\dot{\gamma}}{\gamma} \delta t\right) d\mathbf{r}^2,$$

so that all the distances are seen to vary in the same ratio.

The last eq. (23) determines the field of forces in which a local Maxwellian distribution may occur. In particular, when $\mathbf{u} = 0$, eqs. (23) and (24) lead to

$$(26) \quad \alpha = \alpha(\mathbf{r}), \quad \gamma = \text{const}, \quad \mathbf{b} = 0, \quad \mathbf{a} = 0,$$

and in particular

$$(27) \quad \mathbf{F} = + \frac{m}{2\gamma} \text{grad } \alpha.$$

Thus, in the absence of a local velocity of flow, thermal equilibrium can exist only in potential fields which do not vary with time (*cf.* the barometric formula).

43. Fundamental equations of fluid dynamics

A. SERIES EXPANSION FOR THE DISTRIBUTION FUNCTION

In order to evaluate the collision integral in eq. (41.18) it is necessary to know the distribution function f which results from the solution of the collision equation. It will differ from the local Maxwellian distribution (42.21), because we are not dealing with a state of equilibrium. However, the deviations from the equilibrium distribution are, generally speaking, small. For this reason it is useful to begin the construction of f with the local Maxwellian distribution. Without any essential loss of generality we can put

$$(1) \quad f = \left(1 + a_k \frac{\partial}{\partial \xi_k} + a_{kl} \frac{\partial^2}{\partial \xi_k \partial \xi_l} + a_{klm} \frac{\partial^3}{\partial \xi_k \partial \xi_l \partial \xi_m} + \dots \right) f_0$$

because, in essence, f/f_0 is an expansion in Hermite polynomials in three variables, i. e. a complete system (*cf.* Vol. VI), and we may expect that the coefficients of the expansion decrease rapidly, if the deviations are small.

The subscripts k, l, m, \dots in eq. (1) represent the co-ordinates x, y, z . A summation is implied whenever identical indices occur, so that, for example, the second term denotes the sum

$$a_x \frac{\partial f_0}{\partial \xi_x} + a_y \frac{\partial f_0}{\partial \xi_y} + a_z \frac{\partial f_0}{\partial \xi_z}.$$

The same convention applies to the higher-order terms. Generally speaking, the coefficients $a_k, a_{kl}, a_{klm}, \dots$ depend on \mathbf{r} and t , but they are independent of \mathbf{v} by definition. They form tensors of the first, second, and higher orders, and may be assumed to be symmetrical in all subscripts.

The expansion of f in eq. (1) becomes unique only if the coefficients a , α , γ , and u , which are independent of the velocity, are defined with the aid of the local Maxwellian distribution

$$(2) \quad f_0 = e^{\alpha - \gamma(\mathbf{v} - \mathbf{u})^2} = a e^{-\gamma(\mathbf{v} - \mathbf{u})^2}.$$

This follows from the requirement that it should be possible to evaluate particular integrals exactly with the aid of f_0 alone. Thus the *particle density* is:

$$(3a) \quad n = \int f d\xi = \int f_0 d\xi = a (\pi/\gamma)^{3/2},$$

the *mean velocity* is:

$$(3b) \quad \bar{\mathbf{v}} = \frac{1}{n} \int \mathbf{v} f d\xi = \frac{1}{n} \int \mathbf{v} f_0 d\xi = \mathbf{u},$$

and the *mean, isotropic, thermal-kinetic pressure* from eq. (22.3 a) becomes:

$$(3c) \quad p = \frac{m}{3} \int (\mathbf{v} - \bar{\mathbf{v}})^2 f d\xi = \frac{m}{3} \int (\mathbf{v} - \bar{\mathbf{v}})^2 f_0 d\xi = \frac{n m}{2 \gamma}.$$

It represents the pressure which would be exerted on the walls of a small volume moving with the mean velocity.

Equation (3 a) determines the factor a in eq. (2). If we define the temperature T by putting

$$(4) \quad \gamma = m/2 k T$$

and if we use $\rho = n m$ as the *mass density*, we obtain

$$(4a) \quad a = \frac{\rho}{m} \left(\frac{m}{2\pi k T} \right)^{3/2}.$$

The two remaining equations yield conditions for the coefficients in the expansion (1). The i -th component from eq. (3 b) gives

$$\int \xi_i \left(1 + a_k \frac{\partial}{\partial \xi_k} + \dots \right) f_0 d\xi = \int \xi_i f_0 d\xi.$$

Since f_0 vanishes together with all its derivatives on the sphere at infinity in the velocity space, we may perform as many partial integrations as we like, without having to retain surface integrals. Thus we have

$$\int \xi_i f_0 d\xi - a_i \int f_0 d\xi = \int \xi_i f_0 d\xi.$$

The left-hand side contains only two terms. Since already the first term is identical with that on the right-hand side, we have

$$(5) \quad a_i = 0.$$

In a similar way, inserting f from eq. (1) and integrating by parts, we obtain

$$\int (\xi_i - \bar{\xi}_i)^2 f_0 d\xi - 2 a_i \int (\xi - \bar{\xi}_i) f_0 d\xi + a_{ik} \delta_{ik} \int f_0 d\xi = \int (\xi - \bar{\xi}_i)^2 f_0 d\xi.$$

The first term is cancelled by the first term on the right-hand side, and according to eqs. (3 b) and (5) the second term is identically equal to zero. The third term gives $a_{ik} \delta_{ik} = 0$, where

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases}$$

denotes the Kronecker symbol. Thus the sum of the diagonal elements of a_{ik} must vanish, i. e.

$$(6) \quad a_{ij} = 0.$$

In the following, eq. (1) is replaced by

$$(7) \quad f = \left(1 + \frac{1}{2\rho} \sigma_{kl} \frac{\partial^2}{\partial \xi_k \partial \xi_l} + \frac{1}{6\rho} Q_{klm} \frac{\partial^3}{\partial \xi_k \partial \xi_l \partial \xi_m} + \right. \\ \left. + \frac{1}{24\rho} R_{klmn} \frac{\partial^4}{\partial \xi_k \partial \xi_l \partial \xi_m \partial \xi_n} + \dots \right) f_0.$$

The coefficients in the expansion have now been denoted by symbols which will prove convenient later. In conclusion we can deduce an additional condition from (6): The trace of the tensor σ_{kl} vanishes, or

$$(7') \quad \sigma_{ij} = 0.$$

B. MAXWELL'S TRANSPORT EQUATION

A moment is defined here as the local mean value of a power of velocity calculated in accordance with eq. (41.5). For example

$$\overline{\xi_i \xi_k} = \frac{1}{n} \int \xi_i \xi_k f d\xi.$$

In the following the moments of local relative velocity are more important than the former:

$$(8) \quad \mathbf{c} = \mathbf{v} - \bar{\mathbf{v}} = \mathbf{v} - \mathbf{u}, \quad c_i = \xi_i - \bar{\xi}_i = \bar{\xi}_i - u_i.$$

Putting $dc = d\xi$, we can write the second-order moments as

$$\overline{c_i c_k} = \frac{1}{n} \int c_i c_k f dc.$$

According to (8), the first-order moment, the mean values, can be written:

$$(9) \quad \bar{c}_i = \int (\xi_i - \bar{\xi}_i) f dc = 0.$$

The advantage of the expansion in eq. (1) or (7) consists in the fact that a finite number of coefficients in the expansions is always sufficient to calculate a moment, the number being n for a moment of n -th order.

All moments can be reduced to those calculated with the aid of the Maxwellian distribution. For an arbitrary velocity function $\phi(\mathbf{c})$, we shall make use of eq. (41.5):

$$(10) \quad \bar{\phi} = \frac{1}{n} \int f \phi d\xi \quad \text{together with} \quad \bar{\phi}^0 = \frac{1}{n} \int_0 \phi d\xi.$$

On integrating by parts, we have¹

$$(9a) \quad \overline{\phi(\mathbf{c})} = \bar{\phi}^0 + \frac{1}{2\rho} \sigma_{kl} \frac{\partial^2 \bar{\phi}^0}{\partial c_k \partial c_l} + \frac{1}{6\rho} Q_{klm} \frac{\partial^3 \bar{\phi}^0}{\partial c_k \partial c_l \partial c_m} + \dots$$

All odd moments calculated with the aid of the Maxwellian distribution vanish since the former is symmetrical with respect to the origin ($\mathbf{c} \rightarrow -\mathbf{c}$). Furthermore, since the Maxwellian distribution is invariant with respect to mirror reflections and rotations we have

$$(10a) \quad \overline{c_i c_k}^0 = \frac{\rho}{2} \delta_{ik}, \quad \overline{c_i c_j c_k c_l}^0 = \frac{\rho^2}{2} (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj} + \delta_{kl} \delta_{ij}).$$

¹This transformation can lead to difficulties in the higher-order terms if factors $c = \sqrt{c_i^2}$ occur, because the integrals may diverge, in spite of the fact that the integral (10) is convergent. In such cases it is necessary to abandon the integration by parts or to take the "finite parts" of the integrals (cf. Laurent Schwartz, "Théorie des distributions", Hermann & Cie., Paris, 1950.)

except for numerical factors. These can be calculated from the special integrals

$$\overline{c_z^2} = \sqrt{\left(\frac{\gamma}{\pi}\right)^3} \int c^2 \xi^2 e^{-\gamma c^2} c^2 dc d\xi d\phi = \frac{1}{2\gamma} = \frac{\phi}{\rho},$$

$$\overline{c_x^2 c_y^2} = \overline{c_x^2} \cdot \overline{c_y^2} = (\overline{c_z^2})^2 = \frac{\phi^2}{\rho^2}.$$

In accordance with eq. (9 a), the mean moments are:

$$(9 \text{ b}) \quad \overline{c_i c_k} = \frac{\phi}{\rho} \delta_{ik} + \frac{1}{\rho} \sigma_{ik}, \quad \overline{c_i c_k c_l} = \frac{1}{\rho} Q_{ikl},$$

$$\overline{c_i c_j c_k c_l} = \frac{\phi^2}{\rho^2} (\delta_{il} \delta_{jk} + \delta_{jl} \delta_{ki} + \delta_{kl} \delta_{ij}) + \frac{\phi}{\rho^2} [(\sigma_{il} \delta_{jk} + +) + (\sigma_{jk} \delta_{il} + +)] + \frac{1}{\rho} R_{ijkl},$$

the remaining following from these by cyclic transposition.

Generally speaking the moments vary with time and position. They satisfy characteristic equations which are consequences of the collision equation (41.18). Multiplying this by $\phi(\mathbf{v})$ and integrating over the velocity space we obtain directly the *transport* (or *transfer*) equation for the quantity $\phi(\mathbf{v})$:

$$(11) \quad \frac{\partial}{\partial t} (\rho \overline{\phi}) + \text{div} (\rho \overline{\phi \mathbf{v}}) - \mathbf{f} \cdot \left(\frac{\partial \overline{\phi}}{\partial \mathbf{v}} \right) = J(\phi),$$

where

$$(12) \quad \rho = m n, \quad \mathbf{f} = \frac{\rho}{m} \mathbf{F} = n \mathbf{F}$$

denote the mass and force density respectively; $\rho \overline{\phi}$ and $\rho \overline{\phi \mathbf{v}}$ represent the density and the flux of the quantity ϕ ; $J(\phi)$ denotes the integral

$$(12 \text{ a}) \quad J(\phi) = \frac{m s^2}{2} \int \phi(\mathbf{v}) (f' f_1' - f f_1) \cdot |\mathbf{V} \mathbf{e}| d\omega d\xi d\xi_1$$

which can be transformed to

$$(13) \quad J(\phi) = \frac{m s^2}{8} \int (\phi + \phi_1 - \phi' - \phi_1') (f' f_1' - f f_1) |\mathbf{V} \mathbf{e}| d\omega d\xi d\xi_1,$$

in accordance with (42.13). We shall call it the *collision* moment of the quantity $\phi(\mathbf{v})$.

Substituting the additive collision variables from eq. (42.18) into eq. (11), we find that the right-hand side vanishes. All additive collision invariants have been given in eq. (42.18). Accordingly, we obtain five equations which correspond to the five conservation laws: those for mass, energy and the three components of momentum. They are:

$$(14) \quad \frac{\partial}{\partial t} (\rho \bar{\psi}) + \operatorname{div} (\rho \bar{\psi} \mathbf{v}) = n F \left(\frac{\partial \bar{\psi}}{\partial \mathbf{v}} \right).$$

$J(\psi) = 0$ signifies that the mass, energy and momentum do not change *on collision*. The momentum need not be absolutely constant, since we admit an external field of forces.

C. CONSERVATION OF MASS

This follows from $\psi = \psi_0 = 1$. Substituting this expression into eq. (14), we obtain

$$(15) \quad \frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \mathbf{u}) = 0$$

in view of (3 b). This is the familiar (Vol. II, eq. (5.4')) equation of continuity of fluid dynamics. Its validity is more general than would appear from the assumptions required in the derivation of Boltzmann's equation.

The physical significance of this equation becomes clear on taking an integral over an arbitrary, finite volume V . The expression

$$\int \rho \, dx = M$$

gives the total mass enclosed by the volume V , so that the first term in eq. (15) leads to dM/dt . The integral over the second term can be simplified with the aid of Stokes' theorem:

$$\int \operatorname{div} (\rho \mathbf{u}) \, dx = \int \rho u_n \, d\sigma,$$

where $d\sigma$ denotes an element on the surface area O of volume V ; u_n is the component of the mean velocity in the direction of the outward normal to the surface with its positive direction outwards, and $\rho u_n \, d\sigma$ gives the mass flow through the element $d\sigma$. It is positive when the flow is outwards, and negative when the flow is in the opposite direction. The integral must be extended over the whole surface O of V .

Hence, eq. (15) yields

$$(15\ a) \quad -\frac{dM}{dt} = -\frac{d}{dt} \int \rho \, dx = \int \rho \, u_n \, d\sigma.$$

The decrease in mass in V is equal to the flow of mass outwards. The density of mass flow (flux) is

$$(15\ b) \quad \mathbf{s} = \rho \mathbf{u};$$

it is determined by the transport of mass with the mean velocity \mathbf{u} .

D. CONSERVATION OF MOMENTUM

In order to derive the momentum equation it is necessary to write eq. (14) in component form:

$$(14\ a) \quad \frac{\partial}{\partial t} (\rho \bar{\psi}) + \frac{\partial}{\partial k} (\rho \bar{\psi} \xi_k) = f_k \left(\frac{\partial \bar{\psi}}{\partial \xi_k} \right)$$

Substituting $\psi = \psi_i = \xi_i$ in accordance with eq. (42.18), we obtain

$$(16) \quad \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial k} (\rho \bar{\xi_i \xi_k}) = f_i.$$

The first term can be transformed with the aid of the equation of continuity, giving

$$\begin{aligned} \frac{\partial}{\partial t} (\rho u_i) &= \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} = \rho \dot{u}_i - u_i \frac{\partial}{\partial k} (\rho u_k) \\ &= \rho \left(\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial k} \right) u_i - \frac{\partial}{\partial k} (\rho u_i u_k). \end{aligned}$$

Thus eq. (16) now assumes the following form:

$$(17) \quad \rho \frac{du_i}{dt} \equiv \rho \left(\frac{\partial}{\partial t} + u_k \frac{\partial}{\partial k} \right) u_i = -\frac{\partial}{\partial k} [\rho (\bar{\xi_i \xi_k} - u_i u_k)] + f_i,$$

where $d/dt = (\partial/\partial t + u_k \partial/\partial k)$ denotes the substantive derivative referred to a volume moving with the mean velocity (Vol. II, eq. (11.3)).

According to eq. (9 b) the expression in the square bracket becomes

$$\rho (\bar{\xi_i \xi_k} - u_i u_k) = \rho (\bar{\xi_i} - u_i) (\bar{\xi_k} - u_k) = \rho \bar{c_i c_k} = p \delta_{ik} + \sigma_{ik}.$$

Reverting to vector notation we see that eq. (17) leads rigorously to

$$(18) \quad \rho \frac{d\mathbf{u}}{dt} \equiv \rho \left(\frac{\partial}{\partial t} + \mathbf{u} \nabla \right) \mathbf{u} = -\text{grad } p - \text{Div } \sigma + \mathbf{f}.$$

The tensorial divergence $\text{Div } \sigma$ is a vector whose components are

$$(18a) \quad (\text{Div } \sigma)_i = \partial \sigma_{ik} / \partial k.$$

Equation (18) is identical with the equations of motion of fluid dynamics. Since $\sigma_{jj} = 0$, σ_{ik} represents a stress tensor which leads to shearing stresses only (Vol. II, Sec. 10). The assumption

$$(19a) \quad \sigma_{ik} = 0$$

leads to Euler's equation. Putting

$$(19b) \quad \sigma_{ik} = -\eta \left(\frac{\partial u_i}{\partial k} + \frac{\partial u_k}{\partial i} - \frac{2}{3} \frac{\partial u_j}{\partial j} \delta_{ik} \right),$$

we obtain the Navier-Stokes equations. We shall deduce approximately this form of σ_{ik} from the collision equation (see Sec. 44).

In the case of pure shear flow (Couette flow, Sec. 27) for which $\mathbf{u} = (u(y), 0, 0)$, we obtain eq. (27.4), namely

$$\sigma_{xy} = -\eta \frac{\partial u}{\partial y} = \sigma_{yx}.$$

Reciprocally, eq. (19b) follows from (27.4) when we take into account the transformation properties for the rotation of the system of coordinates.

On integrating over a small (!) volume flowing with the mean velocity, we obtain the following equation which is analogous to eq. (15a):

$$(20) \quad \frac{d}{dt} \int \rho u_i dx = - \int (p n_i + \sigma_{in}) d\sigma + \int f_i dx.$$

Here n_i denotes the i -th component of the unit vector \mathbf{n} in the direction of the external normal, and $\sigma_{in} = \sigma_{ik} n_k$. The increase in momentum per unit time is composed of the flow of momentum through the surface (from outside inwards because of the negative sign) and the total force acting on the volume which results from the force density \mathbf{f} .

E. CONSERVATION OF ENERGY

The energy equation is obtained by putting $\psi = \frac{1}{2} \psi_1 = \frac{1}{2} \mathbf{v}^2$. In this case eq. (14) becomes

$$(21) \quad \frac{\partial}{\partial t} \left(\frac{1}{2} \rho \overline{\mathbf{v}^2} \right) + \operatorname{div} \left(\frac{1}{2} \rho \overline{\mathbf{v}^2 \cdot \mathbf{v}} \right) = f \overline{\mathbf{v}}.$$

According to eqs. (8) and (10)

$$\overline{\mathbf{v}^2} = \overline{(\mathbf{c} + \mathbf{u})^2} = \overline{\mathbf{c}^2} + \mathbf{u}^2 = \frac{3}{\rho} \dot{p} + \mathbf{u}^2$$

so that the first term of (21) becomes:

$$\frac{\partial}{\partial t} \left(\frac{3}{2} \dot{p} + \frac{\rho}{2} \mathbf{u}^2 \right) = \frac{3}{2} \ddot{p} + \frac{\dot{\rho}}{2} \mathbf{u}^2 + \rho \mathbf{u} \frac{\partial \mathbf{u}}{\partial t}.$$

Transforming with the aid of the continuity equation (15) and the momentum equation (18), we obtain:

$$\frac{3}{2} \ddot{p} - \frac{1}{2} \mathbf{u}^2 \operatorname{div} (\rho \mathbf{u}) - \mathbf{u} [\rho (\mathbf{u} \nabla) \mathbf{u} + \operatorname{grad} p - \operatorname{Div} \sigma - \mathbf{f}]$$

or

$$(22) \quad \frac{\partial}{\partial t} \left(\frac{\rho}{2} \overline{\mathbf{v}^2} \right) = \frac{3}{2} \ddot{p} - (\mathbf{u} \nabla) p - \operatorname{div} \left(\frac{\rho}{2} \mathbf{u}^2 \cdot \mathbf{u} + \sigma \times \mathbf{u} \right) + \sigma \varepsilon + \mathbf{u} \mathbf{f}.$$

Here the product $\sigma \times \mathbf{u}$ of the tensor σ with the vector \mathbf{u} is a vector whose components are:

$$(22 \text{ a}) \quad (\sigma \times \mathbf{u})_i = \sigma_{ik} u_k;$$

ε denotes that part of the strain tensor of the flow (Vol. II, Sec. 1) which relates to shear:

$$(22 \text{ b}) \quad \varepsilon = \varepsilon_{kl} = \frac{1}{2} \left(\frac{\partial u_l}{\partial k} + \frac{\partial u_k}{\partial l} - \frac{2}{3} \frac{\partial u_j}{\partial j} \delta_{kl} \right)$$

and $\sigma \varepsilon$ is the scalar product of the two tensors σ and ε , defined as

$$(22 \text{ c}) \quad \sigma \varepsilon = \sigma_{kl} \varepsilon_{kl} = \sigma_{kl} \frac{\partial u_l}{\partial k}.$$

The equivalence of the last two expressions is a consequence of the symmetry $\sigma_{lk} = \sigma_{kl}$ subject to the trace condition that $\sigma_{jj} = 0$.

Correspondingly we can transform the mean value in the second term of (21), and we obtain

$$(23) \quad \overline{v^2 \cdot v} = \overline{(c + u)^2 \cdot (c + u)} = \overline{c^2 \cdot c} + \overline{c^2 \cdot u} + 2 \overline{(c | c)} \times u + u^2 \cdot u.$$

Writing the components of this term in accordance with eq. (9 b), we find

$$\overline{c_j^2 c_i} = \frac{1}{\rho} Q_{iji}.$$

Introducing the vector Q with the components

$$(24) \quad Q_i = \frac{1}{2} Q_{iji}$$

we have

$$(24 \text{ a}) \quad \overline{c^2 \cdot c} = \frac{2}{\rho} Q.$$

By eq. (9 b), the second term becomes

$$(24 \text{ b}) \quad \overline{c^2 \cdot u} = \frac{3}{\rho} p u.$$

In the third term $\overline{(c | c)}$ denotes the tensor $\overline{c_i c_k}$, so that eq. (9 b) gives

$$(24 \text{ c}) \quad 2 \overline{(c | c)} \times u = \frac{2}{\rho} p u + \frac{2}{\rho} \sigma \times u.$$

The fourth term need not be transformed.

Substituting eqs. (22) and (23) into (21) and making use of eqs. (24) to (24 c), we obtain

$$\frac{3}{2} p - (u \nabla) p + \operatorname{div} \left[Q + \frac{5}{2} p u \right] + \sigma \varepsilon + f u = f u$$

An elementary transformation gives

$$(25) \quad \frac{3}{2} \frac{dp}{dt} = \frac{3}{2} \left(\frac{\partial}{\partial t} + u \nabla \right) p = -\operatorname{div} Q - \sigma \varepsilon - \frac{5}{2} p \operatorname{div} u.$$

Interpreting Q defined in eq. (24 a) as the *heat flux* (flow of energy in the moving element), and introducing the *density of internal energy* (kinetic energy in the moving system of co-ordinates)

$$(24 \text{ d}) \quad Q = \frac{1}{2} \rho c^2 = \frac{3}{2} p,$$

we can rearrange eq. (25) to read

$$(26) \quad \frac{dQ}{dt} + \operatorname{div} \mathbf{Q} = \left(\frac{\partial}{\partial t} + \mathbf{u} \nabla \right) Q + \operatorname{div} \mathbf{Q} = -Q \operatorname{div} \mathbf{u} - p \operatorname{div} \mathbf{u} - \sigma \varepsilon.$$

This equation can be transformed in two ways. Transposing the first term on the left-hand side to the right-hand side, we have

$$(27) \quad \frac{\partial Q}{\partial t} + \operatorname{div} (\mathbf{Q} + Q \mathbf{u}) = -p \operatorname{div} \mathbf{u} - \sigma \varepsilon.$$

The right-hand side now contains the work of compression and that due to shearing forces (energy dissipation due to friction, see Sec. 44). The divergence on the left-hand side operates on the term representing the local change in Q due to heat conduction \mathbf{Q} and convection $Q \mathbf{u}$.

If, instead of the energy density, we now introduce the energy per unit mass q , we have $Q = \rho q$, and hence

$$\frac{\partial Q}{\partial t} = \rho \frac{\partial q}{\partial t} + q \frac{\partial \rho}{\partial t} = \rho \frac{\partial q}{\partial t} - q \operatorname{div} (\rho \mathbf{u}) = \rho \frac{dq}{dt} - \operatorname{div} (\rho q \mathbf{u}),$$

and it follows from (27) that

$$(28) \quad \rho \frac{dq}{dt} + \operatorname{div} \mathbf{Q} = \rho \left(\frac{\partial}{\partial t} + \mathbf{u} \nabla \right) q + \operatorname{div} \mathbf{Q} = -\rho \operatorname{div} \mathbf{u} - \sigma \varepsilon.$$

The right-hand side has the same meaning as before. The left-hand side now contains the flow of heat due to conduction, because from the macroscopic standpoint we are now concerned with a definite element of mass and observe it as it moves.

F. ENTROPY THEOREM

We now recall the definition of entropy and entropy flux in eqs. (42.1) and (42.5), as well as eq. (42.14) in which the distribution of entropy sources, G , defined in eq. (42.12) is essentially positive. Let η denote the entropy per unit mass,¹ so that

$$H = \rho \eta.$$

¹The symbol η in this Section should not be confused with the viscosity, η , used elsewhere in the present Chapter (*Transl.*).

Taking into account the equation of continuity (15) we conclude from eq. (42.14) that

$$\begin{aligned}\dot{H} &= \dot{\rho} \eta + \rho \dot{\eta} = \rho \dot{\eta} - \eta \operatorname{div}(\rho \mathbf{u}) \\ &= \rho \left(\frac{\partial}{\partial t} + \mathbf{u} \nabla \right) \eta - \operatorname{div}(\rho \eta \mathbf{u}).\end{aligned}$$

Thus eq. (42.14) can be written in the form

$$(29) \quad \rho \frac{d\eta}{dt} = \rho \left(\frac{\partial}{\partial t} + \mathbf{u} \nabla \right) \eta = -\operatorname{div}(\mathbf{S} - \rho \eta \mathbf{u}) + G.$$

In this case the quantity of entropy $H \cdot \mathbf{u}$ transferred by convection must be subtracted from \mathbf{S} . According to (42.1) and (42.5), we have

$$(29 a) \quad \mathbf{S} - H \mathbf{u} = -k \int \mathbf{c} \log f \cdot f d\xi.$$

We now proceed to calculate approximations for $\mathbf{S} - H \mathbf{u}$ and H . We assume that f differs only little from f_0 , so that we may put

$$\log f \approx \log f_0.$$

Thus in view of eq. (42.21), we have

$$\log f \approx \log a - \gamma c^2.$$

Hence, by eq. (42.1) we find the following expression for H

$$H = -k n \log a + k \gamma \int \mathbf{c}^2 f d\xi$$

or, according to (3 c) and (4 a)

$$H = -\frac{k}{m} \rho \log \rho + \frac{3}{2} \frac{k}{m} \rho \log T + \text{const} \times \rho.$$

It follows that the entropy per unit mass (see eq. (5.10)) can be written:

$$(30) \quad \eta = -\frac{k}{m} \left(\log \rho - \frac{3}{2} \log T \right) + \text{const}$$

or, taking the substantive derivative

$$(30 a) \quad \frac{d\eta}{dt} = -\frac{k}{m} \left(\frac{1}{\rho} \frac{d\rho}{dt} - \frac{3}{2} \frac{1}{T} \frac{dT}{dt} \right).$$

Multiplying by T , we find

$$T \frac{d\eta}{dt} = \frac{dq}{dt} - \frac{n k T}{\rho^2} \frac{d\rho}{dt},$$

because $q = (3/2) n k T / \rho = (3/2) (k/m) T$. Written in differential form with $p = n k T$, we see that the last equation becomes

$$(31) \quad T d\eta = dq + p d\left(\frac{1}{\rho}\right),$$

which is the Second Law with dq denoting the differential of internal energy. It is directly linked with the assumption that $\log f \approx \log f_0$ or, in other words, that the flow is not too far removed from the equilibrium distribution.

The last requirement sounds like that in the definition of reversible processes. But, in fact, it is less stringent and allows G in eq. (29) to be different from zero. This follows from the fact that we have only postulated the equality of $\log f$ and $\log f_0$ which both vary very slowly for large arguments. The leeway left by this requirement has already been discussed in Sec. 21 F. It could be justified on the basis of a more exact solution of the Maxwell-Boltzmann collision equation. We shall refrain from doing this here, referring the reader to published papers.¹

The value of G can be calculated from eq. (29). Substituting Maxwell's expression for $\log f$ into (29 a), we have

$$S - H u = k \gamma \int c^2 \cdot c f d\xi,$$

or, in view of eq. (24 a)

$$n k \gamma \cdot \frac{2}{\rho} Q = \frac{Q}{T}.$$

We thus obtain the thermodynamically plausible result:

$$(32) \quad S - H u = \frac{Q}{T}.$$

Substituting this expression, as well as eq. (31), into (21), we find that

$$G = -\frac{k}{m} \frac{d\rho}{dt} + \frac{1}{T} \left(\rho \frac{dq}{dt} + \text{div } Q \right) - \frac{1}{T^2} (Q V) T.$$

¹D. Enskog, Phys. Z. 12 (1911) 533; J. Meixner, Z. Phys. Chemie 53 (1943) 235.

The middle term can now be transformed with the aid of the energy equation (28). Since $p/T = n k = (k/m) \rho$, we have

$$G = -\frac{1}{T^2} (\mathbf{Q} \cdot \text{grad } T) - \frac{1}{T} (H \sigma) - \frac{k}{m} \left(\frac{d\rho}{dt} + \rho \text{div } \mathbf{u} \right).$$

The last term vanishes (equation of continuity). Thus

$$(33) \quad G = -\frac{1}{T^2} (\mathbf{Q} \cdot \text{grad } T) - \frac{1}{T} (\sigma \epsilon).$$

This is the fundamental relation on which irreversible thermodynamics is based and in this connection reference may be made to Sec. 21. Making use of eq. (19 b) and of Fourier's hypothesis (21.3) for heat conduction

$$(33 \text{ a}) \quad \mathbf{Q} = -\kappa \text{grad } T,$$

which, incidentally, can be justified on the ground of kinetic theory in the same way as eq. (19 c), Sec. 44, we obtain, on inserting into eq. (33), that

$$(33 \text{ b}) \quad G = +\frac{\kappa}{T^2} (\text{grad } T)^2 + \frac{\eta}{T} \epsilon^2 \geq 0$$

where G is seen to be essentially positive.

44. On the integration of the collision equation

A. INTEGRATION WITH THE AID OF MOMENT EQUATIONS

Numerous approximate methods have been developed for the integration of the Maxwell-Boltzmann collision equation (41.18). Concerning the details of the theory of integration reference may be made to the comprehensive review by K. F. Herzfeld¹ and to the paper by H. Grad², already quoted. From among the various approximation we shall make use of only those which are consistent with the expansion (43.1) in terms of the derivatives of the Maxwellian distribution. Moreover, we shall carry the development only far enough to exhibit the systematic character of the method and to justify the relations in eqs. (43.19 b) and (43.33 a) which lead to the Navier-Stokes equations and to the heat conduction equation. In this way we are led to the moment method (see H. Grad²).

¹"Freie Weglänge und Transporterscheinungen in Gasen", Hand- u. Jahrbuch d. Chem. Physik, Vol. III 2, Sec. IV, Leipzig, 1939.

²*l. c.* p. 288.

We can develop a large class of functions $g(\mathbf{v})$ in terms of the derivatives of Maxwell's distribution. If we put

$$(1) \quad g(\mathbf{v}) = \left(A_0 + A_k \frac{\partial}{\partial \xi_k} + A_{kl} \frac{\partial^2}{\partial \xi_k \partial \xi_l} + \dots \right) f_0,$$

we can find the coefficients by calculating the moments

$$\begin{aligned} n G_0 &= \int g(\mathbf{v}) d\xi = n A_0; \\ (2) \quad n G_k &= \int \xi_k g(\mathbf{v}) d\xi = n(A_0 \overline{\xi_k}^0 - A_k); \\ n G_{kl} &= \int \xi_k \xi_l g(\mathbf{v}) d\xi = n(A_0 \overline{\xi_k \xi_l}^0 - A_k \overline{\xi_l}^0 - A_l \overline{\xi_k}^0 + A_{kl}) \end{aligned}$$

etc. The mean values $\overline{\xi_k}^0, \overline{\xi_k \xi_l}^0, \dots$ have been defined in eq. (43.10). The equations (2) constitute recurrence formulae for the coefficients of the expansion. They give the following relations:

$$\begin{aligned} A_0 &= G_0, \\ (3) \quad A_k &= G_0 \overline{\xi_k}^0 - G_k, \\ A_{kl} &= G_0 \overline{\xi_k \xi_l}^0 - G_k \overline{\xi_l}^0 - G_l \overline{\xi_k}^0 + G_{kl}, \quad \text{etc.} \end{aligned}$$

According to these equations the coefficients in the expansions of two functions are identical if their moments are identical. We now apply this proposition to the collision equation which is valid when all equations for moments (43.11) are satisfied. Instead of solving the collision equation we can integrate all moment equations. The equations for moments constitute a suitable starting point for approximations.

We have already considered the first equations for moments in Sec. 43, that is all for which the contribution from the collision integral vanishes. In addition we now proceed to consider the moment equations for $\phi = \xi_i \xi_k$ and $\phi = \xi_i \xi_j \xi_k$. From eq. (43.11) we obtain:

$$(4) \quad \frac{\partial}{\partial t} (\rho \overline{\xi_i \xi_k}) + \frac{\partial}{\partial t} (\rho \overline{\xi_i \xi_k \xi_l}) - (\overline{\xi_i} f_k + \overline{\xi_k} f_i) = J_{ik},$$

and

$$(5) \quad \frac{\partial}{\partial t} (\rho \overline{\xi_i \xi_j \xi_k}) + \frac{\partial}{\partial t} (\rho \overline{\xi_i \xi_j \xi_k \xi_l}) - (\overline{\xi_j \xi_k} f_i + \overline{\xi_k \xi_i} f_j + \overline{\xi_i \xi_j} f_k) = J_{ijk}.$$

The right-hand sides contain the *collision moments*

$$(4 \text{ a}) \quad J_{ik} = \frac{1}{2} m s^2 \int \xi_i \xi_k (f' f'_1 - f f_1) |\mathbf{V} \mathbf{e}| d\omega d\xi d\xi_1$$

and

$$(5 \text{ a}) \quad J_{ijk} = \frac{1}{2} m s^2 \int \xi_i \xi_j \xi_k (f' f'_1 - f f_1) |\mathbf{V} \mathbf{e}| d\omega d\xi d\xi_1.$$

B. TRANSFORMATION OF THE EQUATIONS FOR MOMENTS

When calculating the mean values and collision moments in eqs. (4) and (5) we must insert a suitable approximation to the distribution function f . The simplest non-trivial approximation is obtained when we consider on both sides of the equation only the highest non-vanishing term. This means that in our approximation it is sufficient to use the Maxwell distribution on the left-hand side.

Referring partly to previous calculations, we replace the mean values of the powers of ξ by the following expressions:

$$(6) \quad \begin{aligned} \rho \overline{\xi_i} &= \rho u_i, & \rho \overline{\xi_i \xi_k} &\approx \rho \overline{\xi_i \xi_k}^0 = p \delta_{ik} + \rho u_i u_k, \\ \rho \overline{\xi_i \xi_j \xi_k} &\approx \rho \overline{\xi_i \xi_j \xi_k}^0 = p(u_i \delta_{jk} + u_j \delta_{ki} + u_k \delta_{ij}) + \rho u_i u_j u_k, \\ \rho \overline{\xi_i \xi_j \xi_k \xi_l} &\approx \rho \overline{\xi_i \xi_j \xi_k \xi_l}^0 = \frac{p^2}{\rho} (\delta_{jk} \delta_{il} + \delta_{il} \delta_{jk} + \delta_{ki} \delta_{jl} + \delta_{jl} \delta_{ki} + \delta_{li} \delta_{kj} + \delta_{kj} \delta_{li}) \\ &\quad + (\delta_{ij} u_k u_l + \delta_{kl} u_i u_j + \delta_{ik} u_j u_l + \delta_{il} u_j u_k + \delta_{jk} u_i u_l + \delta_{jl} u_i u_k + \delta_{ki} u_j u_l + \delta_{li} u_j u_k + \delta_{ij} u_k u_l + \delta_{kl} u_i u_j) + \rho u_i u_j u_k u_l. \end{aligned}$$

Hence eq. (4) becomes:

$$\frac{\partial}{\partial t} (p \delta_{ik} + \rho u_i u_k) + \frac{\partial}{\partial l} [p(u_i \delta_{kl} + u_k \delta_{il}) + \rho u_i u_k u_l] - [u_i f_k + u_k f_i] = J_{ik},$$

or, after a simple rearrangement:

$$\begin{aligned} J_{ik} &= \left(\frac{dp}{dt} + \frac{5}{3} p \operatorname{div} \mathbf{u} \right) \delta_{ik} + p \left(\frac{\partial u_i}{\partial k} + \frac{\partial u_k}{\partial i} - \frac{2}{3} \frac{\partial u_l}{\partial l} \delta_{ik} \right) \\ &\quad + [\dot{\rho} + \operatorname{div}(\rho \mathbf{u})] u_i u_k + u_i \left(\rho \frac{du_k}{dt} + \frac{\partial p}{\partial k} - f_k \right) + u_k \left(\rho \frac{du_i}{dt} + \frac{\partial p}{\partial i} - f_i \right). \end{aligned}$$

According to eqs. (43.15), (43.18) and (43.25) all terms on the right-hand side vanish, except the second, if it is taken into account that in the present

approximation $\sigma_{ik} = 0$ and $Q_i = 0$. Introducing the strain tensor ε_{ik} from eq. (43.22 b), we have

$$(7) \quad J_{ik} = 2 p \varepsilon_{ik}.$$

Equation (5) can be transformed in the same way. Introducing the mean values (6) we obtain first in shorthand notation

$$J_{ijk} = \frac{\partial}{\partial t} [p(u_i \delta_{jk} + +) + \rho u_i u_j u_k] + \frac{\partial}{\partial l} \left[\frac{p^2}{\rho} (\delta_{il} \delta_{jk} + +) + p(\delta_{jk} u_i u_l + +) \right. \\ \left. + p(\delta_{il} u_j u_k + +) + \rho u_i u_j u_k u_l \right] - \frac{p}{\rho} (f_i \delta_{jk} + +) - (f_i u_j u_k + +).$$

It is convenient to rearrange terms to obtain the following more lucid form

$$J_{ijk} = \left(\frac{dp}{dt} + p \operatorname{div} \mathbf{u} \right) (u_i \delta_{jk} + +) + \frac{p}{\rho} \left[\delta_{jk} \left(\rho \frac{du_i}{dt} - f_i + \frac{\partial p}{\partial i} \right) + + \right] \\ + p \left[\delta_{jk} \frac{\partial}{\partial i} \left(\frac{p}{\rho} \right) + + \right] + [\dot{\rho} + \operatorname{div}(\rho \mathbf{u})] u_i u_j u_k \\ + \left[u_i u_k \left(\rho \frac{du_j}{dt} - f_j + \frac{\partial p}{\partial j} \right) + + \right] + p \left[\left(\frac{\partial u_j}{\partial i} u_k + \frac{\partial u_k}{\partial i} u_j \right) + + \right].$$

Taking into account eqs. (43.15), (43.18) and (43.25) with $\sigma_{ik} = 0$, $Q_i = 0$ and $p/\rho = (k/m) T$, we find that

$$J_{ijk} = \frac{k p}{m} \left(\frac{\partial T}{\partial i} \delta_{jk} + + \right) + p \left[u_i \left(\frac{\partial u_k}{\partial j} + \frac{\partial u_j}{\partial k} - \frac{2}{3} \frac{\partial u_l}{\partial l} \delta_{jk} \right) + + \right],$$

or, according to eq. (4):

$$(8) \quad J_{ijk} = \frac{k p}{m} \left(\frac{\partial T}{\partial i} \delta_{jk} + + \right) + (u_i J_{jk} + +).$$

We shall see later that J_{ik} and σ_{ik} are proportional. Since on the left-hand side of eqs. (4) and (5) we have assumed $\sigma_{ik} = 0$ we must drop J_{jk} in eq. (8) for reasons of consistency. Thus we obtain finally

$$(9) \quad J_{ijk} = \frac{k p}{m} \left(\frac{\partial T}{\partial i} \delta_{jk} + \frac{\partial T}{\partial j} \delta_{ki} + \frac{\partial T}{\partial k} \delta_{ij} \right).$$

It follows for the trace that

$$(10) \quad J_{ijj} = \frac{5 k p}{m} \frac{\partial T}{\partial i}.$$

C. EVALUATION OF COLLISION MOMENTS

These are given by eqs. (43.12 a) or (43.13) and vanish when f is replaced by the Maxwell distribution. Thus in this case it is necessary to take into account the correction terms. The lowest-order term in the expansion of the product f/f_1 can be written

$$(11) \quad f/f_1 = f_0/f_{01} + \frac{1}{2\rho} \sigma_{rs} \left(f_0 \frac{\partial^2 f_{01}}{\partial \xi_{1r} \partial \xi_{1s}} + f_{01} \frac{\partial^2 f_0}{\partial \xi_r \partial \xi_s} \right) - \\ - \frac{1}{6\rho} Q_{mrs} \left(f_0 \frac{\partial^3 f_{01}}{\partial \xi_{1m} \partial \xi_{1r} \partial \xi_{1s}} + f_{01} \frac{\partial^3 f_0}{\partial \xi_m \partial \xi_r \partial \xi_s} \right) + \dots$$

as seen from eq. (43.7). The collision moments are homogeneous and linear in σ_{rs} and Q_{mrs} , because the first term does not contribute to the collision moments, and the quadratic terms in σ_{rs} and Q_{mrs} occur only from the fourth order onwards, which we do not consider.

The moments in eqs. (4 a) and (5 a) are symmetric tensors in the same way as the coefficients σ_{ik} and Q_{ijk} . We can write down the form of the collision moments because no other tensors than σ_{ik} , Q_{ijk} and the unit tensor δ_{ik} play any part. Thus we must have

$$(12) \quad J_{ik} = a \sigma_{ik} \\ J_{ijk} = b Q_{ijk} + c(Q_{rri} \delta_{jk} + Q_{rrj} \delta_{ki} + Q_{rrk} \delta_{ij}).$$

In the first eq. (12) the term proportional to δ_{jk} has been omitted because the factor which follows from homogeneity, $\sigma_{jj} = 0$.

Making use of eqs. (7) and (9) together with (12), we can now calculate the coefficients σ_{ik} and Q_{ijk} in the expansion. They are proportional to ε_{ik} and $\left(\frac{\partial T}{\partial i} \delta_{jk} + \frac{\partial T}{\partial j} \delta_{ki} + \frac{\partial T}{\partial k} \delta_{ij} \right)$. To be consistent with eqs. (43.19 b) and (43.33 a) we denote the coefficients of proportionality by

$$(13) \quad \sigma_{ik} = -2\eta \varepsilon_{ik}, \quad Q_l = -\kappa \frac{\partial T}{\partial l}.$$

The last equation, as seen from (43.24), is derived from the tensor

$$(13 \text{ a}) \quad Q_{ijk} = -\frac{2\kappa}{5} \left(\frac{\partial T}{\partial i} \delta_{jk} + \frac{\partial T}{\partial j} \delta_{ki} + \frac{\partial T}{\partial k} \delta_{ij} \right).$$

The viscosity, η , and the thermal conductivity, κ , can be calculated from eqs. (12). Substituting eqs. (13) and (13 a) into it, we obtain two equations which must be identical with eqs. (7) and (9) respectively:

$$J_{ik} = -2\eta a \varepsilon_{ik} = 2p \varepsilon_{ik},$$

$$J_{ijk} = -\frac{2\kappa(b+5c)}{5} \left(\frac{\partial T}{\partial i} \delta_{jk} + + \right) = \frac{k p}{m} \left(\frac{\partial T}{\partial i} \delta_{jk} + + \right).$$

It follows that

$$(14) \quad \eta = -\frac{p}{a}, \quad \kappa = -\frac{5kp}{2m(b+5c)} = +\frac{5}{2} \frac{a}{b+5c} \frac{k}{m} \eta.$$

Substituting the expressions (13) and (13 a) into (43.7), we find the distribution function

$$(15) \quad f = f_0 - \frac{\eta}{\rho} \varepsilon_{ik} \frac{\partial^2 f_0}{\partial \xi_i \partial \xi_k} + \frac{\kappa}{5\rho} \left(\frac{\partial T}{\partial i} \frac{\partial}{\partial \xi_i} \right) \frac{\partial^2 f_0}{\partial \xi_i^2}.$$

Equations (13) and (13 a) agree with eqs. (43.19 b) and (43.33 a), respectively, and provide a justification for the Navier-Stokes equations as well as for the heat conduction equation. The source density of entropy (43.33) becomes, as it should, essentially positive on condition that κ and η are positive (*cf.* Sec. D), as already demonstrated in eq. (43.33 b).

D. VISCOSITY AND THERMAL CONDUCTIVITY

It now remains to evaluate the integrals (4 a) and (5 a) starting with the form (43.13) in which the term with $f' f_1'$ can be transformed. We replace the variables \mathbf{v}' , \mathbf{v}_1' with \mathbf{v} , \mathbf{v}_1 , and notice that in accordance with eqs. (41.12 b) and (41.16 a) $d\xi d\xi_1$ and $|\mathbf{V} \mathbf{e}|$ remain unchanged; the first factor changes sign. In this way (43.13) can be replaced by

$$(17) \quad J_\phi = +\frac{1}{4} m s^2 \int (\phi' + \phi_1' - \phi - \phi_1) f f_1 |\mathbf{V} \mathbf{e}| d\omega d\xi d\xi_1$$

and the integration can be carried out in two steps.

Since the distribution functions f and f_1 are independent of the unit vector \mathbf{e} we may split off the integral over \mathbf{e} :

$$(18) \quad I_\phi = I_\phi(\mathbf{v}, \mathbf{v}_1) = \frac{1}{4} m s^2 \int (\phi' + \phi_1' - \phi - \phi_1) \cdot |\mathbf{V} \mathbf{e}| d\omega.$$

We now consider the special value of I_ϕ for $\phi = \xi_i \xi_k$ and $\phi = \xi_i \xi_j \xi_k$, so that

$$(18 \text{ a}) \quad I_{ik} = + \frac{1}{4} m s^2 \int (\xi_i' \xi_k' + \xi_{1i}' \xi_{1k}' - \xi_i \xi_k - \xi_{1i} \xi_{1k}) |\mathbf{V} \mathbf{e}| d\omega$$

and

$$(18 \text{ b}) \quad I_{ijk} = + \frac{1}{4} m s^2 \int (\xi_i' \xi_j' \xi_k' + \xi_{1i}' \xi_{1j}' \xi_{1k}' - \xi_i \xi_j \xi_k - \xi_{1i} \xi_{1j} \xi_{1k}) |\mathbf{V} \mathbf{e}| d\omega.$$

In the above equations \mathbf{V} denotes the relative velocity. Introducing the mean velocity $\mathbf{U} = \frac{1}{2} (\mathbf{v} + \mathbf{v}_1)$, we have

$$(19 \text{ a}) \quad \mathbf{v} = \mathbf{U} - \frac{1}{2} \mathbf{V}, \quad \mathbf{v}_1 = \mathbf{U} + \frac{1}{2} \mathbf{V},$$

and from eq. (41.12) we have

$$(19 \text{ b}) \quad \mathbf{v}' = \mathbf{U} - \frac{1}{2} \mathbf{V}', \quad \mathbf{v}_1' = \mathbf{U} + \frac{1}{2} \mathbf{V}'$$

with

$$(19) \quad \mathbf{V}' = \mathbf{V} - 2 (\mathbf{V} \mathbf{e}) \mathbf{e}.$$

After an elementary rearrangement, we find

$$(20 \text{ a}) \quad \xi_i' \xi_k' + \xi_{1i}' \xi_{1k}' - \xi_i \xi_k - \xi_{1i} \xi_{1k} = 2 (\mathbf{V} \mathbf{e})^2 e_i e_k - (\mathbf{V} \mathbf{e}) (V_i e_k + V_k e_i) = V_{ik}$$

and

$$(20 \text{ b}) \quad \xi_i' \xi_j' \xi_k' + \xi_{1i}' \xi_{1j}' \xi_{1k}' - \xi_i \xi_j \xi_k - \xi_{1i} \xi_{1j} \xi_{1k} = (U_i V_{jk} + U_j V_{ki} + U_k V_{ij}).$$

It is seen from eq. (20 a) that the integral (18 a) depends only on the vector \mathbf{V} . Since the result must be a symmetrical tensor whose trace is zero, we must have

$$(20) \quad I_{ik} = \frac{\gamma m s^2}{4} V \left(V_i V_k - \frac{1}{3} V^2 \delta_{ik} \right).$$

Furthermore it follows from (18 b) that

$$(21) \quad I_{ijk} = (U_i I_{jk} + U_j I_{ki} + U_k I_{ij}).$$

The constant γ can be calculated from the special integral

$$\begin{aligned} I_{ik} V_i V_k &= \frac{\gamma m s^2}{6} V^5 = \frac{m s^2}{2} \int [(\mathbf{V} \mathbf{e})^4 - V^2 (\mathbf{V} \mathbf{e})^2] |\mathbf{V} \mathbf{e}| d\omega \\ &= 2\pi m s^2 \cdot V^5 \int_0^1 (\zeta^5 - \zeta^3) d\zeta = -\frac{\pi m s^2}{6} V^5, \end{aligned}$$

so that $\gamma = -\pi$ and

$$(22) \quad I_{ik} = -\frac{\pi m s^2}{4} V \left(V_i V_k - \frac{1}{3} V^2 \delta_{ik} \right).$$

According to (17) and (18) the equation which determines the collision moments is

$$J_\phi = \int I_\phi f f_1 d\xi d\xi_1,$$

where $f f_1$ is given by eq. (11). The first term gives no contribution, and the two following pairs can be contracted, because eqs. (21) and (22) are symmetrical in \mathbf{v} and \mathbf{v}_1 . Thus

$$(23) \quad J_\phi = \frac{1}{\rho} \int I_\phi \left(\sigma_{rs} \frac{\partial^2 f_0}{\partial \xi_r \partial \xi_s} - \frac{1}{3} Q_{mrs} \frac{\partial^3 f_0}{\partial \xi_m \partial \xi_r \partial \xi_s} \right) f_{01} d\xi d\xi_1.$$

Applying partial integration it is possible to transfer the derivatives to the I_ϕ . The product $f f_1$ is replaced by the function

$$(24) \quad f_0 f_{01} = n^2 \frac{\gamma^3}{\pi^3} e^{-2\gamma(\mathbf{U}-\mathbf{u})^2 - \frac{1}{2}\gamma \mathbf{V}^2}.$$

It follows that integrals extending over odd polynomials of $\mathbf{V}/|\mathbf{V}|$ vanish so that in view of (22) and (21), respectively, eq. (23) leads to

$$(25 \text{ a}) \quad J_{ik} = \frac{\sigma_{rs}}{\rho} \int \frac{\partial^2 I_{ik}}{\partial V_r \partial V_s} f_0 f_{01} d\xi d\xi_1$$

and

$$(25 \text{ b}) \quad J_{ijk} = \frac{1}{2\rho} \left(Q_{irs} \int \frac{\partial^2 I_{jk}}{\partial V_r \partial V_s} f_0 f_{01} d\xi d\xi_1 + + \right).$$

The sum is cyclic in ijk .

The integral in (25 a) is symmetrical in ik and rs and the trace over $i = k$ vanishes. It then follows that

$$(25) \quad \int \frac{\partial^2 I_{ik}}{\partial V_r \partial V_s} f_0 f_{01} d\xi d\xi_1 = I \left(\delta_{ir} \delta_{ks} + \delta_{is} \delta_{kr} - \frac{2}{3} \delta_{ik} \delta_{rs} \right).$$

The numerical factor is calculated for special values of the subscripts, e. g. for $i = r = x$, $k = s = z$. Thus

$$\begin{aligned} I &= -\frac{\pi m s^2}{4} \int \frac{\partial^2 V}{\partial V_x \partial V_z} V_x V_z f_0 f_{01} d\xi d\xi_1 \\ &= -\frac{\pi m s^2}{4} \int V \left(1 + \frac{V_x^2 + V_z^2}{V^2} - \frac{V_x^2 V_z^2}{V^4} \right) f_0 f_{01} d\xi d\xi_1. \end{aligned}$$

It is seen that the integration over \mathbf{U} and over the direction of \mathbf{V} can be carried out at once on substituting the expression from eq. (24). Thus

$$I = -\frac{\pi m s^2}{4} n^2 \frac{\gamma^3}{\pi^3} \left(\frac{\pi}{2\gamma} \right)^{3/2} \cdot 2\pi \cdot \frac{16}{5} \int_0^\infty V^3 e^{-kV^2} dV,$$

or

$$(26) \quad I = -\frac{4}{5} m n^2 s^2 \left(\frac{2\pi}{\gamma} \right)^{1/2} = -\frac{8}{5} n^2 s^2 (\pi m k T)^{1/2}.$$

Substituting eq. (25) into (25 a) and (25 b) we obtain

$$\begin{aligned} J_{ik} &= \frac{2}{\rho} I \sigma_{ik} \\ (27) \quad J_{ijk} &= \frac{3}{\rho} I Q_{ijk} - \frac{1}{3\rho} I (Q_{irr} \delta_{jk} + +), \end{aligned}$$

in agreement with eq. (12). A comparison with the latter yields

$$(27') \quad a = 2 \frac{I}{\rho}, \quad b = 3 \frac{I}{\rho}, \quad c = -\frac{1}{3} \frac{I}{\rho}.$$

Taking into account (26), we can deduce from eq. (14) that

$$(28) \quad \eta = \frac{p\rho}{2I} = \frac{5/16}{\pi s^2} (\pi m k T)^{1/2}$$

and

$$(28a) \quad \frac{\kappa}{\eta} = \frac{5}{2} \frac{a}{b} + \frac{k}{5c} \frac{m}{m} = \frac{15}{4} \frac{k}{m}.$$

Measurements on monatomic gases give the following results

	He	Ne	A	Kr	X
$\frac{4m\kappa}{15k\eta}$	0.98	1.00	0.98	1.02	1.03

and the agreement is seen to be astonishingly good because the assumption of rigid molecules must at first appear to be questionable and useful only in a qualitative way.

A remarkable comparison between the two eqs. (13) is obtained in applying the left-hand equation to shear flow (Couette flow, Vol. II) when we assume that $\mathbf{u} = (0, u_y(x), 0)$. In this case

$$(29) \quad \sigma_{xy} = -\eta \frac{\partial u_y}{\partial x} = -\frac{\eta}{m} \frac{\partial p_y}{\partial x}.$$

Here p_y denotes the mean momentum of a molecule in the y -direction and σ_{xy} is the flux of momentum across an element of area of size 1 at right angles to the x -axis. Inserting eq. (28 a) into the right-hand side eq. (13) for the heat flux, and introducing the local heat energy $Q = 3/2 k T$, we find

$$(29 \text{ a}) \quad Q_x = -\frac{5}{2} \frac{\eta}{m} \frac{\partial Q}{\partial x}.$$

This equation shows that the transfer of heat proceeds more efficiently than the transfer of momentum. This result can be understood in a qualitative way. Large molecular velocities in a given direction enhance transfer in that direction. A change in these velocities exerts no influence on the momentum being transferred, because in the case of friction we are concerned with the transfer of that component of momentum which is normal to the direction of flow of momentum. In the case of energy transfer the conditions are different, because every component of velocity contributes to the energy. It follows that large energies are favored in the process of transfer and that, on the whole, thermal contact is more intimate than momentum contact.

The latter remark does not apply to rotational energy. For this reason, in the case of polyatomic rigid molecules, Eucken replaces the expression $5/2 Q$ in eq. (29 a) by

$$\frac{5}{2} Q_{\text{transl}} + Q_{\text{rot}} = \left(\frac{5}{2} \cdot \frac{3}{2} + \frac{f-3}{2} \right) k T = \left(\frac{f}{2} + \frac{9}{4} \right) k T.$$

This leads to the equation

$$(29 \text{ b}) \quad Q_x = -\left(1 + \frac{9}{2f} \right) \frac{\eta}{m} \frac{\partial Q}{\partial x}$$

where $Q = \frac{1}{2} f k T$ denotes the mean energy of a molecule, and f is the number of degrees of freedom. We thus obtain for

$$f = 3 \quad 5 \quad 6$$

$$1 + \frac{9}{2f} = 2.5 \quad 1.9 \quad 1.75.$$

The results of measurements are as follows:

$$\begin{array}{ccccccc} & \text{H}_2 & \text{O}_2 & \text{CO} & \text{Air} & & \\ 1 + \frac{9}{2f} = & 2.00 & 1.92 & 1.81 & 1.96 & \text{instead of } 1.9 & \end{array}$$

and for

$$\begin{array}{ccccccc} & \text{CH}_4 & \text{CO}_2 & \text{C}_3\text{H}_8 & & & \\ 1 + \frac{9}{2f} = & 1.74 & 1.64 & 1.66 & \text{instead of } 1.75. & & \end{array}$$

The term \sqrt{T} in eq. (28) is proportional to the mean velocity. It is easy to verify that

$$\bar{c} \approx \bar{c}^0 = \frac{2\sqrt{2}}{\pi m} (\pi k T m)^{1/2}$$

so that

$$(30) \quad \eta = \frac{5\pi}{32} \frac{m \bar{c}}{\pi s^2}$$

Substituting $l = 1/n\pi s^2 \sqrt{2}$ from eq. (27.11) for the order of magnitude of the mean free path we obtain

$$(30 a) \quad \eta = \frac{5}{32} \rho l \bar{c} = \frac{\rho l \bar{c}}{2.04}.$$

45. Conductivity and the Wiedemann-Franz law

A. THE COLLISION AND TRANSFER EQUATIONS FOR ELECTRONS IN METALS

The collision equation for metal electrons differs from Boltzmann's collision equation (41.18) in that it is necessary to take into account only collisions between conduction electrons and the ions of the lattice and that the lattice ions are very heavy compared with the electrons. Consequently, on impact there is an exchange of momentum, but, practically speaking, no exchange of energy. Evidently such a statement cannot be strictly true because, as we have already seen in Sec. 39, the electrons participate in thermal

equilibrium. Nevertheless, by way of a first approximation, we can neglect the transfer of energy as compared with the transfer of momentum. Let \mathbf{v} denote the velocity of an electron before it collides with a lattice ion, at first considered to be a rigid sphere. Thus the velocity \mathbf{v}' after collision is

$$(1) \quad \mathbf{v}' = \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{e}) \mathbf{e}.$$

Here the symbol \mathbf{e} denotes, as before, the unit vector in the direction of the central axis. The change in momentum is

$$(2) \quad \Delta \mathbf{p} = m(\mathbf{v}' - \mathbf{v}) = -2m(\mathbf{v} \cdot \mathbf{e}) \mathbf{e},$$

whereas the change in energy is negligible:

$$(3) \quad E = \frac{m}{2} (\mathbf{v}'^2 - \mathbf{v}^2) = 0.$$

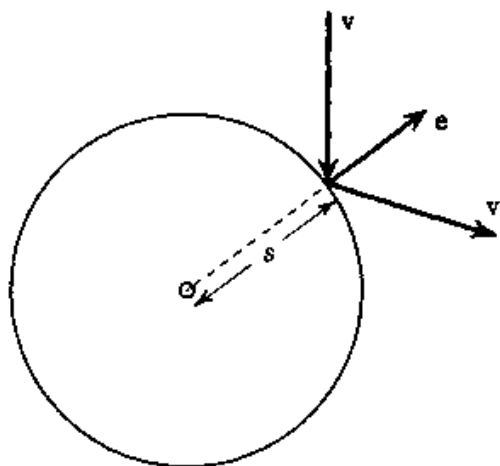


Fig. 37.

Illustrating collision between conduction electrons and lattice ions.

If f denotes the distribution function for the electrons, we can see that the left-hand side of the collision equation (41.18) remains unchanged, but the collision integral is now different. The integrand will contain only one factor f and the second should be replaced by the probability of impinging on a spherical

ion. It is determined by the density, n_0 , of the lattice ions. Denoting the radius of an ion by s , we obtain

$$(4) \quad J_{\text{loss}} = \frac{1}{2} n_0 s^2 \int |\mathbf{v} \cdot \mathbf{e}| f(\mathbf{r}, \mathbf{v}, t) d\omega,$$

$$J_{\text{gain}} = \frac{1}{2} n_0 s^2 \int |\mathbf{v} \cdot \mathbf{e}| f(\mathbf{r}, \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{e}) \mathbf{e}, t) d\omega,$$

in a way similar to Sec. 41 D.

Introducing the mean free path $l = 1/n_0 \pi s^2$, we can write the collision equation in the form

$$(5) \quad \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{1}{m} \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{1}{2\pi l} \int |\mathbf{v} \cdot \mathbf{e}| (f' - f) d\omega.$$

The function f' depends on the argument \mathbf{v}' from eq. (1) in the same way as f depends on \mathbf{v} . In accordance with our model, the mean free path l should be a constant, but the model is certainly very crude. We shall adapt it better

to reality by assuming that l is a function of the velocity of the electron and of the special properties of the lattice, in particular of its temperature. In order to find a more exact expression it would be necessary to resort to wave mechanics, but we shall refrain from doing so here.

The transfer equation for a function $\phi(\mathbf{v})$ can be obtained in a way similar to eq. (43.11). Multiplying eq. (5) by $\phi(\mathbf{v})$, and integrating with respect to \mathbf{v} gives

$$(6) \quad \frac{\partial}{\partial t} (\rho \bar{\phi}) + \operatorname{div} (\rho \bar{\phi} \mathbf{v}) - n F \left(\frac{\partial \bar{\phi}}{\partial \mathbf{v}} \right) = J(\phi).$$

Here $J(\phi)$ denotes the collision moment of the function ϕ :

$$(5 \text{ a}) \quad J(\phi) = \frac{m}{2\pi} \int \frac{\phi}{l} (f' - f) |\mathbf{v} \mathbf{e}| d\omega d\xi = \frac{m}{2\pi} \int \frac{f}{l} d\xi \int (\phi' - \phi) |\mathbf{v} \mathbf{e}| d\omega.$$

Since $\phi' - \phi = 0$ for $\phi = 1$ and for $\phi = \frac{1}{2} v^2$, it is seen that the mass and energy equations remain valid, as before. The momentum equation, however, is now replaced by the equation

$$(7) \quad \frac{\partial}{\partial t} (\rho \bar{\mathbf{v}}) + \operatorname{Div} [\rho (\bar{\mathbf{v}} |\mathbf{v}|)] - n F = \frac{m}{2\pi} \int \frac{f}{l} d\xi \int (\mathbf{v}' - \mathbf{v}) |\mathbf{v} \mathbf{e}| d\omega.$$

According to (1) the integral over $d\omega$ becomes

$$(8) \quad \int (\mathbf{v}' - \mathbf{v}) |\mathbf{v} \mathbf{e}| d\omega = -2 \int |\mathbf{v} \mathbf{e}| (\mathbf{v} \mathbf{e}) \mathbf{e} d\omega = -2\pi v \mathbf{v}.$$

This form is due to the fact that the integral is a vector and a homogeneous function of degree 2 of \mathbf{v} . The numerical factor is obtained by taking the scalar product of (8) with \mathbf{v} and by dividing by v^3 . Thus

$$\frac{\mathbf{v}}{v^3} \int (\mathbf{v}' - \mathbf{v}) |\mathbf{v} \mathbf{e}| d\omega = -2 \int |\xi^2| d\xi d\phi = -2\pi.$$

Consequently, eq. (7) becomes:

$$(9) \quad \frac{\partial}{\partial t} (\rho \bar{\mathbf{v}}) + \operatorname{Div} [\rho (\bar{\mathbf{v}} |\mathbf{v}|)] - n F = -\rho \left(\frac{v \mathbf{v}}{l} \right).$$

B. APPROXIMATE SOLUTION OF THE COLLISION EQUATION

We now propose to consider the solutions of (5) in the neighborhood of the equilibrium solution f_0 , beginning with the entropy principle, as in Sec. 42. Differentiating the entropy density equation (38.9), namely

$$H = -k \int [n \log n + (1-n) \log (1-n)] \cdot \frac{2m^3}{h^3} d\xi$$

with respect to time, we have

$$\frac{\partial H}{\partial t} = -k \int \log \frac{n}{1-n} \cdot \dot{n} \cdot \frac{2m^3}{h^3} d\xi$$

or, according to eq. (5) (with $F = 0$), and in view of the fact that $f = \frac{2m^3}{h^3} n$, that it is equal to

$$\begin{aligned} &+ k \operatorname{div} \int \mathbf{v} [n \log n + (1-n) \log (1-n)] \frac{2m^3}{h^3} d\xi \\ &+ \frac{k}{4\pi} \int \frac{|\mathbf{v} \mathbf{e}|}{l} \cdot \log \frac{n'/(1-n')}{n/(1-n)} \cdot (n' - n) d\omega \frac{2m^3}{h^3} d\xi. \end{aligned}$$

Thus we obtain

$$(10) \quad \frac{\partial H}{\partial t} + \operatorname{div} \mathbf{S} = G,$$

where the flux of entropy is:

$$(11) \quad \mathbf{S} = -k \int \mathbf{v} [n \log n + (1-n) \log (1-n)] \frac{2m^3}{h^3} d\xi$$

and the *source density* is

$$(11a) \quad G = \frac{k}{4\pi} \int \frac{|\mathbf{v} \mathbf{e}|}{l} \log \frac{n'/(1-n')}{n/(1-n)} \cdot (n' - n) d\omega \frac{2m^3}{h^3} d\xi.$$

In the case of equilibrium we must have $G = 0$. Since $n/(1-n)$ increases monotonically with n , the integrand cannot be negative, and $G = 0$ can only occur when $n' = n$. Since the only collision invariants are 1 and \mathbf{v}^2 , f must be a function of energy alone:

$$(12) \quad n = n_0(E), \quad E = \frac{m}{2} \mathbf{v}^2.$$

It is remarkable that the result is indefinite. This is a reflection of the fact that the interactions between electrons have been neglected. We have to imagine that eq. (11 a) contains an additional term for G which is due to collisions between the electrons, in the same way as in eq. (42.12). Normally it would be small but if we substitute (12), this term alone remains. It is then that we establish a special energy function. If G were given by eq. (42.12), f_0 would be identical with Maxwell's distribution, but we know that we have to assume the Fermi distribution in relation to electrons. Leaving aside the problem of correcting the expression for the entropy source density G , we assume with reference to eq. (39.5) that

$$(12\ a) \quad f_0 = \frac{2\ m^3}{h^3} n_0 = \frac{2\ m^3/h^3}{e^{\beta(E-\zeta)} + 1}.$$

Here β and ζ represent parameters which can still depend on time and on space coordinates: $\beta = \frac{1}{kT}$ and ζ is the free enthalpy per electron (cf. eq. (39.4)). Sommerfeld's theory of conductivity differs from that due to Drude and Lorentz only in the assumption (12 a).

Equations (12) and (12 a) differ from previous results in one more respect, namely in that the equilibrium distribution no longer depends upon the local mean velocity. Mathematically this is a consequence of the fact that the momentum of an electron is not preserved after a collision. It is also understandable on physical grounds that the electron distribution changes if we cause the electrons to move collectively while keeping the lattice fixed. It follows that in the expansion of the distribution function, f , in terms of the equilibrium distribution, f_0 , we shall find first-order derivatives as well:

$$(13) \quad f = f_0 - u_k \frac{\partial f_0}{\partial \xi_k} + \frac{1}{2\rho} \sigma_{kl} \frac{\partial^2 f_0}{\partial \xi_k \partial \xi_l} - \frac{1}{6\rho} Q_{klm} \frac{\partial^3 f_0}{\partial \xi_k \partial \xi_l \partial \xi_m} + \dots$$

Following Sommerfeld (and Lorentz) we shall use here a slightly different approximation. Since f_0 depends only on the energy E we can write the first-order term as

$$-u_k \frac{\partial f_0}{\partial \xi_k} = -m(u_k \xi_k) f_0',$$

where primes denote differentiation with respect to E . The higher-order terms of the expansion in (13) also contain terms of the same type. If

$$Q_{klm} = \frac{2}{5} (Q_k \delta_{lm} + Q_l \delta_{mk} + Q_m \delta_{kl})$$

(or when we separate the term of this form from Q_{klm}), we can see that the contribution from the term of the third order will be

$$-\frac{1}{5\rho} Q_k \frac{\partial^3 f_0}{\partial \xi_k \partial \xi_l^2} = -\frac{m^2}{\rho} (Q_k \xi_k) \left(f_0'' + \frac{2}{5} E f_0''' \right).$$

Characteristically these terms are of the form

$$U_k(E) \xi_k$$

and the coefficients U_k depend on energy, apart from their dependence on time and space coordinates; they are, however, independent of the direction of the velocity. If we perform the same transformation with regard to all terms in eq. (13), we obtain a series of the following type:

$$(14) \quad f = f_0(E) + U_k(E) \xi_k + \frac{1}{2} U_{kl}(E) \xi_k \xi_l + \dots$$

The higher coefficients of the expansion can be assumed to be symmetrical tensors whose traces all vanish. We have $U_{kk} = 0$, $U_{kkll} = 0$, etc.

If we had $U_{kkll} \neq 0$, we could represent it in the form

$$U_{klm} = U_{klm}^* + (V_k \delta_{lm} + V_l \delta_{mk} + V_m \delta_{kl})$$

where we could have $U_{kkm}^* = 0$. It would only be necessary to put $U_{kkm} = 5 V_m$. Thus the first term would have the desired form and the second would give a contribution to the series (14) of the form

$$\frac{3}{2} (V_k \xi_k) \xi_l^2 = \frac{3E}{m} V_k(E) \xi_k$$

which could, obviously, be included in the second term of the expansion in (14).

We now substitute eq. (14) into the collision equation (5) and take into account only the first non-vanishing terms, restricting ourselves to steady flows only ($\partial f / \partial t = 0$). Thus we obtain:

$$(15) \quad \left(\mathbf{v} \cdot \frac{\partial f_0}{\partial \mathbf{r}} + \mathbf{F} f_0' \right) = \frac{\mathbf{U}}{2\pi l} \int |\mathbf{v} \cdot \mathbf{e}| (\mathbf{v}' - \mathbf{v}) d\omega.$$

By eq. (8) the right-hand side is equal to

$$-\frac{1}{l} v(\mathbf{U} \cdot \mathbf{v}).$$

Since this equation must apply for arbitrary directions of \mathbf{v} , we may also write

$$(16) \quad \mathbf{U} = -\frac{l}{v} \left(\mathbf{F} f_0' + \frac{\partial f_0}{\partial \mathbf{r}} \right).$$

Hence the first approximation to the distribution function (14) is:

$$(17) \quad f = f_0 - l \left(\frac{\mathbf{v}}{v} \cdot f_0' \mathbf{F} + \frac{\partial f_0}{\partial \mathbf{r}} \right).$$

C. FLUX OF CURRENT AND ENERGY

In terms of the following integrals:

$$(18) \quad \mathbf{W}_n = \int E^n \mathbf{v} f d\xi$$

the *current* and *energy fluxes* are

$$(18 \text{ a}) \quad \mathbf{l} = -e \mathbf{W}_0.$$

$$(18 \text{ b}) \quad \mathbf{W} = \mathbf{W}_1.$$

For reasons of symmetry the first term in (17) gives no contribution to (18). Performing the angular integration we obtain from the second term that

$$\mathbf{W}_n = -\frac{4\pi}{3} \int_0^\infty \left(f_0' \mathbf{F} + \frac{\partial f_0}{\partial \mathbf{r}} \right) l E^n v^3 dv.$$

Substituting into eq. (12) the function

$$(19) \quad g(\epsilon) = -\frac{1}{e^\epsilon + 1}, \quad \epsilon = \beta(E - \zeta)$$

and replacing v by $E = \frac{1}{2} m v^2$, we have

$$(20) \quad \mathbf{W}_n = -\frac{16\pi m \beta}{3 h^3} \int_0^\infty g'(\epsilon) \left(\mathbf{F} - \frac{1}{\beta} \frac{\partial \beta \zeta}{\partial \mathbf{r}} + \frac{E}{\beta} \frac{\partial \beta}{\partial \mathbf{r}} \right) l E^{n+1} dE.$$

Introducing the abbreviations

$$(21) \quad \mathbf{F}_1 = \mathbf{F} - \frac{1}{\beta} \frac{\partial \beta \zeta}{\partial \mathbf{r}}, \quad \mathbf{F}_2 = \frac{1}{\beta} \frac{\partial \beta}{\partial \mathbf{r}}$$

and putting

$$(21 \text{ a}) \quad -\frac{16\pi m \beta}{3 h^3} \int_0^\infty g'(\epsilon) l E^n dE = K_n$$

we can deduce quite generally from eq. (20) that

$$(22) \quad W_n = K_{n+1} F_1 + K_{n+2} F_2,$$

and in particular that

$$(22 \text{ a}) \quad \begin{aligned} l &= -e (K_1 F_1 + K_2 F_2), \\ W &= K_2 F_1 + K_3 F_2. \end{aligned}$$

Eliminating

$$(21 \text{ b}) \quad E' = -\frac{1}{e} F_1 - \frac{\zeta}{e} \frac{1}{\beta} \text{grad } \beta = E + \frac{1}{e} \text{grad } \zeta,$$

we have

$$(23) \quad \begin{aligned} E' &= \frac{1}{e^2 K_1} l + \frac{K_2 - \zeta K_1}{e K_1} \frac{1}{\beta} \text{grad } \beta, \\ W &= -\frac{K_2}{e K_1} l + \frac{K_1 K_3 - K_2^2}{K_1} \frac{1}{\beta} \text{grad } \beta \end{aligned}$$

which agrees with eqs. (21.18 a, b). The *electrical conductivity* becomes

$$(24) \quad \sigma = e^2 K_1$$

and the *Peltier coefficient* and the absolute thermal emf become, respectively,

$$(24 \text{ a}) \quad \Pi = \frac{K_2}{e K_1},$$

$$(24 \text{ b}) \quad \varepsilon = \frac{K_2 - \zeta K_1}{e K_1 T},$$

whereas the thermal conductivity is:

$$(24 \text{ c}) \quad \kappa = \frac{K_1 K_3 - K_2^2}{K_1 T}.$$

D. OHM'S LAW

Ohm's law is obtained on the assumption that ζ and β are constant throughout space. Hence from (21 b) we obtain: $\mathbf{E}' = \mathbf{E}$, and from (23):

$$(25) \quad \mathbf{l} = \sigma \mathbf{E}$$

which is Ohm's law. The electrical conductivity can be calculated from (24) with K_1 from eq. (21 a) assuming complete degeneration (cf. 39 B):

$$\sigma = \frac{16\pi m \zeta_0 l_0}{3 h^3} e^2.$$

Here ζ_0 and l_0 denote the values of ζ and l , respectively, at Fermi's threshold. The particle density can be calculated with the aid of eqs. (39.7 and 39.7 a) and is

$$n = \frac{8\pi m^3}{3 h^3} v_0^3, \quad \zeta_0 = \frac{1}{2} m v_0^2.$$

Consequently σ can be written in the form

$$(25 \text{ a}) \quad \sigma = \frac{n e^2}{m} \cdot \frac{l_0}{v_0}$$

which is Drude's equation (39.2).

In the above equation v_0 denotes Fermi's limiting velocity and l_0 is the value of the mean free path for electrons moving with that velocity. Evidently l_0 can vary with the temperature, since it depends on the lattice properties. On the other hand v_0 is independent of temperature. However, when we evaluate the integrals in accordance with Sec. 39 C for the case of almost complete degeneracy, v shows a weak dependence on temperature. It is of the order $(k T / m v_0^2)$ and cannot be observed.

The mean free path l_0 can also be calculated from measured values of conductivity and so, for example, for silver at room temperature we obtain $l_0 \approx 5 \times 10^{-6}$ cm on the assumption of one conducting electron per atom. This would mean that l_0 is much larger than the distance between lattice ions, a sure sign that the mean free path must be calculated with the aid of wave mechanics.

When evaluating the *Peltier coefficient* it is also sufficient, at least here, to evaluate the integrals K_n for the limiting case of complete degeneration. It follows from eq. (21 a) that in this case

$$(21 \text{ c}) \quad K_n = \frac{16\pi m}{3 h^3} l_0 \zeta_0^n.$$

Consequently

$$(26) \quad \Pi = \frac{m v_0^2}{2 e}.$$

It is noticed that the mean free path cancels and does not appear in the final expression. This is an elementary value of the Peltier coefficient Π . When $\text{grad } \beta = 0$, the first term in eq. (23) gives the energy flux

$$W = n \cdot \frac{m v_0^2}{2} \cdot \bar{v}.$$

It is evident that this is the kinetic energy transferred macrophysically, because the electrons which contribute to the mean value \bar{v} are near the Fermi threshold. Additional remarks concerning the Peltier effect have been given in Sec. 21. In order to determine its numerical value it is necessary to perform a more accurate calculation.

When the current density $I = 0$, but the temperature distribution is not uniform, there is a flow of heat, and an electric field is formed. The flow of heat and the strength of the electric field are determined by the thermal conductivity, κ , and the absolute emf, ε . According to (21 c) both vanish in the case of complete degeneration.

E. THERMAL CONDUCTIVITY AND ABSOLUTE THERMAL ELECTROMOTIVE FORCE

In order to calculate κ and ε it is necessary to obtain a more accurate expression for the numerator. According to Sec. 39 D the integrals in eq. (21 a) can be written

$$K_n = -\frac{16\pi m}{3 h^3} \int_{-\beta\zeta}^{\infty} g'(\varepsilon) F_n\left(\zeta + \frac{\varepsilon}{\beta}\right) d\varepsilon$$

where $F_n = I E^n$ so that

$$(27) \quad K_n = +\frac{16\pi m}{3 h^3} \left[F_n(\zeta) + \frac{\pi^2}{6 \beta^2 \zeta^2} \cdot \zeta^2 F_n''(\zeta) \right].$$

The particle density is given by eq. (39.11 b)

$$n = \frac{8\pi}{3 h^3} (2 m \zeta)^{3/2} \left(1 + \frac{\pi^2}{8 \beta^2 \zeta^2} \right).$$

Consequently

$$(27\ a) \quad K_n = \frac{n}{\sqrt{2m}} l \zeta^{n-3/2} \left[1 + \frac{\pi^2}{6\beta^2 \zeta^2} \left(\frac{\zeta^2 F_n''}{F_n} - \frac{3}{4} \right) \right].$$

In the correction term we can substitute ζ_0 for ζ , i. e. the value at Fermi's threshold. For the first factor it is actually necessary to take into account eq. (39.13), but it is sufficient to substitute the threshold value here as well because by eq. (27 a) we have

$$K_1 K_3 - K_2^2 = \frac{n^2}{2m} l_0^2 \zeta_0 \cdot \frac{\pi^2}{6\beta^2 \zeta_0^2} (1 \times 0 + 3 \times 2 - 2 \times 2 \times 1),$$

so that the factors in front of the brackets in (27 a) are seen to be multiplied by higher-order terms only. Thus we have

$$\frac{K_1 K_3 - K_2^2}{K_1} = \frac{n}{\sqrt{2m}} l_0 \zeta_0^{3/2} \cdot \frac{\pi^2}{3\beta^2 \zeta_0^2}.$$

Putting $\zeta_0 = \frac{1}{2} m v_0^2$, we obtain the *thermal conductivity* from eq. (24 c):

$$(28) \quad \kappa = \frac{\pi^2}{3} \frac{k}{m} \frac{l_0}{v_0} \cdot n k T.$$

It is remarkable that the derivatives of $l(\zeta)$ which occur in eq. (27 a) disappear in the expression for κ .

The absolute thermal emf can be calculated from

$$K_2 - \zeta K_1 = \frac{n}{\sqrt{2m}} l \zeta^{1/2} \frac{\pi^2}{6\beta^2 \zeta^2} \cdot 2 \left(1 + \zeta \frac{l'}{l} \right).$$

Accordingly eq. (24 b) can be written

$$(29) \quad \epsilon = \frac{\pi^2}{3} \frac{k}{e} \cdot \frac{k T}{\zeta_0} \left(1 + \frac{d \log l_0}{d \log \zeta_0} \right).$$

which contains the first derivative of $l(\zeta)$.

F. THE WIEDEMANN-FRANZ LAW

The expressions in Drude's eq. (25 a) and in eq. (28) for thermal conductivity contain only the threshold value l_0/v_0 . On forming the ratio κ/σ the term l_0/v_0 cancels, and we are led to the Wiedemann-Franz law:

$$(30) \quad \frac{\kappa}{\sigma} = \frac{\pi^2}{3} \frac{k^2}{e^2} T.$$

Following Lorenz' method, experimental physicists indicate the relation

$$(31) \quad A = \frac{\kappa}{\sigma T} = \frac{\pi^2 k^2}{3 e^3} \quad (\text{Lorenz number}).$$

We prefer to compare the dimensionless ratio

$$(32) \quad A_0 = \frac{e^2 \kappa}{\sigma k^2 T} = \frac{A e^2}{k^2} = \frac{\pi^2}{3} = 3.29,$$

with experimental results. Thus we have to multiply the Lorenz number by $e^2/k^2 = 1.344 \times 10^8 \text{ deg}^2 \text{ volt}^{-2}$. Experiments show that A_0 is not constant, but decreases with decreasing temperature; however, at high temperatures the curve tends asymptotically to a constant value. At a temperature of 100 C we obtain the following values:

Cu	Au	Pb	Pt
3.15	3.19	3.46	3.51

The curves $A_0(T)$ for copper and gold could tend asymptotically to the value $\pi^2/3$, but in the case of lead and platinum it is distinctly higher. Drude has given the value of $A_0 = 3$ on the basis of a crude estimate. A more accurate calculation on a classical basis performed by H. A. Lorentz gives $A_0 = 2$. Compared with this, the quantum mechanical value appears to constitute a considerable improvement. In actual fact the values for most substances remain below this value which is consistent with the supposition that the temperature is not yet high enough.

However, the values for many substances exceed this value markedly, as seen from the following examples:

W (polycrystalline)	$T = 273 \text{ K},$	$A_0 = 4.11,$
Bi (fine crystals)	$T = 90 \text{ K},$	$A_0 = 5.56,$
	$T = 273 \text{ K},$	$A_0 = 3.62.$

The case of bismuth shows even a temperature anomaly in that A_0 at first increases with decreasing temperature. In this connection it must be remembered that the Lorenz number deals with the electrical current and the heat flux transferred by the electrons. If the lattice itself contributes to the conduction of heat the value of $\kappa/\sigma T$ must be expected to increase. The value $A_0 = 5.56$ would indicate that about 2/3 of the heat flux is conducted by the lattice and the thermal conductivity of the lattice (evaluated from the total

value of $\kappa = 0.06 \text{ cal cm}^{-1} \text{ sec}^{-1} \text{ deg}^{-1}$) would be about $0.024 \text{ cal cm}^{-1} \text{ deg}^{-1}$.¹ This is not entirely unlikely. For example NaCl at 20 C is a good electrical insulator (spec. resistance $\rho = 1 \times 10^{17} \text{ ohm cm}$), but at 25 C its thermal conductivity is $\kappa = 0.02 \text{ cal cm}^{-1} \text{ sec}^{-1} \text{ deg}^{-1}$.

The preceding deviations could be due to purely experimental conditions (insufficiently high temperature, large contribution due to the conductivity of the ion lattice), but the dependence of A_0 on temperature certainly cannot be explained by such secondary influences. In this connection the fact that the collision integral in eq. (5) has been calculated with the aid of classical mechanics makes itself felt, and we are thus led to the consideration of the difficulties which are encountered in an improved theory:

We have assumed originally that the metallic ion is a hard elastic sphere and has a definite radius s , so that the mean free path becomes $1/n\pi s^2$. In fact such an assumption has no sense, and the mean free path must be calculated with the aid of quantum mechanics;² we have assumed that l was an unknown function of the lattice temperature T_g and of the energy of the electron, E . We must, however, stress here that the concept of the mean free path can be justified on the grounds of quantum mechanics only for extremely high, and also, but with certain reservations, for extremely low temperatures.

¹Data from J. D'Ans and E. Lax, *Taschenbuch für Chemiker und Physiker*, 2 ed., Springer 1949, p. 1126.

²H. A. Bethe and A. Sommerfeld: *l. c.* p. 277, footnote 1; Sec. 31—38.

PROBLEMS

CHAPTER I

I.1. It is assumed that the three variables x, y, z satisfy a functional relationship $f(x, y, z) = 0$, or, solved for z , $z = f(x, y)$. Prove the identity:

$$\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -1.$$

Replacing x, y, z by p, V, T in that order, deduce a relation between the coefficient of thermal expansion, the coefficient of tension and the compressibility (for definitions see (1.5) and (1.6)).

I.2. *On the heating problem.*

Calculate the quantity of heat required to increase the room temperature from 0 C to 20 C.

I.3. *Absolute temperature and perfect gas thermometer.*

Prove that the absolute temperature T defined in eq. (6.7 a) is identical with that defined by the perfect-gas law.

I.4. *Application of the Second Law to the proof of an algebraic inequality.*

a) Two bodies whose heat capacities are C_1, C_2 , and whose temperatures are T_1, T_2 exchange heat, both volumes being kept constant. What is the common final temperature of both?

b) Comparing the values of entropy before and after equilibrium has set in ($\Delta S > 0$) for the special case of perfect gases deduce an inequality which is a generalization of that between the arithmetical and geometrical means.

I.5. One mol of a perfect gas expands reversibly until its volume is doubled: a) under constant pressure, b) isothermally, c) isentropically. Calculate the work of expansion, the heat added and the change in entropy for each case.

I.6. Imagine a Carnot cycle with water as the working fluid operating between 2 C and 6 C, so that at 6 C there is isothermal expansion and isothermal compression at 2 C. It is seen that heat is added during both processes, if the pressure is low enough (cf. (7.10)), and so heat is converted completely into work in violation of the Second Law. How is it possible to resolve this contradiction? Make a qualitative sketch of the isentropes and isotherms in a T, v -diagram in the neighborhood of 4 C.

I.7. Show that the ratio of isothermal to isentropic compressibility is always equal to the ratio of specific heats at constant volume and at constant pressure as it is for a perfect gas (*cf.* vol. II). In other words, show that

$$\frac{\kappa_T}{\kappa_S} = \frac{c_p}{c_v} \quad \text{where} \quad \kappa_S = -\frac{1}{v} \left(\frac{\partial v}{\partial p} \right)_S, \quad \kappa_T = -\frac{1}{v} \left(\frac{\partial v}{\partial p} \right)_T.$$

In order to do this express the differential dq in terms of dv and dp and prove that

$$T ds = dq = c_p \left(\frac{\partial T}{\partial v} \right)_p dv + c_v \left(\frac{\partial T}{\partial p} \right)_v dp.$$

I.8. One kilogram of water is compressed isothermally at 20°C from 1 at to 20 at. Calculate the amount of work required, the quantity of heat rejected and the increase in internal energy. Mean compressibility $\kappa = 0.5 \times 10^{-4}/\text{at}$, mean coefficient of thermal expansion $\alpha = 2 \times 10^{-4}/\text{deg}$. Use eq. (7.7) and the relation (2) given in the solution to Problem I.1.

I.9. *Adiabatic equilibrium of the atmosphere.*

During so-called convective (adiabatic) equilibrium of the atmosphere which is particularly well established in the presence of "sirocco" winds the value of $p v^\gamma$ is independent of altitude; v denotes here the molar or the specific volume. Making use of the relation between density and pressure which follows from the conditions of equilibrium in the gravitational field it is possible to show that there is a linear temperature decrease with altitude. Measurements give its value as 1°C/100 m; what is the theoretical value? — Calculate the height of the general polytropic atmosphere (defined by $p v^n = \text{const.}$; n is known as the polytropic exponent). In particular calculate the height of the adiabatic, and of the isothermal atmosphere (for which $n = 1$), for a ground temperature of 0°C.

I.10. *The flow of gases.*

Calculate the final temperature and the maximum value of the flow velocity for superheated steam of 300°C and 5 at pressure which expands isentropically through a suitably shaped nozzle to a back-pressure of 1 at.

In order to perform the calculation make use of the fact that the kinetic energy can at most be equal to the difference in the enthalpies of the compressed and of the expanded gas (*cf.* Sec. 4 B). The same fact can be proved with the aid of Bernoulli's equation for a compressible fluid (*cf.* Vol. II, Sec. 11), assuming the flow to be irrotational and steady.

I.11. *Isothermal equilibrium of the atmosphere.*

A gas is contained in a closed box placed in the gravitational terrestrial field. The internal energy is then augmented by the potential energy; the latter depends on the altitude above ground.

a) Establish the condition of thermodynamic equilibrium by subdividing the gas into i cells of volume V_i at an elevation z_i above ground. Assume a definite

value of specific volume v_i and temperature T_i for each cell and solve the problem by calculating the maximum in entropy for a given total energy, mass and volume.

b) Show that the temperature is independent of altitude.

c) Show that Gibbs' potential is independent of altitude provided that its definition includes the potential energy (compare the electrochemical potential in Sec. 18).

d) Calculate the density in terms of altitude, assuming the validity of the perfect gas law.

e) Calculate the difference in elevation for which the difference in potential energies per mol is equal to $R T$.

I.12. (After H. Einbinder, Phys. Rev. 74, 805 (1948)).

Assuming an equation of state of the form $p v = \alpha u(T, v)$, where $u(T, v)$ is the specific internal energy and α is a constant show:

1. That the specific energy u and the specific entropy s can be expressed in the form

$$u = v^{-\alpha} \Phi(T v^{\alpha}); \quad s = \psi(T v^{\alpha})$$

where Φ is an arbitrary function of the argument, and $\Phi'(x) = x \psi'(x)$.

2. That $u/v = \sigma T^{\frac{\alpha+1}{\alpha}}$ when the energy density u/v depends only on the temperature T . This, for example, is true for black-body radiation with $\alpha = 1/3$ (cf. Sec. 20 B) and for a Bose gas composed of N particles of mass m below the temperature $T_0 \sim \frac{h^2}{m k} (N/v)^{2/3}$ with $\alpha = 2/3$ (Einstein condensation, cf. Sec. 38).

3. Assume that in the neighborhood of absolute zero $\Phi(T v^{\alpha})$ can be represented by the power law

$$\Phi(T v^{\alpha}) = c T^m v^{\alpha m} \quad (m > 0).$$

a) Find the relation between α and m required by the condition of dynamic stability $(\partial p / \partial v)_T \leq 0$.

b) Assume that $u \rightarrow 0$ for $T \rightarrow 0$. According to the uncertainty principle we must also have $v = v(T, p) > 0$ also for $T \rightarrow 0$. Determine the relation between α and m implied by the relation $v(T, p) > 0$.

c) Taking into account the results of questions 3 a and 3 b find an expression for the internal energy and for the equation of state.

4. What result can be deduced for small values of T with the same power law but assuming that $u(T, v)$ tends to a finite limit when $T \rightarrow 0$?

CHAPTER II

II.1. Prove with the aid of (14.11 b) that pV is the thermodynamic potential for the variables T, p, μ_i . (It occurs in the theory of grand canonical ensembles in statistical mechanics; cf. Sec. 40 for a special application).

II.2. *The rate of change of latent heat along the vapor-pressure curve.*

In connection with eq. (16.14 a) we have used an experimental value for dr/dT ; calculate the same quantity theoretically making use of the definition $r = \Delta h$, forming its differential, and utilizing the Clausius-Clapeyron equation together with some of the relations in the Table of Sec. 7.

II.3. A perfectly insulated vessel having a volume of 20 liters contains 1 kg of H_2O at 10 C partly in the liquid and partly in the vapor phase. Calculate the amount of energy required to bring the system to a temperature of 200 C. It is convenient to imagine that the final state is reached as follows: a) isothermal compression until complete liquefaction has been achieved, b) isothermal compression from the saturation pressure of $p = 0.0125$ kg/cm² at 10 C to $p = 15.86$ kg/cm² (= saturation pressure at 200 C), c) heating at constant pressure without evaporation ($c_p \approx 1$ cal g⁻¹ deg⁻¹), d) isothermal expansion (evaporation) until the initial volume has been reached. The energy input for process b) and the work of expansion for process c) may be neglected (Compare Prob. I.8).

At 10 C: latent heat of evaporation $r = 591.6$ cal/g, specific volume of liquid $v_1 = 1.00$ dm³/kg, of steam $v_2 = 106.4$ m³/kg.

At 200 C: $r = 463.5$ cal/g, $v_1 = 1.16$ dm³/kg, $v_2 = 0.127$ m³/kg.

II.4. *Realization of thermodynamic temperature scale.*

Prove that the absolute temperature T can be calculated from the Clausius-Clapeyron equation $T dp/dT = r/(v_{vap} - v_{liq})$ if r , v_{vap} and v_{liq} are known as functions of pressure p .

II.5. The vapor pressure of mercury is:

a) 0.0127 torr at 50 C, and 0.0253 torr at 60 C,

b) 247 torr at 300 C, and 505 torr at 310 C.

Calculate the latent heat of evaporation in each of these intervals assuming r to be constant. Neglect v_{liq} and assume that the vapor behaves like a perfect gas. — Interpolate r linearly between these intervals and determine the vapor-pressure curve for Hg. Extrapolate the curve to temperatures > 300 C and calculate the boiling point at 1 atm (= 760 torr) pressure (accurate value 356.7 C).

II.6. Consider a gas whose molecules are capable of achieving three energy levels $\epsilon_0, \epsilon_1, \epsilon_2$. ($\epsilon_1 - \epsilon_0$ and $\epsilon_2 - \epsilon_0$ are thus excitation energies for the first and second excited energy levels respectively) and suppose that it is a mixture of three gases each consisting of molecules of one internal energy level ϵ_0, ϵ_1 , or ϵ_2 . Deduce the conditions of equilibrium on the assumption that all gases have the same entropy constant and transition is possible between 0 and 1 as well as between 0 and 2. Show that the same conditions apply when transitions from 1 to 2 are also permitted.

II.7. Principle of detailed equilibrium.

a) Calculate the differences in the chemical potentials $\mu_1 - \mu_0$ and $\mu_2 - \mu_0$ for the preceding problem for the case when the molar masses n_i differ only slightly from those at equilibrium, \bar{n}_i .

b) Regarding the variation in n_0, n_1, n_2 make the following assumption: The change per second in, say, n_1 is due to the fact that during that time a fraction $k_{11} n_1$ proportional to the amount present changes into the states 0 and 2. At the same time fractions proportional to n_0 and n_2 , i. e. $k_{10} n_0$ and $k_{12} n_2$ undergo the transition into state 1. Write down the equations for dn_i/dt . The preceding assumption implies that the number of transitions from one state to another depends only on the number of molecules at that state and that it is proportional to it.

c) Show that the law of mass action (see eq. (3) in the solution to the previous problem) follows from this assumption insofar as \bar{n}_1/\bar{n}_0 and \bar{n}_2/\bar{n}_0 no longer depend on \bar{n}_1 and \bar{n}_2 . A comparison with the already mentioned eq. (3) yields two relations which must be satisfied by the coefficients k_{ik} .

d) Making use of the relations deduced in a) substitute $\mu_1 - \mu_0$ and $\mu_2 - \mu_0$ for n_i in the equation for dn_i/dt found in b). What is the meaning of the validity of Onsager's reciprocal relations with respect to the mechanism of the reaction?

CHAPTER III

III.1. A vertical cylinder is fitted with a piston of mass M which can follow the influence of gravity without friction. The cylinder contains a sphere (of mass $m \ll M$) moving up and down in a vertical direction with a velocity c ; it is elastically reflected by the piston and by the cylinder head. Neglect the influence of gravity on the motion of the sphere.

a) Establish the condition of equilibrium for the piston and compare it with the perfect-gas equation, ignoring the dimensions of the sphere.

b) Repeat the calculation taking into account that the radius of the sphere is r and compare the result with the van der Waals equation.

c) Imagine that the piston is being withdrawn slowly with a velocity $V_p \ll c$ and compare the loss in energy suffered by the sphere with the work $dW = P dV$ of a gas.

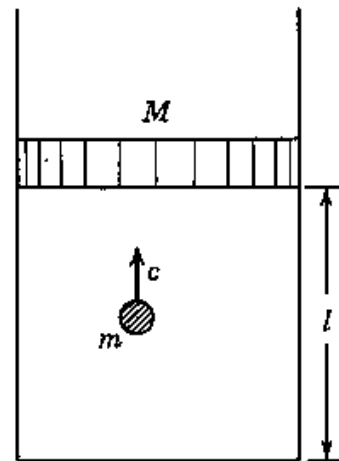


Fig. 38.

One-dimensional gas consisting of a single molecule.

III.2. For the Maxwellian velocity distribution (23.9) calculate a) the most probable velocity, b) the mean velocity, and c) the root-mean-square of the velocity.

III.3. Compute the number of H_2 molecules which impinge on an area $\sigma = 1 \text{ cm}^2$ of a wall in a second with a velocity which exceeds 12000 m/sec, assuming that the temperature is 0 C and that the total number of molecules in 1 cm^3 is 2×10^{19} .

III.4. Calculate the mean value of the number of throws, k , after which the 6 of a dice may be expected to show up. Calculate, further, the mean variance defined as $\overline{(k - \bar{k})^2}$.

III.5. Calculate the pressure exerted by a perfect gas on a wall (assumed placed in the plane $x = 0$ of a rectangular system of coordinates) if the wall attracts the molecules at large distances and repels them at small distances with a force whose potential is $U = -A e^{-x} + B e^{-2ax}$. a) assuming that the influence of the force extends over a distance which is small compared with the mean free path; b) assuming that both are comparable.

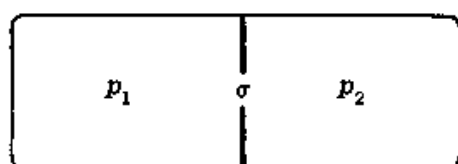


Fig. 39.

The transfer of mass and energy through narrow openings.

Calculate the distance over which the influence of wall forces would have to extend in helium under normal conditions in order to affect the pressure on the wall.

III.6. A perfect gas fills two compartments of a vessel which are connected through a very small opening of area σ . The initial temperature in both compartments is equal to, say, T , and the pressures are p_1 and p_2 respectively.

- Calculate the mass of gas which flows in a unit of time from the compartment with the higher pressure to that with the lower pressure under steady-state conditions ($p_1 = \text{const}$, $p_2 = \text{const}$);
- Calculate the corresponding rate of transfer of energy;
- Calculate the mean quantity of energy transferred per particle;
- Why is it larger than $3/2 k T$?
- What measures is it necessary to take to maintain a steady state?

III.7. A gas at temperature T contains a movable plate B placed between two fixed plates A_1 and A_2 at a distance which is small compared with the mean free path, so that intermolecular collisions may be disregarded. Assume that B and A_1 have the same temperature as the gas and that A_2 is heated to a slightly higher temperature $T' = T + \delta T$.

- Calculate the force acting on the movable plate B assuming that all molecules reach thermal equilibrium with the wall from which they are reflected (infinitely rough wall) and that the plates have equal areas A .
- Calculate the pressure of the gas from this force (ultra-vacuum pressure gage).

CHAPTER IV

IV.1. During a certain experimental measurement the result depends in a random way (i. e. either $+\varepsilon$ or $-\varepsilon$) on a large number n of equal, mutually independent elementary errors. Show that the probability of obtaining an experimental result which deviates from the true one by x is given by $W = a \cdot \exp(-x^2/2n\varepsilon^2)$ for large values of n . In order to make the derivation more lucid the student may visualize Galton's board: the probability that a sphere impinging on a nail will move to the left of it is the same as that for it to roll over to the right.

IV.2. Assume that the experimental error for a single measurement is not $\pm \epsilon$, $\epsilon = \text{const}$, as in the preceding problem, but that it can vary in a certain interval. Let $f_1(x) dx$ denote the probability of the error to fall inside the interval $(x, x + dx)$.

a) Derive an expression for the probability $f_n(x) dx$ when n independent errors of the same kind are superimposed on each other;

b) Prove that all functions $f_n(x)$ are Gaussian if f_1 is Gaussian. Derive an expression for the spread of the Gaussian curve.

c) Derive an expression for f_n for large values of n when $f_1(x) = 1$ for $|x| < \epsilon$ and $= 0$ for $|x| > \epsilon$. Plot the curves f_1 , f_2 , and f_3 and deduce from it the geometrical form of these functions for a polydimensional cube.

IV.3. Compute the number W of permutations of N molecules,

a) when all molecules have the same velocity $+\xi$ as regards magnitude and direction;

b) when one half each of them have the velocities $+\xi$ and $-\xi$ respectively;

c) when one sixth each have the velocities $\pm \xi$, $\pm \eta$, and $\pm \zeta$, respectively. Prove that for $N \rightarrow \infty$ each successive distribution is infinitely more probable than the preceding one.

IV.4. A very small mirror is suspended from a quartz strand whose elastic constant is D , and reflects a beam of light in such a way that the angular amplitudes caused by the impacts due to surrounding molecules (Brownian motion) can be read on a suitable scale. The position of equilibrium is at $\phi = 0$ ($\phi = \text{angular amplitude}$). The probability of finding the mirror at an amplitude between ϕ and $\phi + d\phi$ is given by

$$W d\phi = a e^{-E_{\text{pot}}/kT} d\phi, \quad E_{\text{pot}} = \frac{1}{2} D \phi^2,$$

in accordance with the law of equipartition. From the observed value of $\overline{\phi^2}$ it is possible to determine the Boltzmann constant k . Calculate the numerical value of the Loschmidt-Avogadro number from the following data obtained at $T = 287^\circ \text{K}$: $D = 9.43 \times 10^{-9}$ dyne cm; $\overline{\phi^2} = 4.18 \times 10^{-6}$ using the known value for the universal gas constant, $R = 8.32 \times 10^7$ erg/deg mol (Kappler).

IV.5. Consider a cubic crystal containing $N = 10^{21}$ atoms. The cohesion energy per surface atoms is 9 eV. Calculate the ratio of cohesion energy to thermal energy. Calculate the size of the crystal for which the two are equal.

IV.6. a) Show that the partition function (33.3) of a rotator can be deduced from the definition of the theta function

$$\vartheta_2 \left(z \left| \frac{i q}{\pi} \right. \right) = 2 \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})^2 q} \cos (2n+1) \pi z$$

and that we have

$$Z(q) = -\frac{1}{\pi^2} e^{q/4} \int_0^\infty \frac{\partial \vartheta_2 \left(z \left| \frac{i q}{\pi} \right. \right)}{\partial z} \frac{dz}{z}.$$

b) Making use of the transformation formula

$$\vartheta_2 \left(z \left| \frac{i q}{\pi} \right. \right) = \left(\frac{\pi}{q} \right)^{1/2} e^{-\pi^2 z^2 / q} \vartheta_0 \left(\frac{-i \pi z}{q} \left| \frac{i \pi}{q} \right. \right)$$

(which is related to that in eq. (15.8) of Vol. VI for $\vartheta = \vartheta_3$) where

$$\vartheta_0(z|\tau) = \sum_{n=-\infty}^{+\infty} (-1)^n \exp [i\pi(n^2 \tau + 2nz)]$$

derive the equation

$$Z(q) = \frac{1}{q} e^{q/4} \cdot (\pi q)^{1/2} \oint \frac{t}{\sin t} e^{-t^2/q} dt$$

in which \oint denotes Cauchy's principal value of the integral. In the process of derivation it is necessary to make use of the expansion of $t/\sin t$ into partial fractions known from the theory of meromorphic functions.

c) Calculate the principal value of the integral in (b): α) by expanding $1/\sin t$ in powers of e^{it} , β) by expanding $t/\sin t$ in powers of t . How far can this be regarded as a proof of the transformation formula $\vartheta_2 \rightarrow \vartheta_0$ given in (b)?

d) The expansion (c, β) is a semi-convergent series which gives good approximate values of q . Calculate the molar energy and the molar specific heat of rotation from the first term of this expansion and compare with the result in Sec. 33. Determine the value of q above which c_v is accurate to within 1%.

e) Examine the influence of the higher-order terms on the molar specific heat and compare the two series at $q = 0.458$, $e^{2q} = 2.5$.

IV.7. Bose Einstein distribution.

Let $f_i, f_i', f_i'', \dots, f_i^{(n)}, \dots$ denote the probabilities of finding 0, 1, 2, \dots, n, \dots particles in the i -th phase cell, with the additional conditions:

$$\sum_n f_i^{(n)} = 1, \quad \sum_{n,i} n f_i^{(n)} = N, \quad \sum_{n,i} n f_i^{(n)} \varepsilon_i = U.$$

Calculate the distribution function $n_i = \sum_n f_i^{(n)}$ at equilibrium if the logarithm of the thermodynamic probability is given by

$$\log W = - \sum_i (f_i^0 \log f_i^0 + f_i' \log f_i' + f_i'' \log f_i'' + \dots)$$

i. e. as a sum of Boltzmann terms.

IV.8. *Density fluctuations.*

Calculate the density fluctuations in space in a volume element ΔV of a gas which occupies a vessel of volume V , stipulating that the mean density is constant throughout the volume.

IV.9. *Density fluctuations at the critical point.*

Consider a real gas composed of N molecules occupying a volume V .

a) Write down van der Waals' equation for this case and calculate the critical parameters.

b) Calculate the logarithm of the partition function by a thermodynamical argument, using the thermal and caloric ($c_v = 3/2 R T$) equations of state, and taking into account that a real gas behaves like a perfect gas at high temperatures T and low densities N/V .

c) Determine the relative mean fluctuation of the number of particles in volume ΔV making use of the equation

$$\left(\frac{\Delta n}{\bar{n}} \right)^2 = - \frac{(V/N) \Delta V}{N \partial^2 \log Z / \partial N^2}.$$

Perform the calculation at the critical point assuming small deviations from perfect-gas behavior.

d) Justify this formula with the aid of eq. (40.15) in the text.

CHAPTER V

V.1. Making use of the energy and momentum equations, deduce the transformation equation for the velocities $(\mathbf{v}_1, \mathbf{v}_2)$ before collision and $(\mathbf{v}_1', \mathbf{v}_2')$ after collision when the masses are m_1 and m_2 respectively. Show that Liouville's theorem about the equality of space cells is applicable to this transformation. What is the mean value of the ratio of the energy differences after and before collision?

When evaluating the mean, it is necessary to note that all directions of the central axis are equally probable and that the velocities $\mathbf{v}_1, \mathbf{v}_2$ before the impact are statistically independent.

Hints for the solution of problems

I.1. From $z = z(x, y)$ it follows that

$$(1) \quad dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

for arbitrary changes dx, dy . Assume now that dx and dy are connected in a way to render $dz = 0$. Their ratio must then be equal to $(\partial y / \partial x)_z$ and it follows from (1) that

$$\left(\frac{\partial y}{\partial x} \right)_z = - \left(\frac{\partial z}{\partial x} \right)_y / \left(\frac{\partial z}{\partial y} \right)_x,$$

from which the required proposition can easily be deduced.

Applying this result to the functional relation implied by the equation of state $p = p(T, V)$ we have

$$(2) \quad \left(\frac{\partial p}{\partial T} \right)_V \left(\frac{\partial T}{\partial V} \right)_p \left(\frac{\partial V}{\partial p} \right)_T = -1$$

or, when the definitions in (1.5) and (1.6) are taken into account

$$(3) \quad -p \beta \cdot \frac{1}{V \alpha} \cdot \frac{1}{V \kappa} = -1.$$

In other words $\beta p = \alpha \kappa V^2$.

I.2. The heating of the room from 0°C to 20°C requires the addition of $c_p \times 20$ deg units of heat. Now $(c_p - c_v)_{mol} = R$ and for diatomic gases we have $c_p/c_v = 1.4$. Hence

$$(1) \quad c_{p mol} = 3.5 R, \quad (2) \quad c_p = 3.5 \frac{R}{\mu} = 3.5 \times \frac{2}{29} \frac{\text{cal}}{\text{g deg}}.$$

The value for R was taken from (4.8) and the mean of the molar weights of N_2 and O_2 was used for μ . In order to reduce (2) to a unit of volume it is necessary to multiply by the mean density of the air

$$\rho = 1.25 \text{ kg/m}^3.$$

Multiplying by the temperature increment of 20°C we obtain from (2)

$$(3) \quad \rho \times c_p \times 20 \text{ deg} = \frac{1.25 \times 3.5 \times 2 \times 20}{29} \sim 60 \frac{\text{kcal}}{\text{m}^3}.$$

It will be noted that too many units of volume of air have been heated owing to its expansion, and that some air escapes because $p = \text{const}$. The intensity of

heating required to maintain a temperature of 20 C depends on the tightness of walls and windows and on their heat conductivity and cannot be included in our calculation.

According to Emden (*cf.* Sec. 6 F) the energy imparted to the room on heating is partly lost with the internal energy of the escaping air. Emden's eq. (6.19) can be criticized on the ground that it leaves out of account that the definition of energy leaves a constant undefined and must be replaced by

$$(4) \quad u - u_0 = c_v(T - T_0).$$

Here T_0 denotes some temperature above the point of liquefaction at which the perfect gas law is still valid. Since $u_1 = \rho u$ we have

$$(5) \quad u_1 = c_v \rho T + \rho(u_0 - c_v T_0).$$

The first term on the right-hand side is independent of temperature in view of the relation $\rho T = \mu p/R$, but not the second. The latter is positive because of the large value of the latent heat and considerably exceeds the first term in magnitude. It is seen from this argument that Emden was not justified in restricting himself to the first term in his eqs. (19), (20). It is possible to show that the second term *decreases* with T (because of the relation $\rho = \mu p/RT$). The energy density does not remain constant, as Emden suggests, but even decreases on heating. Thus the remarkable conclusion concerning the preponderance of entropy over energy applies *a fortiori*.

I.3. Assuming that the working fluid in a Carnot cycle is a perfect gas we can write per mol for the isothermal processes $1 \rightarrow 2$ and $3 \rightarrow 4$ (Q denotes heat added):

$$\begin{aligned} du &= 0, & dq &= p dv, & Q_1 &= \int_1^2 p dv = R T_1 \log \frac{v_2}{v_1}, \\ du &= 0, & dq &= p dv, & Q_2 &= \int_3^4 p dv = R T_2 \log \frac{v_3}{v_4} \end{aligned}$$

so that

$$(1) \quad \frac{Q_1}{Q_2} = \frac{T_1}{T_2} \frac{\log v_2/v_1}{\log v_3/v_4}.$$

On the other hand applying eq. (5.3 a) to the isentropic processes $2 \rightarrow 3$ and $4 \rightarrow 1$ we have

$$(2) \quad T_1 v_2^{\gamma-1} = T_2 v_3^{\gamma-1} \quad \text{and} \quad T_2 v_4^{\gamma-1} = T_1 v_1^{\gamma-1}.$$

Eliminating T_1 and T_2 we can prove that the ratios v_2/v_1 and v_3/v_4 are equal. Equation (1) transforms into (6.8). In view of (6.7) we see that in fact $\phi(\theta) = T_1$ as postulated in eq. (6.7 a).

1.4. The heat capacity C is defined as the quantity of heat required to increase the temperature of a body by 1 degree. Since the specific heat is referred to a unit of mass, we have $C = M c_v$ if the volume is constant, M denoting the mass of the body under consideration.

In all problems of heat conduction $dW = 0$ because the volume is constant so that $dU = dQ$. In such cases the material theory of heat may be used. This leads to the "mixing rule"

$$(1) \quad T = \alpha_1 T_1 + \alpha_2 T_2, \quad \alpha_1 = \frac{C_1}{C_1 + C_2}, \quad \alpha_2 = \frac{C_2}{C_1 + C_2}.$$

According to eq. (5.10) the change in the entropy of the perfect gases 1 and 2 at constant volume is given by

$$\Delta S_1 = C_1 \log \frac{T}{T_1}, \quad \Delta S_2 = C_2 \log \frac{T}{T_2}.$$

Since entropies are additive we may write:

$$\Delta S = \Delta S_1 + \Delta S_2 = (C_1 + C_2) \log T - C_1 \log T_1 - C_2 \log T_2.$$

The Second Law states that in an isolated system $\Delta S > 0$. Dividing by $C_1 + C_2$ and making use of (1) we find:

$$(2) \quad \alpha_1 T_1 + \alpha_2 T_2 > T_1^{\alpha_1} \times T_2^{\alpha_2} \quad \text{with} \quad \alpha_1 + \alpha_2 = 1.$$

When $\alpha_1 = \alpha_2 = \frac{1}{2}$ we obtain the well known rule that "arithmetical mean $>$ geometrical mean".

Equation (2) states: If, on forming an arithmetical mean, the two quantities T_1, T_2 are weighted with the factors α_1, α_2 then on forming the geometrical mean it is necessary to take the weighting factors into account exponentially. The sign $>$ must be replaced by $=$ only in the trivial case when $T_1 = T_2$.

For n gases which are free to exchange heat among each other eq. (2) is replaced by

$$(3) \quad \alpha_1 T_1 + \dots + \alpha_n T_n > T_1^{\alpha_1} \dots T_n^{\alpha_n} \quad \text{with} \quad \alpha_1 + \dots + \alpha_n = 1^1.$$

The student may try to apply the same method to other irreversible processes!

1.5. The initial state will be assumed to be given by T_0, v_0, p_0 . Evaluating the integral of $p dV$ between the limits V_0 and $2 V_0$ for

$$a) \quad p = p_0, \quad b) \quad p = p_0 \frac{v_0}{v}, \quad c) \quad p = p_0 \frac{v_0^\gamma}{v^\gamma}$$

¹Algebraic proofs of this proposition can be found in the following — Pólya and Szegő: *Aufgaben und Lehrsätze*, Springer 1925, or Hardy, Littlewood and Pólya: *Inequalities*, Cambridge Univ. Press, 1934.

we obtain, consecutively, the following expressions for work

$$\text{a) } R T_0, \quad \text{b) } R T_0 \log 2, \quad \text{c) } R T_0 \frac{1 - 2^{1-\gamma}}{\gamma - 1}.$$

The quantity of heat required can be found from the integral of $dq = c_v dT + p dv = c_p dT - v dp$. Hence, in the same order as before:

$$\text{a) } R T_0 \frac{c_p}{R}, \quad \text{b) } R T_0 \log 2, \quad \text{c) } 0.$$

The change in entropy in each case is found from $s = c_v \log T + R \log v + \text{const.}$, and is:

$$\text{a) } c_p \log 2, \quad \text{b) } R \log 2, \quad \text{c) } 0.$$

I.6. From (7.8) and (7.8 a) we deduce

$$ds = \frac{1}{T} (du + p dv) = \frac{c_v}{T} dT + \left(\frac{\partial p}{\partial T} \right)_v dv.$$

Expressing $(dp/dT)_v$ with the aid of eq. (2) in the solution of Prob. I.1. and introducing the coefficient of thermal expansion α and that of compressibility κ , we have

$$ds = \frac{c_v}{T} dT + \frac{\alpha}{\kappa} dv.$$

The slope of the isentrope in the T, v -diagram is found by putting $ds = 0$. Hence

$$\left(\frac{\partial v}{\partial T} \right)_s = - \frac{c_v \kappa}{\alpha T}.$$

It is seen that the slope of the isentrope is positive to the left of the minimum in the v vs. T curve because $\alpha < 0$ there. To the right $\alpha > 0$ and the slope is negative, whereas at the minimum itself the isentrope is parallel to the v -axis. Making a qualitative sketch of the isobars in a T, v -diagram it is easy to convince oneself that there are no isentropes which intersect both the 2 and 6 degree isotherms, provided that the process is carried out in a range of pressures where the isobars possess a minimum of volume between 2°C and 6°C.

I.7. At constant v we have $dq = c_v dT = c_v (\partial T / \partial p)_v dp$. At constant pressure p we have $dq = c_p dT = c_p (\partial T / \partial v)_p dv$. Hence in general

$$T ds = dq = c_p \left(\frac{\partial T}{\partial v} \right)_p dv + c_v \left(\frac{\partial T}{\partial p} \right)_v dp.$$

The isentropic compressibility is obtained by putting $ds = 0$ in (1). Hence from (1) we have

$$\left(\frac{\partial v}{\partial p}\right)_s = -\frac{c_v}{c_p} \left(\frac{\partial T}{\partial p}\right)_v \left(\frac{\partial v}{\partial T}\right)_p = +\frac{c_v}{c_p} \left(\frac{\partial v}{\partial p}\right)_T,$$

where eq. (2) from Problem I.1. has been utilized.

I.8. From (7.8) and (7.8 a) we have

$$dq = du + p dv = c_v dT + T \left(\frac{\partial p}{\partial T}\right)_v dv;$$

During isothermal compression $dT = 0$; further we may write $dv = (\partial v / \partial T)_T dp$. Then eq. (2) from Problem I.1. yields

$$dq = T \left(\frac{\partial p}{\partial T}\right)_v \left(\frac{\partial v}{\partial p}\right)_T dp = -T \left(\frac{\partial v}{\partial T}\right)_p dp.$$

Assuming a mean constant coefficient of expansion in the interval of pressure under consideration we find that the heat *added* is:

$$\begin{aligned} \int dq &= -T v \alpha \Delta p = -293 \text{ deg} \times 1 \text{ dm}^3 \times 2 \times (10^{-4}/\text{deg}) \times (19 \text{ kg/cm}^2) = \\ &= -1.113 \text{ liter} \times \text{atm} = -26.1 \text{ cal} \end{aligned}$$

(since $1 \text{ liter} \times \text{atm} = 1 \text{ dm}^3 \times \text{kp cm}^{-2} = 23.43 \text{ cal}$).

The work done on the system is

$$\begin{aligned} -\int p dV &= -\int p \left(\frac{\partial V}{\partial p}\right)_T = +V \alpha \Delta \left(\frac{p^2}{2}\right) \\ &= 1 \text{ dm}^3 \times 0.5 \times (10^{-4}/\text{at}) \times \frac{1}{2} (400 - 1) \text{ at}^2 \\ &= 10^{-2} \text{ liter} \times \text{atm} = 0.23 \text{ cal}. \end{aligned}$$

I.9. See Vol. II, Sec. 7.

I.10. Euler's equation, Vol. II, eq. (11.5) can be written

$$(1) \quad \text{grad} \frac{v^2}{2} = -\frac{1}{\rho} \text{grad } p$$

if we neglect external forces, assume $\partial \mathbf{v} / \partial t = 0$ and $\text{curl } \mathbf{v} = 0$, i. e. steady and irrotational flow, and if we take account of eq. (6) *loc. cit.* The condition of incompressibility, *loc. cit.* (4 a), must now be replaced by the relation between p and ρ for isentropic processes

$$(2) \quad p = p_0(\rho/\rho_0)^\gamma.$$

Eliminating ρ from (1) with the aid of (2) we can integrate (1) to give:

$$(3) \quad \frac{v^2}{2} - \frac{v_0^2}{2} = \frac{p_0}{\rho_0} \frac{\gamma}{\gamma-1} \left[1 - \left(\frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \right] = c_p T_0 \left[1 - \left(\frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \right].$$

According to (5.3 a) we have $(p/p_0)^{(\gamma-1)/\gamma} = T/T_0$. Hence

$$\frac{v^2}{2} - \frac{v_0^2}{2} = c_p T_0 - c_p T.$$

This shows that, in fact, the difference between the kinetic energies per gram is equal to the difference in the specific enthalpies per gram, $c_p T_0$ and $c_p T$, of the gas.

In our example $c_p = 0.49 \text{ cal g}^{-1} \text{ deg}^{-1}$, $\gamma = 1.33$. Hence (3) yields $v = 880 \text{ m/sec}$ if we assume $v_0 \approx 0$. This value may be compared with the mean molecular velocity of 770 m/sec for steam at 300°C (cf. Sec. 22). — The temperature after expansion is 110°C .

I.11. At first we shall assume an arbitrary equation of state. The state of the gas in cell i of volume V_i will be determined by indicating the temperature T_i and the specific volume v_i . The mass enclosed in a cell is then equal to V_i/v_i . The specific internal energy in cell i will be denoted by $u_i(T_i, v_i)$ so that the internal energy of the gas contained by cell i is $\frac{V_i}{v_i} u_i(T_i, v_i)$, its potential energy being $\frac{V_i}{v_i} g z_i$, where g denotes the gravitational acceleration. Let U denote the total energy, N the total mass in the system. Then

$$N = \sum_i \frac{V_i}{v_i}, \quad U = \sum_i \frac{V_i}{v_i} [u_i(T_i, v_i) + g z_i].$$

In a similar way we can calculate the total entropy

$$S = \sum_i \frac{V_i}{v_i} s_i(T_i, v_i),$$

where $s_i(T_i, v_i)$ is the specific entropy for the state prevailing in cell i .

Now, at equilibrium S is a maximum with respect to all variations of the T_i, v_i which are compatible with the constant values of N and U . Introducing two Lagrangian multipliers λ and μ we must also have

$$\delta S \equiv \delta \left[\sum_i \frac{V_i}{v_i} s_i(T_i, v_i) + \lambda \left(N - \sum_i \frac{V_i}{v_i} \right) + \mu \left(U - \sum_i \frac{V_i}{v_i} [u_i(T_i, v_i) + g z_i] \right) \right] = 0$$

for arbitrary variations of the quantities T_i, v_i, λ, μ . This yields

$$(1) \quad \frac{V_i}{v_i} \left(\frac{\partial s_i}{\partial T_i} - \mu \frac{\partial v_i}{\partial T_i} \right) = 0,$$

$$(2) \quad \frac{V_i}{v_i^2} \left(-s_i + v_i \frac{\partial s_i}{\partial v_i} \right) + \lambda \frac{V_i}{v_i^2} + \mu \frac{V_i}{v_i^2} \left(u_i + g z_i - v_i \frac{\partial u_i}{\partial v_i} \right) = 0.$$

It follows from (1) that $\mu = 1/T_i$ if we take into account that $\frac{\partial s_i}{\partial T_i} = \frac{1}{T_i} \left(\frac{\partial u_i}{\partial T_i} \right)_{v_i}$, see (7.1). In other words all T_i 's are equal, and we may put $T_i = T$. From the two thermodynamic relations (7.7) and (7.8) and from (2) we find that

$$\left(\frac{\partial s_i}{\partial v_i} \right)_{T_i} = \left(\frac{\partial p_i}{\partial T_i} \right)_{v_i}, \quad \left(\frac{\partial u_i}{\partial v_i} \right)_{T_i} = T_i \left(\frac{\partial p_i}{\partial T_i} \right)_{v_i} - p_i,$$

and after a short calculation

$$g(T, v_i) \equiv u_i + g z_i - T s_i + v_i p_i = -\lambda T.$$

The left-hand side contains the specific Gibbs potential for the state prevailing in cell i , the potential energy having been added to the internal energy. The right-hand side is independent of i , which must also be true of the left-hand side. In the case of a perfect gas (for the sake of simplicity we shall assume that the specific heat is independent of temperature)

$$u_i = c_v T + u_0, \quad s_i = c_v \log T + \frac{R}{\mu} \log v_i + s_0, \quad p_i = \frac{R}{\mu} \frac{T}{v_i},$$

where now μ denotes molecular weight.

Since $g(T, v_i)$ is independent of i and so of z , we have

$$v(z) = v_0 e^{\mu g z / RT}, \quad p(z) = p_0 e^{-\mu g z / RT}.$$

For a height Δz defined by

$$\mu g \Delta z = R T$$

we have finally

$$\Delta z = \frac{R T}{\mu g} = \frac{p_0 v_0}{g} = \frac{p_0}{\rho_0 g}.$$

It is seen to be equal to the mean height

$$\Delta z = \frac{\int_0^\infty z p(z) dz}{\int_0^\infty p(z) dz}.$$

With $p_0 = 10^6$ dyne cm^{-2} , $\rho_0 = 0.0012939$ g cm^{-3} , $g = 981$ cm sec^{-2} Δz has the numerical value of

$$\Delta z = 8 \times 10^5 \text{ cm} = 8 \text{ km}.$$

When a gas expands in a vertical direction at a height which is considerably below 8 km the decrease in pressure with altitude is small; for a difference in altitude of 80 m it is only 1% but it must be taken into account when barometric readings published by meteorological stations are reduced to sea level.

I.12.

1. Substituting p from the equation of state into the thermodynamic relation $(\partial u / \partial v)_T = -p + T(\partial p / \partial T)_v$, we obtain the following partial differential equation for u :

$$\frac{\partial u}{\partial v} = -\frac{\alpha}{v} u + \frac{\alpha T}{v} \frac{\partial u}{\partial T}$$

whose general solution is:

$$u = v^{-\alpha} \phi(T v^{\alpha});$$

there $\phi(T v^{\alpha})$ is an arbitrary function, as is easy to verify by substitution. The statement about entropy is obtained without difficulty from

$$T ds = du + p dv.$$

2. If u/v depends only on T the expression $v^{-\alpha-1} \phi(T v^{\alpha})$ must be independent of v which means that $\phi(T v^{\alpha}) = \sigma \cdot (T v^{\alpha})^{(\alpha+1)/\alpha}$ where σ is a constant; this proves the assertion. For black body radiation we have $p = \frac{1}{3} u/v$, i. e. $\alpha = \frac{1}{3}$, whence we deduce the Stefan-Boltzmann law $u/v = \sigma T^4$. For $\alpha = 2/3$ we have $u/v = \sigma T^{5/2}$ which is a known result from Einstein's condensation theory, cf. (38.27).

3. a) We have $p = \alpha C T^m v^{\alpha m - \alpha - 1}$; $(\partial p / \partial v)_T \leq 0$ means

$$\alpha(m-1) - 1 \leq 0.$$

b) We have

$$v = \left(\frac{p}{\alpha C} \right)^{\frac{1}{\alpha(m-1)-1}} \times T^{\frac{m}{-\alpha(m-1)+1}}.$$

If we are to have $v > 0$ for $T \rightarrow 0$ and any p , we must also have

$$\alpha(m-1) - 1 \geq 0.$$

c) The two inequalities in a) and b) are then, and only then, satisfied simultaneously if $\alpha(m-1) - 1 = 0$, i. e. if $m = 1 + 1/\alpha$ so that

$$u = C T^{\frac{\alpha+1}{\alpha}} \times v; \quad p = \alpha C \times T^{\frac{\alpha+1}{\alpha}}.$$

From $u = C T^m v^{\alpha(m-1)} > 0$ and $< \infty$ for $T = 0$ we conclude that $m = 0$ so that $u = v^{-\alpha} \times \text{const}$ and $p v^{\alpha+1} = \text{const}$. With $\alpha = 2/3$ we are led to the relation between the pressure and the volume of electron gas in (39.10)

II.1. It follows from (14.11 b) that

$$(1) \quad d(pV) = p dV + V dp = S dT + p dV + \sum_i n_i d\mu_i.$$

From (1) we deduce the relations

$$\left(\frac{\partial S}{\partial V}\right)_{T, \mu_i} = \left(\frac{\partial p}{\partial T}\right)_{V, \mu_i}; \quad \left(\frac{\partial S}{\partial \mu_i}\right)_{T, V, \mu_j} = \left(\frac{\partial n_i}{\partial T}\right)_{V, \mu_j}; \quad \left(\frac{\partial n_i}{\partial \mu_j}\right)_{T, V, \mu_k} = \left(\frac{\partial n_j}{\partial \mu_i}\right)_{T, V, \mu_k}.$$

II.2. From the definition $r = \Delta h$ we find

$$dr = \Delta \left\{ \left(\frac{\partial h}{\partial T}\right)_p dT + \left(\frac{\partial h}{\partial p}\right)_T dp \right\},$$

by taking the differential along the vapor-pressure curve $\phi(p, T) = 0$. Since dp and dT both derive from $\phi = 0$, we obtain

$$(1) \quad \left(\frac{\partial r}{\partial T}\right)_\phi = \Delta c_p + \frac{r}{T \Delta v} \Delta \left(\frac{\partial h}{\partial p}\right)_T,$$

by making use of (4.11) together with the Clapeyron equation. According to the Table in Sec. 7 we have

$$dh = T ds + v dp \quad \text{and hence} \quad \left(\frac{\partial h}{\partial p}\right)_T = T \left(\frac{\partial s}{\partial p}\right)_T + v.$$

From the same Table we take the relation

$$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p; \quad \text{hence} \quad \left(\frac{\partial h}{\partial p}\right)_T = v - T \left(\frac{\partial v}{\partial T}\right)_p = v - T v \alpha,$$

where α denotes the coefficient of thermal expansion. Consequently,

$$(2) \quad \Delta \left(\frac{\partial h}{\partial p}\right)_T = \Delta v - T \Delta(v \alpha).$$

Substitution into (1) yields

$$(3) \quad \left(\frac{dr}{dT}\right)_\phi = \Delta c_p + \frac{r}{T} - r \frac{\Delta(v \alpha)}{\Delta v}.$$

It will be noted that the preceding derivation introduces no approximations as regards the behavior of the liquid or vapor.

II.3. We shall denote the mass of the steam by x , and that of the water by $1 \text{ kg} - x$. The volume of the steam becomes $x \times v_2$ and that of the water is $(1 \text{ kg} - x) v_1$. Thus x is determined by the equation

$$x v_2 + (1 \text{ kg} - x) v_1 = 20 \text{ dm}^3$$

and hence

$$x = (20 \text{ dm}^3 - v_1 \times 1 \text{ kg}) / (v_2 - v_1)$$

so that $x = 0.169 \text{ g}$ at 10°C , $x = 149.7 \text{ g}$ at 200°C .

The quantities of heat added during the three subsidiary processes are:

$$\text{a) Condensing from } 0.169 \text{ g at } 10^\circ\text{C}: \quad -0.169 \text{ g} \times 591.6 \frac{\text{cal}}{\text{g}} = -100 \text{ cal.}$$

$$\text{b) Heating from } 10^\circ\text{C to } 200^\circ\text{C}: \quad 1 \text{ kg} \times 190^\circ\text{C} \times 1 \text{ cal g}^{-1} \text{ deg}^{-1} = 190,000 \text{ cal.}$$

$$\text{c) Evaporation of } 149.7 \text{ g at } 200^\circ\text{C}: \quad 149.7 \text{ g} \times 463.5 \frac{\text{cal}}{\text{g}} = 69,400 \text{ cal.}$$

$$259,300 \text{ cal.}$$

The work of compression or expansion is: a) $+ 19 \text{ dm}^3 \times 0.0125 \text{ kp cm}^{-2} = 0.238 \text{ dm}^3 \text{ kp cm}^{-2}$, c) $- 18.84 \text{ dm}^3 \times 15.86 \text{ kp cm}^{-2} = -298.8 \text{ dm}^3 \text{ kp cm}^{-2}$; in all $-298.6 \text{ dm}^3 \text{ kp cm}^{-2} = -7000 \text{ cal}$. The total energy to be added is thus equal to 252.3 kcal .

II.4. We have

$$\frac{dT}{T} = \frac{v_{\text{sat}} - v_{\text{liq}}}{r} dp.$$

Integrate first between 0 and 100°C . Let T_0 denote the absolute temperature of the ice point, the absolute temperature of the boiling point of water at 760 torr being defined as $T_0 = 100^\circ\text{C}$. Let, further, p_0 and p_{100} denote the vapor pressures of the fluid under consideration at 0°C and 100°C respectively. We then find:

$$(1) \quad \log \frac{T_0 + 100^\circ\text{C}}{T_0} = \int_{p_0}^{p_{100}} \frac{v_{\text{vap}} - v_{\text{liq}}}{r} dp.$$

The integral on the right-hand side can be evaluated numerically on the assumptions given which leads to an equation for T_0 . The absolute temperature T which corresponds to any other vapor pressure is given by

$$(2) \quad \log \frac{T}{T_0} = \int_{p_0}^p \frac{v_{\text{vap}} - v_{\text{liq}}}{r} dp.$$

According to the general propositions of thermodynamics, T_0 is independent of the liquid selected. In the same way T from (2) is the same for different liquids provided that they are in thermal equilibrium.

II.5. The variation in the vapor pressure in each of the intervals of 10°C is too large to permit the substitution of $\Delta p / \Delta T$ for dp / dT . Furthermore, according

to the terms of the problem it is necessary to integrate the Clausius-Clapeyron equation

$$(1) \quad \frac{dp}{dT} = \frac{r}{T(v_{vap} - v_{liq})}$$

exactly, assuming that r is constant in the interval of 10 C, and that the vapor behaves like a perfect gas. The quantities r , v_{vap} , and v_{liq} will be referred to one mol. Hence $v_{vap} = R T/p$, and v_{liq} may be neglected with respect to v_{vap} . Thus we obtain from (1) that $d \log p = -r/R \cdot d(1/T)$ and

$$\log \frac{p_2}{p_1} = \frac{r}{R} \left(\frac{1}{T_1} - \frac{1}{T_2} \right)$$

or

$$\frac{r}{R} = \frac{T_1 T_2}{T_2 - T_1} \left(\log \frac{p_2}{p_1} \right)^{-1}.$$

Substituting the numerical values for the two intervals, we have

$$r = 7413 R \quad \text{and} \quad r = 7045 R.$$

The assumption that the latent heat is constant does not apply over large temperature intervals; the above numerical values must be regarded as average values over each of the intervals; they will be assumed to refer to 55 C and 305 C or to 328 K and 578 K respectively. Linear interpolation now yields:

$$(2) \quad r = (7896 \text{ C} - 1.472 T) R.$$

In order to determine the boiling point at 760 torr we can integrate (1) more accurately by substituting (2). Thus

$$d \log p = + (7896 \text{ C} - 1.472 T) \frac{dT}{T^2}$$

and

$$(3) \quad \log \frac{p}{p_2} = 7896 \text{ C} \left(\frac{1}{T_2} - \frac{1}{T} \right) - 1.472 \log \frac{T}{T_2}.$$

Here p_2 denotes the vapor pressure of 305 torr at 310 C = 583 K. Substituting the value of 760 torr for p we can determine the boiling point T at that pressure from (3). We obtain

$$\frac{1000 \text{ C}}{T} = 1.600 - 0.1864 \log \frac{T}{583 \text{ C}}.$$

This transcendental equation is best solved by the method of successive approximations by assuming an approximate value for the unknown temperature on the right-hand side and by calculating T on the left-hand side. The new value may again be inserted on the right-hand side and the process may be repeated.

The first approximation may be chosen at, say, 630 K leading to $T = 630.7$ K. Repeated substitution yields the same value of T again, including the first decimal. We conclude, therefore, that the boiling point is at 630.7 K or 357.7 C.

II.6. We refer to eq. (13.13) and proceed by forming the variation

$$(1) \quad \delta G = \sum_i \delta n_i \left\{ g_i(T, p) - R T \log \frac{n}{n_i} \right\}.$$

Since transitions $0 \longleftrightarrow 1$ and $0 \longleftrightarrow 2$ are possible we can vary n_1 and n_2 independently, and we must have $\delta n_0 = -\delta n_1 - \delta n_2$. The problem considered in Sec. 13 involved only one chemical reaction and so only one of the n_i 's could be varied arbitrarily; the remaining δn_i 's were determined.

It follows from (1) that

$$(2) \quad g_1 - g_0 = R T \log \frac{n_0}{n_1}; \quad g_2 - g_0 = R T \log \frac{n_0}{n_2}.$$

Since $g_i(T, p) = u_i(T, p) - T s_i(T, p) + p v_i$, where $v_0 = v_1 = v_2$, and the entropy constants are equal, $s_0 = s_1 = s_2$, we find $g_1 - g_0 = u_1 - u_0 = L(\epsilon_1 - \epsilon_0)$; $g_2 - g_0 = u_2 - u_0 = L(\epsilon_2 - \epsilon_0)$, where L denotes the number of molecules in the mass to which g_i has been referred, i. e. per mol in our case. Hence

$$(3) \quad \frac{n_1}{n_0} = \exp \left\{ -\frac{L(\epsilon_1 - \epsilon_0)}{R T} \right\}; \quad \frac{n_2}{n_0} = \exp \left\{ -\frac{L(\epsilon_2 - \epsilon_0)}{R T} \right\}.$$

This is the Maxwell-Boltzmann law, and we have deduced it here from thermodynamical considerations. We shall deduce it once more with the aid of the methods of statistical mechanics in Sec. 29. The inclusion of transitions $1 \longleftrightarrow 2$ makes no change in the argument. The generalization to any number of excited states is immediate.

II.7. a) The n_i 's will now be assumed to refer to our arbitrary composition. The equilibrium values from the preceding problem will be denoted by \bar{n}_i . Thus

$$\mu_1 - \mu_0 = g_1 - g_0 - R T \log \frac{n_0}{n_1} = R T \log \frac{\bar{n}_0}{\bar{n}_1} - R T \log \frac{n_0}{n_1} = R T \left(\log \frac{n_1}{\bar{n}_1} - \log \frac{n_0}{\bar{n}_0} \right)$$

or, for $|n_1 - \bar{n}_1| \ll \bar{n}_1$, i. e. in the neighborhood of equilibrium

$$(1) \quad \mu_1 - \mu_0 = R T \left(\frac{n_1 - \bar{n}_1}{\bar{n}_1} - \frac{n_0 - \bar{n}_0}{\bar{n}_0} \right); \quad \mu_2 - \mu_0 = R T \left(\frac{n_2 - \bar{n}_2}{\bar{n}_2} - \frac{n_0 - \bar{n}_0}{\bar{n}_0} \right).$$

b) The variation in the n_i 's with time is given by the differential equations

$$(2) \quad \left\{ \begin{array}{l} \frac{dn_0}{dt} = -k_{00} n_0 + k_{01} n_1 + k_{02} n_2 \\ \frac{dn_1}{dt} = k_{10} n_0 - k_{11} n_1 + k_{12} n_2 \\ \frac{dn_2}{dt} = k_{20} n_0 + k_{21} n_1 - k_{22} n_2 \end{array} \right.$$

The sum of all n_i 's is constant irrespective of their individual values. Hence

$$(3) \quad k_{00} = k_{10} + k_{20}, \quad k_{11} = k_{01} + k_{21}, \quad k_{22} = k_{02} + k_{12}.$$

The first of these two relations states that the mass lost by state 0 per second, namely $k_{00} n_0$ will be found as $k_{10} n_0$ and $k_{20} n_0$ at states 1 and 2. The same can be said of the remaining two relations.

c) Equilibrium means that $dn_i/dt = 0$, or that

$$(4) \quad \left\{ \begin{array}{l} k_{00} \bar{n}_0 = k_{01} \bar{n}_1 + k_{02} \bar{n}_2 \\ k_{11} \bar{n}_1 = k_{10} \bar{n}_0 + k_{12} \bar{n}_2 \\ k_{22} \bar{n}_2 = k_{20} \bar{n}_0 + k_{21} \bar{n}_1. \end{array} \right.$$

The state of equilibrium is thus seen not to be one in which no more transitions occur. Transitions which cause the mass of one component to increase turn out to be as frequent as those causing an opposite effect. Of the three equations in (4) at most two are mutually independent because of the constraint in (3). They determine the ratios \bar{n}_1/\bar{n}_0 and \bar{n}_2/\bar{n}_0 in terms of the k_{ik} 's which are, in turn, independent of composition as assumed at the outset. Recalling eq. (3) of the preceding solution it is seen that the coefficients k_{ik} satisfy two more relations in addition to the relations (3) above. Having proved that the ratios of the \bar{n}_i are independent of composition we have shown the plausibility of our assumption (2), even if we have not actually proved it. The same conclusion would remain true even if the right-hand sides of (2) were replaced by arbitrary functions, each being a linear combination of the n_i , provided that the functions vanish for the argument 0.

d) It follows from (1) and from $(n - \bar{n}_0) + (n_1 - \bar{n}_1) + (n_2 - \bar{n}_2) = 0$ (preservation of total mass) that

$$\begin{aligned} R T \frac{n_0 - \bar{n}_0}{\bar{n}_0} &= -\frac{\bar{n}_1}{n} (\mu_1 - \mu_0) - \frac{\bar{n}_2}{n} (\mu_2 - \mu_0), \\ R T \frac{n_1 - \bar{n}_1}{\bar{n}_1} &= \left(1 - \frac{\bar{n}_1}{n}\right) (\mu_1 - \mu_0) - \frac{\bar{n}_2}{n} (\mu_2 - \mu_0), \\ R T \frac{n_2 - \bar{n}_2}{\bar{n}_2} &= -\frac{\bar{n}_1}{n} (\mu_1 - \mu_0) - \left(1 - \frac{\bar{n}_2}{n}\right) (\mu_2 - \mu_0); \end{aligned}$$

because $n = n_0 + n_1 + n_2 = \bar{n}_0 + \bar{n}_1 + \bar{n}_2$. We substitute this in the right-hand side of (2) having replaced all n_i 's by $n_i - \bar{n}_i$, this being permissible in view of (4). We then obtain:

$$R T \frac{dn_1}{dt} = -k_{11} \bar{n}_1 (\mu_1 - \mu_0) + k_{12} \bar{n}_2 (\mu_2 - \mu_0)$$

$$R T \frac{dn_2}{dt} = k_{21} \bar{n}_1 (\mu_1 - \mu_0) - k_{22} \bar{n}_2 (\mu_2 - \mu_0),$$

taking (4) into account once more. We now compare this expression with (21.26). Onsager's reciprocal relation $a_{12} = a_{21}$ is equivalent to:

$$(5) \quad k_{12} \bar{n}_2 = k_{21} \bar{n}_1.$$

Making use of (3) and (4) we deduce the further relations

$$(6) \quad k_{01} \bar{n}_1 = k_{10} \bar{n}_0, \quad k_{02} \bar{n}_2 = k_{20} \bar{n}_0.$$

Now, $k_{12} \bar{n}_2$ denotes the mass transferred from component 2 to component 1 per second, $k_{21} \bar{n}_1$ denotes the mass transferred per second from component 1 to component 2. Onsager's reciprocal relation leads to the conclusion that these two expressions are equal at equilibrium. In other words the reaction $1 \rightleftharpoons 2$ is in equilibrium on its own, independently of the fact that the reactions $0 \rightleftharpoons 1$ and

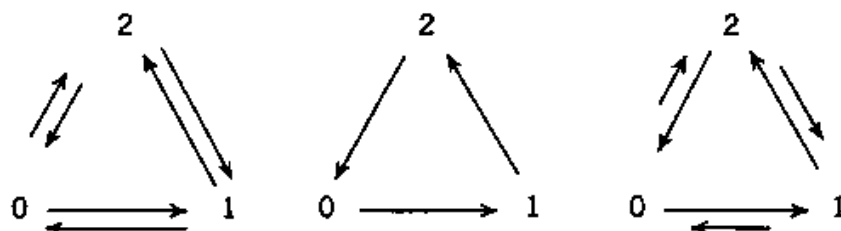


Fig. 40. Illustrating detailed, cyclic, and mixed equilibrium.

$0 \rightleftharpoons 2$ take place simultaneously. In the same way (6) shows that the remaining reactions are in equilibrium on their own. The preceding proposition enjoys a much wider area of validity and is known in chemistry as the principle of detailed equilibrium.

In addition to detailed equilibrium we could also imagine cyclic or mixed equilibrium. The three cases have been shown schematically in Fig. 38. The length of the arrows is assumed to be proportional to the frequency of the individual transitions. The case of detailed equilibrium is the only one of the three which is unaffected when one transition (say 1, 2) is impeded in *both* directions, for example by the introduction of a decatalyzer. In the case of cyclic equilibrium the suppression of, say, the reaction $1 \rightarrow 2$ would, at first, leave the reactions $2 \rightarrow 0$ and $0 \rightarrow 1$ unaffected so that the masses of components 0 and 1 would increase compared with equilibrium. The same is true for the case of mixed equilibrium. According to our laws of thermodynamics a state of equilibrium cannot be affected by the introduction of a catalyzer or decatalyzer, once it has

set in, and provided that no constraints are removed. Hence the principle of detailed equilibrium can also be stated as follows: There can be no decatalyzer which would in different ways affect a reaction and its opposite, such as would, for example, suppress one and not the other.

The ideas expressed here in the form of the principle of detailed equilibrium will be seen to recur in the kinetic theory of gases, or in the electron theory of metals. In the last analysis it is always a consequence of the equality in the contributions of both quantum mechanical matrix elements for the transition between two steady states.

III.1. a) Assume that the velocity of the piston before impact is v . On impact its velocity will change sign and

$$(1) \quad M v = m c.$$

The time of rise t_s for the piston is equal to $t_s = v/g$, the return time is $t_r = 2v/g$ and the latter must be equal to that for the sphere, $t_r' = 2l/c$. Hence

$$(2) \quad v c = l g.$$

The mean force on the piston is

$$(3) \quad K = \frac{2 m c}{t_r} = m g \frac{c}{v} = M g,$$

as seen from (1). Equations (1) and (2) (A = cross-sectional area) also show that

$$m c^2 = M v c = M g l = \frac{M g}{A} \cdot A l,$$

or, introducing the pressure $p = M g/A$ and the volume $V = A l$:

$$(4) \quad p \cdot V = m c^2 = 2 U.$$

instead of the value of $2/3 U$ for a three-dimensional gas.

b) Equation (1), elevation and return time of the piston, and eq. (3) remain unaltered. The length l in (2) should be replaced by $l - 2r$. It follows from the modified eq. (2) that

$$m c^2 = M g(l - 2r)$$

or

$$(5) \quad p(V - 2rA) = 2 U.$$

It is seen that the volume V is decreased by a constant term in the same way as in the van der Waals equation. In the present case the term to be subtracted is equal to a layer of the thickness of the sphere covering the whole cross-section.

c) Imagining that the piston is very heavy (so that the change of its velocity may be ignored), we may apply the law of reflection. If c is the velocity before impact and c' that after it, we have

$$c - V_p = c' + V_p.$$

The change in kinetic energy is

$$\frac{1}{2} m(c'^2 - c^2) \approx -2 m c V_p.$$

During a time interval, Δt , this quantity of energy is transferred $\frac{\Delta t}{t_r} = \frac{c \Delta l}{2 l}$ times; substituting the distance $\Delta x = V_p \Delta t$ traversed by the piston, we have:

$$(6) \quad -\Delta W = -2 m c V_p \cdot \frac{c \Delta t}{2 l} = -\frac{m c^2}{l} \cdot \Delta x = -M g \Delta x = -p \Delta V.$$

III.2. a) Taking the logarithmic derivative of

$$\phi(c) = 4 \pi c^2 \left(\frac{m}{2 \pi k T} \right)^{3/2} e^{-\frac{m c^2}{2 k T}}$$

with respect to c , we have

$$\frac{\phi'}{\phi} = \frac{2}{c} - \frac{2 m c}{2 k T};$$

putting $\phi' = 0$ we find that

$$(1) \quad c_w = \left(\frac{2 k T}{m} \right)^{1/2}$$

which agrees with (23.10).

b) Recalling that $\bar{c} = \int_0^\infty c \phi dc$ and putting $m c^2 / 2 k T = \xi^2$, we have

$$\bar{c} = 4 \left(\frac{2 k T}{\pi m} \right)^{1/2} \int_0^\infty e^{-\xi^2} \xi^3 d\xi.$$

The integral can be reduced to one of Laplace's type and its numerical value is $\frac{1}{2} \times 1!$. Hence

$$(2) \quad \bar{c} = \left(\frac{8 k T}{\pi m} \right)^{1/2}.$$

c) Recalling that $\overline{c^2} = \int_0^\infty c^2 \phi(c) dc$ we have:

$$\overline{c^2} = 4 \cdot \frac{2 k T}{m} \cdot \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\xi^2} \xi^4 d\xi.$$

We compute

$$\int_0^\infty e^{-\gamma \xi^2} \xi^4 d\xi = \frac{\sqrt{\pi}}{2} \frac{d^3}{d\gamma^3} \frac{1}{\sqrt{\gamma}} = \frac{3}{8} \sqrt{\pi} \gamma^{-5/2}.$$

Consequently

$$(3) \quad \overline{c^2} = \frac{3 k T}{m}$$

The velocity squares form the proportion

$$(4) \quad \overline{c^2} : \overline{c_x^2} : c_w^2 = 3 : \frac{8}{\pi} : 2$$

which leads to eq. (23.11).

III.3. The number of molecules per second coming from the cylinder in Fig. 23 on p. 170 is equal to

$$dZ = \sigma c_x \cdot n \left(\frac{m}{2 \pi k T} \right)^{3/2} e^{-\frac{m}{2 k T} (c_x^2 + c_y^2 + c_z^2)} dc_x dc_y dc_z.$$

The total number is obtained by integrating over c_x, c_y from $-\infty$ to $+\infty$ and over c_z from $c_0 = 12000$ m/sec to ∞ . Introducing cylindrical polar co-ordinates and putting

$$\left(\frac{m}{2 k T} \right)^{1/2} \cdot (c_x, c_y, c_z) = (\rho \cos \phi, \rho \sin \phi, \zeta),$$

we have

$$Z = n \sigma \left(\frac{2 k T}{m} \right)^{1/2} \frac{1}{\pi^{3/2}} \int_0^\infty e^{-\rho^2} \rho d\rho d\phi \int_{\left(\frac{m}{2 k T}\right)^{1/2} c_0}^\infty e^{-\zeta^2} \zeta d\zeta = n \sigma \left(\frac{k T}{2 \pi m} \right)^{1/2} \times \exp \left(-\frac{m c_0^2}{2 k T} \right).$$

Introducing the mean velocity

$$\overline{c} = 4 \left(\frac{k T}{2 \pi m} \right)^{1/2}$$

we find

$$Z = \frac{n \sigma \bar{c}}{4} \exp \left(- \frac{m c_0^2}{2 k T} \right).$$

The numerical result is: $\bar{c} = 1690$ m/sec, $1/4 n \sigma \bar{c} = 8.45 \times 10^{23}$; $m c_0^2 / 2 k T = 64$. Hence $Z = 5 \times 10^{-3} \text{ sec}^{-1}$. A particle possessing such a very high velocity arrives about once every 3 minutes.

III.4. See Sec. 27 D, eq. (14). The probability that the 6 will show up for the first time after the

1. 2. 3. ... k 'th... throw

is, respectively

$$W_k = \frac{1}{6}, \frac{5}{6} \times \frac{1}{6}, \left(\frac{5}{6}\right)^2 \times \frac{1}{6}, \dots, \left(\frac{5}{6}\right)^{k-1} \times \frac{1}{6}, \dots$$

The calculations of the mean values of k, k^2, \dots is best performed with the aid of the generating function

$$f(t) = \sum_{k=1}^{\infty} W_k t^k = \frac{(1-p)t}{1-pt}.$$

Here $p = 5/6$ and it follows at once that

$$f(1) = \sum_{k=1}^{\infty} W_k = 1$$

and the required mean values are given by the derivatives

$$f'(1) = \sum k W_k = \bar{k},$$

$$f''(1) = \sum k \cdot (k-1) W_k = \bar{k}^2 - \bar{k}.$$

It is convenient to represent them with the aid of the following logarithmic derivatives

$$\bar{k} = \frac{f'(1)}{f(1)},$$

$$\bar{k}^2 - \bar{k} = \left(\frac{f'(t)}{f(t)} \right)'_{t=1} + \frac{f'(1)}{f(1)}.$$

Hence

$$\bar{k} = \left(\frac{1}{t} + \frac{p}{1-pt} \right)_{t=1} = \frac{1}{1-p} = 6$$

and

$$(\Delta k)^2 = (\overline{k} - \overline{k})^2 = \left(-\frac{1}{t^2} + \frac{p^2}{(1-p)^2} \right)_{t=1} + \frac{1}{1-p} = \frac{p}{(1-p)^2} = 30.$$

Thus the result is:

$$\overline{k} \pm \Delta k = 6 \pm \sqrt{30} = 6 \pm 5.5.$$

III.5. The energy equation can be written

$$(1) \quad \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = E + A e^{-\alpha x} - B e^{-2\alpha x}.$$

The velocity components \dot{y} and \dot{z} are constant. Putting $\frac{1}{2} m (\dot{y}^2 + \dot{z}^2) = E_{tg}$ (tangential energy) and $E - E_{tg} = E_n$, we find that E_{tg} is constant, so that

$$(2) \quad \frac{1}{2} m \dot{x}^2 = E_n + A e^{-\alpha x} - B e^{-2\alpha x}.$$

This equation can be integrated to give

$$t - t_0 = \left(\frac{m}{2} \right)^{1/2} \int (E_n + A e^{-\alpha x} - B e^{-2\alpha x})^{-1/2} dx.$$

Substituting $e^{\alpha x} = \xi$ we have

$$t - t_0 = \frac{1}{\alpha} \left(\frac{m}{2} \right)^{1/2} \int (E_n \xi^2 + A \xi - B)^{-1/2} d\xi.$$

The substitutions $\xi = \zeta - A/2E_n$, $\zeta_0^2 = \frac{B}{E_n} + \frac{A^2}{4E_n^2}$ lead to:

$$t - t_0 = \frac{1}{\alpha} \left(\frac{m}{2E_n} \right)^{1/2} \int (\zeta^2 - \zeta_0^2)^{-1/2} d\zeta.$$

Substituting $\zeta = \zeta_0 \cosh \lambda$, we have

$$\alpha \left(\frac{2E_n}{m} \right)^{1/2} (t - t_0) = \lambda = \cosh^{-1} \frac{\zeta}{\zeta_0}$$

or

$$\zeta = \zeta_0 \cosh \alpha \left(\frac{2E_n}{m} \right)^{1/2} (t - t_0),$$

i. e.

$$\xi = e^{\alpha x} = \zeta_0 \cosh \alpha \left(\frac{2E_n}{m} \right)^{1/2} (t - t_0) - \frac{A}{2E_n}$$

or

$$x = \frac{1}{\alpha} \log \left\{ \zeta_0 \cosh \alpha \left(\frac{2 E_n}{m} \right)^{1/2} (t - t_0) - \frac{A}{2 E_n} \right\}$$

and

$$(3) \quad \dot{x} = \left(\frac{2 E_n}{m} \right)^{1/2} \cdot \frac{\sinh \alpha \left(\frac{2 E_n}{m} \right)^{1/2} (t - t_0)}{\cosh \alpha \left(\frac{2 E_n}{m} \right)^{1/2} (t - t_0) - \frac{A}{2 E_n \zeta_0}}.$$

The change in momentum for case a) is

$$(4) \quad \Delta p_0 = m [\dot{x}(+\infty) - \dot{x}(-\infty)] = 2(2 m E_n)^{1/2}.$$

This is the same expression as that for impact on a rigid wall, so that there is no difference as far as pressure is concerned. Deviations may, however, be expected in case b) when the influence of the wall extends over a distance which is comparable with the mean free path. Let τ denote the average time between two collisions. Then, according to (2), we have

$$(5) \quad \Delta p = m [\dot{x}(t_0 + \tau) - \dot{x}(t_0 - \tau)] = \Delta p_0 \frac{\sinh \alpha \left(\frac{2 E_n}{m} \right)^{1/2} \tau}{\cosh \alpha \left(\frac{2 E_n}{m} \right)^{1/2} \tau - \frac{A}{2 E_n \zeta_0}},$$

or, as a first approximation (for large τ 's):

$$(5 a) \quad \Delta p = \Delta p_0 \left(1 + \frac{A}{E_n \zeta_0} e^{-\alpha \tau (2 E_n / m)^{1/2}} \right).$$

According to eq. (1), $1/\alpha$ is of the order of the range of influence of wall forces. The latter have a marked influence on Δp when the second term in the brackets of (5 a) ceases to be negligible compared with 1. This occurs when

$$\frac{1}{\alpha} \geq \left(\frac{2 E_n}{m} \right)^{1/2} \cdot \tau = \left(\frac{2 E_n}{m c^2} \right)^{1/2} \cdot l$$

i. e. only when the range of influence of the forces is at least equal to the mean free path l .

III.6. a) With reference to Fig. 23 on p. 170 we can write that the number of particles from the cylinder shown in the sketch is

$$dv = \sigma c \cos \theta \cdot m \left(\frac{m}{2 \pi k T} \right)^{3/2} \cdot e^{-mc^2/2kT} c^2 dc \sin \theta d\theta d\phi.$$

Integration over the half-space gives

$$v = \sigma n \left(\frac{k T}{2 \pi m} \right)^{1/2} \cdot 2 \int_0^{\infty} e^{-v^2} v^3 dv.$$

The integral, inclusive of the factor 2, is equal to $\int_0^{\infty} e^{-t} \cdot t dt = 1!$. Expressing n with the aid of pressure, we obtain

$$v = \sigma p (2 \pi m k T)^{-1/2}.$$

If $p_2 > p_1$, more particles will flow from left to right than in the opposite direction. The difference is

$$(1) \quad \Delta v = \sigma (2 \pi m k T)^{-1/2} \cdot \Delta p.$$

b) ΔW can be calculated from $\frac{1}{2} m c^2 dv$. Calculating W in a way analogous to v , we have

$$W = \sigma v \cdot \frac{m}{2} \cdot \frac{2 k T}{m} \cdot 2! = \sigma p \left(\frac{2 k T}{\pi m} \right)^{1/2}.$$

Hence

$$(2) \quad \Delta W = \sigma \left(\frac{2 k T}{\pi m} \right)^{1/2} \cdot \Delta p.$$

c) The average energy transferred per particle is:

$$(3) \quad \frac{\Delta W}{\Delta v} = 2 k T > \frac{3}{2} k T.$$

d) It is larger than $3/2 k T$ because the particles possessing high energy per second arrive from a larger volume.

e) The flow of matter causes the pressure to change. According to the inequality in (3) the right-hand chamber becomes cooled whereas the left-hand compartment is heated. It is, therefore, necessary to control the pressures and to provide for an exchange of heat (temperature bath).

III.7. Let n_1 , n_2 and n_0 denote the densities of molecules reflecting from A_1 , A_2 and B respectively. Mass equilibrium will prevail when $n_1 \bar{c}_x = n_0 \bar{c}_x = n_2 \bar{c}_x'$. Owing to isotropy this is equivalent to

$$(1) \quad n_1 \bar{c} = n_0 \bar{c} = n_2 \bar{c}',$$

where \bar{c} and \bar{c}' denote the mean velocities at temperatures T and T' .

The equilibrium pressure p is given by $p = \frac{1}{3} m n c^2$, where $n = 2 n_1 = 2 n_0$ so that $p = \frac{2}{3} m n_1 \bar{c}^2$. The recoils due to the molecules travelling away from

B are equal on both sides and cancel. The forces which act on B and which are not compensated are solely due to the arriving molecules. The resultant force is

$$(2) \quad F = \frac{2}{3} m A (n_2 \bar{c'^2} - n_1 \bar{c^2}) = p A \left(\frac{n_2}{n_1} \frac{\bar{c'^2}}{\bar{c^2}} - 1 \right).$$

According to (1) it follows that

$$F = p A \left(\frac{\bar{c}}{\bar{c'}} \cdot \frac{\bar{c'^2}}{\bar{c^2}} - 1 \right).$$

Since $\bar{c} \sim \sqrt{T}$, $\bar{c^2} \sim T$, we have

$$(3) \quad F = p A \left[\left(1 + \frac{\delta T}{T} \right)^{1/2} - 1 \right] \approx \frac{p A \delta T}{2 T}$$

or

$$(4) \quad p = \frac{2 F T}{A \delta T}.$$

IV.1. The following table indicates the frequencies of the respective errors:

Error =	-4ε	-3ε	-2ε	$-\varepsilon$	0	$+\varepsilon$	$+2 \varepsilon$	$+3 \varepsilon$	$+4 \varepsilon$
$n = 0$					1				
1				$\frac{1}{2}$		$\frac{1}{2}$			
2			$\frac{1}{4}$		$\frac{2}{4}$		$\frac{1}{4}$		
3		$\frac{1}{8}$		$\frac{3}{8}$		$\frac{3}{8}$		$\frac{1}{8}$	
4	$\frac{1}{16}$		$\frac{4}{16}$		$\frac{6}{16}$		$\frac{4}{16}$		$\frac{1}{16}$

In general the probability of a given error is given by the coefficients of the binomial expansion. If n is the number of individual errors and if k is the number of positive errors, then the probability of this case is given by

$$w_{n,k} = \frac{1}{2^n} \frac{n!}{k!(n-k)!}$$

and the *magnitude* of the error is:

$$f_{n,k} = k \varepsilon + (n-k)(-\varepsilon) = (2k-n)\varepsilon = x.$$

Instead of k we introduce the magnitude of the error, x , so that

$$k = \frac{x}{2\epsilon} + \frac{n}{2}$$

and (with $dk = 1$):

$$dw = \frac{1}{2^n} \frac{n! dk}{k!(n-k)!} = \frac{1}{2^n} \frac{n!}{\left(\frac{n}{2} + \frac{x}{2\epsilon}\right)! \left(\frac{n}{2} - \frac{x}{2\epsilon}\right)!} \frac{dx}{2\epsilon}.$$

The right-hand side is evaluated with the aid of Stirling's formula

$$n! = (2\pi n)^{1/2} \left(\frac{n}{e}\right)^n.$$

Consequently:

$$dw = \frac{1}{\sqrt{2\pi}} \left(\frac{4n}{n^2 - x^2/\epsilon^2}\right)^{1/2} \frac{dx/2\epsilon}{\left(1 + \frac{x}{n\epsilon}\right)^{\frac{n}{2} + \frac{x}{2\epsilon}} \left(1 - \frac{x}{n\epsilon}\right)^{\frac{n}{2} - \frac{x}{2\epsilon}}}.$$

The logarithms of the factors in the denominator are

$$\frac{n}{2} \left(1 \pm \frac{x}{n\epsilon}\right) \log \left(1 \pm \frac{x}{n\epsilon}\right) = \frac{n}{2} \left(1 \pm \frac{x}{n\epsilon}\right) \left(\pm \frac{x}{n\epsilon} - \frac{x^2}{2n^2\epsilon^2}\right) = \pm \frac{x}{2\epsilon} + \frac{x^2}{4n\epsilon^2}.$$

and their sum is:

$$\frac{x^2}{2n\epsilon^2}.$$

Thus the product in the denominator gives $\exp(x^2/2n\epsilon^2)$, so that

$$dw = (2\pi n\epsilon^2)^{-1/2} \exp(-x^2/2n\epsilon^2) \cdot dx,$$

q. e. d. Putting $x = \xi(2n\epsilon^2)^{1/2}$, we see that the integral with respect to dw is

$$\int dw = \frac{1}{\sqrt{\pi}} \int \exp(-\xi^2) d\xi = 1.$$

IV.2. a) When two statistically independent errors of the same kind are superimposed then the probability for the errors to lie within the intervals $(x', x' + dx')$ and $(x'', x'' + dx'')$ is

$$f_1(x') f_1(x'') dx' dx''.$$

The total error is $x' + x'' = x$. Since we are interested in the probability of a definite total error we put $x'' = x - x'$ and integrate with respect to x' . Thus

$$(1) \quad f_2(x) = \int f_1(x') f_1(x - x') dx'.$$

In general, when an n 'th error is superimposed on $(n - 1)$ errors, we have

$$(2) \quad f_n(x) = \int f_1(x') f_{n-1}(x - x') dx'.$$

b) When $f_n = (\pi a_n)^{-1/2} e^{-a_n x^2}$ (with $\int f_n dx = 1$) it follows from (2) that

$$(3) \quad \frac{1}{a_n} = \frac{1}{a_{n-1}} + \frac{1}{a_1} = \frac{n}{a_1}.$$

Gaussian functions are seen to satisfy eq. (2) and the half-width increases like $n^{-1/2}$.

c) In order to answer the question we notice that in this case eq. (3) assumes the form

$$f_n(x) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{+\varepsilon} f_{n-1}(x - x') dx'.$$

Putting

$$f_n(x) = A_n e^{i\lambda x},$$

we find a complete system. It follows that for an arbitrary value of λ we have

$$A_n(\lambda) = A_{n-1}(\lambda) \cdot \frac{1}{2\varepsilon} \int_{-\varepsilon}^{+\varepsilon} e^{-i\lambda x'} dx' = A_{n-1}(\lambda) \left(\frac{\sin \lambda \varepsilon}{\lambda \varepsilon} \right)$$

and

$$(4) \quad A_n(\lambda) = C(\lambda) \left(\frac{\sin \lambda \varepsilon}{\lambda \varepsilon} \right)^{n-1}.$$

The required solution is found by expanding $f_1(x)$ into a Fourier series:

$$f_1(x) = \int C(\lambda) e^{-i\lambda x} d\lambda.$$

Applying Fourier's integral theorem (cf. Vol. VI, Sec. 4, eq. (13)), we have

$$C(\lambda) = \frac{1}{2\pi} \int f_1(x) e^{-i\lambda x} dx = \frac{1}{4\pi\varepsilon} \int_{-\varepsilon}^{+\varepsilon} e^{-i\lambda x} dx = \frac{1}{2\pi} \frac{\sin \lambda \varepsilon}{\lambda \varepsilon}.$$

Hence

$$(5) \quad A_n(\lambda) = \frac{1}{2\pi} \left(\frac{\sin \lambda \varepsilon}{\lambda \varepsilon} \right)^n$$

represent the Fourier components of f_n . The function itself is:

$$(6) \quad f_n(x) = \frac{1}{2\pi} \int \left(\frac{\sin \lambda \varepsilon}{\lambda \varepsilon} \right)^n e^{i\lambda x} d\lambda.$$

For large values of n the expression $\left(\frac{\sin \lambda \varepsilon}{\lambda \varepsilon} \right)^n$ differs markedly from 0 only in the immediate neighborhood of the zero-point. Hence we may replace this factor by the osculating Gaussian bell-curve

$$\left(\frac{\sin \lambda \varepsilon}{\lambda \varepsilon} \right)^n \approx \exp \left(-\frac{n\varepsilon^2}{6} \lambda^2 \right)$$

Now the integral (6) can be evaluated. We have

$$f_n(x) = \frac{1}{2\pi} \int \exp \left[-\frac{n\varepsilon^2}{6} \left(\lambda - \frac{3ix}{n\varepsilon^2} \right)^2 \right] d\lambda \cdot \exp(-3x^2/2n\varepsilon^2)$$

or

$$(7) \quad f_n(x) = \left(\frac{3}{2\pi n\varepsilon^2} \right)^{1/2} \exp(-3x^2/2n\varepsilon^2).$$

The result is a Gaussian distribution, in analogy with the result in IV.1.

We now compute the integrals (6) for $n = 1, 2, 3$. Putting $\lambda \varepsilon = t$, $\xi = x/\varepsilon$, we obtain generally:

$$f_n = \frac{1}{2\pi\varepsilon} \int_{-\infty}^{+\infty} \frac{\sin^n t}{t^n} \cos \xi t dt.$$

On partial integration we find that

$$f_n = \frac{1}{2\pi\varepsilon} \cdot \frac{1}{(n-1)!} \int_{-\infty}^{+\infty} \frac{dt}{t} \frac{d^{n-1}}{dt^{n-1}} (\sin^n t \cos \xi t).$$

The functions operated on by the differential operator are:

$$\sin t \cos \xi t \quad \text{for } n = 1,$$

$$\sin^3 t \cos \xi t = \frac{1}{2} (1 - \cos 2t) \cos \xi t \quad \text{for } n = 2,$$

$$\sin^3 t \cos \xi t = \frac{1}{4} (3 \sin t - \sin 3t) \cos \xi t \quad \text{for } n = 3.$$

The 0-th, 1st and 2nd derivatives of the respective functions are:

$$n = 1 \quad \sin t \cos \xi t,$$

$$2 \quad \sin 2t \cos \xi t - \frac{1}{2} \xi (1 - \cos 2t) \sin \xi t,$$

$$3 \quad -\frac{3}{4} (\sin t - 3 \sin 3t) \cos \xi t - \frac{3}{2} \xi (\cos t - \cos 3t) \sin \xi t$$

$$-\frac{1}{4} \xi^2 (3 \sin t - \sin 3t) \cos \xi t.$$

The remaining integrals are all of the same type (Dirichlet's discontinuous factor). We now write down the results of the integration for the individual terms and for the intervals in which they do *not* vanish:

$$\text{Case } n = 1: \quad = 1 \quad \text{for} \quad |\xi| < 1,$$

which is our starting point.

$$\text{Case } n = 2: \quad = 1 \quad \text{for} \quad |\xi| < 2,$$

$$-\frac{1}{2} |\xi| \quad \text{for} \quad \text{all } \xi\text{'s,}$$

$$+\frac{1}{2} |\xi| \quad \text{for} \quad |\xi| > 2.$$

It follows that

$$2 \varepsilon f_2 = 1 - \frac{1}{2} |\xi| \quad \text{for} \quad |\xi| < 2,$$

$$\text{Case } n = 3: \quad = -\frac{3}{4} \quad \text{for} \quad |\xi| < 1,$$

$$\frac{9}{4} \quad \text{for} \quad |\xi| < 3,$$

$$-\frac{3}{2} |\xi| \quad \text{for} \quad |\xi| > 1,$$

$$+\frac{3}{2} |\xi| \quad \text{for} \quad |\xi| > 3,$$

$$-\frac{3}{4} |\xi|^2 \quad \text{for} \quad |\xi| < 1,$$

$$+\frac{1}{4} |\xi|^2 \quad \text{for} \quad |\xi| < 3.$$

Thus in the given intervals we have:

$$0 \leq |\xi| \leq 1 \quad 4 \varepsilon f_3 = \frac{3}{2} - \frac{1}{2} |\xi|^2,$$

$$1 \leq |\xi| \leq 3 \quad \frac{9}{4} - \frac{3}{2} |\xi| + \frac{1}{4} |\xi|^2,$$

$$3 \leq |\xi| \leq \infty \quad 0.$$

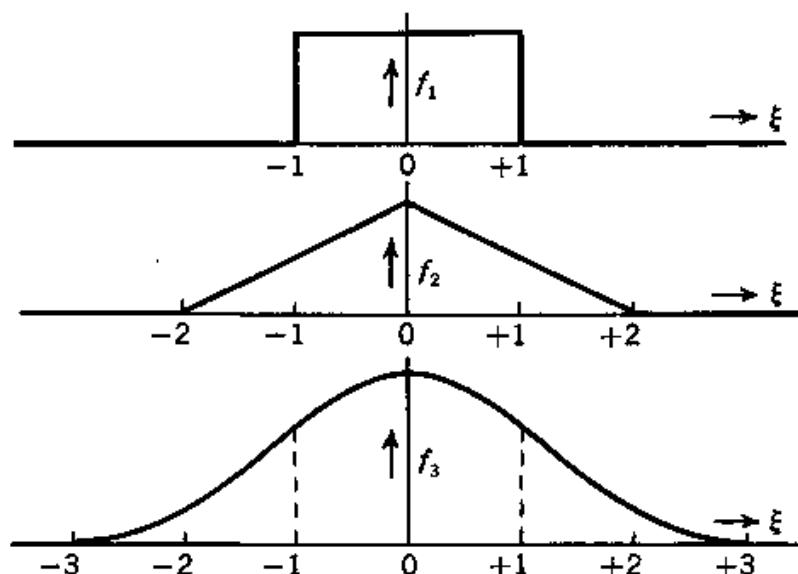


Fig. 41. Principal cross-section functions for polydimensional cubes, drawn for $n = 1, 2, 3$.

The functions f_1, f_2, f_3 are seen plotted in Fig. 41. The function f_1 consists of a horizontal straight segment, f_2 consists of two inclined straight segments, f_3 consists of three parabolic segments, etc. The geometrical interpretation of these functions can be given at once. Since f_2 arises from the superposition of two constant functions f_1 , we see that inside a square whose sides are 2ε we have uniform coverage. We wish to determine the strips of constant total error. They are given by the segments cut out of the square by the straight lines at right angles to the diagonal DD in Fig. 41 a. Their lengths are given by f_2 . Correspondingly f_3 gives sectional areas in a cube at right angles to a principal diagonal, etc.

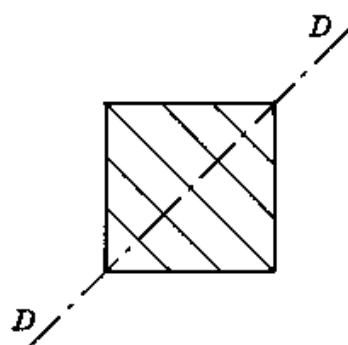


Fig. 41 a.
Principal cross-sections of
a square.

IV.3. If N particles are distributed over n compartments containing N_1, N_2, \dots, N_n particles each, we have

$$W = \frac{N!}{N_1! \dots N_n!}.$$

Thus, with the aid of Stirling's formula (Sec. 294 a) we have for the three cases respectively:

$$\text{a) } W_a = \frac{N!}{N!} = 1,$$

$$\text{b) } W_b = \frac{N!}{(N/2)!^2} = (2\pi N)^{1/2} \frac{2^N}{\pi N},$$

$$\text{c) } W_c = \frac{N!}{(N/6)!^6} = (2\pi N)^{1/2} \left(\frac{3}{\pi N}\right)^3 \cdot 6^N,$$

and hence the ratios:

$$\frac{W_b}{W_a} = \left(\frac{2}{\pi N}\right)^{1/2} \cdot 2^N \rightarrow \infty \quad \text{for } N \rightarrow \infty,$$

$$\frac{W_c}{W_b} = \frac{27}{\pi^2 N^2} \cdot 3^N \rightarrow \infty \quad \text{for } N \rightarrow \infty.$$

IV.4. We have

$$\overline{\phi^2} = -\frac{d}{d\gamma} \log \int_{-\infty}^{+\infty} e^{-\gamma \phi^2} d\phi = \frac{1}{2\gamma}, \quad \gamma = \frac{D}{2kT},$$

and it follows that

$$\overline{\phi^2} = \frac{kT}{D} = \frac{RT}{LD}.$$

The Loschmidt-Avogadro number is

$$L = \frac{RT}{D\overline{\phi^2}} = \frac{8.32 \times 287}{9.43 \times 4.18} \times 10^{23} \text{ mol}^{-1} = 6.05 \times 10^{23} \text{ mol}^{-1}.$$

IV.5. We put $N = Z^3$, $Z = 10^7$. The number of particles on the surface is then $N_S = 6Z^2$. The thermal and surface energies are, respectively

$$U = 3Z^3 kT \quad \text{and} \quad U_S = 6Z^2 \text{ eV}.$$

For $T = 290 \text{ K}$ we have:

$$\frac{U}{U_S} = \frac{ZkT}{2\text{ eV}} = \frac{10^7 \times 400 \times 10^{-23}}{2 \times 1.6 \times 10^{-19} \times 9} = \frac{10^6}{72} \approx 14000.$$

Consequently, $U \approx U_S$ if $Z = 10^7/14000 = 700$; this means that the number of particles is $N = 3.4 \times 10^8$, the linear dimension $l = 2 \times 10^{-8} Z \text{ cm} = 1.4 \times 10^{-5} \text{ cm}$.

IV.6. a) We have

$$\int_0^{\infty} \frac{\partial \vartheta_2}{\partial z} \frac{dz}{z} = -2\pi e^{-1/q} \sum_{n=0}^{\infty} (2n+1) e^{-n(n+1)q} \int_0^{\infty} \frac{\sin(2n+1)\pi z}{z} dz.$$

It can be seen that all integrals under the sign of summation are equal to $\frac{1}{2}\pi$.

b) Substituting the series ϑ_0 into the transformation equation we have

$$\vartheta_2\left(z \left| \frac{i q}{\pi} \right. \right) = \left(\frac{\pi}{q} \right)^{1/2} \sum_{n=-\infty}^{+\infty} (-1)^n \exp \left[-\frac{\pi^2}{q} (z-n)^2 \right].$$

Consequently

$$Z(q) = -\frac{1}{\pi^2} e^{1/q} \left(\frac{\pi}{q} \right)^{1/2} \cdot \left(\frac{-2\pi^2}{q} \right) \cdot \lim_{\varepsilon \rightarrow 0} \sum_{n=-\infty}^{+\infty} (-1)^n \int_{\varepsilon}^{\infty} \frac{z-n}{z} \exp \left[-\frac{\pi^2}{q} (z-n)^2 \right] dz.$$

A simultaneous change in the signs of z and n only changes the limits of integration, so that

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{-\varepsilon} = \frac{1}{2} \int_{-\infty}^{+\infty}.$$

Hence

$$Z(q) = \frac{1}{q} e^{1/q} \cdot \left(\frac{\pi}{q} \right)^{1/2} \cdot \sum_{n=-\infty}^{+\infty} (-1)^n \int_{-\infty}^{+\infty} \frac{z-n}{z} \exp \left[-\frac{\pi^2}{q} (z-n)^2 \right] dz.$$

The transformation $z = t/\pi + n$ gives:

$$Z(q) = \frac{1}{q} e^{1/q} \cdot (\pi q)^{-1/2} \cdot \int_{-\infty}^{+\infty} \left(\sum_{n=-\infty}^{+\infty} (-1)^n \frac{t}{t+n\pi} \right) e^{-t^2/q} dt.$$

This proves the proposition because of the relation:

$$\sum_{n=-\infty}^{+\infty} (-1)^n \frac{t}{t+n\pi} = \frac{t}{\sin t}.$$

c) We have, formally

$$\begin{aligned}\frac{1}{\sin t} &= \frac{2i e^{-it}}{1 - e^{-2it}} = 2i \sum_{k=0}^{\infty} e^{-(2k+1)i} \\ &= \frac{-2i e^{+it}}{1 - e^{+2it}} = -2i \sum_{k=0}^{\infty} e^{+(2k+1)i} \\ &= -i \sum_{k=0}^{\infty} (e^{+(2k+1)i} - e^{-(2k+1)i}) = 2 \sum_{k=0}^{\infty} \sin(2k+1)t.\end{aligned}$$

The integrals in $Z(q)$ [see (b)] can be transformed as follows:

$$2 \int_{-\infty}^{+\infty} t \sin(2k+1)t \cdot e^{-t^2/q} dt = 2 \int_0^{\infty} \sin(2k+1)t \cdot e^{-t^2/q} dt^2,$$

Putting $\tau = t^2$ we obtain an integral of the type of a Laplace transformation, namely:

$$2 \int_0^{\infty} \sin(2k+1) \sqrt{\tau} \cdot e^{-\tau/q} d\tau = q(\pi q)^{1/2} \exp\left(-(k + \frac{1}{2})^2 q\right).$$

The integral can be easily evaluated or taken from tables of Laplace transforms (see e. g. W. Magnus, F. Oberhettinger "Formeln und Sätze für die speziellen Funktionen der mathematischen Physik, Springer, Berlin 1943", or Bateman manuscript project, ed. by A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, McGraw-Hill, New York, 1954/5).

It follows that

$$(\alpha) \quad Z(q) = \sum_{k=0}^{\infty} (2k+1) e^{-(k+\frac{1}{2})^2 q}.$$

In this manner we obtain eq. (34.3) which was derived from the transformed representation. Consequently, it becomes superfluous to prove the transformation equation. It is worth noting that it can also be deduced from the transformation equation in Vol. VI., eq. (15.8).

When q is very small, the only significant contribution to the integral comes from a small interval at $t = 0$ (its length being $\Delta t = q$). In this case we may use the expansion

$$\frac{t}{\sin t} = 1 / \left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \dots \right) = 1 + \frac{1}{6} t^2 + \frac{7}{360} t^4 + \frac{31}{15120} t^6 + \dots$$

Inserting this into $Z(q)$ and putting for the integrals

$$J_n = \int_{-\infty}^{+\infty} e^{-\gamma t^2} t^{2n} dt = \pi^{1/2} \left(-\frac{d}{d\gamma} \right)^n \gamma^{-1/2},$$

or, in particular,

$$J_0 = \left(\frac{\pi}{\gamma} \right)^{1/2}, \quad J_1 = \frac{1}{2\gamma} \left(\frac{\pi}{\gamma} \right)^{1/2}, \quad J_2 = \frac{3}{4\gamma^2} \left(\frac{\pi}{\gamma} \right)^{1/2}, \quad J_3 = \frac{15}{8\gamma^3} \left(\frac{\pi}{\gamma} \right)^{1/2}$$

we have:

$$(\beta) \quad Z(q) = \frac{1}{q} e^{\frac{1}{2}q} \left[1 + \frac{1}{12}q + \frac{7}{480}q^2 + \frac{31}{8064}q^3 + \dots \right].$$

Expanding $e^{\frac{1}{2}q}$ and multiplying the series term by term we obtain:

$$(\beta') \quad Z(q) = \frac{1}{q} \left[1 + \frac{1}{3}q + \frac{1}{15}q^2 + \frac{4}{315}q^3 + \dots \right]$$

and

$$(\beta'') \quad \log Z(q) = -\log q + \frac{1}{3}q + \frac{1}{90}q^2 + \frac{8}{2835}q^3 + \dots$$

d) Differentiating Z we obtain:

$$u = R T^2 \frac{d \log Z}{dT} = -R \Theta \frac{d \log Z}{dq},$$

$$c_v = \frac{\partial u}{\partial T} = -\frac{q^2}{\Theta} \frac{\partial u}{\partial q} = R q^2 \frac{d^2 \log Z}{dq^2}.$$

Hence

$$(\gamma) \quad u = R \Theta \left[\frac{1}{q} - \frac{1}{3} - \frac{1}{45}q - \frac{8}{945}q^2 - \dots \right],$$

$$c_v = R \left[1 + \frac{1}{45}q^2 + \frac{16}{945}q^3 + \dots \right].$$

To a first approximation (except for a constant energy term in the same way as in Sec. 33), we find that:

$$u = R T - \frac{1}{3} R \Theta, \quad c_v = R.$$

The accuracy is better than 1% if $q^2/45 < 1/100$, or if $q < 0.67$.

e) The following table gives an estimate of the accuracy which can be achieved with a small number of terms:

Magnitude of the	First	Second	Third term in c_v
for $q = 0.3$	1	0.002	0.000467
for $q = 0.5$	1	0.008	0.00372
for $q = 0.9$	1	0.018	0.01263
for $q = 1.2$	1	0.032	0.0299

From $q = 0.6$ onwards the third term becomes of the same order as the second and cannot be neglected any more. Thus the asymptotic expansion can be used up to $q = 0.5$ approximately (i. e. for $T > 2 \Theta$). From this value onwards it is preferable to use eq. (α). Equation (β) shows, however, that for large temperatures c_v tends to the value R very fast, and that it approaches it from above.

In conclusion we propose to calculate c_v from the two expansions for $e^{2q} = 2.5$, i. e. for $q = 0.458$, $q^2 = 0.20977$, $q^3 = 0.09607$. The asymptotic series gives

$$c_v = 1.006 R$$

when terms up to and including the third are taken into account.

The value from eq. (33.3) can be calculated from (γ):

$$c_v = R q^3 \frac{d^3 \log Z}{dq^3} = R q^3 \left(\frac{Z'''}{Z} - \frac{Z' Z''}{Z^2} \right).$$

Here

$$\begin{aligned} Z &= 1 + 3e^{-2q} + 5e^{-6q} + 7e^{-12q} + 9e^{-20q} + 11e^{-30q}, \\ -Z' &= 6e^{-2q} + 30e^{-6q} + 84e^{-12q} + 180e^{-20q} + 330e^{-30q}, \\ Z'' &= 12e^{-2q} + 180e^{-6q} + 1008e^{-12q} + 3600e^{-20q} + 9900e^{-30q}. \end{aligned}$$

Substituting the preceding value for q we have

$$Z = 2.549; \quad Z' = 4.683; \quad Z'' = 20.836;$$

and

$$c_v = 1.0064 R.$$

Since the series (α) takes into account all terms which make a marked contribution, it is seen that the fourth term of the asymptotic series (β) would have to be taken into account. However, the two series join to within 0.05% and the molar specific heat is seen to be well represented by taking only a small number of terms in each series.

IV.7. It is necessary to make $\log W$ a maximum under the conditions indicated. Taking these into account with the aid of Lagrange's method of parameters we find that it is necessary to determine the maximum of

$$-\sum_{n,i} \{f_i^{(n)} \log f_i^{(n)} + \lambda_i f_i^{(n)} + \alpha n f_i^{(n)} + \beta n \varepsilon_i f_i^{(n)}\}$$

The derivatives with respect to all $f_i^{(n)}$'s must vanish. Thus

$$\log f_i^{(n)} + 1 + \lambda_i + \alpha n + \beta n \varepsilon_i = 0$$

or

$$(1) \quad f_i^{(n)} = e^{-1-\lambda_i} \cdot e^{-(\alpha+\beta\varepsilon_i)n}.$$

The condition $\sum f_i^{(n)} = 1$ gives:

$$e^{-1-\lambda_i}(1 - e^{-\alpha-\beta\varepsilon_i}) = 1$$

and eq. (1) assumes the following form:

$$(2) \quad f_i^{(n)} = (1 - e^{-\alpha-\beta\varepsilon_i}) e^{-(\alpha+\beta\varepsilon_i)n}.$$

The distribution function follows from

$$n_i = \sum_n n f_i^{(n)} = -(1 - e^{-\alpha-\beta\varepsilon_i}) \frac{d}{d\alpha} \sum_n e^{-(\alpha+\beta\varepsilon_i)n}.$$

Thus we have

$$n_i = -(1 - e^{-\alpha-\beta\varepsilon_i}) \frac{d}{d\alpha} \left(\frac{1}{1 - e^{-\alpha-\beta\varepsilon_i}} \right).$$

This is the Bose-Einstein distribution:

$$(3) \quad n_i = \frac{1}{e^{\alpha+\beta\varepsilon_i} - 1}$$

and proves the assertion.

The constants α and β can be determined from the remaining conditions which now appear in the simpler form:

$$(4) \quad \sum n_i = N, \quad \sum n_i \varepsilon_i = U.$$

IV.8. The volume V of the vessel consists of the parts $V_1 = \Delta V$ and $V_2 = V - \Delta V$. The probability of finding a molecule in V_1 or V_2 is V_1/V or V_2/V , respectively, since equal volume elements are associated with equal probabilities. The probability of finding N_1 particles in V_1 and N_2 in V_2 becomes

$$W(N_1, N_2) = \frac{N!}{N_1! N_2!} \left(\frac{V_1}{V} \right)^{N_1} \left(\frac{V_2}{V} \right)^{N_2}.$$

We calculate, further, in an obvious way that

$$\overline{N_1} = N \frac{V_1}{V}, \quad \overline{N_1(N_1-1)} = N(N-1) \frac{V_1^2}{V^2}.$$

It follows that

$$(\Delta N_1)^2 = \overline{N_1^2} - \overline{N_1}^2 = N \frac{V_1 V_2}{V^2}$$

or

$$\frac{\Delta N_1}{\overline{N_1}} = \sqrt{\frac{V_2}{N V_1}} = \sqrt{\frac{V - \Delta V}{N \Delta V}}.$$

IV.9. a) The van der Waals equation for N particles in volume V can be written:

$$(1) \quad \left[p + A \left(\frac{N}{V} \right)^2 \right] \left(\frac{V}{N} - B \right) = k T.$$

The constants A and B are related to the constants a and b in the original van der Waals equation; we have

$$(1a) \quad A = \frac{a}{L^2}; \quad B = \frac{b}{L}.$$

(L = the Loschmidt-Avogadro number). The critical parameters are obtained in the same way as in Sec. 9 from:

$$\begin{aligned} \left(\frac{\partial p}{\partial V} - \frac{2 A N^2}{V^3} \right) \left(\frac{V}{N} - B \right) + \frac{1}{N} \left(p + \frac{A N^2}{V^2} \right) &= 0, \\ \left(\frac{\partial^2 p}{\partial V^2} + \frac{6 A N^2}{V^4} \right) \left(\frac{V}{N} - B \right) + \frac{2}{N} \left(\frac{\partial p}{\partial V} - \frac{2 A N^2}{V^3} \right) &= 0, \end{aligned}$$

putting $\partial p / \partial V = 0$ and $\partial^2 p / \partial V^2 = 0$.

Starting with the last equation we obtain

$$(2) \quad V_{crit} = 3 N B, \quad p_{crit} = \frac{A}{27 B^2}; \quad k T_{crit} = \frac{8 A}{27 B}.$$

b) The logarithm of the partition function is a thermodynamic parameter of the form

$$(3) \quad d \log Z = - U d\beta + \beta p dV.$$

Substituting the thermal equation of state for p , we obtain:

$$\left(\frac{\partial \log Z}{\partial V} \right)_\beta = \beta p = \frac{N}{V - N B} - \beta A \frac{N^2}{V^2}.$$

On integrating we have:

$$(4) \quad \log Z = N \left[\log (V - N B) + \beta A \frac{N}{V} \right] + N f(\beta).$$

Here $f(\beta)$ is an arbitrary function of β .

Evaluating the energy from this expression with the aid of eq. (3), we obtain, further,

$$U = - \left(\frac{\partial \log Z}{\partial \beta} \right)_v = A \frac{N^2}{V} - N f'(\beta).$$

The heat capacity at constant volume is, therefore,

$$\left(\frac{\partial U}{\partial T} \right)_v = -k \beta^2 \left(\frac{\partial U}{\partial \beta} \right)_v = k \beta^2 N f''(\beta) = \frac{N}{L} c_v \rightarrow \frac{3}{2} N k.$$

The last expression is valid asymptotically for the limiting case of $B \rightarrow 0$. Hence:

$$f''(\beta) \rightarrow \frac{3}{2 \beta^2}$$

and integration yields the asymptotic expressions

$$f'(\beta) \rightarrow -\frac{3}{2 \beta} + C_1,$$

$$f(\beta) \rightarrow -\frac{3}{2} \log \beta + C_1 \beta + C_2.$$

Thus, for $\log Z$, we obtain

$$(5) \quad \log Z \rightarrow N \left[\log (V - N B) + \beta A \frac{N}{V} - \frac{3}{2} \log \beta \right] + N C_1 \beta + N C_2.$$

In the limiting cases of $\beta \rightarrow 0$, $N/V \rightarrow 0$, we have the simplified forms:

$$(5a) \quad \log Z = N \log N + N \left(\log \frac{V}{N} - \frac{3}{2} \log \beta \right) + N C_1 \beta + N C_2.$$

This expression must be identical with that for a perfect gas:

$$(\log Z)_{\text{perf. gas}} = N \left[1 + \log \frac{V}{N} - \frac{3}{2} \log \beta + \log \left(\frac{2 \pi m}{h^2} \right)^{3/2} \right],$$

and it follows that $C_1 = 0$ and that

$$N \log N + N C_2 = N \left[1 + \log \left(\frac{2 \pi m}{h^2} \right)^{3/2} \right].$$

Finally, we obtain

$$(6) \quad \log Z = N \left[1 + A \beta \frac{N}{V} + \log \left(\frac{V}{N} - B \right) \left(\frac{2 \pi m}{\beta h^2} \right)^{3/2} + g(\beta) \right].$$

Here $g(\beta)$ denotes that part of $f(\beta)$ which vanishes for $\beta \rightarrow 0$.

c) Equation (6) gives at once:

$$\frac{\partial \log Z}{\partial N} = \frac{\log Z}{N} + \left(A \beta \frac{N}{V} - \frac{V/N}{V/N - B} \right).$$

Repeated differentiation with respect to N and multiplication by $-N$ gives:

$$(7) \quad -N \frac{\partial^2 \log Z}{\partial N^2} = \frac{\log Z}{N} - \left(\frac{\log Z}{N} + A \beta \frac{N}{V} - \frac{V}{V - N B} \right) - A \beta \frac{N}{V} + \frac{B N / V}{\left(1 - \frac{B N}{V} \right)^2} \\ = \frac{1}{\left(1 - \frac{B N}{V} \right)^2} - \frac{2 B N}{V} \cdot \frac{A}{B h T}.$$

In the limiting case of a perfect gas ($A, B \rightarrow 0$), we have:

$$(7a) \quad -N \frac{\partial^2 \log Z}{\partial N^2} = 1.$$

The relative mean fluctuation for this case agrees with that in eq. (40.18).

When A and B are to be taken into account only in the first approximation, we have

$$(7b) \quad -N \frac{\partial^2 \log Z}{\partial N^2} = 1 + \frac{2 B N}{V} \left(1 - \frac{A}{B h T} \right).$$

Substituting the critical parameters from eq. (2), namely $B N / V = 1/3$, $A / B h T = 27/8$, we have

$$(7c) \quad -N \frac{\partial^2 \log Z}{\partial N^2} = 0.$$

The relative mean fluctuation becomes:

$$(8) \quad \left(\frac{\Delta n}{\bar{n}} \right)^2 = - \frac{V / N \Delta V}{N \partial^2 \log Z / \partial N^2} \rightarrow \infty,$$

i. e. infinitely large. Large fluctuations cause strong scattering of light (*cf.* vol. IV, Sec. 33) which is evidenced by the strong opalescence of real gases near their critical point.

d) According to eq. (40.15), we may write

$$(\Delta n)^2 = \frac{1}{\beta^2} \sum'_{i,k} \left(\frac{\partial^2 \log Z}{\partial \epsilon_i \partial \epsilon_k} \right)_{\alpha, \beta, \epsilon_j} = \frac{1}{\beta^2} \left(\sum'_{i,k} \frac{\partial^2 \Phi}{\partial \epsilon_i \partial \epsilon_k} \right)_{\alpha, \beta, \epsilon_j}$$

because α must be kept constant in addition to β (cf. Sec. 40, in particular, eq. (12)). The symbol Σ' denotes that the sum must be taken over all phase cells in ΔV . According to eq. (38.4) the last equation may be written:

$$(9) \quad (\Delta n)^2 = - \frac{1}{\beta} \sum' \left(\frac{\partial \bar{n}}{\partial \epsilon_i} \right)_{\alpha, \beta, \epsilon_j}.$$

Here \bar{n} denotes the mean number of molecules in ΔV $\left(\bar{n} = \frac{\Delta V}{V} \cdot N \right)$. It follows from eq. (38.7) that

$$d\bar{\phi} = -\bar{n} d\alpha - \bar{u} d\beta - \beta \sum' \bar{n}_i d\epsilon_i$$

($\bar{\phi}$ is equivalent to $\Phi = \log Y$ from Sec. 38, eq. (1), referred to volume ΔV) and

$$(10) \quad \left(\frac{\partial \bar{n}}{\partial \epsilon_i} \right)_{\alpha, \beta, \epsilon_j} = \beta \left(\frac{\partial \bar{n}_i}{\partial \alpha} \right)_{\beta, \epsilon_j}.$$

It follows from (9) that

$$(9a) \quad (\Delta n)^2 = - \sum'_i \frac{\partial \bar{n}_i}{\partial \alpha} = - \frac{\partial \bar{n}}{\partial \alpha}.$$

Reverting to the partition function by means of a Legendre transformation

$$d(\bar{\phi} + \alpha \bar{n}) = \alpha d\bar{n} - \bar{u} d\beta - \beta \sum'_i \bar{n}_i d\epsilon_i = \frac{\Delta V}{V} d \log Z,$$

we have:

$$\alpha = \frac{\Delta V}{V} \cdot \left(\frac{\partial \log Z}{\partial \bar{n}} \right)_{\beta, \epsilon_i} = \left(\frac{\partial \log Z}{\partial N} \right)_{\beta, \epsilon_i}.$$

This shows that eq. (9a) can be written:

$$(\Delta n)^2 = - \frac{1}{\partial \alpha / \partial \bar{n}} = - \frac{V}{\Delta V} \frac{1}{\partial^2 \log Z / \partial N^2}.$$

Dividing by the square of $\bar{n} = N \Delta V / V$, we obtain the fluctuation equation indicated in the statement of the problem.

V.1. It can be verified directly that the momentum and energy equations are satisfied by:

$$(5) \quad \begin{aligned} \mathbf{v}_1' &= \mathbf{v}_1 + \frac{2 m_2}{m_1 + m_2} (\mathbf{V} \cdot \mathbf{e}) \mathbf{e}, \\ \mathbf{v}_2' &= \mathbf{v}_2 - \frac{2 m_1}{m_1 + m_2} (\mathbf{V} \cdot \mathbf{e}) \mathbf{e}. \end{aligned}$$

$\mathbf{V} = \mathbf{v}_2 - \mathbf{v}_1$ denotes the relative velocity. $\mathbf{V} \sim \mathbf{e}$ gives central impact. In order to prove the validity of Liouville's theorem on the equality of the velocity space cells it is necessary to show that the transformation matrix of the six-dimensional space of the combined velocities \mathbf{v}_1 and \mathbf{v}_2 has a determinant -1 (\mathbf{e} should be placed along one axis!).

According to (5) the difference in the energies after impact is

$$(6) \quad \frac{1}{2} m_1 \mathbf{v}_1'^2 - \frac{1}{2} m_2 \mathbf{v}_2'^2 = \frac{1}{2} m_1 \mathbf{v}_1^2 - \frac{1}{2} m_2 \mathbf{v}_2^2 - \frac{4 m_1 m_2}{(m_1 + m_2)^2} (\mathbf{v}_1 - \mathbf{v}_2 \cdot \mathbf{e}) (m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 \cdot \mathbf{e}).$$

Making use of the condition that all directions are equally probable we find that the *mean value* of a product of the form $(\mathbf{A} \cdot \mathbf{e}) (\mathbf{B} \cdot \mathbf{e})$ taken over \mathbf{e} is:

$$\overline{(\mathbf{A} \cdot \mathbf{e}) (\mathbf{B} \cdot \mathbf{e})}^{(e)} = \frac{1}{3} \mathbf{A} \cdot \mathbf{B}.$$

Inserting this value into eq. (6), we have:

$$(7) \quad E_1' - E_2' = E_1 - E_2 - \frac{1}{3} \frac{4 m_1 m_2}{(m_1 + m_2)^2} [(\mathbf{v}_1 - \mathbf{v}_2) \cdot (m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2)].$$

Since the direction of \mathbf{v}_2 is independent of that of \mathbf{v}_1 the mean value with respect to \mathbf{v}_2 becomes:

$$\overline{(\mathbf{v}_1 \cdot \mathbf{v}_2)}^{(v_2/v_1)} = 0.$$

Thus in the last bracket in eq. (7) only the terms $m_1 \mathbf{v}_1^2 - m_2 \mathbf{v}_2^2 = 2 (E_1 - E_2)$ remain. Hence

$$\frac{E_1' - E_2'}{E_1 - E_2} = 1 - \frac{2}{3} \cdot \frac{4 m_1 m_2}{(m_1 + m_2)^2}.$$

For $m_1 = m_2$ the ratio becomes equal to $1/3$, for $m_1 \ll m_2$ it becomes equal to $1 - 8 m_1/3 m_2$. This result means that on the average the difference in the kinetic energy of the two particles taking part in the collision decreases continuously. Consequently the mean kinetic energy of the translational motion becomes equally distributed.

The law of equipartition is valid for any molecules and not only for molecules of one kind.

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Partial Differential Equations in Physics

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FOREWORD

The topic with which I regularly conclude my six-term series of lectures in Munich is the partial differential equations of physics. We do not really deal with mathematical physics, but with *physical mathematics*; not with the mathematical formulation of physical facts, but with the physical motivation of mathematical methods. The oft-mentioned “prestabilized harmony” between what is mathematically interesting and what is physically important is met at each step and lends an esthetic — I should like to say metaphysical — attraction to our subject.

The problems to be treated belong mainly to the classical mathematical literature, as shown by their connection with the names of Laplace, Fourier, Green, Gauss, Riemann, and William Thomson. In order to show that these methods are adequate to deal with actual problems, we treat the *propagation of radio waves* in some detail in Chapter VI.

Chapter V deals with the general method of *eigenfunctions*. The most spectacular domain of application of that method is *wave mechanics*, as we show here with the help of some selected, particularly simple examples. The mathematically rigorous foundation of the existence and the properties of eigenfunctions with the help of theorems about integral equations cannot be given here; the latter are mentioned only occasionally as the counterpart of the corresponding theorems on differential equations.

Chapter IV on *Bessel functions* and *spherical harmonics* is comparatively lengthy despite a development that is as concise as possible. For the sake of brevity we have relegated some proofs to the exercises, as we have also done in other chapters. A special section is dedicated to the beautiful *method of reciprocal radii* and to the demonstration of the fact that it unfortunately cannot be applied to other than potential problems.

Chapter III deals exclusively with the classic problem of heat conduction. In addition to the Fourier method we develop in detail the intuitive method of *reflected images* for regions with plane boundaries. Chapter II deals with the different types of differential equations and boundary value problems; *Green's theorem* and *Green's function* are introduced in considerable generality.

Chapter I about Fourier series and integrals is based throughout on the *method of least squares*. If the latter is complemented by a requirement which we called “the condition of finality,” then we can

replace the more formal computations of the older developments in a complete and generalizable way, not only in the trigonometric case but also for spherical harmonics and general eigenfunctions.

As is seen from this survey, the arrangement of the material is determined not by systematic but by didactic points of view. Chapter I intends to put the reader in the midst of the methodology of the Fourier and the Fourier-like expansions. Only in Chapter II do we start to introduce the concepts from the theory of partial differential equations that are of the greatest importance for the mathematical physicist. From a systematic point of view Chapter III would be subordinated to the general methods of Chapter V but it precedes it for historic and didactic reasons. The lengthiness of Chapter IV may be justified by the fact that a large part of the material contained in the textbooks on Bessel functions and spherical harmonics is at least sketched there, and is put in readiness for application. The formal mathematical part is interrupted for didactic reasons for both classes of functions by typical examples of applications.

It is obvious that this material could not be presented completely in a short summer term. In fact several mathematically more complicated sections have been added in print, some of these in the form of appendixes. In this connection we wish to mention Appendix II to Chapter V, which was added only after the completion of the rest of the manuscript and which is likely to be of fundamental importance for problems dealing with the intermittent range between short waves and long waves, that is, for the passage from geometrical optics to wave optics.

In the preparation of the manuscript I was able to rely on the lecture notes of R. Schlatterer for 1935, as well as on earlier notes of Professor J. Meixner. My friend F. Sauter critically perused the entire manuscript and has also been most generous in giving me his own improved version on many points. I owe him more than I can point out in the text. My colleague, J. Lense, examined the manuscript from the mathematical point of view. Dr. F. Renner collaborated on the last chapter especially; H. Schmidt advised me on the arrangement of the material.

ARNOLD SOMMERFELD.

[*Publisher's note:* This is a translation of Sommerfeld's "Lectures on Theoretical Physics," Volume VI. Translations of Volume I entitled, "Mechanics," and Volume II entitled, "Mechanics of Deformable Bodies," are in preparation. In this text they are referred to as v. I and v. II.]

EDITORS' FOREWORD

This book is the first volume in a projected new series of mathematical books to appear under the title "Pure and Applied Mathematics." The books of the new series will be "advanced" in the sense that they will maintain a standard of scientific maturity. It is not intended, however, to adhere to any rigid pattern of presentation or degree of difficulty. Thus there will be a place for textbooks for first-year graduate students as well as monographs for research workers and possibly an occasional treatise. It is the hope of the Editors that these volumes will find a worthy place in the growing list of excellent scientific works which have appeared in recent years.

P. A. S.
S. E.

New York, 1949.

ERRATUM

"Eigenvalues" (see pp. 166ff.) should be written as one word. The two-word form is incorrect.

CHAPTER I

Fourier Series and Integrals

Fourier's *Théorie analytique de la chaleur*¹ is the bible of the mathematical physicist. It contains not only an exposition of the trigonometric series and integrals named after Fourier, but the general boundary value problem is treated in an exemplary fashion for the typical case of heat conduction.

In mathematical lectures on Fourier series emphasis is usually put on the concept of arbitrary function, on its continuity properties and its singularities (accumulation points of an infinity of maxima and minima). This point of view becomes immaterial in the physical applications. For, the initial or boundary values of functions considered here, partially because of the atomistic nature of matter and of interaction, must always be taken as smoothed mean values, just as the partial differential equations in which they enter arise from a statistical averaging of much more complicated elementary laws. Hence we are concerned with relatively simple idealized functions and with their approximation with "least possible error." What is meant by the latter is explained by Gauss in his "Method of Least Squares." We shall see that it opens a simple and rigorous approach not only to Fourier series but to all other series expansions of mathematical physics in spherical and in cylindrical harmonics, or generally in eigenfunctions.

§ 1. Fourier Series

Let an arbitrary function $f(x)$ be given in the interval $-\pi \leq x \leq +\pi$; this function may, e.g., be an empirical curve determined by sufficiently many and sufficiently accurate measurements. We want to approximate it by the sum of $2n + 1$ trigonometric terms

$$(1) \quad S_n(x) = A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_n \cos nx \\ + B_1 \sin x + B_2 \sin 2x + \cdots + B_n \sin nx$$

¹ Jean Baptiste Fourier, 1768–1830. His book on the conduction of heat appeared in 1822 in Paris. Fourier also distinguished himself as an algebraist, engineer, and writer on the history of Egypt, where he had accompanied Napoleon.

The influence of his book even outside France is illustrated by the following quotation: "Fourier's incentive kindled the spark in (the then 16-year-old) William Thomson as well as in Franz Neumann." (F. Klein, *Vorlesungen über die Geschichte der Mathematik im 19. Jahrhundert*, v. I, p. 233.)

By what criterion shall we choose the coefficients A_k, B_k at our disposal? We shall denote the error term $f(x) - S_n(x)$ by $\varepsilon_n(x)$; thus

$$(2) \quad f(x) = S_n(x) + \varepsilon_n(x).$$

Following Gauss we consider the *mean square error*

$$(3) \quad M = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \varepsilon_n^2 dx$$

and reduce M to a minimum through the choice of the A_k, B_k .

To this we further remark that the corresponding measure of the total error formed with the first power of ε_n would not be suitable, since arbitrarily large positive and negative errors could then cancel each other and would not count in the total error. On the other hand the use of the absolute value $|\varepsilon_n|$ under the integral sign in place of ε_n^2 would be inconvenient because of its non-analytic character.²

The requirement that (3) be a minimum leads to the equations

$$(4) \quad \begin{aligned} -\frac{\partial M}{\partial A_k} &= \frac{1}{\pi} \int_{-\pi}^{+\pi} \{f(x) - S_n(x)\} \cos kx dx = 0, \quad k = 0, 1, 2, \dots, n \\ -\frac{\partial M}{\partial B_k} &= \frac{1}{\pi} \int_{-\pi}^{+\pi} \{f(x) - S_n(x)\} \sin kx dx = 0, \quad k = 1, 2, \dots, n. \end{aligned}$$

These are exactly $2n + 1$ equations for the determination of the $2n + 1$ unknowns A, B . A favorable feature here is that each individual coefficient A or B is determined directly and is not connected recursively with the other A, B . We owe this to the *orthogonality relations* that exist among trigonometric functions:³

$$\begin{aligned} (5) \quad & \int \cos kx \sin lx dx = 0, \\ (5a) \quad & \int \cos kx \cos lx dx \\ (5b) \quad & \int \sin kx \sin lx dx \end{aligned} \left. \vphantom{\begin{aligned} (5) \\ (5a) \\ (5b) \end{aligned}} \right\} = 0, \quad k \neq l.$$

² A completely different approach is taken by the great Russian mathematician Tchebycheff in the approximation named after him. He considers not the *mean* but the *maximal* $|\varepsilon_n|$ appearing in the interval of integration, and makes this a minimum through the choice of the coefficients at his disposal.

³ Here and below all integrals are to be taken from $-\pi$ to $+\pi$. In order to justify the word "orthogonality" we recall that two vectors u, v which are orthogonal in Euclidean three dimensional, or for that matter n -dimensional space, satisfy the condition that their scalar product

In order to prove them it is not necessary to write down the cumbersome addition formulae of trigonometric functions, but to think rather of their connection with the exponential functions $e^{\pm i k x}$ and $e^{\pm i l x}$. The integrands of (5a,b) consist then of only four terms of the form $\exp \{ \pm i (k + l) x \}$ or $\exp \{ \pm i (k - l) x \}$, all of which vanish upon integration unless $l = k$. This proves (5a,b). The fact that (5) is valid even without this restriction follows from the fact that for $l = k$ it reduces to

$$\frac{1}{4i} \int (e^{2ikx} - e^{-2ikx}) dx = 0$$

In a similar manner one obtains the values of (5a,b) for $l = k > 0$ (only the product of $\exp(ikx)$ and $\exp(-ikx)$ contributes to it): this value simply becomes equal to π ; for $l = k = 0$ the value of the integral in (5a) obviously equals 2π . We therefore can replace (5a,b) by the single formula which is valid also for $l = k > 0$

$$(6) \quad \frac{1}{\pi} \int \cos kx \cos lx dx = \frac{1}{\pi} \int \sin kx \sin lx dx = \delta_{kl}$$

with the usual abbreviation

$$\delta_{kl} = \begin{cases} 0 & \dots l \neq k \\ 1 & \dots l = k > 0. \end{cases}$$

Equation (6) for $k = l$ is called the *normalizing condition*. It is to be augmented for the exceptional case $l = k = 0$ by the trivial statement

$$(6a) \quad \frac{1}{2\pi} \int dx = 1.$$

If we now substitute (5),(6) and (6a) in (4) then in the integrals with S_n all terms except the k -th vanish, and we obtain directly *Fourier's representation of coefficients*:

$$(7) \quad \left. \begin{aligned} A_k &= \frac{1}{\pi} \int f(x) \cos kx dx \\ B_k &= \frac{1}{\pi} \int f(x) \sin kx dx \end{aligned} \right\} k > 0, \\ A_0 &= \frac{1}{2\pi} \int f(x) dx.$$

$$(u v) = \sum_1^N u_i v_i = 0$$

vanish. The integrals appearing in (5) can be considered as sums of this same type with infinitely many terms. See the remarks in §26 about so-called "Hilbert space."

Our approximation S_n is hereby determined completely. If, e.g. $f(x)$ were given empirically then the integrations (7) would have to be carried out numerically or by machine.⁴

From (7) one sees directly that for an even function $f(-x) = f(+x)$, all B_k vanish, whereas for an odd function, $f(-x) = -f(+x)$, all A_k , including A_0 , vanish. Hence the former is approximated by a *pure cosine series*, the latter by a *pure sine series*.

The accuracy of the approximation naturally increases with the number of constants A, B at our disposal, i.e., with increasing n . Here the following fortunate fact should be stressed: since the A_k, B_k for $k < n$ are independent of n , the previously calculated A_k, B_k remain unchanged by the passage from n to $n + 1$, and only the coefficients A_{n+1}, B_{n+1} have to be newly calculated. The A_k, B_k , once found, are *final*.

There is nothing to prevent us from letting n grow indefinitely, that is, to perform the passage to the limit $n \rightarrow \infty$. The finite series considered so far thereby goes over into an *infinite Fourier series*. The following two sections will deal with its convergence.

More complicated than the question of convergence is that of the *completeness* of the system of functions used here as basis. It is obvious that if in the Fourier series one of the terms, e.g., the k -th cosine term, were omitted, then the function $f(x)$ could no longer be described by the remaining terms with arbitrary accuracy; even in passing to the limit $n \rightarrow \infty$ a finite error $A_k \cos kx$ would remain. To take an extremely simple case, if one attempted to express $\cos nx$ by an incomplete series of all cosine terms with $k < n$ and $k > n$, then all A_k would vanish because of orthogonality and the error would turn out to be $\cos nx$ itself. Of course it would not occur to anyone to disturb the regularity of a system like that of the trigonometric functions by the omission of one term. But in more general cases such considerations of mathematical esthetics need not be compelling.

What the mathematicians teach us on this question with their *relation of completeness* is in reality no more than is contained in the basis of the method of least squares. One starts, namely, with the remark that a system of functions say $\varphi_0, \varphi_1, \dots, \varphi_k, \dots$, can be complete only if for every continuous function $f(x)$ the mean error formed according to (3) goes to zero in the limit $n \rightarrow \infty$. It is assumed that the system of φ is orthogonal and normalized to 1, that is

$$(8) \quad \int \varphi_k \varphi_l dx = 0, \quad \int \varphi_k^2 dx = 1,$$

⁴ Integrating machines that serve in Fourier analysis are called "harmonic analyzers." The most perfect of these is the machine of Bush and Caldwell; it can be used also for the integration of arbitrary simultaneous differential equations; see *Phys. Rev.* **38**, 1898 (1931).

which implies that the expansion coefficients A_k are simply

$$(9) \quad A_k = \int f(x) \varphi_k(x) dx.$$

Let the limits of integration in this and the preceding integrals be a and b so that the length of the interval of expansion is $b - a$. One then forms according to (3)

$$(b - a) M = \int \left(f - \sum_{k=0}^n A_k \varphi_k \right)^2 dx = \int f^2 dx - 2 \sum_{k=0}^n A_k \int f \varphi_k dx + \sum_{k=0}^n A_k^2.$$

Equation (8) has been used in the last term here. According to (9) the middle term equals twice the last term except for sign. Hence

$$\lim_{n \rightarrow \infty} (b - a) M = \int f^2 dx - \sum_{k=0}^{\infty} A_k^2$$

and one requires, as remarked above, that for every continuous function

$$(10) \quad \sum A_k^2 = \int f^2 dx.$$

This is the mathematical formulation of the *relation of completeness* which is so strongly emphasized in the literature. It is obvious that it can hardly be applied as a practical criterion. Also, since it concerns only the mean error, it says nothing on the question of whether the function f is really represented everywhere by the Fourier series (see also §3, p. 15).

In this introductory section we have followed the historical development in deducing the *finality of the Fourier coefficients* from the *orthogonality of the trigonometric functions*. In §4 we shall demonstrate, for the typical case of spherical harmonics, that, conversely, orthogonality can be deduced quite generally from our requirement of *finality*. From our point of view of approximation this seems to be the more natural approach. In any case it should be stressed at this point that *orthogonality* and *requirement of finality* imply each other and can be replaced by each other.

Finally, we want to translate our results into a form that is mathematically more perfect and physically more useful. We carry this out for the case of infinite Fourier series, remarking however, that the following is valid also for a truncated series — actually the more general and rigorous case.

We write, replacing the variable of integration in (7) by ξ :

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int f(\xi) d\xi + \\
&\quad \frac{1}{\pi} \sum_{k=1}^{\infty} \int f(\xi) \cos k\xi d\xi \cdot \cos kx + \frac{1}{\pi} \sum_{k=1}^{\infty} \int f(\xi) \sin k\xi d\xi \cdot \sin kx \\
&= \frac{1}{2\pi} \int f(\xi) d\xi + \frac{1}{\pi} \sum_{k=1}^{\infty} \int f(\xi) \cos k(x-\xi) d\xi \\
&= \frac{1}{2\pi} \left\{ \int f(\xi) d\xi + \sum_{k=1}^{\infty} \left(\int f(\xi) e^{ik(x-\xi)} d\xi + \int f(\xi) e^{-ik(x-\xi)} d\xi \right) \right\}.
\end{aligned}$$

In the last term we can consider the summation for positive k in $\exp\{-ik(x-\xi)\}$ to be the summation for the corresponding negative values of k in $\exp\{+ik(x-\xi)\}$. We therefore replace this term by

$$\sum_{k=-1}^{-\infty} \int f(\xi) e^{ik(x-\xi)} d\xi = \sum_{k=-\infty}^{-1} \int f(\xi) e^{ik(x-\xi)} d\xi.$$

Then the uncomfortable exceptional position of the term $k=0$ is removed: it now fits between the positive and negative values of k and we obtain

$$(11) \quad f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \int f(\xi) e^{ik(x-\xi)} d\xi.$$

Finally, introducing the Fourier coefficients C_k , which are *complex* even for real $f(x)$:

$$(12) \quad f(x) = \sum_{k=-\infty}^{+\infty} C_k e^{ikx}, \quad C_k = \frac{1}{2\pi} \int f(\xi) e^{-ik\xi} d\xi.$$

The relations among the C 's and the A 's and B 's defined by (7), are given by

$$\begin{aligned}
(13) \quad C_k &= \begin{cases} \frac{1}{2} (A_k - i B_k), & k > 0, \\ \frac{1}{2} (A_{|k|} + i B_{|k|}), & k < 0, \end{cases} \\
C_0 &= A_0.
\end{aligned}$$

Our complex representation (12) is obviously simpler than the usual real representation; it will be of special use to us in the theory of Fourier integrals.

If we extend our representation, originally intended for the interval

$-\pi < x < +\pi$, to the intervals $x > \pi$ and $x < -\pi$ then we obtain continued periodic repetitions of the branch between $-\pi$ and $+\pi$; in general they do not constitute the analytic continuation of our original function $f(x)$. In particular the periodic function, thus obtained will have *discontinuities* for the odd multiples of $\pm\pi$, unless we happen to have $f(-\pi) = f(+\pi)$. The next section deals with the investigation of the error arising at such a point.

§ 2. Example of a Discontinuous Function. Gibbs' Phenomenon and Non-Uniform Convergence

Let us consider the function

$$(1) \quad f(x) = \begin{cases} +1 & \text{for } 0 < x < \pi \\ -1 & \text{for } -\pi < x < 0. \end{cases}$$

We sketch it in Fig. 1 with its periodic repetitions completed by the vertical connecting segments of length 2 at the points of discontinuity $x = 0, \pm\pi, \pm 2\pi, \dots$, whereby it becomes a "meander line." Our function f is odd, its Fourier series consists therefore solely of sine terms as pointed out in (1.7). The coefficients can best be calculated from equation (1.12), which yields

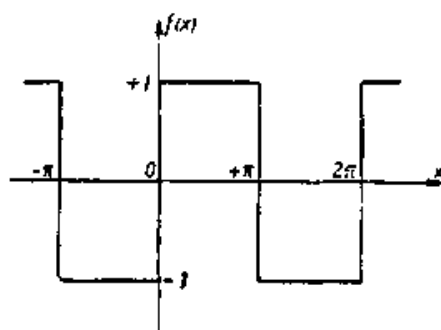


Fig. 1. The chain of segments $y = \pm 1$ for positive and negative $|x| < \pi$ and its periodic repetition represented by the Fourier series.

$$(1a) \quad \begin{aligned} C_k &= \frac{1}{2\pi} \left(\int_0^{\pi} e^{-ik\xi} d\xi - \int_{-\pi}^0 e^{-ik\xi} d\xi \right) \\ &= \frac{1}{2\pi} \left(\frac{e^{-ik\pi} - 1}{-ik} - \frac{1 - e^{+ik\pi}}{-ik} \right) = \frac{(-1)^k - 1}{-i\pi k} = \begin{cases} -\frac{2i}{\pi k} & \dots k \text{ odd} \\ 0 & \dots k \text{ even} \end{cases} \end{aligned}$$

This implies according to (1.13):

$$B_k = \frac{4}{\pi k}, \quad k = 1, 3, 5, \dots$$

We obtain the following sine series:

$$(2) \quad f(x) = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right).$$

One may imagine the upheaval caused by this series when it was first constructed by Fourier. A discontinuous chain formed through the

superposition of an infinite sequence of only the simplest continuous functions! Without exaggeration one may say that this series has contributed greatly to the development of the general concept of real function. We shall see below that it also served to deepen the concept of convergence of series.

In order to understand how the series manages to approximate the discontinuous sequence of steps, we draw⁵ in Fig. 2 the approximating functions S_1, S_3, S_5 defined by (1.1) together with $S_\infty = f(x)$.

$$S_1 = \frac{4}{\pi} \sin x, \quad S_3 = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x \right),$$

$$S_5 = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x \right).$$

S_1 has its maximum value

$$y = 4/\pi = 1.27,$$

at $x = \pi/2$, and hence rises 27% above the horizontal line $y = 1$, which is to be described. S_3 has a minimum value at the same point and hence

$$y = \frac{4}{\pi} \left(1 - \frac{1}{3} \right) = 0.85,$$

stays 15% below the straight line to be described. In addition S_3 also has maxima at $\pi/4$ and $3\pi/4$, which lie 20% above that line. (The reader is invited to check this!) S_5 on the other hand has a maximum of

$$y = \frac{4}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} \right) = 1.10,$$

at $x = \pi/2$ which is too high by only 10%. A flat minimum on either side is followed by two steeper maxima situated near $x = 0$ and $x = \pi$. In general the maxima and minima of S_{2n+1} lie *between* those of S_{2n-1} (see exercise I.1).

All that has been said here about the stepwise approximation of the line $y = +1$, is of course equally valid for its mirror image $y = -1$. It too is approximated by *successive oscillations*, so that the approximating curve S_n swings n times above and $n + 1$ times below the line segment which is to be represented. The oscillations in the *middle part* of the line segment decrease with increasing n ; at the *points of discontinuity*

⁵ In the lectures at this point abundant use was made of colored chalk. Since this unfortunately is impossible in print, both S_∞ and the approximation S_1 , which are the most important for us, are drawn in bolder lines.

$x = 0, \pm\pi, \pm 2\pi, \dots$, where there is no systematic decrease of the maxima, the approximating curves approach the vertical jumps of discontinuity. The picture of an approximating curve of very large n therefore looks the way it has been pictured schematically in Fig. 3.

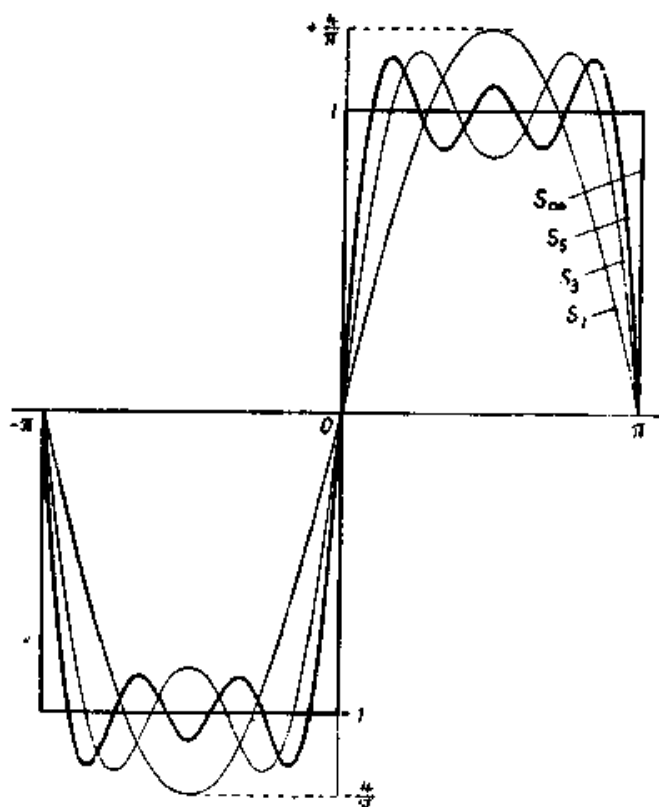


Fig. 2. The approximations of the chain S_∞ ; the maxima and minima lie at equally spaced values of x , respectively between those of the preceding approximation.

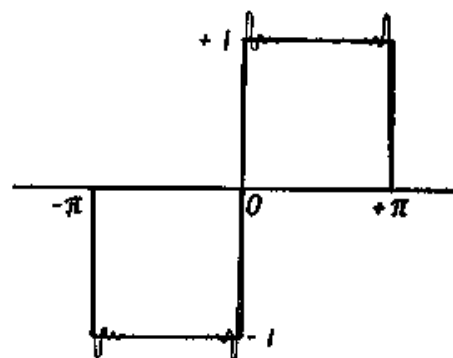


Fig. 3. An approximation S of very high order for the illustration of Gibbs' phenomenon.

We now consider more closely the behavior of $S_{2n+1}(x)$ for large n at one of the jumps, e.g., for $x = 0$. To this end we rewrite the original formula for S_{2n+1} in integral form (an integral usually being easier to discuss than a sum). This is done in the following steps:

$$\begin{aligned} S_{2n+1} &= \frac{4}{\pi} \sum_{k=0}^n \frac{\sin(2k+1)x}{2k+1} = \frac{4}{\pi} \sum_{k=0}^n \int_0^x \cos(2k+1)\xi \, d\xi \\ &= \frac{2}{\pi} \int_0^x \left\{ \sum_{k=0}^n e^{(2k+1)i\xi} + \sum_{k=0}^n e^{-(2k+1)i\xi} \right\} d\xi. \end{aligned}$$

After factoring out $\exp(\pm i\xi)$ from the two sums of the last line they become geometric series in increasing powers of $\exp(\pm 2i\xi)$ which can be summed in the familiar manner. Therefore, one obtains

$$(3) \quad S_{2n+1} = \frac{2}{\pi} \int_0^x \left\{ e^{i\xi} \frac{1 - e^{2i(n+1)\xi}}{1 - e^{2i\xi}} + e^{-i\xi} \frac{1 - e^{-2i(n+1)\xi}}{1 - e^{-2i\xi}} \right\} d\xi.$$

By further factorization these two fractions can be brought to the common form (except for the sign of i):

$$(3a) \quad e^{\pm in\xi} \frac{\sin(n+1)\xi}{\sin\xi}.$$

In this way (3) goes over into

$$(3b) \quad S_{2n+1} = \frac{2}{\pi} \int_0^x \frac{2 \cos(n+1)\xi \sin(n+1)\xi}{\sin\xi} d\xi.$$

Finally for sufficiently small x we can replace $\sin\xi$ in the denominator by ξ ; the corresponding simplification in the numerator would not be permissible since ξ there is accompanied by the large factor $n+1$. We obtain therefore for (3a), if we introduce the new variable of integration u and the new argument v ,

$$(4) \quad S_{2n+1} = \frac{2}{\pi} \int_0^v \frac{\sin u}{u} du \dots \quad \begin{cases} u = 2(n+1)\xi, \\ v = 2(n+1)x. \end{cases}$$

From this the following conclusion may be drawn: if for finite n we set $x = 0$ then v vanishes and $S_{2n+1} = 0$. If now we allow n to increase toward infinity, the relation $S_{2n+1} = 0$ holds in the limit. Hence

$$(4a) \quad \lim_{n \rightarrow \infty} \lim_{x \rightarrow 0} S_{2n+1} = 0.$$

But if for $x > 0$ we first allow n to approach infinity, then v becomes infinite, and, according to a fundamental formula that will be treated in exercise I.5: $S_{2n+1} = 1$. If we then allow x to decrease towards zero, the value $S_{2n+1} = 1$ holds also for the limit $x = 0$; hence

$$(4b) \quad \lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} S_{2n+1} = 1.$$

The two limiting processes therefore are not interchangeable. If the function $f(x)$ to be represented were continuous at the point $x = 0$, then the order

of passage to the limit would be immaterial, and in contrast to (4a,b) one would have

$$(4c) \quad \lim_{n \rightarrow \infty} \lim_{x \rightarrow 0} S_{2n+1} = \lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} S_{2n+1} = f(v).$$

This, however, does not exhaust by any means the peculiarities contained in equation (4); in order to develop them we introduce the frequently tabulated⁶ "integral sine"

$$(5) \quad Si(v) = \int_0^v \frac{\sin u}{u} du$$

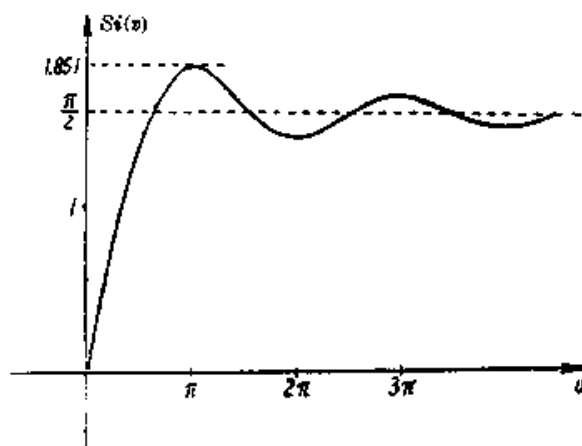


Fig. 4. Graphic representation of the integral sine.

and represent its general form in Fig. 4. It can be described as follows: for small values of v , where $\sin u$ can be set equal to u , we have *proportionality with v* ; for large values of v we have *asymptotic approach to $\pi/2$* ; in between we have *successively decreasing oscillations* with maxima and minima at $v = \pi, 2\pi, 3\pi, \dots$, as can be seen from (5); the ordinate of the first and greatest maximum is 1.851 according to the above mentioned tables. To the associated abscissa of the Si -curve there corresponds in the original variable x , owing to the relation $v = 2(n+1)x$, the infinite sequence of points

$$(6) \quad \dots, x_n = \frac{\pi}{2(n+1)}, \quad x_{n+1} = \frac{\pi}{2(n+2)}, \dots$$

at which according to (4) the approximations $S_{2n+1}, S_{2n+3}, \dots$ have the fixed value:

$$(7) \quad S = \frac{2}{\pi} 1.851 = 1.18.$$

This value, which exceeds $y = 1$ by 18%, is at the same time the *upper limit* of the range of values given by our approximations. Its *lower limit*, $S = -1.18$ is assumed when we approach zero from the negative side in the sequence of points $-x_n, -x_{n+1}, \dots$. Each point of the range

⁶ E.g. B. Jahnke-Emde, *Funktionentafeln*, Teubner, Leipzig, 3d edition, 1938.

between -1.18 and $+1.18$ can be obtained by a special manner of passing to the limit; e.g., the points $S = 0$ and $S = 1$ are obtained in the manner described in (4a) and (4b).

This behavior of the approximating functions, in particular the appearance of an excess over the range of discontinuity ± 1 , is called *Gibbs' phenomenon*. (Willard Gibbs, 1844 to 1906, was one of America's greatest physicists, and simultaneously with Boltzmann, was the founder of statistical mechanics.) Gibbs' phenomenon appears wherever a discontinuity is approximated. One then speaks of the *non-uniform convergence* of the approximation process.

We still want to convince ourselves that actually every point between $S = 1.18$ and $S = -1.18$ can be obtained if we *couple* the two passages to the limit in a suitable fashion. According to (6), this coupling consists in setting $x(n+1)$ or, what comes to the same thing, setting x_n equal to the fixed value $\pi/2$. If instead we take the more general value, q , then from (4) we obtain $v = 2q$, and (4) and (5) together yield

$$S_{2n+1} = \frac{2}{\pi} Si(2q),$$

where $Si(2q)$ can assume all values between 0 and 1.851 with varying positive q , as can be seen directly from Fig. 4. Correspondingly for negative q one obtains all values between 0 and -1.851 . The passages to the limit that have thus been coupled yield not only the approach of our approximating function to the discontinuity from -1 to $+1$, but also an excess beyond it, i.e., Gibbs' phenomenon.

In addition to these basic statements we want to deduce some formal mathematical facts from our Fourier representation (2). In particular we substitute $x=\pi/2$ therein and obtain the famous *Leibniz series*

$$(8) \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

This series converges slowly; we obtain more rapidly convergent representations for the powers of π if we integrate (2) repeatedly. For the following refer to Fig. 5 below.

By restricting ourselves to the interval $0 < x < \pi$, we write

$$(9) \quad \frac{\pi}{4} = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots$$

instead of (2). Integration from 0 to x yields:

$$(10) \quad \frac{\pi}{4} x = 1 - \cos x + \frac{1}{3^2} (1 - \cos 3x) + \frac{1}{5^2} (1 - \cos 5x) + \dots$$

Hence for $x = \pi/2$

$$(11) \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Subtracting (10) from (11) we get:

$$(12) \quad \frac{\pi}{4} \left(\frac{\pi}{2} - x \right) = \cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots$$

By another integration from 0 to x this becomes

$$(13) \quad \frac{\pi}{8} (\pi x - x^2) = \sin x + \frac{1}{3^3} \sin 3x + \frac{1}{5^3} \sin 5x + \dots$$

Hence for $x = \pi/2$, as an *analogue to the Leibniz series*

$$(14) \quad \frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots$$

We integrate (13) once more with respect to x and set $x = \pi/2$:

$$(15) \quad \frac{\pi}{8} \left(\pi \frac{x^2}{2} - \frac{x^3}{3} \right) = 1 - \cos x + \frac{1}{3^4} (1 - \cos 3x) + \frac{1}{5^4} (1 - \cos 5x) + \dots$$

$$(16) \quad \frac{\pi^4}{3 \cdot 32} = 1 - \frac{1}{3^4} + \frac{1}{5^4} - \dots$$

Finally we subtract (15) from (16) and have

$$(17) \quad \frac{\pi}{8} \left(\frac{\pi^3}{12} - \frac{\pi x^2}{2} + \frac{x^3}{3} \right) = \cos x + \frac{1}{3^4} \cos 3x + \frac{1}{5^4} \cos 5x + \dots$$

The series (11) and (16) range only over the *odd* numbers. The series ranging over the *even* numbers are respectively equal to $1/4$ and $1/16$ of the sums ranging over *all* integers. If we denote the latter by Σ_2 and Σ_4 respectively, then we have

$$\frac{\pi^2}{8} + \frac{1}{4} \Sigma_2 = \Sigma_2 \quad \text{and} \quad \frac{\pi^4}{3 \cdot 32} + \frac{1}{16} \Sigma_4 = \Sigma_4,$$

hence

$$(18) \quad \Sigma_2 = \frac{\pi^2}{6} \quad \text{and} \quad \Sigma_4 = \frac{\pi^4}{90}.$$

This value of Σ_4 was needed in the derivation of Stefan's law of radiation or Debye's law for the energy content of a fixed body. The trigonometric series (12), (13), (17) will be useful examples in the following sections. The higher analogues to the "Leibniz series" (8) and (14) as well as those to Σ_2 and Σ_4 will be computed in exercise I.2.

§ 3. On the Convergence of Fourier Series

We are going to prove the following theorem: If a function $f(x)$, together with its first $n - 1$ derivatives is continuous and differentiable between $-\pi$ and $+\pi$ inclusive, and the n -th derivative, is differentiable over the same interval except possibly at a finite number of points $x = x_i$ where it may have bounded discontinuities (i.e., finite jumps), then the coefficients A_k, B_k of its Fourier expansion approach zero at least as fast as k^{-n-1} as $k \rightarrow \infty$.

The stipulation "inclusive" in referring to the boundaries of the interval has here the following meaning: every function which is represented by a Fourier series is periodic in nature. An adequate picture of its argument would therefore not be the straight line segment from $-\pi$ to $+\pi$, but a circle closing at $x = \pm\pi$. It is this fact to which the *continuity* of f and its first $n - 1$ derivatives at the point $x = \pm\pi$ refers. This point is in no way distinguished from the interior points of the interval, just as it is immaterial whether we denote the boundaries of the interval by $-\pi, +\pi$ or, e.g., by $\frac{\pi}{4}, \frac{9\pi}{4}$ etc.

For the proof of this theorem it is convenient to use the complex form (1.12)

$$(1) \quad f(x) = \sum_{-\infty}^{+\infty} C_k e^{ikx}, \quad (1a) \quad 2\pi C_k = \int_{-\pi}^{+\pi} f(\xi) e^{-ik\xi} d\xi$$

From (1a) one obtains through integration by parts

$$(2) \quad 2\pi C_k = \frac{1}{-ik} f(\xi) e^{-ik\xi} \Big|_{-\pi}^{+\pi} + \frac{1}{ik} \int_{-\pi}^{+\pi} f'(\xi) e^{-ik\xi} d\xi.$$

Here the first term on the right side vanishes because of the postulated continuity of f ; the second term can again be transformed by integration by parts. After n iterations of the same process one obtains

$$(3) \quad 2\pi (ik)^n C_k = \int_{-\pi}^{+\pi} f^{(n)}(\xi) e^{-ik\xi} d\xi.$$

Because of the discontinuities of $f^{(n)}(x)$ at $x = x_l$, this integral has to be divided into partial integrals between $x = x_l$ and $x = x_{l+1}$; let the jumps of $f^{(n)}$ at the points of discontinuity be denoted by Δ_l^n . Equation (3) written explicitly then reads:

$$(3a) \quad 2\pi(i k)^n C_k = \sum_l \int_{x_l}^{x_{l+1}} f^{(n)}(\xi) e^{-ik\xi} d\xi,$$

where the point $x = \pm\pi$ may be contained among the points $x = x_l$. By one more partial integration (3a) becomes

$$(4) \quad 2\pi(i k)^n C_k = \frac{1}{-ik} \sum_l \Delta_l^n e^{-ikx_l} + \frac{1}{ik} \sum_l \int_{x_l}^{x_{l+1}} f^{(n+1)}(\xi) e^{-ik\xi} d\xi.$$

Considering the fact that the discontinuities Δ_l^n were assumed to be bounded and that $f^{(n)}$ was assumed to be differentiable between the points of discontinuity, one sees from (4) that C_k vanishes at least to the same order as k^{-n-1} when one lets $k \rightarrow \infty$. For special relations between the Δ_l^n or for special behavior of $f^{(n+1)}(\xi)$, the order of vanishing could become even higher.

This theorem is valid for negative k too. This implies that it is valid also for the *real Fourier coefficients* A_k, B_k ($k > 0$), since according to (1.13) they are expressible in terms of the C_k with positive and negative k .

A special consequence of our theorem is that an *analytic function* of period 2π (such a function is continuous and periodic together with all its derivatives) has Fourier coefficients that decrease faster than any power of $1/k$ with increasing k . An example of this would be an arbitrary polynomial in $\sin x$ and $\cos x$. This is represented by a *finite* Fourier series with as many terms as required by the degree of the polynomial, so that all higher Fourier coefficients are equal to zero. Another example is given by the elliptic ϑ series, which we shall meet in a heat conduction problem in §15; its Fourier coefficients C_k decrease as fast as e^{-ak^2} .

It further follows from our theorem that the sum $\sum A_k^2$ which appears in the relation of completeness converges like $\sum k^{-2}$ for every function $f(x)$ which has a finite number of jumps and which is differentiable everywhere else (case $n = 0$ of our theorem). An example of this is given by our function (2.1) where $\sum A_k^2$ converges, although $\sum A_k$ diverges. This function also shows that the relation of completeness does not insure representability of the function at every point (this has already been noted on p. 5). Namely, if we sharpen definition (2.1) by putting $f = 1$ for $x \geq 0$ and $f = -1$ for $x < 0$,

then f if not represented by the Fourier series (2.2) at the point $x = 0$, for there the series converges to 0.

A further illustration of our theorem is given by the sine and cosine series which were derived at the end of the last section. The expressions of the functions which are represented by these series were valid only for the interval $0 < x < \pi$. We complete these expressions by adjoining the corresponding expressions for the interval $-\pi < x < 0$. The latter are obtained simply from the remark that the cosine series are even functions of x , and the sine series are odd. The expressions thus obtained are written below inside the $\{ \}$ to the right of the semicolon. We therefore complete the equations (2.9), (2.12), (2.15), (2.17) as follows:

$$(5) \left\{ \frac{\pi}{4}; -\frac{\pi}{4} \right\} = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots$$

$$(6) \left\{ \frac{\pi}{4} \left(\frac{\pi}{2} - x \right); \frac{\pi}{4} \left(\frac{\pi}{2} + x \right) \right\} = \cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots$$

$$(7) \left\{ \frac{\pi}{8} (\pi x - x^2); \frac{\pi}{8} (\pi x + x^2) \right\} = \sin x + \frac{1}{3^3} \sin 3x + \frac{1}{5^3} \sin 5x + \dots$$

$$(8) \left\{ \frac{\pi}{8} \left(\frac{\pi^3}{12} - \frac{\pi x^2}{2} + \frac{x^3}{3} \right); \frac{\pi}{8} \left(\frac{\pi^3}{12} - \frac{\pi x^2}{2} - \frac{x^3}{3} \right) \right\} = \cos x + \frac{1}{3^4} \cos 3x + \dots$$

Here the functions which are represented possess successively stronger continuity properties: in (5) the function possesses discontinuities at the points $x = 0$ and $x = \pm \pi$, in (6) the function is continuous but the first derivative is discontinuous, in (7) the function and its first derivative are continuous but the second derivative is discontinuous; in (8) the function and its first two derivatives are continuous but the third derivative is not. The discontinuity arising in each case is the same as that of the function in (5) and it appears at the same points $x = 0$ and $x = \pm \pi$ corresponding to the fact that each succeeding function was obtained from the previous one by integration.

Figure 5 illustrates this. Its curves 0,1,2,3 represent the left sides of (5),(6),(7),(8). The discontinuity of the tangent to the curve 1 at $x = 0$ strikes the eye; the discontinuity of the curvature of 2 at $x = 0$ can be deduced from the behavior of the two mirror image parabolas which meet there. Curve 3 consists of two cubic parabolas, that osculate with continuous curvature. The scale, which for convenience has been chosen differently for the different curves, can be seen by the ordinates of the maximal values which have been inserted on the right hand side.

The *increasing continuity* of our curves 0 to 3 has its counter-

part in the *increasing rate of convergence* of the Fourier series on the right sides of eqs. (5) to (8): in (5) we have a decrease of the coefficients like $1/k$, in general, in accord with our theorem, we have a decrease with k^{-n-1} , where n is the order of the first discontinuous derivative of the represented function.

The convergence of Fourier series stands in a marked contrast to that of Taylor series. The former depends only on the continuity of the function to be represented and its derivatives on the real axis, the latter depends also on the position of the singularities in the complex domain. (Indeed the singular point nearest the origin of expansion in the complex plane determines the radius of convergence of the Taylor series.) Accordingly the principles of the two expansions are basically different: for Fourier series we have an *oscillating approach* over the entire range of the interval of representation, for Taylor series we have an *osculating approach* at its origin. We shall return to this in §6.

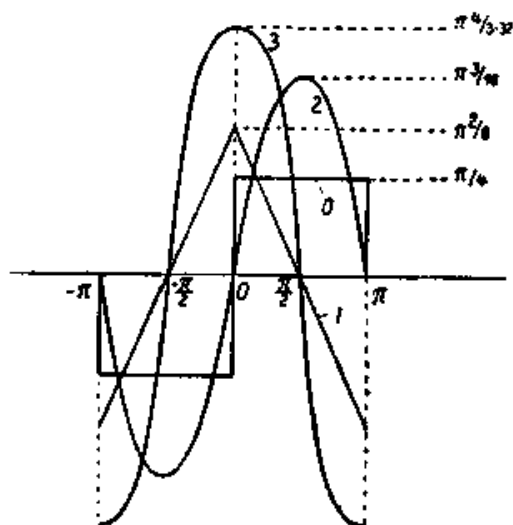


Fig. 5. Four curves 0,1,2,3, obtained by successive integration. Increasing continuity at $x = 0$: 0 discontinuous in the ordinate, 1 in the tangent, 2 in the curvature, 3 in the third derivative.

§ 4. Passage to the Fourier Integral

The interval of representation $-\pi < x < \pi$ can be changed in many ways. Not only can it be displaced, as remarked on p. 14, but also its length can be changed, e.g., to $-a < z < +a$ for arbitrary a . This is done by the substitution

$$(1) \quad x = \frac{\pi z}{a},$$

which transforms (1.7) into

$$(2) \quad \left. \begin{matrix} A_k \\ B_k \end{matrix} \right\} = \frac{1}{a} \int_{-a}^{+a} f(z) \frac{\cos \frac{\pi k z}{a}}{\sin \frac{\pi k z}{a}} dz, \quad A_0 = \frac{1}{2a} \int_{-a}^{+a} f(z) dz.$$

In the more convenient complex way of writing (1.12), one then has

$$(3) \quad f(z) = \sum_{-\infty}^{+\infty} C_k e^{i \frac{\pi}{a} k z}, \quad C_k = \frac{1}{2a} \int_{-a}^{+a} f(\zeta) e^{-i \frac{\pi}{a} k \zeta} d\zeta.$$

We may obviously consider also the more general interval $b < z < c$, by substituting

$$(4) \quad x = \alpha z + \beta, \quad \alpha = \frac{2\pi}{c-b}, \quad \beta = -\pi \frac{c+b}{c-b}$$

The formulas (2) then become

$$(5) \quad \left. \begin{matrix} A_k \\ B_k \end{matrix} \right\} = \frac{2}{c-b} \int_b^c f(z) \frac{\cos}{\sin} k(\alpha z + \beta) dz, \quad A_0 = \frac{1}{c-b} \int_b^c f(z) dz.$$

In this connection we mention some "pure sine and cosine series" that appear in Fourier's work. One considers a function $f(x)$ which is given only in the interval $0 < x < \pi$ say, and which is to be continued to the negative side in an odd or even manner. For example, one gets for odd continuation

$$f(x) = \sum_{k=1}^{\infty} B_k \sin kx, \quad B_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx dx,$$

See also exercise I.3.

Starting from (3) we take a to be very large. The sequence of values

$$\omega_k = \frac{\pi}{a} k$$

then becomes dense, for which reason we shall write ω instead of ω_k from now on. For the difference of two consecutive ω_k we write correspondingly

$$d\omega = \frac{\pi}{a}, \quad \frac{1}{a} = \frac{d\omega}{\pi}.$$

If in (3) we replace the symbols z, ζ by the previous ones x, ξ then we obtain

$$(6) \quad C_k = \frac{d\omega}{2\pi} \int_{-a}^{+a} f(\xi) e^{-i\omega\xi} d\xi.$$

For the moment we avoid calling the limits of this integral $-\infty$ and $+\infty$

Introducing (6) into the infinite series (3) for $f(x)$, replacing the

summation by integration, and denoting the limits of integration for the time being by $\pm \Omega$, we get:

$$(7) \quad f(x) = \lim_{\Omega \rightarrow \infty} \lim_{a \rightarrow \infty} \frac{1}{2\pi} \int_{-\Omega}^{+\Omega} e^{i\omega x} d\omega \int_{-a}^{+a} f(\xi) e^{-i\omega \xi} d\xi.$$

The order of passage to the limit indicated here is obviously necessary: if the passage to the limit $\Omega \rightarrow \infty$ were carried out first, we would obtain the completely meaningless integral

$$\int_{-\infty}^{+\infty} e^{i\omega(x-\xi)} d\omega$$

On the other hand $f(\xi)$ must vanish for $\xi \rightarrow \pm \infty$ in order that the first limit for $a \rightarrow \infty$ have a meaning. We do not have to investigate how fast $f \rightarrow 0$ in order that the other passage to the limit be possible, since for all suitably formulated physical problems this convergence to 0 will be "sufficiently rapid."

After this preliminary discussion we shall further abbreviate the more exact form of (7) by writing:

$$(8) \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\xi) e^{i\omega(x-\xi)} d\xi.$$

From this we pass to the *real* form of the Fourier integral (8) as it is commonly given in the literature. We set

$$e^{i\omega(x-\xi)} = \cos \omega(x-\xi) + i \sin \omega(x-\xi).$$

Here the sine is an odd function of ω , and hence vanishes on integration from $-\infty$ to $+\infty$; the cosine, being even in ω , yields twice the integral taken from 0 to ∞ . We therefore have

$$(9) \quad f(x) = \frac{1}{\pi} \int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\xi) \cos \omega(x-\xi) d\xi,$$

by which we do not wish to imply that the real form is better or simpler than our complex form (8). We can write instead of (9):

$$(10) \quad f(x) = \int_0^{\infty} a(\omega) \cos \omega x d\omega + \int_0^{\infty} b(\omega) \sin \omega x d\omega$$

where

$$(10\ a) \quad a(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(\xi) \cos \omega \xi d\xi, \quad b(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(\xi) \sin \omega \xi d\xi.$$

In particular $b(\omega)$ must vanish if $f(x)$ is even, $a(\omega)$ if $f(x)$ is odd. We then have corresponding to the above "pure cosine or sine series," a "pure cosine or sine integral." One or the other can be produced whenever $f(x)$ is given only for $x > 0$, by continuing $f(x)$ as an even or odd function to the negative side. We then write explicitly:

for even continuation

$$(11a) \quad f(x) = \int_0^{\infty} a(\omega) \cos \omega x d\omega, \quad a(\omega) = \frac{2}{\pi} \int_0^{\infty} f(\xi) \cos \omega \xi d\xi,$$

for odd continuation

$$(11b) \quad f(x) = \int_0^{\infty} b(\omega) \sin \omega x d\omega, \quad b(\omega) = \frac{2}{\pi} \int_0^{\infty} f(\xi) \sin \omega \xi d\xi.$$

The usefulness of this procedure will become apparent to us in some particular problems of heat conduction below.

We denoted the variable of integration by ω deliberately. In general one denotes the *frequency* in oscillation processes by ω . Let us therefore, for the time being, think of x as the *time coordinate*; then in equation (10) we have the *decomposition of an arbitrary process in time, $f(x)$, into its harmonic components*. In the *Fourier integral* one is concerned with a *continuous spectrum*, which ranges over all frequencies from $\omega = 0$ to $\omega = \infty$ in the *Fourier series* with a *discrete spectrum*, consisting of a fundamental tone plus harmonic overtones. Here the following fact must be kept in mind: when a physicist determines the spectrum of a process with a suitable spectral apparatus, he finds only the *amplitude* belonging to the frequency ω , while the phase of the partial oscillations remains unknown to him. In our notation the amplitude corresponds to the quantity

$$c(\omega) = \sqrt{a^2(\omega) + b^2(\omega)},$$

the phase, $\gamma(\omega)$, is given by the ratio b/a . The relation between these various quantities is best given as

$$(12) \quad c(\omega) e^{i\gamma(\omega)} = a(\omega) + i b(\omega).$$

The Fourier integral which describes the process completely uses both

quantities a and b , i.e., both amplitude and phase. The observable spectrum therefore yields, so to speak, only half the information which is contained in the Fourier integral.

This is noted markedly in the "Fourier analysis of crystals," which is so successfully carried out nowadays. Here only the *intensities* of the crystal reflexes, i.e., the squares of the *amplitudes*, can be observed; for a complete knowledge of the crystal structure one would have to know the *phases* too. This defect can only be partially removed by symmetry considerations.

In exercise I.4 we shall deal with the spectra of diverse oscillation processes as examples for the theory of the Fourier integral and at the same time as completion of the spectral theory.

Once more we return to the complex form of the Fourier integral and split it into two parts

$$(13) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(\omega) e^{i\omega x} d\omega, \quad \varphi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx.$$

which together are equivalent to (8). Disregarding the splitting of the denominator 2π into $\sqrt{2\pi} \cdot \sqrt{2\pi}$, which was done mainly for reasons of symmetry, and disregarding the notation of the variable of integration in the second equation, we have $\varphi(\omega)$ identical with the quantity $a(\omega) - ib(\omega)$ defined in (10a); it therefore contains information concerning both the amplitude and the phase of the oscillating process $f(x)$.

Moreover (10) shows that the two functions f and φ have a *reciprocal* relation: one is determined by the other, whether we regard f as known and φ as unknown or conversely, and the determination in each case is by "integral equations" of exactly the same character. One says that one function is the *Fourier transform* of the other. In (13) we have a particularly elegant formulation of Fourier's integral theorem.

So far we have spoken only of functions $f(x)$ of *one* variable. It is obvious that a function of several variables can be developed into a Fourier series or integral with respect to any one of the variables. By developing with respect to x, y, z for example we obtain a triply infinite Fourier series and sixfold Fourier integrals. We do not wish to write here the somewhat lengthy formulas since we shall have ample opportunity to explain them in their applications.

§ 5. Development by Spherical Harmonics

We do not claim that the path we shall pursue is the most convenient approach to the theory of spherical harmonics; but it proceeds immedi-

ately from the discussion of §1, needs no preparation from the theory of differential equations, and leads to interesting points of view on far reaching generalizations.

We consider the problem: Approximate a function $f(x)$ given in the interval $-1 < x < +1$ by a sequence of polynomials $P_0, P_1, P_2, \dots, P_k, \dots, P_n$ of degrees $0, 1, 2, \dots, k, \dots, n$ in the manner which is the best possible from the point of view of the method of least squares. We form an n -th approximation of the form

$$(1) \quad S_n = \sum_{k=0}^n A_k P_k$$

and reduce the mean error

$$(2) \quad M = \frac{1}{2} \int_{-1}^{+1} [f(x) - S_n]^2 dx$$

to a minimum through choice of A_k , just as in (1.3). This leads to the $n+1$ equations:

$$(3) \quad \int_{-1}^{+1} [f(x) - S_n] P_k dx = 0, \quad k = 0, 1, \dots, n.$$

just as in (1.4). This minimal requirement we complete by a requirement concerning the amount of calculation that will be needed: the coefficients A_k which are to be calculated from (3) in the n -th approximation, shall also be valid in the $(n+1)$ -st and in all subsequent approximations; they shall represent the *final* A_k for all $k \leq n$, and the finer approximations are to complete their determination by yielding the A_k for $k > n$. In §1, p. 4 this finality of the A_k resulted from the known orthogonality of the trigonometric functions. Here, conversely, the *requirement of finality* will be seen to imply the *orthogonality* of the P_k .

The proof is very simple. Equation (3), written explicitly, reads (we omit in the following the limits of integration ± 1):

$$(4) \quad A_0 \int P_0 P_k dx + A_1 \int P_1 P_k dx + \dots + A_n \int P_n P_k dx = \int f(x) P_k dx.$$

Since the right side is independent of n and the A_i are to be final, this equation retains its validity for the $(n+1)$ -st approximation S_{n+1} , except that on the left side we add the term

$$A_{n+1} \int P_{n+1} P_k dx$$

Equation (4) implies that this term must vanish, and since A_{n+1} does

not vanish (except for special choice of $f(x)$), the integral must vanish for all k for which (4) is valid, i.e., for all $k \leq n$. But this implies that P_{n+1} is *orthogonal* to P_0, P_1, \dots, P_n for arbitrary n . Hence, if we take P_0 and P_1 orthogonal to each other, our requirement of finality implies the *general condition of orthogonality*

$$(5) \quad \int P_n P_m dx = 0, \quad m \neq n.$$

Using (5) we obtain from (4)

$$(6) \quad A_k \int P_n^2 dx = \int f(x) P_k(x) dx.$$

The A_k are therefore determined individually if we add a convention about the *normalizing integral* on the left side of (6). The most obvious procedure would be to set it directly equal to 1, and indeed we shall do this in the general theory of characteristic functions. Here we prefer to follow historical usage and require instead that

$$(7) \quad P_n(1) = 1.$$

This normalizing condition has an advantage in that, as we shall see, all the coefficients in P_n become rational numbers.

We now pass to the recursive calculation of P_0, P_1, P_2, \dots from (5) and (7). P_0 is a constant, which according to (7), must be set equal to 1. In the linear function $P_1 = ax + b$ we see from (5), after setting $n = 0$ and $m = 1$, that $b = 0$ and from (7) that $a = 1$. After setting $P_2 = ax^2 + bx + c$ we obtain

$$\int P_2 P_0 dx = \frac{2}{3}a + 2c = 0; \quad \text{hence} \quad c = -\frac{a}{3};$$

$$\int P_2 P_1 dx = \frac{2}{3}b = 0; \quad \text{hence} \quad b = 0;$$

Therefore $P_2 = a(x^2 - \frac{1}{3})$ and by (7)

$$a = \frac{3}{2}, \quad P_2 = \frac{3}{2}x^2 - \frac{1}{2}.$$

Correspondingly we find

$$P_3 = \frac{5}{2}x^3 - \frac{3}{2}x, \quad P_4 = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}, \text{ etc.}$$

The P_n are therefore completely determined by our two requirements, the P_{2n} as even, the P_{2n+1} as odd polynomials with rational coefficients.

More transparent than the recursive process is the following explicit representation:

$$(8) \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

We see that $P_n(x)$ as defined by (8) satisfies condition (7) as follows: for $x \rightarrow 1$ we have to carry out the n -fold differentiation solely for the factor $(x - 1)^n$, whereby we obtain $n!$; the factor $(x + 1)^n$ becomes equal to 2^n ; equation (8) therefore does imply that $P_n(1) = 1$.

It remains to be proven that (8) satisfies the orthogonality condition (5), which is equivalent to our "condition of finality." To this end we introduce the notation

$$(9) \quad D_{k,l} = \frac{d^k}{dx^k} (x^2 - 1)^l$$

and write the left side of (5) (suppressing the constant factor which is immaterial here) as

$$\int_{-1}^{+1} D_{n,n} D_{m,m} dx,$$

where we take, say, $m > n$. We now reduce the order of differentiation of the second factor $D_{m,m}$ by integration by parts; this increases the order of differentiation of $D_{n,n}$. The terms which fall outside the integral sign will vanish for $x = \pm 1$, since in $D_{m-1,m}$ according to (9) one factor $x^2 - 1$ remains. Repeating this process we get

$$(10) \quad \begin{aligned} \int D_{n,n} \cdot D_{m,m} dx &= - \int D_{n+1,n} \cdot D_{m-1,m} dx = \\ \int D_{n+2,n} \cdot D_{m-2,m} dx &= \dots = (-1)^n \int D_{2n,n} \cdot D_{m-n,m} dx. \end{aligned}$$

Here according to (9) $D_{2n,n}$ is a constant, namely $(2n)!$. Hence

$$(11) \quad \begin{aligned} \int D_{n,n} \cdot D_{m,m} dx &= (-1)^n (2n)! \int D_{m-n,m} dx \\ &= (-1)^n (2n)! D_{m-n-1,m} \Big|_{-1}^{+1}. \end{aligned}$$

This vanishes, since the number $m - n - 1$ of differentiations that still remain to be carried out is less than the number m of factors $x - 1$ and $x + 1$ which are to be differentiated. This deduction is valid for $m = n + 1$, too, and fails only for $m = n$. The orthogonality is therefore proved for all $m \neq n$.

At the same time the method just used provides a way of calculating the normalizing integral of (6):

$$\int P_k^2 dx = \left(\frac{1}{2^k k!} \right)^2 \int D_{k,k} \cdot D_{k,k} dx.$$

Using the first line of (11) for $m = n = k$, we obtain

$$\int P_k^2 dx = \frac{(-1)^k (2k)!}{(2^k k!)^2} \int D_{0,k} dx = \frac{(2k)!}{(2 \cdot 4 \cdot 6 \dots 2k)^2} \int (1-x^2)^k dx.$$

The numerical factor in front of the last integral is

$$z = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2k},$$

under the substitution $x = \cos \vartheta$ the integral itself goes over into the well known form

$$\int_0^\pi \sin^{2k+1} \vartheta d\vartheta = 2 \cdot \frac{2 \cdot 4 \cdot 6 \dots 2k}{3 \cdot 5 \cdot 7 \dots (2k+1)} = \frac{2}{2k+1} \cdot \frac{1}{z}.$$

Therefore, one obtains

$$(12) \quad \int P_k^2 dx = \frac{2}{2k+1} = \frac{1}{k + \frac{1}{2}}.$$

Equation (6) then gives

$$(13) \quad A_k = (k + \frac{1}{2}) \int f(x) P_k(x) dx.$$

Substituting this in equation (1) of the n -th approximation S_n and letting $n \rightarrow \infty$ we get (assuming convergence and the completeness of the system of functions P):

$$(14) \quad f(x) = \sum_{k=0}^{\infty} (k + \frac{1}{2}) \int_{-1}^{+1} f(\xi) P_k(\xi) d\xi \cdot P_k(x).$$

The two assumptions just mentioned can be justified here, just as in the case of Fourier series, by consideration of the limiting value of the mean square error. The k -th approximating function has k zeros in the interval of approximation just as before, except that now they are not equally spaced. The approach to the given function, f , proceeds, here too, through more and more frequent *oscillations*. Also, we find Gibbs' phenomenon at the points of discontinuity, etc.

**§ 6. Generalizations: Oscillating and Osculating
Approximations. Anharmonic Fourier Analysis.
An Example of Non-Final Determination of Coefficients**

The following question suggests itself: Why are the two series different, despite the identical nature of the approximation processes? Since we saw that the form of the $P_n(x)$ was completely determined by our approximation requirements, we might think, e.g., that the pure cosine series (expansion of an even function) would go over into a series of spherical harmonics, if in the former we set $\cos \varphi = x$, because then $\cos k\varphi$ becomes a polynomial of degree k in x just like $P_k(x)$, and the interval of expansion $0 < \varphi < \pi$ becomes the interval $-1 < x < +1$. But the individual infinitesimal elements of this interval receive a different weight g in each case since

$$d\varphi = -\frac{dx}{\sqrt{1-x^2}}.$$

Whereas in the Fourier approximation we associate the same weight with all $d\varphi$, the endpoints $x = \pm 1$ of the interval in the x scale seem to be favored since $g(x) = 1/\sqrt{1-x^2}$. At these points the function is better approximated than at the middle of the interval. The opposite is obviously the case for approximations by spherical harmonics which, translated to the φ -scale, discriminate against the endpoints of the interval since $g(\varphi) = \sin \varphi$. Pictorially speaking, in the case of Fourier series, one deals with a uniformly weighted unit semicircle between $\varphi = 0$ and π , which, under orthogonal projection on the diameter between $x = -1$ and $+1$, yields a non-uniform density; on the other hand the case of spherical harmonics deals with a uniformly weighted diameter, which corresponds to a non-uniformly weighted semicircle.

A. OSCILLATING AND OSCULATING APPROXIMATION

These different *distributions of weight g* (that is, densities) are the factors that, in conjunction with the delimitation of the interval of expansion, distinguish among the different series expansions common in mathematical physics. Here we only mention the expansions in Hermite- and Laguerre-polynomials because of their importance for wave mechanics. We shall not concern ourselves here with their formal representation — they can be obtained from the requirement of a best possible calculation of the coefficients satisfying a condition of finality, just as in the case of spherical harmonics. (See exercise I.6, where the usual

normalizations are given; orthogonality would again be the necessary result of these requirements.) We restrict ourselves here to a tabulation of the most important characteristics of both polynomial series:

	HERMITE	LAGUERRE
Interval	$-\infty < x < +\infty$	$0 < x < \infty$
Weight $g(x)$	e^{-x^2}	e^{-x}
Orthogonality condition for $m \neq n$	$\int_{-\infty}^{+\infty} H_n H_m e^{-x^2} dx = 0$	$\int_0^{\infty} L_n L_m e^{-x} dx = 0$

For these series, just as for Fourier series and spherical harmonics, the approach to the given function, f , is through closer and closer *oscillations*. However, from the calculus we know a series whose character is *osculating* rather than *oscillating*, namely the *Taylor series*. In the case of Taylor series the consecutive approximations S_n osculate the curve to be represented in such a way that at a given point S_n has the same derivatives as f up to and including $f^{(n)}$. The graphic representation of the power series of $\sin x$ (Fig. 6) demonstrates this without further explanation.

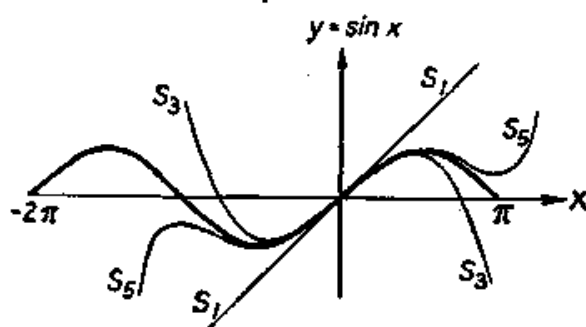


Fig. 6. The Taylor expansion of $\sin x$ (heavy line) and its approximations

$$S_1 = x, \quad S_3 = x - \frac{x^3}{3!},$$

$$S_5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Here the total accuracy is concentrated at a single point. Following Dirac we can express this succinctly as follows: $g(x)$ has degenerated into a δ function. Dirac defines, as an analogue to the algebraic symbol δ_{kl} of (1.6), a highly discontinuous function $\delta(x | x_0)$

$$(1) \quad \delta(x | x_0) = \begin{cases} 0 & x \neq x_0, \\ \infty & x = x_0, \end{cases} \quad \int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x | x_0) dx = 1$$

for arbitrary ϵ . For the Taylor series of Fig. 6, where x_0 has been set equal to 0, we get

$$(1a) \quad g(x) = \delta(x | 0).$$

B. ANHARMONIC FOURIER ANALYSIS

Whereas in §1-3 we considered only Fourier series which proceed according to *harmonic* (integral) overtones of a fundamental tone, we

now consider the problem of expanding an arbitrary function $f(x)$ in the interval $0 < x < \pi$ into a series of the form

$$(2) \quad f(x) = B_1 \sin \lambda_1 x + B_2 \sin \lambda_2 x + B_3 \sin \lambda_3 x + \dots$$

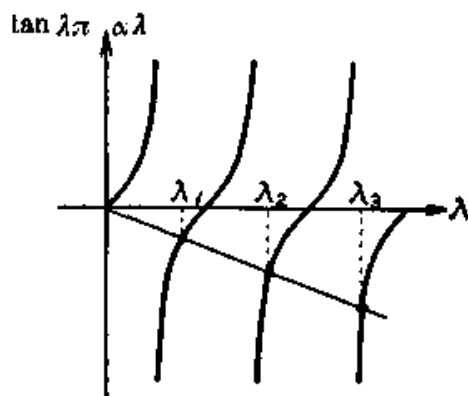


Fig. 7. Diagram of the transcendental equation $\tan \lambda \pi = \alpha \lambda$ $\alpha < 0$. In the ordinate both $y = \tan \lambda \pi$ and $y = \alpha \lambda$ have been drawn. The intersections yield the roots, λ_k , of the equation. $\lambda_0 = 0$ is not to be considered as a root; for $n \rightarrow \infty$ we get asymptotically $\lambda_n = n - \frac{1}{2}$.

where the λ_k are given as the roots of a transcendental equation, e.g.,

$$(2a) \quad \tan \lambda \pi = \alpha \lambda$$

(α being an arbitrary number). We do this for use in problems of heat conduction (see §16). The fact that (2a) has infinitely many roots is seen directly from Fig. 7 where λ has been drawn as the abscissa and both $\tan \lambda \pi$ and $\alpha \lambda$ as ordinates. We shall meet another equation of character similar to (2a) in exercise II.1.

We first show that the functions $\sin \lambda_k x$ form an orthogonal system with weighting factor $g(x) = 1$, i.e., that

$$(3) \quad \int_0^\pi \sin \lambda_k x \sin \lambda_l x dx = 0 \quad k \neq l.$$

In fact, by passing from the product of sines to the cosines of the sums and differences, we obtain for the left hand of (3)

$$\frac{\lambda_k \lambda_l}{\lambda_k^2 - \lambda_l^2} \cos \lambda_k \pi \cos \lambda_l \pi \left(\frac{\tan \lambda_k \pi}{\lambda_k} - \frac{\tan \lambda_l \pi}{\lambda_l} \right),$$

where the expression inside the brackets now vanishes because of (2a). In the same manner we find for $k = l$

$$(3a) \quad \int_0^\pi \sin^2 \lambda_k x dx = \frac{\pi}{2} \left(1 - \frac{1}{\lambda_k \pi} \sin \lambda_k \pi \cdot \cos \lambda_k \pi \right).$$

This calculation of (3) and (3a) which is based on special trigonometric identities, will receive a less formal treatment in §16 where it will be reduced to an application of Green's theorem.

From (3) and (3a) one obtains the following value for the expansion coefficients B_k in (1):

$$(3b) \quad B_k = \frac{2}{\pi} \int_0^{\pi} \frac{f(x) \sin \lambda_k x}{1 - \frac{\sin \frac{2 \lambda_k \pi}{2 \lambda_k \pi}}{2 \lambda_k \pi}} dx.$$

This value for B_k is *final* in the sense of p. 22, since it is independent of n and minimizes the mean square error of the approximation

$$S_n = \sum_{k=1}^n B_k \sin \lambda_k x$$

At the same time this settles the question of convergence and completeness, if for $n \rightarrow \infty$ the mean square error approaches zero.

C. AN EXAMPLE OF A NON-FINAL DETERMINATION OF COEFFICIENTS

As preparation for an optical (or rather "quasi-optical") application, we shall consider a much more involved case in which the requirement of finality is *not* satisfied. Let us consider a metal mirror in the shape of a circular cylinder (see Fig. 8). The electric vector of the total oscillation, which we take as perpendicular to the plane of the drawing, is composed of the incoming wave, represented on the mirror by

$$(4) \quad w = -f(\varphi), \quad -\alpha < \varphi < +\alpha, \quad r = a$$

and of the reflected (refracted, scattered) wave. Let the latter be represented by:

$$u = u(r, \varphi), \quad -\pi < \varphi < +\pi, \quad r < a, \text{ inner field,}$$

$$v = v(r, \varphi), \quad -\pi < \varphi < +\pi, \quad r > a, \text{ outer field.}$$

We then have to demand

$$(5) \quad u + w = v + w = 0 \quad \text{for } r = a \text{ and } |\varphi| < \alpha,$$

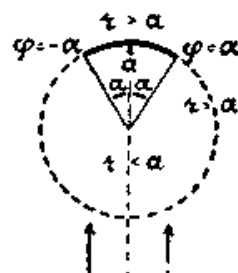


Fig. 8. Reflection of an incoming "quasi-optical" wave on a circular cylinder mirror of opening $\varphi = \alpha$ and radius $r = a$.

$$(6) \quad u = v, \quad \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} \quad \text{for } r = a \text{ and } |\varphi| > \alpha,$$

the former on account of the assumed infinite conductivity of the metal mirror, the latter on account of the required continuous passage from the inner to the outer field.

Assuming w to be symmetric with respect to the axis of the mirror (as, for example in the case of a plane wave proceeding in that direction), we write⁷

$$(7) \quad \begin{aligned} u &= \sum_n C_n g_n(r) \cos n\varphi, \\ v &= \sum_n D_n h_n(r) \cos n\varphi. \end{aligned}$$

g_n and h_n will turn out to be Bessel and Hankel functions, respectively (see §19); they can be chosen so that

$$g_n(a) = h_n(a) = 1$$

Equation (5) and the first equation (6) then imply

$$(8) \quad \sum_n C_n \cos n\varphi = \sum_n D_n \cos n\varphi = f(\varphi) \quad |\varphi| < \alpha$$

and

$$(9) \quad \sum_n C_n \cos n\varphi = \sum_n D_n \cos n\varphi \quad |\varphi| > \alpha.$$

respectively. From these two equations it follows that

$$\sum_n (C_n - D_n) \cos n\varphi = 0 \quad \text{for all } \varphi,$$

hence, whether the preceding summations are extended over all integers n or only over the first N integers (the more general case), we have

$$D_n = C_n.$$

This satisfies (9) while (8) still requires

$$(10) \quad \sum_n C_n \cos n\varphi = f(\varphi) \quad \text{for } |\varphi| < \alpha.$$

In addition to this we have to satisfy the second equation (6) which on account of (7) reads:

⁷ In view of the notations to be used in Chapter IV, it is advisable here to change the index of summation from k to n . For the previous n we shall write N , and instead of l we shall use m .

$$(11) \quad \sum_n C_n \gamma_n \cos n\varphi = 0 \quad \text{for } |\varphi| > \alpha$$

$$(11a) \quad \gamma_n = a \left(\frac{dg_n(r)}{dr} - \frac{dh_n(r)}{dr} \right)_{r=a}$$

We add the factor a before the parentheses here, as we may by (11), in order to make γ_n a pure number. Equations (10) and (11) together determine the C_n .

Here the way is again shown by the method of least squares. We consider the square errors corresponding to the equations (10) and (11)

$$\int_0^\alpha \left(f(\varphi) - \sum_{n=0}^N C_n \cos n\varphi \right)^2 d\varphi \quad \text{and} \quad \int_\alpha^\pi \left(\sum_{n=0}^N C_n \gamma_n \cos n\varphi \right)^2 d\varphi.$$

The sum of these two is to be minimized through choice of the C_n . By differentiation with respect to the C_n this yields a system of $N + 1$ linear equations for $C_0, \dots, C_n, \dots, C_N$, of which the $(m + 1)$ -st equation is:

$$(12) \quad \sum_{n=0}^N C_n \left\{ \int_0^\alpha \cos n\varphi \cos m\varphi d\varphi + \gamma_n \gamma_m \int_\alpha^\pi \cos n\varphi \cos m\varphi d\varphi \right\} \\ = \int_0^\alpha f(\varphi) \cos m\varphi d\varphi.$$

If we pass to the limit $N \rightarrow \infty$ we obtain an *infinite system of linear equations for the infinitely many unknowns C_n* , which are in general of no interest to us. We must postpone further treatment of this problem until appendix I of Chapter IV, for only there shall we have the necessary values of the parameters γ_n . The corresponding spatial problem, where we have a spherical segment instead of a circular cylinder segment, would lead in the limit $N \rightarrow \infty$ to an infinite system of linear equations, in which $P_n(\cos \vartheta)$ would replace $\cos n\varphi$ (by ϑ we denote here the angle measured from the axis of symmetry of the spherical mirror). This problem too will be treated in appendix I of Chapter IV. At present we call attention only to the difference in method between those problems in which the method of least squares leads to a definitive calculation of the *individual coefficients C* , and those problems in which the “requirement of finality” is not satisfied and in which therefore, *the totality of the C_n must be determined from the totality of minimality conditions.*

CHAPTER II

Introduction to Partial Differential Equations

§ 7. How the Simplest Partial Differential Equations Arise

The potential equation

$$(1) \quad \Delta u = 0 \quad \text{or} \quad (1a) \quad \Delta u = -(4\pi) \varrho$$

is known in the *theory of gravitation* as the expression of the field-action approach, as opposed to the action-at-a-distance approach of Newton. The *Laplace operator* is defined as

$$(2) \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \text{div grad.}$$

The same equations (1) and (1a) are fundamental for *electrostatic* and *magnetic fields*, (1) in empty space, (1a) in the presence of a source of density ϱ the factor 4π in (1a) has been put in parentheses since it can be removed by a proper choice of units.

Equation (1) appears also in the *hydrodynamics* of incompressible and irrotational fluids, u standing for the velocity potential. We also mention the two-dimensional potential equation

$$(3) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

as the basis of Riemannian function theory, which we may characterize as the "field theory" of the analytic functions $f(x + iy)$.

Equally well known is the wave equation

$$(4) \quad \Delta u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.$$

It is fundamental in acoustics (c = velocity of sound). It is also fundamental in the electrodynamics of variable fields (c = velocity of light), and therefore in optics. In the special theory of relativity one may write (4) as the four-dimensional potential equation

$$(5) \quad \square u = 0 \quad \text{with} \quad \square = \sum_{k=1}^4 \frac{\partial^2}{\partial x_k^2}$$

by introducing the fourth coordinate x_4 (or x_0) = ict in addition to the three spatial coordinates x_1, x_2, x_3 . For an oscillating membrane we have (4) with two spatial dimensions, for an oscillating string we have one spatial dimension. In the latter case we write

$$(6) \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \text{or sometimes} \quad (6a) \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

setting, for the time being, $y = ct$ (not $y = ict$). Neither membrane nor string has a proper elasticity; the constant c is computed from the tension imposed from outside and from the density per unit of area or of length.

In the general theory of elasticity one has, as a special case, the differential equation for the transverse vibrations of a thin disc

$$(7) \quad \Delta \Delta u = -\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}, \quad \Delta \Delta = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4};$$

for reasons of dimensionality c here does not stand for the velocity of sound in the elastic material, as it does in acoustics, but is computed from the elasticity, density, and thickness of the disc. Analogously, the differential equation of an oscillating elastic rod is

$$(8) \quad \frac{\partial^4 u}{\partial x^4} = -\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.$$

This will be derived in exercise II.1, where the resulting characteristic frequencies will be compared with the acoustic frequencies of open and of covered pipes.

As a third type we add to the differential equations of states of equilibrium ((1) to (3)), and of oscillating processes ((4) to (8)), those of *equalization processes*. As their chief representative we shall here consider *heat conduction* (equalization of energy differences). We remark, however, that *diffusion* (equalization of differences of material densities), *fluid friction* (equalization of impulse differences), and pure *electric conduction* (equalization of differences of potential), follow the same pattern.

Let \mathbf{G} be a vector of the magnitude and direction of the heat flow and let the initial point P be surrounded by an element of volume $d\tau$. Then $\text{div } \mathbf{G} d\tau$ is the outflow of heat energy from $d\tau$ per unit of time. A decrease per unit of time in the amount of heat in $d\tau$, which we

shall denote by $-\partial Q/\partial t$, corresponds to this. We then have

$$(9) \quad \operatorname{div} \mathbf{G} d\tau = -\frac{\partial Q}{\partial t}.$$

Our heat conductor is here considered to be a rigid body so that we can neglect expansion; heat content is then the same as energy content. Now every increase dQ in heat causes an increase in the temperature of $d\tau$, every decrease $-dQ$ in heat causes a decrease in temperature. Denoting the temperature by u , we have

$$(10) \quad dQ = c dm du, \quad dm = \rho d\tau.$$

c being the specific heat (for a rigid body we need not distinguish between c_v and c_p). The factor dm is due to the fact that c is related to the unit of mass.

From (9) and (10) we get

$$(11) \quad \operatorname{div} \mathbf{G} = -c\rho \frac{\partial u}{\partial t}.$$

We now apply *Fourier's law*, which determines the relation between \mathbf{G} and u . It states that for an isotropic medium

$$(12) \quad \mathbf{G} = -\kappa \operatorname{grad} u:$$

the flow of heat is in the direction of decreasing temperature and is proportional to the rate of this decrease. The factor of proportionality κ is called the *heat conductivity*.

Introducing (12) in (11) we get the differential equation of *heat conduction*

$$(13) \quad \Delta u = \frac{1}{k} \frac{\partial u}{\partial t}, \quad k = \frac{\kappa}{c\rho}.$$

k is called the *temperature conductivity*.

Fourier's law was adapted to the case of *diffusion* by the physiologist Fick. Here u stands for the concentration of dissolved matter in the solvent, \mathbf{G} for the *material flow* of the dissolved matter, and k for the *diffusion coefficient*. In the case of *inner friction* of an incompressible fluid, k stands for the *kinematic viscosity*, and (13) is the Navier-Stokes equation for laminar flow (i.e., flow in a fixed direction). Owing to the tensor character of this process equation (12) has no general validity here. The analogue of Fourier's law in the *electric* case is Ohm's law. Here u stands for the *potential*, \mathbf{G} for the *specific electric current* (the current per unit of area of the conductor), and k for the *specific resistance of the*

conductor. Equation (13) is of the type of Maxwell's equations in the case of pure Ohm conduction.

Schrödinger's equation of wave mechanics belongs formally to the same scheme, in particular in the force-free case, to which we restrict ourselves here:

$$(14) \quad \Delta u = \frac{2m}{i\hbar} \frac{\partial u}{\partial t} \quad \left\{ \begin{array}{l} \hbar = \text{Planck's constant divided by } 2\pi \\ m = \text{mass of the particle.} \end{array} \right.$$

However, owing to the fact that the real constant, k , of (13) is replaced here by the imaginary constant $i\hbar/2m$, equation (14) describes an oscillation rather than an equalization process. We see this in the passage to the case of periodicity in time, if we set

$$(14a) \quad u = \psi e^{-i\omega t}, \quad \omega = \frac{W}{\hbar}, \quad W = \text{energy of the state.}$$

Then (14) becomes

$$(15) \quad \Delta \psi + C \psi = 0, \quad C = \frac{2m}{\hbar^2} W.$$

This is the same form as we would obtain from the wave equation (4) if we set $u = \psi \cdot \exp(-i\omega t)$ and let $C = \omega^2/c^2$.

The so-called case of *linear heat conduction*, with the thermal state depending on only one variable x , will be treated in detail in the following chapter. In order to compare its differential equation with (3) and (6a), we write it in the form:

$$(16) \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0, \quad y = k t.$$

Looking back on this sketchy survey one notices a family resemblance among the differential equations of physics. This stems from the *invariance under rotation and translation*, which must be demanded for the case of isotropic and homogeneous media. The differential operator of second order implied by this invariance is just the Laplace Δ . In the case of space-time invariance of relativity this is replaced by the corresponding four-dimensional \square of (15). For the case of an *anisotropic medium*, Δ must be replaced by a sum of all second derivatives with factors determined from the crystal constants. For the case of an *inhomogeneous medium* these factors will also be functions of position. We shall deal with such generalized differential expressions in the beginning of the next section.

The fact that we are dealing throughout with *partial differential*

equations is due to the *field-action approach*, which is the basis of present day physics, according to which only neighboring elements of space can influence each other.

§ 8. Elliptic, Hyperbolic and Parabolic Type. Theory of Characteristics

We restrict ourselves to the case of two independent variables, x and y . The most general form of a *linear partial differential equation of second order* is then:

$$(1) \quad L(u) \equiv A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u = 0.$$

A, B, \dots, F being given functions of x and y having sufficiently many derivatives. For the present we may even consider the far more general equation:

$$(2) \quad A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = \Phi \left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, x, y \right),$$

where Φ need not be linear in $u, \partial u/\partial x, \partial u/\partial y$.

We now investigate the conditions for the solvability of the following problem, which is put first in the mathematical theory of partial differential equations, although in the physical applications it is of secondary importance compared to certain boundary value problems considered later.

Let Γ be a given curve in the xy -plane along which both u and the derivative $\partial u/\partial n$ of u in the direction of the normal are prescribed. Does a solution of (2) that satisfies these initial conditions exist?

Preliminary remark: If u is given on Γ then so is $\partial u/\partial s$; but from $\partial u/\partial s$ and $\partial u/\partial n$ one can calculate $\partial u/\partial x$ and $\partial u/\partial y$. Therefore both u and its first derivatives are known on Γ .

We introduce the following abbreviations, which are common in the theory of surfaces:

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}, \\ r = \frac{\partial^2 u}{\partial x^2}, \quad s = \frac{\partial^2 u}{\partial x \partial y}, \quad t = \frac{\partial^2 u}{\partial y^2}.$$

Written in terms of r, s, t equation (2) reads:

$$(3) \quad A r + 2 B s + C t = \Phi.$$

Furthermore the following relations are valid in general, and therefore hold on Γ

$$(3a) \quad dp = r dx + s dy,$$

$$(3b) \quad dq = s dx + t dy.$$

Now, since p and q are known on Γ , equations (3) and (3a,b) constitute three linear equations for the determination of r, s, t on the curve. The determinant of this system is

$$\Delta = \begin{vmatrix} A & 2B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = A dy^2 - 2B dx dy + C dx^2.$$

Only when this determinant Δ is different from zero can r, s, t be calculated from (3), (3a), and (3b). However, in general, two directions, $dy:dx$, exist for every point (x, y) , for which this is not the case. Therefore two (real or conjugate complex) families of curves exist on which $\Delta = 0$, and which, according to Monge, are called *characteristics*.¹ They are the dotted lines of Fig. 9. Along each of these characteristics it is in general impossible to solve for r, s, t in terms of u, p, q . We shall therefore demand as a necessary condition for the solvability of our problem, that Γ shall be *nowhere tangent to a characteristic*. The opposite case, in which Γ coincides with one of the characteristics, will be discussed in §9A in connection with D'Alembert's solution.

When the condition $\Delta \neq 0$ is satisfied, a solution of the differential equation in the neighborhood of Γ must exist. Then the higher derivatives can be calculated in exactly the same way as the second derivatives. Let us consider, say, the third derivatives:

$$r_x = \frac{\partial^3 u}{\partial x^3}, \quad s_x = \frac{\partial^3 u}{\partial x^2 \partial y} = r_y, \quad t_x = \frac{\partial^3 u}{\partial x \partial y^2} = s_y, \quad t_y = \frac{\partial^3 u}{\partial y^3}.$$

Differentiating (3) and (3a,b) with respect to x , we get:

$$A r_x + 2B s_x + C t_x = \Phi_x + \dots$$

$$r_x dx + s_x dy = dr,$$

$$s_x dx + t_x dy = ds.$$

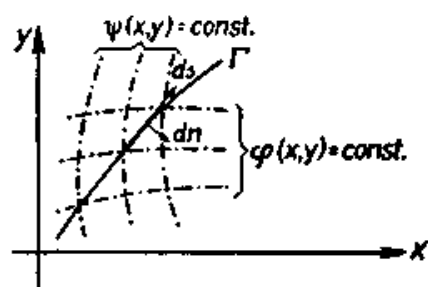


Fig. 9. The curve Γ , along which u and $\partial u / \partial n$ are given, and the two families of characteristics $\xi = \varphi(x, y) = \text{const.}$ and $\eta = \psi(x, y) = \text{const.}$

¹ A geometrically intuitive introduction of characteristics is given, e.g., by B. Baule in v. VI (Partielle Differentialgleichungen) of his *Mathematik des Naturforschers und Ingenieurs*, Hirzel, Leipzig 1944.

On the right . . . represents terms that contain no third derivatives, and therefore contain only known quantities. The determinant of this system is again Δ . The same holds for equations obtained by differentiation with respect to y . Our condition is therefore sufficient for the computability of the third and all higher derivatives. Therefore u can be expanded in a Taylor series at every point of Γ and the coefficients are uniquely determined by the boundary conditions on Γ .

We now turn to the discussion of the equation of characteristics

$$(4) \quad A dy^2 - 2 B dx dy + C dx^2 = 0,$$

where we restrict ourselves to an arbitrarily chosen neighborhood in the xy -plane,² and distinguish between the following cases:

- 1) $A C - B^2 > 0$ *elliptic type* in which the characteristics are conjugate complex.
- 2) $A C - B^2 < 0$ *hyperbolic type* in which the characteristics form two distinct families.
- 3) $A C - B^2 = 0$ *parabolic type* in which only one real family of characteristics exists.

Each of the three types can be brought into a special normal form in which the equations of the characteristics are utilized for the introduction of new coordinates. Let these equations be

$$(4a) \quad \varphi(x, y) = \text{const. and } \psi(x, y) = \text{const.}$$

respectively. Then through the transformation

$$(5) \quad \xi + i\eta = \varphi(x, y), \quad \xi - i\eta = \psi(x, y)$$

one obtains the normal form for the *elliptic* type,

$$(5a) \quad \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = X\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right);$$

through the transformation

$$(6) \quad \xi = \varphi(x, y), \quad \eta = \psi(x, y)$$

one obtains the normal form for the *hyperbolic* type,

$$(6a) \quad \frac{\partial^2 u}{\partial \xi \partial \eta} = X\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right);$$

² When A, B, C depend on x, y , then the equation may obviously be of different types for different neighborhoods of the xy -plane.

and through

$$\xi = \varphi(x, y) = \psi(x, y), \quad \eta = x$$

one obtains the normal form for the *parabolic* type,

$$(7a) \quad \frac{\partial^2 u}{\partial \eta^2} = X\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right).$$

Before proving this, we compare the above forms (5a), (6a) and (7a) with the equations (7.3), (7.6a), and (7.16), i.e., with the two-dimensional potential equation, the equation of the vibrating string, and the equation of linear heat conduction. We observe that the left hand sides of (5a) and (7.3) coincide except for the letters used to denote the independent variables. The analogous relation holds between (7a) and (7.16). In (6a) we only have to perform the simple transformation

$$(8) \quad \xi = \frac{1}{2}(\xi' + \eta'), \quad \eta = \frac{1}{2}(\xi' - \eta')$$

with the inverse

$$(8a) \quad \xi' = \xi + \eta, \quad \eta' = \xi - \eta$$

we obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial^2 u}{\partial \xi'^2} - \frac{\partial^2 u}{\partial \eta'^2},$$

which establishes the essential equality of the left hand sides of (6a) and (7.6a). Hence *the two-dimensional potential equation, the equation of the vibrating string and the equation of linear heat conduction are the simplest examples of the elliptic, the hyperbolic, and of the parabolic types, respectively.*

Starting with the treatment of the *hyperbolic case*, we first show that (6a) is obtained from the initial equation (2) through the transformation (6). From (6) we obtain for the first derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \varphi_x + \frac{\partial u}{\partial \eta} \psi_x, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \varphi_y + \frac{\partial u}{\partial \eta} \psi_y$$

where the subscripts again denote differentiation. From this we obtain for the second derivatives

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2} \varphi_x^2 + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} \varphi_x \psi_x + \frac{\partial^2 u}{\partial \eta^2} \psi_x^2 + \dots \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 u}{\partial \xi^2} \varphi_x \varphi_y + \frac{\partial^2 u}{\partial \xi \partial \eta} (\varphi_x \psi_y + \varphi_y \psi_x) + \frac{\partial^2 u}{\partial \eta^2} \psi_x \psi_y + \dots \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2} \varphi_y^2 + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} \varphi_y \psi_y + \frac{\partial^2 u}{\partial \eta^2} \psi_y^2 + \dots \end{aligned}$$

where the three dots stand for terms containing only first derivatives. Multiplying the last three equations by A , $2B$ and C , respectively, and adding, we obtain for the left side of (2):

$$(9) \quad \begin{aligned} & \frac{\partial^2 u}{\partial \xi^2} (A \varphi_x^2 + 2 B \varphi_x \varphi_y + C \varphi_y^2) \\ & + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} (A \varphi_x \varphi_\eta + B (\varphi_x \varphi_y + \varphi_y \varphi_\eta) + C \varphi_y \varphi_\eta) \\ & + \frac{\partial^2 u}{\partial \eta^2} (A \varphi_\eta^2 + 2 B \varphi_x \varphi_\eta + C \varphi_\eta^2) + \dots \end{aligned}$$

But here the coefficients of $\partial^2 u / \partial \xi^2$ and $\partial^2 u / \partial \eta^2$ vanish, since for the family of characteristics $\varphi = \text{const.}$ we have

$$\varphi_x dx + \varphi_y dy = 0,$$

Hence on introducing the ratio $dx:dy$ into (4) we get

$$(10) \quad A \varphi_x^2 + 2 B \varphi_x \varphi_y + C \varphi_y^2 = 0.$$

The derivatives of ψ must satisfy the same equation. Hence (9) indeed reduces to the hyperbolic normal form (6a) if we transfer the coefficient of $\partial^2 u / \partial \xi \partial \eta$ in (9) to the other side of the equation.

Since in the *parabolic case* we have $\eta = x$, we must substitute in (9)

$$(11) \quad \psi(x, y) = x; \quad \text{and hence} \quad \psi_x = 1, \quad \psi_y = 0,$$

whereas (10) still holds for φ_x, φ_y . The first term in (9) therefore vanishes. Owing to (11) the coefficient of the second term reduces to $A \varphi_x + B \varphi_y$ which also vanishes since $A C - B^2 = 0$ makes the left side of (10) a perfect square, so that (10) can be rewritten as $(A \varphi_x + B \varphi_y)^2 / A = 0$. Considering (9) and (11) the third term finally becomes simply

$$A \frac{\partial^2 u}{\partial \eta^2},$$

which is the parabolic normal form (7a).

The *elliptic case* need not be treated separately. It can be reduced to the hyperbolic case by a transformation analogous to (8a):

$$\xi' = \xi + i\eta, \quad \eta' = \xi - i\eta.$$

§ 9. Differences Among Hyperbolic, Elliptic, and Parabolic Differential Equations. The Analytic Character of Their Solutions

The problem of integration, which is illustrated in Fig. 9, is applied in physics only to the case of hyperbolic differential equations; for elliptic

differential equations it is replaced by an entirely different kind of problem, the *boundary value problem*. For the time being, we shall discuss this profound difference only sketchily and refer the reader to the following sections for a more precise treatment.

A. HYPERBOLIC DIFFERENTIAL EQUATIONS

As the simplest example we use the equation of the vibrating string, which, written in its normal form, is

$$(1) \quad \frac{\partial^2 u}{\partial \xi \partial \eta} = 0, \quad \xi = x + y, \quad \eta = x - y, \quad y = ct.$$

Here the characteristics are the lines $\xi = \text{const.}$, $\eta = \text{const.}$, which in Fig. 10 are drawn at 45° angles with the x - and y -axes. The general solution of (1) is the sum of a function of ξ and a function of η :

$$(2) \quad u = F_1(\xi) + F_2(\eta).$$

Because of the meaning of ξ and η this is *d'Alembert's solution* (see V, II, §13). For the sake of simplicity, let us consider u as being given on segments AB and AD of two of the characteristics. This determines u in the entire rectangle $ABCD$. We could calculate the value of u at P by passing in the directions of the characteristics to P_1 and P_2 , and substituting into (2) the values $F_1(\xi)$, $F_2(\eta)$ which are given at these points. *The values along two intersecting characteristics determine the function everywhere. For example, any discontinuities of the given functions on the characteristics would be continued into the interior of $ABCD$.* Thus the solution need not be an analytic function³ of x and y over its domain of definition.

In physics one is given the values of u and of $\partial u / \partial y$ along a segment of length l on the x -axis (l = length of string):

$$u = u(x, 0) \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{c} \frac{\partial u}{\partial t} = v(x, 0).$$

This segment corresponds to the curve Γ of Fig. 9, on which, too, u and $\partial u / \partial n$ were given, and it satisfies the requirement of not being tangent to any characteristic.

In order to apply the conclusions drawn from (2) to our present problem, we have to calculate F_1 and F_2 from our given $u(x, 0)$, $v(x, 0)$. This is done with the help of the following equations, which are immediate consequences of (2):

³ A function of two real variables x, y is called analytic in a certain domain, if in some neighborhood of each point (x_0, y_0) of this domain it can be represented as a power series in $x - x_0$ and $y - y_0$.

$$\begin{aligned}
 u(x, 0) &= F_1(x) + F_2(x), & F_1(x) &= \frac{1}{2} \left\{ u(x, 0) + \int v(x, 0) dx \right\}, \\
 v(x, 0) &= F_1'(x) - F_2'(x), & F_2(x) &= \frac{1}{2} \left\{ u(x, 0) - \int v(x, 0) dx \right\}.
 \end{aligned}$$

We conclude: the given initial values, together with any possible discontinuities, are continued along the characteristics. The solution, $u(x, y)$ is in general not an analytic function of x and y . It is determined only within the rectangle of characteristics determined by the length of string l as shown in Fig. 10.

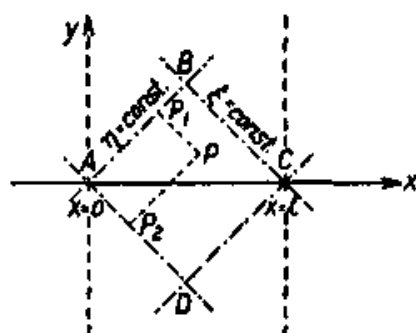


Fig. 10. The vibrating string of length l and the square of characteristics determined by its end point.

However, from a physical point of view, the solution must be determined from the initial time on, i.e., for all $y > 0$. This indicates that, in addition to the initial values, certain boundary values must be prescribed at the ends of the string. These are the stringing conditions $u = 0$ for $x = 0$ and $x = l$. Just as for all x such that $0 < x < l$ two values (u and $\partial u / \partial y$) had to be given, so for all $y > 0$, two values are given. This is due to the fact that our differential equation is of second order in both variables x, y and the only difference is that both values

along the x -axis are given at the same point $(x, 0)$ whereas the values along the y -direction are given at the different points $(0, y)$ and (l, y) . The only exceptions to this rule of two necessary boundary conditions are the characteristics on which, as we saw above, one value (F_1 or F_2) is sufficient.

We shall show in §11 that these results, which we have established for the case of the vibrating string, can be extended to all cases of hyperbolic type.

B. ELLIPTIC DIFFERENTIAL EQUATIONS

Here the characteristics are imaginary and therefore have no direct bearing on the problems we are going to treat. These problems do not deal with an arc F , as in Fig. 9, but rather with a closed region S of the real xy -plane. On the boundary of S , u or $\partial u / \partial n$ (or a linear combination of u and $\partial u / \partial n$) will be given but not both u and $\partial u / \partial n$ as in the hyperbolic case. Discontinuities of the boundary values are not continued into the interior of S , but only into the imaginary domain, and the function u is analytic everywhere in the interior of S .

These are known theorems from the theory of functions (two-

dimensional potential theory). Their proof for arbitrary linear elliptic differential equations will be given in the following section.

The analogue to d'Alembert's solution (2) is given in potential theory by

$$u = f_1(x + i y) + f_2(x - i y),$$

where, in order that u be real, we must set $f_2 = \overline{f_1}$, i.e., f_2 conjugate⁴ to f_1 . We may also write:

$$(3) \quad u = \operatorname{Re}[f(z)],$$

where f is an arbitrary analytic function of the complex variable $z = x + iy$. However, this general solution of the equation $\Delta u = 0$ does not help us (at least not directly) in the general solution of our boundary value problem.

C. PARABOLIC DIFFERENTIAL EQUATIONS

Here the two families of characteristics have degenerated into *one*. In the special case of the normal form of the equation of linear heat conduction this is the family of lines parallel to the x -axis. Only *one* boundary condition should be given on these characteristics just as in the case of hyperbolic differential equations (see p. 42). We can also see this directly from (7.16): here $\partial u / \partial y$ is determined uniquely if u is given as a function of x for some fixed y . From physical considerations one sees this in the following manner: the thermal behavior of a rod of length l is determined once and for all as soon as its initial temperature is given together with conditions for the ends of the rod (the lateral surface of the rod must be considered adiabatically closed, if heat is to flow only in the x -direction).

We shall see in §12, that the temperature distribution of the rod becomes *an analytic function of x and y* for arbitrary — even discontinuous — initial temperature. To this extent, therefore, the parabolic type resembles the elliptic type. However, the problem is not relative to a bounded region, but rather, as in the hyperbolic case, relative to a strip, i.e., a region which is infinite in one direction. The parabolic type, therefore, occupies a middle position between the elliptic and the hyperbolic types.

⁴ We use the notation f^* instead of \overline{f} , which is more common in mathematical literature, since we want to reserve the use of the bar for mean values in time. Re and Im stand for the real and imaginary part respectively.

§10. Green's Theorem and Green's Function for Linear, and, in Particular, for Elliptic Differential Equations

In (8.1) we had the general form of a linear differential equation of second order. In order to retain a common expression for the three types, we shall not transform this system into its canonical form for the time being.

A. DEFINITION OF THE ADJOINT DIFFERENTIAL EXPRESSION

We now have to introduce the seemingly rather formal concept of the differential form $M(v)$ which is adjoint to $L(u)$. It is defined by the requirement that the expression $v L(u) - u M(v)$ be generally integrable or as we may put it, that it be a kind of divergence.

We demand, namely

$$(1) \quad v L(u) - u M(v) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}.$$

The problem is to determine M and X, Y as functions of v and of u, v respectively.⁵

We shall use the following identities:

$$(2) \quad v A \frac{\partial^2 u}{\partial x^2} - u \frac{\partial^2 A v}{\partial x^2} = \frac{\partial}{\partial x} \left(A v \frac{\partial u}{\partial x} - u \frac{\partial A v}{\partial x} \right), \dots$$

$$(2a) \quad v B \frac{\partial^2 u}{\partial x \partial y} - u \frac{\partial^2 B v}{\partial x \partial y} = \frac{\partial}{\partial x} \left(v B \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left(u \frac{\partial B v}{\partial x} \right) = \dots$$

$$(3) \quad D v \frac{\partial u}{\partial x} - u \frac{\partial}{\partial x} (D v) = \frac{\partial}{\partial x} (D u v), \dots$$

Here the three dots (...) indicate the fact that (2) and (3) remain valid if we replace x by y and A, D by C, E respectively, and that on the right side of (2a) we may use the symmetric expression obtained by interchanging x and y . From this we get:

$$(4) \quad M(v) = \frac{\partial^2 A v}{\partial x^2} + 2 \frac{\partial^2 B v}{\partial x \partial y} + \frac{\partial^2 C v}{\partial y^2} - \frac{\partial D v}{\partial x} - \frac{\partial E v}{\partial y} + F v,$$

$$(5) \quad \begin{aligned} X &= A \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) + B \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} \right) + \left(D - \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} \right) u v, \\ Y &= B \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) + C \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} \right) + \left(E - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y} \right) u v. \end{aligned}$$

⁵ The operation of divergence is properly defined only for a vector. Since, as equation (5) will show, X and Y are not vector components, we speak of "a kind of divergence."

Obviously X and Y are determined only up to quantities X_0, Y_0 , whose divergence vanishes. We can therefore change the terms in (5): we may add $-\partial\Phi/\partial y$ to X and $+\partial\Phi/\partial x$ to Y , where Φ is an arbitrary function of x, y as well as of u, v .

We see that the relation between L and M is *reciprocal*: $L(v)$ is the adjoint differential form to $M(u)$.

Of particular importance for mathematical physics are those differential expressions for which $L(u) = M(u)$. They are called *self-adjoint*. By comparing (4) with (8.1) we get the condition of self-adjointness

$$(6) \quad \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} = D, \quad \frac{\partial B}{\partial x} + \frac{\partial C}{\partial y} = E.$$

B. GREEN'S THEOREM FOR AN ELLIPTIC DIFFERENTIAL EQUATION IN ITS NORMAL FORM

We now consider a region S with boundary curve C in the xy -plane and integrate (1) over S . We denote the element of area of S by $d\sigma$, and the line element of C by ds ; let the orientation be counter-clockwise (see Fig. 11).

Applying Gauss' theorem⁶ we get

$$(7) \quad \int_S [v L(u) - u M(v)] d\sigma = \int_S \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) d\sigma \\ = \int_C \{ X \cos(n, x) + Y \cos(n, y) \} ds.$$

This is the *general formulation of Green's theorem* which is valid for all three types. Setting $A = C = 1$, $B = 0$, we specialize it to the case of the *elliptic type in normal form*. We then have:

$$(7a) \quad \int_S [v L(u) - u M(v)] d\sigma = \int_C \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds \\ + \int_C \{ D \cos(n, x) + E \cos(n, y) \} u v ds.$$

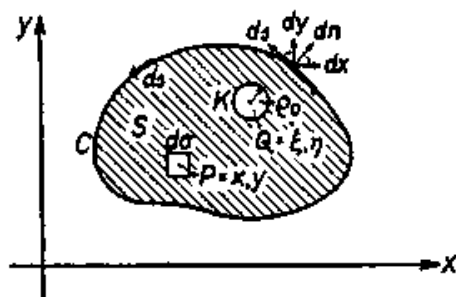


Fig. 11. Illustrating Green's theorem for an elliptic differential equation. The integration with respect to $d\sigma$ is extended over the domain S between the boundary curve C and the circle K of radius ρ_0 which contains the unit source at Q .

⁶ It states, when applied to a two-dimensional vector A with components X, Y , that $\int \operatorname{div} A d\sigma = \int A_n ds$,

This is a generalization of *Green's theorem* of potential theory

$$\int (v \Delta u - u \Delta v) d\sigma = \int \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds,$$

which is obtained from (7a) by setting $D = E = 0$. (The fact that in potential theory also $F = 0$ is of no importance here.)

We shall meet another form of Green's theorem in exercise 112.

If, in the interior of S , u and v satisfy the equations

$$L(u) = 0, \quad M(v) = 0$$

then the left hand sides of the equations (7), (7a) vanish. These equations, therefore, become

$$(7b) \quad 0 = \int_C \{X \cos(n, x) + Y \cos(n, y)\} ds$$

$$(7c) \quad 0 = \int_C \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds + \int_C \{D \cos(n, x) + E \cos(n, y)\} u v ds.$$

However, this holds only if u and v and the derivatives which appear here are *continuous* throughout S . If v has a discontinuity at the point $Q = (\xi, \eta)$, then it must be excluded from the domain of integration, just as in all applications of Green's theorem. We therefore surround Q by a curve K , which we choose to be a circle of arbitrarily small radius ρ_0 . The integration in (7b,c) must then be taken over both boundaries K and C :

$$\int_K \dots ds + \int_C \dots ds = 0,$$

where the orientation is opposite on the two curves and the direction n is to the exterior of S .

If for K we use (7c) and for C we use (7b) we get:

$$(8) \quad \int_K \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds + \int_K u v \{D \cos(n, x) + E \cos(n, y)\} ds \\ = - \int_C \{X \cos(n, x) + Y \cos(n, y)\} ds.$$

or, written in terms of coordinates:

$$\int \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) d\sigma = \int [X \cos(n, x) + Y \cos(n, y)] ds.$$

We apply this formally in (7) to our "pseudovector" X, Y .

C. DEFINITION OF A UNIT SOURCE AND OF THE PRINCIPAL SOLUTION

We shall assume that the discontinuity of v at Q consists of a "unit source." By this we mean the following: the yield q of a source Q is defined as the outward gradient of its field v . If we denote the distance from Q by ϱ , we have

$$(9) \quad q = \int_K \frac{\partial v}{\partial \varrho} ds$$

where K has the same meaning as before. Assuming that in the immediate neighborhood of the source v depends only on ϱ , we get

$$(9a) \quad q = \int_{\varphi=-\pi}^{+\pi} \frac{dv}{d\varrho} \varrho d\varphi = 2\pi \varrho \frac{dv}{d\varrho}.$$

A unit source is therefore given by:

$$(9b) \quad \frac{dv}{d\varrho} = \frac{1}{2\pi\varrho}, \quad v = \frac{1}{2\pi} \log \varrho + \text{const} \quad \text{for } \varrho \rightarrow 0.$$

For arbitrary ϱ we write:

$$(10) \quad v = U \log \varrho + V, \quad \varrho = \sqrt{(x - \xi)^2 + (y - \eta)^2},$$

where U and V are analytic functions of x, y and ξ, η such that U becomes $1/2\pi$ for $(x, y) \rightarrow (\xi, \eta)$.

A function of this kind we call a *principal solution* of the differential equation $M(v) = 0$. In the same way we shall speak of a principal solution of the adjoint equation $L(u) = 0$. Since the latter also corresponds to a unit source it will have the same form (10), although in general U and V will be different functions. Here too we can assume U and V to be analytic as long as the coefficients D, E, F in the differential equation are analytic. In the case of the potential equation $\Delta u = 0$ our principal solution corresponds essentially to the logarithmic potential, where we have for all ϱ

$$(10a) \quad v = \frac{1}{2\pi} \log \varrho.$$

D. THE ANALYTIC CHARACTER OF THE SOLUTION OF AN ELLIPTIC DIFFERENTIAL EQUATION

We return now to equation (8). Substituting (10) in (8), we see that only the term with

$$\frac{\partial v}{\partial n} = -\frac{\partial v}{\partial \varrho} = -\frac{U}{\varrho} + \dots = -\frac{1}{2\pi\varrho} + \dots$$

contributes to the integral over K , while all the other terms on the left side of (8) have zeros of the same order as $\varrho \log \varrho$ or of higher order. Since u is continuous at Q and the perimeter of K is $2\pi\varrho_0$, we obtain for the left side of (8):

$$-\int_K u \frac{\partial v}{\partial n} ds = \frac{u_Q}{2\pi\varrho_0} \int_K ds = u_Q;$$

and equation (8) becomes

$$(11) \quad u_Q = -\int_C \{X \cos(n, x) + Y \cos(n, y)\} ds.$$

The most interesting aspect of this formula is its dependence upon ξ, η which is brought about by the terms $v, \partial u/\partial x, \partial v/\partial y$ that enter in X and Y and are given analytically by (10). When Q lies in the interior of S (not on the boundary), then $\log \varrho$ is a regular analytic function, since the point $P = (x, y)$ in the integration is restricted to the boundary curve C and does not coincide with Q . Therefore $u_Q = u(\xi, \eta)$ is an analytic function of ξ, η in the interior of S . This holds whether or not the boundary values $u, \partial u/\partial x, \partial u/\partial y$ are analytic; in any case the dependence of the integrand on x, y disappears upon integration with respect to ds . Even discontinuities of the boundary values are averaged out. Discontinuities on the boundary are not continued into the interior of S . (The characteristics are imaginary.) This proves the assertion of §9B.

For a self-adjoint differential equation in its normal form we have, according to (6), $D = E = 0$. Using the form (7c) of the line integral we get from (11)

$$(11a) \quad u_Q = \int_C \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds.$$

Using expression (10a) for v , we get for the special case of the potential equation:

$$(11b) \quad u_Q = u(\xi, \eta) = \frac{1}{2\pi} \int_C \left(u \frac{\partial \log \varrho}{\partial n} - \log \varrho \frac{\partial u}{\partial n} \right) ds.$$

E. THE PRINCIPAL SOLUTION FOR AN ARBITRARY NUMBER OF DIMENSIONS

We restrict ourselves here to the case of the potential equation. The three-dimensional analogue of (9b) is

$$\frac{\partial v}{\partial r} = \frac{1}{4\pi r^2}, \quad v = -\frac{1}{4\pi r} + \text{const.}$$

(r = distance from the source at Q , $4\pi r^2$ = surface area of the sphere.) This is essentially⁷ the so-called "Newtonian potential."

In the four-dimensional case we have equation (7.5). This yields the principal solution:

$$\frac{\partial v}{\partial R} = \frac{1}{2\pi^2 R^3}, \quad v = -\frac{1}{4\pi^2 R^2} + \text{const.}$$

where R is the distance from Q and $2\pi^2 R^3$ is the surface of the hypersphere. The following table shows the decreasing orders of infinity at the source with decreasing number of dimensions. For the dimension one v is continuous at the source. In fact, the potential equation in one dimension is $d^2v/dx^2=0$ which yields $dv/dx = \text{const.}$ The constant will have different values C_1 and C_2 on the right and left side of the source respectively. This follows from the condition that it be a unit source so that $C_1 - C_2 = 1$. The discontinuity has passed from v to the gradient of v . (See exercise II.3).

Dimension	4	3	2	1
grad v . . .	$\frac{1}{2\pi^2 R^3}$	$\frac{1}{4\pi r^2}$	$\frac{1}{2\pi \varrho}$	C_1 or C_2
v	$-\frac{1}{4\pi^2 R^2}$	$-\frac{1}{4\pi r}$	$-\frac{1}{2\pi} \log \frac{1}{\varrho}$	continuous

F. DEFINITION OF GREEN'S FUNCTION FOR SELF-ADJOINT DIFFERENTIAL EQUATIONS

We now deal with the boundary value problem of §9. This question is by no means settled by the construction of the principal solution. We first consider the simplest case of self-adjointness. In order to calculate u at the point Q in equation (11a) we must know *both* u and $\partial u/\partial n$ on C , whereas in the boundary value problem we are given *either* u or $\partial u/\partial n$.

⁷ The denominator 4π corresponds to the "rational units" of electrodynamics.

Our problem is now to modify the principal solution v , so as to eliminate $\partial u / \partial n$ (or u) from (11a). We call this modified function of the two pairs of variables x, y and ξ, η Green's function and denote it by $G(P, Q)$. It has to satisfy the following conditions:

- a) $L(G) = 0$ in the interior of C ,
- b) $G = 0$ (or $\partial G / \partial n = 0$) on C ,
- c) $\lim_{P \rightarrow Q} G(P, Q) \rightarrow \frac{1}{2\pi} \log \varrho$ (condition of unit source).

Conditions a) and c) are the same as for the original v , but condition b) has been added. Replacing v by G in (11a) we get

$$(12) \quad u_Q = \int u \frac{\partial G}{\partial n} ds \quad \text{or} \quad \left(u_Q = - \int \frac{\partial u}{\partial n} G ds \right).$$

This solves the boundary value problem in both cases (for given u or $\partial u / \partial n$). However, due to condition b), the construction of G itself requires the solution of a boundary value problem. But this problem is simpler than the general boundary value problem, and we shall see that in special cases it can be solved in an elegant way with the help of a reflection process. On the other hand G , unlike u , is not regular in the interior of C , but like v is a function with a prescribed unit source.

Equation (12) reduces the boundary value problem to a simple quadrature. Green's function plays the same role in the general theory of integral equations. It is called there the "resolving kernel."

Another interesting property of G which follows from the conditions a), b), c) is the *reciprocity relation*

$$d) \quad G(P, Q) = G(Q, P).$$

It expresses the interchangeability of *source-point* and *action-point*, so to speak, the interchangeability of *cause* and *effect*.

In order to prove d) we substitute in (7a)

$$M = L, \quad u = G(I, P), \quad v = G(I, Q).$$

The point $I = (x_1, y_1)$ shall be called "point of integration." Since u becomes infinite for $I = P$, and v for $I = Q$, these points must be excluded from the integration by infinitesimal circles K_P and K_Q . According to a), integration over the region bounded by these circles and by C makes the left side of (7a) equal to 0; also, according to b), the integral over C on the right side of (7a) becomes equal to 0. There only remain the line integrals over K_P and K_Q which, according to c), yield:

$$u_Q \int_{K_Q} \frac{ds}{2\pi \varrho_Q} - v_P \int_{K_P} \frac{ds}{2\pi \varrho_P} = G(Q, P) - G(P, Q).$$

Since this must vanish d) is proved.

Equation (12) is the solution of the boundary problem for the homogeneous equation $L(u) = 0$. We now consider the solution of the non-homogeneous differential equation

$$(13) \quad L(u) = \varrho$$

where $(\varrho x, y)$ is an arbitrary continuous point-function in S with continuous first and second derivatives. Substituting $v = G(P, Q)$ in (7a) we get for the first term on the left side:

$$\int_S \varrho G(P, Q) d\sigma_P$$

which is added to the term in (12). Instead of (12) we get

$$(13a) \quad u_Q = \int_S \varrho G d\sigma + \int_C u \frac{\partial G}{\partial n} ds$$

or, if $\partial u / \partial n$ instead of u is given on C :

$$(13b) \quad u_Q = \int_S \varrho G d\sigma - \int_C \frac{\partial u}{\partial n} G ds.$$

These formulas apply to every self-adjoint differential expression in its normal form $L(u) = \Delta u + Fu$, in particular to the ordinary wave equation ($F = k^2 = \text{Const.}$) and to the potential equation ($F = 0$).

In the case of a *non-self-adjoint* differential form $L(u)$ equations (12) and (13a,b) remain valid. But, as we see from (7a), G must satisfy the adjoint equation $M(G) = 0$ in the variables x, y ; also, condition b) must be changed somewhat. Instead of a) and b) we now have:

$$a') \quad M(G) = 0,$$

$$b') \quad G = 0 \quad \left(\text{or } \frac{\partial G}{\partial n} - G \{D \cos(n, x) + E \cos(n, y)\} = 0 \right).$$

Condition c) remains valid. However the reciprocity law d) now reads

$$d') \quad G(P, Q) = H(Q, P).$$

Here H is Green's function for the *adjoint* equation to $M = 0$, and hence it satisfies the equation $L(H) = 0$ in the coordinates of Q .

§11. Riemann's Integration of the Hyperbolic Differential Equation

The normal form of a linear differential equation of second order of hyperbolic type is obtained from (8.1) by setting $A = C = 0$, $B = 1/2$:

$$(1) \quad L(u) = \frac{\partial^2 u}{\partial x \partial y} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u = 0.$$

Its adjoint differential equation is according to (10.4):

$$(2) \quad M(v) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial D v}{\partial x} - \frac{\partial E v}{\partial y} + F v = 0.$$

At the same time one obtains from (10.5)

$$(3) \quad \begin{aligned} X &= \frac{1}{2} \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} \right) + D u v, \\ Y &= \frac{1}{2} \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) + E u v. \end{aligned}$$

Substituting (1),(2),(3) in (10.7) we get:

$$(4) \quad \int_S \{v L(u) - u M(v)\} d\sigma = \int_C \{X \cos(n, x) + Y \cos(n, y)\} ds.$$

In order to obtain an integration of the hydrodynamic equations Riemann chose as the region S the "triangle" PP_1P_2 of Fig. 12, with a boundary consisting of the segments of characteristics PP_1 and PP_2 and of the arc P_1P_2 of the curve Γ . u and $\partial u / \partial n$ are given on Γ , which implies that $\partial u / \partial x$, $\partial u / \partial y$ are given on Γ (see p. 36). The curve Γ must satisfy the condition that it be tangent to no characteristic (p. 37). The function v in (4) is determined according to Riemann by the conditions:

(5a) $M(v) = 0$ in S with respect to the variables x, y ;

(5b) $v = 1$ at the point P with coordinates $x = \xi$, $y = \eta$;

(5c) $\frac{\partial v}{\partial y} - D v = 0$ on the characteristic $x = \xi$,
 $\frac{\partial v}{\partial x} - E v = 0$ on the characteristic $y = \eta$.

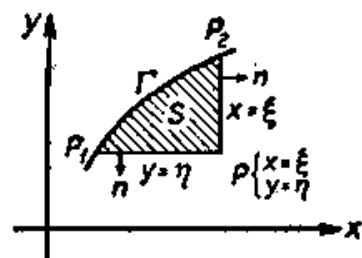


Fig. 12. Riemann's integration of a hyperbolic differential equation in its normal form with the help of the characteristic function v .

We add the following remarks.

1. It would not be possible to replace (5b) by a condition of discontinuity as in (10.10), since a hyperbolic equation does not admit isolated singularities (every singularity is continued along the characteristics). For this reason we no longer call v the "principal solution" or "Green's function" but call it the "characteristic function."

2. The conditions (5c) prescribe only *one* condition each on the characteristics $x = \xi$ and $y = \eta$, whereas on Γ two conditions have to be given for u . This corresponds to the fact that the characteristics are an exception with respect to the boundary conditions that are to be prescribed on them; we saw this in §9 in connection with the equation of the vibrating string. If we call the boundary value problem along two intersecting characteristics a *boundary value problem of the second kind*, in contrast to a *boundary value problem of the first kind* along a general curve Γ , then we can say that Riemann's method consists of *the reduction of a boundary value problem of the first kind to a much simpler boundary value problem of the second kind*.

Substituting condition (5a) in (4) and remembering that $L(u) = 0$ we get

$$(6) \quad 0 = \int_{\Gamma} \cdots + \int_{P_1}^P \cdots + \int_P^{P_1} \cdots$$

In the last integral $\cos(n, y) = 0$ and only the X term remains. We transform the term with $\partial u / \partial y$ by integration by parts:

$$\frac{1}{2} \int_P^{P_1} v \frac{\partial u}{\partial y} dy = \frac{1}{2} v u \Big|_P^{P_1} - \frac{1}{2} \int_P^{P_1} u \frac{\partial v}{\partial y} dy.$$

Combining this with the other terms we get

$$(6a) \quad \int_P^{P_1} X dy = \frac{1}{2} (v u)_{P_1} - \frac{1}{2} (v u)_P - \int_P^{P_1} u \left(\frac{\partial v}{\partial y} - Dv \right) dy.$$

For the middle integral of (6) where $\cos(n, x) = 0$ and $\cos(n, y) = -1$ (n is the outer normal) we get in an analogous manner:

$$(6b) \quad - \int_{P_1}^P Y dx = \frac{1}{2} (v u)_{P_1} - \frac{1}{2} (v u)_P + \int_{P_1}^P u \left(\frac{\partial v}{\partial x} - E v \right) dx.$$

The integrals on the right sides of (6a) and (6b) vanish on account of condition (5c). If we consider (5b) equation (6) becomes,

$$(7) \quad u_P = \int_F \{X \cos (n, x) + Y \cos (n, y)\} ds + \frac{1}{2} \{(v u)_{P_1} + (v u)_{P_2}\}.$$

The value of u at an arbitrary point P is given here in terms of the values of u and its first derivatives on F as they enter in X and Y (u_{P_1} and u_{P_2} are among those values). We state: Equation (7), reduces the boundary value problem of the first kind for u to the problem of the computation of v , that is to a boundary value problem of the second kind which is given by the conditions (5a,b,c).

The computation of v is not difficult. It is particularly easy in the hydrodynamic example that was treated by Riemann. In that case we have⁸

$$(8) \quad D = E = -\frac{a}{x+y}, \quad F = 0.$$

Condition (5c) implies

$$\begin{aligned} x = \xi : \quad \frac{1}{v} \frac{\partial v}{\partial y} &= -\frac{a}{\xi+y}, & v &= C_1(\xi+y)^{-a}, \\ y = \eta : \quad \frac{1}{v} \frac{\partial v}{\partial x} &= -\frac{a}{x+\eta}, & v &= C_2(x+\eta)^{-a}. \end{aligned}$$

Both these conditions and condition (5b) are satisfied if we set:

$$(9) \quad C_1 = C_2 = (\xi+\eta)^a, \quad v = \left(\frac{\xi+\eta}{x+y}\right)^a.$$

In order to satisfy (5a) Riemann modifies (9) as follows:

$$(10) \quad v = \left(\frac{\xi+\eta}{x+y}\right)^a F(a+1, -a, 1, z), \quad z = -\frac{(x-\xi)(y-\eta)}{(x+y)(\xi+\eta)},$$

where

$$(10a) \quad F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \dots$$

is the hypergeometric series. In §24D we shall see some of the function-theoretic properties of this series, and in appendix II of Chapter IV we shall prove that v in (10) satisfies condition (5a), in other words, that it is a solution of the equation $M(v) = 0$. We note that on the characteristics we have $x = \xi$ or $y = \eta$, and therefore $z = 0$ and $F = 1$, which makes (10) identical with (9). Equation (10) for v and equation (7) for u solve our hyperbolic boundary value problem completely.

⁸The constant a which enters in (8) is expressed simply in terms of the exponent in the anisotropic equation of state from which the hydrodynamic problem is derived.

§12. Green's Theorem in Heat Conduction. The Principal Solution of Heat Conduction

The differential equation (7.14) of heat conduction

$$(1) \quad L(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0, \quad y = kt,$$

is not self-adjoint. The adjoint equation of (1) is:

$$(2) \quad M(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y} = 0.$$

We see this from (9.4) if we substitute the values of (1)

$$B = C = D = F = 0, \quad A = 1, \quad E = -1;$$

from (9.5) we get

$$(3) \quad X = v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}, \quad Y = -uv.$$

Just as in the elliptic and the hyperbolic case we get *Green's theorem for linear heat conduction* from (1), (2), (3) by integration over the interior and the boundary of a bounded domain in the x, y -plane. However, since x represents a spatial measurement and y a time measurement, we shall not consider here a region with curved boundary, but only such regions whose boundaries consist of segments parallel to the x - or y -axis, as that shown in Fig. 13.

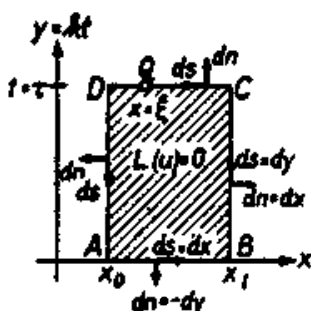


Fig. 13. Reduction of the general boundary value problem of heat conduction to the principal solution V for a rod with endpoints x_0 and x_1 . The unit heat pole is at Q and has the coordinates $x = \xi$, $y = \eta$.

Along the side AB of the figure we have $ds = dx$, $dn = -dy$, $\cos(n, x) = 0$, $\cos(n, y) = -1$ and therefore

$$\int_A^B \{X \cos(n, x) + Y \cos(n, y)\} ds = - \int_A^B Y dx.$$

The same thing holds for the side CD which is also parallel to the x -axis and where the signs of both dx and $\cos(n, y)$ are reversed. Correspondingly we have for the sides BC and AD parallel to the y -axis

$$\int_B^C \{X \cos(n, x) + Y \cos(n, y)\} ds = + \int_B^C X dy.$$

Using the values of L, M, X, Y given in (1), (2), (3) we get the following form of Green's theorem:

$$(4) \quad \int \left\{ v \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} \right) - u \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y} \right) \right\} dx dy \\ = \int u v dx + \int \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) dy,$$

where the first integral on the right side is taken over the two sides of the rectangle that are parallel to the x -axis and the second integral is taken over the other sides.

Formula (4) also represents Green's theorem for *two-dimensional* or *three-dimensional* heat conduction if we perform the following replacements:

$$(5a) \quad dx \quad \text{by} \quad \begin{cases} d\sigma & \text{(two-dimensional case)} \\ d\tau & \text{(three-dimensional case)} \end{cases}$$

$$(5b) \quad \frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial x}, \quad dy \quad \text{by} \quad \frac{\partial u}{\partial n}, \quad \frac{\partial v}{\partial n}, \quad \begin{cases} dy ds & \text{(two-dimensional case)} \\ dy d\sigma & \text{(three-dimensional case)} \end{cases}$$

$$(5c) \quad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 v}{\partial x^2} \quad \text{by} \quad \Delta u, \Delta v.$$

In the three-dimensional case we integrate over a four-dimensional cylinder whose base is the three-dimensional heat conductor and whose generatrix is parallel to the time-axis; the integration in the second term on the right, which is indicated in (5b) by dy and is now replaced by integration with respect to $dy d\sigma = k dt d\sigma$, is extended over the three-dimensional lateral surface of this cylinder.

Before we apply these general formulas we must decide how we want to define the analogue of the "principal solution" of (10.9). We shall see that the "unit source" will have to be replaced by a "heat pole of strength one."

We first consider the case of linear heat conduction and its differential equation $L = 0$; the passage to the adjoint equation $M = 0$ and to the two- and three-dimensional cases will then be easy.

Let the heat conductor be infinite in both directions and let its temperature for $t = 0$ be given as a function of x :

$$u = f(x), \quad -\infty < x < +\infty.$$

We represent $f(x)$ by a Fourier integral as in (4.8):

$$(6) \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\xi) e^{i\omega(x-\xi)} d\xi.$$

In order to obtain a solution of equation (1) we must merely complete $\exp[i\omega(x-\xi)]$ to the product

$$(7) \quad \varphi(y) e^{i\omega(x-\xi)}.$$

Substituting this in (1) we get:

$$-\omega^2 \varphi(y) = \frac{d\varphi(y)}{dy}, \quad \varphi(y) = C e^{-\omega^2 y}.$$

Here $C = 1$ on account of the obvious condition $\varphi(0) = 1$. We therefore replace in (6)

$$(7a) \quad \exp[i\omega(x-\xi)] \quad \text{by} \quad \exp[i\omega(x-\xi) - \omega^2 y].$$

This seemingly complicates the Fourier integral (6) but in reality it makes it much simpler. In (6) $f(x)$ must converge to 0 "sufficiently rapidly" in order that the integral with respect to ξ will converge, and this integration must be performed *before* the much simpler integration with respect to ω , which would otherwise not converge. But now the order of integration is reversible and $f(x)$ is less restricted in its behavior at infinity. The new factor $\varphi(y) = \exp(-\omega^2 y)$ serves as convergence factor³ for all $y > 0$.

Combining (6) and (7) and substituting $y = kt$ we get:

$$(8) \quad u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) d\xi \int_{-\infty}^{+\infty} e^{-\omega^2 kt + i\omega(x-\xi)} d\omega.$$

We abbreviate the exponent in (8) by $-\alpha \omega^2 + \beta \omega$ and complete the square:

$$-\alpha \omega^2 + \beta \omega = -\alpha \left(\omega - \frac{\beta}{2\alpha} \right)^2 + \frac{\beta^2}{4\alpha}.$$

³ See the author's dissertation, Königsberg 1891: "Die willkürlichen Funktionen der mathematischen Physik" where the general case of the limit for $t \rightarrow 0$ of a Fourier integral with a convergence factor is considered. The function $f(x)$ may then have, for example, "an infinity of maxima and minima" or arbitrary discontinuities.

Substituting $p = \omega - \beta/2\alpha$: we get:

$$(9) \quad \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\omega^2 k t + i\omega(x-\xi)} d\omega = \frac{1}{2\pi} e^{-\frac{(x-\xi)^2}{4kt}} \int_{-\infty}^{+\infty} e^{-\alpha p^2} dp.$$

We have the well known formula for the Laplace integral:

$$\int_{-\infty}^{+\infty} e^{-p^2} dp = \sqrt{\pi}, \text{ and therefore } \int_{-\infty}^{+\infty} e^{-\alpha p^2} dp = \sqrt{\frac{\pi}{\alpha}}.$$

Denoting the right side of (9) by U we get

$$(10) \quad U = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\xi)^2}{4kt}}.$$

Equation (8) then becomes

$$(10a) \quad u(x, t) = \int_{-\infty}^{+\infty} f(\xi) U d\xi:$$

We note that the initial temperature at the point $x = \xi$ spreads in space-time independently of the initial temperature at all other points. (This is due to the linearity of the differential equation which permits the superimposition of solutions.) For $t \rightarrow 0$ we have $u(x, t) \rightarrow f(x)$ and (10a) becomes:

$$f(x) = \int_{-\infty}^{+\infty} f(\xi) U d\xi.$$

This shows that U has the "character of a δ function." As on p. 27 this means that U vanishes in the limit $t \rightarrow 0$ for all values of $x \neq \xi$ and becomes infinite at $x = \xi$ so that

$$(10b) \quad \int_{x-\epsilon}^{x+\epsilon} U d\xi = 1$$

(These properties of U are easily seen from (10).) Ignoring the distinction between heat-energy and temperature we may say that U describes the space-time behavior of a *unit heat-source* or of a *heat-pole of strength 1*.

For the case of a general initial time $t = \tau$ we get instead of (10):

$$(10c) \quad U = \{4\pi k(t-\tau)\}^{-\frac{1}{2}} \exp\left\{-\frac{(x-\xi)^2}{4k(t-\tau)}\right\}$$

For the special case of a heat-pole at $\xi = 0$ we get

$$(10d) \quad U = (4\pi kt)^{-\frac{1}{2}} \exp \left\{ -\frac{x^2}{4kt} \right\}.$$

Before discussing the deeper meaning of these formulas we shall generalize them to two and three dimensions.

We noted the possibilities of generalizing Fourier's double integral to quadruple and sextuple integrals at the end of §4. We perform this generalization by writing instead of (6):

$$(11) \quad f(x, y) = \frac{1}{2\pi} \int d\omega \int f(\xi, y) e^{i\omega(x-\xi)} d\xi$$

and

$$(11a) \quad f(\xi, y) = \frac{1}{2\pi} \int d\omega' \int f(\xi, \eta) e^{i\omega'(y-\eta)} d\eta;$$

Combining (11) and (11a) we get

$$(11b) \quad f(x, y) = \frac{1}{(2\pi)^2} \int d\omega \int d\omega' \iint f(\xi, \eta) e^{i\omega(x-\xi) + i\omega'(y-\eta)} d\xi d\eta.$$

The same process which led from (6) to (10a) leads for the two-dimensional case from (11b) to

$$(12) \quad u(x, y, t) = \iint_{-\infty}^{+\infty} f(\xi, \eta) U d\xi d\eta$$

where U is the product of two factors of the form (10):

$$(13) \quad U = (4\pi kt)^{-1} e^{-\frac{(x-\xi)^2 + (y-\eta)^2}{4kt}}$$

In the three-dimensional case U is the product of three factors of the form (10):

$$(14) \quad U = (4\pi kt)^{-\frac{3}{2}} e^{-\frac{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}{4kt}}$$

Equations (13) and (14) stand for *unit heat-poles in the plane and in space*. Equations (10), (13), (14) indicate the connection between *heat-conduction* and *probability*.

We compare (10d) with the Gaussian law of error

$$(15) \quad dW = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} dx.$$

Here dW is the probability of an error between x and $x + dx$ in a measuring process whose precision is given by the "precision factor" α . In our case this factor is $(4kt)^{-1}$; "infinite precision" is given by $t = 0$ which means absolute concentration of heat in the point $x = 0$; "decreasing precision" corresponds to increasing t . Fig. 14 shows the well known bell-shaped curves which for decreasing α give the behavior of U for increasing t . The function U in (10d) is equal to the "probability density" dW/dx .

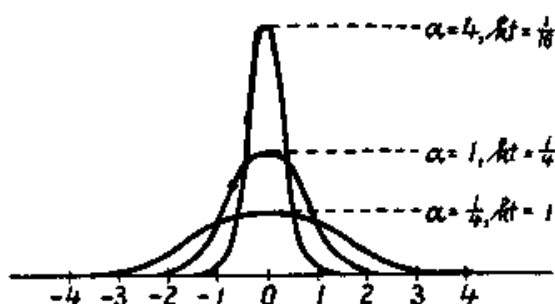


Fig. 14. The Gaussian error curve for the precision factors $\alpha = 4, 1, 1/4$, being at the same time the principal solution of heat conduction for $kt = 1/16, 1/4, 1$.

In an analogous manner we compare (13) to the measuring of a position in the xy -plane whose exact position is given by (ξ, η) ; and (14) with the measurement of a space point with the true position (ξ, η, ζ) . The precision-factor in both cases is $\alpha = (4kt)^{-1}$ as before. This suggests that the physical reason for heat-conduction is of a *statistical* — not *dynamical* — nature. This becomes apparent in the treatment of heat conduction in the kinetic theory

of gases (or, more correctly, the "statistical theory of gases"). Connected with this is the following fact which we discuss for the spatial case of equation (14). For $t = 0$ the total heat-energy is concentrated at the point (ξ, η, ζ) , but after an arbitrarily short time we have a non-vanishing temperature U at a distant point (x, y, z) . Hence *heat expands with infinite velocity*. This is impossible from the point of view of dynamics where no velocity may exceed c .

From §7, p. 34 we know that diffusion, electric conduction, and viscosity satisfy the same differential equation as heat conduction. Here too the statistical approach is clear. Diffusion is based on the *Brownian motion* in a solvent of the individual dissolved molecules, and the statistical origin is ascertained both by theory and experiment. The electron theory of metals shows that upon electrical conduction the electrons are diffused through the grid of metal molecules etc.

The function U of the equations (10), (13), (14) is the *principal solution of the differential equation* $L(u) = 0$. We now wish to transform it into the principal solution V of the *adjoint equation* $M(v) = 0$. Comparing (1) and (2) we see that this is done by reversing the sign of $y = kt$; we shall also reverse the sign of $y_0 = k\tau$ so that the heat pole will again be situated at the point $x = \xi, t = \tau$. Thus we obtain from (10c):

$$(16) \quad V = \{4\pi k(\tau - t)\}^{-\frac{1}{2}} \exp \left\{ -\frac{(x - \xi)^2}{4k(\tau - t)} \right\}$$

V has an essential singularity for $t = \tau$ and is defined only for the past of τ , i.e., for $t < \tau$, in contrast to the principal solution of U which is regular only for the future of τ , i.e., for $t > \tau$.

We return to Green's theorem (4). Setting $v = V$ and taking for u a solution of $L(u) = 0$ we get:

$$(17) \quad \int u V dx + k \int \left(V \frac{\partial u}{\partial x} - u \frac{\partial V}{\partial x} \right) dt = 0.$$

The two integrals are extended over the sides of the rectangle of Fig. 13, the first over the two horizontal sides, the second over the two vertical sides.

Since V too is a “ δ function” the first integral taken over the side $t = \tau$ yields $-u_Q$. If we decompose the second integral into the two components which correspond to the two rod ends x_0 and x_1 and denote this by $\Sigma^{(x_1, x_0)}$ we get:

$$(18) \quad u_Q = \int_{x_0}^{x_1} u V_0 dx + k \Sigma^{(x_1, x_0)} \int_0^\tau \left(V \frac{\partial u}{\partial x} - u \frac{\partial V}{\partial x} \right) dt.$$

Here V_0 is V for $t = 0$.

This representation of u is general since the source point Q can be situated at the *arbitrary* point $x = \xi$, $t = \tau$. However it does not yet solve the boundary value problem of §9C, since in addition to the initial values of u it assumes that the boundary values of u and of $\partial u / \partial x$ are given at the endpoints, whereas in the boundary value problem only u or $\partial u / \partial x$ may be prescribed. In order to solve the boundary value problem we must replace V in (18) by *Green's function* G which *satisfies the condition* $G = 0$ *at the endpoints* and thereby makes the term containing $\partial u / \partial x$ in (18) vanish. We shall see in the next chapter how G can be constructed from V by a reflection process. Exercise II.4 contains an application of (18) to laminar fluid friction.

The above considerations can be transferred immediately to the two- and three-dimensional cases. As remarked above in connection with (5a, b), we merely have to extend the integration in the first integral of (18) over the base, and in the second integral over the lateral surface of the three- or four- dimensional space-time cylinder. However the construction of Green's function G by a reflection process for the two and three dimensional problems will succeed only in exceptional cases (see §17).

On the other hand equation (18) (in terms of V , not G) suffices to insure the analytic character of u , since the coordinates ξ, τ (or ξ, η, τ or ξ, η, ζ, τ) of Q appear on the right side only in the principal solution V , that is, only in analytic form. The solutions of our parabolic differential equation are *analytic* in the interior of their domain just as in the elliptic case. However, the domain here is not bounded as in the elliptic case, but is an infinite strip (as was pointed out at the end of §9). From this latter point of view the parabolic boundary value problem resembles the hyperbolic one.

CHAPTER III

Boundary Value Problems in Heat Conduction

§13. Heat Conductors Bounded on One Side

In the preceding section we treated the equalization process for a linear heat conductor that is infinite in both directions and represented it by equation (12.10a):

$$(1) \quad u(x, t) = \int_{-\infty}^{+\infty} f(\xi) U d\xi, \quad U = (4\pi kt)^{-\frac{1}{2}} \exp \left\{ -\frac{(x-\xi)^2}{4kt} \right\}.$$

By the substitution

$$(1a) \quad \xi = x + \sqrt{4kt} z$$

it goes over into the *Laplace form*

$$(2) \quad u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x + \sqrt{4kt} z) e^{-z^2} dz.$$

It is instructive to compare this with d'Alembert's solution (9.2): in the latter we have two arbitrary functions F_1, F_2 , corresponding to the hyperbolic type of the wave equation, whereas in (1) and (2) we have *one* arbitrary function f corresponding to the parabolic type of the equation of heat conduction.

In the case of a heat conductor which is bounded on one side, $0 < x < \infty$, we have to deal first with the boundary condition at $x = 0$:

a) A given temperature $u(0, t)$; in particular the *isothermal* boundary condition

$$(3a) \quad u = 0.$$

b) A given heat flow $G(0, t)$ (notation as in §7, equations (9) to (12)); in particular the *adiabatic* boundary condition

$$(3b) \quad \frac{\partial u}{\partial x} = 0.$$

c) A linear combination of both which takes into consideration the so-called *outer heat conduction*, written in the conventional form

$$(3c) \quad \frac{\partial u}{\partial n} + hu = 0.$$

Here n stands for the *outer* normal which in our case is in the direction of the *negative* x -axis. The name "outer heat conduction" summarizes the effect of convection, the radiation into the surrounding medium and the heat conduction into that medium which is usually negligible. We note that (3c) is obtained as an approximation of the radiation law of Stefan-Boltzmann which states: the radiation of heat per unit of time and area of a body of absolute temperature T is proportional to T^4 . If we denote the factor of proportionality by a and if the end of the rod¹ is in a neighborhood of temperature T_0 which radiates towards the end of the rod an amount of heat aT_0^4 per unit of time and area, then the energy emitted in the normal direction by a surface element $d\sigma$ of the end of the rod is given by:

$$dQ_n = a(T^4 - T_0^4)d\sigma dt.$$

Since usually both temperatures T and T_0 are far from absolute zero, we get

$$(4) \quad dQ_n \sim 4aT_0^3 u d\sigma dt \quad \text{with } u = T - T_0 \ll T \quad \text{and } \ll T_0.$$

This amount of heat dQ_n must be balanced by the heat flow from the interior of the rod which is given by Fourier's law (7.12). Hence we write:

$$(4a) \quad dQ_n = -\kappa \frac{\partial u}{\partial n} d\sigma dt.$$

By comparison of (4) and (4a) we obtain:

$$4aT_0^3 u = -\kappa \frac{\partial u}{\partial n},$$

and hence

$$(5) \quad \frac{\partial u}{\partial n} + hu = 0, \quad h = \frac{4aT_0^3}{\kappa},$$

which corresponds to (3c) and shows h to be a *positive* constant.

We first treat conditions (3a) and (3b). These conditions are satisfied if we develop f , which is given only for $0 < x < \infty$, into a *pure sine* or

¹ We speak here of a "rod" although the linear heat conductor need not have the form of a thin rod but may have an arbitrary cross-section so long as its state depends only on *one* coordinate; for a real rod one must add the adiabatic condition $\partial u / \partial n = 0$ for the lateral surface (see the end of §16).

cosine integral, or in other words, if we continue f in the negative x -axis as an *even* or *odd* function (see equations (4.11a,b)). If as in (12.8) we append the time dependence factor $\exp(-\omega^2 k t)$ which is required by the equation of heat conduction, and integrate with respect to ω , then (1) becomes

$$(6) \quad u(x, t) = \int_0^{\infty} f(\xi) U(\xi) d\xi \mp \int_0^{\infty} f(\xi) U(-\xi) d\xi.$$

The second integral which was originally taken from $-\infty$ to 0 has been converted by a change of sign of the variable of integration into an integral from 0 to $+\infty$. The principal solution $U(\xi)$ is then transformed into

$$(7) \quad U(-\xi) = (4 \pi k t)^{-\frac{1}{2}} \exp \left\{ -\frac{(x + \xi)^2}{4 k t} \right\}$$

which is the expression for a unit heat pole at $x = -\xi$, $t = 0$. Equation (6) becomes:

$$(8) \quad u(x, t) = \int_0^{\infty} f(\xi) G d\xi, \quad G = U(\xi) \mp U(-\xi).$$

This *Green's function* G satisfies all the conditions of p. 61. It has only one heat pole in the region $0 < x < \infty$, since the additional heat pole at $x = -\xi$ lies outside the region; it also complies with the condition that G satisfy the *adjoint* equation in the variables ξ, τ , since in our case G is independent of τ and a change of sign in τ becomes immaterial.

It would be more intuitive to start from a single heat pole at $x = \xi$ and to reflect it on the boundary $x = 0$, with a negative or positive sign of U depending on whether we impose condition (3a) or (3b). In this manner we would first construct *Green's function* and then reconstruct the given initial temperature $f(x)$ by the successive superposition of the heat poles of strength $f(\xi) d\xi$. From now on we shall use mainly this intuitive process, that is we restrict ourselves to the *construction of Green's function* from which we can write down the solution for arbitrary initial temperature $f(x)$ as in (8). We first use this

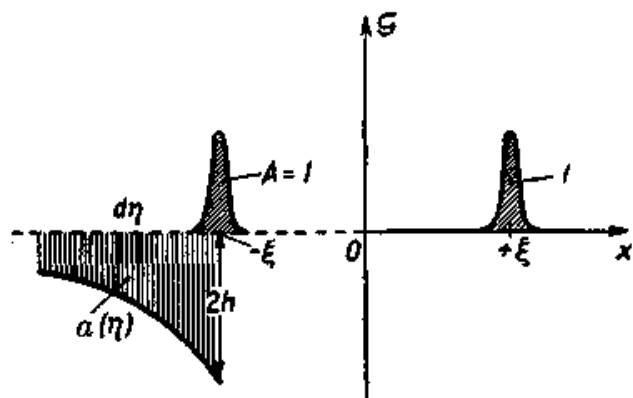


Fig. 15. *Green's function* for a linear heat conductor which is bounded on one side with outer heat conduction. A heat pole is at $x = +\xi$, its mirror image at $x = -\xi$, with an associated continuous spectrum of heat sources.

process for the somewhat more complicated boundary condition (3c). (In problem III.1 we shall treat the same boundary condition according to Fourier's process.)

We notice at once that an isolated reflected point $x = -\xi$ will not be sufficient, but that we also need a continuous sequence of heat sources which we shall place at all points $\eta < -\xi$. Let A and $a(\eta) d\eta$ be the yields of the isolated and the continuous heat sources (see Fig. 15). The corresponding function G is then

$$(9) \quad G = U(\xi) + AU(-\xi) + \int_{-\infty}^{-\xi} a(\eta) U(\eta) d\eta \\ = (4\pi kt)^{-\frac{1}{2}} \left[e^{-\frac{(x-\xi)^2}{4kt}} + Ae^{-\frac{(x+\xi)^2}{4kt}} + \int_{-\infty}^{-\xi} a(\eta) e^{-\frac{(x-\eta)^2}{4kt}} d\eta \right]$$

Hence at $x = 0$:

$$(10) \quad (4\pi kt)^{\frac{1}{2}} G = (1 + A) e^{-\frac{\xi^2}{4kt}} + \int_{-\infty}^{-\xi} a(\eta) e^{-\frac{\eta^2}{4kt}} d\eta.$$

From (9) we form $\partial G / \partial x$. Then if we replace $\partial / \partial x$ by $-\partial / \partial \eta$ under the integral sign we obtain for $x = 0$

$$(10a) \quad (4\pi kt)^{\frac{1}{2}} \frac{\partial G}{\partial x} = \xi \left(\frac{1}{2kt} - \frac{A}{2kt} \right) e^{-\frac{\xi^2}{4kt}} - \int_{-\infty}^{-\xi} a(\eta) \frac{\partial}{\partial \eta} e^{-\frac{\eta^2}{4kt}} d\eta.$$

and after integrating by parts:

$$(11) \quad (4\pi kt)^{\frac{1}{2}} \frac{\partial G}{\partial x} = \left[(1 - A) \frac{\xi}{2kt} - a(-\xi) \right] e^{-\frac{\xi^2}{4kt}} + \int_{-\infty}^{-\xi} a'(\eta) e^{-\frac{\eta^2}{4kt}} d\eta.$$

If we substitute (10) and (11) in condition (3c) with $\partial / \partial n$ replaced by $-\partial / \partial x$, then (3c) must be satisfied identically for all $t > 0$. By setting the terms of different time dependence individually equal to zero we obtain:

$$(12) \quad A - 1 = 0 \quad \dots \quad A = +1,$$

$$(13) \quad a(-\xi) + h(1 + A) = 0 \quad \dots \quad a(-\xi) = -2h,$$

and the differential equation:

$$(14) \quad a'(\eta) - h a(\eta) = 0$$

Considering (13) we see that (14) has the solution

$$a(\eta) = b e^{h\eta} = -2h e^{h(t+\eta)}$$

This determines the constant A and the function $a(\eta)$. The fact that this determination is unique will be demonstrated in §17.

The result is

$$(15) \quad (4\pi kt)^{\frac{1}{2}} G = e^{-\frac{(x-t)^2}{4kt}} + e^{-\frac{(x+t)^2}{4kt}} - 2h e^{ht} \int_{-\infty}^{-t} e^{-\frac{(x-\eta)^2}{4kt}} e^{h\eta} d\eta.$$

For numerical applications this integral can be reduced to the tabulated normal error function.²

Only when the given initial values $f(x)$ are particularly simple will it be more convenient to use equations (1) and (2) instead of the more intuitive method of Green's function. We illustrate this by an example which also shows the translation of problems of heat conduction into the language of *diffusion problems*.

Let the bottom section of a cylindrical vessel, $0 < x < H$, be filled with a concentrated solution (say CuSO_4); above it let there be a layer of pure solvent (water) to an arbitrary height $x = \infty$. Let the concentration of the solution be u and let the initial concentration be 1. At the base of the cylinder we have $\partial u / \partial x = 0$ at all times, since the dissolved salt molecules cannot penetrate the base.

This condition may also be satisfied by extending the vessel downward and by prescribing the reflected initial distribution as above. (For a finite height of water column one would have to use the somewhat more complicated reflection process of §16.) The initial distribution of u is then:

$$f(x) = \begin{cases} 1 \cdots -H < x < +H, \\ 0 \cdots H < |x| < \infty. \end{cases}$$

Equation (2) yields

$$(16) \quad u(x, t) = \frac{1}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-z^2} dz.$$

² E.g., in Jahnke-Emde's tables of functions, 3rd ed., Teubner, Leipzig, 1938, p. 24

The values for the limits of integration z_1 and z_2 are obtained if in (1a) we let $\xi = \pm H$:

$$(17) \quad z_1 = \frac{-H-x}{\sqrt{4kt}}, \quad z_2 = \frac{H-x}{\sqrt{4kt}}.$$

Using the customary notation

$$(18) \quad \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$$

for the error function, we can write our solution (16) with astonishing simplicity:

$$(19) \quad u(x, t) = \frac{1}{2} \{ \Phi(z_2) - \Phi(z_1) \}.$$

§14. The Problem of the Earth's Temperature

We treat the surface of the earth as a plane and assume an averaged *purely periodic* temperature $f(t)$ (annual averaged or daily averaged temperature). In order to determine the temperature in the earth's interior³ we can in general use the method described in Fig. 13, by setting $x_0 = 0$ (surface of the earth), $x_1 = \infty$ (great depth), and $u_0 = f(t)$ for $x = 0$. It is convenient in our case to expand $f(t)$ into a complex Fourier-series:

$$(1) \quad f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{2\pi i n t / T}, \quad T = \text{length of year or day}$$

and to set for the temperature in the interior of the earth at a depth x :

$$(2) \quad u(x, t) = \sum_{n=-\infty}^{+\infty} C_n u_n(x) e^{2\pi i n t / T}.$$

Each individual term of this series must satisfy the basic law of heat conduction. This yields the ordinary differential equation for u_n :

$$(3) \quad \frac{d^2 u_n}{dx^2} = p_n^2 u_n \dots \text{ with } p_n^2 = \frac{2\pi i n}{kT}.$$

In order that (2) go into (1) for $x = 0$, we must have (3a)

$$(3a) \quad u_n(0) = 1.$$

³ The physical problem of "geothermic depth" (increase of temperature in the interior of the earth due to radioactive or nuclear processes) is of course ignored here.

Depending on whether n is positive or negative we set

$$2in = (1 \pm i)^2 |n|$$

and

$$(4) \quad p_n = (1 \pm i) q_n, \quad q_n = \sqrt{\frac{|n| \pi}{kT}} > 0.$$

The general solution of (3) is then

$$(5) \quad u_n(x) = A_n e^{(1 \pm i) q_n x} + B_n e^{-(1 \pm i) q_n x}.$$

Here we must have $A_n = 0$, since otherwise the temperature would become infinite for $x \rightarrow \infty$, and $B_n = 1$, to satisfy (3a). Substituting this in (2) we get

$$(6) \quad u(x, t) = \sum_{n=-\infty}^{+\infty} C_n e^{-(1 \pm i) q_n x} e^{2\pi i n t / T}.$$

In order to translate this into real language we write for $n > 0$

$$C_n = |C_n| e^{i\gamma_n}.$$

According to (1.13) C_n for $n < 0$ has the same absolute value but the negative phase. Equation (6) then becomes

$$(7) \quad u(x, t) = C_0 + 2 \sum_{n=1}^{\infty} |C_n| e^{-q_n x} \cos \left(2\pi n \frac{t}{T} + \gamma_n - q_n x \right).$$

We see that the amplitude $|C_n|$ of the n -th partial wave is *damped exponentially* with increasing depth x , and that this damping increases with increasing n . At the same time the phase of the partial wave is *retarded* increasingly with increasing x and n .

We now consider the numerical values. For an average type of soil we have the approximate temperature conductivity

$$k = 2 \cdot 10^{-3} \frac{\text{cm}^2}{\text{sec}}.$$

For the period of one year $T = 365 \times 24 \times 60 \times 60 = 3.15 \times 10^7$ sec. and for $x = 1 \text{ m} = 100 \text{ cm}$. we have then

$$(8) \quad q_1 x = 0.7 \sim \frac{\pi}{4}, \quad e^{-q_1 x} \sim \frac{1}{2}.$$

At a depth of 4 meters we already have a "phase lag" $q_1 x = \pi$ and an amplitude damping $2^{-4} = 1/16$. Even for the first and principal partial wave of temperature fluctuation it is *winter at a depth of 4 meters when it is summer on the surface; the amplitude is only a fraction of the surface amplitude*. For the higher partial waves $n > 1$ the phase lag and amplitude damping are correspondingly greater owing to the factor $\sqrt{|n|}$ in q_n . We may say that the ocean acts as a *harmonic analyzer* (see p. 4) by singling out the principal (though much weakened) wave from among all partial waves.

As a special example we consider the yearly curve of an "extremely continental climate," namely a uniform summer temperature and the same negative winter temperature which we shall set arbitrarily $= \pm 1$. This year-curve is represented graphically by the meander line of Fig. 1. and analytically by (2.2)

$$(9) \quad u(0, t) = \frac{4}{\pi} \left(\sin \tau + \frac{1}{3} \sin 3\tau + \frac{1}{5} \sin 5\tau + \dots \right), \quad \tau = 2\pi \frac{t}{T}.$$

In order to obtain the corresponding series $u(x, t)$ we must, according to (2.1a), specialize the coefficients C in (7) as follows:

$$C_{2n} = 0, \quad |C_{2n+1}| = \frac{2}{\pi(2n+1)}, \quad \gamma_{2n+1} = -\frac{\pi}{2}.$$

However it is somewhat simpler to apply the calculation process used for (2) directly to equation (9). We immediately get:

$$(9a) \quad u(x, t) = \frac{4}{\pi} \left(e^{-q_1 x} \sin(\tau - q_1 x) + \frac{1}{3} e^{-q_3 x} \sin(3\tau - q_3 x) + \dots \right).$$

Then, substituting the values of the q we get for $x = 100$ cm.

$$(9b) \quad u(x, t) \approx \frac{4}{\pi} \left[\frac{1}{2} \sin\left(\tau - \frac{\pi}{4}\right) + \frac{1}{3 \cdot 3.4} \sin\left(3\tau - \frac{\sqrt{3}\pi}{4}\right) + \frac{1}{5 \cdot 4.8} \sin\left(5\tau - \frac{\sqrt{5}\pi}{4}\right) + \dots \right]$$

and for $x = 400$ cm.

$$(9c) \quad u(x, t) \approx \frac{4}{\pi} \left(\frac{1}{16} \sin(\tau - \pi) + \frac{10^{-3}}{3 \cdot 1.3} \sin(3\tau - \sqrt{3}\pi) + \frac{10^{-3}}{5 \cdot 5.3} \sin(5\tau - \sqrt{5}\pi) + \dots \right).$$

A comparison of (9) with (9b,c) shows clearly the influence of depth on the amplitude and the phase of the temperature process.

This shows the usefulness of a deep cellar. It has not only much smaller temperature fluctuations than the surface of the earth, but is also warmer in winter than in summer (or it would be if there were no air flow).

Our conclusions become even more striking if we pass from the consideration of an averaged yearly temperature to that of an averaged *daily* temperature. The q_n are then increased by the factor $\sqrt{365} \sim 19$. Hence the damping and phase lag which for a yearly period belongs to a depth x occurs now at a depth of $x/19$. The decrease of amplitude to $1/16$ for the principal term (see (9c)) and the reversal of the time of day (midnight instead of noon) will now occur at a depth of only $x = 400/19 = 21$ cm. Hence the daily fluctuations of temperature enter into the earth with noticeable intensity for only a few centimeters; the whole process takes place in a *thin surface layer*.

We deal here with an obvious analogue to the *skin effect* of electricity. The fact that in practice it is particularly observable on cylindrical wires makes no difference here; for a conductor bounded by a plane it occurs quantitatively in almost the same manner. Our daily curve corresponds to an alternating current of high frequency, our yearly curve to one that is 365 times slower. We know from §7 that the differential equations are the same in both cases, but for electricity we interpret the coefficient k as the specific resistance of the conductor.

§15. The Problem of a Ring-Shaped Heat Conductor

We now turn to the case of a heat conductor of finite length 1. However, at its two ends $x = \pm 1/2$ we do not prescribe the boundary conditions a), b) or c) of p. 63, but instead the much simpler *condition of periodicity*. By this we mean that not only u but also all its derivatives

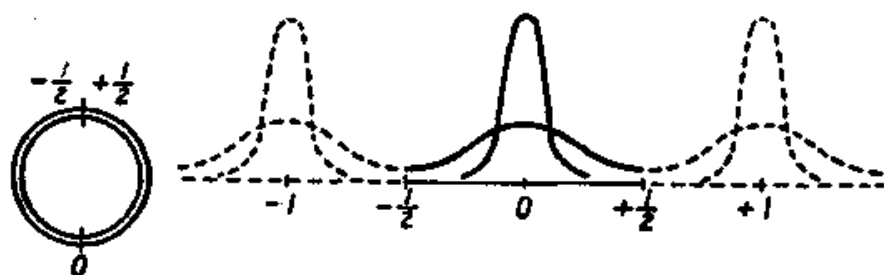


Fig. 16. Heat conduction in a ring. Heat pole at $x = 0$ with periodic repetitions. Temperature distribution for $kt < 1$ (steep curve) and for $kt > 1$ (flat curve).

shall coincide at the ends. We achieve this by bending our rod into a ring so that the two ends coincide. The shape of the ring is of no

importance since, as in all cases of linear heat conduction, we must consider the lateral surface of the ring as adiabatically closed. In Fig. 16 we have drawn a circular ring.

As initial temperature we take $f(x)$ which is arbitrary but symmetric with respect to $x = 0$. Its Fourier expansion is a pure cosine series which automatically satisfies the periodicity condition at the ends. From (4.1) and (4.2) we get, after setting $a = 1/2$:

$$(1) \quad f(x) = \sum_{n=0}^{\infty} A_n \cos(2\pi n x), \quad \begin{aligned} A_0 &= \int_{-1/2}^{+1/2} f(z) dz, \\ A_n &= 2 \int_{-1/2}^{+1/2} f(z) \cos(2\pi n z) dz. \end{aligned}$$

In order to obtain the corresponding solution $u(x, t)$ of the equation of heat conduction we merely must multiply the n -th term by

$$e^{-(2\pi n)^2 k t}$$

We then obtain

$$(2) \quad u(x, t) = \sum_{n=0}^{\infty} A_n e^{-4\pi^2 n^2 k t} \cos(2\pi n x).$$

We now consider $f(x)$ to be a “ δ function” by writing

$$f(x) = 0 \text{ for } x \neq 0, \quad \text{but} \quad \int_{-s}^{+s} f(x) dx = 1.$$

Then in (1) we get $A_0 = 1$, $A_1 = A_2 = \dots = 2$, and if we replace u by the customary ϑ we get:

$$(3) \quad \vartheta(x, t) = 1 + 2 \sum_{n=1}^{\infty} e^{-4\pi^2 n^2 k t} \cos(2\pi n x).$$

The letter ϑ stands for the *theta-function* which was introduced by C. G. J. Jacobi in the theory of elliptic functions and which is of paramount importance in all numerical computations.⁴ The fact that it satisfies the equation of heat conduction is frequently used there as an incidental property, whereas we use this property for the definition of ϑ .

We now have to adjust our notation t to the theory of the ϑ function by setting

⁴ The reason for its special convergence was mentioned in §3, p. 15: since the ϑ series together with all its derivatives is periodic, and therefore has no jumps at $x = +1/2$ and $x = -1/2$, its terms vanish with increasing n more rapidly than any power of n .

$$(4) \quad \tau = 4 \pi i k t.$$

This τ , which of course has nothing to do with the symbol $t - \tau$ of the principal solution U , does not have the dimension of time and is positive-imaginary in our case. (In the theory of elliptic function τ is in general complex with positive imaginary part, namely the ratio of the two periods of these functions). Written in terms of τ (3) becomes:

$$(5) \quad \vartheta(x|\tau) = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi\tau n^2} \cos(2\pi n x).$$

This form converges very well for large τ or, in other words, for large kt . It represents the *later phases* of the damping out of the unit source exceptionally well, but it does not help us for the beginning of this process. We therefore complete Fourier's process which, in analogy to the reflection process, is based on a periodic repetition of the initial state (see the right half of Fig. 16).

We have rolled off the cut ring on the x -axis both to the right and the left in an infinite sequence. From the heat source $U_0(x, t)$ given in the ring we get, at the points $x = n$ ($n = \pm 1, \pm 2, \dots$), the identical heat sources:

$$(6) \quad U_n(x, t) = \frac{1}{\sqrt{4\pi kt}} \exp\left\{-\frac{(x-n)^2}{4kt}\right\}.$$

In the series

$$(7) \quad u(x, t) = \sum_{n=-\infty}^{+\infty} U_n(x, t)$$

we have a second representation of the damping out process which converges excellently for *small values of kt* . This is so because for such values we need consider only U_0 and its immediate successors, the subsequent U_n having no effect on account of the factor $\exp(n^2/4kt)$ of (6). Equation (7) is therefore the desired complement of (5). The figure shows the nature of both representations: the flat curve shows the behavior for large kt according to (5), the steep curve shows the behavior for small kt according to (7).

Oddly enough we can bring (7) into a form very similar to that of the ϑ series in (5). All we must do is put the factor $\exp(-x^2/4kt)$ outside the summation and combine the terms with $\pm n$. Equation (7) then becomes

$$(7a) \quad u(x, t) = (4\pi kt)^{-\frac{1}{2}} \exp\left\{-\frac{x^2}{4kt}\right\} \cdot \left[1 + 2 \sum_{n=1}^{\infty} e^{-\frac{n^2}{4kt}} \cos \frac{inx}{2kt}\right].$$

If we replace t by τ according to (4) then the bracketed term becomes

$$1 + 2 \sum_{n=1}^{\infty} e^{-\frac{\pi n^2}{\tau}} \cos 2\pi n \frac{x}{\tau}.$$

This differs from (5) only in that x/τ has replaced x in the argument of the cosines and that $-1/\tau$ has replaced τ in the exponents. Hence the bracket of (7a) is

$$\vartheta\left(\frac{x}{\tau} \middle| -\frac{1}{\tau}\right).$$

Substituting this in (7) and remembering that both (5) and (7) are solutions of the same problem of heat conduction we obtain

$$\vartheta(x|\tau) = \left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \exp\left\{-\frac{\pi i x^2}{\tau}\right\} \cdot \vartheta\left(\frac{x}{\tau} \middle| -\frac{1}{\tau}\right)$$

or conversely

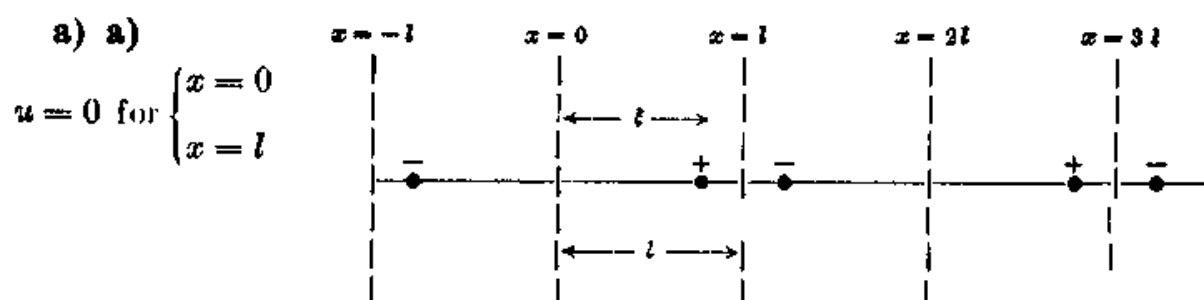
$$(8) \quad \vartheta\left(\frac{x}{\tau} \middle| -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \exp\left\{\frac{\pi i x^2}{\tau}\right\} \cdot \vartheta(x|\tau).$$

This is the famous *transformation formula of the ϑ function*. It is used in the theory of elliptic functions to transform the series $\vartheta(x|\tau)$ which converges slowly for small τ into the very rapidly converging series $\vartheta\left(\frac{x}{\tau} \middle| -\frac{1}{\tau}\right)$. For us it constitutes the passage from Fourier's method to the method of heat poles. In quantum theory formula (8) is of importance for the rotational energy of diatomic molecules and for the calculation of their specific heat for low temperatures.

§16. Linear Heat Conductors Bounded on Both Ends

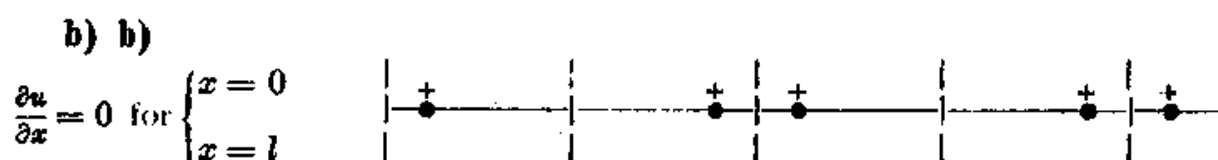
By setting the length of the ring in the preceding section equal to 1, we tacitly introduced a new dimensionless coordinate $x' = x/l$ and wrote x instead of x' . For the case of a rod of length l , which we shall consider now, we must replace x by x/l whenever we apply one of the preceding formulas. The meaning of τ , which has the dimension of x^2 , must then be amended in the manner described on the following page.

We first give a table of the problems corresponding to the boundary conditions a) and b) of p. 63 and of their solutions by both Fourier's method and the method of heat poles. The latter leads to an *infinite* sequence of reflections since not only the primary heat pole but also all its images are reflected at both ends of the rod. Let us consider a room with parallel mirrors as an optical example; the chandelier will be reflected in both mirrors not once but in infinite repetition.



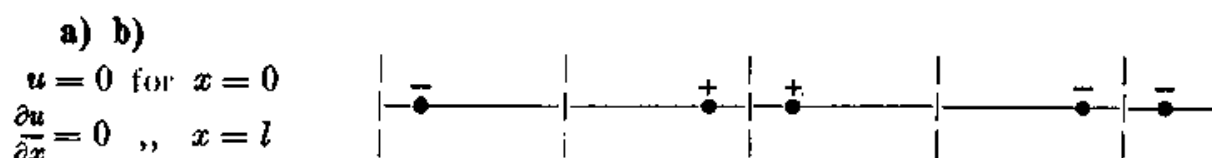
$$f(x) = \sum B_n \sin \pi n \frac{x}{l}, \quad B_n = \frac{2}{l} \int_0^l f(x) \sin \pi n \frac{x}{l} dx,$$

$$G = \vartheta\left(\frac{x-\xi}{2l} \mid \tau\right) - \vartheta\left(\frac{x+\xi}{2l} \mid \tau\right).$$



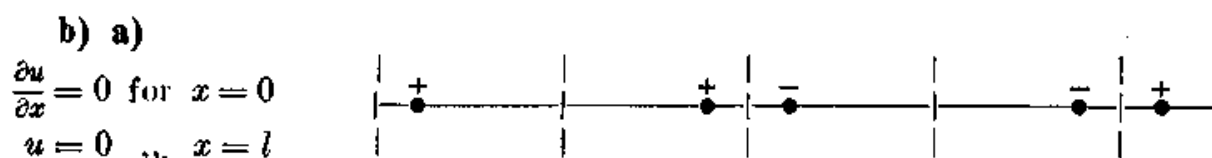
$$f(x) = \sum A_n \cos \pi n \frac{x}{l}, \quad A_n = \frac{2}{l} \int_0^l f(x) \cos \pi n \frac{x}{l} dx, \quad A_0 = \frac{1}{l} \int_0^l f(x) dx,$$

$$G = \vartheta\left(\frac{x-\xi}{2l} \mid \tau\right) + \vartheta\left(\frac{x+\xi}{2l} \mid \tau\right).$$



$$f(x) = \sum B_n \sin \pi(n + \frac{1}{2}) \frac{x}{l}, \quad B_n = \frac{2}{l} \int_0^l f(x) \sin (n + \frac{1}{2}) \frac{x}{l} dx,$$

$$G = \vartheta\left(\frac{x-\xi}{4l} \mid \tau\right) - \vartheta\left(\frac{x+\xi}{4l} \mid \tau\right) + \vartheta\left(\frac{x+\xi-2l}{4l} \mid \tau\right) - \vartheta\left(\frac{x-\xi-2l}{4l} \mid \tau\right).$$



$$f(x) = \sum A_n \cos \pi(n + \frac{1}{2}) \frac{x}{l}, \quad A_n = \frac{2}{l} \int_0^l f(x) \cos (n + \frac{1}{2}) \frac{x}{l} dx,$$

$$G = \vartheta\left(\frac{x-\xi}{4l} \mid \tau\right) + \vartheta\left(\frac{x+\xi}{4l} \mid \tau\right) - \vartheta\left(\frac{x+\xi-2l}{4l} \mid \tau\right) - \vartheta\left(\frac{x-\xi-2l}{4l} \mid \tau\right).$$

We see immediately that the functions $f(x)$ in this table satisfy the boundary conditions a)a) to b)b); these boundary conditions then hold for the corresponding solutions of the boundary value problems $u(x, t)$, which are obtained from $f(x)$ according to *Fourier's method* by multiplying the series of f termwise by

$$e^{-(\pi n/l)^2 kt} \quad \text{or} \quad e^{-[\pi(n+1/2)/l]^2 kt}$$

The diagrams show the positions and the signs of the heat poles according to the *reflection method*. In the first two cases the heat poles are seen to have the period $2l$, in the last two they have the period $4l$. Their summation yields Green's function $G = \Sigma U$ which is expressed here in terms of the ϑ function of the preceding section. In the formulas of the preceding section, where the period was taken equal to 1 and the heat

pole was at $x = 0$, we have to replace x by $\frac{x - \xi_t}{2l}$ for a)a) and b)b) and by $\frac{x - \xi_t}{4l}$ for a)b) and b)a), where ξ_t stands for the position of any heat pole of the sequence which is summed by ϑ . (Due to the periodicity the choice of the heat pole is immaterial.) In our formulas we have chosen for ξ_t the heat pole of the initial region $0 < x < l$ or of one of the adjacent regions. From Green's function we get the solution of the boundary value problem for arbitrary initial values $u(x, 0) = f(x)$ according to the general formula

$$(1) \quad u(x, t) = \int_0^l f(\xi) G(x, \xi; t) d\xi.$$

We now wish to treat the *boundary condition c)* of p. 64, where we particularly consider the combination a)c). In order to satisfy condition a) at $x = 0$ we set

$$(2) \quad f(x) = \sum_{n=1}^{\infty} B_n \sin \lambda_n \pi \frac{x}{l}.$$

That is to say, in the solution for a)a) we replace the sequence of integers n by the sequence

$$\lambda_1, \lambda_2, \dots, \lambda_n, \dots$$

which we wish to determine in such a way that for $x = l$ condition c) is satisfied. This leads to the transcendental equation

$$\lambda_n \frac{\pi}{l} \cos \lambda_n \pi + h \sin \lambda_n \pi = 0.$$

or

$$(3) \quad \tan \lambda_n \pi = -\frac{\pi \lambda_n}{h l}.$$

This is equation (6.2a) with $\alpha = -\pi/h l$; its solution was illustrated in Fig. 7. Hence we are dealing here with a typical case of *anharmonic Fourier analysis*. The values of the coefficients B_n in (2) can be taken directly from (6.3b). We obtain the final solution of our boundary value problem a)c) if we multiply the terms in the sum of (2) by the required time factors:

$$(4) \quad u(x, t) = \sum_{n=1}^{\infty} B_n \sin \lambda_n \pi \frac{x}{l} \cdot \exp \left\{ -\left(\frac{\lambda_n \pi}{l} \right)^2 k t \right\}.$$

We mentioned in §6 that the formal computation of the coefficients B can be replaced by a physically meaningful one. We shall do this now in such a manner that we shall be able to refer to this case in all future expansions in "eigenfunctions."

We consider two arbitrary terms of (2):

$$(5) \quad u_n = \sin \lambda_n \pi \frac{x}{l}, \quad u_m = \sin \lambda_m \pi \frac{x}{l};$$

They satisfy the differential equations:

$$(5a) \quad \frac{d^2 u_n}{dx^2} + k_n^2 u_n = 0, \quad \frac{d^2 u_m}{dx^2} + k_m^2 u_m = 0, \quad \begin{cases} k_n = \lambda_n \frac{\pi}{l}, \\ k_m = \lambda_m \frac{\pi}{l}. \end{cases}$$

Hence

$$(5b) \quad u_m \frac{d^2 u_n}{dx^2} - u_n \frac{d^2 u_m}{dx^2} = (k_m^2 - k_n^2) u_m u_n.$$

The left side is a total derivative (for the case of Green's theorem, p. 44, we spoke of a "divergence"). The integration of (5b) over the fundamental region $0 < x < l$ reduces to the boundary points on the left (in Green's theorem we said "to a boundary integral"). From this we obtain the value of the right side without further calculation:

$$(6) \quad (k_m^2 - k_n^2) \int_0^l u_m u_n dx = u_m \frac{du_n}{dx} - u_n \frac{du_m}{dx} \Big|_{x=0}^{x=l}.$$

Here the right side vanishes for $x = 0$ since according to (5) we have then $u_n = u_m = 0$; but it also vanishes for $x = l$, since the boundary

condition c) holds for every individual term of (2) and hence the du/dx are proportional to the u . Therefore the *condition of orthogonality* (6.3) is satisfied for $k_m \neq k_n$.

We shall show in exercise III.2 that the normalizing integral (6.3b) can be obtained almost without further calculation.

The expressions as well as the mathematical formulations in this section were based on the assumption that the lateral surface of the rod is closed to heat flow, an assumption that is open to reasonable doubt. We shall show now that our formulas can be used also in the case of incomplete closure with respect to heat flow.

Instead of imposing the adiabatic condition b) on an element of area $d\sigma$ of the lateral surface of our rod, which we assume to be a circular cylinder, we impose condition c) which states that an amount of heat

$$-\kappa \frac{\partial u}{\partial n} d\sigma = \kappa h u d\sigma$$

passes out through $d\sigma$ per unit of time. We apply this to the case of an element of a cylindrical rod of altitude dx and radius of cross section b so that the lateral area is $2\pi b dx$ and the outer normal dn is in the direction of the extended radius. The amount of heat passing out of the lateral surface is

$$(7) \quad dQ_1 = \kappa h u \cdot 2\pi b dx dt.$$

The amount of heat flowing out of the bases $x = \text{const.}$ and $x + dx = \text{const.}$ of this element is

$$(8) \quad dQ_2 = -\kappa \frac{\partial^2 u}{\partial x^2} \cdot \pi b^2 dx dt$$

According to (7.9) the total outflux of heat from the rod element is equal to the product of $\text{div } G$ dt with its volume. Hence we have

$$(9) \quad \text{div } G \cdot \pi b^2 dx dt = dQ_1 + dQ_2.$$

and after substituting (7) and (8)

$$(10) \quad \text{div } G = \frac{2\kappa h}{b} u - \kappa \frac{\partial^2 u}{\partial x^2}.$$

According to (7.11) $\text{div } G$ is proportional to $-\partial u/\partial t$. Substituting this and dividing by κ we obtain

$$(11) \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t} + \frac{2h}{b} u.$$

Hence the “outer-heat conduction” through the lateral surface only

modifies our differential equation by the additional term of equation (11). Our derivation depended on the assumption that the linear character of the thermal state, i.e., its sole dependence on x , is not affected by the lateral radiation, an assumption which seems plausible for sufficiently small cross section.

The integration of (11) is very simple. We let:

$$u = v e^{-\lambda t};$$

After division by $\exp(-\lambda t)$ equation (11) becomes:

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{k} \frac{\partial v}{\partial t} + \left(\frac{2h}{b} - \frac{\lambda}{k} \right) v,$$

which is the ordinary equation of heat conduction if we set:

$$(12) \quad \lambda = \frac{2hk}{b}.$$

Hence all the developments of this chapter are valid for a rod with outer heat conduction if we multiply by the factor $\exp\left(-\frac{2hk}{b}t\right)$

In exercises III.3 and III.4 we shall see an elegant experimental determination of the ratios inner to outer heat conduction and heat conduction to electron conduction in metals.

§ 17. Reflection in the Plane and in Space

We finally leave the case of linear heat conduction and turn to spatial regions which are bounded by planes and which can be treated by the simple *reflection method*. The corresponding plane regions bounded by straight lines will be treated in a very similar manner.

The simplest case is that of a *half space* with the boundary conditions $u = 0$ or $\partial u / \partial n = 0$. Since we know the spatial function of a heat pole from equation (12.14), we can write Green's function for the half space directly. If we take the boundary of the half space to be $z = 0$ and the source point to be (ξ, η, ζ) then we have

$$(1) \quad (4\pi k t)^{\frac{3}{2}} G = e^{-\frac{r^2}{4kt}} \mp e^{-\frac{r'^2}{4kt}} \begin{cases} r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \\ r'^2 = (x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2 \end{cases}$$

Since for $z = 0$

$$r^2 = r'^2 \quad \text{and} \quad \frac{\partial r^2}{\partial z} = -\frac{\partial r'^2}{\partial z}$$

we get

$$G = 0 \quad \text{or} \quad \frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z} = 0 \quad \text{for } z = 0.$$

Even for the boundary condition c) of p. 64 we can transfer the solution (13.15) directly to the spatial case. We have

$$(2) \quad (4\pi kt)^{\frac{3}{2}} G = e^{-\frac{r^2}{4kt}} + e^{-\frac{r'^2}{4kt}} - 2h e^{\lambda z} e^{-\frac{r^2}{4kt}} \int_{-\infty}^z e^{-\frac{(z-\beta)^2}{4kt}} e^{\lambda\beta} d\beta.$$

with

$$r^2 = (x - \xi)^2 + (y - \eta)^2,$$

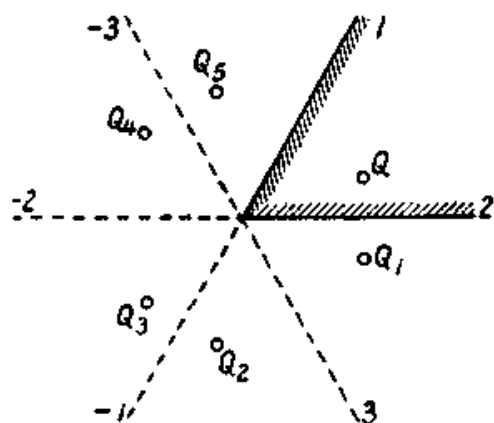


Fig. 17. Wedge with face angle $\pi/3$. Simple and complete covering of space upon successive reflection.

Not all spatial regions bounded by planes can be treated according to the reflection process. It is necessary that, under successive reflections of the original region, space be covered *completely and simply*. We demonstrate this with the example of a *wedge*. If it has an angle of 60° (see Fig. 17) then it is reproduced five times upon successive reflections whereupon the process terminates. Hence Green's function can be represented by a sum of six heat poles where, for the boundary condition

$\varphi = 0$, half the poles (the original pole Q and its images Q_2, Q_4) are positive and the others (Q_1, Q_3, Q_5) negative.

From this figure it becomes apparent that the reflection process may be attempted only for those polyhedrons whose *face angles are all submultiples of π* (not merely of 2π). In the case of wedges the angle $2\pi/3 = 120^\circ$ leads to a double covering of space, $3\pi/2$ leads to a triple covering, and every angle which is incommensurable with π leads to an infinite covering. A particularly interesting case is that of space with a half plane removed, a wedge of angle 2π so to speak. Its treatment according to a reflection process requires the study of the principal solution in a "two-sheeted Riemann space" whose branch line is the edge of the half plane.⁵

⁵ This solution was given by the author in 1894, *Math. Ann.* 45, first for the case of heat conduction and soon thereafter for the refraction of light (*ibid.* 47). For details see Frank-Mises, 2nd ed. (8th ed. of Riemann-Weber), Vieweg, 1935, chapter 20.

Among the polyhedral regions we consider first the interior of a *cube* (the exterior would lead to most complicated ramifications) and as its generalization the *rectangular solid*. The mirror images of the given primary source point form eight superimposed spatial lattices corresponding to the eight combinations of signs $\pm \xi, \pm \eta, \pm \zeta$. Each of these lattices taken separately forms a triply periodic solution of the differential equation, so to speak a higher ϑ function (see below). If the base of a rectangular solid is divided by its diagonals into four isosceles triangles then a rectangular cylinder with one of them as base is also a polyhedron of the required kind. Another example is given by a rectangular cylinder whose base is an equilateral triangle or half an equilateral triangle obtained through bisection by the altitude. A rectangular cylinder whose base is a regular hexagon has face angles of $2\pi/3$ and therefore leads to a double, not a simple, covering of space.

Everything said about the subdivision of rectangular solids is of course true for cubes. In addition, for suitable subdivisions the cube yields permissible tetrahedra: Lamé's "tetrahedra $1/6$ and $1/24$," of which the former fills out the cube upon six reflections, the latter upon twenty-four, and another tetrahedron which was discovered by Schönflies in his general investigations on crystal structure.⁶

For all these regions we can not only solve the problem of heat conduction but *any physical process of isotropic symmetry*, such as an acoustic, optic, or electric process, by the reflection method. The very word "reflection" reminds us of the optical application.

The set of permissible regions is extended very considerably if we no longer impose *boundary conditions*, but require *periodicity*, as in the case of the ring in the beginning of §15. Then instead of a rectangular solid we can treat an *arbitrary parallelepiped*; all we have to do is to repeat periodically the pole of Green's function in the initial domain in all its images under the translation group. The elliptic ϑ function of the ring is then replaced by a higher ϑ function (hyperelliptic Abelian); however we shall not go into this since there are no immediate physical applications.

Everything said here about spatial regions can be transferred directly to plane regions. Half space is replaced by half plane, rectangular solid by rectangle, the rectangular cylinder whose base is an equilateral triangle by that triangle. In formula (1) for Green's function of half space we have to replace the exponent $\frac{3}{2}$ on the left by the exponent 1,

⁶ G. Lamé, *Leçons sur la théorie de la chaleur*, Paris 1861; he does not use the reflection method but Fourier's method with a suitable continuation of the arbitrary initial distribution. For A. Schönflies' tetrahedron see *Math. Ann.* 34.

and the three-dimensional square of distances on the right by the corresponding square of distances in the plane.

Unfortunately this method of reflection for problems of heat conduction can not be applied to spherical (or circular) regions (see §23).

§ 18. Uniqueness of Solution for Arbitrarily Shaped Heat Conductors

The physicist may consider such a proof superfluous; we shall consider it, however, on account of its mathematical elegance and the importance of its method.

It will suffice to use Green's theorem of potential theory which we formulated in exercise II.2 as the "second form." The parabolic character of the equation of heat conduction plays no particular role here; it would become important if, as in Fig. 13, we were to impose time dependent boundary conditions, but we shall restrict ourselves here to the boundary conditions a), b), c) of p. 63.

Our heat conductor may have an arbitrary boundary; as part of this boundary we include the boundaries of any possible inner cavities. On this total boundary surface σ there may be given an arbitrary combination of the boundary conditions

$$\text{a) } u = f_1(\sigma), \quad \text{b) } \frac{\partial u}{\partial n} = f_2(\sigma), \quad \text{c) } \frac{\partial u}{\partial n} + hu = f_3(\sigma)$$

("non-homogeneous" boundary conditions where f_1, f_2, f_3 are arbitrary point functions on σ , in contrast to the previous "homogeneous" boundary conditions where the right sides are zero). In addition we assume the initial temperature u to be given as an arbitrary point function $f(x, y, z)$.

Let u_1 and u_2 be two different solutions of the equation of heat conduction under these initial and boundary conditions. Their difference $u_1 - u_2 = w$ then satisfies same differential equation as u_1, u_2

$$(1) \quad \Delta w = \frac{1}{k} \frac{\partial w}{\partial t}$$

with a distribution over σ of the "homogeneous" boundary conditions

$$(2) \quad \text{a) } w = 0, \quad \text{b) } \frac{\partial w}{\partial n} = 0, \quad \text{c) } \frac{\partial w}{\partial n} + hw = 0$$

and the initial condition

$$(3) \quad w = 0 \quad \text{for} \quad t = 0.$$

Setting both u and v equal to w in Green's theorem of exercise II.2 we get:

$$(4) \quad \int w \Delta w d\tau + \int (\text{grad } w, \text{grad } w) d\tau = \int w \frac{\partial w}{\partial n} d\sigma.$$

Considering (1) and (2) this becomes

$$(5) \quad \frac{1}{2k} \frac{\partial}{\partial t} \int w^2 d\tau = - \int Dw d\tau - \int h w^2 d\sigma_c.$$

where Dw is the so-called first differential parameter:

$$Dw = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2.$$

The last term of (5) is to be considered only over that part of σ on which the boundary condition c) holds, as indicated by the subscript c attached to $d\sigma$.

Equation (5) contains a contradiction: the right side is *negative* since $h > 0$ as established in (13.5). (We no longer have to assume that h is a constant; h may vary on the surface σ_c depending on the local structure of the surface.) The left side of (5) is certainly *positive* for small t , since w^2 is 0 for $t = 0$ and therefore can only increase for increasing t . In order to make this contradiction even more apparent and to extend it to arbitrary t , we integrate (5) with respect to t :

$$(6) \quad \frac{1}{2k} \int w^2 d\tau = - \int_0^t dt \int Dw d\tau - \int_0^t dt \int h w^2 d\sigma_c.$$

The only possibility of removing this contradiction is in setting:

$$(7) \quad w = 0, \quad \text{hence} \quad u_1 = u.$$

This *uniqueness result* can also be expressed as follows: in heat conduction there exist no eigenfunctions for any shape of the conductor (see Chapter V). In this sense heat conduction and all analogous *equalization processes* differ in a characteristic manner from *oscillation processes*.

CHAPTER IV

Cylinder and Sphere Problems

This chapter serves to complete our stock of mathematical tools rather than to solve any new physical problems. A necessary part of the tools of a mathematical physicist are cylindrical and spherical harmonics. We shall develop these tools with the help of simple physical considerations rather than in an abstract mathematical manner. We shall connect spherical harmonics with *potential theory* (in which they first arose) and cylindrical harmonics with the *wave equation* and its simplest solution, the monochromatic wave.

§ 19. Bessel and Hankel Functions

We assume the time dependence in the wave equation (7.4) to be periodic, and write it conveniently in the form

$$(1) \quad e^{-i\omega t}, \quad \omega = \text{circular frequency.}$$

We introduce

$$(2) \quad k = \frac{\omega}{c}, \quad k = \text{wave number;}$$

and then write (7.4) for one and two dimensions:

$$(3a) \quad \frac{d^2 u}{dx^2} + k^2 u = 0, \quad (3b) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0.$$

Equation (3a) has the integrals

$$(4a) \quad u = A e^{ikx} \quad \text{and} \quad u = B e^{-ikx}.$$

Because of our choice of negative sign in (1) the first equation stands for a plane wave which progresses in the direction of the positive x -axis, the second for one which progresses in the direction of the negative x -axis. The fact that it is simpler to operate with a wave which progresses in the positive x -direction is the main reason for the choice of sign in (1). For the two-dimensional case (3b) we get

$$(4b) \quad u = A e^{i(ax+by)}, \quad a^2 + b^2 = k^2, \quad \begin{cases} a = k \cos \alpha, \\ b = k \sin \alpha. \end{cases}$$

Introducing the plane polar coordinates r, φ with

$$x = r \cos \varphi, \quad y = r \sin \varphi,$$

we get from (4b)

$$(5) \quad u = A e^{i k r \cos(\varphi - \alpha)}.$$

Equation (5) represents a plane wave which progresses in the direction $\varphi = \alpha$; for $\alpha = 0$ it becomes (4a). From such solutions we can construct the general solution of (3b) by summation (integration) over with coefficients A which may depend on α .

Written in terms of r, φ equation (3b) reads

$$(6) \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0$$

or if we set $\varrho = k r$,

$$(6a) \quad \frac{\partial^2 u}{\partial \varrho^2} + \frac{1}{\varrho} \frac{\partial u}{\partial \varrho} + \frac{1}{\varrho^2} \frac{\partial^2 u}{\partial \varphi^2} + u = 0.$$

We seek the solutions of this equation which have the form

$$(7) \quad u = Z_n(\varrho) e^{i n \varphi},$$

For this purpose we set

$$A = c_n e^{i n \alpha}, \quad c_n \text{ being a constant independent of } \alpha,$$

and integrate with respect to α between suitable limits β and γ :

$$(8) \quad u = c_n \int_{\beta}^{\gamma} e^{i \varrho \cos(\varphi - \alpha)} e^{i n \alpha} d\alpha.$$

Equation (8), unlike (5), does not represent *one* wave of direction α , but a *bundle* of waves with directions varying from $\alpha = \beta$ to $\alpha = \gamma$, which obviously satisfies the differential equation (6a). In order to bring (8) into the form (7) we write

$$(8a) \quad \alpha = w + \varphi, \quad w_0 = \beta - \varphi, \quad w_1 = \gamma - \varphi,$$

Equation (8) then becomes

$$(9) \quad u = c_n \int_{w_0}^{w_1} e^{i \varrho \cos w} e^{i n w} dw \cdot e^{i n \varphi}.$$

The coefficient here of $e^{i n \varphi}$ is a function of ϱ alone if, and only if, we remove the dependence of w_0 and w_1 on φ . This is done in (8a) by

letting β and γ , and with them w_0, w_1 , increase to infinity in some way. In order to accomplish this we first must investigate the convergence of the integral in (9) in the neighborhood of infinity (see Fig. 18). This is obviously a question of determining those regions of the complex w -plane in which the real part of the exponent $i \varrho \cos w$ of (9) is negative. We assume for the time being that ϱ is real and positive and set

$$w = p + i q, \text{ and hence } \operatorname{Re} \{i \cos w\} = \sinh q \sin p.$$

Hence for the upper half of the w -plane, $q > 0$, we have

$$(10 a) \quad \sin p < 0, \quad -\pi < p < 0 \bmod 2\pi^1$$

and for the lower half of the w -plane, $q < 0$,

$$(10 b) \quad \sin p > 0, \quad 0 < p < \pi \bmod 2\pi$$

Since conditions (10a,b) depend on only the real part p of w and not on q , we know that the regions in question are strips which are parallel to the imaginary axis. The regions for which the passage of w_0 and w_1 to infinity is permissible are shaded in Fig. 18.

If ϱ is not real and positive, say $\varrho = |\varrho|e^{i\theta}$, then the above pattern is maintained and is only shifted by $\pm \theta$ in the direction of the real axis, where the $+$ and $-$ signs are for the upper and lower half planes. In the convergence considerations of (10a,b) we merely have to replace $\sin p$ by $\sin(p \mp \theta)$ (see the beginning of exercise IV.2).

For each choice of the limits w_0, w_1 which satisfies the stated conditions the coefficient of $e^{i n \varphi}$ in (9) is a possible form of the *general cylinder function* $Z_n(\varrho)$ of (7). Substituting (7) in (6a) we see that the functions Z_n satisfy the differential equation:

$$(11) \quad \frac{d^2 Z_n}{d\varrho^2} + \frac{1}{\varrho} \frac{dZ_n}{d\varrho} + \left(1 - \frac{n^2}{\varrho^2}\right) Z_n = 0.$$

A. THE BESSEL FUNCTION AND ITS INTEGRAL REPRESENTATION

Our first special choice is

$$(12) \quad \begin{aligned} w_0 &= a + i\infty, & -\pi < a < 0, \\ w_1 &= b + i\infty, & \pi < b < 2\pi. \end{aligned}$$

The corresponding path of integration is denoted in Fig. 18 by w_0 ; the function obtained is called a *Bessel function* if the factor c_n in (9) is normalized by:

¹Two numbers p and p' are said to be "congruent modulo 2π " (written $p \equiv p' \bmod 2\pi$), if $p - p'$ is an integral multiple of 2π .

$$(13) \quad c_n = \frac{1}{2\pi} e^{-in\pi/2}.$$

Using the common² notation I_n we obtain

$$(14) \quad I_n(\varrho) = \frac{1}{2\pi} \int_{W_0} e^{i\varrho \cos w} e^{in(w-\pi/2)} dw.$$

The normalization (13) has been chosen so that $I_0(\varrho) = 1$ for $\varrho = 0$ and $I_n(\varrho)$ is real for arbitrary n and ϱ . The former follows from (14) if we pass to the rectangular form of W_0 , which is depicted in Fig. 18 by the dotted path. We thereby cause the two partial integrals along the parts parallel to the imaginary axis (which are otherwise divergent for $\varrho \rightarrow 0$) to be complex conjugates. In order to prove this we make the substitution $w - \pi/2 = \beta$

The rectangular path W_0 is then, in terms of β ,

$$-\pi + i\infty \rightarrow -\pi \rightarrow \pi \rightarrow \pi + i\infty,$$

which lies symmetric with respect to the β -axis. For real ϱ and n , $I_n(\varrho)$ decomposes into the two parts:

$$(15) \quad I_n(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i(n\beta - \varrho \sin \beta)} d\beta - \frac{\sin n\pi}{\pi} \int_0^\infty e^{-(n\gamma + \varrho \sinh \gamma)} d\gamma,$$

where the second term is obtained from the integrals over the two paths $\pi \rightarrow \pi + i\infty$ and $-\pi + i\infty \rightarrow -\pi$ by the substitution $\beta = \pm\pi + i\gamma$; it does not vanish in general as it did for $n = 0$. Hence under the normalization of (14) $I_n(\varrho)$ is indeed real for real ϱ and n .

Since our integral representation converges for all values of ϱ it follows that $I_n(\varrho)$ is an *everywhere regular transcendental function* except for a single *essential singularity* at infinity and a *branch point* of order n at $\varrho = 0$ which for negative n is also a pole of the same order.

If n is an integer then the second term in (15) vanishes and we get

$$(16) \quad I_n(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i(n\beta - \varrho \sin \beta)} d\beta.$$

² In the English literature one writes J_n instead of I_n and sets $I_n(\rho) = J_n(i\rho)$. We wish to reserve the letter J to denote "intensity" and we shall need no special symbol for $I_n(i\rho)$.

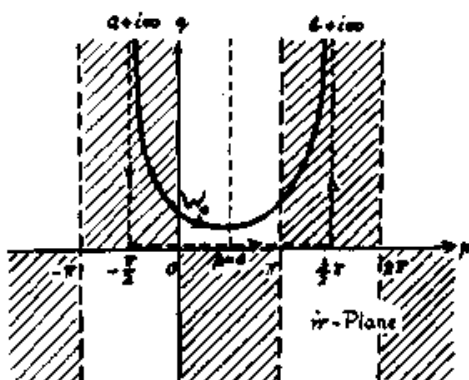


Fig. 18. Regions of the plane $w = p + iq$ in which the real part of $i\varrho \cos w$ is negative are shaded. The path of integration W_0 for the Bessel function I goes from $w_0 = a + i\infty$ to $w_1 = b + i\infty$. In addition to w we use the variable of integration $\beta = w - \pi/2$.

If we express the exponential function in terms of trigonometric functions and consider the odd and even character of the sine and cosine, then we get a representation which was given by Bessel:

$$(17) \quad I_n(\rho) = \frac{1}{\pi} \int_0^{\pi} \cos(\rho \sin \beta - n\beta) d\beta.$$

We can see this directly from our original integral with respect to w . In the rectangular path W_0 the two parts which are parallel to the imaginary axis will cancel for integral values of n , and only the section of the real axis from $-\pi/2$ to $3\pi/2$ remains. Due to the periodicity of the integrand this can be replaced by the path from $-\pi$ to $+\pi$. We thus obtain

$$(18) \quad I_n(\rho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i\rho \cos w} e^{in(w-\pi/2)} dw,$$

which agrees with (16). The integral over a complex path W_0 as in (14) has a great advantage over the real representations in that it is not limited to integral values of n but remains valid for arbitrary n . The integral (14) is first mentioned in Schläfli 1871,³ though only with a rectangular path of integration. The following integrals (22) were first published by the author in 1896.

Since the differential equation (11) depends only on n^2 we know that if I_n is a solution then so is I_{-n} . The general solution can therefore be written in the form

$$(19) \quad Z_n(\rho) = c_1 I_n(\rho) + c_2 I_{-n}(\rho)$$

However this holds only for *non-integral* n . For *integral* n , I_n and I_{-n} are not linearly independent; we have rather

$$(19a) \quad I_{-n}(\rho) = (-1)^n I_n(\rho).$$

This follows directly from (16) if in $I_{-n}(\rho)$ we make the substitution $\beta = \pi - \beta'$.

B. THE HANKEL FUNCTION AND ITS INTEGRAL REPRESENTATION

As limits of integration in (9) we now choose

$$(20) \quad \begin{aligned} w_0 &= a_1 + i\infty, & -\pi < a_1 < 0, \\ w_1 &= b_1 - i\infty, & 0 < b_1 < \pi; \end{aligned}$$

³ For details see G. N. Watson. A Treatise on the Theory of Bessel Functions, Cambridge 1922, p. 176 and 178.

and

$$(20a) \quad \begin{aligned} w_0 &= a_2 - i\infty, & 0 < a_2 < \pi, \\ w_1 &= b_2 + i\infty, & \pi < b_2 < 2\pi. \end{aligned}$$

These paths, which are largely arbitrary and are restricted only asymptotically to the shaded regions, are denoted by W_1 and W_2 in Fig. 19. The fact that they have been drawn through the points $w = 0$ and $w = \pi$ is also arbitrary but will prove convenient later. The constant c_n is now determined by

$$(21) \quad c_n = \frac{1}{\pi} e^{-in\pi/2}.$$

The cylindrical functions thus obtained are called the *first and second Hankel functions*.

$$(22) \quad \begin{aligned} H_n^1(\varrho) &= \frac{1}{\pi} \int_{W_1} e^{i\varrho \cos w} e^{in(w-\pi/2)} dw, \\ H_n^2(\varrho) &= \frac{1}{\pi} \int_{W_2} e^{i\varrho \cos w} e^{in(w-\pi/2)} dw. \end{aligned}$$

They are almost more important to mathematical physicists than the Bessel functions I_n . They differ from the latter by the fact that they become infinite at $\varrho = 0$ even for positive n . This follows from the fact that the integral

$$\int_{W_1, W_2} e^{in(w-\pi/2)} dw$$

obtained from (22) by setting $\varrho = 0$, diverges in the infinite part of the lower half-plane.

The singularities of H^1 and H^2 at $\varrho = 0$ will be discussed in Section C. Due to their construction H^1 and H^2 are again solutions of the differential equation (11). The general integral of (11) can therefore be written in the form

$$(23) \quad Z_n(\varrho) = C_1 H_n^1(\varrho) + C_2 H_n^2(\varrho)$$

We now want to show that the special integral I_n is obtained from this formula by setting

$$C_1 = C_2 = \frac{1}{2}$$

as is seen by looking at Fig. 19. If we traverse the paths W_1 and W_2 in

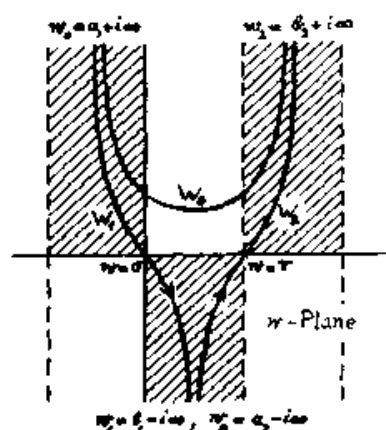


Fig. 19. The paths of integration W_1 and W_2 for H^1 and H^2 . Combined in succession they are equivalent to the path W_0 .

succession then the lower parts cancel and the whole path contracts to W_0 . Considering the new determination of c_n in (21) we have twice the amount obtained with the previous definition (13). Hence we indeed have

$$(24) \quad I_n(\varrho) = \frac{1}{2} \{H_n^1(\varrho) + H_n^2(\varrho)\}.$$

With this we compare the difference $H_n^1 - H_n^2$, which, written in terms of the variables of integration β and γ of (15), is purely imaginary for real ϱ and n . We denote this difference by $2iN_n$ and call $N_n(\varrho)$ Neumann's function:

$$(25) \quad N_n(\varrho) = \frac{1}{2i} \{H_n^1(\varrho) - H_n^2(\varrho)\}.$$

From (24) and (25) we have

$$(26) \quad \begin{aligned} H_n^1(\varrho) &= I_n(\varrho) + iN_n(\varrho), \\ H_n^2(\varrho) &= I_n(\varrho) - iN_n(\varrho). \end{aligned}$$

This decomposition of H^1, H^2 is completely analogous to the decomposition of the exponential function into its trigonometric components, as indicated in the following arrangement:

$$\begin{array}{cccc} e^{iz} & e^{-iz} & \cos x & \sin x \\ H^1(\varrho) & H^2(\varrho) & I(\varrho) & N(\varrho). \end{array}$$

We shall see in Section D that this is not only a qualitative analogy, but that asymptotically (for $\varrho \rightarrow \infty$) it holds quantitatively also. Just as we prefer the exponential imaginary representation to the trigonometric real one in descriptions of wave phenomena, so as a rule we prefer a representation in terms of Hankel functions to one in terms of Bessel and Neumann functions, especially since our complex integrals are equally convenient for all three.

For non-integral n the H^1, H^2 must be expressible in the form (19). In order to determine the coefficients c_1, c_2 we make the following observation: according to (14)

$$(27) \quad 2\pi e^{i\pi n/2} I_n(\varrho) = \int_{W_0} e^{i\varrho \cos w + inw} dw$$

and if we replace n by $-n$ and w by $-w$ (or W_0 by $-W_0$),

$$(28) \quad -2\pi e^{-i\pi/2} I_{-n}(\varrho) = \int_{-W_0} e^{i\varrho \cos w + i\pi w} dw.$$

In Fig. 20 we have W_0 and $-W_0$, drawn for convenience in rectangular shape, with their proper orientation. Their central parts from $w = -\pi/2$ to $w = +\pi/2$ cancel. There remain two rectangular paths, which, for convenience, we have deformed into paths of the type W_2 in Fig. 19.

The right hand path from $\frac{\pi}{2} - i\infty$ to $\frac{3\pi}{2} + i\infty$ coincides with W_2 . Let the left hand path from $-\frac{\pi}{2} + i\infty$ to $-\frac{3\pi}{2} - i\infty$ be denoted by W_2' . Adding (27) and (28) we obtain then

$$(29) \quad 2\pi \{e^{i\pi/2} I_n(\varrho) - e^{-i\pi/2} I_{-n}(\varrho)\} = \left(\int_{W_2} + \int_{W_2'} \right) e^{i\varrho \cos w + i\pi w} dw.$$

Here according to (22) the integral over W_2 equals

$$(29a) \quad \pi e^{i\pi/2} H_n^2(\varrho).$$

The integral over W_2' differs from this only in the orientation of the path and in its translation by -2π . This integral, according to (22), is then

$$(29b) \quad -\pi e^{-i\pi/2} H_n^2(\varrho).$$

Substituting (29a,b) in (29) we obtain

$$2[I_n(\varrho) - e^{-i\pi} I_{-n}(\varrho)] = (1 - e^{-2i\pi}) H_n^2(\varrho)$$

and hence

$$(30) \quad H_n^2(\varrho) = \frac{e^{i\pi} I_n(\varrho) - I_{-n}(\varrho)}{i \sin \pi}.$$

The corresponding representation for H^1 is obtained from (24):

$$(31) \quad H_n^1(\varrho) = 2 I_n(\varrho) - H_n^2(\varrho) = \frac{e^{-i\pi} I_n(\varrho) - I_{-n}(\varrho)}{-i \sin \pi}.$$

The coefficients c_1, c_2 for Hankel functions in equation (19) are thereby determined. We note that for real n and complex ϱ

$$(32) \quad H_n^1(\varrho^*) = [H_n^2(\varrho)]^*, \quad \text{hence} \quad H_n^2(\varrho^*) = [H_n^1(\varrho)]^*.$$

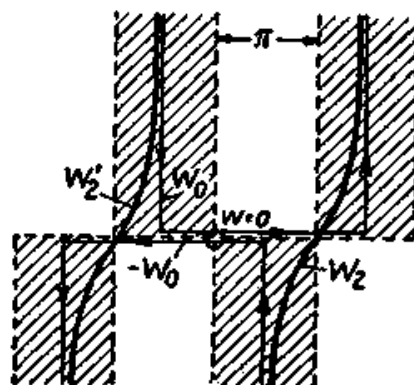


Fig. 20. The paths W_0 and $-W_0$ for I_n and I_{-n} are equivalent to the paths W_2 and W_2' which belong to the type H_n^r .

Here the asterisk * stands as usual for the passage to the complex conjugate. In the derivation of (32) from (30) and (31) we use the relation $I_n(\varrho^*) = [I_n(\varrho)]^*$ which follows from (34). We further deduce from (30) and (31) that

$$(32a) \quad H_{-n}^1(\varrho) = e^{in\pi} H_n^1(\varrho), \quad H_{-n}^2(\varrho) = e^{-in\pi} H_n^2(\varrho).$$

and from (25), (30) and (31) that

$$(33) \quad N_n(\varrho) = \frac{\cos n\pi I_n(\varrho) - I_{-n}(\varrho)}{\sin n\pi}.$$

C. SERIES EXPANSION AT THE ORIGIN

We have seen that $I_n(\varrho)$ is regular in the entire finite plane. It can therefore be expanded in ascending powers of ϱ . Indeed we see directly that the differential equation (11) is satisfied by the series:

$$(34) \quad I_n(\varrho) = \left(\frac{\varrho}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n+m+1)} \left(\frac{\varrho}{2}\right)^{2m}.$$

For $n = 0$ it assumes the particularly elegant form

$$(35) \quad I_0(\varrho) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! m!} \left(\frac{\varrho}{2}\right)^{2m}.$$

which was known to Fourier. We shall demonstrate in exercise IV.1 that these series agree with the integral representation (14).

In order to obtain the series for the general cylinder function Z_n and to investigate the *singularity* at $\varrho = 0$, we proceed as in the case of ordinary linear differential equations: we write

$$(36) \quad Z_n = \varrho^\lambda (a_0 + a_1 \varrho + a_2 \varrho^2 + \cdots + a_k \varrho^k + \cdots)$$

and substitute this in the differential equation (11); the resulting power series must vanish term by term. The lowest power $\varrho^{\lambda-2}$ yields the determination of λ , the general term $\varrho^{\lambda+k-2}$ yields a recursion formula for a_k . We obtain

$$(37) \quad \lambda(\lambda-1) + \lambda - n^2 = 0, \quad \lambda = \pm n,$$

and

$$(37a) \quad \{(\lambda+k)(\lambda+k-1) + \lambda + k - n^2\} a_k + a_{k-2} = 0,$$

By the use of (37) equation (37a) can be simplified to

$$(37b) \quad (k^2 + 2k\lambda)a_k + a_{k-2} = 0.$$

By repeated application of this recursion formula we get for $k = 2m$

$$a_{2m} = \frac{-a_{2m-2}}{4m(m+\lambda)} = \frac{(-1)^2}{2^4} \frac{a_{2m-4}}{m(m-1)(m+\lambda)(m+\lambda-1)} = \dots$$

If, as in (34), we choose $a_0 = 1/2^n \Gamma(\lambda + 1)$ and set $a_1 = 0$ then we obtain

$$(38) \quad a_{2m} = \frac{(-1)^m}{2^{2m+n}} \frac{1}{m! \Gamma(m + \lambda + 1)}, \quad a_{2m+1} = 0.$$

This establishes the validity of (34). According to (37) it is equally valid for $\lambda = -n$ and for $\lambda = +n$. As mentioned on p. 88, equation (11) has the solution $I_{-n}(\varrho)$ in addition to $I_n(\varrho)$. If n has a positive real part the latter vanishes for $\varrho = 0$ with the same rapidity as ϱ^n , whereas the former becomes *infinite* with the same rapidity as ϱ^{-n} .

What we have said so far in Section C holds only when n is *non-integral*. For *integral* n , or more generally for the cases in which the difference of the two roots of (37) is integral,⁴ we encounter in the solution belonging to the smaller λ a difficulty that is well known from the general theory of linear differential equations, namely, that in addition to powers with negative exponents we have *logarithmic terms*. We demonstrate this as follows.

Substituting in (37b) $\lambda = -n$ and $k = 2n$, we obtain $a_{2n-2} = 0$. Hence, tracing the recursion formula for a_{2n} backward, we see that in the series (36) for $Z_n = I_{-n}$ all the terms $a_k = a_{2n}$ vanish. This implies the relation (19a) previously established between I_n and I_{-n} .

The problem is to find a second solution of Bessel's differential equation (11) which is linearly independent of I_n . We do this by a limit consideration in which n is taken as a positive number which is arbitrarily close to an integer. Instead of applying this to the Hankel function H we apply it directly to the Neumann function N of equation (33), the decisive function for the singularity under discussion. Before passing to the limit N is given by (33); in the limit it becomes 0/0 due to equation (19a). The limiting value is determined according to De l'Hospital's rule. Denoting the integral limit of n by \bar{n} , we get for the denominator of (33)

$$\frac{\partial}{\partial n} \sin n\pi = \pi \cos n\pi = \pi (-1)^{\bar{n}}$$

⁴ This is the case for Bessel functions in which n is half an integer. The fact that, in spite of this, the complications which are discussed in the text do not arise, will be explained in §21 C.

and for the numerator

$$\begin{aligned} & -\pi \sin n\pi I_n(\varrho) + \cos n\pi \frac{\partial}{\partial n} I_n(\varrho) - \frac{\partial}{\partial n} I_{-n}(\varrho) \\ & = (-1)^n \left\{ \frac{\partial}{\partial n} I_n(\varrho) - (-1)^n \frac{\partial}{\partial n} I_{-n}(\varrho) \right\}_{n \rightarrow \bar{n}}, \end{aligned}$$

hence

$$(39) \quad \pi N_{\bar{n}}(\varrho) = \text{Lim}_{n \rightarrow \bar{n}} \left\{ \frac{\partial}{\partial n} I_n(\varrho) - (-1)^n \frac{\partial}{\partial n} I_{-n}(\varrho) \right\}.$$

Here the limit sign indicates that the differentiation with respect to n must be performed before the passage to the limit, i.e., for non-integral n . Since we are primarily interested in the neighborhood of $\varrho = 0$, we naturally use the series (34) which (for non-integral n) represents not only I_n but also I_{-n} . We compute the two parts of the right side of (39) separately.

Using the well known formula

$$\frac{d}{dx} a^x = a^x \log a$$

we obtain from the first term of the series (34)

$$(40) \quad \text{Lim}_{n \rightarrow \bar{n}} \frac{\partial}{\partial n} I_n = \frac{1}{\Gamma(\bar{n}+1)} \left(\frac{\varrho}{2}\right)^{\bar{n}} \left\{ \log \frac{\varrho}{2} - \frac{\Gamma'(\bar{n}+1)}{\Gamma(\bar{n}+1)} \right\} + \dots,$$

where the three dots indicate terms of higher order than $\varrho^{\bar{n}}$. We use the abbreviation introduced by Gauss

$$(41) \quad \Psi(z) = \frac{\Gamma'(z+1)}{\Gamma(z+1)} = -C + \sum_{v=1}^{\infty} \left(\frac{1}{v} - \frac{1}{z+v} \right)$$

where C is Euler's constant

$$C = \text{Lim}_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right) = 0.5772.$$

If we set

$$(41a) \quad C = \log \gamma, \quad \gamma = 1.781,$$

then using (41) and (41a) we can write for the term $\{ \}$ in (40)

$$(41b) \quad \log \frac{\varrho}{2} - \Psi(\bar{n}) = \log \frac{\gamma \varrho}{2} - \sum_{v=1}^{\bar{n}} \frac{1}{v}.$$

The coefficient of the term $\{ \}$ in (40) is equal to $I_n(\varrho)$ except for higher powers than ϱ^n . Using (41b) we can rewrite (40) as

$$(42) \quad \text{Lim } \frac{\partial}{\partial n} I_n = \left\{ \log \frac{\gamma \varrho}{2} - \sum_{\nu=1}^{\bar{n}} \frac{1}{\nu} \right\} I_{\bar{n}}(\varrho) \dots$$

The three dots indicate that equation (42) is exact only up to terms of order $\varrho^{\bar{n}}$.

The computation of the second term on the right side of (39) is somewhat different. We start from

$$(43) \quad I_{-n} = \left(\frac{\varrho}{2}\right)^{-n} \left\{ \frac{1}{\Gamma(-n+1)} - \frac{1}{1! \Gamma(-n+2)} \left(\frac{\varrho}{2}\right)^2 + \dots \right. \\ \left. + \frac{(-1)^{\bar{n}}}{\bar{n}! \Gamma(-n+\bar{n}+1)} \left(\frac{\varrho}{2}\right)^{2\bar{n}} + \dots \right\}.$$

By first differentiating only the term $(\varrho/2)^{-n}$ with respect to n we get as in (40)

$$-\log \frac{\varrho}{2} \left(\frac{\varrho}{2}\right)^{-n} \left\{ \frac{1}{\Gamma(-n+1)} - \dots + \frac{(-1)^{\bar{n}}}{\bar{n}! \Gamma(-n+\bar{n}+1)} \left(\frac{\varrho}{2}\right)^{2\bar{n}} \right\} + \dots$$

For $n \rightarrow \bar{n}$, all the Γ 's become infinite except the last. We have then:

$$(44) \quad -\log \frac{\varrho}{2} \frac{(-1)^{\bar{n}}}{\bar{n}!} \left(\frac{\varrho}{2}\right)^{\bar{n}} = (-1)^{\bar{n}+1} \log \frac{\varrho}{2} I_{\bar{n}}(\varrho) + \dots$$

On the other hand the differentiation of the term $\{ \}$ in (43) yields⁵

$$(44a) \quad \left(\frac{\varrho}{2}\right)^{-n} \left\{ \frac{\Psi(-n)}{\Gamma(-n+1)} - \frac{\Psi(-n+1)}{1! \Gamma(-n+2)} \left(\frac{\varrho}{2}\right)^2 + \dots + \frac{(-1)^{\bar{n}} \Psi(-n+\bar{n})}{\bar{n}! \Gamma(-n+\bar{n}+1)} \left(\frac{\varrho}{2}\right)^{2\bar{n}} \right\} + \dots$$

The function $\Psi(z)$ has simple poles at the points $z = -1, -2, -3, \dots$ just like $\Gamma(z+1)$. According to (41) we have in the neighborhood of the ν -th pole

$$(45) \quad \Psi(z) = -\frac{1}{z+\nu}.$$

The development of $\Gamma(z+1)$ at the same point is⁶

$$(45a) \quad \Gamma(z+1) = \frac{(-1)^{\nu-1}}{(\nu-1)!} \frac{1}{z+\nu}.$$

⁵ In (44a) two minus signs have cancelled each other. Namely for $z = -n+1, -n+2, \dots$ we have

$$\frac{d}{dn} \frac{1}{\Gamma(z)} = -\frac{\Gamma'(z)}{[\Gamma(z)]^2} \frac{dz}{dn} = (-1)^2 \frac{\Gamma'(z)}{[\Gamma(z)]^2} = \frac{\Psi(z-1)}{\Gamma(z)}.$$

⁶ For this and the previous formulas see Jahnke-Emde's tables of functions, 3rd ed., Teubner, Leipzig, 1938, p. 10, 11, and 18.

Hence

$$(45b) \quad \frac{\Psi(z)}{\Gamma(z+1)} = (-1)^{\nu} (\nu-1)! \quad \text{for } z = -\nu.$$

Since $\Gamma(1) = 1$ and $\Psi(0) = -C$ we have, in the neighborhood of $z = 0$

$$(45c) \quad \frac{\Psi(z)}{\Gamma(z+1)} = -C = -\log \gamma.$$

After these preparations we can pass to the limit in (44a). According to (45) and (45a) all the terms Ψ/Γ , with the exception of the last, have the form ∞/∞ which according to (45b) can be replaced by $(-1)^{\nu} (\nu-1)!$ where $\nu = \bar{n}$ in the first term, $\nu = \bar{n} - 1$ in the subsequent terms. For the last term we apply (45c) and obtain

$$- \frac{(-1)^{\bar{n}}}{\bar{n}!} \log \gamma \left(\frac{\varrho}{2}\right)^{2\bar{n}} = (-1)^{\bar{n}} \left(\frac{\varrho}{2}\right)^{\bar{n}} \log \gamma I_{\bar{n}}(\varrho) + \dots$$

We have as the limiting value of (44a) (instead of \bar{n} we now write n , which is still an integer)

$$(46) \quad (-1)^n \left\{ (n-1)! \left(\frac{\varrho}{2}\right)^{-n} + \frac{(n-2)!}{1!} \left(\frac{\varrho}{2}\right)^{-n+2} + \dots - \log \gamma I_n \right\} + \dots$$

The sum of (46) and (44) now yields the second term in $\{ \}$ of (39)

$$\begin{aligned} -(-1)^n \operatorname{Lim} \frac{\partial}{\partial n} I_{-n} &= - (n-1)! \left(\frac{\varrho}{2}\right)^{-n} - \frac{(n-2)!}{1!} \left(\frac{\varrho}{2}\right)^{-n+2} \\ &\quad - \dots + \log \frac{\gamma \varrho}{2} I_n + \dots \end{aligned}$$

Combining this with (42) we obtain in (39) for $n > 0$

$$(47) \quad \begin{aligned} \pi N_n(\varrho) &= - (n-1)! \left(\frac{\varrho}{2}\right)^{-n} - \frac{(n-2)!}{1!} \left(\frac{\varrho}{2}\right)^{-n+2} \\ &\quad - \dots + 2 \log \frac{\gamma \varrho}{2} I_n - \sum_{\nu=1}^n \frac{1}{\nu} + \dots \end{aligned}$$

The terms on the right are written in decreasing order, the term with $(\varrho/2)^{-n}$ having highest order and the logarithmic term having lowest order. This implies a simple logarithmic singularity for $n = 0$; we have then:

$$(48) \quad \frac{\pi}{2} N_0(\varrho) = \log \frac{\gamma \varrho}{2} I_0 + \dots,$$

or the complete form, which we state without proof

$$(48a) \quad \frac{\pi}{2} N_0(\varrho) = \log \frac{\gamma \varrho}{2} I_0(\varrho) + 2 \left(I_2(\varrho) - \frac{1}{2} I_4(\varrho) + \frac{1}{3} I_6(\varrho) - \dots \right).$$

According to (26) this logarithmic singularity arises in H just as it does in N . We see from this that the H_n have branch points at the origin of the complex ϱ -plane even for integral n . From (26) and (47) we see that upon continuation around the origin H_n increases by $\mp 4I_n(\varrho)$ (for details see exercise IV.2). In exercise IV.3 we shall deduce the existence of the logarithmic singularity of H_0 in a more direct, though mathematically less satisfactory, way.

D. RECURSION FORMULAS

The $Z_n(\varrho)$ satisfy a differential equation in ϱ and a difference equation in n , for arbitrary, not necessarily integral, n . We can deduce this from our integral representation for the H and hence for arbitrary linear combinations of the H , in particular for the I and N .

Remembering that the paths of integration W_1 and W_2 in (22) are independent of n we form:

$$(49) \quad \frac{\pi}{2} (H_{n+1} + H_{n-1}) = \int e^{i\varrho \cos w} e^{in(w-\pi/2)} \{ \}_1 dw,$$

$$(50) \quad \frac{\pi}{2} (H_{n+1} - H_{n-1}) = \int e^{i\varrho \cos w} e^{in(w-\pi/2)} \{ \}_2 dw,$$

where

$$\{ \}_1 = \frac{1}{2} (e^{i(w-\pi/2)} + e^{-i(w-\pi/2)}) = \sin w,$$

$$\{ \}_2 = \frac{1}{2} (e^{i(w-\pi/2)} - e^{-i(w-\pi/2)}) = -i \cos w.$$

We may therefore write for the integrals on the right of (49), (50)

$$(49a) \quad -\frac{1}{i\varrho} \int \frac{\partial}{\partial w} (e^{i\varrho \cos w}) \cdot e^{in(w-\pi/2)} dw,$$

$$(50a) \quad -\frac{\partial}{\partial \varrho} \int e^{i\varrho \cos w} \cdot e^{in(w-\pi/2)} dw;$$

and by integration by parts (49a) becomes

$$(49b) \quad \frac{n}{\varrho} \int e^{i\varrho \cos w} \cdot e^{in(w-\pi/2)} dw.$$

We now can express the right sides of (49) and (50) in terms of Hankel functions of index n . These formulas are valid for both H^1 and H^2 depending on the path of integration; we may write them directly for the general cylinder function Z , which is a linear combination of the two. We have

$$(51) \quad Z_{n+1} + Z_{n-1} = \frac{2n}{\varrho} Z_n,$$

and

$$(52) \quad Z_{n+1} - Z_{n-1} = -2 \frac{dZ_n}{d\varrho}.$$

These are the recursion formulas we were seeking. They hold for n integral or non-integral, positive or negative.

For $n = 0$ we get as a special case

$$(51a) \quad Z_{-1} = -Z_{+1}$$

and

$$(52a) \quad Z_1 = -\frac{dZ_0}{d\varrho},$$

and by further specialization of (52a) we get the relation

$$(52b) \quad I_1(\varrho) = -\frac{d}{d\varrho} I_0(\varrho).$$

which could also have been obtained directly from the series (27) and (27a).

E. ASYMPTOTIC REPRESENTATION OF THE HANKEL FUNCTIONS

The integrand in our representations (14) and (22) oscillates more and more rapidly with increasing ϱ , for the non-shaded regions of the w -plane with increasing amplitude, for the shaded region with amplitude decreasing to zero. As shown in Fig. 19, the paths W_1 and W_2 for H^1 and H^2 can be drawn completely in the shaded regions for real ϱ . The figures illustrating exercise IV.2 show that this is no longer the case for complex ϱ . We also see from Fig. 19 that the points $w = 0$ and $w = \pi$, at which the paths touch two non-shaded regions, will play an important role for the asymptotic computation of H^1 and H^2 .

We shall develop here the *method of saddle points* in an intuitive, so to speak topographical, manner, and leave all analytic refinements and generalizations for §21. We assume

$$(53) \quad \varrho \text{ real} > 1 \quad \text{and} \quad n < \varrho.$$

For H^1 the path W_1 begins and ends in the shaded "low lands," and the same holds for H^2 and W_2 . The deciding exponent has its extremum at

$$\sin w = 0, \quad w = \begin{cases} 0 & \text{on } W_1 \\ \pi & \text{on } W_2. \end{cases}$$

This extremum, like all extrema of real or imaginary parts of complex functions, is not a maximum or a minimum but a saddle point. To the right and left of W_1 and W_2 at these points there tower steeply rising mountain ranges. Between them run W_1 and W_2 as *mountain passes*. The saddle point method is therefore also called the *pass method*. The altitude of the paths at $w = 0$ and $w = \pi$ is

$$|e^{i\theta}| = 1 \quad \text{and} \quad |e^{-i\theta}| = 1.$$

What path should a mountain climber take in order to surmount the pass in the fastest possible manner? The answer is, the path of steepest ascent and descent, the so-called "drop lines." However this prescription is not binding and it may be amended for reasons of convenience (analytic reasons⁷ or mountain climber's reasons). The English name "method of steepest descent" instead of pass method is therefore not entirely appropriate.

We consider a short segment of the path W_1 in the neighborhood of the crest of the path: let ds denote the arc element on this path with the orientation W_1 , and let the crest itself be given by $s = 0$. We write:

$$(54) \quad w = s e^{i\gamma}, \quad i \cos w = i \left(1 - \frac{s^2}{2} e^{2i\gamma} \right) = \frac{s^2}{2} \sin 2\gamma + i \left(1 - \frac{s^2}{2} \cos 2\gamma \right).$$

The level lines of the real part are perpendicular to both the level lines of the imaginary part and to the drop lines, therefore the level line of the imaginary part is at the same time the drop line of the real part which determines the altitude of the pass. In our case the level line of the imaginary part of (54) is given by

$$1 - \frac{s^2}{2} \cos 2\gamma = \text{const.}$$

with the constant equal to one, since the line must pass through the crest $s = 0$. Hence we have

$$(54a) \quad \cos 2\gamma = 0, \quad \gamma = \mp \frac{\pi}{4}.$$

For H^1 we must choose the minus sign for γ (see Fig. 19) whereby (54) becomes

$$(54b) \quad w = e^{-i\pi/4} s, \quad dw = e^{-i\pi/4} ds, \quad i \cos w = i - \frac{s^2}{2}.$$

We substitute this in (22) and at the same time set $s = 0$ in the "slowly varying" factor $\exp \{in(w - \pi/2)\}$; the integration can obviously

⁷ G. Faber, Bayr. Akad. 1922, p. 285.

be restricted to the immediate neighborhood of the pass, say to distances $< \varepsilon$. We obtain

$$(54c) \quad H_n^1(\varrho) = \frac{1}{\pi} e^{i[\varrho - (n + \frac{1}{2})\pi/2]} \int_{-\varepsilon}^{+\varepsilon} e^{-\frac{\varrho}{2}s^2} ds.$$

This integral can be reduced to the Laplace integral with the help of the substitution $s = \sqrt{\frac{2}{\varrho}} t$, which, at the limits of integration, becomes

$\sqrt{\frac{\varrho}{2}}\varepsilon \rightarrow \infty$ and $-\sqrt{\frac{\varrho}{2}}\varepsilon \rightarrow -\infty$. We therefore have the final result:

$$(55) \quad H_n^1(\varrho) = \sqrt{\frac{2}{\varrho\pi}} e^{i[\varrho - (n + \frac{1}{2})\pi/2]}.$$

For H^2 , where we have to use the path W_2 with the saddle point at $w = \pi$ and where in (54a) we have to choose the plus sign for γ , we obtain correspondingly

$$(56) \quad H_n^2(\varrho) = \sqrt{\frac{2}{\varrho\pi}} e^{-i[\varrho - (n + \frac{1}{2})\pi/2]}.$$

By taking half the sum of (55) and (56) we get

$$(57) \quad I_n(\varrho) = \sqrt{\frac{2}{\varrho\pi}} \cos \left[\varrho - \left(n + \frac{1}{2} \right) \frac{\pi}{2} \right].$$

These asymptotic representations, though derived for real ϱ , can be continued analytically in the complex ϱ -plane (for the representation of the two H 's this plane must be cut along a suitable half line because of the branching discussed at the end of Section C). On the basis of equations (55) and (56) we state: H^1 vanishes asymptotically for $\text{Im } \varrho \rightarrow +\infty$, H^2 for $\text{Im } \varrho \rightarrow -\infty$. This is the reason for the particular suitability of Hankel functions for the treatment of problems of damped oscillations. On the other hand both the Bessel function I and the Neumann function N become asymptotically infinite in both half planes.

We shall show in §21 how our asymptotic limits can be extended into asymptotic expansions and how the condition $n \ll \varrho$ of (53) can be dropped. The factor $\varrho^{-\frac{1}{2}}$ in (55) and (56) is connected with the fact that H_0^1 (or, for another choice of time dependence, H_0^2) represents, upon introduction of a coordinate z which is perpendicular to the r, φ -plane, an *expanding cylindrical wave* with source $r = 0$. Since the energy $2\pi r |H_0|^2$ passing through a cylinder of radius r must be inde-

pendent of r (in the absence of absorption), we see that H_0 is proportional to $r^{-1/2}$ or in other words to $\varrho^{-1/2}$.

One may think of the real and imaginary parts of H_0^1 and I_0 as defining surfaces over the complex ϱ -plane. The surface of $\text{Re} \{H_0^1\}$ osculates the positive ϱ -half-plane exponentially and in the negative half plane it has exponentially rising mountain ranges separated by correspondingly deepening valleys. The surface of $\text{Im} (H_0^1)$ behaves similarly and in addition has a narrow funnel at the origin which corresponds to the logarithmic singularity of H_0 (also of N_0 : see (48)), as well as a discontinuity along the negative real axis corresponding to the branching discussed above. The surface of $\text{Re}(I_0)$ consists of a mildly undulating depression flanked on both sides by rising mountain country. The undulating nature of the depression follows from the asymptotic equation (57) and indicates the existence of an infinity of roots of the equation $I_0 = 0$ along the real axis. These roots are represented in Fig. 21. The surface of $\text{Im} (I_0)$ is very similar in appearance except that the bottom of the depression is level throughout, corresponding to the fact that I_0 is real along the real axis.

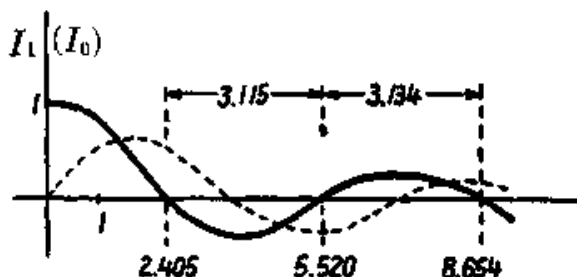


Fig. 21. Representation of I_0 (heavy line) and of I_1 (dotted line) along the real axis. The first three roots of $I_0(\varrho) = 0$.

§ 20. Heat Equalization in a Cylinder

As an excellent example for the application of the theory of Bessel functions we again consider a special problem of heat conduction. The problem was treated by Fourier, who, in fact, mentioned the functions with integral n whence they are sometimes referred to as *Fourier-Bessel functions*.

We shall treat our problem in three steps:

- A. For an infinitely long cylinder and an axially symmetric initial state $f = f(r)$.
- B. For an initial state which depends also on the argument $f = f(r, \varphi)$.
- C. For a cylinder of finite length and general initial state $f = f(r, \varphi, z)$.

The boundary condition shall, for the sake of simplicity, always be that of isothermy

$$(1) \quad u = 0 \quad \text{for } r = a = \text{radius of cylinder.}$$

For the complete cylinder this is augmented by the further "boundary condition" of finality along the axis:

$$(1a) \quad u \neq \infty \quad \text{for} \quad r = 0.$$

A. ONE-DIMENSIONAL CASE $f = f(r)$

The equation of heat conduction is:

$$(2) \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{1}{k} \frac{\partial u}{\partial t}.$$

Making the special substitution

$$(3) \quad u = R(r) e^{-\lambda^2 k t}$$

we get the differential equation for R

$$(3a) \quad \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \lambda^2 R = 0.$$

This is Bessel's differential equation (19.11) with $n = 0$ and $\rho = \lambda r$. Its general solution can be written as:

$$Z_0 = A I_0(\lambda r) + B N_0(\lambda r).$$

However, the condition of finality (1a) requires that we set $B = 0$; because of (1) we must further demand that

$$(4) \quad I_0(\lambda a) = 0.$$

We already know that this equation has an infinity of roots such that the distance between consecutive roots approaches π ; from (19.57) we get for the m -th root

$$(4a) \quad \lambda_m a - \frac{\pi}{4} = \left(m - \frac{1}{2}\right)\pi, \quad \text{hence} \quad \lambda_m a = \left(m - \frac{1}{4}\right)\pi.$$

This approximation is valid down to $m = 2$ with an accuracy of about 1%; for $m = 1$ we get

$$(4b) \quad \lambda_1 a = 2.40$$

as compared to 2.36 from (4a) (see Fig. 21).

We have then at our disposal an infinity of solutions of (3a):

$$R(r) = A_m I_0(\lambda_m r), \quad m = 1, 2, \dots$$

Correspondingly we get from (3) as the general solution of our problem

$$(5) \quad u = \sum_{m=1}^{\infty} A_m I_0(\lambda_m r) e^{-\lambda_m^2 k t}.$$

We now merely need to satisfy the initial condition

$$(6) \quad f(r) = \sum_{m=1}^{\infty} A_m I_0(\lambda_m r).$$

A way of doing this is indicated by the treatment of the anharmonic sine series in §16. In order to emphasize the complete analogy with the equations (5) and (6) of §16 we write

$$u_n = I_0(\lambda_n r), \quad u_m = I_0(\lambda_m r)$$

and then write our present equation (3a) in the form:

$$\frac{d}{dr} \left(r \frac{du_n}{dr} \right) + \lambda_n^2 r u_n = 0, \quad \frac{d}{dr} \left(r \frac{du_m}{dr} \right) + \lambda_m^2 r u_m = 0.$$

Multiplying by u_m and u_n and subtracting we get as an analogue to (16.5a)

$$(7) \quad u_m \frac{d}{dr} \left(r \frac{du_n}{dr} \right) - u_n \frac{d}{dr} \left(r \frac{du_m}{dr} \right) = (\lambda_m^2 - \lambda_n^2) r u_m u_n.$$

Integrating over the fundamental domain $0 < r < a$ we get as an analogue to (16.6)

$$(7a) \quad (\lambda_m^2 - \lambda_n^2) \int_0^a r u_m u_n dr = r \left(u_m \frac{du_n}{dr} - u_n \frac{du_m}{dr} \right) \Big|_0^a.$$

This is *Green's theorem* applied to the two-dimensional circular region $r = a$.

The right side of (7a) vanishes for the upper limit $r = a$ on account of equation (1), for the lower limit $r = 0$ on account of the factor r and equation (1a). Since $\lambda_m \neq \lambda_n$, for $m \neq n$, we have the *orthogonality condition*:

$$(8) \quad \int_0^a u_m u_n r dr = 0 \text{ for } m \neq n.$$

The "weighting factor" r is due to the two-dimensional element of area $r dr d\varphi$ in Green's theorem.

From (7a) we may also deduce the normalizing integral

$$N_m = \int u_m^2 r dr$$

if we drop the assumption that λ_n is a root of (4). We consider λ_n

rather as a continuous variable which in the limit coincides with λ_m . Equation (7a) then represents N_m as a fraction which for $\lambda_n \rightarrow \lambda_m$ assumes the form 0/0. By differentiating the numerator and denominator with respect to λ_n and substituting $r = a$ and $r = 0$ we find, because of (1), that

$$(9) \quad N_m = \frac{a}{2\lambda_n} \left(\frac{du_n}{d\lambda_n} \frac{du_m}{dr} \right)_{r=a} \rightarrow \frac{a}{2\lambda_m} \left(\frac{du_m}{d\lambda_m} \frac{du_m}{dr} \right)_{r=a}.$$

But for $r = a$

$$\frac{du_m}{d\lambda_m} = a I'_0(\lambda_m a), \quad \frac{du_m}{dr} = \lambda_m I'_0(\lambda_m a).$$

Substituting this in (9) we get

$$(9a) \quad N_m = \frac{a^2}{2} [I'_0(\lambda_m a)]^2.$$

The coefficients A_m of the series (6) can now be calculated from (8) and (9) in the Fourier manner:

$$(10) \quad A_m N_m = \int_0^a f(r) I_0(\lambda_m r) r dr,$$

We substitute this in the series (5), thereby completing the solution of problem A.

B. TWO-DIMENSIONAL CASE $f = f(r, \varphi)$

We first develop $f(r, \varphi)$ in the complex Fourier series (1.12)

$$(11) \quad f(r, \varphi) = \sum_{n=-\infty}^{+\infty} C_n e^{in\varphi}, \quad C_n = C_n(r) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(r, \varphi) e^{-in\varphi} d\varphi.$$

Due to the two-dimensional equation (2)

$$(12) \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = \frac{1}{k} \frac{\partial u}{\partial t}$$

and the generalized substitution (3)

$$(13) \quad u = R_n(r) e^{in\varphi} e^{-\lambda^2 k t}$$

we have the differential equation for $R_n(r)$

$$(14) \quad \frac{d^2 R_n}{dr^2} + \frac{1}{r} \frac{dR_n}{dr} + \left(\lambda^2 - \frac{n^2}{r^2} \right) R_n = 0.$$

This is Bessel's differential equation (19.11) with $\varrho = \lambda r$. Equation (1a) requires that the only permissible solutions be of the form $A_n I_n(\lambda r)$. On account of (1) λ must satisfy the equation $I_n(\lambda a) = 0$ which, just like $I_0(\lambda a) = 0$, has an infinity of roots:

$$\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,m}, \dots$$

Each of these roots yields a particular solution of the form (13):

$$(15) \quad u_{n,m} = A_{n,m} I_n(\lambda_{n,m} r) e^{in\varphi} e^{-\lambda_{n,m}^2 t},$$

and these solutions satisfy the differential equation (12). Through superposition we can construct from them the general solution of (14) which at the same time satisfies our boundary conditions:

$$(16) \quad u = \sum \sum u_{n,m} = \sum_{n=-\infty}^{+\infty} \sum_{m=1}^{\infty} A_{n,m} I_n(\lambda_{n,m} r) e^{in\varphi} e^{-\lambda_{n,m}^2 t}.$$

Here the constants $A_{n,m}$ must be chosen so that for $t = 0$ and every integer $-\infty < n < +\infty$ we have the equation

$$(17) \quad C_n(r) = \sum_{m=1}^{\infty} A_{n,m} I_n(\lambda_{n,m} r)$$

where according to (11) the left side is a known function of r . Equation (17) necessitates the development of this function in Bessel functions I_n . This is possible due to the *orthogonality* of the latter, which follows from Bessel's differential equation (14) and Green's theorem as in (7) and (7a)⁸. Using the abbreviations

$$v_m = I_n(\lambda_{n,m} r), \quad v_l = I_n(\lambda_{n,l} r)$$

we obtain as generalization of (7)

$$(\lambda_{n,m}^2 - \lambda_{n,l}^2) \int_0^a r v_m v_l dr = r \left(v_m \frac{dv_l}{dr} - v_l \frac{dv_m}{dr} \right) \Big|_0^a$$

Here, too, the right side vanishes. We thus have for $l \neq m$

$$(18) \quad \int_0^a v_m v_l r dr = 0.$$

At the same time we obtain by a passage to the limit as described in (9)

⁸ In order to avoid the trivial result $0 = 0$, in the application of Green's theorem to the circle $r = a$ in the r, φ -plane we must use the two functions

$$v_{nm} = I_n(\lambda_{n,m} r) e^{+in\varphi} \quad \text{and} \quad v_{nl} = I_n(\lambda_{n,l} r) e^{-in\varphi}$$

$$(19) \quad N_{n,m} = \int_0^a v_m^2 r dr = \frac{a^4}{2} [I_n'(\lambda_{n,m} a)]^2.$$

The $A_{n,m}$ in (17) can now be calculated in the Fourier manner from the given $C_n(r)$ by (18) and (19) in analogy to (10). Substituting these expressions for $C_n(r)$ in (11) we obtain

$$(20) \quad 2\pi N_{n,m} A_{n,m} = \int_0^a \int_{-\pi}^{+\pi} f(r, \varphi) I_n(\lambda_{n,m} r) e^{-i n \varphi} r dr d\varphi.$$

which concludes the solution of (16).

C. THREE-DIMENSIONAL CASE $f = f(r, \varphi, z)$

Let the cylinder have the finite length h and let $0 < z < h$. We first develop $f(r, \varphi, z)$ in a Fourier series with respect to z , which, due to the boundary conditions $u = 0$ for $z = 0$ and $z = h$, becomes a pure sine series:

$$(21) \quad f(r, \varphi, z) = \sum_{\mu=1}^{\infty} B_{\mu} \sin \mu \pi \frac{z}{h}, \quad B_{\mu} = \frac{2}{h} \int_0^h f(r, \varphi, z) \sin \mu \pi \frac{z}{h} dz;$$

We then develop $B_{\mu} = B_{\mu}(r, \varphi)$ in a series of $\exp(in\varphi)$:

$$(22) \quad B_{\mu}(r, \varphi) = \sum_{n=-\infty}^{+\infty} C_{\mu,n} e^{in\varphi}, \quad C_{\mu,n} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} B_{\mu}(r, \varphi) e^{-in\varphi} d\varphi.$$

Finally we represent $C_{\mu,n} = C_{\mu,n}(r)$ as a series in the Bessel functions $I_n(\lambda r)$, which progresses according to the roots of

$$I_n(\lambda a) = 0, \quad \lambda = \lambda_{n,m}, \quad m = 1, 2, \dots$$

Due to the three-dimensional equation of heat conduction

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k} \frac{\partial u}{\partial t}$$

the time factor has the form

$$e^{-\alpha^2 k t} \quad \text{with} \quad \alpha^2 = \lambda_{n,m}^2 + \left(\frac{\mu \pi}{h}\right)^2.$$

The complete solution is given by the triply infinite sum

$$(23) \quad u = \sum_{\mu=1}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} A_{\mu n m} I_n(\lambda_{n,m} r) e^{i n \varphi} \sin \mu \pi \frac{z}{h} e^{-[\lambda_{n,m}^2 + (\frac{\mu \pi}{h})^2] k t}.$$

The coefficients A are calculated from the $N_{n,m}$ of (19) in analogy to (10) and (20)

$$(24) \quad \pi h N_{n,m} A_{\mu, n, m} = \int_0^{a+\pi h} \int_{-\pi}^{\pi} \int_0^h f(r, \varphi, z) I_n(\lambda_{n,m} r) e^{-i n \varphi} \sin \mu \pi \frac{z}{h} r dr d\varphi dz.$$

This completes the solution of (23).

In the case of a *hollow cylinder* the condition of finality (1a) is dropped. Hence in the expansion of the solution there may appear terms with N_n in addition to those with I_n (or, in other words, terms in H_n^1 and H_n^2). Heat conduction through a heating pipe is an example of this.

§ 21. More About Bessel Functions

A. GENERATING FUNCTION AND ADDITION THEOREMS

In §19 we started from the two-dimensional wave equation $\Delta u + k^2 u = 0$ and its simplest solution, the *plane wave*

$$(1) \quad u = e^{i k x} = e^{i \varrho \cos \varphi}, \quad \varrho = k r, \quad k = \text{the wave number.}$$

If we develop this into a Fourier series then, due to the origin of Bessel's differential equation (19.11), the coefficients must be Bessel functions, and because of the regularity of (1) for $r = 0$ only the I functions will appear. Hence, we set the coefficient of $\exp(i n \varphi)$ equal to $c_n I_n$ and according to (1.12) we then have

$$c_n I_n(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i \varrho \cos \varphi} e^{-i n \varphi} d\varphi.$$

If we compare this with (19.18), in which we may replace w by $-w$, we obtain

$$c_n = e^{i n \pi/2}.$$

Hence we have the Fourier series

$$(2) \quad e^{i\rho \cos \varphi} = \sum_{n=-\infty}^{+\infty} e^{i n \pi/2} I_n(\rho) e^{i n \varphi}$$

or upon the substitution $\psi = \varphi + \pi/2$

$$(2a) \quad e^{i\rho \sin \psi} = \sum_{n=-\infty}^{+\infty} I_n(\rho) e^{i n \psi}.$$

In the older literature (2) is usually written in the less symmetric form

$$(2b) \quad e^{i\rho \cos \varphi} = I_0(\rho) + 2 \sum_{n=1}^{\infty} i^n I_n(\rho) \cos n\varphi.$$

The left sides of (2) and (2a) are called *generating functions of the Bessel functions with integral index*.

We now pass from the case of a plane wave to that of a *cylindrical wave* with its logarithmic source at the origin, which, according to p. 100, is represented by $H_0(\rho)$. (We omit the upper index since the following is valid for both functions H , i.e., both for radiated and for absorbed waves.) We now shift the origin from $\rho = 0$ to $\rho = \rho_0$, $\varphi = \varphi_0$ whereby $H_0(\rho)$ goes over into

$$H_0(R), \quad R = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\varphi - \varphi_0)}.$$

If we develop this into a Fourier series with respect to $\varphi - \varphi_0$, then the coefficients must again be cylinder functions, namely functions $H_n(\rho)$ for $\rho < \rho_0$ and functions $I_n(\rho)$ for $\rho > \rho_0$. The latter follows from the fact that $\rho = 0$ is now a regular point, the former from the fact that each term in the series must have the same type of radiation or absorption for $\rho \rightarrow \infty$ as $H_0(R)$ itself. For reasons of symmetry the same consideration holds for the dependence of our coefficients on the variable ρ_0 , except that the functions I_n and H_n are interchanged since the condition $\rho \geq \rho_0$ is the same as $\rho_0 \leq \rho$. Hence the n -th Fourier coefficient must be

$$c_n \begin{cases} I_n(\rho_0) H_n(\rho) & \text{for } \rho > \rho_0, \\ H_n(\rho_0) I_n(\rho) & \text{for } \rho < \rho_0. \end{cases}$$

The numerical factor c_n is independent of ρ and ρ_0 and is the same for both expansions, since the two series must go into each other continuously for $\rho = \rho_0$ (unless at the same time $\varphi = \varphi_0$, in which case both series diverge); it turns out to be equal to 1 if, in the case $\rho < \rho_0$, we pass to the limiting case of a plane wave $\rho_0 \rightarrow \infty$, setting

$\varphi_0 = \pi$ and comparing the resulting asymptotic formula with equation (2). We thus obtain the *addition theorem*:

$$(3) \quad H_0(R) = \begin{cases} \sum_{n=-\infty}^{+\infty} I_n(\varrho_0) H_n(\varrho) e^{in(\varphi - \varphi_0)} & \varrho > \varrho_0, \\ \sum_{n=-\infty}^{+\infty} H_n(\varrho_0) I_n(\varrho) e^{in(\varphi - \varphi_0)} & \varrho < \varrho_0. \end{cases}$$

If we consider this written for *both* Hankel functions and take half the sum then we obtain the *addition theorem for Bessel functions*:

$$(3a) \quad I_0(R) = \sum_{n=-\infty}^{+\infty} I_n(\varrho_0) I_n(\varrho) e^{in(\varphi - \varphi_0)} \quad \varrho \geq \varrho_0.$$

In the same manner we get from half the difference the addition theorem for Neumann functions, where we again have to distinguish between the cases $\varrho > \varrho_0$ and $\varrho < \varrho_0$.

Concerning (3) we note that the series in I_n corresponds to *Taylor's series* in the theory of complex functions, whereas the series in H_n corresponds to *Laurent's series*.⁹ This is illustrated by the following example, in which one may replace z and z_0 by $\varrho e^{i\varphi}$ and $\varrho_0 e^{i\varphi_0}$:

$$(4) \quad \begin{aligned} \frac{z}{z - z_0} &= \sum_{n=0}^{\infty} z_0^n z^{-n-1} & |z| > |z_0|, \\ \frac{z_0}{z_0 - z} &= \sum_{n=0}^{\infty} z_0^{-n-1} z^n & |z| < |z_0|. \end{aligned}$$

In §24 we shall develop corresponding addition theorems for spherical waves in space; there will also be a counterpart to the representation (2) of a plane wave.

B. INTEGRAL REPRESENTATIONS IN TERMS OF BESSEL FUNCTIONS

We shall give here the development of a given function $f(r)$ in terms of Bessel functions which is analogous to a representation by a Fourier integral. According to (12.11b) a function of two variables can be represented by the Fourier integral

$$(5) \quad f(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d\omega \, d\omega' \int_{-\infty}^{+\infty} d\xi \, d\eta \, f(\xi, \eta) e^{i\omega(x - \xi) + i\omega'(y - \eta)}.$$

⁹ This is further discussed, together with questions of convergence, in §2 of the author's work which was cited on p. 80. We also refer to the great work of H. Weber, *Math. Ann.* I, p. 1, which was a fitting beginning for that journal. It is the problem of adapting the methods of Riemann's dissertation, i.e., of adapting the theory of the differential equation $\Delta u = 0$, to the differential equation $\Delta u + k^2 u = 0$.

We introduce the polar coordinates:

$$\begin{aligned} x &= r \cos \varphi & \xi &= \varrho \cos \psi & \omega &= \sigma \cos \alpha \\ y &= r \sin \varphi & \eta &= \varrho \sin \psi & \omega' &= \sigma \sin \alpha \\ & & d\xi d\eta &= \varrho d\varrho d\psi & d\omega d\omega' &= \sigma d\sigma d\alpha. \end{aligned}$$

We assume the special angular dependence of $f(x, y)$:

$$(6) \quad f(x, y) = f(r) e^{in\varphi} \quad (n \text{ integer}).$$

By using the relations

$$\begin{aligned} \omega x + \omega' y &= \sigma r \cos(\alpha - \varphi) \\ \omega \xi + \omega' \eta &= \sigma \varrho \cos(\psi - \alpha) \end{aligned}$$

we can transform (5) into

$$(7) \quad f(r) e^{in\varphi} = \int_0^\infty \sigma d\sigma \int_0^\infty f(\varrho) \varrho d\varrho \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i\sigma r \cos(\alpha - \varphi)} d\alpha \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{in\psi} e^{-i\sigma \varrho \cos(\psi - \alpha)} d\psi.$$

In order to compare these integrals with respect to α and ψ to the representation (19.18)

$$\begin{aligned} I_n(z) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{iz \cos \beta} e^{in(\beta - \pi/2)} d\beta \\ &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-iz \cos \beta'} e^{in(\beta' + \pi/2)} d\beta' \quad (\beta' = \beta - \pi) \end{aligned}$$

we multiply under the integrals of (7) by

$$e^{in(\alpha - \pi/2)} \text{ and } e^{in(-\alpha + \pi/2)},$$

and divide through by $e^{in\varphi}$. We thus obtain the simple representation

$$(8) \quad f(r) = \int_0^\infty \sigma d\sigma \int_0^\infty f(\varrho) I_n(\sigma r) I_n(\sigma \varrho) \varrho d\varrho.$$

In analogy to the form (4.13) of the Fourier integral theorem we can write this relation in the symmetric form:

$$\begin{aligned} (8a) \quad f(r) &= \int_0^\infty \sigma d\sigma \varphi(\sigma) I_n(\sigma r), \\ \varphi(\sigma) &= \int_0^\infty \varrho d\varrho f(\varrho) I_n(\sigma \varrho). \end{aligned}$$

The transition from rectangular coordinates to polar coordinates in the passage from (5) to (7) requires certain conditions about the behavior of f at infinity which we do not discuss here. Equation (7) will be useful in the treatment of spherical waves in §24.

We obtain a further application of equation (8) if we let $f(r)$ degenerate to a δ -function, namely,¹⁰ let

$$(9) \quad f(r) = \delta(r|s) = \begin{cases} 0 & \text{for } r \neq s \\ \infty & \text{for } r = s \end{cases}, \quad \text{with } \int_{-\infty}^{+\infty} f(r) r dr = 1.$$

We then obtain from (8)

$$(9a) \quad \int_0^{\infty} I_n(\sigma r) I_n(\sigma s) \sigma d\sigma = \delta(r|s).$$

This equation represents the *orthogonality* of the two functions I_n at the points r and s of the continuous domain $0 < r, s < \infty$; it is a counterpart to (20.18) in which we deal with two points m and l of the discrete λ -sequence. We shall return to the important relation (9a) in §36.

C. THE INDICES $n + 1/2$ AND $n \pm 1/3$

Substituting $n = 1/2$ in (19.34) we get $I'(\frac{1}{2}) = \sqrt{\pi}$, $I'(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, ...

$$(10) \quad \begin{aligned} I_{\frac{1}{2}}(\varrho) &= \sqrt{\frac{\varrho}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{2 \cdot 2}{1 \cdot 3 \sqrt{\pi}} \left(\frac{\varrho}{2}\right)^2 + \frac{2 \cdot 2 \cdot 2}{2! \cdot 1 \cdot 3 \cdot 5 \cdot \sqrt{\pi}} \left(\frac{\varrho}{2}\right)^4 - \dots \right] \\ &= \sqrt{\frac{2\varrho}{\pi}} \left(1 - \frac{\varrho^2}{3!} + \frac{\varrho^4}{5!} - \dots \right) = \sqrt{\frac{2\varrho}{\pi}} \frac{\sin \varrho}{\varrho}; \end{aligned}$$

in the same manner we find for $n = -1/2$

$$(10a) \quad I_{-\frac{1}{2}}(\varrho) = \sqrt{\frac{2\varrho}{\pi}} \frac{\cos \varrho}{\varrho}.$$

We write generally

$$(11) \quad I_{n+\frac{1}{2}}(\varrho) = \sqrt{\frac{2\varrho}{\pi}} \psi_n(\varrho), \quad \text{in particular } \psi_0 = \frac{\sin \varrho}{\varrho}.$$

From Bessel's differential equation (19.11) we get the differential equation for ψ_n

¹⁰ We draw the reader's attention to the weighting factor r in the integral of (9). Because of this factor we no longer have $\int \delta(r|s) ds = 1$, but instead $\int \delta(r|s) r dr = 1$, as in equation (9).

$$(11a) \quad \frac{1}{\varrho} \frac{d^2(\varrho \psi_n)}{d\varrho^2} + \left(1 - \frac{n(n+1)}{\varrho^2}\right) \psi_n = 0.$$

We shall meet this equation again in the theory of spherical harmonics. We now wish to show that the solutions which are finite for $\varrho = 0$ can be obtained from ψ_0 by the following rule:

$$(12) \quad \psi_n = (-\varrho)^n \left(\frac{d}{\varrho d\varrho}\right)^n \psi_0.$$

We start from the series (19.34). Let p be an arbitrary index (in our case we have $p = 1/2$) then we have

$$(13) \quad \frac{I_p(\varrho)}{(\varrho/2)^p} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(p+m+1)} \frac{(\varrho^2/2)^m}{2^m}.$$

We differentiate this equation n times with respect to $\varrho^2/2$. Then the right side becomes

$$\sum_{m=n}^{\infty} \frac{(-1)^m}{(m-n)! \Gamma(p+m+1)} \frac{(\varrho^2/2)^{m-n}}{2^m}.$$

By introducing a new index of summation ($u = m - n$, $m = n + u$) we get

$$(13a) \quad \frac{(-1)^n}{2^n} \sum_{u=0}^{\infty} \frac{(-1)^u}{u!} \frac{(\varrho^2/4)^u}{\Gamma(u+n+p+1)} = \frac{(-1)^n}{2^n} \frac{I_{n+p}(\varrho)}{(\varrho/2)^{n+p}}.$$

At the same time the left side of (13) becomes (because of $d(\varrho^2/2) = \varrho d\varrho$)

$$(13b) \quad \left(\frac{d}{\varrho d\varrho}\right)^n \frac{I_p(\varrho)}{(\varrho/2)^p}.$$

Comparing (13a) and (13b) we get

$$(13c) \quad \frac{I_{n+p}(\varrho)}{\varrho^p} = (-\varrho)^n \left(\frac{d}{\varrho d\varrho}\right)^n \frac{I_p(\varrho)}{\varrho^p};$$

due to (10); this coincides with (12) for $p = 1/2$.

If instead of the *ratio* of $I_p(\varrho)$ and $(\varrho/2)^p$ we differentiate their *product* n times with respect to $\varrho^2/2$, then instead of (13c) we get

$$(13d) \quad I_{-n+p}(\varrho) \cdot \varrho^p = \varrho^n \left(\frac{d}{\varrho d\varrho}\right)^n \{I_p(\varrho) \cdot \varrho^p\}.$$

If we again set $p = 1/2$ and apply definition (11) we get as the complement of (12)

$$(13e) \quad \psi_{-n}(\varrho) = \varrho^{n-1} \left(\frac{d}{\varrho d\varrho} \right)^n \{ \varrho \psi_0 \}.$$

From (12) and (13e) we deduce the *recursion formulas* for $\psi_{\pm n}$

$$(14) \quad \begin{aligned} \psi_{n+1} &= -\frac{d\psi_n}{d\varrho} + \frac{n}{\varrho} \psi_n, \\ \psi_{-n-1} &= \frac{d\psi_{-n}}{d\varrho} - \frac{n-1}{\varrho} \psi_{-n}. \end{aligned}$$

The corresponding formulas for $Z_{\pm n}$ were discussed in §19 D.

According to (14) we get successively from $\psi_0 = \sin \varrho / \varrho$:

$$(14a) \quad \begin{aligned} \psi_1 &= \frac{\sin \varrho - \varrho \cos \varrho}{\varrho^2}, & \psi_2 &= \frac{3(\sin \varrho - \varrho \cos \varrho) - \varrho^2 \sin \varrho}{\varrho^3}, \dots \\ \psi_{-1} &= \frac{\cos \varrho}{\varrho}, & \psi_{-2} &= -\frac{\cos \varrho + \varrho \sin \varrho}{\varrho^2}, \\ \psi_{-3} &= \frac{3(\cos \varrho + \varrho \sin \varrho) - \varrho^2 \cos \varrho}{\varrho^3}, \dots \end{aligned}$$

We see from this that for integral n all $\psi_{\pm n}$ can be expressed in an *elementary form with the help of the sine and cosine functions*. This representation confirms the non-logarithmic character of the half-index Bessel functions, as stated in footnote 4 of this chapter.

The "Hankel functions" ζ_n which correspond to the ψ_n are given by the equation

$$(15) \quad H_{n+\frac{1}{2}}(\varrho) = \sqrt{\frac{2\varrho}{\pi}} \zeta_n(\varrho)$$

analogous to (11) (the upper indices 1 and 2 on both sides have been omitted). We are particularly interested in the functions ζ_0 . We obtain them from the functions $H_{\frac{1}{2}}$, which we get from (19.31), (19.30), and (11):

$$H_{\frac{1}{2}}^1 = \frac{-i I_{\frac{1}{2}} - I_{-\frac{1}{2}}}{-i} = \sqrt{\frac{2\varrho}{\pi}} \frac{e^{i\varrho}}{i\varrho}, \quad H_{\frac{1}{2}}^2 = \frac{i I_{\frac{1}{2}} - I_{-\frac{1}{2}}}{i} = \sqrt{\frac{2\varrho}{\pi}} \frac{e^{-i\varrho}}{\varrho}.$$

Hence according to (15) we have

$$(15a) \quad \zeta_0^1 = \frac{e^{i\varrho}}{i\varrho}, \quad \zeta_0^2 = \frac{e^{-i\varrho}}{-i\varrho}.$$

Concerning the notation we make the following remark: our notation coincides with the original definition of the ψ_n in Heine's *Handbook of Spherical Harmonics* and with that used by the author in Frank-Mises. However it differs by a factor ϱ from that of other authors¹¹ who instead

¹¹ P. Debye, *Ann. Physik* 30 (1909), B. van der Pol and H. Bremmer, *Phil. Mag.* 24 (1937). Further references in G. N. Watson, *Theory of Bessel Functions*, p. 56.

of (11) write:

$$(16) \quad I_{n+\frac{1}{2}}(\varrho) = \sqrt{\frac{2}{\pi\varrho}} \psi_n(\varrho),$$

which is sometimes convenient.

In analogy to the equations (14a) for the ψ_n we can express the ζ_n in *elementary form with the help of* $\exp(\pm i\varrho)$. This states the fact, which will be derived in Section D, that Hankel's asymptotic expansions break off in the case of half-indices.

The differential equation for $Z_{\pm\frac{1}{2}}(kr)$ assumes an unexpectedly elegant form if we replace the independent and the dependent variable¹² by

$$(17) \quad kr = \frac{2}{3}\varrho^{\frac{3}{2}}, \quad Z_{\pm\frac{1}{2}}(kr) = \varrho^{-\frac{1}{2}}\Phi(\varrho).$$

The functions ψ_n in (16) are denoted by S_n in acoustical engineering; the corresponding C_n would, in our terminology, have to be called a "Neumann function," since it is proportional to $\zeta_n^1 - \zeta_n^2$.

¹² The direct computation would lead to lengthy transformations. We avoid them, and at the same time recognize the generalizability of the relations (17) to (20), if we start from the conformal mapping

$$x + iy = f(\xi + i\eta), \quad \Delta_{xy} = \frac{1}{|f'(\xi + i\eta)|^2} \Delta_{\xi\eta},$$

through which

$$(1) \quad \Delta_{xy} u + k^2 u = 0$$

goes over into

$$(2) \quad \Delta_{\xi\eta} v + k^2 |f'(\xi + i\eta)|^2 v = 0;$$

where $u(x, y) = v(\xi, \eta)$ (see equation (23.17) below). If in (2) we set f proportional to a power of $\xi + i\eta$, e.g.,

$$kf(\xi + i\eta) = \frac{2}{\mu}(\xi + i\eta)^{\mu/2}, \quad \begin{cases} x + iy = r e^{i\varphi} \\ \xi + i\eta = \varrho e^{i\psi} \end{cases}$$

then we get

$$(3) \quad k|x + iy| = kr = \frac{2}{\mu}|\xi + i\eta|^{\mu/2} = \frac{2}{\mu}\varrho^{\mu/2}, \quad \varphi = \frac{\mu}{2}\psi,$$

and

$$k|f'(\xi + i\eta)| = \varrho^{\mu/2}/\varrho.$$

The solution of (1)

$$u = I_{\mp 1/\mu}(kr) e^{i\varphi/\mu}$$

then goes over into the solution of (2)

$$(4) \quad v = I_{\mp 1/\mu}\left(\frac{2}{\mu}\varrho^{\mu/2}\right) e^{i\psi/2};$$

It then becomes

$$(18) \quad \Phi''(\varrho) + \varrho \Phi(\varrho) = 0.$$

If we write its solutions as a series with undetermined coefficients starting with ϱ^0 and ϱ^1 , we get

$$(19) \quad \begin{aligned} \Phi_0 &= 1 - 1 \cdot \frac{\varrho^3}{3!} + 1 \cdot 4 \cdot \frac{\varrho^6}{6!} - 1 \cdot 4 \cdot 7 \cdot \frac{\varrho^9}{9!} + \dots, \\ \Phi_1 &= \varrho - 2 \cdot \frac{\varrho^4}{4!} + 2 \cdot 5 \cdot \frac{\varrho^7}{7!} - 2 \cdot 5 \cdot 8 \cdot \frac{\varrho^{10}}{10!} + \dots \end{aligned}$$

Considering (19.34) we see that the first is proportional to $I_{-\frac{1}{2}}$, the second to $I_{+\frac{1}{2}}$, namely, that we have

$$(20) \quad \begin{aligned} \Phi_0(\varrho) &= 3^{-\frac{1}{2}} \Gamma\left(1 - \frac{1}{3}\right) \varrho^{\frac{1}{2}} I_{-\frac{1}{2}}\left(\frac{2}{3} \varrho^{\frac{3}{2}}\right), \\ \Phi_1(\varrho) &= 3^{+\frac{1}{2}} \Gamma\left(1 + \frac{1}{3}\right) \varrho^{\frac{1}{2}} I_{+\frac{1}{2}}\left(\frac{2}{3} \varrho^{\frac{3}{2}}\right). \end{aligned}$$

We shall meet the functions $I_{\pm\frac{1}{2}}$ again at the end of Section D. and therefore

$$\frac{1}{\varrho} \frac{\partial}{\partial \varrho} \left(\varrho \frac{\partial v}{\partial \varrho} \right) + \frac{1}{\varrho^2} \frac{\partial^2 v}{\partial \varphi^2} + \varrho^{\mu-2} v = 0.$$

Substituting $v = \varrho^{-\frac{1}{2}} \Phi(\varrho) e^{i\varphi/2}$, here we get:

$$(5) \quad \frac{d^2 \Phi}{d\varrho^2} + \varrho^{\mu-2} \Phi = 0,$$

with the easily verifiable series representations for its solutions:

$$(6) \quad \begin{aligned} \Phi_0 &= 1 - \frac{\varrho^\mu}{\mu(\mu-1)} + \frac{\varrho^{2\mu}}{\mu(\mu-1)2\mu(2\mu-1)} - \dots, \\ \Phi_1 &= \varrho - \frac{\varrho^{\mu+1}}{(\mu+1)\mu} + \frac{\varrho^{2\mu+1}}{(\mu+1)\mu(2\mu+1)2\mu} - \dots. \end{aligned}$$

We can relate these solutions to the solutions (4) for v ; namely we have:

$$(7) \quad \begin{Bmatrix} \Phi_0 \\ \Phi_1 \end{Bmatrix} = C_{\mp} \varrho^{\frac{1}{2}} I_{\mp 1/\mu} \left(\frac{2}{\mu} \varrho^{\mu/2} \right),$$

where C_{\mp} are constant factors. Substituting the power series for I from (19.34), and comparing with (6) we get

$$(7a) \quad C_- = \mu^{-1/\mu} \Gamma\left(1 - \frac{1}{\mu}\right), \quad C_+ = \mu^{1/\mu} \Gamma\left(1 + \frac{1}{\mu}\right).$$

For $\mu = 3$ the equations (5), (6), (7), (7a) go over into the equations (18), (19), (20) of the text.

For $\mu = 2$ equation (5) reduces to the differential equation for the trigonometric functions, and Φ_0, Φ_1 become $\cos \varrho, \sin \varrho$. For $\mu = 1$ our series representation breaks down, since then $\varrho = 0$ is a singular point of the differential equation (5).

D. GENERALIZATION OF THE SADDLE-POINT METHOD ACCORDING TO DEBYE

Although in later applications we shall in general apply only the asymptotic limiting value of the Bessel functions as determined at the end of §19, we wish to discuss here certain more general expansions due to Hankel, which progress according to negative powers of ϱ and in which the first term is the above mentioned asymptotic limiting value. Actually these series are *divergent*, being developments at an essential singularity, but they are frequently called *semi-convergent*. The first terms decrease rapidly, but from a certain term on they increase to infinity. We obviously must break off at that term in order to obtain approximation formulas.

The shortest way of obtaining these series is from the differential equation for the Hankel function, by substituting formal power series, and then computing the coefficients by setting the factor of each power equal to zero (this is obviously not completely rigorous). Considering (19.55) and (19.56) we write

$$(22) \quad H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi\varrho}} e^{\pm i(\varrho - (n+\frac{1}{2})\pi/2)} \left(a_0 + \frac{a_1}{\varrho} + \frac{a_2}{\varrho^2} + \cdots + \frac{a_m}{\varrho^m} + \cdots \right)$$

and after dividing out the factor $\sqrt{2/\pi} \exp \{ \pm i(\varrho - (n+\frac{1}{2})\pi/2) \}$ from the differential equation (19.11) we find the terms with ϱ^{-m-1} to be

$$\begin{aligned} & -a_{m+1} \mp 2i(m+\tfrac{1}{2})a_m + (m+\tfrac{1}{2})(m-\tfrac{1}{2})a_{m-1}, \\ & \qquad \qquad \qquad \pm ia_m \qquad \qquad \qquad -(m-\tfrac{1}{2})a_{m-1}, \\ & + a_{m+1} \qquad \qquad \qquad -n^2 a_{m-1}. \end{aligned}$$

where the consecutive rows correspond to the consecutive terms

$$\frac{d^2 Z}{d\varrho^2}, \quad \frac{1}{\varrho} \frac{dZ}{d\varrho}, \quad \left(1 - \frac{n^2}{\varrho^2}\right) Z$$

in (19.11). Summing the three rows we get the following *first order recursion formula*

$$(23) \quad \mp 2i m a_m = (n^2 - \{m - \tfrac{1}{2}\}^2) a_{m-1}.$$

Setting $a_0 = 1$ we get

$$(24) \quad \frac{a_1}{\varrho} = \frac{4n^2 - 1}{2^2 (\mp 2i\varrho)}, \quad \frac{a_2}{\varrho^2} = \frac{(4n^2 - 1)(4n^2 - 9)}{2^4 2! (\mp 2i\varrho)^2}, \dots$$

Using the symbol

$$(25) \quad (n, m) = \frac{(4n^2 - 1)(4n^2 - 9) \cdots (4n^2 - (2m - 1)^2)}{2^{2m} m!}, \quad (n, 0) = 1$$

which was introduced by Hankel, we get the general formula:

$$(26) \quad \frac{a_m}{\varrho^m} = \frac{(n, m)}{(\mp 2i\varrho)^m}.$$

The series in (22) for H^1 and H^2 then assume the final form:

$$(27) \quad H_n^1(\varrho) = \sqrt{\frac{2}{\pi\varrho}} e^{i(\varrho - (n + \frac{1}{2})\pi/2)} \sum_{m=0,1,2,\dots} \frac{(n, m)}{(-2i\varrho)^m},$$

$$(28) \quad H_n^2(\varrho) = \sqrt{\frac{2}{\pi\varrho}} e^{-i(\varrho - (n + \frac{1}{2})\pi/2)} \sum_{m=0,1,2,\dots} \frac{(n, m)}{(+2i\varrho)^m}.$$

Taking half their sum we get:

$$(29) \quad \begin{aligned} I_n &= \sqrt{\frac{2}{\pi\varrho}} \cos(\varrho - (n + \frac{1}{2})\pi/2) \sum_{m=0,2,4,\dots} (-1)^{\frac{m}{2}} \frac{(n, m)}{(2\varrho)^m} \\ &\quad - \sqrt{\frac{2}{\pi\varrho}} \sin(\varrho - (n + \frac{1}{2})\pi/2) \sum_{m=1,3,5,\dots} (-1)^{\frac{m-1}{2}} \frac{(n, m)}{(2\varrho)^m}. \end{aligned}$$

In exercise IV.5 we shall apply a similar method in order to determine the leading terms of the series (27), (28) (which here were borrowed from the saddle-point method) from Bessel's differential equation with large ϱ . This method does not include the normalizing factor which remains undetermined by the differential equation.

Extensive mathematical investigations about the domain of validity of such asymptotic series exist, starting with a great work of Poincaré,¹³ which we cannot discuss here. Exceptions to the divergence are the series with half-integer index $n = \nu + \frac{1}{2}$, which, according to the definition of the symbol (m, n) break off with the ν -th term and represent the Bessel function in question *exactly*. We then obtain the elementary expressions for ζ_n, ψ_n which were given in Section C.

Our considerations so far are essentially restricted by the condition $n \ll \varrho$; they fail if n becomes infinite with ϱ . The latter is the case in all optical problems which are on the border line between geometrical optics (optics of very short wavelengths) and wave optics. It was in connection with the investigation of a problem of this type, namely that of the rainbow (radius of water droplet approximately equal to

¹³ *Acta Math.* 8, 1886.

wavelength of light) that Debye¹⁴ discovered his fundamental generalization of Hankel's asymptotic series. In order to understand its origin we first have to generalize the saddle-point method.

The exponent

$$(30) \quad f(w) = i [\varrho \cos w + n (w - \pi/2)]$$

in our representation (19.22) of the Hankel functions now depends on two large numbers ϱ and n . For convenience we take ϱ and n to be real and positive. Depending on whether n is smaller or larger than ϱ we set

$$(30a) \quad n = \varrho \cos \alpha \quad \text{or} \quad (30b) \quad n = \varrho \cosh \alpha;$$

in addition we use the variable of integration

$$(30c) \quad \beta = w - \pi/2.$$

as in Fig. 18.

a) For $n < \varrho$ we have

$$(31) \quad f(w) = F(\beta) = -i \varrho (\sin \beta - \beta \cos \alpha).$$

The saddle point $F'(\beta) = 0$ is given by

$$\cos \beta - \cos \alpha = 0;$$

it lies at $\beta_0 = \mp \alpha$, for H^1 and H^2 respectively. This corresponds to the previous values for the saddle points $w_0 = 0, \pi$, which by (30c) go into $\beta_0 = \mp \pi/2$. From (31) we get

$$F''(\beta_0) = \mp i \varrho \sin \alpha$$

which yields as the expansion of $F(\beta)$ up to the quadratic term

$$(31a) \quad F(\beta) = \pm i \varrho (\sin \alpha - \alpha \cos \alpha) \mp \frac{1}{2} i \varrho \sin \alpha (\beta - \beta_0)^2.$$

Instead of β we introduce the arc length s measured from the saddle point $\beta_0 = \mp \alpha$ and set $(\beta - \beta_0)^2 = (\beta \pm \alpha)^2 = \mp i s^2$, $d\beta = e^{\mp i\pi/4} ds$. Concerning the \mp -sign in the last equation we refer the reader to the discussion in (19.54b). Integrating over a neighborhood of the saddle point we get

$$(31b) \quad H_n^{1,2}(\varrho) = \frac{1}{\pi} \int e^{F(\beta)} d\beta = e^{\pm i \varrho (\sin \alpha - \alpha \cos \alpha)} \frac{1}{\pi} \int_{-\epsilon}^{+\epsilon} e^{-\frac{\varrho}{2} \sin \alpha \cdot s^2} e^{\mp \frac{i\pi}{4}} ds.$$

¹⁴ *Math. Ann.* 67, 1909 and *Bayr. Akad.* 1910.

This again can be reduced to the Laplace integral. We have:

$$(32) \quad H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi \varrho \sin \alpha}} e^{\pm i \varrho (\sin \alpha - \alpha \cos \alpha) \mp i \pi/4}.$$

In the limit $\alpha \rightarrow \pi/2$ our form (32) goes into the previous representation (19.55), (19.56).

b) The same calculation holds in the case $n > \varrho$ if in (30b) we replace $\cos \alpha$ by $\cosh \alpha$ and hence (31) by

$$F(\beta) = -i \varrho (\sin \beta - \beta \cosh \alpha)$$

That one of the two saddle points $\beta_0 = \pm i \alpha$, which yields the dominant term, is the one with greater altitude, namely $\beta_0 = -i \alpha$. At this point $F''(\beta_0) = \varrho \sinh \alpha$. Instead of (32) we now get

$$(33) \quad H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi \varrho \sinh \alpha}} e^{\varrho (\alpha \cosh \alpha - \sinh \alpha) \mp i \pi/2}.$$

From these limiting values (32), (33) Debye deduced series developments of the Hankel type, which we may omit here.

c) The only remaining case is the transition case $n \sim \varrho$ in which according to (30a,b) we have $\alpha \sim 0$ and hence the representations (32), (33) fail on account of the denominators $\sqrt{\sin \alpha}$ and $\sqrt{\sinh \alpha}$. This indicates that now $F''(\beta_0)$, also approaches zero and that only the third term of the Taylor series for $F(\beta)$ is appreciably different from zero. We therefore need a better approximation in the neighborhood of the saddle point. This was carried out by Watson,¹⁵ who instead of (31a) used an expansion which goes up to the third order in $(\beta - \beta_0)$. The Laplace integrals of the Airy type (see end of this section) which arise there can also be computed rigorously. We thus find:
in the case $n \leq \varrho$,

$$(34) \quad H_n^{1,2}(\varrho) = \frac{\tan \alpha}{\sqrt{3}} e^{\pm i \pi (\tan \alpha - \frac{1}{3} \tan^3 \alpha - \alpha) \pm i \pi/6} H_{\frac{1}{3}}^{1,2} \left(\frac{1}{3} n \tan^3 \alpha \right);$$

in the case $n > \varrho$ (where $n = \varrho \cosh \alpha$ as in (30b)),

$$(35) \quad H_n^{1,2}(\varrho) = \frac{\tanh \alpha}{\sqrt{3}} e^{-n(\tanh \alpha + \frac{1}{3} \tanh^3 \alpha - \alpha) \mp 2i \pi/3} H_{\frac{1}{3}}^{2,1} \left(\pm \frac{i n}{3} \tanh^3 \alpha \right).$$

Taken together the Watson formulas (34) and (35) cover the entire

¹⁵ Chap. VIII, and in particular p. 252 of his *Theory of Bessel Functions*, Cambridge 1922.

asymptotic range of the Bessel-Hankel functions including the border case $n \sim \varrho$, we are now treating. In this case we are in the neighborhood of $\alpha = 0$. Hence we may replace $H_{\frac{1}{3}}$ by its limiting value for small arguments, which according to (19.31) and (19.30), is (since we may neglect $I_{\frac{1}{3}}$ as compared to $I_{-\frac{1}{3}}$)

$$H_{\frac{1}{3}}^{1,2}(z) = \mp \frac{i}{\sin \pi/3} I_{-\frac{1}{3}}(z).$$

Since in (34) we have $z = \frac{1}{3}n \tan^3 \alpha$ we get $I_{-\frac{1}{3}}$ proportional to $1/\tan \alpha$, which cancels with the factor $\tan \alpha$ on the right side of (34); hence after the necessary contractions we get from (34)

$$(36) \quad H_{\varrho}^{1,2}(\varrho) = \frac{2}{\Gamma(\frac{2}{3})} \left(\frac{2}{9\varrho}\right)^{\frac{1}{3}} e^{\mp i\pi/3}.$$

The same expression is obtained from (35). As the corresponding limiting value of I we get

$$(37) \quad I_{\varrho}(\varrho) = \frac{1}{\Gamma(\frac{2}{3})} \left(\frac{2}{9\varrho}\right)^{\frac{1}{3}}.$$

This coincides with the original results of Debye.

We also see that if n is not too near ϱ , equations (34), (35) coincide with the Debye formulas (32), (33). For, in this case we may substitute its Hankel limiting value (19.55,56) for the function $H_{\frac{1}{3}}$ (large argument and small index):

$$H_{\frac{1}{3}}^{1,2}\left(\frac{1}{3}n \tan^3 \alpha\right) = \sqrt{\frac{2 \cdot 3}{\pi n \tan^3 \alpha}} e^{\pm \frac{in}{3} \tan^3 \alpha \mp i(\frac{1}{3} + \frac{1}{3})\pi/2},$$

whereby (34) simplifies to

$$H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi n \tan \alpha}} e^{\pm in(\tan \alpha - \alpha) \mp i\pi/4}.$$

Due to $n = \varrho \cos \alpha$ this coincides with (32). In the same way one shows that (35) and (33) coincide.

Finally we have to consider the problem of the roots of the equations $H_n^{1,2}(\varrho) = 0$ for large n and ϱ . However, while up to now we assumed n and ϱ to be real, we now have to admit arbitrary complex values for n ; we still may assume ϱ to be real on account of its physical meaning ($\varrho = k r$). Concerning the parameter α , whose sign is undetermined in (30a), we agree that its real part is to be positive.

It would seem from (32), (33) that no roots could exist even for

complex n , since the *exponential function* vanishes for no finite value of the exponent. However these representations were obtained (see p. 119 under b) by considering only that one of the two saddle points which has the *greater altitude*. If they are of *equal altitude* and if the required path of integration (leading from depression to depression) can be made to lead over both passes, then as sum of the two exponential expressions we get a *trigonometric function*, which makes the existence of roots possible.

Using the notations of equations (30) to (32) we represent the saddle points by $\beta = \mp \alpha$ and the corresponding exponential functions which appear in H by

$$e^{\pm i \varrho (\sin \alpha - \alpha \cos \alpha) \mp i \pi/4}$$

The altitude of the passes is determined by the real part of the exponent. Equal altitudes therefore mean equal real parts of the exponents, and since ϱ was assumed real, this means equal imaginary parts of $\pm (\sin \alpha - \alpha \cos \alpha)$, or in other words,

$$(38) \quad \text{Im} (\sin \alpha - \alpha \cos \alpha) = 0.$$

For small α this yields:

$$\text{Im} (\alpha^3) = 0,$$

This means that α lies on one of three curves that pass through the origin and intersect there at angles of $\pi/3$, one being the real α -axis. The real axis remains a solution of (38) even for infinite α , while the other branches are continued into curves that are mirror images with respect to the imaginary α -axis.

Considering the path of integration, described in Fig. 19 for the Hankel function, we see that the path of integration for H^1 can be taken meaningfully over the two saddle points only if they lie on the branch of (38) which leads from the second to the fourth quadrants, i.e., if α has a positive real part whenever it has a negative imaginary part; hence $n = \varrho \cos \alpha$ (with real ϱ) has a positive imaginary part. On the other hand the path of integration for H^2 must lead from the third to the first quadrant, so that α has a positive, and n a negative, imaginary part.

Superimposing the contributions of both saddle points according to (32) we get the following representation for H^1

$$(39) \quad \begin{aligned} H_n^1(\varrho) &= \sqrt{\frac{2}{\pi \varrho \sin \alpha}} (e^{i \varrho (\sin \alpha - \alpha \cos \alpha) - i \pi/4} - e^{-i \varrho (\sin \alpha - \alpha \cos \alpha) + i \pi/4}) \\ &= 2i \sqrt{\frac{2}{\pi \varrho \sin \alpha}} \sin [\varrho (\sin \alpha - \alpha \cos \alpha) - \pi/4]. \end{aligned}$$

From this we obtain the roots of the equations $H_n^1(\varrho) = 0$ directly: they are the roots of the following transcendental equation in α

$$(40) \quad \varrho (\sin \alpha - \alpha \cos \alpha) - \pi/4 = -m\pi, \quad m = 1, 2, 3, \dots,$$

where we have to choose the negative sign on the right in order to satisfy our requirement that α be in the fourth quadrant.

For small α we obtain from (40)

$$\varrho \frac{\alpha^3}{3} = -(4m-1)\pi/4$$

and after the correct choice of the cube root of unity, we have

$$\alpha = \left[\frac{3\pi}{4\varrho} (4m-1) \right]^{1/3} e^{-i\pi/3}.$$

Now α is related to ϱ by equation (30a), which for small α , after we make the substitution $\cos \alpha = 1 - \frac{\alpha^2}{2}$, yields:

$$(41) \quad n = \varrho + \frac{1}{2} \varrho^{\frac{1}{2}} \left[\frac{3\pi}{4} (4m-1) \right]^{2/3} e^{i\pi/3}.$$

The roots of $H_n^1(\varrho) = 0$ lie in the *positive-imaginary n -half-plane*, a fact that we shall apply later, and they are *infinite in number*. If we solve (41) with respect to ϱ then we get values in the *negative-imaginary ϱ -half-plane*. According to (41) n and ϱ are of the same order of magnitude, as assumed in the beginning. Hence (41) is the solution of the root problem in question.

We see that the saddle-point method is very general. It can be transferred from the treatment of the Hankel functions to that of arbitrary integrals of the form

$$(42) \quad \int e^{F(w, \varrho, n, \dots)} dw,$$

where F depends on several large numbers ϱ, n, \dots in addition to the variable of integration w , and where the path of integration W starts in a complex region in which $\lim \exp F(w, \dots) = 0$ and leads to a similar region. In the integrals of the type (42) when the saddle point $F' = 0$ approaches a point $F'' = 0$ one encounters the same peculiarities that we encountered in the case of Hankel functions for the border line case $n \sim \varrho$. This is the case of the Airy diffraction theory of the rainbow. The phenomenon of the rainbow is in fact linked to the appearance of a turning point ($F'' = 0$) in the wave front, which in the asymptotic approximation coincides with the saddle point $F' = 0$. The calculation¹⁶

¹⁶ W. Wirtinger, *Berichte des Naturw.-mediz. Vereins in Innsbruck* 23, 97 (1896); J. W. Nicholson, *Phil. Mag.* 18, (1909).

of the "Airy integral" in question leads then to the functions $H_{\frac{1}{2}}$ or, what is the same thing, to the functions $I_{\pm\frac{1}{2}}$ just as in (34) and (35).

§ 22. Spherical Harmonics and Potential Theory

A. THE GENERATING FUNCTION

The simplest approach to the theory of spherical harmonics is given by potential theory. We start from the so-called Newtonian potential $1/r$ and, after shifting the origin from $x = y = z = 0$ to (x_0, y_0, z_0) , obtain

$$(1) \quad \frac{1}{R} = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} = \frac{1}{\sqrt{r^2 - 2r r_0 \cos \vartheta + r_0^2}}.$$

The polar coordinates r, ϑ, φ have been chosen so that the polar axis $\vartheta = 0$ goes through the point (x_0, y_0, z_0) . We then have $x_0 = y_0 = 0$, $z_0 = r_0$ and

$$(2) \quad \begin{aligned} x &= r \sin \vartheta \cos \varphi, \\ y &= r \sin \vartheta \sin \varphi, \\ z &= r \cos \vartheta. \end{aligned}$$

We may expand (1) in ascending or descending powers of r depending on whether $r < r_0$ or $r > r_0$. If we denote the coefficient of the n -th ascending or descending power by P_n we have:

$$(3) \quad \frac{1}{R} = \begin{cases} \frac{1}{r_0} \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^n P_n(\cos \vartheta) & r < r_0, \\ \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^n P_n(\cos \vartheta) & r > r_0. \end{cases}$$

The P_n must be the same in both expansions since they must coincide when $r = r_0$ and $\vartheta \neq 0$. The point $r = r_0, \vartheta = 0$ is a singular point, the sphere $r = r_0$ playing the role here that is played by the circle of convergence of the Taylor series in the two-dimensional case.

The polynomials P_n defined by (3) are of n -th degree in $\cos \vartheta$. They are called *spherical harmonics* and we shall show that they coincide with the polynomials P_n which were introduced in §5. The function $1/R$ is called the *generating function* of spherical harmonics.

B. DIFFERENTIAL AND DIFFERENCE EQUATION

First we want to find the differential equation of *spherical harmonics*. The fundamental equation of potential theory $\Delta u = 0$, which is

satisfied by $1/R$, can be written in the form

$$(4) \quad \frac{1}{r} \frac{\partial^2(ru)}{\partial r^2} + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial u}{\partial \vartheta} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 u}{\partial \varphi^2} = 0.$$

Since this equation must be satisfied by each term in the series (3), we obtain from the n -th term of the first line of (3) after dividing out the common factor r^{n-2}/r_0^{n+1} ,

$$(5) \quad n(n+1) P_n + \frac{1}{\sin \vartheta} \frac{d}{d\vartheta} \left(\sin \vartheta \frac{dP_n}{d\vartheta} \right) = 0;$$

the same follows from the second line after factoring out r_0^n/r^{n+3} . We introduce the abbreviation

$$\cos \vartheta = \zeta = \frac{z}{r}$$

and note that

$$-\sin \vartheta d\vartheta = d\zeta, \quad \sin \vartheta \frac{d}{d\vartheta} = \sin^2 \vartheta \frac{d}{\sin \vartheta d\vartheta} = -(1-\zeta^2) \frac{d}{d\zeta}.$$

Equation (5) can then be written

$$(6) \quad \frac{d}{d\zeta} \left\{ (1-\zeta^2) \frac{dP_n}{d\zeta} \right\} + n(n+1) P_n = 0,$$

or

$$(6a) \quad \left\{ (1-\zeta^2) \frac{d^2}{d\zeta^2} - 2\zeta \frac{d}{d\zeta} + n(n+1) \right\} P_n = 0.$$

We consider (6) with n replaced by l and then, following the scheme of Green's theorem, we multiply by P_l and P_n and subtract:

$$(7) \quad \begin{aligned} & P_l \frac{d}{d\zeta} \left\{ (1-\zeta^2) \frac{dP_n}{d\zeta} \right\} - P_n \frac{d}{d\zeta} \left\{ (1-\zeta^2) \frac{dP_l}{d\zeta} \right\} \\ &= \frac{d}{d\zeta} \left\{ (1-\zeta^2) \left(P_l \frac{dP_n}{d\zeta} - P_n \frac{dP_l}{d\zeta} \right) \right\} = \{l(l+1) - n(n+1)\} P_l P_n. \end{aligned}$$

The physical range of the variable ζ is from $\zeta = -1$, to $\zeta = +1$, ($\vartheta = 0$). Integrating over this range we get the condition of orthogonality for $l \neq n$

$$(8) \quad \int_{-1}^{+1} P_l P_n d\zeta = 0,$$

since the integral of the second line of (7) with respect to ζ vanishes

unless the P become singular for $\zeta = \pm 1$ which is excluded by equation (3).

We now show that our P_n satisfy the normalizing condition (5.7)

$$(9) \quad P_n(1) = 1$$

For $\cos \vartheta = \pm 1$ we get from (1):

$$\frac{1}{|r \mp r_0|} = \begin{cases} \frac{1}{r_0} \sum (\pm 1)^n \left(\frac{r}{r_0}\right)^n & r < r_0, \\ \frac{1}{r} \sum (\pm 1)^n \left(\frac{r_0}{r}\right)^n & r > r_0. \end{cases}$$

Comparing this with (3) we even get the somewhat more general equation:

$$(9a) \quad P_n(\pm 1) = (\pm 1)^n.$$

Now we saw in §5 that the P_n were uniquely determined by the orthogonality (8) and the normalization (9). Hence our present definition, with the help of a generating function, leads to the same functions P_n as did the method of least squares in §5. In particular we have the representation (5.8)

$$(10) \quad P_n(\zeta) = \frac{1}{2^n n!} \frac{d^n}{d\zeta^n} (\zeta^2 - 1)^n$$

and as a result, according to (5.12)

$$(10a) \quad \int P_n^2(\zeta) d\zeta = \frac{1}{n + \frac{1}{2}}.$$

The P_n are even or odd functions of ζ according as n is even or odd:

$$(10b) \quad P_n(-\zeta) = (-1)^n P_n(\zeta).$$

In addition to a differential equation with respect to the variable ζ , our generating function yields a *difference equation with respect to the index n* . We rewrite, say, the first line of equation (3) with the abbreviation $\alpha = r/r_0$:

$$(11) \quad \frac{1}{\sqrt{\alpha^2 - 2\alpha\zeta + 1}} = \sum \alpha^n P_n;$$

By logarithmic differentiation with respect to α we obtain

$$(11a) \quad \frac{\zeta - \alpha}{\alpha^2 - 2\alpha\zeta + 1} = \frac{\sum n \alpha^{n-1} P_n}{\sum \alpha^n P_n}$$

and after cross-multiplication

$$(\zeta - \alpha) \Sigma \alpha^n P_n = (\alpha^2 - 2\alpha\zeta + 1) \Sigma n \alpha^{n-1} P_n.$$

If we compare the coefficients of the same power of α on both sides, say those of α^n , we obtain

$$\zeta P_n - P_{n-1} = (n-1) P_{n-1} - 2\zeta n P_n + (n+1) P_{n+1},$$

or

$$(11b) \quad (n+1) P_{n+1} - (2n+1) \zeta P_n + n P_{n-1} = 0.$$

The same *recursion formula* is, of course, obtained from the second line of (3).

By the logarithmic differentiation of (11) with respect to ζ we obtained a mixed *differential difference equation*.

Instead of (11a) we now obtain

$$(11c) \quad \frac{\alpha}{\alpha^2 - 2\alpha\zeta + 1} = \frac{\Sigma \alpha^n P'_n}{\Sigma \alpha^n P_n}, \quad P'_n = \frac{dP_n(\zeta)}{d\zeta}$$

and after cross-multiplication we get, from the coefficient of α^{n+1} ,

$$(11d) \quad P_n - P'_{n-1} + 2\zeta P'_n - P'_{n+1} = 0.$$

Multiplying this equation by $2n+1$ and adding twice the equation obtained from (11b) by differentiation with respect to ζ , we obtain

$$-(2n+1) P_n - P'_{n-1} + P'_{n+1} = 0.$$

We rewrite this as the *differential recursion formula*:

$$(11e) \quad \frac{d}{d\zeta} (P_{n+1} - P_{n-1}) = (2n+1) P_n.$$

C. ASSOCIATED SPHERICAL HARMONICS

The potential equation (4) suggests that in addition to the particular solutions

$$(12) \quad u_n = \begin{Bmatrix} r^n \\ r^{-n-1} \end{Bmatrix} P_n(\cos \vartheta)$$

which depend only on r and ϑ we might consider also the particular solutions

$$(12a) \quad u_{nm} = \begin{Bmatrix} r^n \\ r^{-n-1} \end{Bmatrix} P_n^m (\cos \vartheta) e^{im\varphi}$$

which depend on r , ϑ and φ , by associating to P_n certain *spherical harmonics* P_n^m (where m is an integer assumed positive for the time being) defined by the differential equation

$$(13) \quad \frac{1}{\sin \vartheta} \frac{d}{d\vartheta} \left(\sin \vartheta \frac{dP_n^m}{d\vartheta} \right) + \left\{ n(n+1) - \frac{m^2}{\sin^2 \vartheta} \right\} P_n^m = 0,$$

which follows from (4). Written in analogy to (6) and (6a) we have:

$$(13a) \quad \frac{d}{d\zeta} \left\{ (1-\zeta^2) \frac{dP_n^m}{d\zeta} \right\} + \left\{ n(n+1) - \frac{m^2}{1-\zeta^2} \right\} P_n^m = 0,$$

$$(13b) \quad \left\{ (1-\zeta^2) \frac{d^2}{d\zeta^2} - 2\zeta \frac{d}{d\zeta} + n(n+1) - \frac{m^2}{1-\zeta^2} \right\} P_n^m = 0.$$

According to Thomson and Tait our original P_n are called *zonal spherical harmonics* and the associated ones are called *tesseral*. The lines of zeros of the former divide the surface of the sphere into latitudinal regions of different signs, those of the latter divide it into quadrangles (tesserae) of different signs which are bounded by lines of latitude and of longitude. The associated or tesseral spherical harmonics are *orthogonal* for different lower but *equal upper* indices; namely as in (7) we conclude from our differential equation, which now is (13a), that

$$(14) \quad \int_{-1}^{+1} P_l^m P_n^m d\zeta = 0 \quad \text{for} \quad l \neq n.$$

In order to obtain an analytic expression for P_n^m we expand at the points $\zeta = \pm 1$ (north and south poles of the unit sphere) in powers of $\zeta \mp 1$, :

$$P_n^m = (\zeta \mp 1)^\lambda [a_0 + a_1 (\zeta \mp 1) + a_2 (\zeta \mp 1)^2 + \dots].$$

This is analogous to (19.36). The determination of λ in analogy to (19.37), is obtained from the differential equation (13b):

$$(15) \quad \lambda(\lambda-1) + \lambda - \frac{m^2}{4} = 0, \quad \lambda = +\frac{m}{2}.$$

(The other root $\lambda = -m/2$ must be excluded for reasons of continuity.) We unite the branches at each of the points $\zeta = \pm 1$ into

$$(1-\zeta^2)^{m/2}$$

and write

$$(16) \quad P_n^m = (1 - \zeta^2)^{m/2} v = \sin^m \vartheta \cdot v.$$

For the v which we introduced here we obtain from (13b)

$$(17) \quad \left\{ (1 - \zeta^2) \frac{d^2}{d\zeta^2} - 2(m+1)\zeta \frac{d}{d\zeta} + [n(n+1) - m(m+1)] \right\} v = 0,$$

which now must be solved in terms of series which contain only *integral* powers of $\zeta \mp 1$. However we do not have to investigate these series since the required integral of (17) can be obtained in closed form from (6a). Namely if we differentiate (6a) m times with respect to ζ and apply the well known rule of differentiation

$$(17a) \quad \frac{d^m}{d\zeta^m} \zeta \frac{d}{d\zeta} = \left\{ \zeta \frac{d}{d\zeta} + \binom{m}{1} \right\} \frac{d^m}{d\zeta^m},$$

then we obtain exactly the expression $\{ \}$ of (17) applied to the m -th derivative of P_n . Hence we see that we obtain a solution of (17) by setting

$$(17b) \quad v = \frac{d^m P_n}{d\zeta^m}.$$

With the use of (16) and (10) we obtain a simple representation for our associated spherical harmonics and at the same time a determination of the normalizing factor which has been free up to this point:

$$(18) \quad P_n^m = \frac{(1 - \zeta^2)^{m/2}}{2^n n!} \frac{d^{n+m} (\zeta^2 - 1)^n}{d\zeta^{n+m}}.$$

Hence for even m , P_n^m is, like P_n , a polynomial of degree n ; for odd m , P_n^m is $\sqrt{1 - \zeta^2}$ times a polynomial of degree $n - 1$. We further see from (18) that

$$(18a) \quad P_n^0 = P_n, \quad P_n^m = 0 \quad \text{for } m > n.$$

The last statement follows from the fact that for $m > n$ the order of differentiation in (18) is greater than the degree of the differentiated polynomial.

D. ON ASSOCIATED HARMONICS WITH NEGATIVE INDEX m

Up to now we had to assume a positive index m , for example, in (17b) we made use of differentiation of order m with respect to ζ . However our final representation (18) can be extended directly to

negative m with $m \geq -n$. We therefore extend (18) to the $2n + 1$ values $|m| \leq n$. For negative m too, the function P_n^m is a polynomial of degree n (in the same sense as for positive m). This is because the pole at $\zeta = \pm 1$ given by the factor $(1 - \zeta^2)^{m/2}$ for negative m is cancelled by the second factor of (18), which has a zero there of corresponding order due to the fact that the order of differentiation has been lowered by $|m|$. In addition the P_n^m satisfy the differential equation (13) for negative m too (since (18) depends only on m^2). Hence the P_n^m for negative m can differ from the $P_n^{|m|}$ only by a constant factor C , which is best determined by the comparison of the highest powers of ζ in P_n^{-m} and P_n^{+m} as calculated from (18). We then have:

$$\frac{P_n^{-m}}{P_n^{+m}} = (1 - \zeta^2)^{-m} \frac{d^{n-m}}{d\zeta^{n-m}} \zeta^{2n} / \frac{d^{n+m}}{d\zeta^{n+m}} \zeta^{2n} \\ \sim (-1)^m \zeta^{-2m} \frac{(2n)!}{(n+m)!} \zeta^{n+m} / \frac{(2n)!}{(n-m)!} \zeta^{n-m} = (-1)^m \frac{(n-m)!}{(n+m)!};$$

hence:

$$(18b) \quad P_n^{-m} = C \cdot P_n^{+m}, \quad C = (-1)^m \frac{(n-m)!}{(n+m)!}.$$

This equation holds for both positive and negative m .

Our definition of the P_n^m , which departs from the older mathematical literature, has been justified by wave mechanics and will also serve to unify our expressions.¹⁷

Hence we have exactly $2n + 1$ adjoined P_n^m , one of which coincides with P_n , the rest being pairwise equal except for the constant factor C ; another difference is in the factor $\exp(im\varphi)$ by which they are multiplied in (12a).

¹⁷ In the older literature the upper index of P_n^m is assumed positive throughout and the φ -dependence is given by $\cos m\varphi$ or $\sin m\varphi$. It is much simpler to assume this dependence exponential, as we have done in (12a), where we also dropped the restriction to positive m .

An even greater departure from the customary definition is suggested by C. G. Darwin (*Proc. Roy. Soc. London* 115, 1927) who appends the factor $(n - m)!$ to the right side of (18). Then (18b) simplifies to

$$P_n^m = (-1)^m P_n^{|m|} \quad \text{for } m < 0.$$

But this definition implies a change in the classical expression for the Legendre polynomials $P_n = P_n^0$, which we want to avoid.

Moreover, some authors, in particular E. W. Hobson in his *Theory of Spherical and Ellipsoidal Harmonics*, Cambridge 1931, use the factor $(-1)^m$ in the definition of P_n^m in (18), but this is immaterial for our purposes.

E. SURFACE SPHERICAL HARMONICS AND THE REPRESENTATION OF ARBITRARY FUNCTIONS

By the most general "*surface spherical harmonic*" (introduced by Maxwell) we mean the expression

$$(19) \quad Y_n = \sum_{m=-n}^{+n} A_m P_n^m(\cos \vartheta) e^{im\varphi},$$

which contains $2n + 1$ arbitrary constants. Multiplied by r^n (or r^{-n-1}) Y_n yields the most general potential of order n (or $-n-1$) which is *homogeneous* in the coordinates x, y, z (Maxwell's solid harmonics). It is a combination of the special u_{nm} of (12a) which also are homogeneous in x, y, z :

$$(19a) \quad \left. \begin{matrix} r^n \\ r^{-n-1} \end{matrix} \right\} Y_n = \sum_{m=-n}^{+n} A_m u_{nm},$$

and the general non-homogeneous solution of the potential equation (4) is represented as a sum of its homogeneous parts:

$$(19b) \quad u = \sum_{n=0}^{\infty} \left\{ \begin{matrix} r^n \\ r^{-n-1} \end{matrix} \right\} Y_n.$$

By restricting this representation to the case of a sphere of radius 1 and giving the value of u on the surface as $f(\varphi, \vartheta)$ we obtain

$$(20) \quad f(\vartheta, \varphi) = \sum_{n=0}^{\infty} Y_n = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} A_{nm} P_n^m(\cos \vartheta) e^{im\varphi}.$$

By using the notation A_{nm} instead of the A_m of (19) we emphasize the fact that the free constants in each Y_n are independent of and different from the constants in any other Y_n . The series (19b) and (20) express the fact that the *boundary value problem of potential theory for a sphere* is solvable for an arbitrary value $f(\vartheta, \varphi)$ of the potential on the surface of the sphere, both for the interior of the sphere (factor r^n in (19b)) and for the exterior (factor r^{-n-1}). In the following section we shall treat this problem by direct construction of Green's function and thereby derive the above series again. The first rigorous proof of (20) under very general assumptions on the nature of $f(\vartheta, \varphi)$ was given by Dirichlet in 1837.

F. INTEGRAL REPRESENTATION OF SPHERICAL HARMONICS

We now consider a special homogeneous function of degree n in x, y, z

$$(21) \quad (z + i x)^n = r^n (\cos \vartheta + i \sin \vartheta \cos \varphi)^n = r^n (\zeta + \sqrt{\zeta^2 - 1} \cos \varphi)^n,$$

which, like every function of the form $f(z + ix)$ or $f(z + iy)$ etc., obviously satisfies the equation $\Delta u = 0$. Hence the coefficient of r^n in (21) is a surface spherical harmonic Y_n . If we average it over φ , making it a pure function of ζ , we obtain our zonal spherical harmonic

$$(22) \quad P_n(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} (\zeta + \sqrt{\zeta^2 - 1} \cos \varphi)^n d\varphi.$$

If on the other hand we construct the m -th Fourier coefficient of Y_n as in (1.12) then we obtain the associated (tesseral) spherical harmonic

$$(23) \quad P_n^m(\zeta) = \frac{C}{2\pi} \int_{-\pi}^{+\pi} (\zeta + \sqrt{\zeta^2 - 1} \cos \varphi)^n e^{-im\varphi} d\varphi.$$

The integral representation (22) is first mentioned by Laplace in his *Mécanique Céleste*, Vol. V. The fact that the denominator 2π provides the correct normalization is seen by setting $\zeta = 1$, for then the integrand becomes 1 and hence $P_n(1) = 1$. On the other hand we still have to determine the normalizing factor C in (23). By comparison with the normalization of (18) we find¹⁸

$$(23a) \quad C = \frac{(n+m)!}{n!} e^{-im\pi/2}.$$

If in (21) we replace n by $-n-1$, as we know is possible, we get, equivalent to (22), the representation

$$(23b) \quad P_n(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{d\varphi}{(\zeta + \sqrt{\zeta^2 - 1} \cos \varphi)^{n+1}},$$

which is also seen to be normalized.

¹⁸ For instance by passing to the limit $\zeta \rightarrow \infty$ in both (23) and (18), whereby, except for the factors ζ^n and $\exp(-im\varphi)$, the integrand in (23) reduces to

$$(1 + \cos \varphi)^n = 2^n \cos^{2n} \varphi/2 = 2^{-n} e^{in\varphi} (1 + e^{-i\varphi})^{2n}.$$

In the binomial expansion of this expression we have to consider only the term with $e^{im\varphi}$ since all the other terms vanish upon integration with respect to φ . The factor $\exp\{-im\pi/2\}$ in (23a) is due to the factor $\sin^m \vartheta$ in (18). See also the similar passage to the limit $\zeta \rightarrow \infty$ in (18b).

G. A RECURSION FORMULA FOR THE ASSOCIATED HARMONICS

Starting from the recursion formulas (11b) and (11e) for the zonal spherical harmonics, we differentiate (11b) m times with respect to ζ , apply rule (17a) to the middle term and multiply each term by $\sin^m \vartheta$. As a result of (18) we obtain

$$(24) \quad (n+1) P_{n+1}^m - (2n+1) \zeta P_n^m - m(2n+1) \sin \vartheta P_n^{m-1} + n P_{n-1}^m = 0.$$

On the other hand we obtain from (11e) upon $(m-1)$ -fold differentiation with respect to ζ and multiplication by $\sin^m \vartheta$:

$$(25) \quad P_{n+1}^m - P_{n-1}^m = (2n+1) \sin \vartheta P_n^{m-1}.$$

Eliminating the term with $\sin \vartheta$ from (24) and (25) we obtain the recursion formula

$$(26) \quad (n+1-m) P_{n+1}^m - (2n+1) \zeta P_n^m + (n+m) P_{n-1}^m = 0.$$

which is a generalization of (11b). Apparently this equation holds only for positive m , due to its derivation through m -fold differentiation. However we can verify it for negative m if we consider our general definition (18) of P_n^m and the relation (18b).

H. ON THE NORMALIZATION OF ASSOCIATED HARMONICS

From (10a) we know the value of the normalizing integral for $m=0$. We denote it by N_n or also by N_n^0 . Its computation in §5 is based on the symbol $D_{k,l}$ of (5.9). We first consider the generalized form of the normalizing integral

$$(27) \quad N_n^{\pm m} = \int_{-1}^{+1} P_n^m(\zeta) P_n^{-m}(\zeta) d\zeta.$$

Written in terms of the symbol $D_{k,l}$ we have as a result of our general definition (18)

$$(28) \quad N_n^{\pm m} = \frac{1}{2^{2n} n! n!} \int_{-1}^{+1} D_{n+m,n} \cdot D_{n-m,n} d\zeta.$$

Through m -fold integration by parts we obtain, since the terms outside the sign of integration vanish for $\zeta = \pm 1$:

$$(29) \quad N_n^{\pm m} = \frac{(-1)^m}{2^{2n} n! n!} \int_{-1}^{+1} D_{n,n} \cdot D_{n,n} d\zeta = (-1)^m N_n^0 = \frac{(-1)^m}{n + \frac{1}{2}}.$$

Here in the last equation we have substituted the value of N_n^o from (10a). The normalizing integral is usually taken as

$$(30) \quad N_n^m = \int_{-1}^{+1} P_n^m(\xi) P_n^m(\xi) d\xi.$$

This can be deduced directly from (29) by using the relation (18b) which yields

$$(31) \quad N_n^m = \frac{1}{C} N_n^{\pm m} = \frac{1}{n + \frac{1}{2}} \frac{(n+m)!}{(n-m)!}.$$

Its direct computation in the manner of (28) would have been somewhat more cumbersome.

We remark that in the following chapter we shall always carry out a "normalization of the eigenfunction to 1." If we denote the associated harmonic with this normalization by Π_n^m , then we have:

$$(31 a) \quad \int_{-1}^{+1} \Pi_n^m(\xi) \Pi_n^m(\xi) d\xi = 1,$$

and comparing this with (30) we get

$$(31 b) \quad \Pi_n^m = P_n^m / \sqrt{N_n^m} = P_n^m \cdot \left[(n + \frac{1}{2}) \frac{(n-m)!}{(n+m)!} \right]^{\frac{1}{2}}.$$

J. THE ADDITION THEOREM OF SPHERICAL HARMONICS

The proof of this theorem is based on a lemma, which we shall be able to prove only with the methods of the following chapter and which shall be assumed here without proof, namely: *The surface spherical harmonic*

$$(32) \quad Y_n = \sum_{m=-n}^{+n} \Pi_n^m(\cos \vartheta) e^{im\varphi} \cdot \Pi_n^m(\cos \vartheta_0) e^{-im\varphi_0}$$

depends only on the relative position of the two points (ϑ, φ) and (ϑ_0, φ_0) on the surface of the sphere, in other words it has an invariant meaning independent of the coordinate system. If we now change the coordinate system of ϑ, φ by letting the polar axis of a new coordinate system Θ, Φ pass through the point (ϑ_0, φ_0) , then the latter has the coordinate $\Theta_0 = 0$ (its Φ_0 becomes undetermined), while the Θ coordinate of the former point (ϑ, φ) is now given by

$$(33) \quad \cos \Theta = \cos \vartheta \cos \vartheta_0 + \sin \vartheta \sin \vartheta_0 \cos (\varphi - \varphi_0)$$

For $\Theta_0 = 0$ all the terms in (32) vanish except those with $m = 0$. Hence the right side of (32) becomes the product of the zonal spherical harmonics $\Pi_n(\cos \Theta)$ and $\Pi_n(1)$. Due to the stated invariance of Y_n we then have

$$(34) \quad \Pi_n(\cos \Theta) \Pi_n(1) = \sum_{m=-n}^{+n} \Pi_n^m(\cos \vartheta) \Pi_n^m(\cos \vartheta_0) e^{im(\varphi - \varphi_0)}.$$

This is the *symmetric form* of the addition theorem which expresses its structure in a convincingly simple form. The form which is common in the literature is obtained by expressing the Π_n^m in terms of P_n^m with the help of (31b) and (31). Equation (34) then becomes

$$(35) \quad P_n(\cos \Theta) = \sum_{m=-n}^{+n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos \vartheta) P_n^m(\cos \vartheta_0) e^{im(\varphi - \varphi_0)},$$

or written in real form

$$(36) \quad \begin{aligned} P_n(\cos \Theta) &= P_n(\cos \vartheta) P_n(\cos \vartheta_0) \\ &+ 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos \vartheta) P_n^m(\cos \vartheta_0) \cos m(\varphi - \varphi_0). \end{aligned}$$

It is however evident that the true structure of the addition theorem is gradually lost in the passage (34) \rightarrow (35) \rightarrow (36).

Another rather transparent form of the addition theorem is obtained from (35) by replacing one of the upper indices m by $-m$ and applying (18b):

$$(37) \quad P_n(\cos \Theta) = \sum_{m=-n}^{+n} (-1)^m P_n^m(\cos \vartheta) P_n^{-m}(\cos \vartheta_0) e^{im(\varphi - \varphi_0)}.$$

§ 23. Green's Function of Potential Theory for the Sphere. Sphere and Circle Problems for Other Differential Equations

We superimpose two principal solutions u, u' of the potential equation $\Delta u = 0$,

$$(1) \quad \begin{aligned} u &= \frac{e}{R}, \quad R^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2, \\ u' &= \frac{e'}{R'}, \quad R'^2 = (x - \xi')^2 + (y - \eta')^2 + (z - \zeta')^2 \end{aligned}$$

and seek the level surfaces of the function $G = u - u'$, in particular the

surface $G = 0$. According to (1) the latter is given by the equation $R'^2 = (e'/e)^2 R^2$, or written explicitly:

$$(2) \quad \begin{aligned} & \left(1 - \frac{e'^2}{e^2}\right) (x^2 + y^2 + z^2) \\ & - 2 \left\{ \left(\xi' - \frac{e'^2}{e^2} \xi\right) x + \left(\eta' - \frac{e'^2}{e^2} \eta\right) y + \left(\zeta' - \frac{e'^2}{e^2} \zeta\right) z \right\} \\ & + \xi'^2 - \frac{e'^2}{e^2} \xi^2 + \eta'^2 - \frac{e'^2}{e^2} \eta^2 + \zeta'^2 - \frac{e'^2}{e^2} \zeta^2 = 0. \end{aligned}$$

This is the equation of a *sphere*. The position of its center and the length of its radius can be calculated from (2): the center O lies on the connecting line of the "source points" $Q = (\xi, \eta, \zeta)$ and $Q' = (\xi', \eta', \zeta')$, the radius a is obtained as the *mean proportional* of $OQ = \varrho$ and $OQ' = \varrho'$,

$$(3) \quad \varrho \varrho' = a^2,$$

so that one of the source points lies in the interior, the other in the exterior of the sphere of radius a .

For further discussion we shall not use the cumbersome formula (2), but rather the elementary geometric Fig. 22.

A. GEOMETRY OF RECIPROCAL RADII

Fig. 22 illustrates the method of *reciprocal radii*¹⁹ formulated in (3). We call Q' the *inverse image* of Q with respect to the sphere of radius a , or also the *electric image* (Maxwell); the notations e and e' in (1) are connected with the electric point of view. The relation between Q and Q' is symmetric: Q is the inverse image of Q' . From (3) we see in the well known manner that the points Q, Q' are harmonic with respect to the points of intersection P_1, P_2 of the line OQQ' and the sphere.

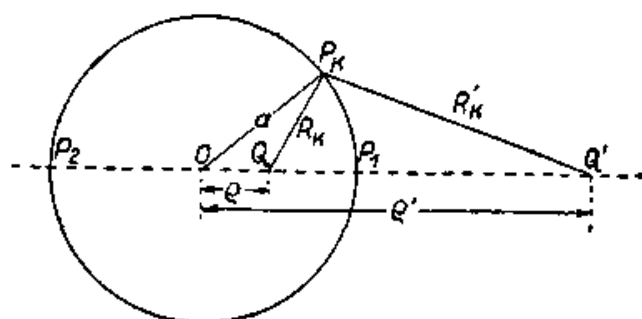


Fig. 22. Geometry of reciprocal radii. Q and Q' are transformed into each other by inversion on the sphere of radius a around the center O . The triangles OQP_k and OP_kQ' are similar.

The *method of reciprocal radii* was developed through the pioneering

¹⁹ The term "reciprocal" arises from the (bad) habit of setting $a = 1$, in which case $\varrho' = 1/\varrho$. For reasons of dimensionality we consider it better to retain the radius a as a length.

work of William Thomson²⁰ who applied it to a wide range of problems in electro- and magnetostatics. Transformation by reciprocal radii is called *inversion* for short.

From (3) it follows that the triangle OP_kQ of the figure is similar to the triangle $OQ'P_k$, hence we have:

$$(4) \quad \frac{e}{R_k} = \frac{a}{R'_k}, \text{ where } R_k = P_kQ \text{ and } R'_k = P_kQ'.$$

Here P_k denotes a special point on the surface of the sphere while the symbols P, R, R' are reserved for an arbitrary point $P: (x, y, z)$ and its distances from Q and Q' . In order to determine the "image charge" e' which was introduced in (1) we compare (4) with the relation

$$\frac{e}{R_k} = \frac{e'}{R'_k}$$

which follows from (1), and is valid for every point P_k . We obtain

$$(5) \quad \frac{e'}{e} = \frac{a}{e} = \frac{e'}{a} \quad (\text{the latter due to equation (3)}).$$

B. THE BOUNDARY VALUE PROBLEM OF POTENTIAL THEORY FOR THE SPHERE, THE POISSON INTEGRAL

Equation (5) brings us back to our starting point, the condition $G = u - u' = 0$; and we can now justify the notation G , which signifies *Green's function*. We have in fact

$$(6) \quad G = G(P, Q) = \frac{e}{R} - \frac{e'}{R'}$$

as Green's function for the "interior boundary value problem" which is: to find a potential U which has no singularities in the interior of the sphere for a given boundary value \bar{U} on the surface of the sphere. In the same manner

$$(6a) \quad G = G(P, Q') = \frac{e'}{R'} - \frac{e}{R}$$

is Green's function for the corresponding "exterior boundary value problem." Of the three conditions a), b), c) (see p. 50) that serve to define Green's function, we have b) satisfied on account of (5) and a) satisfied because the potential equation is self-adjoint and therefore the

²⁰ *Journal de Math.* 10 (1845), 12 (1847). Maxwell in his *Treatise*, vol. I, Chap. XI, quotes a paper in the *Cambridge and Dublin Math. Journ.* of 1848.

differential equation of G coincides with that of U . In order to satisfy condition c) of the unit source we merely have to make

$$e \text{ or } e' = -1/4\pi.$$

for the inner or outer boundary value problem, as seen from the table on p. 49.

We write (6) explicitly by introducing the spherical coordinates r, ϑ, φ for P ($r = 0$ is the center of the sphere, $\vartheta = 0$ an arbitrary direction). Let the corresponding coordinates for Q and Q' be:

$$\begin{array}{ll} r_0, \vartheta_0, \varphi_0 & \text{with } r_0 = \varrho, \\ r'_0, \vartheta'_0, \varphi'_0 & \text{with } r'_0 = \varrho', \vartheta'_0 = \vartheta_0, \varphi'_0 = \varphi_0; \end{array}$$

as in (22.33) we let

$$\cos \Theta = \cos \vartheta \cos \vartheta_0 + \sin \vartheta \sin \vartheta_0 \cos (\varphi - \varphi_0);$$

for e'/e we use the first of the values given by (5), and we let $e = -1/4\pi$. Equation (6) then becomes

$$(7) \quad -4\pi G = \frac{1}{\sqrt{r^2 + \varrho^2 - 2r\varrho \cos \Theta}} - \frac{a/\varrho}{\sqrt{r^2 + \frac{a^2}{\varrho^2} - 2r\frac{a^2}{\varrho} \cos \Theta}}.$$

The solution for the interior boundary value problem according to the scheme of (10.12) is then

$$(8) \quad U(Q) = \int \bar{U} \frac{\partial G}{\partial n} d\sigma.$$

The integration on the right is with respect to the point P and is taken over the surface of the sphere, $\partial G / \partial n = \partial G / \partial r$ for $r = a$ and $d\sigma = a^2 \sin \vartheta d\vartheta d\varphi$; Q is an arbitrary point in the interior of the sphere. From (7) we get the general relation

$$4\pi \frac{\partial G}{\partial r} = \frac{r - \varrho \cos \Theta}{R^3} - \frac{a}{\varrho} \frac{r - \frac{a^2}{\varrho} \cos \Theta}{R'^3},$$

where R and R' are as before. Hence, for the surface of the sphere, where according to (4) we have $R'_k = \frac{a}{\varrho} R_k$, we get

$$4\pi \frac{\partial G}{\partial r} = \frac{1}{R_k^3} \left\{ a - \varrho \cos \Theta - \frac{\varrho^2}{a^2} \left(a - \frac{a^3}{\varrho} \cos \Theta \right) \right\} = \frac{a}{R_k^3} \left(1 - \frac{\varrho^2}{a^2} \right).$$

Therefore, if we set $\bar{U} = f(\vartheta, \varphi)$ equation (8) becomes

$$(9) \quad 4\pi U(Q) = a^3 \left(1 - \frac{\varrho^2}{a^2} \right) \iint \frac{f(\vartheta, \varphi) \sin \vartheta d\vartheta d\varphi}{(a^2 + \varrho^2 - 2a\varrho \cos \Theta)^{\frac{3}{2}}}.$$

This representation was deduced by Poisson in a very circuitous manner through the development of $f(\theta, \varphi)$ in spherical harmonics. Here we see that the direct way is through Green's function (7).

The corresponding formula for the outer boundary value problem is obtained from (6a) by setting $e' = -1/4\pi$ and taking the second value in (5) for e'/e . We have:

$$(9a) \quad 4\pi U(Q') = a^3 \left(\frac{\rho'^2}{a^2} - 1 \right) \iint \frac{f(\theta, \varphi) \sin \theta d\theta d\varphi}{(a^2 + \rho'^2 - 2a\rho' \cos \Theta)^{3/2}}.$$

The so-called "second boundary value problem," in which we set $\partial G/\partial n = 0$ on the surface of the sphere, can not be solved with the method of reciprocal radii.

We now wish to gain a clearer geometrical understanding of the way in which formula (9); which is analytic throughout, can, on the surface, represent an arbitrary function $f(\theta, \varphi)$, which is in general not analytic. For this purpose we have to consider the passage to the limit $Q \rightarrow K$ as $\rho \rightarrow a$. In this limit the factor $1 - \rho^2/a^2$ in front of the integral in (9) vanishes and hence only those elements of area $d\sigma$ contribute to the integral for which the denominator R^3 vanishes. The latter approaches zero for $\rho \rightarrow a$ only when $\cos \Theta = 1$, and therefore $\theta = \theta_0, \varphi = \varphi_0$. Thus the only determining part is a neighborhood of that element of area which approaches Q , in other words the special value $f(\theta_0, \varphi_0)$ on this element of area will alone determine the limiting value. We indicate this fact by rewriting (9) in the form

$$(10) \quad 4\pi \lim_{Q \rightarrow K} U(Q) = a^3 \left(1 - \frac{\rho^2}{a^2} \right) f(\theta_0, \varphi_0) \int_0^\pi \int_0^{2\pi} \frac{\sin \Theta d\Theta d\Phi}{(a^2 + \rho^2 - 2a\rho \cos \Theta)^{3/2}}.$$

In the numerator of the integrand we are allowed to replace $\sin \theta d\theta d\varphi$ by $\sin \Theta d\Theta d\Phi$; after this is done the integration can be carried out explicitly. We obtain

$$\frac{2\pi}{a\rho} \left(\frac{1}{a-\rho} - \frac{1}{(a^2 + \rho^2 - 2a\rho \cos \epsilon)^{1/2}} \right).$$

Substituting this in (10) we observe that the contribution of the second term in the parentheses vanishes for $\rho \rightarrow a$. From the first term, after dividing the denominator $a - \rho$ into the factor $1 - \rho^2/a^2$, we get the desired value:

$$\lim_{Q \rightarrow K} U = f(\theta_0, \varphi_0).$$

In the two-dimensional case (circle instead of sphere) we can carry

out a simplified consideration in close analogy to (9) and (9a), where instead of (9) we get

$$(11) \quad 2\pi U(Q) = a^2 \left(1 - \frac{\varrho^2}{a^2}\right) \int \frac{f(\varphi) d\varphi}{a^2 + \varrho^2 - 2a\varrho \cos(\varphi - \varphi_0)}.$$

C. GENERAL REMARKS ABOUT TRANSFORMATIONS BY RECIPROCAL RADII

Returning to the three-dimensional case we now wish to consider the transformation by reciprocal radii from more general viewpoints. We choose an arbitrary point O to be the *center of inversion* and, at the same time, the origin of a spherical polar coordinate system; we then select a sphere with a center O and an arbitrary radius a as the *sphere of inversion*. An arbitrary point $P: (r, \vartheta, \varphi)$ is transformed into a point $P': (r', \vartheta', \varphi')$. Between these points we have the relations:

$$(12) \quad \begin{aligned} r r' &= a^2, & \vartheta' &= \vartheta, & \varphi' &= \varphi; \\ dr &= -\frac{a^2}{r^2} dr', & d\vartheta &= d\vartheta', & d\varphi &= d\varphi'. \end{aligned}$$

For the sake of completeness we also give the corresponding relations between the rectangular coordinates x, y, z and x', y', z' . Using the scheme of (22.2) we obtain from (12)

$$x' = r' \sin \vartheta' \cos \varphi' = \frac{a^2}{r} \sin \vartheta \cos \varphi = \frac{a^2}{r^2} x, \text{ etc.}$$

or, written in summarized form,

$$(12a) \quad (x', y', z') = \frac{a^2}{r^2} (x, y, z);$$

and conversely

$$(12b) \quad (x, y, z) = \frac{a^2}{r'^2} (x', y', z').$$

We now seek the transformation in polar coordinates of the line element ds^2 into ds'^2 . According to (12) we have

$$(13) \quad \begin{aligned} ds^2 &= dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \\ &= \left(\frac{a}{r'}\right)^4 (dr'^2 + r'^2 d\vartheta'^2 + r'^2 \sin^2 \vartheta' d\varphi'^2) = \left(\frac{a}{r'}\right)^4 ds'^2. \end{aligned}$$

Hence the transformations by reciprocal radii are *conformal*: every infinitesimal triangle with sides ds is transformed into a *similar* triangle

with sides ds' (ratio of sides $= (a/r')^2 = (r/a)^2$). According to a theorem by Liouville these mappings are the only non-trivial conformal transformations in three-dimensional space.

The geometric characterization of our transformation consists of the fact that it transforms spheres into spheres, where the plane has to be considered as a sphere of infinite radius. A *sphere* that passes through the center of inversion is transformed into a *plane*, since the center of inversion is transformed into infinity. (Infinity in this "geometry of spheres" is a *point* and not a *plane* as it is in projective geometry.) Conversely, a plane that does not pass through the center of inversion is transformed into a sphere.

D. SPHERICAL INVERSION IN POTENTIAL THEORY

The next point of interest to us is the transformation of the differential parameter Δu : we start from a function $u(r, \vartheta, \varphi)$ and transform the *product* ru by reciprocal radii (r = distance from center of inversion). We denote the new function by

$$(14) \quad v(r', \vartheta', \varphi') = \frac{a^2}{r'} u\left(\frac{a^2}{r'}, \vartheta', \varphi'\right).$$

In other words, we transfer the value ru from the original point P (r, ϑ, φ) to the point P' with the coordinates

$$(14a) \quad r' = \frac{a^2}{r}, \vartheta' = \vartheta, \varphi' = \varphi$$

and we want to show that the differential parameter $\Delta'v$ which is calculated in terms of the coordinates r', ϑ', φ' is given by

$$(15) \quad \Delta'v = \left(\frac{a}{r'}\right)^4 r \Delta u.$$

Again, the reason for this relation lies in the conformality of the mapping, as indicated by the appearance in (15) of the square of the ratio of dilation $(r/a)^2$ from equation (13). Equation (15) can be proven as follows: we define the operators Δ and D as in (22.4) by the formulas

$$(15a) \quad r^2 \Delta = \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + D,$$

$$(15b) \quad D = \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2}.$$

Then, if Δ' and D' stand for the same expressions in r', ϑ', φ' , we obtain from (14) and (14a)

$$(15\ c) \quad D'v = r Du, \quad \frac{\partial r}{\partial r'} = \frac{\partial(ru)}{\partial r} \frac{dr}{dr'} = -\frac{\alpha^2}{r'^2} \frac{\partial(ru)}{\partial r},$$

$$(15\ d) \quad \frac{\partial}{\partial r'} \left(r'^2 \frac{\partial v}{\partial r'} \right) = -\alpha^2 \frac{\partial^2(ru)}{\partial r^2} \frac{dr}{dr'} = \frac{\alpha^4}{r'^2} \frac{\partial^2(ru)}{\partial r^2};$$

and hence according to (15a,c,d)

$$(15\ e) \quad r'^2 \Delta'v = \frac{\alpha^4}{r'^2} \left(\frac{\partial^2(ru)}{\partial r^2} + \frac{r r'^2}{\alpha^4} Du \right) = \frac{\alpha^4}{r'^2} \left(\frac{\partial^2(ru)}{\partial r^2} + \frac{1}{r} Du \right).$$

According to equation (22.4) the expression in the last parentheses is just $r \Delta u$. Hence (15e) becomes

$$(16) \quad \Delta'v = \left(\frac{\alpha}{r} \right)^4 r \Delta u,$$

which coincides with (15).²¹

If we start from a function u which satisfies the differential equation $\Delta u = 0$ in the coordinates x, y, z , then the function $v = ru$ after transformation by reciprocal radii satisfies the differential equation $\Delta v = 0$ in the coordinates x', y', z' .

This theorem (William Thomson) enables us to transfer solutions of potential problems obtained for a certain region of space S to the transformed region S' . In particular this holds for Green's function: if it is known for a region S bounded by planes with the boundary condition $G = 0$, then our theorem gives Green's function G' for an arbitrary region S' bounded by spheres where the boundary condition $G' = 0$ remains valid. Depending on the position of the center of inversion, the region S' may have diverse shapes. The totality of those regions which were treated in §17 with the help of elementary reflections now becomes a richer manifold of regions bounded by spheres, thus permitting our generalized reflection by inversion. As before, this more general reflection leads to a simple and complete covering of space. The previous condition that all face angles must be submultiples of π remains valid owing to the conformality of the mapping. Where there was an infinity of image points (e.g., plane plate) there will still be an infinity of image points (e.g., the region between spheres tangent at the center of inversion, which is the image of the plane plate). Where there was a finite number of image points (e.g., for the wedge of 60° in Fig. 17), the inversion process for the spherical problem again terminates after a finite number of steps.

Examples will be given in exercises IV.6 and IV.7, where we shall also discuss the problem of a suitable choice of the center of inversion.

²¹ Here the reason for the retention of α becomes apparent: if we had $\alpha = 1$ the dimensionality of the factor $1/r'^4$ in (16) would not be understood, whereas now the dimensional consistency is clear.

Obviously all that has been said above can be transferred to two-dimensional potential theory, where inversion in a sphere becomes inversion in a circle. At the same time the range of possible mappings is increased tremendously since every transformation $z' = f(z)$ where f is an analytic function of the complex variable $z = x + iy$ leads to a conformal mapping. The dilation ratio of the line elements is then $|df/dz|$ and (16) is replaced by

$$(17) \quad \Delta'v = \left| \frac{df}{dz} \right|^2 \Delta u.$$

E. THE BREAKDOWN OF SPHERICAL INVERSION FOR THE WAVE EQUATION

Unfortunately, these mapping methods for the two- and three-dimensional case are *entirely restricted to potential theory*. If we were to perform a transformation by reciprocal radii on the wave equation

$$(18) \quad \Delta u + k^2 u = 0$$

then according to (16) the factor $(a/r')^4 r$ would appear and (18) would become:

$$(19) \quad \Delta'v + k^2 \left(\frac{a}{r'} \right)^4 v = 0.$$

Only in the potential equation ($k = 0$) does this disturbing factor $(a/r')^4$ disappear. In the wave equation this factor means that the originally homogeneous medium (k constant) appears transformed into a highly inhomogeneous medium, which, in the neighborhood of the point $r' = 0$, shows a lens-like singularity of the index of refraction. The same holds for the equation of heat conduction, which, written in our customary form with u as temperature and k as temperature conductivity, would go into

$$(19a) \quad \Delta'v = \frac{1}{k} \left(\frac{a}{r'} \right)^4 \frac{\partial v}{\partial t}.$$

This form of the equation certainly can not serve to simplify the boundary value problem for the sphere. Instead, we have to rely on the much more cumbersome method of series expansion as applied in the corresponding two-dimensional case of §20 A.

§ 24. More About Spherical Harmonics

A. THE PLANE WAVE AND THE SPHERICAL WAVE IN SPACE

The simplest solution of the three-dimensional wave equation

$$(1) \quad \Delta u + k^2 u = 0$$

is the plane wave, e.g., a purely periodic sound wave which progresses in the z -direction

$$(2) \quad u = e^{ikz} = e^{i\varrho \cos \vartheta}, \quad \varrho = kr, \quad k = \text{wave number.}$$

If we develop this solution in zonal spherical harmonics $P_n(\cos \vartheta)$ then the coefficients will be the $\psi_n(\varrho)$ of §21 C. This follows from the wave equation on the one hand, and the differential equation of the P_n on the other hand. Using the left side of (22.4) and the postulated independence from φ , equation (1) becomes

$$\frac{1}{r} \frac{\partial^2 r u}{\partial r^2} + \frac{1}{r^2} \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial u}{\partial \vartheta} + k^2 u = 0.$$

Hence u can be separated into a product of P_n by a function $R(r)$ which depends only on r . Due to the differential equation (22.5) of P_n the function R must satisfy the equation

$$\frac{1}{r} \frac{d^2 r R}{dr^2} + \left(k^2 - \frac{n(n+1)}{r^2} \right) R = 0$$

which, in terms of the ϱ of equation (2), can be rewritten as:

$$(3) \quad \frac{1}{\varrho} \frac{d^2 \varrho R}{d\varrho^2} + \left(1 - \frac{n(n+1)}{\varrho^2} \right) R = 0.$$

This is the same differential equation as (21.11a); the solutions, which were continuous for $\varrho = 0$, were defined as ψ_n . Neglecting a multiplicative constant we get from (21.11):

$$(4) \quad R(r) = \psi_n(\varrho) = \sqrt{\frac{\pi}{2\varrho}} I_{n+\frac{1}{2}}(\varrho).$$

Similarly we obtain the linear combinations of the $\zeta^{1,2}(\varrho)$, defined in (21.15), as solutions of (3) discontinuous for $\varrho = 0$. Since the latter do not enter into the expansion of the plane wave, we have to write:

$$(5) \quad e^{i\varrho \cos \vartheta} = \sum_{n=0}^{\infty} c_n \psi_n(\varrho) P_n(\cos \vartheta).$$

Here the coefficients c_n are still undetermined. They are determined

from the orthogonality of the P_n . Namely, according to (22.8) and (22.10a) we get, if we again denote the variable of integration by $\xi = \cos \vartheta$:

$$(6) \quad c_n \psi_n(\varrho) = (n + \frac{1}{2}) \int_{-1}^{+1} e^{i\varrho\xi} P_n(\xi) d\xi.$$

We now compare the asymptotic values for $\varrho \rightarrow \infty$ of the two sides. Due to the relation of ψ_n to $I_{n+\frac{1}{2}}$, we get for the left side from equation (19.57)

$$(6a) \quad c_n \frac{\cos [\varrho - (n + \frac{1}{2})\pi/2]}{\varrho}.$$

The integral on the right side can be expanded into a series in $1/\varrho$ through successive integrations by parts. Ignoring all higher powers of $1/\varrho$ for this integral we obtain:

$$(6b) \quad \frac{e^{i\varrho}}{i\varrho} P_n(1) - \frac{e^{-i\varrho}}{i\varrho} P_n(-1) = \frac{1}{i\varrho} [e^{i\varrho} - (-1)^n e^{-i\varrho}] = 2i^n \frac{\sin(\varrho - n\pi/2)}{\varrho}.$$

The coefficient of $2i^n$ here is the same as the coefficient of c_n in (6a). Substituting (6a,b) in (6) we therefore get

$$(6c) \quad c_n = (2n + 1) i^n.$$

Hence the expansion (5) of the plane wave assumes the final form

$$(7) \quad e^{i\varrho \cos \vartheta} = \sum_{n=0}^{\infty} (2n + 1) i^n \psi_n(\varrho) P_n(\cos \vartheta).$$

This should be compared with the Fourier expansion (21.2b) of the two-dimensional plane wave. Just as we considered the latter as generating function of the I_n , so we may consider the three-dimensional plane wave as the *generating function of the ψ_n* . At the same time (6) and (6c) yield the following integral representation of the ψ_n :

$$(7a) \quad 2i^n \psi_n(\varrho) = \int_{-1}^{+1} e^{i\varrho\xi} P_n(\xi) d\xi,$$

The next simple solution of the wave equation (1) is the *spherical wave*

$$(8) \quad u = \frac{e^{ikr}}{ikr} = \frac{e^{i\varrho}}{i\varrho}.$$

This represents a *radiated* wave which progresses in the positive r -

direction if we give its time dependence by $\exp(-i\omega t)$. According to (21.15a) the solution (8) is identical with the solution of (3):

$$(8a) \quad \zeta_0^1 = \sqrt{\frac{\pi}{2\rho}} H_{\frac{1}{2}}^1(\rho).$$

which is singular at the point $r = 0$. We now transfer the source point $r = 0$ to the arbitrary point

$$Q = (r_0, \vartheta_0, \varphi_0)$$

Then (8) becomes

$$(8b) \quad u = \frac{e^{ikR}}{ikR} = \zeta_0^1(kR) \begin{cases} R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \Theta}, \\ \cos \Theta = \cos \vartheta \cos \vartheta_0 + \sin \vartheta \sin \vartheta_0 \cos(\varphi - \varphi_0). \end{cases}$$

This function too can be expanded in spherical harmonics $P_n(\cos \Theta)$. Here the coefficients must again be solutions of the differential equation (3), namely,

$$\psi_n(\rho) \quad \text{for} \quad r < r_0, \quad \zeta_n^1(\rho) \quad \text{for} \quad r > r_0,$$

the former, since the point $r = 0$ is now a regular point of the spherical wave, the latter, since the type of the *radiated* wave must be preserved in every term of the expansion. Owing to the symmetry of R in r and r_0 the reverse holds for the dependence on r_0 . Hence in the coefficients of $P_n(\cos \Theta)$ we must have the factors

$$\zeta_n^1(\rho_0) \quad \text{for} \quad r < r_0, \quad \psi_n(\rho_0) \quad \text{for} \quad r > r_0,$$

so that the expansion reads

$$(9) \quad \frac{e^{ikR}}{ikR} = \begin{cases} \sum_{n=0}^{\infty} c_n \zeta_n^1(\rho_0) \psi_n(\rho) P_n(\cos \Theta) & r < r_0, \\ \sum_{n=0}^{\infty} c_n \psi_n(\rho_0) \zeta_n^1(\rho) P_n(\cos \Theta) & r > r_0. \end{cases}$$

The numerical factors c_n must be the same in both rows, since for $r = r_0$ the two rows coincide (except for the point Q , where we have $\Theta = 0$ and both series diverge). The situation here is the same as for the cylindrical wave in §21, equation (4): in the interior of the sphere we have a "Taylor series," in the exterior a series of the "Laurent type." The c_n can again be determined by passing to the limit $r \rightarrow \infty$. We get

$$R = r \left(1 - \frac{r_0}{r} \cos \Theta + \dots \right) \rightarrow r - r_0 \cos \Theta, \\ e^{ikR} \rightarrow e^{ikr} e^{-ikr_0 \cos \Theta}.$$

Using (7) with $-i \varrho_0 \cos \Theta$ instead of $+i \varrho \cos \vartheta$ the left side of (9) becomes

$$\frac{e^{i\varrho}}{i\varrho} \sum (2n+1) (-i)^n \psi_n(\varrho_0) P_n(\cos \Theta).$$

Due to (21.15) and (19.55) the second line on the right side of (9) becomes in the limit

$$\sum c_n \psi_n(\varrho_0) \sqrt{\frac{\pi}{2\varrho}} \sqrt{\frac{2}{\pi\varrho}} e^{i[\varrho - (n+1)\pi/2]} P_n(\cos \Theta).$$

This will correspond term for term with the left side if we set

$$(9a) \quad c_n = 2n + 1.$$

We may also consider this representation of the spherical wave as an *addition theorem for the function*

$$\zeta_0^1(kR) = \zeta_0^1(\sqrt{\varrho^2 + \varrho_0^2 - 2\varrho\varrho_0 \cos \Theta}).$$

If, on the left side of (9), we pass from the radiated to the absorbed spherical wave

$$\frac{e^{-ikR}}{-ikR} = \zeta_0^2(\sqrt{\varrho^2 + \varrho_0^2 - 2\varrho\varrho_0 \cos \Theta})$$

then throughout the right side ζ^2 must be replaced by ζ^1 . From half the sum of both representations we obtain the *addition theorem* for the regular "standing wave"

$$(10) \quad \psi_0(kR) = \frac{\sin kR}{kR} = \sum (2n+1) \psi_n(\varrho_0) \psi_n(\varrho) P_n(\cos \Theta),$$

here the distinction between $r \leq r_0$ is unnecessary.

B. ASYMPTOTIC BEHAVIOR

If in the differential equation (22.13) of the associated spherical harmonics we pass to the limit

$$(11) \quad n \rightarrow \infty, \quad \vartheta \rightarrow 0, \quad n\vartheta \rightarrow \eta, \quad P_n^m(\cos \vartheta) \rightarrow O_m(\eta),$$

then we obtain

$$(11a) \quad \frac{1}{\eta} \frac{d}{d\eta} \left(\eta \frac{dO_m}{d\eta} \right) + \left(1 - \frac{m^2}{\eta^2} \right) O_m = 0.$$

This is the differential equation (19.11) of the cylindrical harmonic Z_m .

Since P_n^m and hence O_m is finite for $\vartheta \rightarrow 0$, the only permissible solution of (11a) is the Bessel function I_m . Hence we have

$$(12) \quad O_m(\eta) = C_m I_m(\eta) \quad \text{with} \quad C_0 = 1.$$

The latter follows from the fact that for $m = 0$ and $\eta = 0$ we have $I_0(\eta) = 1$ on one hand and (due to (11)) $\vartheta = 0$ on the other, and hence $P_n(\cos \vartheta) = 1$ and $O_0(\eta) = 1$. In order to determine C_m for $m > 0$ too, we use (22.18), which for $\vartheta \rightarrow 0$ yields:

$$(12a) \quad P_n^m \rightarrow \frac{\vartheta^m}{2^n n!} \lim_{\zeta \rightarrow 1} \frac{d^{n+m}}{d\zeta^{n+m}} (\zeta - 1)^n (\zeta + 1)^n.$$

We rewrite the function under differentiation in the form

$$(\zeta - 1)^n 2^n \left(1 + \frac{\zeta - 1}{2}\right)^n = \dots + 2^n (\zeta - 1)^n \binom{n}{m} \left(\frac{\zeta - 1}{2}\right)^m + \dots$$

In this binomial expansion we have written only one term since the terms of lower degree vanish upon differentiation and those of higher degree vanish in the limit $\zeta \rightarrow 1$. The $(n + m)$ -fold differentiation of this term yields

$$2^{n-m} (n + m)! \binom{n}{m} = 2^{n-m} \frac{n!}{m!} \frac{(n + m)!}{(n - m)!}.$$

Upon substitution in (12a) we obtain

$$(12b) \quad P_n^m \rightarrow \frac{1}{m!} \left(\frac{\vartheta}{2}\right)^m \frac{(n + m)!}{(n - m)!}.$$

Here the last fraction has $2m$ more factors in its numerator than in its denominator; since $m \ll n$, we may identify all these factors with n to obtain

$$(12c) \quad P_n^m \rightarrow \frac{1}{m!} \left(\frac{n\vartheta}{2}\right)^m n^m = \frac{1}{m!} \left(\frac{\eta}{2}\right)^m n^m.$$

Comparing this with (12), where we replace $I_m(\eta)$ by the first term of its power series (19.34), we obtain

$$(13) \quad C_m = n^m.$$

Hence for $m > 0$ we must, in order to obtain I_m , divide P_n^m by n^m before passing to the limit.

The geometrical meaning of our result is as follows: The surface of the sphere can be replaced by its tangent plane for the neighborhood of the north pole $\vartheta \rightarrow 0$. The solution of the spatial wave equation, whose behavior on the sphere is determined by $P_n^m e^{im\varphi}$, thereby goes into a solution of the wave equation for the tangent plane, namely,

$I_m(\eta) e^{im\varphi}$, provided we perform the passage to the limit on P_n^m/n^m instead of P_n^m . The same obviously also holds for the south pole of the sphere $\vartheta \rightarrow \pi$.

Having thus treated the special cases $\vartheta \rightarrow 0$ and $\vartheta \rightarrow \pi$ we now wish to investigate the asymptotic value of P_n^m as $n \rightarrow \infty$ for a general $0 < \vartheta < \pi$. To this end we apply the saddle-point method to the integral (22.23), which we rewrite in the following complex form:

$$(14) \quad P_n^m(\zeta) = \frac{C}{2\pi i} \oint e^{nf(w)} dw, \quad \text{with } w = e^{i\varphi}, \quad C = \frac{(n+m)!}{n!} e^{-i m \pi/2},$$

the latter due to (22.23a). The integration is to be taken over the unit circle of the w -plane in the positive (counterclockwise) sense; the function $f(w)$ stands for

$$(15) \quad f(w) = \log \left\{ \cos \vartheta + \frac{i}{2} \sin \vartheta \cdot (w + 1/w) \right\} - \frac{m+1}{n} \log w.$$

Hence

$$f'(w) = \frac{\frac{i}{2} \sin \vartheta (1 - 1/w^2)}{\cos \vartheta + \frac{i}{2} \sin \vartheta \cdot (w + 1/w)} - \frac{m+1}{nw}.$$

We therefore have two saddle points w_0 , which, for $m \ll n$ and $\sin \vartheta \neq 0$, lie on the unit circle, namely,

$$w_0 = \pm 1$$

and we get

$$(15a) \quad f''(w_0) = \sin \vartheta e^{\mp i(\vartheta - \pi/2)}.$$

With the same assumptions we get

$$(15b) \quad e^{nf(w_0)} = e^{\pm i n \vartheta} (\pm 1)^{m+1}.$$

As in (19.54) we set for the two saddle points $w \mp 1 = s e^{i\gamma}$ and after applying (15a) we obtain

$$(15c) \quad f(w) - f(w_0) = f''(w_0) \frac{(w \mp 1)^2}{2} + \dots = \frac{\delta^2}{2} \sin \vartheta e^{2i\gamma \mp i(\vartheta - \pi/2)}.$$

If we let

$$(15d) \quad 2i\gamma \mp i(\vartheta - \pi/2) = \pm i\pi \quad \text{and hence} \quad \gamma = \pm (\vartheta/2 + \pi/4),$$

then $f(w) - f(w_0)$ becomes real and $= -\frac{\delta^2}{2} \sin \vartheta$. This choice of γ

means that for the saddle points we shall integrate along the line of

steepest descent whose direction, according to (15d), still depends on ϑ . The two integrals then assume the common value

$$(16) \quad \int_{-s}^{+s} e^{-\frac{n s^2}{2} \sin \vartheta} ds,$$

Due to (15d) and the relation $dw = e^{i\gamma} ds$ this must be multiplied by the factor

$$(16a) \quad e^{i\gamma} = e^{\pm i(\vartheta/2 + \pi/4)}$$

In the limit $n \rightarrow \infty$ the integral (16) can be reduced to the Laplace integral by a simple substitution and we obtain

$$(16b) \quad \sqrt{\frac{2\pi}{n \sin \vartheta}}.$$

Due to (15c) and (16a,b) equation (14) becomes

$$(16c) \quad P_n^m = \frac{C}{2\pi i} \sqrt{\frac{2\pi}{n \sin \vartheta}} \left(e^{i\{(n+\frac{1}{2})\vartheta + \pi/4\}} + e^{-i\{(n+\frac{1}{2})\vartheta + \pi/4 + (m+1)\pi\}} \right).$$

Since we have made the assumption $m \ll n$ throughout, the value of C in (14) can be reduced to $C = n^m \exp\{-im\pi/2\}$ by the same reasoning that led from (12b) to (13). Hence C/i becomes

$$n^m \exp\{-i(m+1)\pi/2\}$$

Combining this with the two exponential functions in (16c) we obtain:

$$(17) \quad P_n^m = n^m \sqrt{\frac{2}{\pi n \sin \vartheta}} \cos \left\{ (n + \frac{1}{2})\vartheta - \frac{m\pi}{2} - \frac{\pi}{4} \right\}.$$

Therefore P_n^m for real n is a rapidly oscillating function of varying amplitude; the amplitude is small in the neighborhood of $\vartheta = \pi/2$ and increases symmetrically for decreasing or increasing ϑ . For $\vartheta = 0$ and π equation (17) breaks down since, according to (15a), $f''(w_0)$ vanishes and the series for $f(w) - f(w_0)$ starts with the third term (compare with the limiting case on p. 122 that led to the Airy integral). Equation (17) is then replaced by (12c).

We shall apply (17) in the appendix of Chapter VI for the case of complex n with a positive real part in which our derivation remains valid.

C. THE SPHERICAL HARMONIC AS AN ELECTRIC MULTIPOLE

In this section we return to potential theory. Since in §22 E we were able to define the surface spherical harmonics of degree n as homo-

geneous potentials of degree n (or better of degree $-n-1$), it must be possible to generate them with the help of repeated differentiation "with respect to n -directions" of the elementary potential $1/R$. This is the point of view of Maxwell in Chapter IX of his treatise. We express this by the Maxwell rule:

$$(18) \quad Y_n = \frac{1}{n!} \frac{\partial}{\partial h_1} \frac{\partial}{\partial h_2} \dots \frac{\partial}{\partial h_n} \left(\frac{1}{R} \right) \dots \quad \begin{cases} R^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2, \\ \text{Lim } x, y, z \rightarrow 0, \quad \text{Lim } R \rightarrow r, \\ r^2 = \xi^2 + \eta^2 + \zeta^2 = 1. \end{cases}$$

The "action point" $P = (\xi, \eta, \zeta)$ is to lie on a sphere of radius 1, the "source point" $Q = (x, y, z)$ is to lie in the neighborhood of the origin. The "directional differentiations" h_1, h_2, \dots, h_n can be performed both on the coordinates of P and on the coordinates of Q . We do the latter and then pass to the limit $x, y, z \rightarrow 0, R \rightarrow r$. In this way we obtain a *multipole* at Q whose order increases with the order of differentiation.

We start with the simplest case in which the directions h_1, h_2, \dots coincide, say, with the z -direction. The surface spherical harmonic which is obtained in this way is symmetric with respect to the z -axis and hence is a *zonal* spherical harmonic of the Legendre type P_n . We follow its genesis from line to line denoting the limit process of (18) by \rightarrow :

$$\begin{aligned} 1) \quad & \frac{\partial}{\partial h_1} \frac{1}{R} = \frac{\partial}{\partial z} \frac{1}{R} = \frac{\zeta - z}{R^3} \rightarrow \zeta = P_1, \\ 2) \quad & \frac{1}{2!} \frac{\partial}{\partial h_1} \frac{\partial}{\partial h_2} \frac{1}{R} = \frac{1}{2} \frac{\partial^2}{\partial z^2} \frac{1}{R} = \frac{1}{2} \frac{\partial}{\partial z} \frac{\zeta - z}{R^3} = -\frac{1}{2} \frac{1}{R^3} + \frac{3}{2} \frac{(\zeta - z)^2}{R^5} \\ & \rightarrow \frac{3}{2} \zeta^2 - \frac{1}{2} = P_2, \\ 3) \quad & \frac{1}{3!} \frac{\partial}{\partial h_1} \frac{\partial}{\partial h_2} \frac{\partial}{\partial h_3} \frac{1}{R} = \frac{1}{3!} \frac{\partial^3}{\partial z^3} \frac{1}{R} = \frac{1}{3} \frac{\partial}{\partial z} \left(-\frac{1}{2} \frac{1}{R^3} + \frac{3}{2} \frac{(\zeta - z)^2}{R^5} \right) \\ & = -\frac{3}{2} \frac{\zeta - z}{R^5} + \frac{5}{2} \frac{(\zeta - z)^3}{R^7} \rightarrow \frac{5}{2} \zeta^3 - \frac{3}{2} \zeta = P_3, \\ 4) \quad & \frac{1}{4!} \frac{\partial}{\partial h_1} \frac{\partial}{\partial h_2} \frac{\partial}{\partial h_3} \frac{\partial}{\partial h_4} \frac{1}{R} = \frac{1}{4!} \frac{\partial^4}{\partial z^4} \frac{1}{R} = \frac{1}{4} \frac{\partial}{\partial z} \left(-\frac{3}{2} \frac{\zeta - z}{R^5} + \frac{5}{2} \frac{(\zeta - z)^3}{R^7} \right) \\ & = \frac{8}{8} \frac{1}{R^5} - \frac{15}{4} \frac{(\zeta - z)^2}{R^7} + \frac{35}{8} \frac{(\zeta - z)^4}{R^9} \rightarrow \frac{35}{8} \zeta^4 - \frac{15}{4} \zeta^2 + \frac{3}{8} = P_4. \end{aligned}$$

This sequence P_1, \dots, P_4 , which can be completed by the zeroth derivative $P_0 = 1$ of $1/R$, coincides with the values obtained from the original definition on p. 23 (the variable x being replaced by ξ). This follows by necessity from the relation between spherical harmonics and homogeneous potentials, so that in (18) we were free to determine the normalizing

factor $1/n!$ only. We note the connection between this rule and the second equation (22.3) which, after the substitution $r_0 = z$ (Q on the z -axis) and $r = 1$ (P on the unit sphere), can be written:

$$\frac{1}{R} = \sum_{m=0}^{\infty} z^m P_m(\cos \vartheta);$$

and hence for $z \rightarrow 0$ we indeed have

$$\frac{1}{n!} \frac{d^n}{dz^n} \frac{1}{R} = P_n(\cos \vartheta) = P_n(\zeta).$$

We list the names and symbols for the successive multipoles. In order to avoid the limit process $Q \rightarrow 0$ we replace the differentiation with respect to z (coordinate of source point) by a differentiation with respect to ζ (coordinate of action point) but with opposite sign, and interpret r as the distance $OP = \sqrt{\xi^2 + \eta^2 + \zeta^2}$.

Unipole	charge scheme	\oplus	potential $\frac{1}{r}$,
Bipole	charge scheme	$\oplus \quad \ominus$	potential $-\frac{1}{1!} \frac{d}{d\zeta} \frac{1}{r}$,
Quadrupole	charge scheme	$\oplus \quad \ominus \quad \oplus \quad \ominus$	potential $+\frac{1}{2!} \frac{d^2}{d\zeta^2} \frac{1}{r}$.

By contracting two quadrupoles of opposite scheme we obtain the "octupole" with potential

$$-\frac{1}{3!} \frac{d^3}{d\zeta^3} \frac{1}{r}.$$

(The determination of the corresponding charge scheme is left to the reader.) By n -fold differentiation we obtain the

$$2^n\text{-pole and its potential } \frac{(-1)^n}{n!} \frac{d^n}{d\zeta^n} \frac{1}{r}.$$

In wireless telegraphy one uses the term dipole instead of bipole. Quadrupole and octupole radiations occur in atomic physics.

We now have to consider examples of differentiations with respect to *different directions*. In addition to differentiations in the z -direction we now consider differentiations in the (x, y) -plane. In order to preserve a certain degree of symmetry we consider differentiation with respect to, say, m equally spaced directions in the (x, y) -plane (in "star form," with an angle of π/m between two adjacent directions) together with $n-m$ differentiations still taken with respect to the z -direction. We thus

obtain the *tesseral surface spherical harmonics*

$$(19) \quad P_n^m(\zeta) \Phi_m(\varphi), \quad \Phi_m(\varphi) = e^{\pm im\varphi},$$

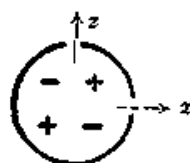
where instead of the two exponential functions given here we may have a linear combination, e.g., $\frac{\cos}{\sin} m\varphi$. To this category (19) belong the so-called *sectorial surface spherical harmonics* (this notation too is Maxwell's) with $m = n$, which according to (22.18) is represented by

$$(19a) \quad P_n^n \Phi_n = \sin^n \vartheta \frac{d^n P_n}{d\zeta^n} \Phi_n = \frac{(2n)!}{2^n n!} \sin^n \vartheta e^{\pm in\varphi}.$$

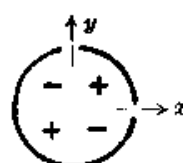
We discuss this further for $n = 2$. The star formed arrangement is obtained here if we take h_1 and h_2 in the x - and y -direction. Equation (18) and what follows then yield

$$\begin{aligned} P_2^2 &= \frac{1}{2} \frac{\partial^2}{\partial x \partial y} \frac{1}{R} = \frac{1}{2} \frac{\partial}{\partial x} \frac{\eta - y}{R^3} = \frac{3}{2} \frac{(\xi - x)(\eta - y)}{R^5} \\ &\rightarrow \frac{3}{2} \xi \eta = \frac{3}{4} \sin^2 \vartheta \sin 2\varphi, \end{aligned}$$

which is indeed of the type (19a). In this case too we speak of a *quadrupole* (see the right hand side of the diagram below; the left side, which is placed differently in space belongs to P_2^1).



Quadrupole P_2^1



Quadrupole P_2^2 .

The fact that (18) yields the complete system of the $2n + 1$ surface spherical harmonics of degree n , follows from the number of constants in (18): two directional constants for every differentiation h and one multiplicative factor.

D. SOME REMARKS ABOUT THE HYPERGEOMETRIC FUNCTION

The hypergeometric function is best defined by its differential equation:

$$(20) \quad z(1-z)y'' + [\gamma - (\alpha + \beta + 1)z]y' - \alpha\beta y = 0$$

From this equation we deduce the Gaussian series representation (11.10a) according to the procedure of §19 C. We set

$$(21) \quad y = z^2 (a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots),$$

with undetermined exponent λ and coefficients a_k . We substitute this in (20) and set the coefficients of the initial term $z^{\lambda-1}$ and the general term $z^{\lambda+k}$ equal to zero. In this way on one hand we obtain:

$$(21a) \quad \lambda(\lambda-1+\gamma) = 0,$$

and on the other hand

$$(21b) \quad \begin{aligned} & [(\lambda+k+1)(\lambda+k) + \gamma(\lambda+k+1)] a_{k+1} \\ & = [(\lambda+k)(\lambda+k-1) + (\alpha+\beta+1)(\lambda+k) + \alpha\beta] a_k. \end{aligned}$$

Equation (21a) has the solutions

$$(22a) \quad \lambda = 0 \quad \text{and} \quad \lambda = 1 - \gamma;$$

We first consider the former solution and by substituting it in (21b) obtain

$$(22b) \quad a_{k+1} = \frac{k(k-1) + (\alpha+\beta+1)k + \alpha\beta}{(k+1)k + \gamma(k+1)} a_k = \frac{(\alpha+k)(\beta+k)}{(k+1)(\gamma+k)} a_k.$$

Hence if we set $a_0 = 1$ we obtain the Gauss series

$$(23) \quad y = y_1 = F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{1 \cdot \gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} z^2 + \dots$$

The other solution of (22a) yields:

$$(23a) \quad y = y_2 = z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z).$$

There are also a large number of related representations for altered parameters α, β, γ and linearly transformed z , which coincide term-wise with (23). They have been compiled lovingly by Gauss as 'relationes inter contiguas.'

If we compare the differential equation (22.6a) of the zonal spherical harmonics

$$(24) \quad (1 - \zeta^2) P'' - 2\zeta P' + n(n+1)P = 0$$

with equation (20) then we see that it is obtained from the latter by the substitution

$$z = \frac{1-\zeta}{2}, \quad \alpha = -n, \quad \beta = n+1, \quad \gamma = 1$$

From this we see that P_n must coincide both with y_1 and y_2 up to a factor. Namely, we have:

$$(24a) \quad \begin{aligned} P_n(\zeta) &= F\left(-n, n+1, 1, \frac{1-\zeta}{2}\right) \\ &= (-1)^n F\left(-n, n+1, 1, \frac{1+\zeta}{2}\right), \end{aligned}$$

which also yields the correct normalization $P_n(1) = 1$ for $\zeta = +1$. The series (24a) for P_n breaks off as does every hypergeometric series with negative integral α or β : since $\alpha = -n$ we have that P_n is a polynomial of degree n (the coefficients of $(1 \mp \zeta)^{n+1}$ and of all subsequent powers contain the factor $\alpha + n = -n + n = 0$ in the numerator). We remark that the series for P_n in terms of $1 - \zeta$ is simpler (since it is hypergeometric) than the series in terms of ζ . The latter reads

$$\begin{aligned} P_n &= 1 + \frac{(-n)(n+1)}{1 \cdot 2} \frac{1-\zeta}{2} \\ (24b) \quad &+ \frac{(-n)(-n+1)(n+1)(n+2)}{2!2!} \left(\frac{1-\zeta}{2}\right)^2 + \cdots + (-1)^n \frac{(2n)!}{n!n!} \left(\frac{1-\zeta}{2}\right)^n \\ &= \sum_{p=0}^n (-1)^p \frac{(n+p)!}{(n-p)!} \frac{1}{p!p!} \left(\frac{1-\zeta}{2}\right)^p. \end{aligned}$$

The associated P_n^m can also be represented by a hypergeometric series. We merely have to consider the general relation which is obtained from (23) by termwise differentiation:

$$(25) \quad \frac{d}{dz} F(\alpha, \beta, \gamma, z) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1, z).$$

Hence for positive m we obtain the representation

$$\begin{aligned} P_n^m(\zeta) &= C (1-\zeta^2)^{m/2} F\left(m-n, m+n+1, m+1, \frac{1-\zeta}{2}\right), \\ (26) \quad C &= \frac{(n+m)!}{2^m m! (n-m)!}. \end{aligned}$$

from (22.18). For the negative integral m this representation breaks down, but can be extended to that case by a limit process; the result then coincides with our general definition (22.18).

From the Gaussian hypergeometric function we derive the *confluent hypergeometric function*, which is of the utmost importance in wave mechanics. It depends only on two parameters α and γ since the third parameter β is subjected to the following limit process:

$$(27) \quad \beta \rightarrow \infty, \quad z \rightarrow 0, \quad \beta z \rightarrow \varrho \quad (\varrho = \text{arbitrary finite number}).$$

We then obtain from (23)

$$(28) \quad F(\alpha, \gamma, \varrho) = 1 + \frac{\alpha}{\gamma} \frac{\varrho}{1} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{\varrho^2}{2!} + \cdots$$

and considering the fact that

$$\frac{d}{dz} = \frac{d}{d\varrho} \cdot \frac{d\varrho}{dz} = \beta \frac{d}{d\varrho}$$

we obtain the corresponding differential equation from (20) after dividing by β :

$$(29) \quad \varrho \frac{d^2 F}{d\varrho^2} + (\gamma - \varrho) \frac{dF}{d\varrho} - \alpha F = 0.$$

We shall encounter this equation again in connection with the eigenfunctions of hydrogen in wave mechanics.

E. SPHERICAL HARMONICS OF NON-INTEGRAL INDEX

Our representation must still be completed in two directions. We have been restricted so far to the case of *integral numbers* n and m and to *functions* P_n^m , which were *finite throughout*. Both these restrictions were suggested by the connection with potential theory.

Concerning the first point, we see that for non-integral n the hypergeometric series at the points $\zeta = \pm 1$ in ascending powers of $1 \mp \zeta$ does not break off as it does in the case of integral n . The solution, which is regular at the north pole $\zeta = +1$, diverges at the south pole $\zeta = -1$ and vice versa. Thus (24a) is valid for integral n only. The "requirement of finiteness all over the sphere" can therefore be satisfied only for integral n . The possibility of non-integral m is excluded by the requirement of uniqueness with respect to the φ -coordinate.

The type of singularity of $P_n(\zeta)$ for non-integral n can be deduced from the general theory of hypergeometric series. We prefer, however, to deduce it by direct calculation.

According to their original definition in (22.3) the P_n are the coefficients of a Taylor series which progresses in powers of $t = r/r_0$; hence for integral n :

$$(30) \quad P_n(\zeta) = \frac{1}{n!} \frac{d^n}{dt^n} \frac{1}{\sqrt{1-2\zeta t+t^2}} \quad \text{at } t=0.$$

According to Cauchy's theorem this can be written as:

$$(30a) \quad P_n(\zeta) = \frac{1}{2\pi i} \oint \frac{dt}{t^{n+1}} \frac{1}{\sqrt{1-2\zeta t+t^2}}.$$

This representation also holds for non-integral n , except that due to the many-valued character of t^{-n-1} we have to perform a branch cut, e.g., from $t = -\infty$ to $t = 0$, and that the path of integration is now a loop which starts on the negative side of the cut at $t = -\infty$, then circles the point $t = 0$ in a counterclockwise direction and ends on the positive side of the cut at $t = -\infty$. It is clear that for this definition the differential equation of P_n is satisfied regardless of whether or not n is integral. Equation (30a) defines that particular solution of the differen-

tial equation which is regular at $\zeta = 1$ and satisfies the normalizing condition $P_n(1) = 1$. Namely for $\zeta = 1$ we obtain from (30a)

$$(30b) \quad P_n(1) = -\frac{1}{2\pi i} \oint \frac{dt}{t^{n+1}} \frac{1}{t-1}.$$

The integrand now has a simple pole at $t = 1$. The loop described above can now be deformed into a path which circles the pole in a clockwise direction. According to Cauchy's theorem the integral then has the value $-2\pi i$, and hence the right side of (30b) has the required value $+1$.

For $\vartheta > 0$ the integrand of (30a) has two further branch points that are due to the square root in the denominator and lie on the unit circle of the t -plane at

$$t = e^{i\vartheta} \quad \text{and} \quad t = e^{-i\vartheta}$$

We connect these branch points by a branch cut, e.g., along the unit circle. The path of integration may not cross this cut either. For $\vartheta = \pi - \delta$, $\delta \ll 1$, the endpoints of the cut approach the negative real axis and restrict the path of integration between them. This explains why (30a) becomes singular for $\delta \rightarrow 0$, or in other words for $\vartheta \rightarrow \pi$, $\zeta \rightarrow -1$.

In order to discuss this singularity we write

$$t = e^{i\pi} (1 + \tau) \quad \text{and} \quad t = e^{-i\pi} (1 + \tau);$$

in the neighborhood of the point $t = -1$ on the upper and lower edge of our branch cut respectively. Then, except for terms of higher order in τ and δ , the square root in (30a) becomes

$$\sqrt{1 - 2\zeta t + t^2} = \sqrt{\tau^2 + \delta^2}.$$

Hence for small δ only the neighborhood of $\tau = 0$ contributes to our limiting value as $\vartheta \rightarrow \pi$. We may, therefore, restrict the integration over the upper and lower edges to the small region between

$$\tau = +\varepsilon \quad \text{and} \quad \tau = -\varepsilon$$

and considering the orientation on the two edges we may write:

$$\frac{dt}{t^{n+1}} = \begin{cases} e^{+i\pi n} d\tau & \text{lower edge} \\ -e^{-i\pi n} d\tau & \text{upper edge,} \end{cases}$$

hence:

$$(31) \quad P_n(\zeta) = \frac{e^{+i\pi n} - e^{-i\pi n}}{2\pi i} \int_{+\varepsilon}^{-\varepsilon} \frac{d\tau}{\sqrt{\tau^2 + \delta^2}} = \frac{\sin n\pi}{\pi} \int_{+\varepsilon}^{-\varepsilon} \frac{d\tau}{\sqrt{\tau^2 + \delta^2}}.$$

Now according to a well known formula we have

$$\int \frac{d\tau}{\sqrt{\tau^2 + \delta^2}} = \log(\tau + \sqrt{\tau^2 + \delta^2})$$

for undetermined upper and lower limits of integration. Hence for the definite integral in (31) we have

$$\log(-\varepsilon + \sqrt{\varepsilon^2 + \delta^2}) - \log(+\varepsilon + \sqrt{\varepsilon^2 + \delta^2})$$

and for $\delta \ll \varepsilon$

$$\log \frac{1}{2} \frac{\delta^2}{\varepsilon} - \log \left(2\varepsilon + \frac{1}{2} \frac{\delta^2}{\varepsilon} \right).$$

In the limit $\delta \rightarrow 0$ we have $\log \delta^2$ as the leading term; hence we obtain from (31)

$$(32) \quad \lim_{\zeta \rightarrow -1} P_n(\zeta) = \frac{\sin n\pi}{\pi} \log \delta^2 + \dots,$$

The terms²² . . . which have been omitted here reduce for $\delta \rightarrow 0$ to a finite constant which is of course independent of ε .

F. SPHERICAL HARMONICS OF THE SECOND KIND

At the beginning of Section E we saw that for non-integral n two different solutions P_n of the hypergeometric differential equation exist. Only for integral n do these solutions coincide. But in this latter case, too, a second solution must exist in addition to the everywhere regular solution found above. This solution will be singular at the points $\zeta = \pm 1$. We call it a *spherical harmonic of the second kind* and denote it by Q_n .

The type of singularity can be determined from general theorems. In (21a) we saw that the quadratic equation for the exponent λ for the case of spherical harmonics ($\gamma = 1$) has the *double root* $\lambda = 0$. By a passage to the limit we see that this indicates a logarithmic singularity for $\zeta = \pm 1$. Just as in the case of spherical harmonics of the first kind, we obtain detailed information about the *spherical harmonics*

²² They are computed in Hobson's textbook, equation (53), p. 225, which was quoted on p. 129 above.

of the second kind Q_n from a generating function²³ (C. Neumann):

$$(33) \quad \frac{1}{\eta - \zeta} = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) Q_n(\eta) P_n(\zeta).$$

Hence the $Q_n(\eta)$ are defined as coefficients in the expansion of $1/(\eta - \zeta)$ in the P_n , and therefore they are given by the following *integral representation* (F. Neumann):

$$(34) \quad Q_n(\eta) = A.M. \int_{-1}^{+1} P_n(\zeta) \frac{d\zeta}{\eta - \zeta}.$$

In this formula the path of integration is to avoid the singular point $\zeta = \eta$ by going around it through the complex domain both to the right and to the left, and the symbol *A.M.*, which we shall omit in the future, indicates that we have to take the arithmetic mean of the two values obtained (this is identical with the so-called "principal value" of the integral). The fact that this avoidance of the singularity is not possible at the limits $\zeta = \pm 1$ implies the above-mentioned logarithmic singularity.

That $Q_n(\eta)$ satisfies the differential equation (24) (written in terms of η), is to be expected from the symmetry of the defining equation in η, ζ and Q, P , but it can also be demonstrated directly as follows: we abbreviate equation (24) to

$$L_{\zeta}\{P\} = 0, \quad L_{\zeta} = \frac{d}{d\zeta} (1 - \zeta^2) \frac{d}{d\zeta} + n(n+1),$$

and by $L_{\eta}\{Q\}$ we mean the analogous expression written in terms of η and Q ; we further note the identity

$$L_{\eta}\left\{\frac{1}{\eta - \zeta}\right\} = L_{\zeta}\left\{\frac{1}{\eta - \zeta}\right\}.$$

Then from (34) we have

$$(35) \quad L_{\eta}\{Q_n\} = \int_{-1}^{+1} P_n(\zeta) L_{\zeta}\left\{\frac{1}{\eta - \zeta}\right\} d\zeta$$

We integrate by parts twice, then the terms which are due to the limits $\zeta = \pm 1$ vanish on account of the factor $1 - \zeta^2$ in L_{ζ} and we obtain

$$(36) \quad L_{\eta}\{Q_n(\eta)\} = \int_{-1}^{+1} L_{\zeta}\{P_n(\zeta)\} \frac{d\zeta}{\eta - \zeta} = 0, \text{ q.e.d.}$$

²³ Usually double this function is used, so that in (33) we have $2n + 1$ instead of $n + \frac{1}{2}$. Correspondingly our $Q_n(\eta)$ differ from the customary ones by the factor 2.

It is now easy to compute the first $Q_n(\eta)$ in terms of the known $P_n(\eta)$ with the help of (34). We deduce for $|\eta| < 1$

$$\begin{aligned} \text{from } P_0 = 1: \quad Q_0 &= \log \frac{1+\eta}{1-\eta}, \\ \text{from } P_1 = \zeta: \quad Q_1 &= -2 + \eta \log \frac{1+\eta}{1-\eta}. \end{aligned}$$

The general law is (Christoffel):

$$(37) \quad Q_n(\eta) = \Pi + P_n(\eta) \log \frac{1+\eta}{1-\eta},$$

where Π is a polynomial of degree $n-1$ which is composed additively from all those P_{n-2k-1} for which the index is non-negative. Finally we obtain from (34), through m -fold differentiation with respect to η and multiplication by $(1-\eta^2)^{m/2}$ (which is analogous to $\sin^m \vartheta$),

$$(38) \quad Q_n^m(\eta) = (-1)^m m! (1-\eta^2)^{m/2} \int_{-1}^{+1} \frac{P_n(\zeta)}{(\eta-\zeta)^{m+1}} d\zeta.$$

Appendix I

REFLECTION ON A CIRCULAR-CYLINDRICAL OR SPHERICAL MIRROR

Referring back to Fig. 8 and the notations defined there we continue the treatment of the problem which we started in §6.

a) *Circular-cylindrical metal mirror.* The incoming wave (electric vector which is perpendicular to the plane of the drawing) is

$$(1) \quad w = e^{ikr \cos \varphi} \approx I_0(kr) + 2 \sum_{n=1}^N i^n I_n(kr) \cos n\varphi.$$

(see (21.2b)). This representation holds for the entire r, φ -plane and for $r = a$ it defines the function $-f(\varphi)$ in (6.4). The sum of $N+1$ terms on the right is the best approximation to w that can be obtained by the method of least squares; the fact that the coefficients in this sum are the same as those in the exact non-truncated series (21.2b) follows from the "finality" of the Fourier series. We write the radiation which is reflected (diffracted, scattered) by the mirror as the sum of $N+1$ particular solutions of the differential equation

$$\Delta u + k^2 u = 0$$

for $r < a$ in the form:

$$(2) \quad u = \sum_0^N C_n \frac{I_n(kr)}{I_n(ka)} \cos n\varphi;$$

(since the solution must be continuous for $r = 0$ only the I_n can occur in the representation; the sine terms disappear on account of the symmetry of the incoming wave with respect to $\varphi = 0$). The denominator $I_n(ka)$ is used for the sake of convenience and merely influences the meaning of the constants C_n which are as yet undetermined. The same holds for the denominators in equation (3) below.

We write the radiation which is scattered by the mirror to the outside $r > a$ as the following sum of $N + 1$ particular solutions of the wave equation:

$$(3) \quad v = \sum_0^N D_n \frac{H_n^1(kr)}{H_n^1(ka)} \cos n\varphi.$$

The time dependence of the whole process should be thought of as given by $\exp(-i\omega t)$; hence only the H^1 occur; the H^2 would correspond to absorbed waves. As we saw in §6 the boundary conditions (6.8) to (6.11) imply $C_n = D_n$ and, according to the method of least squares, the system of linear equations (6.12). The constant γ_n which occurs there is determined from (6.7), (6.11a) and the equations (3), (4) above as

$$(4) \quad \gamma_n = ka \left(\frac{I_n'(ka)}{I_n(ka)} - \frac{H_n^{1'}(ka)}{H_n^1(ka)} \right).$$

According to a well known theorem in the theory of linear differential equations we can rewrite this in the simpler form (see exercise IV.8)

$$(5) \quad \gamma_n = \frac{-2i/\pi}{I_n(ka) H_n^1(ka)}.$$

We introduce the notation:

$$(6) \quad a_{nm} = \int_{-\alpha}^{\alpha} \cos n\varphi \cos m\varphi d\varphi$$

and obtain by a simple transformation

$$(7) \quad \int_0^{\alpha} \cos n\varphi \cos m\varphi d\varphi = \frac{\pi}{(2)} \delta_{nm} - a_{nm}.$$

Here and in the following the symbol (2) stands for the number 2 when $n > 0$ and for the number 1 when $n = 0$. Then the left side of (6.12) becomes

$$(8) \quad \frac{\pi}{(2)} C_m + \sum_{n=0}^N a_{nm} (\gamma_n \gamma_m - 1) C_n$$

Setting $f(\varphi) = -w$, where w is as in equation (1), and $r = a$, we obtain for the right side of (6.12):

$$(8a) \quad -\pi i^m I_m(ka) + \sum_{n=0}^N a_{nm} (2) i^n I_n(ka).$$

Hence the system (6.12) becomes

$$(9) \quad C_m + (2) i^m I_m(ka) - \frac{(2)}{\pi} \sum_{n=0}^N a_{nm} \{C_n + (2) i^n I_n(ka) - \gamma_n \gamma_m C_n\} = 0,$$

which must be satisfied for all $m = 0, 1, \dots, N$.

In order to discuss this system we first set $\alpha = \pi$, in other words we consider a complete circle (spatially speaking a closed, totally conductive, cylinder). According to (6) we then have $a_{nm} = 0$ and (9) yields

$$(10) \quad C_m = -(2) i^m I_m(ka).$$

This result is somewhat trivial. For, by substituting the value (10) of C_n for D_n in (3), we obtain the rigorous solution v of the corresponding scattering problem for $r > a$:

$$(11) \quad v = - \sum_{n=0}^N (2) i^n I_n(ka) \frac{H_n^1(kr)}{H_n^1(ka)} \cos n\varphi,$$

a radiated wave which, on the cylinder $r = a$, exactly cancels the incoming wave w of equation (1) and hence for $N \rightarrow \infty$ yields the rigorous solution of the scattering problem. In the same manner we obtain for u :

$$(12) \quad u = - \sum_{n=0}^N (2) i^n I_n(ka) \cos \varphi = -w.$$

This result is also trivial since in the interior of the closed circle $r = a$ we must have $u + v = 0$.

We now wish to investigate whether our equation (10) yields a useful approximation in the case $\alpha < \pi$, too. To this end we set

$$(13) \quad C_n = -(2) i^n I_n(ka) + \beta_n$$

where β_n is a correction term, and obtain from (9):

$$(14) \quad \beta_m - \frac{(2)}{\pi} \sum_{n=0}^N a_{nm} \{\beta_n (1 - \gamma_n \gamma_m) + \gamma_n \gamma_m (2) i^n I_n(ka)\} = 0.$$

As in (6.12) this is a system of $N + 1$ (and in the limit infinitely many) linear equations with which we can do practically nothing. However if we assume that $\pi - \alpha$ is small, that is that the cylinder has only a *narrow slit*, then a_{nm} becomes small and the product $\beta_n a_{nm}$ becomes *small of the second order*. If we neglect this term then (14) becomes simply

$$(15) \quad \frac{\beta_m}{\gamma_m} = - \frac{(2)}{\pi} \sum_{n=0}^N (2) i^n a_{nm} \gamma_n I_n(ka),$$

which is an explicit value for β_m and from (13) yields an explicit value for C_m .

The narrowness of the slit necessary for this consideration can be estimated by a physical consideration: its width must be *small compared to the wavelength* of the incoming radiation; only in this case does the interior field go over continuously into the zero field of the closed cylinder. Hence we must have

$$(16) \quad \frac{(\pi - \alpha)a}{\lambda} < 1.$$

This condition can be satisfied only approximately for Hertz waves. In the properly optical case this approximation breaks down. This is the reason we spoke of a "quasi-optical case" on p. 29. In the well known Hertz experiment with concave mirrors, for which we have, say, $\alpha = \pi/2$, $\lambda = 200$ cm., $a = 50$ cm., equation (16) is approximately satisfied so that our system of approximation is justified.

b) *The sphere segment as an acoustic reflector.* In order to avoid discussions on vectors we deal with scalar acoustic waves instead of directed optical radiation. By w, u, v we mean the *velocity potentials* of the primary and secondary (reflected) radiation in the interior ($r < a$) and exterior ($r > a$) of the sphere. Let the sphere segment be given by $r = a$, $0 < \theta < \alpha$. According to (24.7) we write w in the form

$$(17) \quad w = \sum_{n=0}^N (2n+1) i^n \psi_n(kr) P_n(\cos \theta);$$

this is the best possible approximation of a plane wave $\exp(i k r \cos \theta)$ by $N+1$ spherical harmonics according to the method of least squares. We further write

$$(18) \quad u = \sum C_n \frac{\psi_n(kr)}{\psi_n(ka)} P_n(\cos \theta) \quad r < a,$$

$$(19) \quad v = \sum D_n \frac{\zeta_n(kr)}{\zeta_n(ka)} P_n(\cos \theta) \quad r > a,$$

where the constants C_n, D_n are as yet undetermined. Concerning the denominators see the remark to equation (2). The function ζ_n is the Bessel function with half-integral index which was defined in (21.15) and corresponds to the Hankel function H_n^1 . On the sphere segment (which was assumed rigid) we have

$$\frac{\partial}{\partial n} (u + w) = \frac{\partial}{\partial n} (v + w) = 0 \quad \text{for } 0 \leq \theta < \alpha \text{ and } r = a,$$

and for reasons of continuity we must have

$$u = v, \quad \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} \quad \text{for } \alpha < \theta \leq \pi \text{ and } r = a.$$

This condition again leads to $D_n = C_n$ and to the equations:

$$(20) \quad \begin{aligned} \sum C_n P_n(\cos \theta) &= f(\theta), & 0 < \theta < \alpha, \\ \sum D_n \gamma_n P_n(\cos \theta) &= 0, & \alpha < \theta < \pi. \end{aligned}$$

which are analogous to (6.10) and (6.11). Here $f(\theta)$ stands for the value of $-\partial w / \partial (kr)$ for $r = a$,

$$(20a) \quad f(\theta) = - \left(\frac{\partial w}{\partial (kr)} \right)_{r=a}, \quad w = e^{i k r \cos \theta}$$

and in analogy to (5)

$$(20b) \quad \gamma_n = \frac{\psi_n(ka)}{\psi'_n(ka)} - \frac{\zeta_n(ka)}{\zeta'_n(ka)}.$$

Introducing the abbreviation

$$a_{nm} = \int_{\alpha}^{\pi} P_n P_m \sin \theta d\theta$$

we obtain

$$(21a) \quad \int_0^{\alpha} P_n P_m \sin \theta d\theta = \frac{\delta_{nm}}{m + \frac{1}{2}} - a_{nm}$$

and from (20a)

$$(21b) \quad \begin{aligned} \int_0^{\alpha} f(\theta) P_m(\cos \theta) \sin \theta d\theta &= - \sum_{n=0}^N (2n+1) i^n \psi'_n(ka) \int_0^{\alpha} P_n P_m \sin \theta d\theta \\ &= -2 i^m \psi'_m(ka) + \sum_{n=0}^N a_{nm} (2n+1) i^n \psi'_n(ka). \end{aligned}$$

In analogy to (6.12) the method of least squares now yields the following system of equations for the C_n :

$$(22) \quad \sum_{n=0}^N C_n \left\{ \int_0^{\alpha} P_n P_m \sin \theta d\theta + \gamma_n \gamma_m \int_{\alpha}^{\pi} P_n P_m \sin \theta d\theta \right\} = \int_0^{\alpha} f(\theta) P_m \sin \theta d\theta,$$

which holds for $m = 0, 1, \dots, N$; due to (21a,b) this becomes

$$(22a) \quad \begin{aligned} \frac{C_m}{m + \frac{1}{2}} - \sum_{n=0}^N a_{nm} C_n (1 - \gamma_n \gamma_m) \\ = -2 i^m \psi'_m(ka) + \sum_{n=0}^N a_{nm} (2n+1) i^n \psi'_n(ka). \end{aligned}$$

which, as in equation (9), can be rearranged to

$$(23) \quad \begin{aligned} \frac{1}{m + \frac{1}{2}} [C_m + (2m+1) i^m \psi'_m(ka)] \\ - \sum_{n=0}^N a_{nm} \{C_n + (2n+1) i^n \psi'_n(ka) - \gamma_n \gamma_m C_n\} = 0. \end{aligned}$$

We again start with the limiting case $\alpha = \pi$ of a closed sphere in which $a_{nm} = 0$. Then (23) yields

$$(24) \quad C_m = -(2m+1) i^m \psi'_m(ka).$$

Substituting this value in (19) we obtain the rigorous solution v of our acoustic problem for $r > a$, namely, the reflection of an incoming wave on a closed sphere. In the interior $r < a$ of the sphere we obtain, by substituting C_m in (18), a field u which, as it should be, is the negative of the field w of the incoming wave.

The next problem is that of a spherical surface with a circular hole in the neighborhood of $\theta = \pi$. By setting

$$(25) \quad C_m = -(2m+1) i^m \psi'_m(ka) + \beta_m,$$

and ignoring the product term of second order $a_{nm}\beta_n$ we obtain from (23)

$$(26) \quad \frac{1}{m+\frac{1}{2}} \frac{\beta_m}{\gamma_m} = \sum_{n=0}^N (2n+1) i^n a_{nm} \gamma_n \psi'_n(ka),$$

which is an explicit computation of the correction term β_m and hence of the coefficients C_m . The reader should compare this result with the analogous result for the problem of the cylinder in equation (15). Just as the width of the slit there, so the diameter of the circular hole here must be small compared to the wavelength of the incoming radiation. Hence, here too, we can treat only a "quasi-acoustical" problem, the problem of *infra-sound*, which is very far from the more interesting problem of *ultra-sound*.

Our aim in this somewhat sketchy appendix has been to show that the method of least squares may be applied successfully even in some cases in which our condition of finality for the computation of the coefficients C_m is not satisfied.

Appendix II

ADDITIONS TO THE RIEMANN PROBLEM OF SOUND WAVES IN §11

The purpose of this appendix is to fill a gap which we left in §11; namely we shall prove that the expression (11.10)

$$(1) \quad v = \left(\frac{\xi + \eta}{x + y} \right)^a F(a+1, -a, 1, z), \quad z = -\frac{(x-\xi)(y-\eta)}{(x+y)(\xi+\eta)},$$

where F stands for the hypergeometric series, satisfies the differential equation

$$(2) \quad M(v) = \frac{\partial^2 v}{\partial x \partial y} + \frac{a}{x+y} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{2av}{(x+y)^2} = 0$$

which is derived from (11.2) and (11.8). Riemann was able to prove

this by his general transformation theory for hypergeometric functions. However we shall proceed in an elementary fashion, by considering the function F in (1) as an unknown function and then by substituting (1) in (2) and deducing a differential equation for $F(z)$. By proving the latter identical with the differential equation (24.20) of the hypergeometric function we verify equation (1) and the determination of the parameters of the hypergeometric function which are contained in it.

First we deduce from (1)

$$\begin{aligned}\frac{\partial v}{\partial x} &= \left(\frac{\xi + \eta}{x + y}\right)^a \left(\frac{-a}{x + y} F(z) + \frac{\partial z}{\partial x} F'(z)\right), \\ \frac{\partial v}{\partial y} &= \left(\frac{\xi + \eta}{x + y}\right)^a \left(\frac{-a}{x + y} F(z) + \frac{\partial z}{\partial y} F'(z)\right)\end{aligned}$$

and hence we obtain as the sum of the last two terms in (2)

$$(3) \quad \frac{a}{x + y} \left(\frac{\xi + \eta}{x + y}\right)^a \left[-\frac{2a + 2}{x + y} F(z) + \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) F'(z)\right].$$

As the first term of (2) we then obtain

$$(4) \quad \begin{aligned} &\frac{a(a + 1)(\xi + \eta)^a}{(x + y)^{a+2}} F(z) - \frac{a(\xi + \eta)^a}{(x + y)^{a+1}} \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) F'(z) \\ &+ \left(\frac{\xi + \eta}{x + y}\right)^a \left(\frac{\partial^2 z}{\partial x \partial y} F'(z) + \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} F''(z)\right). \end{aligned}$$

The first two terms of (4) combine and cancel respectively with the two terms of (3). Hence (2) becomes

$$(5) \quad \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} F''(z) + \frac{\partial^2 z}{\partial x \partial y} F'(z) - \frac{a(a + 1)}{(x + y)^2} F = 0.$$

The derivatives of z here can be expressed as follows:

$$(6) \quad \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = \frac{1}{(x + y)^2} (z^2 - z),$$

$$(7) \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{2z - 1}{(x + y)^2}.$$

Due to (6) and (7) equation (5) becomes

$$z(1 - z) F'' + (1 - 2z) F' + a(a + 1) F = 0.$$

This is indeed the same as equation (24.20) if we substitute

$$\alpha = a + 1, \quad \beta = -a, \quad \gamma = 1.$$

as in (1). This completes the discussion of the problem of §11.

CHAPTER V

Eigenfunctions and Eigen Values

In this chapter we shall develop Fourier's methods to their greatest generality and thereby open up the boundary value problems of physics to mathematical treatment. The most striking demonstration of the power of these methods was given in 1926 when Erwin Schrödinger recognized the quantum numbers as eigen values of his wave equation and thereby put the tools of modern analysis at the service of atomic physics. It was fortunate that he had the aid of his Zürich colleague Hermann Weyl who had, as the greatest pupil and later the successor of Hilbert in Göttingen, an essential part in the development of the theory of integral equations. However we should note that, while the viewpoint of *integral equations* is important for the rigorous mathematical foundation, in particular for the existence proofs for the eigenfunctions and their eigen values, the older viewpoint of *partial differential equations* leads to the same concepts in a natural manner. We shall start by demonstrating this with an example which was known long before integral equations.

§ 25. Eigen Values and Eigenfunctions of the Vibrating Membrane

The subject of the following consideration is a membrane without proper elasticity (see p. 33) which is clamped into a frame whose resistance to distortion is entirely due to the stresses working on its edge. We consider these stresses as perpendicular to the edge in the plane of the membrane. For the deformed membrane this results in a pressure N which acts perpendicular to the surface and is equal to T times the mean curvature of the membrane, and hence is equal to $T \Delta u$ for a small deformation u . The wave equation (7.4) for a pure harmonic oscillation of frequency ω then yields

$$-\sigma \omega^2 u = T \Delta u, \quad \sigma = \text{surface density.}$$

This we rewrite in the customary form

$$(1) \quad \Delta u + k^2 u = 0, \quad k^2 = \frac{\sigma \omega^2}{T}.$$

If we do not consider k^2 as constant but as an arbitrary function $F(x, y)$,

then according to (10.6) this is the general linear *self-adjoint* elliptic differential equation of second order in two variables in its normal form.

The non-trivial solutions of (1) which satisfy the boundary condition $u = 0$ are called *eigenfunctions* and the corresponding k are called the *eigen values* of the problem. If k^2 or $F(x, y)$ were negative then no eigen values would exist, as we saw in the introduction to exercise II.2. The fact that eigen values do exist for positive k^2 , namely, an *infinite number*, can be shown first for the simplest examples.

a) *The rectangle* $0 \leq x \leq a$, $0 \leq y \leq b$. The boundary conditions are satisfied by

$$(2) \quad u = u_{nm} = \sin n\pi \frac{x}{a} \sin m\pi \frac{y}{b}, \quad \begin{cases} n = 1, 2, \dots, \infty, \\ m = 1, 2, \dots, \infty. \end{cases}$$

From the differential equation we then have

$$(2a) \quad k = k_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}.$$

We shall ignore the constant factor by which the solutions can be multiplied. We first assume a and b to be *incommensurable*. Then all the k_{nm} are different and only *one* eigenfunction u corresponds to each k . The number of eigen values is *infinite*.

b) *Circle, circular ring, circular sector*. For the *full circle* $0 \leq r \leq a$ we can write:

$$(3) \quad u = I_m(kr) e^{\pm im\varphi}, \quad m = 0, 1, 2, \dots, \infty,$$

where k satisfies the boundary condition

$$(3a) \quad I_m(ka) = 0.$$

Since this equation has infinitely many roots (see Fig. 21) there are again *infinitely many eigen values* $k = k_{nm}$. The roots of (3a) are all different, but for $m > 0$ there are two eigenfunctions for each eigen value corresponding to the different signs in (3), or, in other words, corresponding to the double possibility $\frac{\cos}{\sin} m\varphi$. We say that the problem is *degenerate* for $m > 0$; in our case it is *simply degenerate*. According to (20.4b)

the (non-degenerate) basic tone of the circular membrane is $k_{10} = 2.40/a$.

For the *circular ring* $b \leq r \leq a$ we write

$$(4) \quad u = [I_m(kr) + c N_m(kr)] e^{\pm im\varphi}.$$

Here we need both particular solutions I and N of the Bessel differential

equation (we could, of course, consider H^1 and H^2 instead) in order to be able to satisfy the two boundary conditions:

$$(4a) \quad \begin{aligned} I_m(ka) + c N_m(ka) &= 0, \\ I_m(kb) + c N_m(kb) &= 0. \end{aligned}$$

Here too there exists an infinite number of different k_{nm} with their associated c_{nm} . This problem, too, is simply degenerate when $m > 0$, since, according to (4), there are then two different u_{nm} for each k_{nm} .

For the circular sector $0 \leq r \leq a$, $0 \leq \varphi \leq \alpha$ we set

$$(5) \quad u = I_\mu(kr) \sin \mu \varphi, \quad \mu = m \frac{\pi}{\alpha},$$

where the k are determined by the condition $I_\mu(ka) = 0$. Infinitely many eigen values $k = k_{nm}$ exist; the problem is not degenerate.

The most general region which can be treated in this manner is the circular ring sector $b \leq r \leq a$, $0 \leq \varphi \leq \alpha$, which is bounded by two circular arcs and two radii.

c) *Ellipse and elliptic-hyperbolic curvilinear quadrangle.* The wave equation (1) written in elliptic coordinates ξ, η can be separated (see v.II exercise IV.3) and leads to a so-called Mathieu equation in each coordinate. The solution $\xi = \text{const}$ yields the ellipses which belong to the family of curves; $\eta = \text{const}$ yields the hyperbolas of the family. For the full ellipse we have, in addition to the boundary condition $u = 0$, a condition of continuity for $\xi = 0$ (focal line) and the condition of periodicity for $\eta = \pm \pi$. The determination of the eigen values leads to complicated transcendental equations which we cannot discuss here. The most general region of this kind is the curvilinear quadrangle whose boundary consists of two elliptic arcs and two arcs of hyperbolas which are confocal with the former.

The simple examples which we considered here are special cases of the *fundamental theorem* of the theory of oscillating systems with infinitely many degrees of freedom and their eigenfunctions: *For an arbitrary region an infinite sequence of eigen values k exists for which there is a solution of the corresponding differential equation $\Delta u + k^2 u = 0$ which is continuous in the interior of the region and satisfies the boundary condition $u = 0$ (or any of the other boundary conditions on p. 63).*¹ The problem of finding a rigorous proof for this theorem has repeatedly challenged the ingenuity of mathematicians, starting with Poincaré's great work (*Rendic. Circ. Math. di Palermo*, 1894) and culminating in the

¹ The same theorem holds for the eigen value λ of the general self-adjoint differential equation $\Delta u + \lambda F(x, y) u = 0$, $F > 0$.

Fredholm-Hilbert theory of integral equations. Here we must be satisfied with proving the related theorem for mechanical systems with a finite number of degrees of freedom: *A system with f degrees of freedom which is in stable equilibrium, can have exactly f linearly independent small (or more precisely, infinitely small) sine-like oscillations about this state.*

We write the kinetic energy for the neighborhood of a state of equilibrium $q_1 = q_2 = \dots = q_f = 0$ in the form:

$$T = \frac{1}{2} \sum \sum a_{nm} \dot{q}_n \dot{q}_m.$$

Because q is so small we consider the a_{nm} as constants. At the same time the potential energy V becomes a quadratic form in the q_n with constant coefficients since the linear terms $\partial V / \partial q_n$ vanish in the expansion of V in terms of the q_i around the state of equilibrium

$$V - V_0 = \frac{1}{2} \sum \sum b_{nm} q_n q_m.$$

Now it is always possible to transform *both* the above quadratic forms simultaneously into sums of squares by a linear transformation (transformation to principal axes of quadratic surfaces). Performing this transformation we obtain:

$$T = \frac{1}{2} \sum a_n \dot{x}_n^2, \quad V - V_0 = \frac{1}{2} \sum b_n x_n^2.$$

The new coordinates x_n are called *normal coordinates* of the system. According to the Lagrange equation we then have

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_n} = - \frac{\partial V}{\partial x_n}, \quad \text{hence} \quad a_n \ddot{x}_n = - b_n x_n.$$

T is a positive definite quadratic form and so is $V - V_0$ for a stable equilibrium; hence the a_n and b_n are positive. Thus for every normal coordinate we obtain a stable oscillation

$$x_n = c_n e^{i\omega_n t} \quad \text{with} \quad \omega_n^2 = \frac{b_n}{a_n} > 0,$$

which gives as many oscillations as there are degrees of freedom. In the limit $f \rightarrow \infty$ there corresponds an eigen value k_n to every ω_n , and to the totality of q_1, \dots, q_n that belong to the individual x_n there now corresponds the eigenfunction u_f . The k and the ω_n are both *real*.

We point out that the fact that the k are real can also be proved directly from the differential equation without passing to the limit. If a k were complex then the corresponding u would be complex and the conjugate function u^* would have to satisfy the conjugate differential equa-

tion $\Delta u^* + k^{*2} u^* = 0$ with the boundary condition $u^* = 0$. From Green's theorem

$$(6) \quad \int (u \Delta u^* - u^* \Delta u) d\sigma = \int \left(u \frac{\partial u^*}{\partial n} - u^* \frac{\partial u}{\partial n} \right) ds,$$

where the right side vanishes due to the boundary conditions; it follows that

$$(k^2 - k^{*2}) \int u u^* d\sigma = 0.$$

But uu^* is always ≥ 0 ; hence the integral cannot vanish and hence we must have $k = k^*$, and k must be real. The physical meaning of the real character of the k is that under our conditions the oscillation process is always free from absorption.

Up to now we have assumed our problem to be *non-degenerate*. However, for the perturbation theory of wave mechanics the degenerate cases are of special interest. We return to our example of the rectangle and no longer assume the sides a, b , to be incommensurable. This is certainly the case for the *square* $a = b$. Then we obtain from (2a)

$$k_{nm} = \frac{\pi}{a} \sqrt{n^2 + m^2}, \quad \text{hence} \quad k_{nm} = k_{mn};$$

but according to (2) we have $u_{nm} \neq u_{mn}$, unless $n = m$, namely,

$$u_{nm} = \sin n \pi \frac{x}{a} \sin m \pi \frac{y}{a},$$

but

$$u_{mn} = \sin m \pi \frac{x}{a} \sin n \pi \frac{y}{a},$$

All oscillations with $n \neq m$ are therefore (at least) *simply degenerate*, since two different types of oscillations u_{nm} and u_{mn} correspond to the same k_{mn} . Only the basic oscillation k_{11} and its overtones $k_{nn} = n k_{11}$ (which in this special case are harmonic) are *non-degenerate*.

Let us examine somewhat more closely the cases $n = 1, m = 2$ and $n = 2, m = 1$ (hence $k_{12} = k_{21} = \sqrt{5} \pi/a$). In Figs. 23 and 24 we characterize the corresponding eigenfunctions by their nodal lines. These are the lines $u = 0$ in which powder strewn on the membrane would collect. Together with u_{12} and u_{21} we have, belonging to $k_{12} = k_{21}$, the eigenfunctions

$$(7) \quad u = u_{12} + \lambda u_{21},$$

where λ is an arbitrary constant. By a continuous

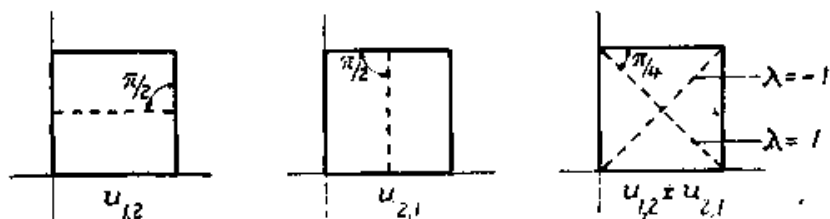


Fig. 23. Simple degeneration in the case of a quadratic membrane for $n = 1$, $m = 2$ or $m = 1$, $n = 2$. The diagonals are the nodal lines for $\lambda = \pm 1$.

deformation of λ the form of the nodal lines within the family (7) is continuously deformed. We compute the linear combinations with $\lambda = \pm 1$:

$$\begin{aligned} u &= \sin \pi \frac{x}{a} \sin 2\pi \frac{y}{a} \pm \sin 2\pi \frac{x}{a} \sin \pi \frac{y}{a} \\ &= 2 \sin \pi \frac{x}{a} \sin \pi \frac{y}{a} \left(\cos \pi \frac{y}{a} \pm \cos \pi \frac{x}{a} \right). \end{aligned}$$

From the last expression here we see that the diagonal $y = x$ is a nodal line of $\lambda = -1$ while the other diagonal $y = a - x$ belongs to $\lambda = +1$. Fig. 24 shows the behavior of the lines for arbitrary values of the parameter λ .

Under certain conditions higher degenerations occur in the case of the quadratic membrane. For example, if we have

$$n_1^2 + m_1^2 = n_2^2 + m_2^2;$$

then for the eigen value

$$k = \frac{\pi}{a} \sqrt{n_1^2 + m_1^2} = \frac{\pi}{a} \sqrt{n_2^2 + m_2^2}$$

we have four linearly independent eigenfunctions

$$u_{n_1, m_1}, u_{n_2, m_1}, u_{m_1, n_1}, u_{m_2, n_1}.$$

Hence we have a case of triple degeneration. The higher degeneration here depends on whether or not a number can be expressed as the sum of two squares in more than one way, as for example

$$65 = 1^2 + 8^2 = 4^2 + 7^2.$$

According to Gauss' *Disquisitiones Arithmeticae* this is the case for every sum of two squares among whose prime factors there are at least two

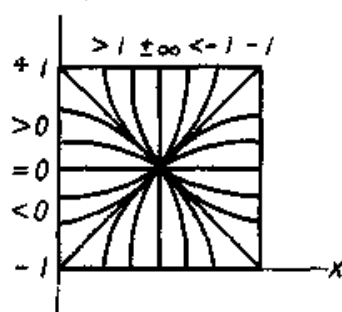


Fig. 24. Total picture of the possible nodal lines for the quadratic membrane. The numbers on the left and on top are the values of the parameter λ in (7).

different ones of the form $4n + 1$. Such primes permit the complex decomposition

$$4n + 1 = (a + bi)(a - bi)$$

with integral a, b ; and the different groupings of the complex factors lead to different representations as a sum of squares. In our example $65 = 5 \cdot 13$ we have

$$5 = (1 + 2i)(1 - 2i) \text{ and } 13 = (2 + 3i)(2 - 3i)$$

and hence

$$65 = \begin{cases} (1 + 2i)(2 + 3i) \cdot (1 - 2i)(2 - 3i) = (-4 + 7i)(-4 - 7i) \\ \quad = 4^2 + 7^2, \\ (1 + 2i)(2 - 3i) \cdot (1 - 2i)(2 + 3i) = (8 + i)(8 - i) \\ \quad = 8^2 + 1^2. \end{cases}$$

For any two eigenfunctions u, u' with $k \neq k'$ we have the *orthogonality theorem*

$$(8) \quad \int u u' d\sigma = 0$$

as a result of Green's theorem. The proof is the same as in (6) if we replace u^* by u' . But this deduction fails if u and u' belong to the same degenerate state, in other words if $k = k'$.

In order to avoid cumbersome considerations of special cases, it is desirable to force orthogonality also in the degenerate cases. It will prove convenient to introduce the abbreviation of Courant-Hilbert² for the integral in (8):

$$(8a) \quad \int u u' d\sigma = (u, u').$$

In §26 we shall return to a discussion of the connection between this expression and the scalar product in ordinary vector analysis. We call the integral in (8a) the "scalar product" of u and u' .

We first prove the theorem that n continuous, real, mutually orthogonal but otherwise arbitrary functions $u_1, u_2, \dots, u_r, \dots, u_n$ are *linearly independent*. For if there existed an equation of the form

$$\sum_{r=0}^n c_r u_r = 0 \quad \text{with} \quad (u_\mu, u_\nu) = 0 \quad \text{for all} \quad \mu \neq \nu,$$

then by the "scalar multiplication" of the equation by u_μ we would obtain

² Courant-Hilbert, *Methoden der mathematischen Physik*, 2nd ed., Springer, Berlin 1931, Chapter II.

$$c_\mu(u_\mu, u_\mu) = 0, \text{ hence } c_\mu = 0 \text{ for all } \mu,$$

which contradicts the assumption of linear dependence.

We now proceed step by step and first treat the case of *simple degeneracy*. Let u_1, u_2 be continuous, real, not necessarily orthogonal functions belonging to the same eigen value. We consider the family

$$u = c_1 u_1 + c_2 u_2$$

and consider the member which is orthogonal to u_1 . This member is given by the condition

$$0 = (u_1, u) = c_1 (u_1, u_1) + c_2 (u_1, u_2).$$

We satisfy this condition by setting

$$(9) \quad c_1 = -(u_1, u_2), \quad c_2 = (u_1, u_1)$$

where $c_2 \neq 0$ and hence $u \neq 0$. In u_1 and u we have two mutually orthogonal eigenfunctions of the family, which we choose as the representatives of the family instead of u_1, u_2 . We now can normalize u by multiplication with a constant factor such that

$$(9a) \quad (u, u) = (u_1, u_1).$$

For *twofold degeneracy* let u_1, u_2 be two functions that are normalized according to (9) and (9a), and let u_3 be a function of the same eigen value that is not necessarily orthogonal to the first two. We consider the family

$$u = c_1 u_1 + c_2 u_2 + c_3 u_3$$

and select the member of the family that is orthogonal to both u_1 and u_2 , thus obtaining the conditions

$$0 = (u_1, u) = c_1 (u_1, u_1) + c_3 (u_1, u_3),$$

$$0 = (u_2, u) = c_2 (u_2, u_2) + c_3 (u_2, u_3).$$

We satisfy both conditions by setting

$$(10) \quad c_1 = -(u_1, u_3), \quad c_2 = -(u_2, u_3), \quad c_3 = (u_1, u_1) = (u_2, u_2).$$

The functions u_1, u_2, u are mutually orthogonal and hence linearly independent; furthermore we can normalize u so as to obtain:

$$(10a) \quad (u, u) = (u_1, u_1) = (u_2, u_2).$$

We thereby obtain the desired orthogonalization for twofold degeneracy.

This process can obviously be continued in the case of higher

degeneracy. The degenerate eigenfunctions are thus made mutually orthogonal; due to (8) they are already orthogonal to the eigenfunctions which belong to different k .

To the orthogonality condition (8) we add the *normality condition*

$$(11) \quad (u, u) = \int u^2 d\sigma = 1$$

This "normalization to 1" leads to a certain simplification of the orthogonalization process above (see, e.g. (10a)). We shall see in §26 that (11) also has its vector-analytic analog. We still must mention that for complex u equation (11) must be amended to read

$$(11a) \quad (u u^*) = \int u u^* d\sigma = 1$$

and that in separable problems the normalization is best carried out for each individual factor. Thus in (2) we have to multiply the sine functions by the factors

$$(12) \quad \sqrt{\frac{2}{a}} \quad \text{and} \quad \sqrt{\frac{2}{b}} \quad \text{respectively}$$

and in (3) we have to multiply the exponential function and the Bessel function by the factors

$$(12a) \quad \frac{1}{\sqrt{2\pi}} \quad \text{and} \quad \frac{\sqrt{2}}{a I'_m(ka)} \quad \text{respectively}$$

(the latter is due to (20.9a)). Thus our solutions in (2) and (4) at the beginning of this section are determined also with respect to their amplitudes.

From the above-mentioned examples we deduce two theorems concerning nodal lines, which we shall prove now for membranes with arbitrary boundaries:

1. If several nodal lines intersect at a point then they intersect at equal angles (isogonally): for two such lines the angle is $\pi/2$, for ν lines it is π/ν .

2. The larger the eigen value k , the finer the subdivision of the membrane into regions of alternating signs; for $k \rightarrow \infty$ the nodal lines become everywhere dense.

In order to demonstrate that theorem 1 holds for our special examples we refer to Figs. 23 and 24, where the boundary itself must be considered a nodal line and the angles are $\pi/2$ and $\pi/4$ as shown. In the case of the full circle we see from (3) that there are m radial lines intersecting at its center at an angle of π/m . In order to show that theorem 2 is satisfied in our cases it suffices to note that for the case of the rectangle and the eigen value k_{nm} the rectangle is subdivided into sub-

rectangles of sides a/n and b/m , so that for $k \rightarrow \infty$ at least one side approaches zero.

For the proof of theorem 1 we develop u in the neighborhood of the point O in a Fourier series. We use an r, φ - coordinate system whose origin is at O . For any shape of the membrane we obtain the expansion

$$(13) \quad u = \sum_n I_n(kr) (a_n \cos n\varphi + b_n \sin n\varphi)$$

which converges in a certain neighborhood of O , where the a, b are determined coefficients which can be computed from the given u . The fact that the radial functions in the Fourier expansion must be the Bessel functions I_n follows from differential equation (1) and the regularity of u at O . Now if there is to exist *at least* one nodal line through the point O ($r = 0$), then according to (13)

$$0 = I_0(0) a_0, \text{ and hence } a_0 = 0.$$

Then if a_1 and b_1 are not both zero there is *only* one line through O whose direction is determined by the equation:

$$0 = I_1(kr) (a_1 \cos \varphi + b_1 \sin \varphi).$$

Hence for $r > 0$

$$\tan \varphi = -\frac{a_1}{b_1}.$$

This determines the direction of the nodal line *uniquely*.

If there is to be more than one nodal line through O then we must have $a_1 = b_1 = 0$. If we do not at the same time have $a_2 = b_2 = 0$ then according to (13) we have

$$0 = I_2(kr) (a_2 \cos 2\varphi + b_2 \sin 2\varphi)$$

or, if there are to be ν nodal lines through O , and hence all a, b up to but not including a_ν, b_ν vanish, then we have

$$0 = I_\nu(kr) (a_\nu \cos \nu\varphi + b_\nu \sin \nu\varphi).$$

In the latter case we have for $r > 0$

$$(13a) \quad \tan \nu\varphi = -\frac{a_\nu}{b_\nu}.$$

The right side of this equation is given by our Fourier expansion and shall be denoted by $\tan \alpha$. The general solution of (13a) is then:

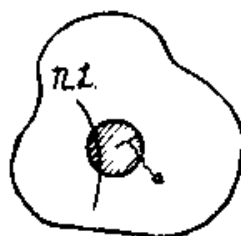
$$(13b) \quad \varphi = \alpha, \quad \alpha + \frac{\pi}{\nu}, \quad \alpha + \frac{2\pi}{\nu}, \quad \dots, \quad \alpha + \frac{(\nu-1)\pi}{\nu}.$$

These angles differ by the constant amount π/ν , which proves the isogonality.

Passing to the proof of theorem 2, we consider two functions u, v where u is a solution of (1) that satisfies the given boundary condition and v is the special solution

$$v = I_0(kr).$$

Fig. 25. With increasing k the nodal lines become denser and denser regardless of the shape of the membrane. The proof is given by considering a small disc anywhere on the membrane whose radius a decreases to zero for increasing k . N.L. stands for a nodal line which intersects the disc.



The value of k which is common to u and v is assumed to be large. With the help of this large k we define a small length a by setting $ka = \varrho_1$ where ϱ_1 is the first root of the equation $I_0(\varrho) = 0$. We consider a circular disc of radius a situated anywhere on the nodal line pattern of the eigenfunction u (see Fig. 25). With this disc as our domain of integration we apply Green's theorem:

$$(14) \quad \int (u \Delta v - v \Delta u) d\sigma = \int \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds.$$

The left side vanishes since both u and v satisfy the differential equation (1) with the same k . On the right side we have for $r = a$

$$v = 0 \text{ and } \frac{\partial v}{\partial n} = k I'_0(\varrho_1) \neq 0.$$

If we set $ds = a d\varphi$ then equation (14) becomes

$$\varrho_1 I'_0(\varrho_1) \int_0^{2\pi} u d\varphi = 0,$$

and hence

$$(15) \quad \int_0^{2\pi} u d\varphi = 0.$$

According to this u assumes both positive and negative values on the circumference of the disc. Hence there must be at least two zeros of u on the circumference; that is, our disc must be intersected by at least one nodal line. The disc becomes smaller as k becomes larger and hence for increasing eigen value k the nodal lines become *arbitrarily dense*. This holds for every part of the nodal line pattern.

§ 26. General Remarks Concerning the Boundary Value Problems of Acoustics and of Heat Conduction

The eigenfunctions of the oscillating membrane can be adapted directly to the spatial case. Here we do not think of an oscillating rigid body, but (in order to avoid all complications involving vectors and tensors) rather of an oscillating air mass in the interior of a closed rigid hull of finite extension. Just as on p.166, we interpret the scalar function u as the velocity potential of the air oscillations and we again set the boundary condition $\partial u / \partial n = 0$.

For the rectangular solid with side lengths a, b, c we have, in analogy to (25.2),

$$(1) \quad u = u_{nml} = \cos n \pi \frac{x}{a} \cos m \pi \frac{y}{b} \cos l \pi \frac{z}{c}$$

with eigen value

$$(1a) \quad k = k_{nml} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{l^2}{c^2}}.$$

This state is non-degenerate if a, b, c are incommensurable.

For a sphere of radius a we obtain the general eigenfunction in analogy to (25.3):

$$(2) \quad u = u_{nlm} = \psi_n(k_{nl}r) P_n^m(\cos \vartheta) e^{im\varphi}$$

Under our boundary condition the eigen value is given by

$$(2a) \quad \psi'_n(k_{nl}a) = 0,$$

where k_n is the l -th root of this equation. This state is $2n$ -fold degenerate, since k_{nl} is independent of m and the different states P_n^m for upper index $-n \leq m \leq +n$ belong to the same k_{nl} .

Also in this category are the eigenfunctions of the circular cylinder ($0 < r < a, 0 < z < h$) which we derived in §20 C. With the boundary condition $\partial u / \partial n = 0$ they are given by

$$(3) \quad u_{nlm} = I_n(\lambda r) e^{\pm i n \varphi} \cos m \pi \frac{z}{h};$$

the corresponding eigen value is determined from the equation $I'_n(\lambda a) = 0$, the l -th solution of which we denote by λ_{nl} . Therefore

$$(3a) \quad k_{nlm}^2 = \lambda_{nl}^2 + m^2 \pi^2 / h^2.$$

Due to the factor $\exp(\pm i n \varphi)$ in (3) this state is simply degenerate for $n > 0$.

We now consider these eigenfunctions "normalized to 1" where we have to keep in mind the remarks on pp. 173, 174. Then for example in (1) we have to replace $\cos n\pi x/a$ by

$$\cos n\pi \frac{x}{a} \sqrt{\frac{2}{a}}$$

and according to (22.31b) we have to replace P_n^m in (2) by

$$P_n^m = P_n^m \sqrt{(n + \frac{1}{2}) \frac{(n - m)!}{(n + m)!}}$$

etc. (see exercise V.1).

We now generalize the fundamental theorem on p. 169 and its (mathematically non-rigorous) proof to the case of an arbitrary spatial region S . The theorem now reads: *There exists an infinite system of eigenfunctions*

$$u_1, u_2, \dots, u_n, \dots,$$

whose elements are regular in the interior of S and satisfy the differential equation

$$\Delta u_n + k_n^2 u_n = 0,$$

as well as a homogeneous boundary condition. The corresponding eigenvalues

$$k_1, k_2, \dots, k_n, \dots,$$

ordered in an increasing sequence, are infinite in number and increase to infinity; if S is bounded then they form a "discrete spectrum" and they are real since the differential equation was assumed free from absorption.

This system of eigenfunctions satisfies the conditions of orthogonality and of normality:

$$(4) \quad \int u_n u_m d\tau = \delta_{nm},$$

which according to (25.8a) can be written as

$$(4a) \quad (u_n, u_m) = \delta_{nm}$$

or for complex eigenfunctions

$$(4b) \quad (u_n, u_m^*) = \delta_{nm}.$$

If the system of u_n is complete (see p. 5) then we claim that any continuous point function f given on S can be expanded in the u_n :

$$(5) \quad f = \sum A_n u_n.$$

If this expansion is possible then, according to (4b), we obtain from (5) through termwise integration

$$(5a) \quad A_n = \int \int u_n^* d\tau.$$

That this expansion is possible is postulated by the *Ohm-Rayleigh principle*, which we shall assume in the following discussion without presenting its mathematical proof. In connection with the name of this principle we remark: Georg Simon Ohm was not only the discoverer of the basic law of Galvanic conduction, but also did profound research in acoustics. He found that the differences in the tone-color of different musical instruments are the result of differences in the mixture of basic tone and overtones. Since, according to (25.1), the overtones ω_n are related to the k_n , and since they are harmonic with the basic tone only for strings and organ pipes, so the construction of an arbitrary tone-color means the construction of an arbitrary function from the (in general anharmonic) eigen values. In Lord Rayleigh's classic book, *Theory of Sound*, this principle is generalized in the sense of equation (5) and is applied in many directions.

We shall now make some remarks about so-called *Hilbert space*, not only to justify the notation (u_n, u_m) of (4a,b) which is reminiscent of vector analysis, but also to give the Ohm-Rayleigh principle an elegant geometric interpretation, which in the hands of the Hilbert school has even been worked out as a means of proving this principle.

In accord with Courant-Hilbert (see p. 172) we define, in a space of N dimensions, the basis vectors e_1, e_2, \dots, e_N (corresponding to the i, j, k of three-dimensional vector analysis) which lie in the coordinate directions x_1, x_2, \dots, x_N and whose scalar product is to satisfy the condition

$$(6) \quad (e_n e_m) = \delta_{nm}$$

We further consider a vector which forms the angles $\alpha_1, \alpha_2, \dots$ with the coordinate axes

$$(7) \quad \mathbf{a} = \cos \alpha_1 e_1 + \cos \alpha_2 e_2 + \dots + \cos \alpha_N e_N$$

and we call it a *unit vector* if the scalar product of \mathbf{a} with itself has the value 1:

$$(7a) \quad (\mathbf{a}, \mathbf{a}) = \sum_{n=1}^N \cos^2 \alpha_n = 1.$$

A second unit vector \mathbf{b} with direction angles β_n is called *orthogonal* to

a if the scalar product of \mathbf{a} and \mathbf{b} vanishes:

$$(7b) \quad (\mathbf{a}, \mathbf{b}) = \sum_{n=1}^N \cos \alpha_n \cos \beta_n = 0.$$

Equations (7a,b) are seen to be generalizations of well-known formulas from three-dimensional analytic geometry.

In the limit $N \rightarrow \infty$ we now obtain Hilbert space. Here we note a formal analogy between the basis vectors \mathbf{e}_n and the elements u_n of our system of eigenfunctions. The relations between the latter as written in the form (4a) are formally the same as the relations (6) between the \mathbf{e}_n . The system u_n , if it is complete, can serve as substitute for the basis \mathbf{e}_n . The same is true for the u_n^* in the case of complex u_n . Every other system of functions that is orthogonalized and normalized to 1 can be composed from the u_n in the sense of equation (7) and can be visualized as a *vector* in Hilbert space. Two such vectors can be transformed into each other by a *rotation* of Hilbert space. But according to (5) any function f is composed of the u_n . With the system of coordinates which is formed by the u_n the function f is associated by (5) to a certain *point* of Hilbert space. The coordinates of this point as measured in the system u_n are the expansion coefficients A_n . Hilbert space thus becomes a *function space*. The association between the arbitrary functions and the points of the space of infinitely many dimensions is one-to-one. If we join the point which represents the function f to the origin of the coordinate system of the u_n , then this infinite dimensional vector represents the function f . According to (5a), which we can write in the form $A_n = (f, u_n^*)$, the coordinates of the representative point are the projections of the representative vector on the axes of the system of u_n^* .

From these highly abstract generalizations we return to the physical applications. For the time being we restrict ourselves to the simple problems of acoustics and heat conduction in their historical form. We defer the questions of wave mechanics to the end of this chapter.

The general problem of *acoustics* for the interior of an arbitrary shell S is the following: the wave equation

$$(8) \quad \frac{\partial^2 v}{\partial t^2} = c^2 \Delta v, \quad c = \text{speed of sound},$$

is to be solved with the boundary condition $\partial v / \partial n = 0$ so that for $t = 0$ the functions v and $\partial v / \partial t$ become equal to arbitrary prescribed functions v_0 and v_1 in S . This problem is solved by:

$$(9) \quad v = \sum A_n u_n \cos \omega_n t + \sum B_n u_n \sin \omega_n t,$$

where the A_n and B_n are to be determined so that

$$(9a) \quad v_0 = \sum A_n u_n \quad \text{and} \quad v_1 = \sum B_n \omega_n u_n$$

Due to the relation of the ω_n with the eigen values k_n (namely $c = \omega_n/k_n$) the second equation can be rewritten as:

$$(9b) \quad \frac{v_1}{c} = \sum B_n k_n u_n.$$

From this we obtain as in (5a)

$$(9c) \quad A_n = \int v_0 u_n^* d\tau, \quad B_n = \frac{1}{k_n c} \int v_1 u_n^* d\tau.$$

We see that this is an *initial value problem*; the *boundary value problem* has been shifted to the u_n .

The general *heat conduction problem* can be solved in the same manner. The difference is that now *one* arbitrary function v_0 suffices to describe the initial state, the initial temperature variation $\partial v/\partial t$ being determined by the differential equation of heat conduction. As a boundary condition we may use any one of the conditions a), b), c) on p. 63, to which we then also subject the eigenfunctions u_n .

We now set

$$(10) \quad v = \sum A_n u_n e^{-\kappa k_n^2 t}$$

where κ stands for temperature (not heat) conductivity. The coefficients A_n are again determined by the initial condition $v = v_0$:

$$(10a) \quad A_n = \int v_0 u_n^* d\tau.$$

In addition to this initial condition the function v satisfies the differential equation (25.1) and the boundary condition to which the u_n are subjected.

The potential equation $\Delta u = 0$ has *no* eigenfunctions, or rather every solution which is regular in the interior of S and which satisfies the boundary condition $u = 0$ or $\partial u/\partial n = 0$ must be zero or constant in the interior of S . Hence there can be here no closed "nodal surfaces" $u = 0$ or $\partial u/\partial n = 0$. However, in the next section we shall construct a solution of the general potential boundary value problem (given values $u = U$ on the boundary) from the eigenfunctions of the wave equation.

A solution of the potential equation which is regular in S can also have no maximum or minimum in the interior of S . Extremal values of u can be assumed only on the boundary of S . This follows from Gauss' theorem on the *arithmetic mean* which can be deduced from Green's theorem (see exercise V.2).

Also, no eigenfunctions exist for the differential equation $\Delta u - k^2 u = 0$ or the more general $\Delta u - F u = 0$ for positive $F(x, y, z)$ (see exercise II.2).

§ 27. Free and Forced Oscillations. Green's Function for the Wave Equation

The eigenfunctions correspond to free oscillations; in a non-absorbing medium they need no energy supply. We now wish to consider *forced* oscillations, which must be stimulated in the rhythm of their period in order to be able to continue in their purely periodic state. Just as the free oscillations, they are to satisfy a homogeneous surface condition, e.g., $u = 0$; the region S will be assumed to be bounded in the discussions in this section. The measure of stimulation shall, for the time being, be assumed to be a continuous point function in the interior of S , and in analogy to the Poisson equation of potential theory we denote it by ϱ .³ Correspondingly we write the differential equation of forced oscillations as:

$$(1) \quad \Delta u + k^2 u = \varrho.$$

Here $k = \omega/c$, as we remarked in (26.9a), where ω is the circular frequency of the stimulation and c is the speed of sound. We assume

$$(2) \quad k \neq k_n,$$

i.e., k is different from every eigen value of the region S for the same boundary condition. The case of "resonance" $k = k_n$ will be treated at the end of this section.

According to the Ohm-Rayleigh principle we can expand ϱ in terms of the normalized u_n as in (26.5) and (26.5a):

$$(3) \quad \varrho = \sum A_n u_n, \quad A_n = \int \varrho u_n^* d\tau,$$

we also write the solution u of (1) in the same form:

$$(3a) \quad u = \sum B_n u_n.$$

Substituting these expansions in (1) and considering the differential equations $\Delta u_n + k_n^2 u_n = 0$, which differ from (1) and which are satisfied by the eigenfunctions u_n , by equating the coefficients of u_n on both sides we obtain

³ The function ϱ does not represent charge density as in potential theory, but is of dimension sec^{-1} if u stands for an acoustic velocity potential.

$$(4) \quad B_n = \frac{A_n}{k^2 - k_n^2}, \quad u = \sum \frac{A_n u_n}{k^2 - k_n^2}.$$

We now consider the special case in which ϱ is a δ -function⁴ and hence the stimulation is limited to a simple source point Q of yield 1 (see §10 C). We then have

$$\int_Q \varrho \, d\tau = 1,$$

for a domain of integration which contains the point Q , and

$$\int \varrho \, d\tau = 0.$$

for a domain of integration which does not contain Q . Hence we obtain from (3)

$$(4a) \quad A_n = u_n^*(Q) \int_Q \varrho \, d\tau = u_n^*(Q)$$

and from (4)

$$(5) \quad G(P, Q) = \sum \frac{u_n(P) u_n^*(Q)}{k^2 - k_n^2}.$$

where the u of (4) is now denoted by the more suggestive $G(P, Q)$. Indeed this solution is *Green's function of our differential equation (1) for arbitrary positions of the action point P and the source point Q and an arbitrary region S* . We assume only the complete system of eigenfunctions and eigen values for the region S . It should be noted that the Ohm-Rayleigh principle has not been applied to the singular δ -function, but only to the continuous function ϱ of (3), which may, e.g., be taken as a regular Gauss error function. Hence in our derivation we do not need the expansion in terms of the u_n of an arbitrary function but only of certain special everywhere regular functions. In the same manner the termwise differentiation which was needed in the derivation of (4) has been carried out on the regular function (3a) before passage to the limit and not on the limit (5).

Green's function is also the solution of an *integral equation*. In order to demonstrate this we recall equation (10.13a), which holds for every self-adjoint differential expression $L(u)$ and hence in particular for the wave equation $\Delta u + k^2 u$. For the three-dimensional case and the boundary value $u = 0$ it reads:

$$(6) \quad u_Q = \int \varrho(P) G(P, Q) \, d\tau_P.$$

⁴ We have dropped the name "peak function" ("Zackenfunktion") which was introduced by the author (see *Jahresber. Deutschen Math. Vereinigung* 21, 312, 1912) in favor of Dirac's notation " δ -function."

The function $G(P, Q)$ is called the “kernel” of the integral equation. Corresponding to the reciprocity theorem d) on p. 1, which, for complex G , has to be rewritten as

$$(6a) \quad G(P, Q) = G^*(Q, P),$$

we call G a “symmetric kernel.” From the structure of (5) we see directly that (6a) is satisfied.

The convergence of the series in (5) is absolute only in the one-dimensional case; in the two- or more dimensional case the convergence is conditioned by the alternation of signs of the eigenfunctions for a suitable arrangement of the series. This is the reason equation (5) does not appear explicitly in Hilbert’s theory of integral equations, but in an integrated form in which it converges absolutely. In the one-dimensional case equation (5) has been rigorously proven by Erhardt Schmidt.⁵

The non-absolute convergence of (5) becomes apparent if we try to show by termwise differentiation that the differential equation (1) is satisfied. For then we obtain from the n -th term

$$\Delta u_n + k^2 u_n = \Delta u_n + k_n^2 u_n + (k^2 - k_n^2) u_n = (k^2 - k_n^2) u_n$$

and cancelling the factor $k^2 - k_n^2$ with the denominator and summing with respect to n we obtain

$$(6b) \quad \Delta G + k^2 G = \sum u_n(P) u_n^*(Q).$$

For $P = Q$ the sum on the right side consists of positive terms and diverges, as it should; the fact that it converges for $P \neq Q$ and vanishes throughout is caused by the alternating signs and cannot be proven from this representation. The order of increase for $P \rightarrow Q$ can be deduced directly from the differential equation (1) as follows. We consider a sphere with small radius r and center Q , and integrate (1) over its interior. Due to the δ -character of ϱ the right side becomes equal to 1. According to Gauss’ theorem the first term on the left side becomes

$$\int \frac{\partial G}{\partial r} d\sigma = 4\pi r^2 \frac{\partial G}{\partial r},$$

while the second term vanishes. Hence we have

$$(7) \quad \frac{\partial G}{\partial r} = \frac{1}{4\pi r^2}, \quad G = -\frac{1}{4\pi r} + \text{Const} \quad \text{for } r \rightarrow 0.$$

This expresses the fact that $G(P, Q)$ has a *unit source* in the point $P = Q$.

The above formulas can be interpreted best in Hilbert space (see

⁵ In his famous dissertation, Göttingen, 1905.

p. 179). Namely, equation (6b) states that $\Delta G + k^2 G$ is the scalar product of the two unit vectors $u(P)$ and $u^*(Q)$. Hence these unit vectors are orthogonal if $u(P)$ and $u(Q)$ are different ($P \neq Q$); if $u(P)$ and $u(Q)$ are equal ($P = Q$) then orthogonality is of course excluded; instead the product becomes infinite. The expression (5) is constructed from the individual terms of the same product with the "resonance denominator" $k^2 - k_n^2$ as weighting factor.

Despite its poor convergence equation (5) has frequently been found useful in wave mechanical computations (see §30). For the time being we apply it in order to close a gap in the theory of spherical harmonics. But first we make a few preparatory remarks:

1. If the system of eigenfunctions is *separable* then the summation in (5) decomposes into three summations corresponding to the three coordinates. For the rectangular solid we should have:

$$(8) \quad \Sigma = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty}$$

where n, m, l are as in (26.1).

2. Green's function depends only on the position of the points P, Q relative to the boundary surface σ and on their distance R . It is independent of the orientation of the coordinates in space. A transformation of the coordinate system which transforms the surface σ into itself and leaves R fixed leaves $G(P, Q)$ *invariant*.

3. If σ is the surface of a sphere then the condition of invariance is satisfied for every rotation of the *spherical polar system* r, ϑ, φ with $r = 0$ as the center of the sphere. The coordinates r, ϑ, φ shall be those of P , $r_0, \vartheta_0, \varphi_0$ those of Q .

4. In the latter case we face the additional fact that the system of eigenfunctions (26.2) is *degenerate*, since the eigen value k_n as defined by (26.2a) is independent of m . Writing G as a triple sum in analogy to (8), we can take the denominator $k^2 - k_n^2$ and the radial part of the eigenfunctions in front of the summation over m . Hence we have

$$(9) \quad G(P, Q) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Psi_n(k_n, r) \Psi_n(k_n, r_0)}{k^2 - k_n^2} Y_n,$$

$$(9a) \quad Y_n = \sum_{m=-n}^{+n} \Pi_n^m(\cos \vartheta) \Pi_n^m(\cos \vartheta_0) e^{im(\varphi - \varphi_0)}.$$

where Ψ_n stands for the function ψ_n of (26.2) normalized to 1, and Π_n for the spherical harmonic P_n normalized in the same manner. The function Y_n is a surface spherical harmonic. In (9a) we have used the fact that, due to the real character of Ψ_n and Π_n^m the *conjugate*

complex of the eigenfunction

$$\Psi_n(k_{nl}r) \Pi_n^m(\cos \vartheta) e^{im\varphi}$$

for the argument $Q = (r_0, \vartheta_0, \varphi_0)$ can be written as

$$\Psi_n(k_{nl}r_0) \Pi_n^m(\cos \vartheta_0) e^{-im\varphi_0},$$

for all values of m between $-n$ and $+n$.

From remark 2 concerning the invariance of G , and from representation (9), we now see that the surface spherical harmonic (9a) has an invariant meaning which is independent of the rotation of the polar coordinate system. But this is the very theorem which we assumed as an axiom for the proof of the addition theorem of spherical harmonics on p. 133. That proof is now completed.

Up to now we have assumed that stimulation of forced oscillation takes place in the *interior* of the region S . We now wish to assume that stimulation takes place from the *surface*. This is the case if, instead of the homogeneous boundary condition $u = 0$, we prescribe the *inhomogeneous* boundary condition

$$(10) \quad u = U.$$

The surface is then held in pulsation with the rhythm ω of the forced oscillation and with the amplitude U which may vary from point to point, while in the interior of S the differential equation (1) holds throughout with $\varrho = 0$. From (10.12) we know that this boundary value problem can be solved with the help of Green's function by the formula

$$(11) \quad u_Q = \int U \frac{\partial G}{\partial \nu_P} d\sigma_P,$$

where the variable of integration on the right side is P and the domain of integration is the surface of S ($d\sigma_P$ = element of surface, $d\nu_P$ = element of normal at the point P). According to (5) equation (11) becomes

$$(11a) \quad u_Q = \sum \frac{u_n^*(Q)}{k^2 - k_n^2} \int U \frac{\partial u_n(P)}{\partial \nu_P} d\sigma_P.$$

This formula contains the general solution of the famous *Dirichlet problem of potential theory*, for by setting $k = 0$ we obtain

$$(12) \quad u_Q = - \sum \frac{u_n^*(Q)}{k_n^2} \int U \frac{\partial u_n(P)}{\partial \nu_P} d\sigma_P.$$

The remarkable fact about this solution is that it is not expanded in

particular solutions of the differential equation $\Delta u = 0$ concerned, but rather in the eigenfunctions of the *wave equation* (there are no eigenfunctions of the potential equation). Equation (12) remains valid if instead of the boundary condition (10) we prescribe the more general condition

$$\frac{\partial u}{\partial n} + h u = U$$

except that in this case we must subject the eigenfunctions u_n to the corresponding homogeneous condition

$$\frac{\partial u}{\partial n} + h u = 0$$

In the special case of a sphere of radius a we obtain from (12) and the boundary condition (10)

$$(13) \quad \frac{2\pi}{a^2} u(r_0, \vartheta_0, \varphi_0) = - \sum_n \sum_l \sum_m \frac{A_{nlm}}{k_{nl}} \Psi_n(k_{nl} r_0) \Psi'_n(k_{nl} a) \Pi_n^m(\cos \vartheta_0) e^{-im\varphi_0},$$

$$(13a) \quad A_{nlm} = \iint U \Pi_n^m(\cos \vartheta) e^{im\varphi} \sin \vartheta d\vartheta d\varphi.$$

where Ψ_n and Π_n have the same meaning as before. The extra factor 2π on the left side of (13) is due to the fact that, as with the Bessel functions and the spherical harmonics, we have to normalize the two functions $\exp(-im\varphi_0)$ and $\exp(im\varphi)$ to 1.

Written in terms of the same variables $Q = (r_0, \vartheta_0, \varphi_0)$ and expanded in terms of particular solutions of $\Delta u = 0$ our solution reads:

$$(14) \quad 2\pi u(r_0, \vartheta_0, \varphi_0) = \sum_n \sum_m A_{nm} \left(\frac{r_0}{a}\right)^n \Pi_n^m(\cos \vartheta_0) e^{-im\varphi_0}.$$

By comparing these solutions we obtain remarkable summation formulas (see exercise V.3).

Finally, we must consider the exceptional case $k = k_m$. From the mechanics and the electrodynamics of oscillating systems we know the "resonance catastrophe": if the rhythm of the stimulating force equals a proper frequency of the system the oscillations increase to infinity. The condition for this event is $\omega = \omega_m$, and hence $k = k_m$. Equation (1) then assumes the form:

$$(15) \quad \Delta u + k_m^2 u = q.$$

Here we have an inhomogeneous equation whose left side coincides with the homogeneous equation of a free oscillation.

For simplicity we first consider the two-dimensional case of the membrane of §25, which now, however, is subjected to a periodically changing transversal pressure⁶ $\varrho = \varrho(x, y)$ with an arbitrary distribution over the membrane. Do pressure distributions exist for which the resonance catastrophe is avoided, that is, for which equation (15) has continuous solutions throughout (for the boundary condition $u = 0$)? The answer to this question is physically evident: for such a solution the pressure on the membrane may do *no work*. Hence we must have:

$$(16) \quad \int \varrho u_m d\sigma = 0.$$

The pressure distribution must be orthogonal to the eigenfunction $u = u_m$ with which it is in resonance, e.g., it may have equal magnitude in oppositely oscillating sectors of the membrane; in particular the pressure along a nodal line may be of arbitrary strength.

This orthogonality theorem is a corner stone in the theory of integral equations and has important applications in the perturbation theory of wave mechanics. Here we must be content with uncovering its physical basis.

The orthogonality theorem can be adapted directly to the three-dimensional case if in (16) we replace the surface integral with respect to $d\sigma$ by a volume integral with respect to $d\tau$. Then we see that the expansion coefficients A_n and B_n in (3) and (4) vanish for $n = m$. By passing from the continuous distribution ϱ to a δ -function we obtain information about Green's function in the case of resonance. From $A_m = 0$ and equation (4a) we have $u_m^*(Q) = 0$. In other words: *The singularity of Green's function must lie on a nodal surface of the critical proper oscillation u_m .*

For this position and only for this position of the source point Q an everywhere regular Green's function exists. The special form of Green's function for the case of resonance is obtained from the general form (5) by omitting the term involving k_m ; it therefore reads:

$$(17) \quad G(P, Q) = \sum_{n \neq m} \frac{u_n(P) u_n^*(Q)}{k_m^2 - k_n^2}$$

§ 28. Infinite Domains and Continuous Spectra of Eigen Values. The Condition of Radiation

With increasing domain the eigen values become closer and closer; for an infinite domain they are dense everywhere; we then deal with a *continuous spectrum of eigen values*.

⁶ More precisely: pressure divided by surface tension T (see equation (25.1)). The dimension of ϱ is not that of pressure dyn/cm.², but $\frac{\text{dyn}}{\text{cm}^2} / \frac{\text{dyn}}{\text{cm}} = \text{cm}^{-1}$.

Let us consider, e.g., the interior of a sphere of radius a for vanishing boundary values. For the case of purely radial oscillations its eigen values are given by the equation

$$(1) \quad \psi_0(k, a) = 0, \quad \psi_0(\varrho) = \frac{\sin \varrho}{\varrho}.$$

Hence $k_\nu a = \nu\pi$ and the difference of successive eigen values is

$$\Delta k_\nu = \frac{\pi}{a} \rightarrow 0 \quad \text{for} \quad a \rightarrow \infty.$$

We may therefore consider the function $\psi_0(kr)$ which is everywhere regular and vanishes at infinity as an *eigenfunction of infinite space*. Thus, if we have an acoustical or an optical problem in which the prescribed sources are in the finite domain (with a discrete or a continuous distribution), and which is to be solved for a given wave number k , then we can always add the function ψ_0 to the solution. Hence oscillation problems (in contrast to potential problems) are *not* determined *uniquely* by their prescribed sources in the finite domain. This paradoxical result shows that the condition of *vanishing* at infinity is not sufficient, and that we have to replace it by a stronger condition at infinity. We call it the *condition of radiation*: the sources must be *sources*, not *sinks*, of energy. The energy which is radiated from the sources must scatter to infinity; *no energy may be radiated from infinity into the prescribed singularities of the field* (plane waves are excluded since for them even the condition $u = 0$ fails to hold at infinity).

For our special eigenfunctions

$$\psi_0(kr) = \frac{1}{2i} \left(\frac{e^{ikr}}{r} - \frac{e^{-ikr}}{r} \right)$$

the state of affairs is simple: for the time dependence $\exp(-i\omega t)$ e^{ikr}/r is a *radiated*, e^{-ikr}/r an *absorbed*, $\psi_0(kr)$ a *standing wave* (nodal surfaces $kr = \nu\pi$). By excluding absorption from infinity we exclude the addition of the eigenfunction $\psi_0(kr)$. Hence the permissible singularities are restricted to the form

$$(1a) \quad u = C \frac{e^{ikr}}{r}$$

For these singularities we have the condition

$$(2) \quad \lim_{r \rightarrow \infty} r \left(\frac{\partial u}{\partial r} - iku \right) = 0,$$

It is called the *general condition of radiation* and we shall apply it to all

acoustic and electrodynamic oscillation problems that are generated by sources in the finite domain.

In fact, condition (2) holds not only for the spherical wave (1a) which radiates from $r = 0$, but it also holds for a stimulation which acts at the point $x = x_0, y = y_0, z = z_0$

$$u = C \frac{e^{ikR}}{R}, \quad R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

Hence, for a continuous stimulation of the spatial density $\varrho = \varrho(x_0, y_0, z_0)$ we have:

$$u = \int \varrho \frac{e^{ikR}}{R} dx_0 dy_0 dz_0.$$

This holds not only for unbounded space, but also in the case where there are bounded surfaces σ on which arbitrary linear boundary conditions are prescribed, whether homogeneous, e.g., $u = 0$, or inhomogeneous, e.g., $u = U$. In the former case we have scattered or reflected radiation emanating from the surface σ , whereas in the latter case we have radiation that is stimulated by the pulsating surface σ itself (see p. 186).

As counterpart to the radiation condition (2) we have what may be called the "absorption condition":

$$(2a) \quad \lim_{r \rightarrow \infty} r \left(\frac{\partial u}{\partial r} + ik u \right) = 0.$$

We demonstrate the general validity of the radiation condition by showing that it guarantees the *uniqueness* of solution of the above general oscillation problem. We may then be convinced that the unique solution of the *mathematical* problem is identical with the *solution that is realized in nature*. Our problem is the following:

- a) In the exterior of a surface σ , which may consist of several surfaces $\sigma_1, \sigma_2, \dots$, the function u is to satisfy the differential equation

$$\Delta u + k^2 u = \varrho$$

The function ϱ measures the yield of the sources which may be continuously distributed or concentrated in single points. The function ϱ is given and must vanish at infinity with sufficient rapidity.

- b) On σ the function u is to satisfy $u = U$, where U is a given point function on σ . The surface σ lies entirely in the finite domain.
- c) In the finite domain u satisfies the condition (2). The quantity r in (2) stands for the distance from any fixed finite point $r = 0$. Around

this point we draw a sphere Σ of radius $r \rightarrow \infty$, which does not intersect the surface σ . The surface element on the sphere is $d\Sigma = r^2 d\omega$, where $d\omega$ is the solid angle seen from $r = 0$. The region between Σ and σ is called S .

- d) Except at possible prescribed sources the function u is to satisfy those conditions of continuity which we prescribed in the derivation of the differential equation.

We assume that two solutions of this problem u_1 and u_2 exist and, as usual, form

$$(3) \quad w = u_1 - u_2,$$

as well as the conjugate function w^* . These functions satisfy the conditions a) to d) with $\varrho = 0$ and $U = 0$. Then in Green's theorem

$$(4) \quad \int_S (w \Delta w^* - w^* \Delta w) d\tau = \left\{ \int_\sigma d\sigma + \int_\Sigma r^2 d\omega \right\} \left(w \frac{\partial w^*}{\partial n} - w^* \frac{\partial w}{\partial n} \right)$$

the integral on the left and the first integral on the right vanish. Hence, the integral over the sphere Σ must also vanish.

For the further discussion we write:

$$(5) \quad w = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{f_n(\theta, \varphi)}{r^n},$$

which is shown to be sufficiently general by the following: we consider w expanded in surface spherical harmonics $Y_n(\theta, \varphi)$. According to §24 A the coefficients must be of the form

$$C_n \zeta_n^1(kr) + D_n \zeta_n^2(kr)$$

where ζ is connected with the half-index Hankel functions by equation (21.15). But here we must have $D_n = 0$, because of the behavior of ζ_n^2 for large values of the argument (see §21 D, p. 117). At the same place we learned that the ζ_n^1 are composed of a finite number of terms of the form $e^{ikr}/(kr)^m$, $m < n$. By arranging this expansion in spherical harmonics according to powers of r^{-n} we obtain (5), where the $f_n(\theta, \varphi)$ turn out to be finite sums of surface spherical harmonics.

The f_n satisfy a simple recursion formula. According to (22.4) the differential equation $\Delta w + k^2 w = 0$ written in terms of r, θ, φ , yields the equation

$$(6) \quad \frac{\partial^2(rw)}{\partial r^2} + \frac{1}{r^2} D(rw) + k^2 r w = 0.$$

where D is the differential symbol of (23.15b) in the coordinates ϑ, φ . Applying (6) to (5) we obtain

$$e^{ikr} \sum_{n=0}^{\infty} \left(-\frac{2ikn}{r^{n+1}} + \frac{n(n+1)}{r^{n+2}} + \frac{D}{r^{n+2}} \right) f_n = 0.$$

Replacing the index of summation n in the first term of the parentheses by $n+1$ we obtain

$$e^{ikr} \sum_{n=0}^{\infty} \frac{1}{r^{n+2}} [-2ik(n+1)f_{n+1} + \{n(n+1) + D\}f_n] = 0$$

and hence the recursion formula:

$$(6a) \quad 2ik(n+1)f_{n+1} = \{n(n+1) + D\}f_n.$$

Hence: if $f_0 = 0$ then all $f_1 = f_2 = \dots = 0$.

We now investigate the remaining integral in (4). Since we are interested in the limit for $r \rightarrow \infty$ we can replace w by the first term of its expansion (5), ignoring the higher powers of $1/r$, whence:

$$\begin{aligned} w &= \frac{e^{ikr}}{r} f_0, & w^* &= \frac{e^{-ikr}}{r} f_0^*; \\ \frac{\partial w}{\partial n} &= ik \frac{e^{ikr}}{r} f_0, & \frac{\partial w^*}{\partial n} &= -ik \frac{e^{-ikr}}{r} f_0^*. \end{aligned}$$

Thus we obtain:

$$\int r^2 d\omega \left(w \frac{\partial w^*}{\partial n} - w^* \frac{\partial w}{\partial n} \right) = -2ik \int f_0 f_0^* d\omega.$$

The integrand is *positive* as long as $f_0 \neq 0$. But we saw in (4) that this integral must vanish. Hence

$$f_0 = 0, \text{ and due to (6a) } f_1 = f_2 = \dots = 0.$$

Therefore

$$w = 0 \text{ and } u_2 = u_1.$$

The author's original proof⁷ of this uniqueness theorem assumed, in addition to the conditions a), b), c) for u , the existence of Green's function for the exterior of the surface and an additional "finality condition." The fact that the latter is superfluous has been rigorously proven by F. Rellich⁸ even for the case of an arbitrary number of dimensions h

⁷ See footnote on p. 183 and Frank-Mises II, chap. XIX, §5. The form of the proof given in the text is essentially F. Sauter's.

⁸ *Jahresber Deutschen Math. Vereinigung* 53, 57 (1943), which also treats the case in which the surface σ stretches to infinity.

where the radiation condition reads

$$(7) \quad \lim_{r \rightarrow \infty} r^{\frac{h-1}{2}} \left(\frac{\partial u}{\partial r} - i k u \right) = 0.$$

In the two-dimensional case $h = 2$, where, as we know, the spherical wave e^{ikr}/r is replaced by the cylindrical wave $H_0^1(kr)$, equation (7) becomes

$$(7a) \quad \lim_{r \rightarrow \infty} r^{\frac{1}{2}} \left(\frac{\partial u}{\partial r} - i k u \right) = 0,$$

which actually is satisfied by $u = H_0^1(kr)$. In the one-dimensional case, where the radiating wave is given by $\exp(i k |x|)$, equation (7) becomes

$$(7b) \quad \lim_{|x| \rightarrow \infty} \left(\frac{\partial u}{\partial |x|} - i k u \right) = 0.$$

Following Rellich, we stress the fact that no radiating solution u of the wave equation can exist which, in every direction, approaches zero more rapidly than $1/r$. For such a function u we would have $f_0 = 0$ in (5) and, as we have seen, this causes u to vanish identically. In this respect the wave equation differs from the potential equation. For the latter solutions exist which, for increasing r , decrease more rapidly than $1/r$, the so-called dipole, quadrupole, and octupole fields of §24 C. For the wave equation such an r -dependence, which implies a pole of higher order than $1/r$ at $r = 0$, can happen only in the so-called "near zone" ($r < \lambda$, $\lambda =$ wavelength); in the "far zone" ($r > \lambda$) every solution of the wave equation behaves like the spherical wave e^{ikr}/r . Potential theory is the limiting case $\lambda = \infty$, as for this case, the near zone reaches to infinity, so to speak.

We now come to the problem of *Green's function for a continuous spectrum*. We first consider in detail the very simplest one-dimensional example ($-\infty < x < +\infty$), in which the radiation condition is the only boundary condition prescribed. Green's function is then identical with the "principal solution" introduced on p. 47, and therefore has a "unit source" at an arbitrary prescribed point $x = x_0$ (see exercise II.3). It must satisfy the conditions:

- a) $\frac{d^2 G}{dx^2} + k^2 G = 0 \quad \text{for } x \neq x_0$
- b) $\left(\frac{dG}{dx} \right)_{x_0+0} - \left(\frac{dG}{dx} \right)_{x_0-0} = 1, \quad (\text{definition of unit source})$
- c) $\frac{dG}{d|x|} - i k G = 0 \quad \text{for } x = \pm \infty.$

The solution is seen to be

$$(8) \quad G = \begin{cases} \frac{1}{2ik} e^{ik(x-x_0)} & \text{for } x > x_0, \\ \frac{1}{2ik} e^{-ik(x-x_0)} & \text{for } x < x_0. \end{cases}$$

We compare this to the representation (27.5) first for the finite region $-l < x < +l$, but with the usual boundary conditions replaced by the radiation condition. In preparation for a continuous spectrum we change the name k_n of the eigen values to λ ; the eigenfunction $u = u_\lambda$ which belongs to λ is then defined by

$$\begin{aligned} \text{a)} \quad & \frac{d^2 u}{dx^2} + \lambda^2 u = 0 & -l < x < +l, \\ \text{b)} \quad & \frac{du}{d|x|} - iku = 0 & |x| = l. \end{aligned}$$

If we write the solution of a) as:

$$(9) \quad u = A e^{i\lambda x} + B e^{-i\lambda x},$$

then according to b) we must have

$$\begin{aligned} A(\pm \lambda - k) e^{\pm i\lambda l} + B(\mp \lambda - k) e^{\mp i\lambda l} &= 0, \\ \frac{A}{B} = \frac{\lambda + k}{\lambda - k} e^{-2i\lambda l} &= \frac{\lambda - k}{\lambda + k} e^{+2i\lambda l}. \end{aligned}$$

From this we obtain the equation for λ :

$$\left(\frac{\lambda - k}{\lambda + k} \right)^2 e^{4i\lambda l} = 1.$$

This equation splits into the equations

$$(9a) \quad \frac{\lambda - k}{\lambda + k} e^{2i\lambda l} = +1, \quad B = A, \quad u = 2A \cos \lambda x,$$

$$(9b) \quad \frac{\lambda - k}{\lambda + k} e^{2i\lambda l} = -1, \quad B = -A, \quad u = 2iA \sin \lambda x.$$

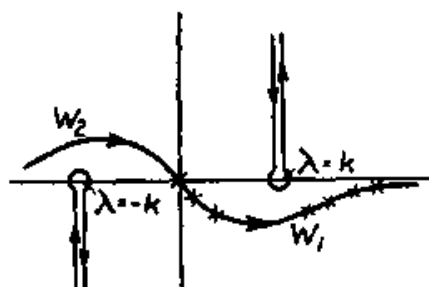
From (9a) we obtain as first and second approximation

$$(10a) \quad \begin{aligned} & \text{for } \lambda \gg k, \lambda = \frac{\pi}{l} m \quad \text{and} \quad = \frac{\pi}{l} m \left(1 - \frac{ikl}{\pi^2 m^2} \right), \quad m \rightarrow \infty, \\ & \text{for } \lambda \ll k, \lambda = \frac{\pi}{l} (m + \frac{1}{2}) \quad \text{and} \quad = \frac{\pi}{l} (m + \frac{1}{2}) \left(1 - \frac{i}{kl} \right), \quad m = 0, 1, 2, \dots \end{aligned}$$

where m is an integer. In the same manner we obtain from (9b)

$$(10b) \quad \begin{aligned} &\text{for } \lambda \gg k, \lambda = \frac{\pi}{l} (m + \frac{1}{2}) \text{ and } = \frac{\pi}{l} (m + \frac{1}{2}) \left[1 - \frac{ikl}{\pi^2 (m + \frac{1}{2})^2} \right], m \rightarrow \infty, \\ &\text{for } \lambda \ll k, \lambda = \frac{\pi}{l} m \quad \text{and } = \frac{\pi}{l} m \left(1 - \frac{i}{kl} \right), m = 0, 1, 2, \dots \end{aligned}$$

Fig. 26. The path of integration W_1 is completed by the path W_2 to an infinite closed path $W = W_1 + W_2$. For positive $x - x_0$ this path can be deformed so that it runs in the positive imaginary λ -half plane.



We see that the values of λ as calculated from (9a) and (9b) form a sequence (marked by x in Fig. 26) that, starting with $\lambda = 0$, first descends linearly into the negative imaginary λ -half plane⁹ and finally for large λ (large m) osculates the real λ -axis from below. According to (9a,b) the successive points alternatingly belong to cosine and sine eigenfunctions. After normalization to 1 these eigenfunctions are

$$(11) \quad u = \begin{cases} \frac{1}{\sqrt{l(1+A)}} \cos \lambda x, \\ \frac{1}{\sqrt{l(1-A)}} \sin \lambda x, \end{cases} \quad A = \frac{\sin \lambda l \cos \lambda l}{\lambda l}.$$

In the limit $l \rightarrow \infty$ the λ -points of Fig. 26 will be everywhere dense on the right half curve denoted by W_1 . The difference between two successive points of the sequence (10a) or (10b) then always becomes

$$(11a) \quad d\lambda = \frac{\pi}{l} \rightarrow 0.$$

We now return to the representation (27.5) of Green's function. For $u(P)$ and $u(Q)$ we substitute their expression (11) in the variables x and x_0 respectively, and combine the pairs of successive cosine and sine terms, i.e., the terms which belong to successive eigen values λ . The numerator of (27.5) then becomes

$$u(P)u^*(Q) = \frac{\cos \lambda x \cos \lambda x_0}{l(1+A)} + \frac{\sin \lambda x \sin \lambda x_0}{l(1-A)}.$$

According to (11) and (11a) we have for $l \rightarrow \infty$

⁹ The fact that the eigen values are complex in contrast to the theorem on p. 169 is due to the fact that our present boundary condition is of a complex nature.

$$A = 0, \quad \frac{1}{l} = \frac{d\lambda}{\pi}.$$

Hence the numerator in (27.5) becomes $\cos \lambda (x - x_0) d\lambda/\pi$, while the denominator in our present notation is $k^2 - \lambda^2$. Equation (27.5) then becomes

$$(12) \quad G = \frac{1}{\pi} \int_{W_1} \frac{\cos \lambda (x - x_0)}{k^2 - \lambda^2} d\lambda = \frac{1}{2\pi} \int_W \frac{e^{i\lambda (x - x_0)}}{k^2 - \lambda^2} d\lambda.$$

where W in the last term is the path $W_1 + W_2$ of Fig. 26. The fact that the integration over W_1 is equal to one half the integral over the whole path W follows from the fact that in the integral over W_1 both the numerator and the denominator are even functions of λ . The fact that in the last term we can replace the cosine by the exponential function follows from fact that the sine part of the exponential function is odd in λ and hence vanishes upon integration. The path W is much more convenient than W_1 since it can be deformed away from the origin by the methods of complex integration.

The manner in which this deformation should be performed can be seen from Fig. 26. For positive $x - x_0$ the path W can be drawn over into the positive imaginary λ half plane, for negative $x - x_0$ it can be drawn into the negative imaginary half plane. In the one case the path can not be transformed across the pole $\lambda = +k$ of the integrand in (12), in the other case it can not be transformed across the pole $\lambda = -k$. Forming the residues and combining the two cases we obtain from (12):

$$(13) \quad G = \frac{1}{2ik} e^{ik|x-x_0|}.$$

This is exactly the same as (8).

Hence we see: *The general representation (27.5) of Green's function remains valid for a continuous eigen value spectrum if, in accordance with the radiation condition, we consider a complex path of integration. If instead we have the "absorption condition" (i replaced by $-i$ in (1a) and (2)), then instead of W we have to consider its reflected image on the real λ -axis; we then obtain equation (13) with i replaced by $-i$.*

If instead of the one-dimensional case we consider the two- or three-dimensional case and correspondingly replace the coordinate x by the polar coordinates r, φ and r, θ, φ , then the spectrum of the eigen values becomes *continuous only in the r -coordinate* but remains *discontinuous* in the angle coordinates. For example in the case of unbounded

three-dimensional space we start from the following representation of Green's function

$$(14) \quad 2\pi G(P, Q) = \sum_{nm} \Pi_n^m(\cos \vartheta) \Pi_n^m(\cos \vartheta_0) e^{im(\varphi - \varphi_0)} \int_{W_1} \frac{F d\lambda}{k^2 - \lambda^2},$$

$$(14a) \quad F = \Psi_n(\lambda r) \Psi_n(\lambda r_0).$$

Here, as in the preceding section, Π and Ψ stand for the spherical harmonic and Bessel functions normalized to 1; and in the following the Z^1, Z^2 correspond to the Hankel functions ζ^1, ζ^2 . The factor 2π on the left side is due to the normalization of the functions $\exp\{im\varphi\}$ and $\exp\{-im\varphi_0\}$. As in Fig. 26 the path W_1 lies in the complex λ -plane from $\lambda = 0$ to $\lambda = \infty$, and again avoids the pole $\lambda = k$. We first give a brief discussion of the way in which this representation can be treated in analogy to the one-dimensional case. This will yield a representation of spherical and cylindrical waves which we have met before.

In order to transform W_1 into the path W of Fig. 26, we write

$$\Psi_n(\lambda r) = \frac{1}{2}(Z_n^1(\lambda r) + Z_n^2(\lambda r)),$$

For the convergence problems which arise in connection with the normalization we refer the reader to Appendix I. Due to the properties of Hankel functions (see exercise IV.2, in particular equation (12), and also the discussion in connection with equation (32.13)), we can transform the integral over W_1 , which involves the function F of (14a), into the integral over W involving

$$(14b) \quad F_1 = \frac{1}{2} Z_n^1(\lambda r) \Psi_n(\lambda r_0) \quad r > r_0,$$

and

$$(14c) \quad F_2 = \frac{1}{2} \Psi_n(\lambda r) Z_n^1(\lambda r_0) \quad r < r_0$$

Since the integrand $\frac{1}{2} F_{1,2}/(k^2 - \lambda^2)$ vanishes at infinity in the positive imaginary λ -plane for both cases $r \gtrless r_0$, the integral of (14) reduces to the residue at the pole $\lambda = k$:

$$(15) \quad \int F_{1,2} \frac{d\lambda}{k^2 - \lambda^2} = \frac{\pi}{2ik} \begin{cases} Z_n^1(kr) \Psi_n(kr_0) & r > r_0, \\ \Psi_n(kr) Z_n^1(kr_0) & r < r_0. \end{cases}$$

Applying the addition theorem of spherical harmonics (22.34) we obtain from (14):

$$(16) \quad G(P, Q) = \frac{1}{4\pi k} \sum_{n=0}^{\infty} H_n(\cos \Theta) H_n(1) \begin{cases} Z_n^1(kr) \Psi_n(kr_0) & r > r_0, \\ \Psi_n(kr) Z_n^1(kr_0) & r < r_0. \end{cases}$$

For reasons of symmetry $G(P, Q)$ in unbounded space is a pure function of the distance

$$R = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \Theta},$$

between P and Q ; namely, due to the definition of the unit source on p. 47 we have

$$(16a) \quad G(P, Q) = -\frac{1}{4\pi} \frac{e^{ikR}}{R} = \frac{1}{4\pi i} \zeta_0^1(kR),$$

where ζ_0 is given by (21.15a). If, on the right side of (16), we pass from H, Ψ, Z to P, ψ, ζ (see Appendix I equation (9a)), we obtain the addition theorem (24.9a).

The corresponding series for the two-dimensional case are contained in (21.3).

More important than the derivation of these known formulas is the generalization to the case in which space is not unbounded but is bounded by a finite closed surface σ (or, in the two-dimensional case, by a curve s) with prescribed boundary conditions. We are then dealing with the *proper problem of Green's function*: to find a function $G(P, Q)$ having a unit source in Q , satisfying the radiation condition at infinity and the given boundary condition on σ (or s).

We choose the special case in which the surface σ is a sphere $r = a$, and the boundary condition is

$$(17) \quad u = 0.$$

The point Q is to lie on the ray

The eigenfunction which belongs to the eigen value λ is no longer $\psi_n(\lambda r)$, but can be written in the (non-normalized) form

$$(18) \quad u_n(\lambda, r) = \psi_n(\lambda r) + A \zeta_n^1(\lambda r)$$

Due to (17) the function A becomes¹⁰

$$(18a) \quad A = -\frac{\psi_n(\lambda a)}{\zeta_n^1(\lambda a)}.$$

For the construction of Green's function we shall not follow the general method of equation (14). Instead we shall use a shorter though

¹⁰ The fact that A depends on λ made it necessary to write $u(\lambda, r)$, instead of $u(\lambda, r)$.

less systematic approach based on equation (24.9) for unbounded space:

$$\begin{aligned} (19a) \quad & \left\{ \begin{aligned} & \frac{e^{ikR}}{ikR} = \sum_{n=0}^{\infty} (2n+1) P_n(\cos \vartheta) \begin{cases} \psi_n(kr_0) \zeta_n^1(kr) & r > r_0, \\ \zeta_n^1(kr_0) \psi_n(kr) & r < r_0. \end{cases} \end{aligned} \right. \\ (19b) \quad & \end{aligned}$$

Here (19b) will not yet satisfy condition (17) for $r = a$; in order to satisfy (17) we complete the right side of (19b) by adding

$$- \sum_{n=0}^{\infty} (2n+1) P_n(\cos \vartheta) \zeta_n^1(kr_0) \psi_n(ka) \frac{\zeta_n^1(kr)}{\zeta_n^1(ka)},$$

Due to (18) the right side of (19b) becomes

$$(20) \quad \sum_{n=0}^{\infty} (2n+1) P_n(\cos \vartheta) \zeta_n^1(kr_0) u_n(k, r).$$

If we make the same adjunction to (19a) then the continuous passage from (19a) to (19b) for $r = r_0$ is preserved, as is the radiation condition for $r \rightarrow \infty$. The right side of (19a) becomes

$$(21) \quad \sum_{n=0}^{\infty} (2n+1) P_n(\cos \vartheta) \zeta_n^1(kr) u_n(k, r_0).$$

From (20) and (21) we obtain Green's function by adjoining the factor $k/4\pi i$ which, as in (16a), is due to the condition of a unit source. We then have:

$$(22) \quad G(P, Q) = \frac{k}{4\pi i} \sum_{n=0}^{\infty} (2n+1) P_n(\cos \vartheta) \begin{cases} \zeta_n^1(kr) u_n(k, r_0) & r > r_0, \\ \zeta_n^1(kr_0) u_n(k, r) & r < r_0. \end{cases}$$

This way of writing reveals the connection with our general method in (14). The function F of (14a) is now represented by

$$F = u_n(\lambda, r) u_n(\lambda, r_0);$$

except for a constant normalizing factor. The corresponding functions F_1, F_2 of (14b,c) become

$$F_{1,2} = \frac{1}{2} \begin{cases} \zeta_n^1(\lambda r) u_n(\lambda, r_0) & r > r_0, \\ \zeta_n^1(\lambda r_0) u_n(\lambda, r) & r < r_0 \end{cases}$$

By forming the residues for $\lambda = k$ we then obtain equation (22).

In Appendix II of this chapter we shall introduce a novel method of constructing Green's function, which not only improves the convergence

of the series in the most important cases, but also reveals new aspects of the method's applicability.

In the appendix to the following chapter we shall further show that this method would solve the problem of wireless telegraphy on a spherical earth (for infinitely conductive soil and a vertical "dipole antenna") if it were not for the decisive role of the ionosphere.

Finally we remark: a representation of the form (14) remains valid if as the surface σ we choose an ellipsoid instead of a sphere. Instead of the r, θ, φ we then have to use the coordinate system of confocal ellipsoids and hyperboloids. The spectrum of eigen values for the exterior of the ellipsoid will then remain discrete in the parameters of the one piece and two piece hyperboloids, but becomes continuous in the parameter of the ellipsoids. By integration over this last parameter we would obtain a simplification similar to that of (22). Even in the most general case where there are no separating coordinates, in which the eigenfunctions can be decomposed into products, we can still use equation (27.5) as a starting point for the representation of Green's function.

§ 29. The Eigen Value Spectrum of Wave Mechanics. Balmer's Term

The Schrödinger equation of wave mechanics for the simple case of the hydrogen atom reads

$$(1) \quad \Delta\psi + \frac{2m}{\hbar^2} (W - V) \psi = 0.$$

This is our equation (7.15), with the difference that the symbol of energy W has been replaced by the difference of the total energy W and the potential energy, V or, mechanically speaking, by the kinetic energy. The Rutherford model for the H-atom consists of a nucleus, the proton with a $+e$ charge, and of an electron with a $-e$ charge that moves in the proton field. Its potential (Coulomb) energy measured in electrostatic units is

$$(2) \quad V = -\frac{e^2}{r},$$

where r is the distance from the proton and V is normalized so that at infinity we have $V = 0$. The mass energy m_0c^2 of the electron at rest is not to be counted in the total energy. In the following we may consider the proton at rest at the point $r = 0$.

Equation (1) differs from the wave equation we have treated so far because the constant k^2 has been replaced by a *point function* which

becomes *singular* at the point $r = 0$. Whereas we have used k to denote eigen value, we shall now use \overline{W} as an eigenparameter. Hence, we shall seek those values of W for which (1) has a solution which is continuous in the entire space. These solutions are the eigenfunctions of our “Kepler problem,” where the nucleus plays the role of the sun and the electron the role of the planets. Since the electron may move in unbounded space, the spectrum of eigen values will be continuous in the r -coordinate as in equation (28.14). More important for us is the fact that the spectrum also has *discrete* components.

The spectral apparatus gives us the discrete spectrum by measuring the *line spectrum*, which, in the case of hydrogen, is given in the visible range by the Balmer series $H_\alpha, H_\beta, H_\gamma, H_\delta, \dots$. The lines of this spectrum cumulate at the limit given by the Rydberg constant R . The adjoining *continuum* lies in the near ultraviolet range. Both the discrete and the continuous spectrum are given by the Schrödinger equation. This equation reduces to a simple mathematical formula the enigma of the spectral lines, with their finite cumulation point, the behavior of which differs so fundamentally from that of all mechanical systems.

Niels Bohr gave a general explanation of the Balmer series and its limiting frequencies twelve years before Schrödinger, by endowing the Rutherford model with certain quantum theoretical traits. However the concept of orbits he used lead to diverse contradictions and had to be abandoned in favor of the analytic model of equation (1). The fact that (1) is also based on quantum theory is indicated by the entrance of Planck's constant $\hbar = h/2\pi$.

What is the physical meaning of the eigenfunction ψ ? The answer to this question shows the complete revolution in the concept of nature that quantum theory has brought about: $|\psi|^2 dx dy dz$ stands for the *probability* with which we may expect to find the hydrogen electron at the point (x, y, z) within an error of dx, dy, dz . Hence, in wave mechanics the concept of probability takes the place of the concept of strict determinism which rules in classical mechanics. The measure of indeterminacy in the atomic range is Planck's h (Heisenberg).

The “normalization of the eigenfunctions to 1,” which so far had been introduced only for mathematical simplicity, thereby acquires a fundamental meaning. Namely, the equation

$$(3) \quad \int |\psi|^2 d\tau = 1$$

asserts the *certainty* that the electron is somewhere in space; this condition is necessary from the point of view of wave mechanics. Equation (3) holds for a discrete spectrum; for the continuum it must be modified according to the prescription of Appendix I to this chapter.

We now turn to the integration of (1), introducing the coordinates r, ϑ, φ . If we write the wave equation in the form (22.4) and let¹¹

$$(4) \quad \psi = \chi(r) P_l^m(\cos \vartheta) e^{im\varphi}$$

then according to the differential equation (22.13) we obtain

$$(5) \quad \frac{d^2 \chi}{dr^2} + \frac{2}{r} \frac{d\chi}{dr} + \left\{ \frac{2m}{\hbar^2} \left(W + \frac{e^2}{r} \right) - \frac{l(l+1)}{r^2} \right\} \chi = 0.$$

We first consider the case in which the electron is *tied* to the nucleus. Then W must be negative since the energy of the electron at rest at infinity is normalized to zero. If it is absorbed by the nucleus and stably tied there then its energy is decreased. If, on the other hand, $W > 0$ then even for an infinite distance from the nucleus the electron has positive kinetic energy and, mechanically speaking, has a hyperbolic orbit.

The asymptotic behavior of χ for $r \rightarrow \infty$ is obtained from (5) by neglecting all terms with $1/r$ and $1/r^2$:

$$\frac{d^2 \chi}{dr^2} + \frac{2m}{\hbar^2} W \chi = 0.$$

For negative W we write

$$(5a) \quad \frac{d^2 \chi}{d\rho^2} = \frac{1}{4} \chi, \quad \chi = e^{-\rho/2}, \quad \rho = \frac{2r}{\hbar} \sqrt{-2mW}.$$

The other solution of (5a), namely, $\chi = \exp(+\rho/2)$, must be neglected since χ is to be finite everywhere.

In order to obtain an exact solution of (5) we write

$$(6) \quad \chi = e^{-\rho/2} v(\rho)$$

and obtain as the differential equation for v

$$(6a) \quad v'' + \left(\frac{2}{\rho} - 1 \right) v' + \left[\frac{n-1}{\rho} - \frac{l(l+1)}{\rho^2} \right] v = 0$$

with the abbreviation

$$(6b) \quad n = \frac{m e^2 / \hbar}{\sqrt{-2mW}}.$$

¹¹ Here we denote the lower index of P by l instead of n , corresponding to wave mechanical usage: l = azimuthal quantum number, n_r = radial quantum number, $n = n_r + l + 1$ = total quantum number, m = magnetic quantum number, where we now have $-l \leq m \leq +l$.

In order to discuss (6a) we use the method of equation (19.36). We write

$$(7) \quad v = \varrho^\lambda w, \quad w = a_0 + a_1 \varrho +$$

and in analogy with (19.37) we obtain:

$$(7a) \quad \lambda(\lambda + 1) = l(l + 1), \text{ and hence } \lambda = +l.$$

The other root of (7a) $\lambda = -l - 1$ must be excluded, since v as well as χ must remain finite for $\varrho = 0$. The recursion formula for the a_k is obtained in analogy with (19.37a) by equating to zero the coefficients of $\varrho^{\lambda+k-1}$ in the power series obtained from (6a) and (7). Thus we find:

$$(7b) \quad a_{k+1} [(\lambda + k + 1)(\lambda + k) + 2(\lambda + k + 1) - l(l + 1)] + a_k [n - 1 - \lambda - k] = 0.$$

If in this equation we make the coefficient of a_k equal to zero by setting

$$(8) \quad n = k + \lambda + 1,$$

then a_{k+1} vanishes and so do all the subsequent terms in w : *the series breaks off, that is, w becomes a polynomial of degree k , whose further properties we shall treat later.* For the time being we shall stress only the following facts: 1. Due to the factor $\exp(-\varrho/2)$ in (6), we see that as $r \rightarrow \infty$ the function χ tends to zero with sufficient rapidity to make possible the normalization of ψ according to (3), no matter what the degree of the polynomial w . 2. If the series did not break off then from (7b) we should obtain an asymptotic behavior of a_k which would make w become infinite to the order $\exp(+\varrho)$ for $\varrho \rightarrow \infty$, and the normalization of ψ would be impossible. Hence the requirement that the series for w break off is a wave mechanical *necessity*.

We now consider equation (8). We denote the value of k there by n_r (radial quantum number) and for λ we substitute its value from (7b) (azimuthal quantum number). Hence, according to (8) n is *integral*:

$$(8a) \quad n = n_r + l + 1.$$

This number n is called the "total quantum number." From equation (6b) we obtain:

$$(8b) \quad W = W_n = -\frac{m e^4}{2 \hbar^2 n^2}.$$

Setting W equal to the energy quantum $\hbar \nu$ we obtain

$$(9) \quad \nu = \frac{m e^4}{2 \hbar^2 n^2} = \frac{R}{n^2}$$

where

$$(9a) \quad R = \frac{2 \pi^2 m e^4}{h^3}.$$

This R is the above mentioned "Rydberg frequency"; it can be measured spectroscopically with extraordinary precision and hence can lead to an improvement of our knowledge of the fundamental constants e, m, h . The number ν of (9) is called the *Balmer term*.

The observable frequency of a spectral line is obtained by the passage of the atom from an initial state 1 to a final state 2 and is computed as the difference of the associated terms ν_2 and ν_1 . Hence for the hydrogen spectrum we have

$$(10) \quad \nu = \nu_2 - \nu_1 = R \left(\frac{1}{n_2^2} - \frac{1}{n_1^2} \right).$$

The Balmer series corresponds to the passage into the final state $n_2 = 2$; the Lyman series which lies in the ultraviolet range corresponds to the passage into the fundamental state of the hydrogen atom $n_2 = 1$; in both cases the passage is from an arbitrary initial state $n_1 > n_2$. Hence we have

$$(10a) \quad \nu = R \left(\frac{1}{2^2} - \frac{1}{n^2} \right), \quad n = 3, 4, 5, \dots \quad \text{Balmer series,}$$

$$(10b) \quad \nu = R \left(\frac{1}{1^2} - \frac{1}{n^2} \right), \quad n = 2, 3, 4, \dots \quad \text{Lyman series.}$$

The series with $n_2 = 3, n_2 = 4, \dots$ lie in the infrared domain.

After having learned about the *eigen values* of the H-atom we wish to consider the analytic character of its *eigenfunctions*. With the use of (7), (7a) and (8a) we obtain from (6a)

$$(11) \quad \varrho w'' + [2(l+1) - \varrho] w' + (n-l-1) w = 0.$$

This equation is obtained through $(2l+1)$ -fold differentiation from the simpler differential equation

$$(12) \quad \varrho L'' + (1 - \varrho) L' + \mu L = 0 \quad \text{with} \quad \mu = n + l.$$

For every integer μ this equation has one and only one polynomial solution of degree μ . With a suitable normalization we obtain the solutions:

$$\begin{array}{ll}
\mu = 0, & L = 1, \\
\mu = 1, & L = -\varrho + 1, \\
\mu = 2, & L = \varrho^2 - 4\varrho + 2, \\
\mu = 3, & L = -\varrho^3 + 9\varrho^2 - 18\varrho + 6. \\
\vdots & \vdots \\
\vdots & \vdots
\end{array}$$

These are precisely the expressions we denoted in exercise I.6 as Laguerre polynomials; equation (12) is the Laguerre differential equation, as indicated by the choice of the letter L . This differential equation coincides with the differential equation (24.29) of the confluent hypergeometric function for the parameters $\alpha = -\mu = -n - l$.

Hence we have

$$(13) \quad L = F(-n-l, 1, \varrho) \quad \text{and} \quad w = \frac{d^{2l+1} L}{d\varrho^{2l+1}}.$$

Hence from (4), (6), (7) and (7a) we obtain the representation of the hydrogen eigenfunction

$$(14) \quad \psi = N \varrho^l e^{-\varrho/2} \frac{d^{2l+1} L}{d\varrho^{2l+1}} P_l^m(\cos \vartheta) e^{im\varphi},$$

where N is a normalization factor due to (3). From (5a) and (8b) we obtain ϱ :

$$(14a) \quad \varrho = \frac{2r}{na}, \quad a = \frac{\hbar^2}{me^2} \sim \frac{1}{2} 10^{-8} \text{ cm.}$$

where, as is customary, a denotes the "hydrogen radius."

In order to justify this notation, and as a single special application of the above, we compute the "probability density" in the "fundamental state" $n = 1$ of the H-atom. For $n = 1$ we have according to (8a) $l = m = 0$, $n_r = 0$ and hence from (14)

$$\psi = -N_1 e^{-\varrho/2} = -N_1 e^{-r/a}, \quad |\psi|^2 = N_1^2 e^{-2r/a},$$

where from (3) we obtain $N_1 = (\pi a^3)^{-1/2}$. Hence, the probability of finding the electron is distributed spherically over the nucleus. For $r = 0$ this probability assumes its maximum N_1^2 , for $r = a$ its value is only $(N_1/e)^2$, but it only vanishes at infinity. The charge density is proportional to this probability. From the point of view of wave mechanical statistics we do not have an electron which is concentrated at a point, but instead we have a *charge cloud* whose principal part is in the interior of a sphere of radius a .

From the older point of view of orbits we must ascribe a *disc-like*

form to the H-atom. The fundamental state (circular orbit of radius a) then corresponds to a circular disc. In a magnetic field all the circular discs of an H-atom gas would have to be parallel to each other and perpendicular to the magnetic force lines; a light ray passing through this gas would have to show "magnetic double-diffraction." Precise measurements by Schütz, though performed not on an H-atom gas but on the analogous Na-vapor, showed no trace of this phenomenon. This is one of the contradictions which have been cleared up by wave mechanics.

A behavior similar to that of the fundamental state of the H-atom is obtained for all states with $l = 0$, the so-called "s-terms" of spectroscopy. For $l = 0$ we obtain from (14)

$$\psi = N_n e^{-\varrho/2} L'_n(\varrho), \quad \varrho = \frac{2r}{na}, \quad n = n_r + 1,$$

which again means spherical symmetry. Such s-terms are the fundamental states of the alkali atoms Li, Na, K, The same holds for all completed shells, e.g., the so-called eight-shells of rare gases. The proof is based on the addition theorem of spherical harmonics. This spherical symmetry of the closed shells is obviously of great importance for all chemical applications.

We have to add a few remarks about the continuous spectrum of hydrogen, that is, about the states $W > 0$ (the "hyperbolic orbits" of the older theory). The electron is then no longer tied to the nucleus but is still in the field of the proton.

According to (5a) and (6b) ϱ and n become *purely* imaginary for $W > 0$. In the asymptotic solution (5a) the two signs of ϱ are equivalent; both solutions $\exp \{\pm \varrho/2\}$ can be used. It is unnecessary, and due to the imaginary character of ϱ it is also impossible, to make the series (7) break off. Hence every value of W is permissible. *The W -spectrum becomes continuous* and reaches from $W = 0$ to $W = \infty$. Since, according to (8b) $W = 0$ corresponds to the limit $n = \infty$ of the discrete spectra, we see that to each of these spectra there adjoins a continuous spectrum in the short wave direction. The analytic form of the representation (14) remains valid; but L is now no longer a Laguerre polynomial but a *confluent hypergeometric series which does not break off*, since the parameter $\alpha = -n - l$ in (13) is no longer negative integral but general complex.

§ 30. Green's Function for the Wave Mechanical Scattering Problem. The Rutherford Formula of Nuclear Physics

Nuclear physics originated with Rutherford's experiments on the scattering of α -rays by heavy atoms. Since the electron shells of the

atom are immaterial for the case of α -rays, we may treat the scattering problem in terms of the continuous H-spectrum. We are dealing, in fact, with a *two-body problem*: a *nucleus* (of charge Ze , where Z is the atomic number, $Z = 1$ for the H-spectrum) and a particle interacting with it (in this case an α -particle with mass m_α and charge $Z'e$ where $Z' = 2$; in the preceding case an electron of mass m and charge $-e$ corresponding to the charge number $Z' = -1$). First we want to find that point of the continuous spectrum that corresponds to the energy constant W_α of the incoming α -rays. For an infinite distance between the α -particle and the nucleus the kinetic energy of the α -particle is

$$W_\alpha = \frac{m_\alpha}{2} v^2, \quad \text{hence} \quad 2 m_\alpha W_\alpha = (m_\alpha v)^2 = p^2,$$

where $\vec{p} = m_\alpha \vec{v}$ is the kinetic momentum of the α -particle.

If we now pass from the *corpuscular interpretation* of the α -rays to the "complementary" *wave interpretation*, then p/\hbar is, at the same time, the *wave number*¹² k_α of the α -rays.

Hence we have

$$(1) \quad k_\alpha = \frac{m_\alpha v}{\hbar} = \sqrt{\frac{2 m_\alpha W_\alpha}{\hbar^2}}.$$

We can, therefore, rewrite the variable ϱ of (29.5a) in the form

$$(2) \quad \varrho = 2 i k_\alpha r.$$

For an arbitrary point of the continuous spectrum (i.e., for an arbitrary value W different from W_α) we replace k_α by λ as in §28. Equations (1) and (2) then generalize to

$$(2a) \quad \lambda = \sqrt{\frac{2 m_\alpha W}{\hbar^2}}, \quad \varrho = 2 i \lambda r.$$

If, as before, we assume the nucleus at rest then the wave equation (29.1) becomes

$$(3) \quad \Delta\psi + \frac{2 m_\alpha}{\hbar^2} \left(W - \frac{ZZ'e^2}{r} \right) \psi = 0.$$

For the time being we replace (3) by:

¹² In fact the formula $k_\alpha = p/\hbar$ is the equation of L. de Broglie: " \hbar times the reciprocal of the wavelength equals the momentum," which in turn is the relativistic completion of Planck's equation: " \hbar times the reciprocal of the time of oscillation equals the energy."

$$(3a) \quad \Delta\psi + K^2\psi = 0, \quad K^2 = \lambda^2 - \frac{2m_\alpha ZZ'e^2}{\hbar^2 r};$$

for the point $\lambda = k_\alpha$ of the spectrum we then have

$$(3b) \quad K_\alpha^2 = k_\alpha^2 - \frac{2m_\alpha ZZ'e^2}{\hbar^2 r}.$$

We note the important fact that in the difference $K^2 - K_\alpha^2$ the potential term, which is a function of position, is eliminated, so that this difference becomes *independent of position*:

$$(4) \quad K^2 - K_\alpha^2 = \lambda^2 - k_\alpha^2.$$

The reader should convince himself that all our previous deductions from Green's theorem, such as the orthogonality of the eigenfunctions in §26 or the representation of Green's function for *constant* k^2 in §27, remain valid for our generalized wave equation $\Delta\psi + K^2\psi = 0$ with K^2 a *function of position* given by (3a,b).

We now return to the Rutherford scattering experiment. If we consider the source of the α -rays (the radium particle) to be point-like and in the finite domain, then we have a *spherical wave* of corpuscular rays, which is modified by the presence of the nucleus in the manner prescribed by the wave equation (3). However if we remove the source to infinity, which is more natural and at the same time simpler, then we have to treat the same problem for the *plane wave*. In both cases the solution is given by Green's function of §28; in the first case for a general position of the source point Q , in the second case for the limit $Q \rightarrow \infty$. Since Green's function was to be summed over the complete system of eigenfunctions, we have to consider the discrete as well as the continuous eigen value spectrum for a finite Q . However in the limit $Q \rightarrow \infty$ $u(Q)$ vanishes for all eigen values of the discrete spectrum; hence, in this case we have to carry out the integration over only the continuous spectrum. We may retain the expression (29.14) for the eigenfunctions $u(P)$ in question, if we replace ϱ by $2i\lambda r$ in accordance with (2a). If in addition we let $\vartheta = 0$ be in the direction of the line which joins Q to the position O of the nucleus, then the scattering problem becomes symmetric with respect to the axis $\vartheta = 0$, and hence independent of φ , so that the eigenfunctions $u(P)$ must be independent of φ . Therefore according to (29.14) we have

$$(5) \quad u(P) = \chi_i(\varrho) P_i(\cos \vartheta), \quad \chi_i = N \varrho^i e^{-\varrho/2} \frac{d^{2i+1} L}{d\varrho^{2i+1}}.$$

The corresponding expression for $u(Q)$ is obtained from (5) by replacing $P_i(\cos \vartheta)$ by $P_i(\cos \vartheta_0) = P_i(-1) = (-1)^i$, and $\chi_i(\varrho)$ by $\chi_i(\varrho_0)$

and then passing to the limit $\varrho_0 \rightarrow \infty$. We then obtain the representation of the plane wave from (28.14). By performing the integration over the path W of Fig. 26 and forming the residue at the pole $\lambda = k_a$ we obtain a series representation of the form

$$(6) \quad \sum C_l \chi_l(\varrho) P_l(\cos \vartheta), \quad \varrho = 2ik_a r,$$

where the coefficients C_l are determined in a somewhat cumbersome fashion in terms of the normalizing factors of the χ and P and of the asymptotic behavior of $\chi(\varrho_0)$. This representation was first derived by W. Gordon.¹³

A much simpler representation is obtained if we replace the polar coordinates r, ϑ by the *parabolic coordinates* ξ, η . We thus obtain as the wave function of the scattering process (see Appendix III of this chapter):

$$(7) \quad \psi = e^{ikx} L_n(ik\eta), \quad \eta = r - x = r(1 - \cos \vartheta).$$

Here k is the wave number of (1)

$$(7a) \quad k = k_a = \frac{m_a v}{\hbar};$$

and n is the total quantum number, which, becomes purely imaginary, for the continuous spectrum as mentioned at the end of §29. This total quantum number is computed from (29.13) where, as in equation (3), we have to replace e^2 by $-ZZ'e^2$:

$$(7b) \quad n = \frac{ie^2 ZZ'}{\hbar v}.$$

The function L_n is the confluent hypergeometric function of (29.13) for $l = 0$:

$$(7c) \quad L_n(\varrho) = F(-n, 1, \varrho).$$

The variable η is the parabolic coordinate defined in (7), the other coordinate is $\xi = r + x = r(1 + \cos \vartheta)$. In the following ϑ will be called the "scattering angle."

From (7) we obtain the asymptotic value for $r \rightarrow \infty$

$$(8) \quad \psi = C_1 e^{ikx} + C_2 \frac{e^{ikr}}{r}$$

with the abbreviations

¹³ *Z. Physik* (1928). See also the excellent book by Mott and Massey, *The Theory of Atomic Collisions*, Oxford, 1933, chapter III.

$$(8a) \quad C_1 = \frac{(-ik\eta)^n}{\Gamma(n+1)}, \quad C_2 = C_1^* \frac{in/k}{1-\cos\theta}.$$

The first term on the right side of (8) represents the incoming *plane wave*, the second term represents the *spherical wave* scattered from the nucleus. The quantities C_1, C_2 are not constants but depend on η ; however, since n is purely imaginary only their phases depend on η . We are interested only in the absolute value of the ratio C_2/C_1 , which is independent of η and hence of r , and depends only on the scattering angle θ . Namely, from (8a) and (7a,b) we obtain

$$(9) \quad \left| \frac{C_2}{C_1} \right| = \frac{|in|}{k(1-\cos\theta)} = \frac{e^2 Z Z'}{m_\alpha v^2 (1-\cos\theta)} = \frac{e^2 Z Z'}{2 m_\alpha v^2 \sin^2 \theta/2}.$$

According to the wave mechanical definition (29.3) of probability density, the square of this quantity is the *ratio of the number of scattered particles per unit of spatial scattering angle and the number of incoming particles per unit of area on a surface perpendicular to the incoming direction*. This law was deduced by Rutherford through geometric consideration of the classical hyperbolic orbits without the help of quantum theory. This was possible owing to the fact that the constant \hbar canceled in (9). Rutherford's law holds not only for α -rays but also for any other particles (protons, electrons, . . .) which are in Coulomb interaction with the nucleus, of course with a correspondingly different meaning of Z' and m_α . The interesting "exchange effect" that occurs for the equality of scattering and the scattered particle will not be discussed here. For very high velocities of the scattering particle we should have to use relativity theory.

Appendix I

NORMALIZATION OF THE EIGENFUNCTIONS IN THE INFINITE DOMAIN

In passing from a bounded to an unbounded domain we encounter certain difficulties in convergence which can be removed only by a change in the normalization process. This modified process was introduced by H. Weyl in the theory of integral equations and has been adapted to wave mechanics.

As an example we choose the function $I_\nu(kr)$, where ν is arbitrary and k is a root of the equation $I_\nu(ka) = 0$. According to (20.19) its normalizing integral would be:

$$(1) \quad N = \int_0^a [I_\nu(kr)]^2 r dr = \frac{a^2}{2} [I'_\nu(ka)]^2;$$

Due to the asymptotic behavior of aI_r this integral becomes divergent in the limit $a \rightarrow \infty$ (undetermined between the limits $\pm \infty$). In order to obtain a normalization of I_r for $a \rightarrow \infty$ we start from the more general integral

$$(1a) \quad N' = \lim_{a \rightarrow \infty} \int_0^a I_r(kr) I_r(k'r) r dr$$

According to (21.9a) N' behaves like the function $\delta(k|k')$:¹⁴ namely, it vanishes for $k \neq k'$ and becomes infinite for $k = k'$ so that

$$(2) \quad \int_{\Delta} N' k' dk' = 1$$

where Δ is an arbitrary interval containing the critical point $k' = k$. Since in particular Δ can be chosen arbitrarily small, so that k' can be considered constant in Δ , we may replace (2) by

$$(2a) \quad \int_{\Delta} N' dk' = \frac{1}{k},$$

and similarly

$$(2b) \quad \int_{\Delta} N' \sqrt{k'} dk' = \frac{1}{\sqrt{k}}.$$

We now change the normalizing integral (1) in the limit $a \rightarrow \infty$ to

$$(3) \quad N = \int_0^{\infty} r dr I_r(kr) \int_{\Delta} dk' I_r(k'r),$$

that is, in (1) we replace one¹⁵ of the two proper oscillations I_r by the group of neighboring proper oscillations

$$(3a) \quad \int_{\Delta} dk' I_r(k'r)$$

and thus eliminate the above indeterminacy by averaging, so to speak, by interference within the wave group. From (1a) and (2a) we then obtain

$$(4) \quad N = \int_{\Delta} N' dk' = \frac{1}{k}.$$

In order to make clear the mathematical essence of the above

¹⁴ The symbols σ, r, s of (21.9a) have been replaced here by r, k', k .

¹⁵ According to the original method of Weyl we could also replace *both* eigenfunctions I by "*eigendifferentials*" of the form (3a). Instead of the expressions "group of proper oscillations" or "wave group" used in the text, the less attractive term "wave bundle" is customarily used in wave mechanics.

process, we were somewhat careless in the interchanging of the limiting processes $a \rightarrow \infty$ and $\Delta \rightarrow 0$ (or in other words $k' \rightarrow k$). It would therefore be desirable to deduce equation (4) once more on the basis of Green's theorem. Writing

$$(5) \quad u = I_\nu(kr) e^{i\nu\varphi}, \quad v = I_\nu(k'r) e^{-i\nu\varphi}, \quad d\sigma = r dr d\varphi, \quad ds = a d\varphi$$

we compute¹⁶ $\int (u \Delta v - v \Delta u) d\sigma$ in the known manner both as a surface integral and by Green's theorem as a contour integral. We then obtain:

$$(5a) \quad (k^2 - k'^2) \int_0^a u v r dr = a \left(u \frac{\partial v}{\partial r} - v \frac{\partial u}{\partial r} \right)_{r=a}.$$

If we divide by $k^2 - k'^2$ and integrate with respect to k' (under the integral sign) from $k' = k - \Delta/2$ to $k' = k + \Delta/2$, and then pass to the limit $a \rightarrow \infty$, the left side becomes the normalizing integral N of (3). On the right side we choose a so large that we can compute the I_ν asymptotically. If we choose half the sum of (19.55) and (19.56), the constants $\mp (\nu + \frac{1}{2}) \pi/2$ in the exponents partly cancel and partly are of no consequence in the following consideration, and so may be omitted. Then we obtain as the right side of (5a):

$$(5b) \quad \frac{1}{2\pi} \int_{k-\Delta/2}^{k+\Delta/2} \frac{1}{\sqrt{k k'}} \frac{dk'}{k^2 - k'^2} \{ (e^{ik'a} + e^{-ik'a}) i k' (e^{ik'a} - e^{-ik'a}) \\ - (e^{ik'a} + e^{-ik'a}) i k (e^{ik'a} - e^{-ik'a}) \}.$$

After multiplying out and collecting terms we obtain

$$(6) \quad \frac{1}{\pi} \int_{k-\Delta/2}^{k+\Delta/2} \frac{dk'}{\sqrt{k k'}} \left\{ \frac{\sin(k-k')a}{k-k'} + \frac{\sin(k+k')a}{k+k'} \right\}.$$

For sufficiently small Δ we can replace $\sqrt{k k'}$ by k and put it in front of the integral. If in the first term of the integral we make the substitution $x = (k - k')a$, then it becomes

$$- \int_{x_1}^{x_2} \sin x \frac{dx}{x} \quad \text{with} \quad \begin{cases} x_1 = a \Delta/2, \\ x_2 = -a \Delta/2, \end{cases}$$

In the limit $a \rightarrow \infty$ this assumes the value π (see exercise I.5). In the second term of the integral (6) we make the substitution $y = (k + k')a$.

¹⁶The domain of integration of Green's theorem is not the complete circle of radius a , but the domain of periodicity of u and v as in equation (25.5), namely a circular sector of angle $\alpha = 2\pi/\nu$. In the integration with respect to φ we obtain a factor α on both sides of (5a) that, in the text, has already been canceled.

The limits of integration then become

$$\left. \begin{aligned} y_1 &= (2k - \Delta/2)a \\ y_2 &= (2k + \Delta/2)a \end{aligned} \right\} \rightarrow \infty \text{ for } a \rightarrow \infty.$$

It is easy to see that this second term vanishes. Hence (6) becomes $1/k$ and (5a) becomes

$$(7) \quad N = \frac{1}{k}$$

which coincides with (4). From this it follows that the Bessel function which is normalized to 1 in the above manner is given by

$$(7a) \quad \frac{1}{\sqrt{N}} I_\nu(kr) = \sqrt{k} I_\nu(kr).$$

From the relation (21.11)

$$(8) \quad I_{n+\frac{1}{2}}(kr) = \sqrt{\frac{2kr}{\pi}} \varphi_n(kr)$$

we further see that the function φ_n normalized to 1, which we denoted by Ψ_n on p. 197, is related to φ_n by

$$(8a) \quad \Psi_n(kr) = \sqrt{\frac{2}{\pi}} k \varphi_n(kr).$$

Indeed, from (7a) and (8) we have

$$1 = \int_0^\infty r dr \sqrt{k} I_{n+\frac{1}{2}}(kr) \int_{\Delta} dk' \sqrt{k'} I_{n+\frac{1}{2}}(k'r) = \frac{2}{\pi} \int_0^\infty r^2 dr k \varphi_n(kr) \int_{\Delta} dk' k' \varphi_n(k'r)$$

substituting this in (8a) we obtain

$$(9) \quad 1 = \int_0^\infty r^2 dr \Psi_n(kr) \int_{\Delta} dk' \Psi_n(k'r),$$

which according to (3) means normalization to 1.

Equation (8) relates not only $I_{n+\frac{1}{2}}$ and φ_n , but also the associated functions $H_{n+\frac{1}{2}}^{(1,2)}$ and $\zeta_n^{(1,2)}$; hence equation (8a) holds also for the ζ_n normalized to 1 which we denoted by Z_n on p. 197.

For a general three-dimensional eigenfunction equation (3) holds in the form

$$(10) \quad N = \int d\tau u_n \int_{\Delta} dk' u'_n,$$

where u_n and u'_n are the eigenfunctions belonging to the eigen values k and k' . The eigenfunctions normalized to 1 are then

$$(10a) \quad U_n = u_n / \sqrt{N}.$$

Appendix II

A NEW METHOD FOR THE SOLUTION OF THE EXTERIOR BOUNDARY
VALUE PROBLEM OF THE WAVE EQUATION PRESENTED FOR THE
SPECIAL CASE OF THE SPHERE

The "exterior boundary value problem" consists of the construction of a solution of the wave equation $\Delta u + k^2 u = 0$, which is continuous throughout the exterior of the given bounded surface σ , assumes arbitrarily prescribed boundary values $u = U$ on σ , and satisfies the radiation condition at infinity. We know that this solution is best represented by Green's function which vanishes on σ , satisfies the radiation condition at infinity, and at a prescribed point Q has a discontinuity of the character of a unit source.

In the case of a sphere of radius a and a source point Q with the coordinates $r = r_0$, $\vartheta = 0$, we constructed G in the form of equation (28.22):

$$(1) \quad G = \frac{k}{4\pi i} \begin{cases} \sum C_n u_n(k, r_0) \zeta_n(kr) P_n(\cos \vartheta) & \text{for } r > r_0, \\ \sum C_n \zeta_n(kr_0) u_n(k, r) P_n(\cos \vartheta) & \text{for } r < r_0, \end{cases} \quad C_n = 2n + 1.$$

The radiation condition is satisfied by the factor $\zeta_n(kr)$, or more precisely $\zeta_n^1(kr)$, in the first line, the boundary condition for $r = a$ is satisfied by the factor in the second line

$$(1b) \quad u_n(k, r) = \psi_n(kr) + A \zeta_n(kr), \quad A = -\frac{\psi_n(ka)}{\zeta_n(ka)}.$$

where n is a positive integral and hence $P_n(\cos \vartheta)$ is continuous for all values $0 \leq \vartheta \leq \pi$.

We now attempt to solve this problem in a more economical manner, by subjecting $\zeta_n(kr)$ not only to the radiation condition, but also to the boundary condition

$$(2) \quad \zeta_n(ka) = 0$$

Then ν must be integral, since the roots of $\zeta_\nu(\rho) = 0$ coincide with those of $H_{\nu+\frac{1}{2}}^1(\rho) = 0$. According to (21.41) these roots are non-integral and complex (of large absolute value). We denote the consecutive roots which lie in the first quadrant of the complex ν -plane by ν_1, ν_2, \dots and the general root by ν_m . We use Σ to denote summation over the complete system of these roots, and write

$$(3) \quad G = \Sigma D_\nu \zeta_\nu(kr) P_\nu(-\cos \vartheta).$$

The function P_ν here is not a Legendre polynomial but the hyper-

geometric series of (24.24a)

$$(3a) \quad P_\nu(-\cos \vartheta) = F\left(-\nu, \nu+1, 1, \frac{1+\cos \vartheta}{2}\right).$$

The fact that in (3) we used $P_\nu(-\cos \vartheta)$ instead of $P_\nu(+\cos \vartheta)$ is due to the fact that G is to be regular on the ray $\vartheta = \pi$ whereas the ray $\vartheta = 0$ is to contain the singular point Q . According to (3a) the function $P_\nu(-\cos \vartheta)$ for $\vartheta = \pi$ has the value $F(-\nu, \nu+1, 1, 0) = 1$; for $\vartheta = 0$ we obtain from (24.32), after replacing ζ by $-\zeta = -\cos \vartheta$, the value

$$(4) \quad F(-\nu, \nu+1, 1, 1) \rightarrow \frac{\sin \nu \pi}{\pi} \log \vartheta^2.$$

We now turn to the determination of the coefficients D_ν , to which F. Sauter has made such valuable contributions. We remark: the functions ζ_ν are mutually orthogonal, that is, we have

$$(5) \quad \int_{ka}^{\infty} \zeta_\nu(\varrho) \zeta_\mu(\varrho) d\varrho = 0, \quad \mu \neq \nu$$

This is a consequence of the differential equation for ζ_ν , which in analogy to (21.11) reads:

$$\varrho \frac{d^2 \varrho \zeta_\nu}{d\varrho^2} + [\varrho^2 - \nu(\nu+1)] \zeta_\nu = 0.$$

If we also write the same equation for ζ_μ and multiply these equations by ζ_μ and ζ_ν respectively, then by integrating the difference of these equations over the fundamental domain $ka \leq \varrho < \infty$ we obtain:

$$(5a) \quad \{\nu(\nu+1) - \mu(\mu+1)\} \int_{ka}^{\infty} \zeta_\mu \zeta_\nu d\varrho = \varrho \left(\zeta_\mu \frac{d\varrho \zeta_\nu}{d\varrho} - \zeta_\nu \frac{d\varrho \zeta_\mu}{d\varrho} \right) \Big|_{ka}^{\infty}.$$

The right side of (5a) vanishes at the lower limit on account of (2), at the upper limit on account of the asymptotic behavior of the ζ according to (19.55). Thus (5) is proven. At the same time (5a) yields the normalizing integral

$$N = \int_{ka}^{\infty} [\zeta_\nu(\varrho)]^2 d\varrho = \lim_{\mu \rightarrow \nu} \varrho \frac{\zeta_\mu \frac{d\varrho \zeta_\nu}{d\varrho} - \zeta_\nu \frac{d\varrho \zeta_\mu}{d\varrho}}{\nu(\nu+1) - \mu(\mu+1)} \Big|_{ka}^{\infty}$$

Differentiating the numerator and denominator with respect to μ and considering (2) we obtain

$$(6) \quad N_\nu = \frac{(ka)^2}{2\nu+1} \left(\frac{\partial \zeta_\nu}{\partial \nu} \frac{d\zeta_\nu}{d\varrho} \right)_{\varrho=ka},$$

Introducing the abbreviations

$$(6a) \quad \eta_\nu(\varrho) = \frac{\partial \zeta_\nu(\varrho)}{\partial r}, \quad \zeta'_\nu(\varrho) = \frac{d \zeta_\nu(\varrho)}{d \varrho}.$$

we can rewrite (6) in the shorter form

$$(6b) \quad N_\nu = \frac{(ka)^2}{2\nu+1} \eta_\nu(ka) \zeta'_\nu(ka).$$

If we now multiply (3) by $\zeta_{\bar{\nu}}(kr) k dr$ and integrate from $r = a$ to $r = \infty$ then, on account of the orthogonality and the normalization, we obtain:

$$(7) \quad D_{\bar{\nu}} N_{\bar{\nu}} P_{\bar{\nu}}(-\cos \vartheta) = \int_a^\infty G \zeta_{\bar{\nu}}(kr) k dr.$$

where $\bar{\nu}$ is any one of the roots ν .

However this determination of the D is not yet satisfactory since we do not know G . We therefore perform the following limit process: we divide (7) by $P_{\bar{\nu}}(-\cos \vartheta)$ (we now omit the bars over the ν) and then let ϑ approach zero. We also split off the source point singularity from G by writing

$$(7a) \quad -4\pi G = \frac{e^{ikR}}{R} + g, \quad R^2 = r^2 + r_0^2 - 2rr_0 \cos \vartheta.$$

Then equation (7) becomes

$$(7b) \quad -4\pi D_\nu N_\nu = \lim_{\vartheta \rightarrow 0} \int_a^\infty \left(\frac{e^{ikR}}{R} + g \right) \zeta_\nu(kr) k dr / P_\nu(-\cos \vartheta).$$

Now g remains finite whenever $a < r < \infty$ and ϑ is arbitrary, whereas $P_\nu(-\cos \vartheta)$ becomes infinite as $\vartheta \rightarrow 0$; hence the contribution of g on the right side of (7b) vanishes. For the same reason we may restrict the contribution of e^{ikR}/R to an integration over the immediate neighborhood of the source point coordinate r_0 by writing

$$\begin{aligned} r &= r_0(1 + \eta), & -\varepsilon < \eta < +\varepsilon, & & dr &= r_0 d\eta, \\ \zeta_\nu(kr) &= \zeta_\nu(kr_0), & e^{ikR} &= 1, \end{aligned}$$

while the denominator R is approximated by:

$$R = r_0 \sqrt{(1 + \eta)^2 + 1 - 2(1 + \eta)\left(1 - \frac{\vartheta^2}{2}\right)} \sim r_0 \sqrt{\eta^2 + \vartheta^2}.$$

Thus (7b) becomes

$$(7c) \quad 4\pi D_\nu N_\nu = -k \zeta_\nu(k r_0) \lim_{\vartheta \rightarrow 0} \int_{-\varepsilon}^{+\varepsilon} \frac{d\eta}{\sqrt{\eta^2 + \vartheta^2}} / P_\nu(-\cos \vartheta).$$

The limit on the right side is known from (24.31) to be $\pi/\sin \nu \pi$. Hence with the help of (6) we obtain from (7c) the completely determined value

$$(8) \quad D_\nu = -\frac{1}{4} \frac{2\nu+1}{k a^2 \sin \nu \pi} \frac{\zeta_\nu(k r_0)}{\eta_\nu(k a) \zeta'_\nu(k a)},$$

which no longer depends on ε . Equation (3) for Green's function then assumes the form

$$(9) \quad G = -\frac{1}{4} \frac{1}{k a^2} \sum_\nu \frac{2\nu+1}{\sin \nu \pi} \frac{\zeta_\nu(k r_0)}{\eta_\nu(k a)} \frac{\zeta_\nu(k r)}{\zeta'_\nu(k a)} P_\nu(-\cos \vartheta).$$

This formula becomes considerably simpler if, instead of the ζ , we introduce the normalized eigenfunctions

$$Z_\nu(k r) = \frac{\zeta_\nu(k r)}{\sqrt{N_\nu}}, \quad Z_\nu(k r_0) = \frac{\zeta_\nu(k r_0)}{\sqrt{N_\nu}}$$

Indeed, with the help of (6b) we can then rewrite (9) as

$$(9a) \quad G = -\frac{k}{4} \sum_\nu Z_\nu(k r_0) Z_\nu(k r) \frac{P_\nu(-\cos \vartheta)}{\sin \nu \pi}.$$

Equation (9a) will prove useful later on; for the time being we shall use equation (9), which has the following advantages and disadvantages:

1. In (9) Green's function is represented by the *same* formula both for $r > r_0$ and for $r < r_0$, not by two *different* formulas as in (1).

2. The general requirement of the reciprocity of Green's function is satisfied owing to the fact that (9) is symmetric in r and r_0 . On the other hand in equation (1) the reciprocity of G was expressed by the fact that by interchanging r and r_0 we interchange the two representations for $r > r_0$ and $r < r_0$. The reciprocity of G with respect to the angles ϑ and ϑ_0 (we considered the case in which the latter is zero), can be expressed both in (9) and in (1) by replacing $\cos \vartheta$ by $\cos \vartheta_0$ which is symmetric in ϑ and ϑ_0 .

3. The orthogonality relation (5) is essentially different from our previous formulations: in (5) ζ_ν is multiplied by ζ_μ and not by ζ_μ^* as in (25.11a); also, in (5) $\zeta_\nu \zeta_\mu$ is multiplied by the one-dimensional interval dr not by $r^2 dr$ as in the application of Green's theorem in exercise V.1b.

4. It is remarkable that our representation (9) seems to break

down for $\vartheta = 0$, since then according to (4) every term of the series becomes infinite like $\log \vartheta^2$, whereas the function it represents is to be regular for $\vartheta = 0$ and $r \neq r_0$. Hence the *whole* ray $\vartheta = 0$ (not only the *point* $\vartheta = 0, r = r_0$ on it) must be considered a *singularity* of the representation (9). Hence in the neighborhood of this singularity, that is to say in a more or less narrow cone around this ray, our representation will become more or less useless. The question of the completion of our representation for the interior of such a cone shall be postponed to p. 221.

5. On the other hand equation (9) simplifies for the neighborhood of $\vartheta = \pi$, say for $\vartheta = \pi - \delta$, where in the original form (3) it reads:

$$(10) \quad G = \sum D_\nu \zeta_\nu(kr) P_\nu(\cos \delta).$$

We now let r_0 increase to ∞ , that is to say we pass from the primary *spherical* wave G to a *plane* wave u coming in the direction $\vartheta = 0$. Then for $H_{\nu+1/2}(kr_0)$ we may use Hankel's approximation (19.55) no matter what the index ν since now the argument is large compared to the index. Then according to (21.15) we may write:

$$(10a) \quad \zeta_\nu(kr_0) = \frac{1}{kr_0} e^{i(kr_0 - (\nu+1)\pi/2)}.$$

By combining all the factors which are independent of ν into the amplitude A we obtain from (8),

$$(10b) \quad D_\nu = A \frac{2\nu+1}{\sin \nu \pi} e^{-i\nu\pi/2} / \eta_\nu(ka) \zeta'_\nu(ka).$$

Substituting this value in (10) we obtain the diffraction field of the plane wave u in the rear of a sphere of radius a under the angle of diffraction δ . For the time being we set the boundary condition $u = 0$ for $r = a$; later on we shall discuss a boundary condition that is adapted to electromagnetic optics.

6. The great advantage of (9) as compared to (1) lies in its *rapid convergence for large values of ka* . In order to test this convergence we compute the factors in the denominator of (8) for large ka and ν . If as in (21.30a) we set

$$(11) \quad \cos \alpha = \frac{\nu}{e},$$

then we have according to (21.39)

$$(11a) \quad \zeta_\nu(e) = \frac{i}{e\sqrt{\sin \alpha}} \sin z;$$

where

$$(11b) \quad z = \varrho (\sin \alpha - \alpha \cos \alpha) - \pi/4.$$

The roots of $\zeta_\nu = 0$ are given by

$$(11c) \quad \sin z = 0, \quad z = z_m = -m\pi, \quad \cos z_m = (-1)^m, \quad \cos^2 z_m = 1.$$

Hence according to (11a) we obtain for $z = z_m$, if we ignore the slowly varying factor $\sin \alpha$,

$$\frac{\partial \zeta_\nu}{\partial \alpha} = \frac{i}{\varrho \sqrt{\sin \alpha}} \left(\cos z \frac{dz}{d\alpha} \right)_{z=z_m} = i\alpha \sqrt{\sin \alpha} \cos z_m,$$

therefore by (11)

$$(11d) \quad \eta_\nu = \frac{\partial \zeta_\nu}{\partial \nu} = \frac{\partial \zeta_\nu}{\partial \alpha} \frac{d\alpha}{d\nu} = \frac{-i\alpha}{\varrho \sqrt{\sin \alpha}} \cos z_m.$$

On the other hand, for $z = z_m$ we obtain from (11a,b), if we remember that $\sin z_m = 0$,

$$(11e) \quad \begin{aligned} \zeta'_\nu &= \frac{d\zeta_\nu}{d\varrho} = \frac{\partial \zeta_\nu}{\partial \varrho} + \frac{\partial \zeta_\nu}{\partial \alpha} \frac{d\alpha}{d\varrho} = \frac{i}{\varrho \sqrt{\sin \alpha}} \cos z_m \left(\frac{\partial z}{\partial \varrho} + \frac{\partial z}{\partial \alpha} \frac{d\alpha}{d\varrho} \right)_{z=z_m} \\ &= \frac{i}{\varrho} \sqrt{\sin \alpha} \cos z_m. \end{aligned}$$

Finally we obtain from (11d,e) for $\varrho = ka$

$$(12) \quad \eta_\nu(ka) \zeta'_\nu(ka) = \alpha \left(\frac{\cos z_m}{ka} \right)^2 = \frac{\alpha}{(ka)^2}.$$

If we substitute this in (8) we obtain

$$(13) \quad D_\nu = - \frac{k}{4\alpha} \frac{2\nu+1}{\sin \nu\pi} \zeta_\nu(kr_0).$$

According to (21.41) ν in the first approximation is equal to ka , but with increasing m it increases in the positive imaginary direction. Hence $\sin \nu\pi$ increases exponentially in the sequence ν_1, ν_2, \dots and hence D_ν decreases exponentially due to the denominator $\sin \nu\pi$.

The same thing occurs in (13) due to the factor $\zeta_\nu(kr_0)$. This latter factor is to be computed according to Debye's formula (21.32) (the higher saddle point is the determining one) and not according to (11a) (where the saddle points are approximately of the same height). Hence the auxiliary angle now has a meaning different from that in (11): $\cos \alpha$ is equal to ν/kr_0 and not to ν/ka as before. Hence, $\zeta_\nu(kr_0)$

also decreases exponentially on the ν -sequence. The same thing holds for the additional factor $\zeta_\nu(kr)$ in (9).

In the special case of wireless telegraphy (appendix to Chapter VI) we would need about 1000 terms for a representation of the type (1), whereas, as we shall see, we need only one or two terms of the corresponding series of the type (9). In this appendix we shall also discuss how one type is obtained from the other by a *purely mathematical transformation* (in the complex plane of the index ν).

7. The *structure* of Green's function and its *singular behavior* for $\vartheta = 0$ becomes particularly clear in our representation (9a). In order to obtain a rough estimate for the behavior of Green's function for small ϑ we use the approximation formula (24.32)

$$P_\nu(-\cos \vartheta) \rightarrow \frac{\sin \nu\pi}{\pi} \log \vartheta^2 \quad \text{for } \vartheta \rightarrow 0$$

and then obtain from (9a)

$$(14) \quad G = -\frac{k}{4\pi} \log \vartheta^2 \sum_\nu Z_\nu(kr_0) Z_\nu(kr).$$

Here \sum_ν has a " δ -like character." Namely, if we expand a function in the fundamental interval $a < r < \infty$ in an arbitrary normalized orthogonal system of functions $Z_\nu(kr)$:

$$\delta(r, r_0) = \begin{cases} 0 & \dots & r \neq r_0 \\ \infty & \dots & r = r_0 \end{cases} = \sum A_\nu Z_\nu(kr) \dots \text{with } \int_{r_0-\epsilon}^{r_0+\epsilon} \delta(r, r_0) dr = 1,$$

then we obtain formally:

$$A_\nu = \int_0^\infty \delta(r, r_0) Z_\nu(kr) dr = Z_\nu(kr_0) \int_{r_0-\epsilon}^{r_0+\epsilon} \delta(r, r_0) dr = Z_\nu(kr_0),$$

but as yet we know nothing about the convergence of this general δ -series

$$(14a) \quad \delta(r, r_0) = \sum Z_\nu(kr_0) Z_\nu(kr) \quad \text{for } r \neq r_0$$

Only the divergence for $r = r_0$ (all terms positive) is apparent. The "representation of zero" for $r \neq r_0$ is obtained by an infinitely rapid "oscillation around zero." Hence (14a) is not suitable for the actual computation of G for $\vartheta \rightarrow 0$.

We obtain this representation from the defining equation (7a) of the function g , which is continuous throughout, and which for $r = a$ assumes the boundary values

$$(14b) \quad g = g_a = -\frac{e^{i k R_a}}{R_a}, \quad R_a^2 = a^2 + r_0^2 - 2 a r_0 \cos \vartheta_a;$$

here ϑ_a is the polar distance on the sphere $r = a$. In addition, g must satisfy the differential equation $\Delta g + k^2 g = 0$ in the exterior of the sphere $a < r < \infty$, and the radiation condition at infinity. Hence g can be computed as a solution of the exterior boundary value problem on p. 214, which can be represented in terms of Green's function by

$$g = \int g_a \frac{\partial G}{\partial n} d\sigma_a, \quad \frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial r}\right)_{r=a}.$$

Using the representation (9) of G we obtain for g a series summed over ν , which, on the ray $\vartheta = 0$, $a < r < \infty$, reads:

$$(14c) \quad g(r, \vartheta = 0) = -\frac{\pi}{2} \sum \frac{2\nu+1}{\sin \nu\pi} C_\nu \frac{\zeta_\nu(kr)}{\eta_\nu(ka)},$$

$$(14d) \quad C_\nu = \int_0^\pi g_a P_\nu(-\cos \vartheta_a) \sin \vartheta_a d\vartheta_a.$$

Since the singularity $\vartheta = 0$ of $P_\nu(-\cos \vartheta)$ now occurs only under the integral in (14d) and is only of logarithmic order, the coefficients C_ν are all finite; however their explicit computation¹⁷ does not seem to be easy.

In this appendix we have introduced an entirely new kind of "singular eigenfunctions," which are essentially different from the "regular eigenfunctions" that we have used so far in this chapter. Our singular eigenfunctions

$$\zeta_\nu(kr) P_\nu(-\cos \vartheta)$$

are not *free* oscillations, but require a *stimulation* along the ray $\vartheta = 0$. On the other hand each of the particular solutions

$$u_n(k, r) P_n(\cos \vartheta) \quad \text{and} \quad \zeta_n(kr) P_n(\cos \vartheta)$$

in (1) is *source free* throughout the physical domain $a \leq r < \infty$, $0 \leq \vartheta \leq \pi$. Their stimulation, if we should speak of one, takes place in the *exterior* of this domain, namely in the center $r = 0$, and for u_n at infinity.

In the author's 1912 paper, quoted on p. 183, our "regular" eigenfunctions were called "*improper*" and our "singular" eigenfunctions were called "*proper*." The following discussion may serve to justify this apparently paradoxical notation.

¹⁷ See the discussion of "Whittaker's integral," which is a limiting case of our C_ν , in the textbook by Watson, pp. 239-240.

We start from the fact that for all oscillation problems, whether free or forced, periodic or damped, we have the relation

$$(15) \quad c = \frac{\omega}{k}$$

where we may assume c (the velocity of sound or light) to be real and the carrying medium to be absorption free. Heretofore we have assumed ω to be real and the time dependence to be in the form $\exp\{-i\omega t\}$. The equation

$$(15a) \quad a \cos \alpha = \frac{\nu}{k}$$

which follows from (11), and our condition $\zeta_\nu(ka) = 0$, then yielded a complex value of ν with positive imaginary part, so that for real k the quantity $a \cos \alpha$ also had the same character and hence was of the form

$$(15b) \quad a \cos \alpha = A + iB \quad \text{with } B > 0.$$

Now however, while preserving the relations (15), (15a), (15b), we set ν equal to a positive integer, say $= n$. Then from these relations we obtain complex values for k and ω with negative imaginary parts:

$$k = k_1 - i k_2 = \frac{n}{A + iB} = n \frac{A - iB}{A^2 + B^2}, \quad \omega = \omega_1 - i \omega_2 = c(k_1 - i k_2).$$

The boundary condition $\zeta_\nu(ka) = 0$ now becomes

$$(15c) \quad \zeta_n\{(k_1 - i k_2)a\} = 0$$

and the above purely periodic time dependence factor $\exp\{-i\omega t\}$ becomes

$$\exp\{-i\omega_1 t\} e^{-\omega_2 t} = \exp\{-i c k_1 t\} e^{-c k_2 t}.$$

Appending this time factor and considering the regular character of $P_n(-\cos \vartheta)$ for integral n we obtain from our singular eigenfunction (14) the damped oscillation

$$(16) \quad \zeta_n[(k_1 - i k_2)r] P_n(-\cos \vartheta) \exp\{-i c k_1 t\} e^{-c k_2 t},$$

which is regular throughout the region $r > a$. The amplitude is infinite at $t = -\infty$, decreases at a constant rate, and vanishes at $t = +\infty$, whereas the frequency remains constant. (The only exception is the surface $r = a$, since there we have $\zeta_n(ka) = 0$ throughout.) There exist ∞^2 such oscillations of zonal character. Their parameters are the integer n and the complex roots of the transcendental equation $\zeta_n(ka) = 0$

which are infinite in number. (For tesseral spherical harmonics the number of possible oscillations would increase to ∞^3 .) These damped oscillations are obviously the physically simplest particular solutions of our sphere problem for the boundary condition $u = 0$ and hence deserve the name "proper eigenfunctions." Their close connection with our singular eigenfunctions (16) explain how the latter led to the simplest solution of the boundary value problem.

In the following discussion we apply these eigenfunctions, which so far have been developed only for scalar fields, to the case of the electromagnetic-optic field. We shall see in Chapter VI that this can be done without difficulty. We have to keep in mind the following facts:

1. The boundary condition $u = 0$ must be replaced in the electromagnetic case by

$$(17) \quad \frac{\partial}{\partial r} (r u) = 0,$$

as will be shown in Chapter VI.

2. In the transcendental equation $\zeta_n(ka) = 0$, or, as we wrote in (15c), $\zeta_n(ka) = 0$, we have to replace the function $\zeta_n(\varrho)$ by

$$\xi_n(\varrho) = \frac{\partial}{\partial r} \{r \zeta_n(\varrho)\},$$

which, due to $\varrho = kr$, is the same as

$$(17a) \quad \xi_n(\varrho) = \zeta_n(\varrho) + \varrho \zeta'_n(\varrho).$$

In the same manner we have to replace the function η in (6a) by

$$(17b) \quad \eta_n(\varrho) = \frac{\partial \xi_n(\varrho)}{\partial n}.$$

3. While in the scalar case we had *one* field function u , in the electromagnetic case we have *two* such functions u and v . The function v satisfies the same differential equation as u and a similar boundary condition.

Our damped electromagnetic eigen-oscillations have long been known in the literature. In the case of the *sphere* they were investigated by J. J. Thomson¹⁸ in 1884 as the simplest case of the *Hertz oscillator* which was then the center of interest. They were generalized by M. Abraham¹⁹ to the case of the elongated *ellipsoid of revolution* (rod-like oscillator) and the *Paraboloid of revolution* (wire with free ends). Indeed,

¹⁸ *London Math. Soc. Proc.* 15, 197, and the textbook *Recent Researches in Electricity and Magnetism*, Oxford 1893, the so-called "third volume of Maxwell."

¹⁹ *Ann. Physik* 66, 435 (1898); 67, 834 (1899); *Math. Ann.* 52, 81 (1899). For further literature see *Enzykl. d. Math. Wiss.* v. V. 2, Abraham's article, p. 508.

our entire development can, with the introduction of elliptic coordinates, be adapted without fundamental change from the cylindrical and spherical harmonics of the representation (16) to the domain of the Lamé wave functions. We can use this method in order to construct Green's function for the exterior of an ellipsoid or paraboloid and thus obtain general solutions to the associated boundary value problems.

We finally indicate some problems for which the method of this appendix is helpful.

a) *Dispersion by colloidal particles.* In a 1908 paper G. Mie deduced the impressive color phenomena seen in the ultramicroscope from the dielectricity constant and the conductivity of the individual scattering particles. The particle was assumed to be spherical with a diameter *small compared to the wavelength*, i.e., $ka \ll 1$. In this case the series of type (1) converge sufficiently rapidly. In the opposite case $ka \gg 1$ the use of geometrical optics suffices; but the intermediate case gives rise to difficulties. In this intermediate case we have to use series of the type (9) as specialized for a source at infinity. The fact that in our case the sphere was assumed to be infinitely conductive, while in Mie's case it was assumed to be an arbitrary dispersive medium, does not make an important difference. We must merely replace the *boundary condition* (equation (17)) for the complete conductor by a *transition condition* between the interior and the exterior. The convergence of the series will be the better the nearer we are to the limiting case of geometrical optics.

b) *The reflection of a plane wave on the surface of a completely conductive sphere.* The diffraction field in the *rear* of a sphere was discussed schematically (i.e., with the simplified boundary condition $u = 0$ and for a scalar field) under 5 above and was represented by the equations (10), (10a) for $ka \gg 1$. On the *front* of the sphere, especially for $\vartheta = 0$, we know from experience of strange interference phenomena, which so far have not been amenable to the usual treatment by series of type (1). The analytical difficulties which arise here are expressed by the singularity of the ray $\vartheta = 0$ in series of the type (9). However, we claim that this problem can be treated in the manner indicated on p. 221, if we take into consideration the actual conditions of the reflection problem.

c) *The rainbow.* With this classic problem we return to the starting point of Debye's asymptotic investigations (see p. 117) and all subsequent advances in the domain of short waves ($ka \gg 1$). The rainbow problem has since been brought to a beautiful conclusion by B. Van der Pol and H. Bremmer.²⁰ However, from the viewpoint of method there remains a gap between the wave-optical and geometric-optical method.

It was the task of this appendix to bridge such gaps mathematically.

²⁰ *Phil. Mag.* 24, 141, 825 (1937).

Appendix III

THE WAVE MECHANICAL EIGENFUNCTIONS OF THE SCATTERING PROBLEM
IN PARABOLIC COORDINATES

In the following discussion we outline the steps which lead to the representation (30.7). For details the reader is referred to textbooks on wave mechanics.²¹

The parabolic coordinates $\xi = r + x$, $\eta = r - x$ define, in a plane which passes through the x -axis, a double system of confocal parabolas which have the point $r = 0$ as a common focus. The degenerate parabolas $\xi = 0$, $\eta = 0$ coincide with the negative and positive x -axis respectively; the parabolas $\xi = \infty$, $\eta = \infty$ limit the plane in the direction of large positive and negative x respectively.

If we rotate the plane around the x -axis then ξ , η together with the rotation angle φ form a spatial coordinate system which bears the following relation to the Cartesian coordinates x, y, z :

$$x = \frac{1}{2}(\xi - \eta), \quad y = \sqrt{\xi \eta} \cos \varphi, \quad z = \sqrt{\xi \eta} \sin \varphi.$$

From this we obtain the line element

$$(1) \quad ds^2 = \frac{1}{4}(\xi + \eta) \left(\frac{d\xi^2}{\xi} + \frac{d\eta^2}{\eta} \right) + \xi \eta d\varphi^2.$$

With its help $\Delta\psi$ is transformed into

$$\Delta\psi = \frac{4}{\xi + \eta} \left(\frac{\partial}{\partial \xi} \xi \frac{\partial \psi}{\partial \xi} + \frac{\partial}{\partial \eta} \eta \frac{\partial \psi}{\partial \eta} \right) + \frac{1}{\xi \eta} \frac{\partial^2 \psi}{\partial \varphi^2}.$$

The wave equation (29.1) for an interaction energy

$$V = \frac{ZZ'e^2}{r} = \frac{2ZZ'e^2}{\xi + \eta}$$

and for independence from φ , becomes

$$(2) \quad \frac{\partial}{\partial \xi} \xi \frac{\partial \psi}{\partial \xi} + \frac{\partial}{\partial \eta} \eta \frac{\partial \psi}{\partial \eta} + \frac{m_\alpha}{2\hbar^2} [(\xi + \eta) W - 2ZZ'e^2] \psi = 0.$$

This can be separated by setting $\psi = \psi_1(\xi) \psi_2(\eta)$; with β as the separation constant we then obtain:

²¹ For example, the author's *Atombau und Spektrallinien*, v. II, Chapter V. §6 and Chapter II, §9. There, in addition, the asymptotic representation (30.8) is derived with the help of a complex integral representation of L that we cannot discuss here.

$$(3) \quad \frac{d}{d\xi} \xi \frac{d\psi_1}{d\xi} + \left(\frac{m_\alpha W}{2\hbar^2} \xi - \frac{m_\alpha Z Z' e^2}{2\hbar^2} + \beta \right) \psi_1 = 0$$

$$(4) \quad \frac{d}{d\eta} \eta \frac{d\psi_2}{d\eta} + \left(\frac{m_\alpha W}{2\hbar^2} \eta - \frac{m_\alpha Z Z' e^2}{2\hbar^2} - \beta \right) \psi_2 = 0.$$

The function ψ_1 must satisfy the *radiation condition* (28.2) for $\xi \rightarrow \infty$ (large positive x). Written in parabolic coordinates (according to (1)) ds_ξ is equal to $\frac{1}{2} d\xi$ for large ξ , whence $\partial/\partial r$ in (28.2) becomes $\partial/\partial s_\xi = (d\xi/ds_\xi) \partial/\partial \xi \equiv 2 \partial/\partial \xi$, and we have:

$$(5) \quad \frac{\xi}{2} \left(2 \frac{d\psi_1}{d\xi} - i k \psi_1 \right) = 0 \text{ with } k = \frac{m_\alpha v}{\hbar}, \text{ as in (30.7a).}$$

Hence we set $\psi_1 = \exp \left(\frac{i k}{2} \xi \right)$ and get from (3)

$$(6) \quad \left(-\frac{k^2}{4} \xi + \frac{i k}{2} + \frac{m_\alpha W}{2\hbar^2} \xi - \frac{m_\alpha Z Z' e^2}{2\hbar^2} + \beta \right) \psi_1 = 0.$$

The terms with ξ cancel because of the meaning of k and W in equation (30.1). Therefore (6) is satisfied by choosing

$$(7) \quad \beta = \frac{m_\alpha Z Z' e^2}{2\hbar^2} - \frac{i k}{2}.$$

We see that for this ψ_1 equation (3) is satisfied not only asymptotically but for all ξ .

Due to (7) equation (4) becomes

$$(8) \quad \frac{d}{d\eta} \eta \frac{d\psi_2}{d\eta} + \left(\frac{m_\alpha W}{2\hbar^2} \eta - \frac{m_\alpha Z Z' e^2}{\hbar^2} + \frac{i k}{2} \right) \psi_2 = 0.$$

The function ψ_2 must satisfy the *absorption condition* for $\eta \rightarrow \infty$ (large negative x), which, written in analogy to (5), reads:

$$\frac{\eta}{2} \left(2 \frac{d\psi_2}{d\eta} + i k \psi_2 \right) = 0.$$

Hence for large η we have the first approximation

$$\psi_2 = \exp (-i k \eta / 2).$$

However, this is not an exact solution of (8). Hence we set the more general

$$\psi_2 = e^{-i k \eta / 2} f(\eta).$$

From (8) we obtain the equation for $f(\eta)$

$$(9) \quad \eta \frac{d^2 f}{d\eta^2} + (1 - i k \eta) \frac{df}{d\eta} - m_\alpha Z Z' \frac{e^2}{\hbar^2} f = 0.$$

Equation (9) is the differential equation (29.12) of the Laguerre function $L_\mu(\varrho)$, if we set

$$\varrho = i k \eta, \quad \mu = -\frac{m_\alpha Z Z' e^2}{i k \hbar^2}.$$

The last value coincides with the imaginary total quantum number n of (30.7b). Hence we have

$$f(\eta) = L_n(i k \eta), \quad \psi_2 = e^{-i k \eta/2} L_n(i k \eta)$$

and finally

$$(10) \quad \psi = \psi_1(\xi) \psi_2(\eta) = e^{i k (\xi - \eta)/2} L_n(i k \eta).$$

Thus we have a quick proof of (30.7.)

Appendix IV

PLANE AND SPHERICAL WAVES IN UNLIMITED SPACE OF AN ARBITRARY NUMBER OF DIMENSIONS

After having treated plane and spherical waves in three-dimensional space and plane and cylinder waves in two-dimensional space, we cannot resist the temptation to adapt these formulas to the many-dimensional case. In this connection we shall encounter remarkable generalizations of the ordinary spherical harmonics, the *Gegenbauer polynomials*, and generalized addition theorems of the *Bessel functions*. A systematic approach to these generalizations is again given by our theorem in §27 about the representation of Green's function in terms of the eigenfunctions for the space in question.

A. COORDINATE SYSTEM AND NOTATIONS

Let the number of dimensions be $p + 2$ so that $p = 0$ represents two-dimensional, and $p = 1$ represents three-dimensional space. On the one hand we use the Cartesian coordinates x_1, x_2, \dots, x_{p+2} , and on the other hand the polar coordinates $r, \vartheta, \varphi_1, \varphi_2, \dots, \varphi_p$. The connection shall be given by

$$(1) \quad \begin{aligned} x_1 &= r \cos \vartheta \\ x_2 &= r \sin \vartheta \cos \varphi_1 \\ x_3 &= r \sin \vartheta \sin \varphi_1 \cos \varphi_2 \\ &\vdots \\ x_{p+1} &= r \sin \vartheta \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{p-1} \cos \varphi_p \\ x_{p+2} &= r \sin \vartheta \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{p-1} \sin \varphi_p. \end{aligned}$$

In order to cover the whole space $-\infty < x_i < +\infty$ the coordinates $r, \theta, \dots, \varphi_p$ must vary between the limits

$$0 < r < \infty, \quad 0 < \theta < \pi, \quad \begin{cases} 0 < \varphi_i < +\pi, & i = 1, 2, \dots, p-1, \\ -\pi < \varphi_p < +\pi. \end{cases}$$

By forming the sum of the squares in (1) we obtain

$$(1a) \quad \sum_{i=1}^{p+2} x_i^2 = r^2.$$

The definition of the $(p+2)$ -dimensional line element is

$$(1b) \quad ds^2 = \sum_{i=1}^{p+2} dx_i^2.$$

If for every direction of the coordinates r, θ, φ_i we compute the corresponding ds from (1) then we obtain (in unified form)

$$(2) \quad ds = (dr, r d\theta, r \sin \theta d\varphi_1, r \sin \theta \sin \varphi_1 d\varphi_2, \dots, r \sin \theta \sin \varphi_1 \dots \sin \varphi_{p-1} d\varphi_p).$$

The coefficients of $dr, d\theta, d\varphi_1, \dots, d\varphi_p$ on the right side of (2) will be denoted by g_1, g_2, \dots, g_{p+2} . We then have

$$(2a) \quad g_1 = 1, \quad g_2 = r, \quad g_3 = r \sin \theta, \quad g_4 = r \sin \theta \sin \varphi_1, \\ \dots, g_{p+2} = r \sin \theta \sin \varphi_1 \dots \sin \varphi_{p-1}.$$

From (2) and (2a) we obtain the $(p+2)$ -dimensional volume element

$$(2b) \quad d\tau = \tilde{g} dr d\theta d\varphi_1 \dots d\varphi_p,$$

$$(2c) \quad \tilde{g} = \prod_{i=1}^{p+2} g_i = r^{p+1} \sin^p \theta \sin^{p-1} \varphi_1 \sin^{p-2} \varphi_2 \dots \sin \varphi_{p-1}.$$

We denote the surface element of the unit sphere in $(p+2)$ -dimensional space by $d\omega$, its total surface by Ω and set

$$(2d) \quad \Omega = \int d\omega = \Omega_\theta \Omega_\varphi,$$

where Ω_θ and Ω_φ are the components obtained through integration of $d\omega$ with respect to θ and $\varphi_1, \varphi_2, \dots, \varphi_p$ respectively. From (2b,c) we obtain

$$(2e) \quad \Omega_\theta = \int_0^\pi \sin^p \theta d\theta = \frac{\pi}{2^p} \frac{\Gamma(p+1)}{\Gamma(\frac{p}{2}+1) \Gamma(\frac{p}{2}+1)} = \frac{\pi}{p} \frac{2^{-p+2} \Gamma(p)}{\Gamma(p/2) \Gamma(p/2)}.$$

$$(2f) \quad \Omega_\varphi = \int_0^\pi \sin^{p-1} \varphi_1 d\varphi_1 \int_0^\pi \sin^{p-2} \varphi_2 d\varphi_2 \dots \int_0^\pi \sin \varphi_{p-1} d\varphi_{p-1} \int_{-\pi}^{+\pi} d\varphi_p \\ = 2\pi \frac{\frac{p+1}{2}}{\Gamma(\frac{p+1}{2})}.$$

We denote the Laplace operator in our space by Δ_p (thus in three-dimensional space we would denote it by Δ_1) and we write for a function u which depends on r alone

$$(3) \quad \Delta_p u = \frac{1}{g} \frac{d}{dr} \left(\frac{g}{g_1^2} \frac{du}{dr} \right) = \frac{1}{r^{p+1}} \frac{d}{dr} \left(r^{p+1} \frac{du}{dr} \right).$$

The potential equation $\Delta_p u = 0$ then becomes

$$(3a) \quad \frac{d^2 u}{dr^2} + \frac{p+1}{r} \frac{du}{dr} = 0.$$

Except for additive and multiplicative constants of integration we obtain the solution:

$$(4) \quad u = r^{-p}.$$

We generalize this solution to

$$(4a) \quad u = R^{-p}, \quad R^2 = \sum (x_i - y_i)^2.$$

If we place the second point introduced here on the axis $\theta = 0$ and denote its distance from the origin by r_0 then according to (1) we have $y_1 = r_0, y_2 = y_3 = \dots = y_{p+2} = 0$ and $R^2 = r^2 - 2r r_0 \cos \theta + r_0^2$. Hence

$$(4b) \quad u = \frac{r_0^p}{R^p} = \frac{1}{\left[1 + \left(\frac{r}{r_0} \right)^2 - 2 \frac{r}{r_0} \cos \theta \right]^{p/2}}$$

is also a solution of $\Delta_p u = 0$.

As in (22.3) we expand (4b) in ascending (or descending) powers of r/r_0 and call the coefficients p -dimensional zonal spherical harmonics

$$P_n(\cos \theta | p)$$

or also *Gegenbauer polynomials*.²² The *Legendre polynomials* may thus be denoted by

$$P_n(\cos \theta | 1).$$

Hence we write²³

$$(5) \quad \left[1 + \left(\frac{r}{r_0} \right)^2 - 2 \frac{r}{r_0} \cos \theta \right]^{-p/2} = \sum_{n=0}^{\infty} \left(\frac{r}{r_0} \right)^n P_n(\cos \theta | p),$$

and deduce from this

²² Gegenbauer's original notation (see e.g., *Wien. Akad.* 70 (1875)) is $C_n^p(\cos \theta)$, where $p = p/2$.

²³ The defining equation (5) is not limited to integral p ; equation (5) breaks down for $p = 0$ since in that case (4b) has to be replaced by the two-dimensional logarithmic potential.

$$(5a) \quad \begin{array}{l|l} \cos \vartheta = 1 & P_n(1|p) = (-1)^n \binom{-p}{n} = \frac{(p+n-1)!}{n!(p-1)!} \\ \cos \vartheta = -1 & P_n(-1|p) = (-1)^n P_n(1|p) \\ & P_{2s+1}(0|p) = 0 \\ \cos \vartheta = 0 & P_{2s}(0|p) = \binom{-p/2}{s} = (-1)^s \frac{\Gamma(\frac{p}{2} + s)}{\Gamma(s+1)\Gamma(\frac{p}{2})}. \end{array}$$

For the particular solution of the potential equation $\Delta_p u = 0$ which depends only on r and ϑ

$$(5b) \quad u = r^n P_n(\cos \vartheta|p)$$

we then obtain the differential equation

$$\frac{1}{\bar{g}} \left\{ \frac{\partial}{\partial r} \left(\frac{\bar{g}}{g_1^2} \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \vartheta} \left(\frac{\bar{g}}{g_2^2} \frac{\partial u}{\partial \vartheta} \right) \right\} = 0;$$

and after dividing out the factor $r^{n+p-1}\bar{g}$ we obtain the ordinary differential equation for $P_n(\cos \vartheta|p)$

$$(5c) \quad \left[\frac{d}{d\vartheta} \sin^p \vartheta \frac{d}{d\vartheta} + n(n+p) \sin^p \vartheta \right] P_n(\cos \vartheta|p) = 0.$$

The reader is asked to check the connection of these and the following formulas with the formulas from the theory of ordinary spherical harmonics.

The Gegenbauer polynomials can be expressed in terms of hypergeometric series in a manner similar to that of the Legendre polynomials in (24.24a); we have

$$(5d) \quad P_n(\cos \vartheta|p) = P_n(1|p) F\left(-n, n+p, \frac{p+1}{2}, \frac{1-\cos \vartheta}{2}\right).$$

B. THE EIGENFUNCTIONS OF UNLIMITED MANY-DIMENSIONAL SPACE

From the potential equation we pass to the wave equation. For a function which depends only on r the wave equation is, according to (3),

$$(6) \quad \frac{d^2 u}{dr^2} + \frac{p+1}{r} \frac{du}{dr} + k^2 u = 0.$$

If we set $u = r^{-p/2} w$, then we obtain the Bessel differential equation with index $p/2$ for w . Hence (6) is integrated by

$$(6a) \quad u = r^{-p/2} I_{p/2}(kr),$$

and also by

$$(6b) \quad u = r^{-p/2} H_{p/2}^1(kr), \quad (6c) \quad u = r^{-p/2} H_{p/2}^2(kr).$$

The function in (6b) behaves asymptotically like

$$C e^{i k r} / r^{\frac{p+1}{2}}, \quad C = \sqrt{\frac{2}{k \pi}} e^{-\frac{p+1}{2} \frac{i \pi}{2}}$$

and satisfies the radiation condition (28.7)

$$\lim_{r \rightarrow \infty} r^{\frac{p+1}{2}} \left(\frac{\partial u}{\partial r} - i k u \right) = 0;$$

In the same manner (6c) satisfies the absorption condition. Hence (6b,c) represent the radiated and absorbed *spherical waves in* $(p+2)$ -dimensional space. This remains valid for a general position of the source point with the solutions

$$(7) \quad U = R^{-p/2} H_{p/2}(kR), \quad R^2 = r^2 - 2r r_0 \cos \theta + r_0^2.$$

The function in (6a) may be called "eigenfunction of *spherical symmetry*." We now want to find the general eigenfunctions of *zonal symmetry*. They are of the form

$$(8) \quad u_n(r, \theta) = v_n(r) P_n(\cos \theta | p).$$

From the equation (5c) of P_n we find the differential equation of v_n

$$\left(\frac{d^2}{dr^2} + \frac{p+1}{r} \frac{d}{dr} + k^2 - \frac{n(n+p)}{r^2} \right) v_n = 0.$$

If we treat this equation as we did (6) by setting $v_n = r^{-p/2} w$ then for w we obtain the Bessel differential equation with index $n + p/2$, and hence as the solution which is finite for $r = 0$

$$w = I_{n+p/2}(kr).$$

Hence the eigenfunction becomes

$$(8a) \quad u_n = r^{-p/2} I_{n+p/2}(kr) P_n(\cos \theta | p).$$

According to §26 any two of these eigenfunctions are mutually *orthogonal*, both in the continuous spectrum $0 < k < \infty$, and in the discrete spectrum $n = 0, 1, 2, \dots$

For two eigenfunction u_n, u_m with equal k but different indices we obtain from (2b,c) and (8):

$$(9) \quad \int u_n u_m d\tau = \int_0^\infty I_{n+p/2}(kr) I_{m+p/2}(kr) r dr \int_0^\pi P_n(\cos \theta | p) P_m(\cos \theta | p) \sin^p \theta d\theta \Omega_\varphi.$$

where Ω_φ is as in (2f). Due to the fact that neither Ω_φ nor the integral with respect to r vanish and due to the orthogonality of u_n and u_m we obtain:

$$(10) \quad \int_0^\pi P_n(\cos \theta | p) P_m(\cos \theta | p) \sin^p \theta d\theta = 0, \quad m \neq n.$$

Note the characteristic factor $\sin^p \theta$ in (10), which in the three-dimensional case ($p = 1$) becomes the customary factor $\sin \theta$ for the Legendre polynomials. While in the customary analytic derivation of (10) this factor might appear artificial, it follows in our many-dimensional approach directly from the meaning of $d\tau$.

We also note the corresponding normalizing integral for $m = n$

$$(11) \quad N = \int_0^\pi [P_n(\cos \theta | p)]^2 \sin^p \theta d\theta = \frac{\Gamma(n+p)}{2^{p-1} (n+p/2) n! \Gamma(p/2)} \frac{\pi}{\Gamma(p/2)}$$

which is a generalization of the normalizing integral for ordinary zonal spherical harmonics: $N = 1/(n + \frac{1}{2})$ for $p = 1$. The proof of (11) starts from the defining equation (5) of the Gegenbauer polynomials.

With the help of (2e) we can replace (11) by:

$$(11a) \quad N = \frac{p}{2} \frac{\Gamma(n+p)}{(n+p/2) n! \Gamma(p)} \Omega_p.$$

C. SPHERICAL WAVES AND GREEN'S FUNCTION IN MANY-DIMENSIONAL SPACE

The spherical wave of zonal symmetry has been described by equation (7). From this function we obtain Green's function of $(p+2)$ -dimensional unlimited space by adding a factor f such that the source Q of U becomes a unit source. According to §10 C this means

$$(12) \quad 1 = \int \frac{\partial G}{\partial n} d\sigma = f \int \frac{\partial U}{\partial R} d\sigma = f \int \frac{\partial}{\partial R} [R^{-p/2} H_{p/2}^1(kR)] R^{p+1} d\omega,$$

where the integration is to be taken over a sphere of radius $R \rightarrow 0$; $d\sigma$ denotes the surface element on this sphere; $d\omega$, as in (2d), denotes the surface element on the unit sphere. Hence we obtain from (12)

$$(12a) \quad 1 = f \Omega \lim_{R \rightarrow 0} R^{p+1} \frac{\partial}{\partial R} [R^{-p/2} H_{p/2}^1(kR)].$$

For odd p we can use the formula (19.31) for H , which yields

$$\lim_{R \rightarrow 0} \dots = \frac{-i R^{p+1} \partial}{\sin p\pi/2 \partial R} [R^{-p/2} I_{-p/2}(kR)] = \frac{i p}{\sin p\pi/2} \frac{(k/2)^{-p/2}}{\Gamma(-p/2+1)}.$$

Using a well-known Γ -relation we can replace this by

$$\frac{i p}{\pi} \left(\frac{k}{2}\right)^{-p/2} \Gamma\left(\frac{p}{2}\right).$$

For even p we obtain the same value from (19.26) and (19.47). Hence we obtain from (12a)

$$(12b) \quad f = \frac{\pi}{\Omega} \left(\frac{k}{2}\right)^{p/2} / i p \Gamma\left(\frac{p}{2}\right)$$

and from (7) upon multiplication by f

$$(13) \quad G(P, Q) = \frac{\pi}{\Omega} \left(\frac{k}{2}\right)^{p/2} \left(\frac{kR}{2}\right)^{-p/2} H_{p/2}^1(kR) / i p \Gamma\left(\frac{p}{2}\right).$$

On the other hand we want to construct $G(P, Q)$ as in §28 from the eigenfunctions $u(P)$ in (8a)

$$(13a) \quad u(P) = r^{-p/2} I_{n+p/2}(\lambda r) P_n(\cos \theta|p)$$

and the associated $u(Q)$ for a point Q with the coordinates $r = r_0$, $\theta_0 = 0$:

$$(13b) \quad u(Q) = r_0^{-p/2} I_{n+p/2}(\lambda r_0) P_n(1|p).$$

In both representations (13a,b) λ (see equation (28.14)) denotes the variable of integration in the continuous part of the eigen value spectrum. In a similar manner we perform the integration over λ in the complex λ -plane and obtain in analogy to (28.15)

$$(14) \quad \int u(P) u(Q) \frac{d\lambda}{k^2 - \lambda^2} = \frac{\pi i}{2} (r r_0)^{-p/2} \sum_n P_n(\cos \theta|p) P_n(1|p) \begin{cases} I_{n+p/2}(k r_0) H_{n+p/2}(k r) & r > r_0, \\ I_{n+p/2}(k r) H_{n+p/2}(k r_0) & r < r_0. \end{cases}$$

In order to be able to apply this formula to Green's function we still must normalize the functions $u(P)$ and $u(Q)$ to one. The general term on the right side of (14) must therefore be: 1) divided by the normalizing factor N of (11a) due to the dependence on θ ; 2) divided by Ω_θ of (2f) due to its independence of the coordinates $\varphi_1, \varphi_2, \dots, \varphi_p$, and 3) multiplied by k due to the r -dependence according to Appendix I, equation (4). Altogether this yields the factor (see also (5a))

$$(14a) \quad \frac{k}{\Omega} \frac{2n+p}{p} \frac{n! \Gamma(p)}{\Gamma(n+p)} = \frac{k}{\Omega} \frac{2n+p}{p} / P_n(1|p),$$

which has to be introduced under the Σ -sign of (14). Thus, according to our general theorem of §28 we obtain Green's function of unlimited space. Comparing this with (13) we obtain:

$$(15) \quad \frac{H_{p/2}(kR)}{(kR)^{p/2}} = 2^{p/2} \Gamma\left(\frac{p}{2}\right) \sum_{n=0}^{\infty} \left(n + \frac{p}{2}\right) P_n(\cos \theta|p) \left\{ \right\},$$

$$\left\{ \right\} = \begin{cases} \frac{I_{n+p/2}(k r_0)}{(k r_0)^{p/2}} \frac{H_{n+p/2}(k r)}{(k r)^{p/2}} \dots & r > r_0, \\ \frac{I_{n+p/2}(k r)}{(k r)^{p/2}} \frac{H_{n+p/2}(k r_0)}{(k r_0)^{p/2}} \dots & r < r_0. \end{cases}$$

This *general addition theorem of Bessel functions* holds both for $H = H^1$ and for $H = H^2$, and hence for any linear combination of the two, so that in (15) we may replace H on both sides by

$$Z = c_1 H^1 + c_2 H^2,$$

hence, in particular by

$$I = \frac{1}{2}(H^1 + H^2),$$

in which latter case the distinction $r \geq r_0$ becomes immaterial. The theorem holds, under more general conditions than those assumed in the derivation, if, as in footnote 22, we replace $p/2$ by an arbitrary number, say ν .

D. PASSAGE FROM THE SPHERICAL WAVE TO THE PLANE WAVE

For $r_0 \rightarrow \infty$ we deduce a representation of the plane wave in many-dimensional space from the last line of (15)

First we obtain on the right side, according to Hankel's approximation (19.55),

$$H_{\pi + p/2}^1(k r_0) = a e^{-i \pi/2}, \quad a = \sqrt{\frac{2}{\pi k r_0}} \exp \left\{ i \left(k r_0 - \frac{p+1}{2} \frac{\pi}{2} \right) \right\}.$$

Correspondingly, on the left side we obtain

$$H_{p/2}^1(k R) = \sqrt{\frac{2}{\pi k R}} \exp \left\{ i \left(k R - \frac{p+1}{2} \frac{\pi}{2} \right) \right\}.$$

However for $r_0 \rightarrow \infty$ we have

$$R = r_0 \left(1 - 2 \frac{r}{r_0} \cos \theta + \dots \right)^{\frac{1}{2}} = r_0 - r \cos \theta + \dots;$$

therefore,

$$\begin{aligned} H_{p/2}^1(k R) &= \sqrt{\frac{2}{\pi k r_0}} \exp \left\{ i \left(k r_0 - k r \cos \theta - \frac{p+1}{2} \frac{\pi}{2} \right) \right\} \\ &= a \exp \{ -i k r \cos \theta \}, \end{aligned}$$

so that the left side of (15), with the corresponding approximation for the denominator, becomes:

$$\frac{a}{(k r_0)^{p/2}} \exp \{ -i k r \cos \theta \}.$$

After canceling the common factor on both sides we obtain

$$(16) \quad e^{-i k r \cos \theta} = 2^{p/2} \Gamma\left(\frac{p}{2}\right) \sum_{n=0}^{\infty} \left(n + \frac{p}{2}\right) e^{-i n \pi/2} P_n(\cos \theta | p) \frac{I_{n+p/2}(k r)}{(k r)^{p/2}}.$$

This represents an incoming wave in the direction of the positive axis $\theta = 0$, or, in other words, a wave which proceeds in the negative direction of this axis. The wave which proceeds in the positive direction is obtained from (16) by replacing $+i$ with $-i$. The reader is asked to verify that this formula coincides for $p = 1$ with the three-dimensional representation (24.7). In the two-dimensional case in which (4b) breaks down (see footnote 22) equation (16) is replaced by the representation (21.2b).

Through a suitable averaging or with the help of an "addition theorem"²⁴ of Gegenbauer polynomials" we obtain from (16) remarkable relations between Bessel functions of integral and of fractional indices.²⁵

²⁴ See the lucid collection of Gegenbauer's results in the book by Magnus and Oberhettinger, *Formeln und Sätze über die speziellen Funktionen der Mathematischen Physik*, Springer, 1943, particularly p. 77.

²⁵ G. Bauer, *Sitzungsber. bayr. Akad.*, 1875, p. 247; generalization to higher dimensions, A. Sommerfeld, *Math. Ann.* 119, 1 (1943).

CHAPTER VI

Problems of Radio

The problems of signals with electric waves have been in the foreground of applied physics since the beginning of the century. Can we understand the remarkable range of radio signals from the otherwise completely reliable Maxwell theory? The answer is both yes and no. Yes, in so far as only the known electrodynamic laws are applied. No, in so far as the ionosphere (Kenelly-Heaviside layer) plays an essential role in overcoming the curvature of the earth, and has to be added to the Maxwell wave propagation as a *deus ex machina*.

Unfortunately we shall be unable to treat the reflection processes in the ionosphere, and shall restrict ourselves to *questions of propagation in the homogeneous atmosphere and in the earth which is also assumed to be homogeneous*. We shall also have to omit the questions of the construction of transmitters and receivers, which are of such great importance for the engineer, since they do not properly belong to the domain of partial differential equations. Instead, we shall idealize the transmitter to the utmost and treat it as a *Hertz dipole* (§31). On the other hand the questions of propagation definitely belong to our domain and they will give us a complete demonstration of the usefulness of the methods which we have developed above and which we have so far applied mainly to rather artificial problems of heat conduction and of potential theory. Further demonstrations of this usefulness are given by problems in general electrodynamic oscillations. They are treated with some completeness in the textbook by Frank-Mises, Chapter XXIII, and the reader is referred to that book. If, among these problems, we again consider radio, it is because the previous representation was simplified so drastically that it could not be reconciled with practical problems. Now we shall not place our antenna-dipole on the surface of the earth, but at some distance from it, we shall treat the radiation of the horizontal antenna in more detail and demonstrate its asymptotic identity with the radiation of the vertical antenna for increasing distance from the origin, and we shall treat the radiation characteristic with respect to the terms of second order in $1/r$, etc. The energy conditions (required energy supply for prescribed antenna current, heat loss in the earth) will be discussed in the final section. We shall almost always consider the earth as a *plane*. The analytically interesting problem of the earth's

curvature, which opens a further domain of application to the method of eigenfunctions, can be treated only in an appendix, since even an only moderately complete treatment of the problem of a plane earth is almost too long for us here.

§ 31. The Hertz Dipole in a Homogeneous Medium Over a Completely Conductive Earth

We assume that the reader has a knowledge of the concepts of electrodynamics and their interconnection through Maxwell's equations. Since we are not dealing with atomic physics but only with the phenomenological Maxwell theory, we shall use the system of the four units, M (meter), K (kilogram mass), S (second), Q (charge, measured in Coulombs). In this system the specific inductive capacity and the permeability are definite quantities; as usual their values in a vacuum are denoted by ϵ_0 and μ_0 . We then have $\epsilon_0 \mu_0 = 1/c^2$. The parasite factor 4π , which mars the customary electromagnetic equations, is suppressed in our system through the suitable choice of units, wherever it is not implied by the spherical symmetry of the problem.

A. INTRODUCTION OF THE HERTZ DIPOLE

In the electrostatic case we deduce the potential of the dipole by an oriented differentiation from the fundamental potential $\Phi = 1/r$ (see §24 C); the field \mathbf{E} of the dipole is then obtained from this potential by another differentiation. In the electrodynamic case Φ is replaced by the function of the spherical wave

$$(1) \quad \Pi = \frac{1}{r} e^{ikr}, \quad \text{or more completely} \quad \Pi = \frac{1}{r} e^{i(kr - \omega t)}.$$

The notation Π is due to Hertz¹ himself. As shown by the second form of equation (1), we assume the oscillation to be *purely periodic* and *undamped in time* (this is realized for the tube transmitter).

In the abbreviated first form of (1), which we shall use in the following discussion, we have to remember that

$$(2) \quad \Pi = -i\omega\Pi = -ikc\Pi.$$

where

¹ In his fundamental work "Die Kräfte elektrischer Schwingungen," collected works II, p. 147, which also contains the well-known force lines of the oscillating dipole.

$$\begin{aligned}
 \omega &= \text{circular frequency} \\
 (2a) \quad k &= 2\pi/\lambda = \omega/c = \text{wave number} \\
 c &= \omega/k = \text{velocity of light in a vacuum.}
 \end{aligned}$$

As we know, Π satisfies the oscillation equation (7.4), which for purely periodic processes becomes the wave equation:

$$(3) \quad \Delta \Pi + k^2 \Pi = 0.$$

In the electrodynamic case Π is not a *scalar* but a *vector*. Hence in the future we shall speak of the *Hertz vector* $\vec{\Pi}$. It is connected with the vector potential \mathbf{A} by the simple relation

$$(3a) \quad \vec{\Pi} = \mathbf{A}.$$

Just as the individual elements of which \mathbf{A} is composed have the direction of the corresponding elements of current, so our $\vec{\Pi}$ in empty space (i.e., in the absence of the earth) for a single antenna would have the direction of the antenna current. Here we assume the antenna to be *short compared to the wavelength*, that is, with both ends loaded with capacities so that the current can be considered in the same phase along the whole antenna. In representation (1) we could express the vector character of Π by multiplying the right side of (1) by a constant vector which has the direction of the antenna and, as we shall show later, the dimension of an electric momentum (charge \times length). However, we shall refrain from doing this in order not to make the formulas unnecessarily cumbersome; hence we retain equation (1), although it is inconsistent from a vectorial and even a dimensional point of view. Only in §36 shall we correct this flaw. However, we wish to stress now that, due to the vector character of $\vec{\Pi}$, we have to give the Laplace operator Δ in (3) its general vector-analytic meaning

$$(3b) \quad \Delta \vec{\Pi} = \text{grad div } \vec{\Pi} - \text{curl curl } \vec{\Pi}.$$

(see v.II, equation (3.10a)). This will be used in §32. Only in this and the following section, where we deal with one Cartesian component Π_x or Π_y at a time, can we use the ordinary Δ .

We now claim that the field \mathbf{E}, \mathbf{H} can be obtained from $\vec{\Pi}$ by the following differentiation process:

$$(4) \quad \mathbf{E} = k^2 \vec{\Pi} + \text{grad div } \vec{\Pi}, \quad \mathbf{H} = \frac{k^2}{\mu_0 i \omega} \text{curl } \vec{\Pi}.$$

In order to prove this we must show that Maxwell's equations in a vacuum

$$(5) \quad \begin{aligned} \mu_0 \dot{\mathbf{H}} + \text{curl } \mathbf{E} &= 0, \\ \epsilon_0 \dot{\mathbf{E}} - \text{curl } \mathbf{H} &= 0 \end{aligned}$$

are satisfied, where as in (2) we have to replace

$$(5a) \quad \dot{\mathbf{H}} \text{ by } -i\omega \mathbf{H}, \quad \dot{\mathbf{E}} \text{ by } -i\omega \mathbf{E}$$

Due to (4) and (5a) the left sides of (5) become:

$$\text{curl } (-k^2 + k^2 + \text{grad div}) \vec{\Pi}$$

and

$$-i\omega \epsilon_0 (k^2 + \text{grad div} - \text{curl curl}) \vec{\Pi}.$$

Both vanish, the first due to $\text{curl grad} = 0$, the second due to (3) and (3b). Hence, if for $\vec{\Pi}$ we substitute (1) and determine the free constant in terms of the strength of the alternating current in the antenna, then, according to Maxwell, we have in (4) the field radiated from the antenna, valid for all distances that are large compared to $\lambda = 2\pi/k$. For the immediate neighborhood of the antenna our description breaks down owing to the excessive idealization of our antenna model. Following Hertz, we call our model an oscillating or pulsating *dipole*, since the ends of the antenna (both in this picture and in reality) carry alternating opposite charges. This extreme simplification of the antenna, which in reality is of complicated construction, may serve as an example of the degree to which physical data can be idealized in order to make them accessible to fruitful mathematical treatment.

We now pass from the case of vacuum to that of a *medium* "earth" of *general electromagnetic behavior*: it is still homogeneous but with arbitrary dielectric constant ϵ and conductivity σ ; also its permeability μ will be arbitrary for the time being. The equations (1) and (3) for $\vec{\Pi}$ remain formally valid; however the wave number k is no longer determined by (2a) but by

$$(6) \quad k^2 = \epsilon\mu\omega^2 + i\mu\sigma\omega.$$

At the same time (4) is replaced by:

$$(7) \quad \mathbf{E} = k^2 \vec{\Pi} + \text{grad div } \vec{\Pi}, \quad \mathbf{H} = \frac{k^2}{\mu i\omega} \text{curl } \vec{\Pi}.$$

As before, we prove that the corresponding generalized Maxwell equations

$$(7a) \quad \begin{aligned} \mu \dot{\mathbf{H}} + \text{curl } \mathbf{E} &= 0, \\ \varepsilon \dot{\mathbf{E}} + \sigma \mathbf{E} - \text{curl } \mathbf{H} &= 0 \end{aligned}$$

are satisfied. The oscillation equation, from which we obtained the wave equation by the elimination of time dependence, is obtained in analogy to (7.4):

$$(7b) \quad \Delta \Pi = \left(\varepsilon \mu \frac{\partial^2}{\partial t^2} + \sigma \mu \frac{\partial}{\partial t} \right) \Pi.$$

B. INTEGRAL REPRESENTATION OF THE PRIMARY STIMULATION

We first wish to bring the representation (1) of Π into the form of a *superposition of eigenfunctions*. Since we are dealing with cylindrical polar coordinates r, φ, z , we shall use the eigenfunctions u and eigenvalues λ of (26.3) and (26.3a) that are independent of φ ; we denote the quantity $m\pi/\hbar$ by μ .²

We then have:

$$(8) \quad u = I_0(\lambda r) \cos \mu z, \quad k^2 = \lambda^2 + \mu^2.$$

However, whereas the λ has previously been restricted to a discrete spectrum corresponding to the boundary conditions on the cylinder of finite radius, we now have a continuous spectrum $0 \leq \lambda < \infty$ corresponding to the unlimited medium (see §28). Thus, according to (8) the μ also have continuous, and in general, complex values. Furthermore, since we no longer have the boundary condition for the bases of the cylinder, we shall replace $\cos \mu z$ by $\exp(\pm \mu z)$. Hence we are looking for a representation of Π of the form

$$(9) \quad \Pi = \int_0^\infty F(\lambda) I_0(\lambda r) e^{\pm \mu z} d\lambda, \quad \mu = \sqrt{\lambda^2 - k^2},$$

where $F(\lambda)d\lambda$ represents the arbitrary amplitude constant by which any eigenfunction may be multiplied. Due to the altered meaning of r (cylindrical coordinate r instead of the spherical polar coordinate r in (1)) we have to rewrite the expression (1) for Π as

$$(10) \quad \Pi = \frac{e^{ikz}}{R}, \quad R^2 = r^2 + z^2.$$

Our condition (9) then reads for $z = 0$:

² A confusion between this μ and the above magnetic constant μ is unlikely. The latter, moreover, will soon disappear from our formulas.

$$(11) \quad \frac{e^{ikr}}{r} = \int_0^{\infty} F(\lambda) I_0(\lambda r) d\lambda.$$

In order to satisfy this condition we use the integral representation of an arbitrary function by the Bessel functions of §21 B. We employ equation (8a) of that section, which for

$$f(r) = \frac{e^{ikr}}{r}, \quad n = 0$$

becomes

$$\begin{aligned} \frac{e^{ikr}}{r} &= \int_0^{\infty} \sigma \varphi(\sigma) I_0(\sigma r) d\sigma, \\ \varphi(\sigma) &= \int_0^{\infty} e^{ik\rho} I_0(\sigma \rho) d\rho. \end{aligned}$$

The first of these equations becomes identical with (11), if we make the following changes in notation

$$\sigma = \lambda, \quad \sigma \varphi(\sigma) = F(\lambda), \quad \text{hence} \quad \varphi(\sigma) = F(\lambda)/\lambda;$$

The second equation then becomes

$$(11b) \quad F(\lambda) = \lambda \int_0^{\infty} e^{ik\rho} I_0(\lambda \rho) d\rho,$$

which is the *solution of the integral equation* (11). The integration in (11b) can be performed in an elementary fashion, if we use the representation (19.14) for I_0 with the limits of integration $\pm \pi$; namely, by reversing the order of integration we obtain

$$(12) \quad F(\lambda) = \frac{\lambda}{2\pi} \int_{-\pi}^{+\pi} dw \int_0^{\infty} e^{i\rho(k + \lambda \cos w)} d\rho = -\frac{\lambda}{2\pi i} \int_{-\pi}^{+\pi} \frac{dw}{k + \lambda \cos w}.$$

The last expression arises from the lower limit $\rho = 0$ in the preceding integration with respect to ρ ; the term arising from the upper limit $\rho = \infty$ can be made to vanish by a small deformation of the path of integration into the "shaded" region of the w -plane (see Fig. 18). The remaining integration with respect to w yields

$$(12a) \quad \frac{2\pi}{\sqrt{k^2 - \lambda^2}}.$$

Hence (12) becomes:

$$(13) \quad F(\lambda) = \frac{\lambda}{\sqrt{\lambda^2 - k^2}} = \frac{\lambda}{\mu}$$

and (11) becomes:

$$(13a) \quad \frac{e^{ikr}}{r} = \int_0^\infty I_0(\lambda r) \frac{\lambda d\lambda}{\mu}.$$

From (10) we now obtain a corresponding representation for Π . Namely, we can complete (13a) to a function of r and z , which satisfies the differential equation (1) by setting:

$$(14) \quad \Pi = \frac{e^{ikz}}{R} = \int_0^\infty I_0(\lambda r) e^{-\mu|z|} \frac{\lambda d\lambda}{\mu}$$

where $\mu = \sqrt{\lambda^2 - k^2}$ is to be taken with *positive real part*, in order to insure the convergence of the integral and its vanishing in the limit $z \rightarrow \pm \infty$. The fact that (14) coincides with (13a) for $z = 0$ insures that it also gives the correct representation of e^{ikR}/R for $z \neq 0$.

In the following section we shall transform (14) into

$$(14a) \quad \Pi = \frac{1}{2} \int_{-\infty}^{+\infty} H_0^1(\lambda r) e^{-\mu|z|} \frac{\lambda d\lambda}{\mu}$$

with a more exact determination of the path of integration, which will then be complex. Due to the asymptotic character of H_0^1 , equation (14a) has the advantage over (14) in that it demonstrates that the radiation condition is satisfied, just as in (1) where the factor $\exp(+ikr)$ is adapted to the radiation condition.

C. VERTICAL AND HORIZONTAL ANTENNA FOR INFINITELY CONDUCTIVE EARTH

Up to now we have dealt only with unlimited space, whether empty or filled by a homogeneous medium with the constants ε, μ, σ . We now pass to the case of the half-space $z > 0$, which, at $z = 0$, is bounded by an *infinitely conductive earth* ($\sigma \rightarrow \infty$), in which $\mathbf{E} = 0$. Hence, due to the equality of the tangential field strength, which is required by the Maxwell theory, we know that \mathbf{E}_{tang} must vanish also on the positive side of $z = 0$. According to (7) this means

$$(15) \quad (k^2 \vec{\Pi} + \text{grad div } \vec{\Pi})_{\text{tang}} = 0 \quad \text{for } z = 0.$$

We satisfy this condition by adjoining the mirror images of opposite sign to the two single poles of the given dipole: Figure 27a,b serves to illustrate this.

a) *Vertical antenna at a distance h above $z = 0$.* The arrows leading from the negative to the positive charge are in the *same direction* for the original dipole and for its mirror image. Hence we write:

$$(16) \quad \Pi = \Pi_z = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \quad \begin{cases} R^2 = r^2 + (z-h)^2, \\ R'^2 = r^2 + (z+h)^2. \end{cases}$$

The parallelogram on the left side of the drawing shows that charges of the two dipoles equidistant from $z = 0$ act on a hypothetical unit charge situated in the plane $z = 0$ so that the resulting force is in the z -direction. This means $\mathbf{E}_{\text{tang}} = 0$.

b) *Horizontal antenna at a distance h above $z = 0$.* The arrow of the reflected dipole has the *opposite* direction to that of the original dipole. Hence we write

$$(17) \quad \Pi = \Pi_x = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}.$$

where R and R' are as before.³ The parallelogram on the right side of the drawing shows that two associated charges of the two dipoles act on a positive unit charge in the plane $z = 0$

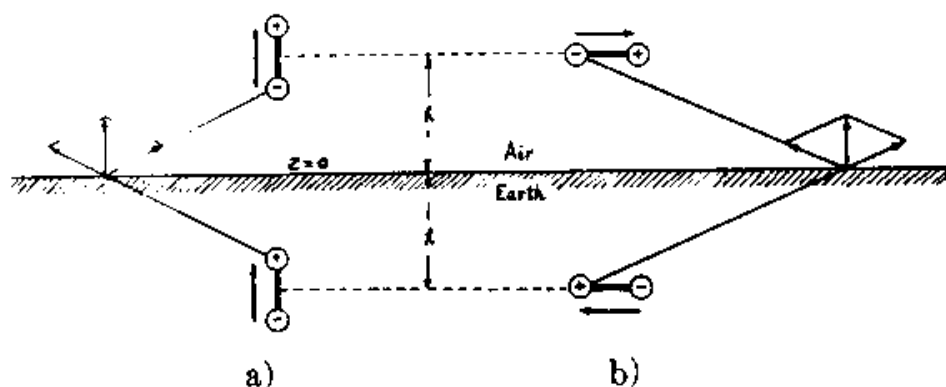


Fig. 27. Reflection by infinitely conductive earth. a) The vertical dipole. The auxiliary construction on the left shows that the horizontal components of the forces exerted by a pair of mirror image poles on a particle on the boundary plane cancel. b) The same thing is shown by the auxiliary construction on the right for the horizontal dipole. For the latter the orientations of the arrows in the original and its mirror image are *opposite*, for the vertical dipole they are *equal*.

so that the resulting force is perpendicular to the plane $z = 0$. Hence we again have $\mathbf{E}_{\text{tang}} = 0$.

³ The opposite choice of signs in (16) and (17) indicates the vector character of Π , which is suppressed in equation (1).

In exercise VI.1 we shall compute this using the vector formula (15) for both cases a) and b).

In one respect a) and b) differ fundamentally. Namely, if we pass to the limit $h \rightarrow 0$, we obtain

$$\Pi = 2 \frac{e^{ikR}}{R}, \text{ from (16) but } \Pi \equiv 0 \text{ from (17)}$$

Hence: a *vertical antenna located directly on the earth* for a sufficient conductivity of the soil generates the field which would be generated by the same antenna in empty space in complete absence of the earth. On the other hand a *horizontal antenna located directly on the earth* for a complete conductivity of the soil is canceled by its mirror image. The former made it possible to adapt the formulas and figures of Hertz' original work, which were relative to empty space, to the case of a grounded antenna (Max Abraham). In fact, we can cut the Hertz pattern of force lines of the oscillating dipole along its central plane and replace that plane by the surface of the earth. The force lines are then perpendicular to this plane and hence satisfy condition (15). The latter, that is, the disappearance of the horizontal antenna field for $h = 0$ as expressed by (17), decreases rapidly in importance for $h > 0$ (see the figures in §36). Indeed, the horizontal antenna is an effective means of communication even when $h < \lambda$ and the medium is sea water (a very good conductor for the comparatively long radio waves). Thus we see that for the horizontal antenna the nature of the ground and the distance from the ground play a greater role than for the vertical antenna. The formula $\vec{\Pi} = \Pi_x$ in (17) is then no longer adequate and must be generalized (see §33).

D. SYMMETRY CHARACTER OF THE FIELDS OF ELECTRIC AND MAGNETIC ANTENNAS

As we have just seen, the vertical antenna gives the field of a Hertz *dipole* of strength 2 for the limit $h \rightarrow 0$, and the horizontal antenna yields a zero field. However, if in the latter case we let the antenna current increase at the rate at which h decreases, then we obtain the field of a *quadrupole*. In fact under this limit process Fig. 27b goes over into the quadrupole scheme as seen on p. 152. Replacing the amplitude factors 2 and zero by A and B we can write:

$$(18) \quad \begin{array}{ll} \text{Vertical antenna:} & \Pi_z = A \frac{e^{ikR}}{R} \quad \text{Dipole,} \\ \text{Horizontal antenna:} & \Pi_x = B \frac{\partial}{\partial x} \frac{e^{ikR}}{R} \quad \text{Quadrupole.} \end{array}$$

The latter representation corresponds in Fig. 27b to the combination of the pairs of poles which lie on the same vertical line to a vertical dipole and to their relative translation in the horizontal direction. This means that the horizontal antenna in the x -direction is equivalent to vertical antennas with opposite current that are mutually translated in the x -direction. We shall discuss this more closely in connection with Fig. 30. Written in polar coordinates $x = r \cos \varphi$, $y = r \sin \varphi$ the second formula (18) reads

$$(18a) \quad \Pi_x = B \frac{x}{r} \frac{\partial}{\partial r} \frac{e^{ikR}}{R} = B \cos \varphi \frac{r}{R} \frac{d}{dR} \frac{e^{ikR}}{R}.$$

Hence, the directions $\varphi = 0$ and $\varphi = \pi$ parallel to the antenna are preferred directions for Π_x ; in the perpendicular directions $\varphi = \pm \pi/2$, Π_x vanishes. The associated direction characteristics of the horizontal antenna will be described in Fig. 29, where we shall also compute the constant B (which vanishes with increasing conductivity). On the other hand the field of the vertical antenna is symmetric with respect to the z -axis and hence its direction characteristic is a circle. From this follows the particular suitability of the horizontal antenna for directed broadcasts (see §33).

Rod antennas of vertical or horizontal direction are called *electric transmitters*. A coil traversed by an alternating current or any (circular, rectangular, etc.) closed conductor is called a *magnetic transmitter*, because then the magnetic field is concentrated in the axis of the coil (the normal of the wire loop); the customary notation is "frame antenna." In the central perpendicular of the frame a magnetic alternating current pulsates, while along the rod antenna there pulsates an electric alternating current. While the *magnetic* force lines are circles around the rod axis in the electric transmitter, in the magnetic transmitter the *electric* force lines are circles around the normal of the frame antenna (at least for distances that are large compared to the frame). These statements are correct only for the vertical electric or magnetic dipole; for an oblique or horizontal position the circular symmetry is disturbed by the conductive ground. Generally speaking the data for the magnetic transmitter are deduced from those for the electric transmitter by replacing \mathbf{E}, \mathbf{H} by $\mathbf{H}, -\mathbf{E}$, (for details see §35). Due to the boundary conditions for \mathbf{E} (not for \mathbf{H}) in the case of an infinitely conductive ground, the signs in (16) and (17) are interchanged. Namely, for the magnetic Π_z (horizontal position of the plane of the frame), we have

$$(19) \quad \Pi_z = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}, \quad \Pi_z \equiv 0 \quad \text{for } h \rightarrow 0$$

and for the magnetic Π_x (vertical position of the plane of the frame) we have

$$(20) \quad \Pi_x = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \quad \Pi_x = 2 \frac{e^{ikR}}{R} \quad \text{for } h \rightarrow 0.$$

The proof will be given in exercise VI.1. The frame antenna of type (19) is of no practical importance, the antenna of type (20) will be treated in §35. As a transmitter this latter antenna shows a marked direction in the plane of the frame (e.g., for Π_x the y,z -plane) with the same characteristic as the electric rod antenna of (18). As a receiver it is arranged so that it can be rotated around the vertical line; if it is then oriented for maximal reception its plane points to the origin of the signal and it is therefore particularly suited for range finding (see §34).

§ 32. The Vertical Antenna Over an Arbitrary Earth

Let ε and σ be the electric constants of the ground. As regards its magnetic behavior we may assume $\mu = \mu_0$, which is sufficiently close to reality and simplifies the following calculations. We write

$$(1) \quad n^2 = \left(\varepsilon + i \frac{\sigma}{\omega} \right) / \varepsilon_0$$

and, as in optics, we call n the “complex refractive index.” The wave number k of (31.6) will, in the following discussion, be called k_E in order to distinguish it from the wave number of air for which we keep the notation k . Then according to (31.6) and (31.2a) we have

$$(2) \quad k_E = nk.$$

We denote the altitude of the dipole antenna above the ground by h , as in (31.16).

We have to distinguish three regions:

I. Air $z > h$. In addition to the primary stimulation that becomes singular at the dipole $z = h$, $r = 0$, we have a secondary stimulation that is regular throughout due to currents induced in the ground. We write according to (31.14) and in analogy to (31.9)

$$(3) \quad \Pi_{\text{prim}} = \int_0^\infty I_0(\lambda r) e^{-\mu(z-h)} \frac{\lambda d\lambda}{\mu}, \quad \Pi_{\text{sec}} = \int_0^\infty F(\lambda) I_0(\lambda r) e^{-\mu(z+h)} d\lambda.$$

where $F(\lambda)$ is, so to speak, the spectral distribution in the λ -continuum of the eigenfunctions, and is as yet undetermined. The factor $\exp(-\mu h)$ in the representation of Π_{sec} is convenient for what follows, and it is

permissible since it is a pure function of λ and thus merely alters the meaning of $F(\lambda)$.

II. Air layer $h > z > 0$. Here, too, we have a primary and a secondary stimulation. Due to $z < h$ and according to the rule of signs of (31.14) we must write the former with a sign opposite to that in (3); the latter, being an analytic continuation, has the same form as in (3):

$$(4) \quad \Pi_{\text{prim}} = \int_0^{\infty} I_0(\lambda r) e^{+\mu(z-h)} \frac{\lambda d\lambda}{\mu}, \quad \Pi_{\text{sec}} = \int_0^{\infty} F(\lambda) I_0(\lambda r) e^{-\mu(z+h)} d\lambda.$$

Equations (3) and (4) insure the continuous behavior of the Π -field at the boundary between I and II for an arbitrary choice of $F(\lambda)$.

III. Earth $0 > z > -\infty$. Here there is no primary stimulation; the Π -field — denoted by Π_E — must be continuous throughout. In order to satisfy the differential equation for earth (31.3) with k_E^2 instead of k^2 , we write:

$$(5) \quad \Pi_E = \int_0^{\infty} F_E(\lambda) I_0(\lambda r) e^{+\mu_E z - \mu h} d\lambda, \quad \mu_E^2 = \lambda^2 - k_E^2.$$

According to our general rule we must choose the sign of $\mu_E z$ positive since $z < 0$. The factor $\exp(-\mu h)$ is adjoined for reasons of convenience; again, this merely influences the arbitrary function $F_E(\lambda)$. The functions $F_E(\lambda)$ and $F(\lambda)$ are determined from the boundary conditions on the surface of the earth.

According to Maxwell we must require the continuity of the tangential components of \mathbf{E} and \mathbf{H} . These are merely

$$\mathbf{E}_\varphi \text{ and } \mathbf{H}_\varphi.$$

Indeed, the electric force lines are in the planes through the dipole axis, the magnetic force lines are circles around this axis, and hence \mathbf{E}_r and \mathbf{H}_r vanish. (This follows from the fact that $\Pi = \Pi_E$ is a function of r and z alone.) Now according to (31.4) and (31.7) we have

$$(6) \quad \begin{aligned} \mathbf{E}_r &= \frac{\partial}{\partial r} \frac{\partial \Pi}{\partial z}, & \mathbf{H}_\varphi &= \frac{-k^2}{\mu_0 i \omega} \frac{\partial \Pi}{\partial r} & \text{for } z > 0, \\ \mathbf{E}_r &= \frac{\partial}{\partial r} \frac{\partial \Pi_E}{\partial z}, & \mathbf{H}_\varphi &= \frac{-k_E^2}{\mu_0 i \omega} \frac{\partial \Pi_E}{\partial r} & \text{for } z < 0. \end{aligned}$$

Hence the continuity conditions for $z = 0$ are:

$$\frac{\partial}{\partial r} \frac{\partial \Pi}{\partial z} = \frac{\partial}{\partial r} \frac{\partial \Pi_E}{\partial z}, \quad k^2 \frac{\partial \Pi}{\partial r} = k_E^2 \frac{\partial \Pi_E}{\partial r}.$$

We can integrate these conditions with respect to r and the constants of

integration must be zero since all expressions vanish for $r \rightarrow \infty$. If in the second equation above we replace k_E^2 by $n^2 k^2$ according to (2) then we obtain:

$$(7) \quad \frac{\partial \Pi}{\partial z} = \frac{\partial \Pi_E}{\partial z}, \quad \Pi = n^2 \Pi_E \quad \text{for } z = 0.$$

On the right side of this equation we have to substitute the value of Π_E from (5) and on the left side we have to substitute the sum of Π_{prim} and Π_{sec} from (4). We thus obtain the conditions:

$$(7a) \quad \int_0^\infty I_0(\lambda r) e^{-\mu h} (\lambda - \mu F - \mu_E F_E) d\lambda = 0,$$

$$(7b) \quad \int_0^\infty I_0(\lambda r) e^{-\mu h} (\lambda + \mu F - n^2 \mu F_E) \frac{d\lambda}{\mu} = 0.$$

They are satisfied if we set

$$\begin{aligned} \mu F + \mu_E F_E &= \lambda, \\ \mu F - n^2 \mu F_E &= -\lambda. \end{aligned}$$

Hence

$$(8) \quad F = \frac{\lambda}{\mu} \left(1 - \frac{2\mu_E}{n^2\mu + \mu_E} \right), \quad F_E = \frac{2\lambda}{n^2\mu + \mu_E}.$$

Thus, we have demonstrated that equations (3), (4), (5) do indeed lead to a solution of our problem with its boundary conditions. The fact that there can be no other solution is deduced from the uniqueness axiom of physical boundary value problems, which always proves reliable. Due to the meaning of n, μ, μ_E equation (8) can be written in the more symmetric form:

$$(8a) \quad \begin{aligned} F &= \frac{\lambda}{\sqrt{\lambda^2 - k^2}} \frac{k_E^2 \sqrt{\lambda^2 - k^2} - k^2 \sqrt{\lambda^2 - k_E^2}}{k_E^2 \sqrt{\lambda^2 - k^2} + k^2 \sqrt{\lambda^2 - k_E^2}}, \\ F_E &= \frac{2\lambda k^2}{k_E^2 \sqrt{\lambda^2 - k^2} + k^2 \sqrt{\lambda^2 - k_E^2}}. \end{aligned}$$

We again write the primary stimulation in its original form e^{ikR}/R with $R^2 = r^2 + (z - h)^2$ to show that the contribution to Π_{sec} that is due to the first term of F in (8) differs from Π_{prim} only by the fact that we have to replace $-h$ by $+h$, and hence R^2 by $R'^2 = r^2 + (z + h)^2$. Then representations (3) and (4) for regions I and II can be contracted and we obtain as the general solution of our problem for $z > 0$ and $z < 0$:

$$(9) \quad \begin{aligned} H &= \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'} - 2 \int_0^\infty I_0(\lambda r) e^{-\mu(z+\lambda)} \frac{\mu_E}{n^2 \mu + \mu_E} \frac{\lambda d\lambda}{\mu}, \\ H_E &= 2 \int_0^\infty I_0(\lambda r) e^{\mu_E z - \mu n} \frac{\lambda d\lambda}{n^2 \mu + \mu_E}. \end{aligned}$$

If in particular we have $h = 0$ so that we can use equation (4) for the coinciding expressions e^{ikR}/R and $e^{ikR'}/R'$, then the first line of (9) can be rewritten in an elegant manner. According to previous work by the author we then have:

$$(10) \quad \begin{aligned} H &= \int_0^\infty I_0(\lambda r) e^{-\mu z} \frac{2 n^2 \lambda d\lambda}{n^2 \mu + \mu_E} \\ H_E &= \int_0^\infty I_0(\lambda r) e^{\mu_E z} \frac{2 \lambda d\lambda}{n^2 \mu + \mu_E}. \end{aligned}$$

If, on the other hand, we consider the special case $|n| \rightarrow \infty$ of a completely conductive earth then μ_E can be neglected as compared to n^2 and the integrands in (9) will vanish. This confirms the result of the elementary reflection process in §31, equation (16):

$$(10a) \quad H = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \quad H_E = 0.$$

It is profitable to consider this limit process with respect to n somewhat further. To this end we replace $n^2 \mu + \mu_E$ by $n^2 \mu$ in the denominator of the integrand in the first equation (9), and in the numerator we write, for all values of λ that are not too large,

$$(10b) \quad \mu_E = \sqrt{\lambda^2 - k_E^2} = k_E \sqrt{-1 + \frac{\lambda^2}{k_E^2}} \sim -i k_E,$$

$$\text{and hence} \quad \frac{\mu_E}{n} = -i k$$

(concerning the sign of μ_E see the figure below). Then the first equation (9) becomes:

$$(10c) \quad H = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'} + \frac{2ik}{n} \int_0^\infty I_0(\lambda r) e^{-\mu(z+\lambda)} \frac{\lambda d\lambda}{\mu^2}.$$

An intuitive interpretation of the latter integral⁴ can be obtained as

⁴ Since the denominator μ^2 vanishes at $\lambda = k$ the path of integration must be chosen in the complex λ -plane so as to avoid the point $\lambda = k$. This remark holds for the following λ -integrals, too. In the preceding integrals, starting with (31.14), we had the denominator μ , which did not destroy the convergence.

follows: corresponding to

$$R' = \sqrt{r^2 + (z + h)^2}, \quad \frac{e^{i k R'}}{R'} = \int_0^\infty I_0(\lambda r) e^{-\mu(z+h)} \frac{\lambda d\lambda}{\mu}$$

we write

$$R'' = \sqrt{r^2 + (z + h')^2}, \quad \frac{e^{i k R''}}{R''} = \int_0^\infty I_0(\lambda r) e^{-\mu(z+h')} \frac{\lambda d\lambda}{\mu}$$

and compute

$$\begin{aligned} \int_h^\infty \frac{e^{i k R''}}{R''} dh' &= \int_h^\infty dh' \int_0^\infty \dots d\lambda = \int_0^\infty I_0(\lambda r) e^{-\mu z} \frac{\lambda d\lambda}{\mu} \int_h^\infty e^{-\mu h'} dh' \\ &= \int_0^\infty I_0(\lambda r) e^{-\mu(z+h)} \frac{\lambda d\lambda}{\mu^2}. \end{aligned}$$

hence the integral in (10c) stands for the action of an imaginary continuous covering of the ray $h < h' < \infty$ with dipoles that reach from the image point $z = -h$, $r = 0$ to $z = -\infty$, $r = 0$. Hence the approximating equation (10c) can also be written as:

$$(10d) \quad \Pi = \frac{e^{i k R}}{R} + \frac{e^{i k R'}}{R'} + \frac{2 i k}{n} \int_h^\infty \frac{e^{i k R''}}{R''} dh'.$$

In this connection we should remember a similar covering of a ray with imaginary source points that we used in a heat conduction problem in Fig. 15. While there we required exact satisfaction of the simple boundary condition $\partial u / \partial n + h u = 0$ (the h there, of course, had nothing to do with the h here), we now require approximate satisfaction of the complicated boundary conditions that arise from the juxtaposition of the air with the highly conductive earth.

From the above formulas we can deduce the field \mathbf{E} , \mathbf{H} by differentiation. However, we shall not write this somewhat cumbersome representation since we shall need it only in connection with the energy considerations of §36.

The integrals in (9) and (10) are not yet uniquely determined because of the square roots

$$(11) \quad \mu = \sqrt{\lambda^2 - k^2}, \quad \mu_E = \sqrt{\lambda^2 - k_E^2}$$

that appear in them. Corresponding to the four combinations of signs

of μ and μ_E , the integrand is four-valued, and its Riemann surface has four sheets. By our rule of signs in (31.14), which refers to the real part of λ and also applies to the real part of μ_E , one of the four sheets is singled out as a "permissible sheet." In order to insure the convergence of our integrals we demand that the path of integration at infinity shall be on the permissible sheet only. We achieve this by joining the "branch points"

$$(11a) \quad \lambda = k \quad \text{and} \quad \lambda = k_E$$

by two (essentially arbitrary) "branch cuts," which may not be intersected by the path of integration. Referring to Fig. 28 we therefore do not integrate along the real axis over the branch point $\lambda = k$, but avoid it by going into the negative imaginary half-plane and from there to infinity in, say, a direction parallel to the real λ -axis. Thus, along the path denoted by W_1 in Fig. 28 we integrate from $\lambda = 0$ to $\lambda = \infty$; this makes the meaning of the integrals in (9) and (10) precise.

But even then the representations (9) and (10) suffer from a mathematical inelegance: they are integrals with the fixed initial point $\lambda = 0$, not integrals along closed paths in the λ -plane,

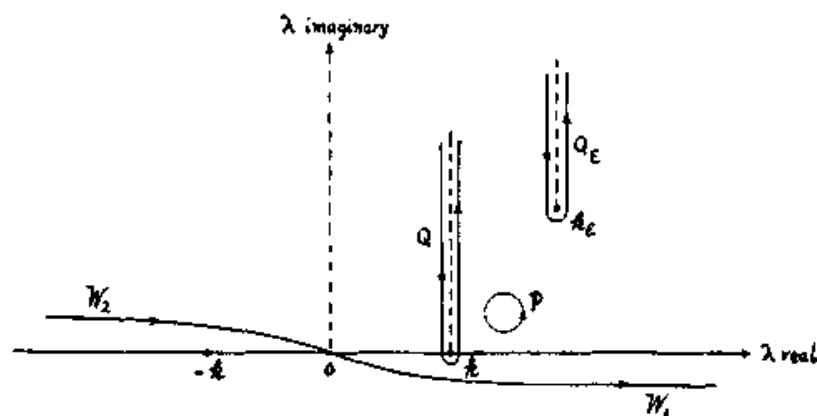


Fig. 28. The paths W_1 and $W = W_1 + W_2$ in equation (13); deformation of the path W into the loops Q and Q_E around the branch cuts and into the closed path P around the pole.

which, due to their deformability, would be much more useful. We remove this flaw by using the relation

$$I_0 = \frac{1}{2} (H_0^1 + H_0^2)$$

and the "semi-circuit relation" (10) of the introduction to exercise IV.2. If in the latter we set $\varrho = \lambda r_1$, then the preceding equation becomes

$$(12) \quad I_0 = \frac{1}{2} [H_0^1(\lambda r) - H_0^1(\lambda e^{i\pi} r)].$$

We imagine (12) multiplied by an arbitrary function of λ^2 (indicated in the following by . . .) and by $\lambda d\lambda$ and integrated over W_1 . Then, if we write $\lambda' = \lambda e^{i\pi}$, we obtain from the subtrahend on the right side of (12)

$$(12a) \quad \int_{W'} H_0^1(\lambda' r) \dots \lambda' d\lambda'.$$

Here W' is the path obtained from W_1 through reflection on the origin taken in the direction $\lambda' = 0 \rightarrow \lambda' = -\infty$, which, except for sign, is identical with the path W_2 in Fig. 28. Hence (12a) is the same as

$$(12b) \quad \int_{W_2} H_0^1(\lambda' r) \dots \lambda' d\lambda';$$

and from (12), if we denote the variable of integration throughout by λ and combine the paths W_1 and W_2 to $W = W_1 + W_2$, we obtain

$$(13) \quad \int_{W_1} I_0(\lambda r) \dots \lambda d\lambda = \frac{1}{2} \int_W H_0^1(\lambda r) \dots \lambda d\lambda.$$

Thus we have achieved our purpose to replace the seemingly real integration that starts at $\lambda = 0$ in representations (9) and (10) by a complex integration over a path which closes at infinity. We consider this transformation (13) performed on all the integrals in (9) and (10). In particular we write, e.g., the primary stimulation of (31.14) and the first line of (10) in the new form

$$(14) \quad \Pi_{\text{prim}} = \frac{1}{2} \int_W H_0^1(\lambda r) e^{-\mu|z|} \frac{\lambda d\lambda}{\mu},$$

$$(14a) \quad \Pi = \int_W H_0^1(\lambda r) e^{-\mu z} \frac{n^2 \lambda d\lambda}{n^2 \mu + \mu_z}.$$

The attentive reader must have noticed long ago that Fig. 28 coincides with Fig. 26 (even with respect to the notation of the paths W, W_1 and the variable of integration λ), and that the present problem (determination of the function Π in space as subdivided by the surface of the earth for prescribed singularities at the dipole antenna) is summarized under the general *problem of Green's function*. Here we constructed the solution from eigenfunctions that satisfy the *radiation condition* at infinity. The fact that this condition is satisfied in the

present case is made evident by the fact that in (14) and (14a) only the first Hankel function H^1 enters.⁵

We now consider the upper part of Fig. 28. Since we know that $H^1(\lambda r)$ vanishes in the infinite part of the positive imaginary half-plane, we can deform the path W into that half-plane. The path cannot be deformed across the branch cuts (11a), which it avoids by the loops Q and Q_E . However there is a further singularity of the integrand in (14) and in the analogous integrals, namely the point at which the denominator $n^2 \mu + \mu_E$ vanishes. We denote it by

$$\lambda = \dot{p}.$$

This corresponds to a *pole* of the integrand and must be avoided by the path of integration in a circuit P . We have not drawn the paths which join P to infinity since in the integration they cancel each other.

Of the three components Q, Q_E, P of the integral we can ignore the contribution of Q_E for large $|k_E|$, since $H^1(\lambda r)$ vanishes exponentially for great distances from the real axis. We first consider P separately, but we shall soon see that P and Q can hardly be separated.

From the defining relation for p

$$(15) \quad n^2 \mu + \mu_E = 0$$

we have

$$(16) \quad \sqrt{\frac{p^2 - k^2}{p^2 - k_E^2}} = -\frac{k^2}{k_E^2}, \quad p^2 = \frac{k^2 k_E^2}{k^2 + k_E^2},$$

which we can also write as

$$(16a) \quad \frac{1}{p^2} = \frac{1}{k^2} + \frac{1}{k_E^2}.$$

Due to $|k_E| \gg k$ we have approximately

$$(16b) \quad p = k \left(1 - \frac{1}{2} \frac{k^2}{k_E^2} \right), \quad k - p = \frac{k}{2} \frac{k^2}{k_E^2}.$$

However, we wish to stress the fact that the precise value of p given by (16) or (16a) is symmetric in k and k_E .

⁵ Here we have assumed a time dependence of the preferred form $\exp(-i\omega t)$. For a time dependence of the form $\exp(+i\omega t)$ we would have to make the transition from I to H^2 in (12) with the help of the semi-circuit relation (10a) in exercise (IV.2). Thus we would obtain a representation that, e.g., in (14a) is constructed from elements of the form

$$H_0^2(\lambda r) e^{-\mu_E z} e^{+i\omega t}$$

and hence also has the type of radiated waves.

We now compute the integral over P by applying the method of residues to (14a). Here we can let $\lambda = p$ in all the factors of the integrand of (14a), but we must replace the denominator which vanishes for $\lambda = p$ by

$$\frac{d}{d\lambda}(n^2 \mu + \mu_E) = \lambda \left(\frac{n^2}{\sqrt{\lambda^2 - k^2}} + \frac{1}{\sqrt{\lambda^2 - k_E^2}} \right),$$

taken for $\lambda = p$. We thus obtain:

$$(17) \quad \frac{p}{k^2} K, \quad K = \frac{k_E^2}{\sqrt{p^2 - k^2}} + \frac{k^2}{\sqrt{p^2 - k_E^2}};$$

here the new quantity K is symmetric in k and k_E . Hence, as the contribution of P to (14a) we obtain:

$$(18) \quad \Pi = 2 \pi i \frac{k_E^2}{K} H_0^1(p r) e^{-\sqrt{p^2 - k^2} z}.$$

In the same manner, for $z < 0$ (earth, interchange of k and k_E , and reversal of the sign of z) we obtain

$$(18a) \quad \Pi_E = 2 \pi i \frac{k^2}{K} H_0^1(p r) e^{+\sqrt{p^2 - k_E^2} z}.$$

Except for the immediate neighborhood of the transmitter, namely, for all distances $|p r| \gg 1$ we can replace H by the asymptotic value (19.55). We then obtain

$$(19) \quad \Pi = 2 \sqrt{\frac{2 \pi i}{p r}} \frac{k_E^2}{K} e^{i p r - \sqrt{p^2 - k^2} z} \quad z \geq 0,$$

$$(19a) \quad \Pi_E = 2 \sqrt{\frac{2 \pi i}{p r}} \frac{k^2}{K} e^{i p r + \sqrt{p^2 - k_E^2} z} \quad z \leq 0.$$

These formulas bear all the marks of "surface waves," which are mentioned in v.II in connection with the water waves or the seismic Rayleigh waves, and which have the following properties:

1. They are tied to the surface $z = 0$ and decrease in both directions from that surface; in the direction of the earth they decrease rapidly due to the coefficient $(p^2 - k_E^2)^{\frac{1}{2}}$ of z ; in the direction of the air the decrease is slow at first but exponential for large z .

2. The propagation along $z = 0$ is given by

$$\frac{dr}{dt} = \frac{\omega}{p},$$

and hence depends in a symmetric manner on the material constants air and earth, as must be the case for a surface wave.

3. If for the time being we neglect the absorption in the radial direction, then the amplitude of the expressions (19), (19a) decreases as $1/\sqrt{r}$, with increasing distance from the transmitter, whereas the intensity decreases as $1/r$. This too is a criterion for the essentially two-dimensional propagation of energy in the surface $z = 0$ (see p. 100).

4. For the sake of completeness we also mention the exponential absorption in the radial direction; it is given by the real part of ipr and according to (16b) and (1), (2) it is given by

$$-\frac{kr}{2} \operatorname{Re} \left(\frac{i}{n^2} \right) = -\frac{kr}{2} \frac{\epsilon_0 \sigma}{\omega} / \left(\epsilon^2 + \frac{\sigma^2}{\omega^2} \right),$$

which is valid both for $z > 0$ and for $z < 0$.

For sufficiently large r , where the relative change of $r^{-1/2}$ is small, we can consider (19), (19a) as waves whose origin is at infinity, e.g., in the direction of the negative x -axis. These equations then become

$$(20) \quad \Pi = A k_E^2 e^{ipx - \sqrt{p^2 - k^2} z},$$

$$(20a) \quad \Pi_E = A k^2 e^{ipx + \sqrt{p^2 - k_E^2} z}$$

where A is a slowly varying amplitude factor, and hence represent the so-called "Zenneck waves." As early as 1907 Zenneck,⁶ in great graphical and numerical detail, investigated the fields \mathbf{E} , \mathbf{H} derived from (20), (20a), and discussed the material constants of the different types of soil (also fresh and salt water). It was the main point of the author's⁷ work of 1909 to show that these fields are automatically contained in the wave complex, which, according to our theory, is radiated from a dipole antenna. This fact has, of course, not been changed. What has changed is the weight which we attached to it. At the time it seemed conceivable to explain the overcoming of the earth's curvature by radio signals with the help of the character of the *surface waves*; however, we know now that this is due to the ionosphere (see the introduction to this chapter). In any case the recurrent discussion in the literature on the "reality of the Zenneck waves" seems immaterial to us.

Epstein⁸ has recently shown that the surface wave P taken by itself is a solution of our problem, and hence in principle does not have to be accompanied by the wave complex represented by Q . The latter, generally speaking, has the character of spatial waves and, in contrast to (20), is represented by the formal type

$$\Pi = B \frac{e^{ikr}}{r}.$$

⁶ *Ann. Physik* 23, 846.

⁷ *Ann. Physik* 28, 665.

⁸ P. S. Epstein, *Proc. Natl. Acad. Sci. U. S.*, June 1947.

Under the actual circumstances of radio communication $P + Q$ is best represented by one contour integral which goes around the near points $\lambda = p$ and $\lambda = \bar{k}$, and which must be discussed with the help of the saddle-point method. This has been carried out most completely by H. Ott.⁹ However, we have to forego the presentation of his results in order not to get lost in the details of the problem.

We shall consider one more general aspect and one special formula which is convenient for numerical computations.

The general aspect concerns a kind of similarity relation of radio, the introduction of "numerical distance." Measured in terms of wavelengths the radial distance traversed by a spatial wave in the time t is kr (except for a factor 2π), the distance traversed by the surface wave in the same time is equal to the real part of pr . We form the difference of these distances and introduce the quantity

$$(21) \quad \varrho = i(k - p)r.$$

The absolute value of ϱ is called the *numerical distance*. The quantity ϱ is a pure number whose absolute value is small compared to kr . In fact, according to (16a) we have

$$(21a) \quad |\varrho| \sim \frac{kr}{2} \frac{k^2}{|k_x|^2} = \frac{kr}{2|n|^2}.$$

Hence, for small values of ϱ the spatial-wave type predominates in the expression of the reception intensity; in this case the ground peculiarities have no marked influence and we can make computations using an infinite ground conductivity without introducing great errors, as was done by Abraham (see §31). For larger ϱ the rivalry between spatial and surface waves becomes apparent, as the value of ϱ in (21) was defined in terms of the difference of the two propagations. In this case the material constants of the ground are important, and indeed not only σ , but also ϵ . Generally speaking, equal ϱ 's imply equal wave types and equal reception strengths. Thus ϱ indicates a similarity relation. The fact that for sea water, due to its relatively high conductivity, ϱ is much smaller according to (21a) (for the same absolute distance r) than it is for fresh water or for an equally level dry soil, explains the good reception at sea (the difference in reception during the day and night is, of course, due to the ionosphere).

An expansion in ascending powers of ϱ led the author, in his first investigation (1909), to a convenient approximation formula, which since has been rededuced by different authors (B. Van der Pol, K. F.

⁹ *Ann. Physik* **41**, 443 (1942).

Niessen, L. H. Thomas, F. H. Murray) partly in a simpler manner. In its final form the approximation formula reads:

$$(22) \quad \Pi = 2 \frac{e^{ikr}}{r} \left(1 + i \sqrt{\pi \varrho} e^{-\tau} - 2 \sqrt{\varrho} e^{-\tau} \int_0^{\sqrt{\tau}} e^{\alpha^2} d\alpha \right),$$

It is valid for the air close to the earth.

In order to confirm the preceding remark we note: The first term, which is the dominating term for small ϱ , is of the spatial-wave type, and due to the factor 2 it corresponds to an infinitely conductive ground; the second term is of the surface-wave type and corresponds qualitatively, and for the purpose of our approximation even quantitatively, to the first equation (19); the third term represents the correction for larger ϱ . The generalization of (22) to the case of small finite distances z above the ground is

$$(23) \quad \Pi = 2 \frac{e^{ikr}}{R} \left(1 + i \sqrt{\pi \varrho} e^{-\tau} - 2 \sqrt{\varrho} e^{-\tau} \int_0^{\sqrt{\tau}} e^{\alpha^2} d\alpha \right);$$

where

$$\tau = i(k-p)r \left(1 + n \frac{z}{r} \right)^2;$$

for $z = 0$ we have $\tau = \varrho$ and (23) becomes the same as (22).

§ 33. The Horizontal Antenna Over an Arbitrary Earth

For a horizontal antenna lying in the x -direction it seems advisable to set the Hertz vector $\vec{\Pi}$ equal to Π_x . However, as we remarked at the end of §31 C, this is possible only for an infinitely conductive ground. We start by proving this fact.

For $\vec{\Pi} = \Pi_x$ we obtain from (31.4) and (31.7)

$$(1) \quad \begin{aligned} \mathbf{E}_x &= k^2 \Pi_x + \frac{\partial^2 \Pi_x}{\partial x^2}, & \mathbf{E}_y &= \frac{\partial^2 \Pi_x}{\partial x \partial y} & z &\geq 0, \\ \mathbf{E}_x &= k_E^2 \Pi_{xE} + \frac{\partial^2 \Pi_{xE}}{\partial x^2}, & \mathbf{E}_y &= \frac{\partial^2 \Pi_{xE}}{\partial x \partial y} & z &\leq 0. \end{aligned}$$

where \mathbf{E}_x and \mathbf{E}_y must be continuous at the boundary $z = 0$. From the above formulas for \mathbf{E}_y we then deduce the continuity of Π_x , which implies the continuity of $\partial^2 \Pi_x / \partial x^2$. But then the formulas for \mathbf{E}_x imply the equality of k^2 and k_E^2 , which is a contradiction.

We resolve this contradiction by writing the Hertz vector with *two* components:

$$(2) \quad \vec{\Pi} = (\Pi_x, \Pi_z);$$

then instead of (1) we have

$$(3) \quad \begin{aligned} \mathbf{E}_x &= k^2 \Pi_x + \frac{\partial}{\partial x} \operatorname{div} \vec{\Pi}, & \mathbf{E}_y &= \frac{\partial}{\partial y} \operatorname{div} \vec{\Pi} & z \geq 0, \\ \mathbf{E}_x &= k_E^2 \Pi_{xE} + \frac{\partial}{\partial x} \operatorname{div} \vec{\Pi}_E, & \mathbf{E}_y &= \frac{\partial}{\partial y} \operatorname{div} \vec{\Pi}_E & z \leq 0. \end{aligned}$$

Hence for $z = 0$:

$$(4) \quad \operatorname{div} \vec{\Pi} = \operatorname{div} \vec{\Pi}_E$$

and

$$(5) \quad k^2 \Pi_x = k_E^2 \Pi_{xE}.$$

For the magnetic components, according to (31.4) and (31.7), we have

$$\begin{aligned} \mathbf{H}_x &= \frac{k^2}{i\mu_0\omega} \frac{\partial \Pi_z}{\partial y}, & \mathbf{H}_y &= \frac{k^2}{i\mu_0\omega} \left(\frac{\partial \Pi_x}{\partial z} - \frac{\partial \Pi_z}{\partial x} \right) & z \geq 0, \\ \mathbf{H}_x &= \frac{k_E^2}{i\mu_0\omega} \frac{\partial \Pi_{zE}}{\partial y}, & \mathbf{H}_y &= \frac{k_E^2}{i\mu_0\omega} \left(\frac{\partial \Pi_{xE}}{\partial z} - \frac{\partial \Pi_{zE}}{\partial x} \right) & z \leq 0. \end{aligned}$$

From the continuity of \mathbf{H}_x it follows that

$$(6) \quad k^2 \Pi_z = k_E^2 \Pi_{zE}$$

and from the continuity of \mathbf{H}_y it follows that

$$(7) \quad k^2 \frac{\partial \Pi_x}{\partial z} = k_E^2 \frac{\partial \Pi_{xE}}{\partial z}.$$

Hence we have two conditions (5) and (7) for Π_x which we can write in the form

$$(8) \quad \Pi_x = n^2 \Pi_{xE}, \quad \frac{\partial \Pi_x}{\partial z} = n^2 \frac{\partial \Pi_{xE}}{\partial z},$$

After we have determined Π_x we obtain the two conditions (6) and (4) for Π_z :

$$(9) \quad \Pi_z = n^2 \Pi_{zE}, \quad \frac{\partial \Pi_z}{\partial z} - \frac{\partial \Pi_{xE}}{\partial z} = \frac{\partial \Pi_{zE}}{\partial z} - \frac{\partial \Pi_x}{\partial x}.$$

The computation of Π_x is carried out by the methods of §32. We again distinguish the three regions:

$$\text{I. } \infty > z > h, \quad \text{II. } h > z > 0, \quad \text{III. } 0 > z > -\infty.$$

In I and II the function Π_z is composed of a primary and a secondary stimulation, which can be expressed exactly as in (32.3); in III we have only the secondary stimulation of the form (32.5). The conditions (8) then yield, in analogy to (32.7a,b),

$$(10a) \quad \int_0^\infty I_0(\lambda r) e^{-\mu h} (\lambda - \mu F - n^2 \mu_E F_E) d\lambda = 0,$$

$$(10b) \quad \int_0^\infty I_0(\lambda r) e^{-\mu h} (\lambda + \mu F - n^2 \mu F_E) \frac{d\lambda}{\mu} = 0$$

and by setting the parentheses equal to zero we obtain:

$$(11) \quad F = \frac{\lambda}{\mu} \left(-1 + \frac{2\mu}{\mu + \mu_E} \right), \quad F_E = \frac{1}{n^2} \frac{2\lambda}{\mu + \mu_E}.$$

This expression for F is written so that the second term vanishes for $n \rightarrow \infty$, which is the same as $\mu_E \rightarrow \infty$, so that in the limit only the first term $F = -\lambda/\mu$ remains. If we substitute (11) in the equations (32.3,4,5), then in analogy to (32.9) we obtain the representation of the Π_z -field:

$$(12) \quad \begin{aligned} \Pi_z &= \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'} + 2 \int_0^\infty I_0(\lambda r) e^{-\mu(z+h)} \frac{\lambda d\lambda}{\mu + \mu_E}, \\ \Pi_{zE} &= \frac{2}{n^2} \int_0^\infty I_0(\lambda r) e^{+\mu_E z - \mu h} \frac{\lambda d\lambda}{\mu + \mu_E}. \end{aligned}$$

where $R^2 = r^2 + (z - h)^2$, $R'^2 = r^2 + (z + h)^2$.

If in particular $h = 0$ then we have $R' = R$ and (12) simplifies to:

$$(12a) \quad \begin{aligned} \Pi_z &= 2 \int_0^\infty I_0(\lambda r) e^{-\mu z} \frac{\lambda d\lambda}{\mu + \mu_E}, \\ \Pi_{zE} &= \frac{2}{n^2} \int_0^\infty I_0(\lambda r) e^{+\mu_E z} \frac{\lambda d\lambda}{\mu + \mu_E}. \end{aligned}$$

If on the other hand we consider the special case $n \rightarrow \infty$, then we also have $|\mu_E| \rightarrow \infty$ and the integrals in (12) vanish, so that (12) reduces to

$$(12b) \quad \Pi_z = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}, \quad \Pi_{zE} = 0$$

in agreement with (31.17).

The integration in equations (12) and (12a) is to be taken over the path W_1 of Fig. 28. Here again we can profitably replace this path by the closed path $W = W_1 + W_2$. If at the same time we replace I_0 by $\frac{1}{2} H_0^1$ then for vanishing h but finite k_E we obtain, in analogy to (32.14a),

$$(12c) \quad \Pi_x = \int_W H_0^1(\lambda r) e^{-\mu z} \frac{\lambda d\lambda}{\mu + \mu_E}, \quad \Pi_{xE} = \frac{1}{n^2} \int_W H_0^1(\lambda r) e^{+\mu_E z} \frac{\lambda d\lambda}{\mu + \mu_E}.$$

We now turn to the determination of Π_x and first consider the second condition (9). Since Π_x and Π_{xE} do not depend on x and y separately but only on $r = \sqrt{x^2 + y^2}$, we have

$$\frac{\partial \Pi_x}{\partial x} = \frac{\partial \Pi_x}{\partial r} \frac{\partial r}{\partial x} = \cos \varphi \frac{\partial \Pi_x}{\partial r}, \quad \varphi = \angle(x, \vec{r}),$$

and a corresponding relation for $\partial \Pi_{xE} / \partial x$. From the second equation (9) it follows that Π_x must also contain the factor $\cos \varphi$. Hence we deduce that Π_x can no longer be constructed from the eigenfunctions $I_0(\lambda r) e^{\mp \mu z}$; it is necessary to use Bessel functions with the next higher index 1

$$I_1(\lambda r) \cos \varphi e^{\mp \mu z}$$

Considering the fact that Π_x should contain no primary stimulation we write:

$$(13) \quad \begin{aligned} \Pi_x &= \cos \varphi \int I_1(\lambda r) e^{-\mu(z+h)} \Phi(\lambda) d\lambda, \\ \Pi_{xE} &= \cos \varphi \int I_1(\lambda r) e^{+\mu_E z - \mu h} \Phi_E(\lambda) d\lambda. \end{aligned}$$

where Φ and Φ_E are still to be determined. The first condition (9) then yields

$$(13a) \quad \Phi = n^2 \Phi_E.$$

The second condition (9) yields

$$(13b) \quad \begin{aligned} & -\cos \varphi \int I_1(\lambda r) e^{-\mu h} (\mu \Phi + \mu_E \Phi_E) d\lambda \\ & = \cos \varphi \left(\frac{1}{n^2} - 1 \right) \int I_0'(\lambda r) e^{-\mu h} \frac{2 \lambda^2 d\lambda}{\mu + \mu_E}. \end{aligned}$$

In fact, in the representation (12) the terms not under the integral signs vanish for $z = 0$. If we multiply the numerator and denominator of the integrand on the right side by $\mu - \mu_E$ and consider the fact that

$$\mu^2 - \mu_E^2 = k_E^2 - k^2 = k_E^2 \left(1 - \frac{1}{n^2} \right)$$

and that according to (19.52b) we have $I'_0(\varrho) = -I_1(\varrho)$, then we can contract (13b) to

$$(13c) \quad -\cos \varphi \int I_1(\lambda r) e^{-\mu h} \left(\mu \Phi + \mu_E \Phi_E + \frac{2}{k_E^2} (\mu - \mu_E) \lambda^2 \right) d\lambda = 0.$$

From this we deduce a further relation between Φ and Φ_E :

$$(13d) \quad \mu \Phi + \mu_E \Phi_E = -\frac{2}{k_E^2} (\mu - \mu_E) \lambda^2.$$

This, together with (13a) and (32.2), yields

$$(14) \quad \Phi = -\frac{2\lambda^2}{k^2} \frac{\mu - \mu_E}{n^2\mu + \mu_E}, \quad \Phi_E = \frac{\Phi}{n^2}.$$

According to (13) the final representation of Π_z is then

$$(15) \quad \begin{aligned} \Pi_z &= -\frac{2}{k^2} \cos \varphi \int I_1(\lambda r) e^{-\mu(z+h)} \frac{\mu - \mu_E}{n^2\mu + \mu_E} \lambda^2 d\lambda, \\ \Pi_{zE} &= -\frac{2}{k_E^2} \cos \varphi \int I_1(\lambda r) e^{+\mu_E z - \mu h} \frac{\mu - \mu_E}{n^2\mu + \mu_E} \lambda^2 d\lambda. \end{aligned}$$

The path of integration is W_1 of Fig. 28, or, if we replace I_1 by $\frac{1}{2} H_1^1$, the path $W = W_1 + W_2$. Since the denominator in (15) coincides with that of Π_z in §32, "surface waves" also exist for the Π_z which is induced by a horizontal antenna. These surface waves correspond to the pole P in Fig. 28 and they are superimposed on the "spatial waves" or merge with them. Under the assumption $k_E \rightarrow \infty$, which implies $n \rightarrow \infty$, $\mu_E \rightarrow \infty$, the component Π_{zE} vanishes. Hence the induced vertical component is strongly dependent on the nature of the ground and thus does not appear in the previous elementary treatment of §31.

A principal distinction of the horizontal antenna as compared to the vertical antenna is its *directed radiation*, which is implied by the factor $\cos \varphi$ in (15). The same factor is contained in the electric and magnetic field components which determine the radiation and it is a quadratic factor of the radiated energy. Later on we shall see that the component Π_z , which is free of $\cos \varphi$, in general gives no essential contribution to distant transmissions, and hence it can be neglected in the following discussion.

The solid curve in Fig. 29 represents the "direction characteristic" of the horizontal antenna. In order to obtain this curve we plot the radiated energy $\sqrt{E} = M \cos \varphi$ in a polar diagram, where M is the maximum of \sqrt{E} radiated in the direction $\varphi = 0$. This curve is

symmetric with respect to the direction $\varphi = \pm \pi/2$ in which there is no radiation; the radiation in the forward direction $\varphi = 0$ and in the

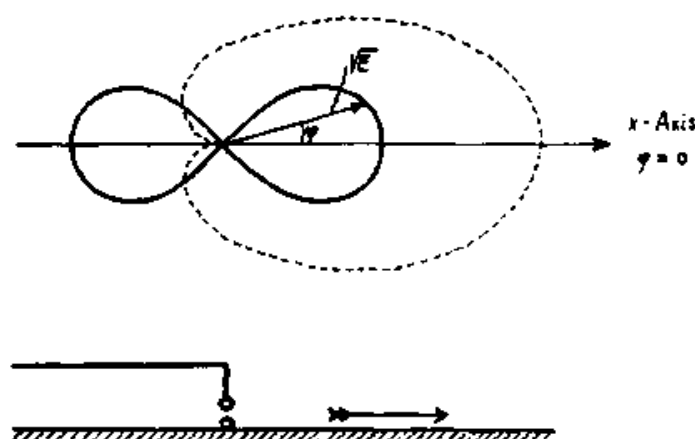


Fig. 29. Upper half: solid curve = direction characteristic of the horizontal antenna; broken curve = direction characteristic of the Marconi antenna.

Lower half: diagram of the Marconi antenna.

backward direction $\varphi = \pi$ is the same. If we combine the horizontal antenna *coherently* with a vertical antenna so that the vertical antenna alone would give the same radiation M as would be given by the horizontal antenna in the direction $\varphi = 0$ (the polar diagram would be a circle of radius M), then, we obtain as the total characteristic the curve

$$\sqrt{E} = M(1 + \cos \varphi) = \begin{cases} 2M & \text{for } \varphi = 0 \\ M & \text{for } \varphi = \pi/2 \\ 0 & \text{for } \varphi = \pi. \end{cases}$$

This characteristic is represented by the broken curve and shows a stronger directedness than the solid horizontal antenna curve.

In the lower half of Fig. 29 we sketched an arrangement by which such a combination of horizontal and vertical antennas was realized on a large scale by Marconi (about 1906) for transatlantic communication (station Clifden in Ireland). The preferred radiation in the direction of the arrow in Fig. 29 aroused general amazement and raised the problem studied by H. von Hörschmann,¹⁰ in which the above theory was developed (Marconi worked only with the instinct of the ingenious experimenter). However the Clifden arrangement was somewhat cumbersome, and it has since been replaced by a more convenient combination of two or more vertical antennas (see Fig. 30).

In Fig. 30 we have drawn a horizontal antenna of the effective

¹⁰ Dissertation, Munich 1911, *Jahresber. f. drahtl. Tel.* 5, 14, 158 (1912).

length l , together with the current which flows through and the influx and outflux through the earth. The last two are equivalent to two

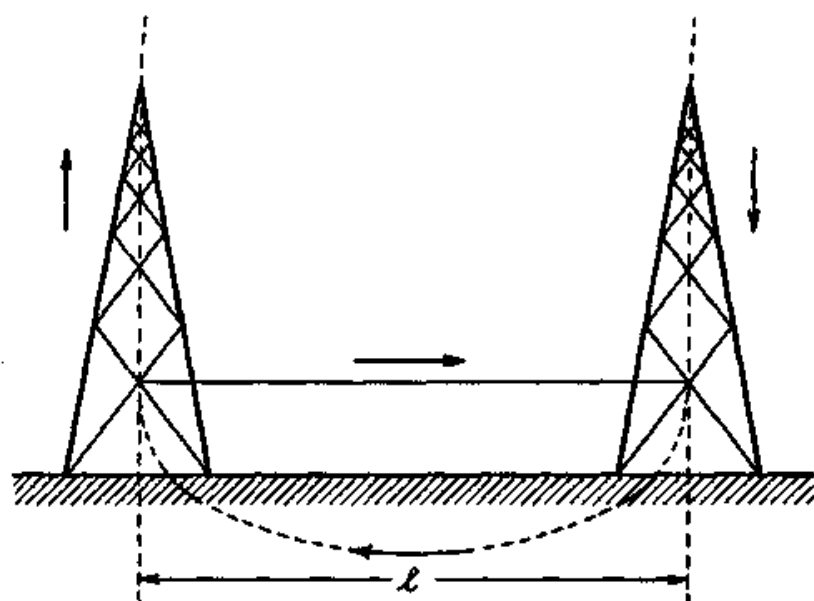


Fig. 30. Horizontal antenna with the accompanying earth currents; at a distance the effect of the two vertical antennas is the same as the effect of the horizontal antenna.

coherent vertical antennas of opposite phase, which we have indicated by towers. Their action at a distance is represented by a formula of the type

$$(16) \quad l \cos \varphi \frac{\partial \Pi_1}{\partial r}, \quad \Pi_1 = \frac{e^{i k R}}{R},$$

where Π_1 is the Hertz vector of the individual tower. We want to show that our theory in a rough approximation really leads to a formula of this type.

Since we are interested only in action at a distance we set $h = 0$ in (15) and in analogy to (32.10b) write

$$\mu - \mu_E \sim -\mu_E = i k n, \quad n^2 \mu + \mu_E \sim n^2 \mu.$$

In addition we have the relation

$$\frac{\partial}{\partial r} I_0(\lambda r) = -\lambda I_1(\lambda r).$$

The first equation (15) thus becomes

$$(16a) \quad \Pi_z = \frac{2i}{kn} \cos \varphi \frac{\partial}{\partial r} \int_0^\infty I_0(\lambda r) e^{-\mu z} \frac{\lambda d\lambda}{\mu},$$

so that we now have the primary stimulation e^{ikR}/R under the integral sign. Hence, we have actually obtained the form of equation (16); for the length of the antenna l we obtain

$$(16b) \quad |l| = \frac{2}{k|n|} = \frac{2\sqrt{\epsilon_0}}{k|\sqrt{\epsilon} + i\sigma/\omega|}.$$

Due to the meaning of k this length $|l|$ is of the order of magnitude of the wave length λ , but it also depends strongly on the nature of the ground; in the limit $\sigma \rightarrow \infty$ we have $l = 0$ as has been stressed before.

The same approximation method leads to an estimate of the order of magnitude of Π_x . We start from the first equation (12) and set $h = 0$ as well as $\mu + \mu_E \sim \mu_E \sim -ikn$. We then obtain

$$(16c) \quad \begin{aligned} \Pi_x &= \frac{2i}{kn} \int_{W_1} I_0(\lambda r) e^{-\mu z} \lambda d\lambda = -\frac{2i}{kn} \frac{\partial}{\partial z} \int_{W_1} I_0(\lambda r) e^{-\mu z} \frac{\lambda d\lambda}{\mu} \\ &= -\frac{2i}{kn} \frac{\partial}{\partial z} \frac{e^{ikR}}{R}. \end{aligned}$$

Now we have

$$\frac{\partial}{\partial z} \frac{e^{ikR}}{R} = \frac{z}{R} \frac{d}{dR} \frac{e^{ikR}}{R}, \quad \frac{\partial}{\partial r} \frac{e^{ikR}}{R} = \frac{r}{R} \frac{d}{dR} \frac{e^{ikR}}{R}.$$

The ratio of these latter quantities is z/r , and hence is very small in the neighborhood of the surface of the earth at a great distance from the transmitter. According to (16c) and (16a) $-\Pi_x$ and Π_z have the same ratio. Hence we have

$$(17) \quad |\Pi_x| \ll |\Pi_z|.$$

This fact has been mentioned before but is proved here for the first time.

The result is very remarkable: *The primary stimulation Π_x serves only to give rise to the secondary stimulation Π_z . The transmission at a distance is caused by Π_z alone. Only in the immediate neighborhood of the transmitter, due to the prescribed pole of Π_x , does Π_x have an effect which outweighs that of Π_z . At a great distance the field of transmission of a horizontal antenna has the same character as the field of transmission of a vertical antenna, except for the φ -dependence which indicates the primary origin from a horizontal antenna. In both cases the signals for large distances are best received with a vertical antenna; a horizontal antenna would be unsuited as a receiver, since the horizontal component of the induced field is always small compared to the vertical component, even for a moderately conductive ground.*

These results are generally known in practice, but they can hardly

be understood without our theory which takes the nature of the soil into account.

We note that approximations (16a) and (16c) can be considered as the first terms of an expansion in ascending powers of the numerical distance ϱ . Just as we had to complement the term e^{ikR}/R by terms dependent on ϱ for the vertical antenna, so now we must correct (16a) and (16c) by terms dependent on ϱ .

§ 34. Errors in Range Finding for an Electric Horizontal Antenna

In navigation, range finding means the location of that direction from which a signal reaches the receiver. As an ideal *receiver* for radio signals we have the frame antenna, which was described at the end of §31, and which will be investigated in greater detail in §35. We consider the receiving antenna rotatable around a vertical axis. As for navigation, we assume the receiver to be at sea, near the surface of the earth. We assume the *transmitter* to be a horizontal antenna. Then, corresponding to the directional characteristic of Fig. 29, we not only expect maximal reception on all points of the x -axis for an x -directed transmitter, but at every point (x, y) on the earth we expect a maximal reception in the r -direction from which the signal comes, and no reception in the φ -direction. In reality things are not that simple because, in addition to the principal radiation of the order $1/r$, the horizontal antenna also emits radiation of the order $1/r^2$.

In order to prove this last fact we have to carry the approximation of the field one step further than we did in the equations of the preceding section. Namely, equations (33.16a) and (33.16c) yield $\operatorname{div} \vec{H} = 0$ and hence, since $\Pi_z = 0$, for $h = 0$ and $z = 0$ they yield a field \vec{E}_s perpendicular to the surface of the earth. We now compute $\operatorname{div} \vec{H}$ with greater precision. We obtain Π_z from (33.12c) with $h = 0$, and Π_r from (33.15) by setting $h = 0$ and replacing I_1 by $\frac{1}{2} H_1^1$. Then we obtain

$$\begin{aligned}\frac{\partial \Pi_s}{\partial x} &= -\cos \varphi \int_W H_1^1(\lambda r) e^{-\mu z} \frac{\lambda^2 d\lambda}{\mu + \mu_s}, \\ \frac{\partial \Pi_s}{\partial z} &= \cos \varphi \int_W H_1^1(\lambda r) e^{-\mu z} \frac{\mu}{k^2} \frac{\mu - \mu_s}{n^2 \mu + \mu_s} \lambda^2 d\lambda, \\ \operatorname{div} \vec{H} &= -\cos \varphi \int_W H_1^1(\lambda r) e^{-\mu z} \left(\frac{1}{\mu + \mu_s} - \frac{\mu}{k^2} \frac{\mu - \mu_s}{n^2 \mu + \mu_s} \right) \lambda^2 d\lambda;\end{aligned}$$

and by a simple contraction:

$$\begin{aligned}
 \operatorname{div} \vec{\Pi} &= -\cos \varphi \int_{\vec{W}} H_1^1(\lambda r) e^{-\mu z} \frac{\lambda^2 d\lambda}{n^2 \mu + \mu_0} \\
 (1) \qquad &= \cos \varphi \frac{\partial}{\partial r} \int_{\vec{W}} H_0^1(\lambda r) e^{-\mu z} \frac{\lambda d\lambda}{n^2 \mu + \mu_0}.
 \end{aligned}$$

According to (32.14a) the last integral is nothing else than the Π -field of a *vertical* antenna divided by n^2 . Since we assume $z = 0$ we may represent this field by (32.22). It even suffices to use the first term of (32.22), which we can write as

$$(2) \qquad \operatorname{div} \vec{\Pi} = \frac{2}{n^2} \cos \varphi \frac{\partial}{\partial r} \frac{e^{ikr}}{r}.$$

It is now profitable to use polar coordinates. Then we obtain from (2), if we neglect the terms with $(kr)^{-3}$,

$$\begin{aligned}
 \operatorname{grad}_r \operatorname{div} \vec{\Pi} &= \frac{\partial}{\partial r} \operatorname{div} \vec{\Pi} = \frac{2}{n^2} \cos \varphi \frac{\partial^2}{\partial r^2} \frac{e^{ikr}}{r} \\
 &= -\frac{2k^2}{n^2} \cos \varphi \left(1 - \frac{2}{ikr}\right) \frac{e^{ikr}}{r}, \\
 (3) \qquad \operatorname{grad}_\varphi \operatorname{div} \vec{\Pi} &= \frac{1}{r} \frac{\partial}{\partial \varphi} \operatorname{div} \vec{\Pi} = -\frac{2}{n^2} \sin \varphi \frac{1}{r} \frac{\partial}{\partial r} \frac{e^{ikr}}{r} \\
 &= +\frac{2k^2 \sin \varphi}{n^2} \frac{e^{ikr}}{ikr} \frac{1}{r};
 \end{aligned}$$

and from (31.4) we obtain:

$$\begin{aligned}
 (4) \qquad \mathbf{E}_r &= \cos \varphi \mathbf{E}_x + \sin \varphi \mathbf{E}_y = k^2 \cos \varphi \Pi_x + \operatorname{grad}_r \operatorname{div} \vec{\Pi}, \\
 \mathbf{E}_\varphi &= -\sin \varphi \mathbf{E}_x + \cos \varphi \mathbf{E}_y = -k^2 \sin \varphi \Pi_x + \operatorname{grad}_\varphi \operatorname{div} \vec{\Pi}.
 \end{aligned}$$

We still have to estimate Π_x . With the approximation in (33.16c) we would obtain $\Pi_x = 0$ for $z = 0$; a more exact computation yields, if we again ignore the terms with $(kr)^{-3}$,

$$(5) \qquad k^2 \Pi_x = \frac{2}{n^2} \frac{1}{r} \frac{\partial}{\partial r} \frac{e^{ikr}}{r} = -\frac{2k^2}{n^2} \frac{1}{ikr} \frac{e^{ikr}}{r}.$$

Hence we obtain from (3), (4), (5)

$$\begin{aligned}
 \mathbf{E}_r &= -\frac{2k^2}{n^2} \cos \varphi \left(1 - \frac{1}{ikr}\right) \frac{e^{ikr}}{r}, \\
 \mathbf{E}_\varphi &= +\frac{4k^2 \sin \varphi}{n^2} \frac{e^{ikr}}{ikr} \frac{1}{r};
 \end{aligned}$$

and thus, due to $kr \gg 1$,

$$(6) \quad \begin{aligned} \mathbf{E}_r &= -\frac{2k^2}{n^2} \cos \varphi \frac{e^{ikr}}{r}, \\ \mathbf{E}_\varphi &= +\frac{4k^2}{n^2} \frac{\sin \varphi}{ikr} \frac{e^{ikr}}{r}. \end{aligned}$$

From this we conclude that for $\varphi = 0$ we have $\mathbf{E}_\varphi = 0$ and that the horizontal antenna field is in the r -direction. Therefore —

A rotatable frame antenna situated *on the extension of the transmitting antenna* shows the strongest reception *in the direction of the transmitting antenna*, as we had expected from the start.

On the other hand for $\varphi = \pm \pi/2$ we have

$$(7) \quad \mathbf{E}_r = 0, \quad \mathbf{E}_\varphi = \frac{4k^2}{n^2} \frac{e^{ikr}}{ikr^2}.$$

A rotatable receiving antenna situated *on the perpendicular to the transmitting antenna* shows a misdirection. In the position of maximal reception the receiving antenna does not point in the direction of the transmitting antenna, but in a direction perpendicular to it, which is *parallel to the transmitting antenna*. However, the reception is very weak, being of the order $1/r^2$; this explains the fact that in Fig. 29, where we considered only terms of the order $1/r$, this reception was zero.

Generally we may denote \mathbf{E}_r as “correct direction” and \mathbf{E}_φ as “misdirection.” The latter, as in (7), is entirely due to terms of the order $1/r^2$.

For an arbitrary φ the “relative misdirection” in our approximation is, according to (6),

$$\left| \frac{\mathbf{E}_\varphi}{\mathbf{E}_r} \right| = \frac{2}{kr} \tan \varphi.$$

It increases to infinity as φ approaches $\pi/2$, which means that for that value the correct direction vanishes, corresponding to $\mathbf{E}_r = 0$ in (7).

The practical engineer is in error if he considers such misdirections the result of mistakes in the construction of the transmitting or the receiving antenna. As we have seen these misdirections are in the nature of things. Certain other misdirections called “after effects,” which are due to reflections on the ionosphere, will not be discussed here.

§ 35. The Magnetic or Frame Antenna

The *frame antenna* can be used not only for range finding but also for directed transmission. In both cases the plane of the loop is taken

perpendicular to the surface of the earth and the normal to this plane will be taken as the x -axis. For rectangular forms the loop consists of two pairs of coherent vertical and horizontal antennas of opposite phase, similar to the scheme in Fig. 30.

In §31 D we called such an antenna *magnetic*, no matter what the shape of the loop. Our frame antenna, which is situated in the y,z -plane, is equivalent to a *magnetic dipole* in the x -direction; its primary action can be represented by a Hertz vector $\vec{\Pi}_{\text{prim}} = \Pi_x$. Due to the presence of the earth this Hertz vector becomes a general vector $\vec{\Pi}$.

The relation between $\vec{\Pi}$ and the electromagnetic field in a vacuum is the same as in (31.4), but we must replace \mathbf{E} , \mathbf{H} , ϵ_0 , μ_0 by \mathbf{H} , $-\mathbf{E}$, μ_0 , ϵ_0 . In fact this interchange transforms the Maxwell equations (31.5) into themselves. Thus, in a vacuum, as counterpart to (31.4) we have:

$$(1) \quad \mathbf{H} = k^2 \vec{\Pi} + \text{grad div } \vec{\Pi}, \quad -\mathbf{E} = \frac{k^2}{\epsilon_0 i \omega} \text{curl } \vec{\Pi} = -\mu_0 i \omega \text{curl } \vec{\Pi}$$

and in the earth, as counterpart to (31.7) we have:

$$(2) \quad \mathbf{H} = k_E^2 \vec{\Pi} + \text{grad div } \vec{\Pi}; \quad -\mathbf{E} = \frac{k_E^2}{\epsilon i \omega} \text{curl } \vec{\Pi},$$

where we have as before:

$$(2a) \quad k_E^2 = \epsilon \mu_0 \omega^2 + i \sigma \mu_0 \omega, \quad k^2 = \epsilon_0 \mu_0 \omega^2 = \omega^2/c^2.$$

The vector $\vec{\Pi}$ again satisfies the differential equation (31.3).

The boundary conditions for $z = 0$ force us to consider $\vec{\Pi}$ as a vector with two components

$$\vec{\Pi} = (\Pi_x, \Pi_z).$$

just as in the case of the electric horizontal antenna. Indeed, we have

$$(3) \quad \Pi_z = \Pi_{zE}, \quad (4) \quad \frac{\partial \Pi_x}{\partial z} = \frac{\partial \Pi_{xE}}{\partial z}$$

due to the continuity of \mathbf{E}_{tang} ,

$$(5) \quad \text{div } \vec{\Pi} = \text{div } \vec{\Pi}_E, \quad (6) \quad k^2 \Pi_x = k_E^2 \Pi_{xE}$$

due to the continuity of \mathbf{H}_{tang} .

Hence, we have two conditions (4) and (6) for Π_x , and two further conditions (3) and (5) that determine Π_z from the known Π_x . Condi-

tions (4) and (6) are exactly the same as the conditions (32.7) for the vertical antenna. Hence, we can apply the previous representations (32.9) *et seq.* directly to our Π_x . Written in the form (32.14a) as specialized for $h = 0$, these representations read:

$$(7) \quad \begin{aligned} \Pi_x &= \int_W H_0^1(\lambda r) e^{-\mu z} \frac{n^2 \lambda d\lambda}{n^2 \mu + \mu_E}, \\ \Pi_{xE} &= \int_W H_0^1(\lambda r) e^{+\mu_E z} \frac{\lambda d\lambda}{n^2 \mu + \mu_E}. \end{aligned}$$

For the same reasons as in the case of the electric horizontal antenna, we write Π_z in the form (33.13) that contains $\cos \varphi$. However, due to condition (3), the functions Φ and Φ_E will now be equal; their common value is determined from (5):

$$\Phi = \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \frac{\lambda^2}{k^2}.$$

Hence by setting $h = 0$ and replacing I by H we obtain from (33.15)

$$(8) \quad \begin{aligned} \Pi_z &= - \frac{\cos \varphi}{k^2} \int_W H_1^1(\lambda r) e^{-\mu z} \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \lambda^2 d\lambda, \\ \Pi_{zE} &= - \frac{\cos \varphi}{k^2} \int_W H_1^1(\lambda r) e^{+\mu_E z} \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \lambda^2 d\lambda. \end{aligned}$$

However, in contrast to the electric horizontal antenna, we may now neglect Π_z as compared to Π_x , so that in the discussion of the field and of its directional characteristic we shall consider the component Π_x alone.

According to (1) we then have

$$(9) \quad \mathbf{E}_x = 0, \quad \mathbf{E}_y = \mu_0 i \omega \frac{\partial \Pi_x}{\partial z}, \quad \mathbf{E}_z = -\mu_0 i \omega \frac{\partial \Pi_x}{\partial y}.$$

Now we obtained the first line of (7) from the representation (32.14a), which for small numerical distances was approximated by (32.23). Applying the latter to (7) we obtain

$$(10) \quad \Pi_x = 2 \frac{e^{ikR}}{R} (1 + \dots), \quad R = \sqrt{r^2 + z^2}.$$

This agrees with the representation (31.20) for an infinitely conductive ground. From (9) and (10) for $z = 0$, we now obtain

$$\mathbf{E}_x = \mathbf{E}_y = 0, \quad \mathbf{E}_z = -2 \mu_0 i \omega \frac{y}{r} \frac{d}{dr} \frac{e^{ikr}}{r} = 2 \mu_0 \omega k \sin \varphi \frac{e^{ikr}}{r}.$$

For the *directional characteristic* in the sense of p. 261 we obtain

$$(11) \quad \sqrt{E} = M \sin \varphi,$$

where E is the radiated energy and M is the maximum of \sqrt{E} that is radiated in the direction $\varphi = \pm \pi/2$ (M is proportional to r^{-1}).

We compare (11) with the elongated directional characteristic for the horizontal antenna in Fig. 29. The two curves are identical except for the interchange of $\sin \varphi$ and $\cos \varphi$, in accordance with the remark at the beginning of this section about the current in the frame and the horizontal antenna. The interchange of $\sin \varphi$ and $\cos \varphi$ is obviously due to the fact that while our horizontal antenna had the direction of the x -axis the plane of our frame antenna was situated perpendicular to the x -axis.

Hence, the frame antenna has its *maximal radiation in the plane of its frame* ($\varphi = \pm \pi/2$), just as the horizontal antenna has the maximal radiation in its own direction ($\varphi = 0$ and $\varphi = \pi$). Correspondingly, the frame antenna has *maximal reception* if its plane is situated *in the direction of the incoming wave*. Since this plane was assumed throughout to be the y,z -plane, the signal for maximal reception comes from the y -direction with dominating electric z -component (perpendicular to the ground) and magnetic x -component (perpendicular to the plane of the frame). Then the *electric z -component induces an electric current in the frame* or, as we may also put it, *the magnetic x -component stimulates the magnetic dipole of the frame*. Thus, the frame acts as a *magnetic receiver*, just as previously it acted as a *magnetic transmitter*.

Incidentally, in range finding we do not try for *maximal* reception but for *minimal* reception, which yields the more precise measurements, as in all zero methods of measuring in physics. The frame is then in the x,z -plane instead of the y,z -plane. The normal to the frame then points in the y -direction, i.e., *in the direction of the incoming signal*.

§ 36. Radiation Energy and Earth Absorption

In discussing certain energy questions we abandon the domain \mathbf{E}, \mathbf{H} of the field strengths that permit superposition, and turn to the quadratic quantity of energy flow

$$\mathbf{S} = [\mathbf{E} \mathbf{H}]$$

It now no longer suffices to consider the complex representation of the field under omission of the time factor $\exp(-i\omega t)$; instead we must multiply the real field components themselves. However, the complications which this brings with it can be eliminated by *averaging* over space and time. The mean values will be even simpler than our representation of the field so far, since due to the *orthogonality of the eigenfunctions*,

the Bessel functions drop out of the representation and are replaced by more or less elementary functions.

Most important for our purposes is the total energy flow, integrated over a horizontal plane in the air:

$$(1) \quad S = \int S_z d\sigma = \int (\mathbf{E}_r H_\varphi - \mathbf{E}_\varphi H_r) d\sigma.$$

Corresponding to whether this plane lies above ($z > h$) or below ($z < 0$) the dipole antenna ($z = 0$), we denote the energy flow (1) by S_+ or S_- . Both S_- and S_+ are taken relative to the positive z -direction. The energy that effectively¹¹ enters the earth in the *negative* z -direction is then given by $-S_-$, which, for the time being, is to be taken over the plane $z = 0$. However, we see that instead of this plane $z = 0$ we can use an arbitrary plane $z < h$, and in particular the planes $z = h - \varepsilon$, $\varepsilon \rightarrow 0$, for the computation of S_- (the space between two such planes is free from absorption and there is no noticeable energy loss in the direction of infinity). Since all energy which effectively enters the earth is transformed into Joule heat, the function $-S_-$ at the same time represents the total *thermal absorption of the earth* per unit of time. On the other hand S_+ taken over the planes $z = h + \varepsilon$, $\varepsilon \rightarrow 0$, measures the total radiation into the air above the plane $z = h$ per unit of time. We call S_+ the *effective radiation*. Hence

$$(1a) \quad W = S_+ - S_-$$

is the energy needed by the antenna per unit of time if we can neglect all energy losses in the antenna; or, in other words, it is the *power* needed by the antenna (the letter W reminds us of "watt"). In the following discussion we shall have to do mainly with this quantity W .

A. For the vertical antenna we had $\mathbf{E}_\varphi = 0$ and $\mathbf{H}_r = 0$. If we denote the expressions for \mathbf{E}_r and \mathbf{H}_φ , which so far were complex, by E_r and H_φ and adjoin the time dependence, then (1) written explicitly becomes:

$$S = \frac{1}{4} \iint (E_r e^{-i\omega t} + E_r^* e^{+i\omega t}) (H_\varphi e^{-i\omega t} + H_\varphi^* e^{+i\omega t}) r dr d\varphi.$$

Upon averaging over time the terms involving $\exp(\pm 2i\omega t)$ drop out, and, if from now on we understand S to be the mean value, we obtain

$$S = \frac{1}{4} \iint (E_r H_\varphi^* + E_r^* H_\varphi) r dr d\varphi.$$

¹¹ "Effective entry" means "excess of influx over outflux." The outgoing reflected radiation is of course automatically included in S_- .

Owing to the independence of the field from the ϕ -coordinate we can write this in the form

$$(2) \quad S = \frac{\pi}{2} \int_0^{\infty} (E_r H_{\phi}^* + E_r^* H_{\phi}) r dr = \pi \operatorname{Re} \left\{ \int_0^{\infty} E_r^* H_{\phi} r dr \right\}.$$

For the computation of S_+ we take E_r and H_{ϕ} from (32.3) and for S_- we take them from (32.4). These expressions differ only by the signs of Π_{prim} in (32.3) and (32.4). We obtain

$$(3) \quad E_r = \frac{\partial^2 \Pi}{\partial r \partial z} = \int_0^{\infty} I_1(\lambda r) f_1(\lambda, z) \lambda d\lambda.$$

$$(4) \quad H_{\phi} = \frac{-k^2}{\mu_0 i \omega} \frac{\partial \Pi}{\partial r} = \frac{-k^2}{\mu_0 i \omega} \int_0^{\infty} I_1(l r) f_2(l, z) l dl,$$

with

$$(5) \quad f_1(\lambda, z) = \pm \lambda e^{-\mu |z-h|} + \mu F(\lambda) e^{-\mu(z+h)},$$

$$(6) \quad f_2(l, z) = -\frac{l}{\mu_l} e^{-\mu_l |z-h|} - F(l) e^{-\mu_l(z+h)},$$

where $F(\lambda)$ and $F(l)$ are determined by (32.8). The fact that in (4) and (6) we used a variable of integration different from λ and hence had to replace μ by $\mu_l = \sqrt{l^2 - k^2}$, will prove useful in what follows. Using (3) and (4), equation (2) can be rewritten as follows:

$$(7) \quad \frac{\mu_0 \omega}{\pi k^2} S_{\pm} = \operatorname{Re} \left\{ i \int_0^{\infty} f_1^*(\lambda, z) \lambda d\lambda \int_0^{\infty} f_2(l, z) l dl \int_0^{\infty} I_1(\lambda r) I_1(l r) r dr \right\}.$$

Here we can apply the *orthogonality relation* (21.9a), which we write in our present notation for the special case $n = 1$:

$$(8) \quad \int_0^{\infty} I_1(\lambda r) I_1(l r) r dr = \delta(\lambda | l).$$

Hence, the right-most integral in (7) vanishes for all values of l except for $l = \lambda$, so that the middle integration in (7) yields $f_2(\lambda, z)$ (see the footnote on p. 111). Thus (7) reduces to the simple integral

$$(9) \quad \frac{\mu_0 \omega}{\pi k^2} S_{\pm} = \operatorname{Re} \left\{ i \int_0^{\infty} f_1^*(\lambda, z) f_2(\lambda, z) \lambda d\lambda \right\}.$$

A further simplification is obtained if we let the planes $z = h \pm \varepsilon$

approach the position of the dipole antenna, that is

$$|z - h| = \varepsilon \ll h, \quad z + h \sim 2h$$

Then instead of (5) and (6) we have

$$(10) \quad f_1(\lambda) = \pm \lambda e^{-\mu \varepsilon} + \mu F(\lambda) e^{-2\mu h},$$

$$(11) \quad f_2(\lambda) = -\frac{\lambda}{\mu} e^{-\mu \varepsilon} - F(\lambda) e^{-2\mu h}$$

The product $f_1^*(\lambda) f_2(\lambda)$ in (9) is thus the sum of four terms. However, when we pass to the difference $S_+ - S_-$ only two terms remain, namely, those that correspond to the two signs of $f_1^*(\lambda)$ in (10). Applying the definition (1a) we obtain

$$(12) \quad \frac{\mu_0 \omega}{2\pi k^2} W = \operatorname{Re} \left\{ -i \int_0^\infty e^{-(\mu + \mu^*) \varepsilon} \frac{\lambda^2 d\lambda}{\mu} \right\} + \operatorname{Re} \left\{ -i \int_0^\infty F(\lambda) e^{-2\mu h} \lambda^2 d\lambda \right\},$$

where due to $\varepsilon \ll h$ we may neglect $\mu^* \varepsilon$ as compared to $2\mu h$ in the exponential function under the second integral sign. The first integral in (12) is easily evaluated. For $\lambda > k$ both μ and, of course, $\mu + \mu^*$ are real. Hence the real part of $-i$ times the integral from k to ∞ vanishes. Only the integral from 0 to k in which we may pass to the limit $\varepsilon = 0$ remains. Using the variable of integration μ instead of λ we obtain¹²

$$(13) \quad \operatorname{Re} \left\{ -i \int_0^k \frac{\lambda^2 d\lambda}{\mu} \right\} = \operatorname{Re} \left\{ -i \int_{-ik}^0 (\mu^2 + k^2) d\mu \right\} = \frac{2}{3} k^3.$$

Concerning the second term in (12) we first consider the term $F(\lambda) = \lambda/\mu$ in (32.8) which does not vanish for $|k_E| \rightarrow \infty$, and thus compute:

$$(14) \quad \operatorname{Re} \left\{ -i \int_0^\infty e^{-2\mu h} \frac{\lambda^2 d\lambda}{\mu} \right\}.$$

Due to the real character of μ for $\lambda > k$ we again need consider only the integral from $\lambda = 0$ to $\lambda = k$. Written in terms of the variable μ , with the abbreviation $\zeta = 2kh$ (14) becomes

$$\operatorname{Re} \left\{ -i \int_{-ik}^0 e^{-\mu \zeta/k} (k^2 + \mu^2) d\mu \right\} = k^3 \operatorname{Re} \left\{ \left(1 + \frac{d^2}{d\zeta^2} \right) \frac{e^{i\zeta} - 1}{i\zeta} \right\}.$$

¹² Due to the sign of μ we must follow the prescriptions concerning the "permissible sheet of the Riemann surface" in Fig. 28.

By evaluating the real part we obtain:

$$(15) \quad k^2 \left(\frac{\sin \zeta}{\zeta} + \frac{d^2}{d\zeta^2} \frac{\sin \zeta}{\zeta} \right) = 2 k^2 \frac{\sin \zeta - \zeta \cos \zeta}{\zeta^3}.$$

Combining (12), (13) and (15) we have

$$(16) \quad W = \frac{2\pi k^2}{\mu_0 \omega} \left(\frac{2}{3} + 2 \frac{\sin \zeta - \zeta \cos \zeta}{\zeta^3} + K \right), \quad \zeta = 2 k h,$$

where K stands for the remaining contribution of $F(\lambda)$ for $|k h| \neq \infty$, which was not yet considered in (14), namely,

$$(17) \quad K = \frac{1}{k^2} \operatorname{Re} \left\{ i \int_0^\infty \frac{2 \mu_g}{n^2 \mu + \mu_g} e^{-2 \mu h} \frac{\lambda^2 d\lambda}{\mu} \right\}.$$

In connection with (16) we note that the first two terms on the right, which are independent of the nature of the ground, could have been deduced with the help of the apparatus of §31. However, the correction term K can be computed only with the help of our complete theory. We defer the discussion of these formulas to Section C.

B. For the *horizontal antenna* the formulas become more complicated due to the combined action of Π_x and Π_z , but with the help of the orthogonality relation (8) we finally obtain simplifications similar to those obtained for the vertical antenna. We shall merely outline the necessary computations. Instead of (2) we now have

$$(2a) \quad S = \frac{1}{4} \operatorname{Re} \iint (E_r H_\varphi^* - E_\varphi H_r^*) r dr d\varphi$$

and instead of (3) and (4) we have for the time being

$$(3a) \quad E_r = k^2 \cos \varphi \Pi_x + \frac{\partial}{\partial r} \operatorname{div} \vec{\Pi}, \quad E_\varphi = -k^2 \sin \varphi \Pi_x + \frac{1}{r} \frac{\partial}{\partial \varphi} \operatorname{div} \vec{\Pi},$$

$$(4a) \quad H_\varphi = \frac{-k^2}{i \mu_0 \omega} \left(-\cos \varphi \frac{\partial \Pi_z}{\partial z} + \frac{\partial \Pi_x}{\partial r} \right), \quad H_r = \frac{-k^2}{i \mu_0 \omega} \left(\sin \varphi \frac{\partial \Pi_z}{\partial z} + \frac{1}{r} \frac{\partial \Pi_x}{\partial \varphi} \right).$$

Since, according to (34.1) and (33.15), $\operatorname{div} \vec{\Pi}$ and Π_z are proportional to $\cos \varphi$ while according to (33.12) Π_x is independent of φ , we conclude from (3a) and (4a) that E_r and H_φ contain the factor $\cos \varphi$, while E_φ and H_r contain the factor $\sin \varphi$. We then can carry out the integration with respect to φ in (2a), and instead of (7) we obtain a triple integral with respect to λ , l and r that has a somewhat complicated structure. However, if we form the difference $W = S_+ - S_-$ then the formulas become much simpler, since only the term that arises

from the primary stimulation II_x has alternate signs. If we also use the Bessel differential equation for the elimination of the derivatives of I_0 , then we obtain

$$(5a) \quad \frac{W}{\pi k c} = \operatorname{Re} \left\{ -i \int_0^\infty \lambda d\lambda \Lambda \int_0^\infty l dl e^{-\varepsilon \mu l} \int_0^\infty r dr I_0(\lambda r) I_0(l r) \right\},$$

$$(6a) \quad \Lambda = \frac{2k^2 - \lambda^2}{\mu} (e^{-\varepsilon \mu} - e^{-2h\mu}) + 2 \frac{\lambda^2 - 2\mu\mu_x}{n^2\mu + \mu_x} e^{-2h\mu}.$$

where ε is as before. Due to the orthogonality relation (8) this reduces to the simple integral:

$$(7a) \quad \frac{W}{\pi k c} = \operatorname{Re} \left\{ -i \int_0^\infty e^{-\varepsilon \mu} \Lambda \lambda d\lambda \right\}.$$

The integration of the term in (6a) that is independent of k_x can again be carried out as in (13) and (14), and again we need consider only the interval $0 < \lambda < k$. Thus, instead of (16) we obtain

$$(16a) \quad W = \frac{2\pi k^5}{\mu_0 \omega} \left(\frac{2}{3} - \frac{\sin \zeta}{\zeta} + \frac{\sin \zeta - \zeta \cos \zeta}{\zeta^3} + L \right)$$

where ζ is as before and

$$(17a) \quad L = \frac{1}{k^3} \operatorname{Re} \left\{ i \int_0^\infty e^{-2\mu h} \frac{2\mu\mu_x - \lambda^2}{n^2\mu + \mu_x} \lambda d\lambda \right\}.$$

The expression (16a) is free of Bessel functions, just as (16) is (see the beginning of this section). F. Renner has drawn my attention to the fact that the expressions (16) and (16a) can also be obtained by a process that may be more familiar to practical engineers, and that we shall discuss in exercise VI.3. However this process yields only the power $W = S_+ - S_-$ and not the values of S_+ and S_- separately, and the latter are of considerable practical interest.

C. *Discussion.* We first consider the principal terms of the equations (16) and (16a), neglecting for the time being the correction terms K and L :

$$(18) \quad \begin{aligned} & \frac{2}{3} + 2 \frac{\sin \zeta - \zeta \cos \zeta}{\zeta^3}, \\ & \frac{2}{3} - \frac{\sin \zeta}{\zeta} + \frac{\sin \zeta - \zeta \cos \zeta}{\zeta^3}. \end{aligned}$$

For $\zeta \rightarrow \infty$ they assume the common value $2/3$. Due to $\zeta = 2kh$ the limit $\zeta = \infty$ is the same as $h = \infty$. Indeed, for $h = \infty$ the

earth has no influence on the radiation of the antenna and the vertical and horizontal antennas must act in the same way. In both cases the total power is transformed into radiation. Correspondingly the equations (16) and (16a) yield the common limit

$$(18a) \quad W = \frac{4\pi}{3} \frac{k^5}{\mu_0 \omega} = \frac{4\pi}{3} \frac{k^4}{\mu_0 c}.$$

This is identical with a formula given by Hertz¹³ for the radiation of his dipole (freely oscillating in space). We note that the factor k^4 corresponds to the reciprocal fourth power of the wavelength in Rayleigh's law of scattering, which does actually arise from the superposition of a large number of distant dipoles that are distributed over the atmosphere and that are stimulated to radiation by the incoming sun rays.

If we expand the expressions (18) in ascending powers of ζ and then pass to the limit $h = 0$ so that the expansion breaks off with the term ζ^0 , then we obtain:

$$(18b) \quad \frac{2}{3} + \frac{2}{3} + \dots = \frac{4}{3} = 2 \cdot \frac{2}{3} \text{ for the vertical antenna,}$$

$$\frac{2}{3} - 1 + \frac{1}{3} \dots = 0 \cdot \frac{2}{3} \text{ for the horizontal antenna.}$$

We can better understand the factors 2 and 0 on the right here with the help of Fig. 27 in §31: through reflection on an infinitely conductive earth the radiation of the vertical antenna doubles for $h = 0$, the radiation of the horizontal antenna is canceled by its mirror image. However we must remember that we have neglected the correction terms K and L in (18). This disregard of K and L means that simultaneously with passage to the limit $h \rightarrow 0$ we also let $k_E \rightarrow \infty$.

Figure 31 gives a general representation of the expressions (18). Above the axis of abscissas we have marked the values of ζ , below it the corresponding values of h . The figure shows that both for the vertical and the horizontal antenna the passage to the limit $2/3$ is through continued oscillation around this limit. The distance between the abscissas of two consecutive extrema measured in the h scale is, for both curves, approximately equal to half a wavelength; this corresponds to the interference between the incoming radiation and the radiation which is reflected by the infinitely conductive ground.

In addition, for both curves we have traced a first correction by

¹³ In the work quoted on p. 237; the formula can be found on p. 160 of his collected works v.II. In comparing (18a) with Hertz' formula we have to take into consideration the dimensionality factor which will be determined in (22) below.

broken lines as given by the terms K and L in (17) and (17a). The value $k/|k_E| = 1/100$ that we use corresponds to the case of sea water

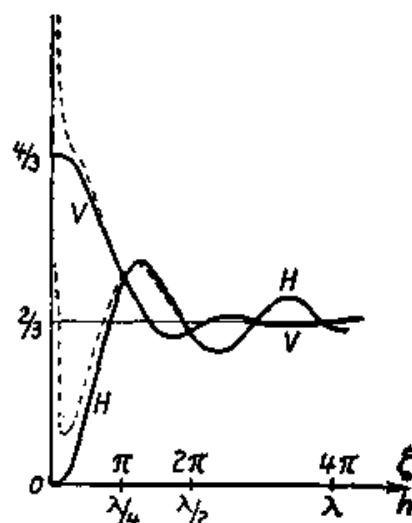


Fig. 31. The power needed by a dipole antenna for different altitudes h above the ground. V = vertical, H = horizontal antenna, the solid curves are for an infinitely conductive ground, the broken curves are for sea water.

for a 40-m. wavelength. We note that the ordinates of the broken curves increase steeply as h tends to zero; of course for finite h the difference in the ordinates from the limiting case $k_E \rightarrow \infty$ increases as k_E decreases. We are dealing here with a quite complicated double limit process, which reminds us of the double limit process in the Gibbs phenomenon of §2: if we first let $k_E = \infty$ and then $h \rightarrow 0$, we end up with the finite ordinates $4/3$ and 0 . However, if we stop at a finite value of k_E and first let $h \rightarrow 0$, then we end with an infinite ordinate which remains the same if afterwards we let $k_E \rightarrow \infty$.

What is the physical meaning of the infinite increase of W ? It does not add to the effective radiation S_+ but gets lost as earth heat $-S_-$. In fact the Joule heat generated in the ground per unit of volume¹⁴ for a fixed antenna current increases with increasing k_E , whereas the effective radiation remains finite. In order to prove this fact we should have to discuss the formulas for S separately, together with the correction terms K and L , and this would lead us too far afield.¹⁵

D. *Normalization to a given antenna current.* We have developed the entire theory of this chapter without consideration of the physical dimensions of the quantities introduced. This omission must be corrected now.

¹⁴ Since the volume in which Joule heat is generated decreases with increasing $|k_E|$ (skin effect), we see that despite the statement in the text there is no heat loss in the limit $|k_E| \rightarrow \infty$.

¹⁵ We refer the reader to the investigation by A. Sommerfeld and F. Renner, *Strahlungsenergie und Erdabsorption bei Dipolantennen*, *Ann. Physik* 41 (1942), where one also finds details concerning the concepts of radiation resistance and the form factor for a finite length of the antenna, which are customary in technology.

In the formula (31.1) we made the Hertz dipole factor equal to 1. In reality this factor is a denominate number, whose dimension is obtained from the relation between \mathbf{H} and \mathbf{E} in (31.4). According to (31.4) \mathbf{H} has the dimension $\mathbf{E} \times M^2$. Since, according to (31.1), \mathbf{H} would have the dimension $1/r$, which is the same as M^{-1} , we obtain for the coefficient of \mathbf{H} , which we set equal to 1, the dimension $\mathbf{E} \times M^3$. We compare this factor with the Maxwell dielectrical translation $\mathbf{D} = \epsilon \mathbf{E}$, which has the dimension of charge per unit of area, that is Q/M^2 , where Q is the dimensional symbol for charge (see p. 237). Hence, written for the special case of the vacuum, we have the dimensional equation:

$$(19) \quad \epsilon_0 \mathbf{E} = \frac{Q}{M^2}, \quad \text{hence} \quad \mathbf{E} M^3 = \frac{QM}{\epsilon_0},$$

where QM is an electric momentum that we set equal to el . In Hertz' original model e was the charge of one particle which oscillated with respect to a resting charge $-e$ and beyond it.

Now what takes the place of this momentum in the case of the short antenna described on p. 237 that is loaded with end capacities? The current j_t that flows in the antenna must by assumption be constant over the whole antenna at every moment. We write it in the form:

$$(20) \quad j_t = j \sin \omega t = j \operatorname{Re} \{i e^{-i\omega t}\}.$$

The corresponding charges of the end capacities shall be

$$e_t = e \cos \omega t \quad \text{and} \quad = -e \cos \omega t.$$

According to the general relation

$$j_t = \frac{d}{dt} e_t$$

we must have $e = j/\omega$. At the time $t = 0$, when the current is zero, the charges of the end capacities are $\pm e$. Since these capacities are at the distance l of the length of the antenna they represent an electric momentum of magnitude

$$(21) \quad el = \frac{j l}{\omega}.$$

We have to substitute this product el for the momentum QM in (19). In addition we have to append to (19) the factor $1/4\pi$ obtained from the comparison of the field (31.4) in the neighborhood of the dipole with the

field of the antenna current. Thus we obtain for the dimensionality factor to be appended to our Π :

$$(22) \quad \frac{j l}{4 \pi \omega \epsilon_0}.$$

Both the radiation S and the power W must be multiplied by the square of this factor. Using the relation

$$\frac{\omega}{k} = c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

we obtain instead of (16)

$$(23) \quad W = \frac{1}{8\pi} k^2 l^2 \sqrt{\frac{\mu_0}{\epsilon_0}} j^2 \left(\frac{2}{3} + 2 \frac{\sin \zeta - \zeta \cos \zeta}{\zeta^3} + K \right).$$

This formula gives the power in watts in a dimensionally consistent manner. Indeed, $\sqrt{\frac{\mu_0}{\epsilon_0}}$ has the dimension of resistance and the numerical value $120 \pi = 377 \Omega$. In our system, which is based on the unit of electricity $Q = 1$ coulomb, j is to be measured in amperes. Since kl has the dimension zero, W is expressed directly in units of power: $\Omega A^2 = \text{watt}$.

After multiplication by the same factor, equation (16a) for the horizontal antenna becomes dimensionally correct; we find

$$(23a) \quad W = \frac{1}{8\pi} k^2 l^2 \sqrt{\frac{\mu_0}{\epsilon_0}} j^2 \left(\frac{2}{3} - \frac{\sin \zeta}{\zeta} + \frac{\sin \zeta - \zeta \cos \zeta}{\zeta^3} + L \right).$$

Appendix

RADIO WAVES ON THE SPHERICAL EARTH

The earth will be assumed to be totally conductive (e.g., everywhere covered with sea water). We are dealing with a vertical antenna near the surface of the earth. The direction of the antenna is taken as the axis $\phi = 0$ of a polar coordinate system r, θ, ϕ ; the distance of the antenna from the center of the earth is denoted by r_0 ; the radius of the earth by $a < r_0$. The field then consists of the components $\mathbf{E}_r, \mathbf{E}_\theta, \mathbf{H}_\phi$ and is independent of ϕ . We now want to deduce this field from a scalar solution u of the wave equation.

The Hertz vector Π is not suitable for the representation of this field, since it satisfies not the simple wave equation $\Delta \Pi + k^2 \Pi = 0$, but the more complicated form (31.3b) which holds for curvilinear coordinates. It is more convenient to start from the magnetic component $H_\phi = H e^{-i\omega t}$.

Using the second equation (31.5) we compute $E_{r,\theta} = E_{r,\theta} e^{-i\omega t}$ from H

$$(1) \quad -i\omega\epsilon_0 E_r = \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta H), \quad i\omega\epsilon_0 E_\theta = \frac{1}{r} \frac{\partial}{\partial r} (rH);$$

then, according to this scheme and from the φ -component of the first equation (31.5) we obtain

$$i\omega\mu_0 r H = \frac{\partial}{\partial r} (r E_\theta) - \frac{\partial}{\partial\theta} E_r = \frac{1}{i\omega\epsilon_0} \left\{ \frac{\partial^2}{\partial r^2} (rH) + \frac{1}{r} \frac{\partial}{\partial\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta H) \right\}.$$

Hence H satisfies the differential equation

$$(2) \quad \frac{\partial^2}{\partial r^2} (rH) + \frac{1}{r} \frac{\partial}{\partial\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta H) + k^2 r H = 0.$$

We can transform this equation into the wave equation $\Delta u + k^2 u = 0$ by making H proportional to $\partial u / \partial\theta$; for convenience we set in particular

$$(2a) \quad H = i\omega\epsilon_0 \frac{\partial u}{\partial\theta}.$$

Then (2) becomes

$$(3) \quad i\omega\epsilon_0 r \frac{\partial}{\partial\theta} \left\{ \frac{1}{r} \frac{\partial^2 r u}{\partial r^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial u}{\partial\theta} \right) + k^2 u \right\} = 0.$$

The first two terms in $\{ \}$ are equal to Δu according to the above mentioned scheme. Hence if we choose u as a solution of

$$(4) \quad \Delta u + k^2 u = 0$$

then according to (1) and (2a) the electromagnetic field is completely described by

$$(5) \quad E_r = -\frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial u}{\partial\theta} \right), \quad E_\theta = \frac{1}{r} \frac{\partial^2 (r u)}{\partial\theta \partial r}, \quad H = i\omega\epsilon_0 \frac{\partial u}{\partial\theta}.$$

The boundary condition $E_\theta = 0$ on the surface of the completely conductive earth is satisfied if we set

$$(5a) \quad \frac{\partial r u}{\partial r} = 0 \quad \text{for} \quad r = a.$$

In addition we have the condition that u is to behave as a unit source at the point $r = r_0, \theta = 0$, which means the existence of a radially directed dipole of the E-field.

In this form the problem can be solved according to the method of §28 with the equation (22) of that section for $G(P, Q)$, the only difference being that then we had the boundary condition $u = 0$ instead of

our present condition (5a). However, this implies only a change in the constant A . Whereas from (28.18) and the condition $u = 0$ we obtained the value (28.18a) for A , we now obtain from (5a)

$$(6) \quad A = - \left\{ \frac{\partial}{\partial r} r \psi_n(kr) / \frac{\partial}{\partial r} r \zeta_n(kr) \right\}_{r=a}.$$

where here and in what follows ζ_n stands for ζ_n^1 .

From the solution (28.22) modified in this way we first deduce the simplified formula for the limiting case $r_0 = a$, in which the antenna is directly on the ground. From (6) and (28.18) we obtain for this case

$$\begin{aligned} u_n(kr_0) = u_n(ka) &= \psi_n(ka) - \frac{\psi_n(ka) + ka \psi'_n(ka)}{\zeta_n(ka) + ka \zeta'_n(ka)} \zeta_n(ka) \\ &= ka \frac{\psi_n(ka) \zeta'_n(ka) - \zeta_n(ka) \psi'_n(ka)}{\zeta_n(ka) + ka \zeta'_n(ka)}. \end{aligned}$$

According to exercise IV.8, equation II, the numerator of this fraction reduces to $i/(ka)^2$, so that if we abbreviate the denominator by $\xi_n(ka)$ we obtain

$$(6a) \quad u_n(ka) = \frac{i/ka}{\xi_n(ka)}, \quad \xi_n(ka) = \zeta_n(ka) + ka \zeta'_n(ka).$$

Substituting this in the first line of (28.22) we obtain for G , which is our present u :

$$(7) \quad u = \frac{k}{4\pi i} \sum_{n=0}^{\infty} (2n+1) P_n(\cos \vartheta) \frac{\zeta_n(kr)}{\xi_n(ka)},$$

This equation holds for all values $a < r < \infty$ and $0 \leq \vartheta \leq \pi$; the domain of validity of the lower line of the same equation (28.22) has now reduced to zero. This result (7) agrees with the previous treatment of this case by Frank-Mises (except for a factor which depends on our present definition of the unit source). In addition, the results there for arbitrary earth can be deduced here by a suitable extension of §28 (continuation into the interior of the sphere instead of the boundary condition on the surface).

If, further, we wished to treat the *horizontal* antenna on the spherical earth, then we should have to introduce, in addition to u , a function v which arises from the interchange of \mathbf{H} and \mathbf{E} , and we should obtain a representation for v that is similar to (7) but somewhat more complicated.¹⁶

However, the convergence of the series (7) is very poor, like that

¹⁶ This was done by P. Debye in his dissertation, Munich 1908; *Ann. Physik* 30, 67 (1909). See also Frank-Mises, Chapter XX, §4.

of Green's general representation in Chapter V. In order to see this for the present case we merely have to note that because of the ratio of earth radius to wavelength the numbers ka and kr are >1000 . As long as n is of moderate magnitude, Hankel's asymptotic values for ζ_n are valid and they show that the ratio ζ_n/ξ_n in (7) is nearly independent of n . We should have to use more than 1000 terms of the series until the Debye asymptotic approximations (21.32) became valid; and only the latter can bring about a real convergence of the series.

In order to obtain a usable computation of u , we apply a method which was first applied successfully to our problem by G. N. Watson,¹⁷ and which we shall find to be connected with the method developed in Appendix II to Chapter V. Namely, we transform the sum (7) into a complex integral.

To this end and on the basis of the relation

$$P_n(\cos \theta) = (-1)^n P_n(-\cos \theta),$$

which is valid for *integral* (and *only for integral*) n , we first rewrite the series in (7) in the form

$$(8) \quad \sum_{n=0}^{\infty} (2n+1) (-1)^n P_n(-\cos \theta) \frac{\zeta_n(kr)}{\xi_n(ka)}$$

We then replace n by a complex variable ν and we trace a loop **A** in the ν -plane of Fig. 32 that surrounds all the points

$$(8a) \quad \nu = 0, 1, 2, 3, \dots, n, \dots$$

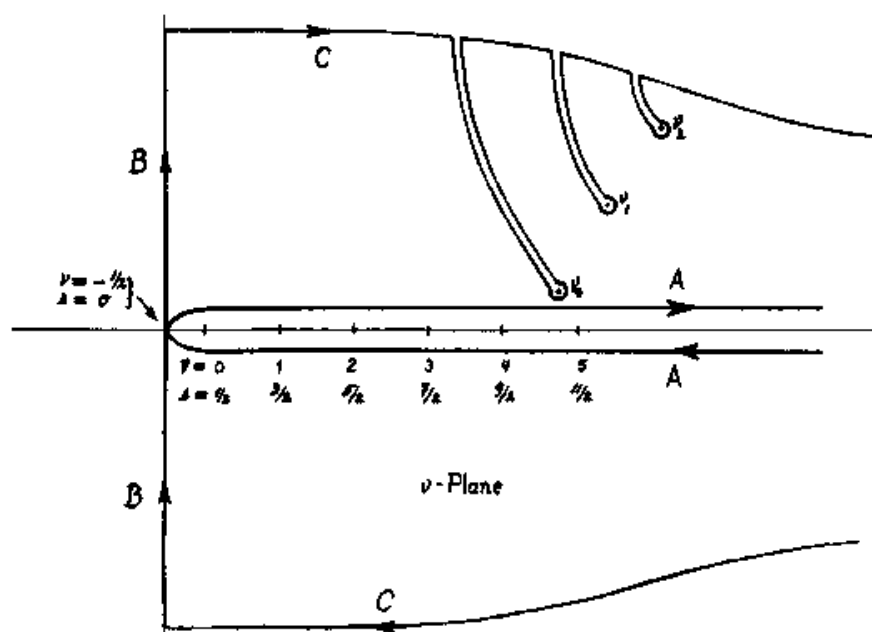


Fig. 32. Deformation of the loop **A** around the real axis. The curve **B** runs parallel to the imaginary axis of the ν -plane. The connection **C** between **B** and **A** must be considered situated at infinity.

¹⁷ *Proc. Roy. Soc. London* 95 (1918).

in a clockwise direction. Over this loop we take the integral

$$(9) \quad \int \frac{2\nu+1}{2i\sin\nu\pi} P_\nu(-\cos\theta) \frac{\zeta_\nu(kr)}{\xi_\nu(ka)} d\nu,$$

which is obtained from the general term in (8) by interchanging n and ν , suppressing the factor $(-1)_n$, and appending the denominator $\sin\nu\pi$. As on p. 215, P_ν does not stand for the Legendre polynomial, but for the hypergeometric function

$$(9a) \quad P_\nu(x) = F\left(-\nu, \nu+1, 1, \frac{1-x}{2}\right)$$

(which is identical with the Legendre polynomial only for integral ν). Now the integrand of (9) has poles of first order at all the zeros of $\sin\nu\pi$; the zeros that lie inside the loop A are the points (8a) and in the neighborhood of the point $\nu=n$ we have as a first approximation:

$$\sin\nu\pi = \sin n\pi + (\nu-n)\pi \cos n\pi = (-1)^n \pi (\nu-n).$$

Hence the residue of the first fraction in (9) becomes

$$\frac{2n+1}{2i\pi} (-1)^n$$

and by computing the integral (9) as $-2\pi i$ times the sum of all residues we obtain

$$(10) \quad - \sum_{n=0}^{\infty} (2n+1) (-1)^n P_n(-\cos\theta) \frac{\zeta_n(kr)}{\xi_n(ka)},$$

which is identical with (8) except for sign.

The next step consists in a deformation of the path A. We note that the hypergeometric series in (9a) is a *symmetric* function of its first two arguments. Hence we have for all (including complex) indices ν :

$$(11) \quad P_\nu = P_{-\nu-1}.$$

With the notation

$$(11a) \quad \nu = s - \frac{1}{2}$$

equation (11) becomes

$$(11b) \quad P_{s-\frac{1}{2}} = P_{-s-\frac{1}{2}}.$$

Hence $P_{s-\frac{1}{2}}$ is an *even* function of s .

This also holds for the last factor of the integrand in (9). In order to prove this we start from the representation (19.22) of H^1 , which is valid for arbitrary indices; if we denote the index by s we have:

$$H_s^1(\varrho) = \frac{1}{\pi} \int_{W_1} e^{i\varrho \cos w} e^{is(w-\pi/2)} dw.$$

If we replace w by $-w$, s by $-s$ and reverse the orientation of W_1 we obtain:

$$H_{-s}^1(\varrho) = \frac{1}{\pi} \int_{W_1} e^{i\varrho \cos w} e^{is(w+\pi/2)} dw = e^{i\pi s} H_{+s}^1(\varrho).$$

If we multiply this equation by $\sqrt{\pi/2\varrho}$ and use (21.15) in order to pass from H to ζ then we obtain

$$(11c) \quad \zeta_{-s-\frac{1}{2}}(\varrho) = e^{i\pi s} \zeta_{s-\frac{1}{2}}(\varrho).$$

The same relation holds for the quantity $\xi(ka)$ of (6a):

$$(11d) \quad \xi_{-s-\frac{1}{2}}(ka) = e^{i\pi s} \xi_{s-\frac{1}{2}}(ka).$$

By division we find that the quotient

$$(12) \quad \frac{\zeta_{s-\frac{1}{2}}(k\tau)}{\xi_{s-\frac{1}{2}}(ka)}$$

is also an *even* function of s .

Finally the first fraction in the integrand of (9) written in terms of s is

$$(13) \quad \frac{2s}{-2i \cos s\pi},$$

and therefore is an *odd* function of s .

We now deform the loop A into a straight line B (which is parallel to the imaginary axis of the ν -plane and passes through the point $s=0$, i.e., $\nu=-\frac{1}{2}$) and two paths C (which are at a great distance from the real axis and, so to speak, join the ends of B with those of A). The poles that have to be considered in this connection will be discussed later. For the moment we show that the integrals over the paths B and C *vanish*.

For the path B this follows directly from the odd character of the integrand of (9) as written in terms of the variable s . In order to show the same thing for the path C we investigate the factor ζ_ν/ξ_ν of the integrand for large values of ν . We start from the series (19.34)

$$I_\nu(\varrho) = \frac{(\varrho/2)^\nu}{\Gamma(\nu+1)} (1 - \dots),$$

where all the terms indicated by \dots can be neglected for $|\nu| > \varrho$. According to Stirling's formula we have

$$\Gamma(\nu+1) \sim \sqrt{2\pi\nu} e^{-\nu} \nu^\nu,$$

hence

$$I_\nu(\varrho) = \frac{1}{\sqrt{2\pi\nu}} \left(\frac{\varrho}{2\nu} \right)^\nu;$$

and for general complex ν :

$$(14) \quad I_{-\nu}(\varrho) = \frac{1}{\sqrt{-2\pi\nu}} \left(\frac{e\varrho}{-2\nu} \right)^{-\nu}.$$

From this follows:

$$\frac{I_{\nu}(\varrho)}{I_{-\nu}(\varrho)} = (-1)^{\nu+\frac{1}{2}} \left(\frac{e\varrho}{2\nu} \right)^{2\nu}.$$

This last quantity approaches zero if the real part of ν approaches plus infinity. Hence, in the representation (19.31) we can neglect I_{ν} as compared to $I_{-\nu}$. If from H_{ν} we pass to

$$\zeta_{\nu} = \sqrt{\frac{\pi}{2\varrho}} H_{\nu+\frac{1}{2}}$$

and from $\zeta_{\nu}(kr)$ we pass to the quotient of two ζ -functions, then, from (14) we obtain:

$$(15) \quad Z = \frac{\zeta_{\nu}(kr)}{\zeta_{\nu}(ka)} = \left(\frac{a}{r} \right)^{\nu+1}.$$

Since $a/r < 1$ the quantity in (15) vanishes if the real part of $\nu+1$ approaches plus infinity, as is the case on both parts of \mathbf{C} . The same statement holds for the quotient $\zeta_{\nu}(kr)/\zeta_{\nu}(ka)$, which according to (6a) and (14) can be written in the form

$$\xi_{\nu}(ka) = \zeta_{\nu}(ka) \left\{ 1 + \varrho \frac{\zeta'_{\nu}(\varrho)}{\zeta_{\nu}(\varrho)} \right\}_{\varrho=ka} = \zeta_{\nu}(ka) \{-\nu\}.$$

From this we see that the third factor of the integrand in (9) vanishes. The first factor vanishes due to the denominator $\sin \nu \pi$. The fact that the second factor vanishes follows from (24.17) which holds for an arbitrary complex index of the spherical harmonic. Hence our original path \mathbf{A} can indeed be deformed through the infinite part of the half plane in which the real part of ν is positive.

However, in this deformation the path cannot cross the poles of the integrand:

$$(15a) \quad \xi_{\nu}(ka) = 0, \quad \nu = \nu_0, \nu_1, \nu_2, \dots$$

We shall now investigate their position more closely. For the neighborhood of the m -th root we write:

$$(15b) \quad \xi_{\nu}(ka) = (\nu - \nu_m) \eta_{\nu}(ka), \quad \eta_{\nu} = \left(\frac{\partial \xi_{\nu}}{\partial \nu} \right)_{\nu=\nu_m}.$$

Then by forming residues we obtain from (9)

$$(16) \quad \pi \sum_{\nu=\nu_0, \nu_1, \nu_2, \dots} \frac{2\nu+1}{\sin \nu \pi} P_{\nu}(-\cos \vartheta) \frac{\zeta_{\nu}(kr)}{\eta_{\nu}(ka)}.$$

Now, except for sign, the integral (9) is identical with the series (10) and the latter in turn is identical, except for a constant factor, with the solution (7) of the sphere problem. Hence the series (16) also represents the solution of the sphere problem, and suppressing the immaterial constant factor we may write:

$$(17) \quad u = \sum_{\nu} \frac{2\nu+1}{\sin \nu \pi} P_{\nu}(-\cos \theta) \frac{\zeta_{\nu}(kr)}{\eta_{\nu}(ka)}.$$

We see that the passage from the series (7), which is summed over integral n , to the series (17), which is summed over the complex ν , is obtained by *forming residues in a complex integral twice*.

Before we proceed with the discussion of (17) we return for a moment to Appendix II of Chapter V. There, too, we were dealing with a series summed over integral n and one summed over complex non-integral ν , namely, the series (1) and (3). We now show that there, too, the identity of the two series can be proved by forming residues in a complex integral twice. Written in analogy to (9) this integral is

$$(18) \quad \begin{aligned} & \int \frac{2\nu+1}{2i \sin \nu \pi} P_{\nu}(-\cos \theta) u_{\nu}(k, r_0) \zeta_{\nu}(kr) d\nu \quad r > r_0, \\ & \int \frac{2\nu+1}{2i \sin \nu \pi} P_{\nu}(-\cos \theta) \zeta_{\nu}(kr_0) u_{\nu}(k, r) d\nu \quad r < r_0, \end{aligned}$$

with
$$u_{\nu}(k, r) = \frac{\psi_{\nu}(kr) \zeta_{\nu}(ka) - \psi_{\nu}(ka) \zeta_{\nu}(kr)}{\zeta_{\nu}(ka)}.$$

We see that the poles of the integrands under consideration are the zeros of the denominator

$$(18a) \quad \zeta_{\nu}(ka) = 0, \quad \nu = \nu_1, \nu_2, \nu_3, \dots, \nu_m, \dots$$

which is common to the functions $u_{\nu}(k, r)$ and $u_{\nu}(k, r_0)$. The corresponding residues of $u_{\nu}(k, r)$ and $u_{\nu}(k, r_0)$ are

$$(18b) \quad -\frac{\psi_{\nu}(ka) \zeta_{\nu}(kr)}{\eta_{\nu}(ka)} \quad \text{and} \quad -\frac{\psi_{\nu}(ka) \zeta_{\nu}(kr_0)}{\eta_{\nu}(ka)}$$

where, in contrast to (15b), we have

$$(18c) \quad \eta_{\nu} = \left(\frac{\partial \zeta_{\nu}}{\partial \nu} \right)_{\nu = \nu_m}.$$

The original path of integration for the integrals in (18) is again the path A of Fig. 32. As in that figure, we can deform the path into the sum of the contours around the points $\nu = \nu_1, \nu_2, \dots$, since here too the

paths B, C make no contribution to the integral. Thus we obtain as the *common representation for the two integrals* (18)

$$(19) \quad - \sum_{\nu} \frac{2\nu + 1}{2i \sin \nu \pi} P_{\nu}(-\cos \vartheta) \frac{\psi_{\nu}(ka)}{\eta_{\nu}(ka)} \zeta_{\nu}(kr_0) \zeta_{\nu}(kr).$$

It can be seen that this coincides with the series (9), Appendix II to Chapter V, by considering the relation (II) from exercise (IV.8)

$$\zeta_{\nu}(\varrho) \psi'_{\nu}(\varrho) - \zeta'_{\nu}(\varrho) \psi_{\nu}(\varrho) = i/\varrho^2,$$

This relation yields the fact that for $\varrho = ka$ and $\zeta_{\nu}(ka) = 0$ the quantity $\psi_{\nu}(ka)$ is inversely proportional to $\zeta'_{\nu}(ka)$.

Thus the novel series of Appendix II can be derived by complex integration from ordinary series summed over integral n . In particular, this derivation shows the mathematical reason for the remarkable fact stressed on p. 217 paragraph 1, that the two n -series which are different for $r > r_0$ merge into the same ν -series (19).

In order to conclude our discussion of the spherical earth problem we must first show that the roots (15a) lie in the first quadrant of the ν -plane (just as the roots of (18a)). In (15a) we had the roots of the transcendental equation:

$$(20) \quad \xi_{\nu}(ka) = (\zeta_{\nu} + \varrho \zeta'_{\nu})_{\varrho=ka} = 0.$$

For ζ_{ν} we again must use the special trigonometric form (11a) on p. 218 (both saddle points of equal altitude), since, for the general exponential form of ζ_{ν} , equation (20) can have no roots at all. Hence we can take $d\zeta_{\nu}/d\varrho$ from (11e), p. 219. We then obtain

$$(20a) \quad \xi_{\nu}(ka) = \frac{i}{\varrho \sqrt{\sin \alpha}} (\sin z + \varrho \sin \alpha \cos z), \quad z = \varrho (\sin \alpha - \alpha \cos \alpha) - \frac{\pi}{4}.$$

Due to the quantity $\varrho = ka$ the second term in the parentheses (20a) is dominant. Hence, the roots of $\xi_{\nu} = 0$ (in contrast to the roots of $\zeta_{\nu} = 0$ on p. 219) are given with sufficient accuracy by

$$(20b) \quad \cos z = 0, \quad z = -\left(m + \frac{1}{2}\right)\pi \quad \sin z = (-1)^m.$$

From this we obtain, in analogy to (21.40),

$$(21) \quad \nu = \varrho \cos \alpha = ka \left(1 - \frac{\alpha^2}{2}\right) = ka \left\{1 + \frac{1}{2} (4m + 1)^{\frac{2}{3}} \left(\frac{3\pi}{4ka}\right)^{\frac{2}{3}} e^{i\pi/3}\right\},$$

so that to each $m = 0, 1, 2, \dots$ a root ν_m does indeed correspond in the first quadrant of the ν -plane.

Since the absolute values of the ν_m are large numbers (because $ka \gg 1$), we can compute $P_\nu(-\cos \theta)$ in (17) from the asymptotic equation (24.17). This equation can be written in the form

$$P_\nu(-\cos \theta) = \sqrt{\frac{1}{2\pi\nu \sin \theta}} e^{-i(\nu + \frac{1}{2})(\pi - \theta) + i\pi/4},$$

if we neglect an exponentially small part. With the same accuracy we have

$$\sin \nu \pi = e^{-i\nu\pi/2} i.$$

Hence we can write in (17)

$$(22) \quad \frac{P_\nu(-\cos \theta)}{\sin \nu \pi} = \sqrt{\frac{2i}{\pi\nu \sin \theta}} e^{i(\nu + \frac{1}{2})\theta}.$$

We now specialize the factor ζ/η in (17) to the neighborhood of the surface of the earth, that is, we set $r = a$. According to equation (11a) on p. 218 with $\sin z = (-1)^m$ we now have

$$\zeta_\nu = \frac{i}{\varrho \sqrt{\sin \alpha}} (-1)^m$$

and if we restrict ourselves to the principal term of (20a) we obtain from (15b)

$$\eta_\nu = \frac{-i}{\varrho \sqrt{\sin \alpha}} (-\varrho \sin \alpha (-1)^m) \frac{\partial z}{\partial \nu}.$$

In analogy to p. 219 we have $\partial z / \partial \nu = -\alpha$. Hence,

$$(23) \quad \frac{\zeta_\nu}{\eta_\nu} = \frac{1}{\varrho \alpha \sin \alpha} = \frac{1}{ka \alpha \sin \alpha} \sim \frac{1}{ka \alpha^2}.$$

Substituting (22) and (23) in (17) we finally obtain

$$(24) \quad u = \frac{\sqrt{2i}}{ka} \sum_{\nu_m} \frac{2\nu + 1}{\sqrt{\pi\nu \sin \theta}} e^{i(\nu + \frac{1}{2})\theta} \alpha^{-2}.$$

The last factor α^{-2} depends on the index of summation m ; in fact we have as in (21)

$$\alpha^2 = (4m + 1)^{\frac{1}{2}} \left(\frac{3\pi}{4ka} \right)^{\frac{1}{2}} e^{-2i\pi/8}.$$

In the first factor under the summation sign of (24) we may replace ν by the first term of the last member of (21), which is independent

of m . Thus (24) simplifies to

$$(25) \quad u = \cdots (\sin \vartheta)^{-\frac{1}{2}} \sum_{\nu_m} (4m+1)^{-\frac{1}{2}} e^{i(\nu_m + \frac{1}{2})\vartheta}.$$

where the terms which are independent of m and ϑ are denoted by In our original series (7), summed over n , we would have had to consider more than 1000 terms. In our present series, summed over m , convergence is so rapid, due to the exponential dependence of the terms on $i\nu_m\vartheta$ and to the increase of ν_m indicated in (21), that we may break off at the first or second term. Because of the positive imaginary part of ν the increase of ν_m indicates an exponential damping of the radio signals with increasing distance along the surface of the earth; the factor $(\sin \vartheta)^{-\frac{1}{2}}$ indicates an increase of intensity at the antipodal point $\vartheta = \pi$ of the transmitter.¹⁸

We have omitted all numerical details since our formulas are of no importance for radio communication due to the predominant role of the ionosphere. However, our formulas are of interest for the general method of Green's function in Appendix II to Chapter V and they show the power of this method for a special example.¹⁹

¹⁸ The apparent infinity of (22) for $\vartheta = \pi$ contradicts our general condition of continuity, but this need not disturb us since the equation (24.17) which was used in (22) loses its validity at the points $\vartheta = \pi$ and $\vartheta = 0$. A more precise investigation of the point $\vartheta = \pi$ leads to a kind of Poisson diffraction phenomenon of finite intensity (see J. Gratiatos, Dissertation, Munich; *Ann. Physik* 86 (1928)).

¹⁹ I wish to mention the fact, communicated to me by Mr. Whipple on the occasion of a friendly visit by English physicists, that Watson's results can be deduced directly without the use of complex integration. I may venture the guess that the particular physical considerations which were made for this case are contained in the general method of our Appendix II to Chapter V.

EXERCISES FOR CHAPTER I

I.1. The position of the maxima and minima in special Fourier approximations. Show that the extrema of the Fourier approximation

$$S_{2n-1} = \sin x + \frac{1}{3} \sin 3x + \cdots + \frac{1}{2n-1} \sin (2n-1)x$$

lie at equal distances, and that except for the extremum $x = \pi/2$ which is common to all S , they lie between the extrema of S_{2n+1} .

I.2. Summation of certain arithmetic series. Compute the higher analogues to the Leibniz series (2.8) and to the series Σ_2, Σ_4 in (2.18).

I.3. Expansion of $\sin x$ in a cosine series. Expand $\sin x$ between 0 and π in a cosine series,

a) by considering $\sin x$ continued as an even function in the interval $-\pi < x < 0$, or

b) by substituting $b = 0$ and $c = \pi$ in (4.5).

I.4. Spectral resolutions of certain simple time processes. Compute the spectra of the time processes which are indicated in Fig. 33a and 33b from their Fourier integral and represent them graphically. In the same manner compute the spectrum of a sine wave $\sin 2\pi t/\tau$, which is bounded on both sides and which ranges from $t = -T$ to $t = +T$ where $T = n\tau$ (Fig. 33c), and deduce from this the fact that the width of a spectral line varies inversely with its life span. An absolutely sharp, completely monochromatic spectral line would therefore need a completely unperturbed sine wave that extends to infinity in both directions.

I.5. Examples of the method of complex integration. Give the reasons for the result of exercise I.4a (*Dirichlet discontinuous factor*) by the method of complex integration; also, resolve the spectrum of the sine curve bounded on both sides into the spectra of two waves that are bounded on one side.

I.6. Compute the first Hermite and Laguerre polynomials from their orthogonality condition by the method applied to spherical harmonics, normalizing so that the leading term of $H_n(x)$ is $(2x)^n$ and the constant term of $L_n(x)$ is $n!$.

For the definition of these polynomials see the table on p. 27.

EXERCISES FOR CHAPTER II

II.1. Elastic rod, open and covered pipe. Compute the transversal proper oscillations of a cylindrical rod of length l , which is clamped at $x = 0$ and oscillates freely at $x = l$, and compare them to the proper oscillations of an open and a covered pipe.

II.2. Second form of Green's theorem. Develop the analogue to Green's theorem, see v.II. equation (3.16), for the general elliptic differential expression $L(u)$,

a) where $L(u)$ is brought to the normal form,

b) in the general case.

c) Investigate the conditions under which the boundary value problem becomes unique for a self-adjoint differential expression L .

We may restrict ourselves to the case of two independent variables, for which Green's theorem is

$$\int u \Delta v \, d\sigma + \int (\text{grad } u, \text{grad } v) \, d\sigma = \int u \frac{\partial v}{\partial n} \, ds.$$

II.3. One-dimensional potential theory. Determine the one-dimensional Green's function from the conditions

$$\text{a) } \frac{d^2 G}{dx^2} = 0 \quad \text{for} \quad \begin{cases} 0 \leq x < \xi, \\ \xi < x \leq l, \end{cases}$$

$$\text{b) } G = 0 \quad \text{for } x = 0 \text{ and } x = l,$$

$$\text{c) } \frac{dG_+}{dx} - \frac{dG_-}{dx} = 1 \quad \text{and } G \text{ continuous for } x = \xi,$$

and apply it to the (obviously trivial) solution of the boundary value problem:

$$\frac{d^2 u}{dx^2} = 0, \quad u \text{ continuous for } 0 \leq x \leq l \quad \begin{cases} u = u_0 & \text{for } x = 0, \\ u = u_l & \text{for } x = l. \end{cases}$$

Condition c) means "yield 1" of the source of G which is situated at $x = \xi$; G_+ is the branch $x > \xi$, G_- the branch $x < \xi$ of G .

II.4. Application of Green's method which was developed for heat conduction to the so-called laminar plate flow of an incompressible viscous fluid. We assume the flow to be planar and rectilinear throughout; this means that it is to be independent of the third coordinate z and directed in the direction of the y -axis. The velocity \mathbf{v} then has the single component $\mathbf{v}_y = v$, which, due to our assumption of incompressibility, is

independent not only of z , but also of y , so that the quadratic convection terms $(\mathbf{v} \cdot \text{grad}) \mathbf{v}$ vanish. The Navier-Stokes equation for v is then according to v.II, equation (16.1)

$$(1) \quad \frac{\partial v}{\partial t} - k \frac{\partial^2 v}{\partial x^2} = -\frac{1}{\rho} \frac{\partial p}{\partial y};$$

where k is the kinematic viscosity; the right side is independent of x due to the corresponding equation for the vanishing x -component of velocity, hence it is a function of t only, say $a(t)$.

The flow is to be bounded at $x = 0$ by a fixed plate, which is at rest up to the time $t = 0$ and thereafter is in motion with the velocity $v_0(t)$. Due to the adhesion of the fluid to the plate we have for $x = 0$:

$$(2) \quad v = \begin{cases} 0 & \dots t \leq 0, \\ v_0(t) & \dots t > 0. \end{cases}$$

For the linear Couette flow (see v.II, Fig. 19b) we would have to add further boundary conditions on a plane that is at rest at a finite distance from $x = 0$. However, for the sake of simplicity, we shall consider this plate situated at infinity. The limiting case obtained in this manner is known in fluid dynamics as *plate flow*. For this flow we have, in addition to (2), the condition for $x = \infty$:

$$(3) \quad v = 0 \quad \text{and} \quad p = p_0 \text{ (independent of } y \text{)}.$$

From this it follows that $a(t) = 0$, so that (1) goes over into the equation of heat conduction.

We are thus led to a boundary value problem, which is a specialization of the problem illustrated by Fig. 13 only in that we now have $x_1 = \infty$ and $x_0 = 0$, and is a simplification of that problem because in the initial state in which the plate and hence the fluid are at rest, we have:

$$(4) \quad v = 0 \quad \text{for} \quad t = 0 \quad \text{and} \quad \text{all } x > 0.$$

The solution is obtained as in (12.18), if the principal solution V is replaced by a suitable Green's function. Discuss the resulting velocity profile $v(x)$ for increasing values of t .

EXERCISES FOR CHAPTER III

III.1. *Linear conductor with external heat conduction according to Fourier.* Let the initial temperature for $x > 0$ be $u(x, 0) = f(x)$. How must this function be continued for $x < 0$ so that the condition is

$$\frac{\partial u}{\partial n} + h u = 0$$

satisfied for $x = 0$?

III.2. Deduce the normalization condition in anharmonic analysis by specialization from Green's theorem.

III.3. *Experimental determination of the ratio of outer and inner heat conductivity.* A rod is to be kept at the fixed temperatures u_1 and u_3 at its ends $x = 0$ and $x = l$ and is to be in a stationary state after the effect of an arbitrary initial state dies out. The flow of temperature would then be linear if the lateral surface of the rod were adiabatically closed. Hence, at the middle section of the rod $x = l/2$ we would have temperature $u_2 = (u_1 + u_3)/2$. Hence from the measurement of

$$q = \frac{u_1 + u_3}{2 u_2}$$

we can determine the ratio of outer and inner heat conduction (essentially our constant h). Deduce the relation between q and h needed for the evaluation of the measurement; according to the above $q = 1$ corresponds to $h = 0$.

III.4. *Determination of the ratio of heat conductivity κ to electric conductivity σ .* A metal rod is to be heated electrically, where the current i per unit of length gives the rod the Joule heat $i^2/q\sigma$ (q = cross-section of the rod); the rod is to be insulated against lateral heat conduction. Write the differential equation of the stationary state and adapt it to the boundary conditions $u = 0$ for $x = 0$ and $x = l$ that are realized in water baths. The potential difference V at the ends of the rod and the maximal temperature U at the mid-section of the rod are to be measured. From them we are to compute the ratio κ/σ . For pure metals this ratio has a universal value (Wiedemann-Franz law).

EXERCISES FOR CHAPTER IV

IV.1. *Power series expansion of $I_n(\varrho)$.* Compute this expansion from the integral representation (19.18)

a) for integral n ,

b) for arbitrary n

with the help of a general definition of the F -function.

IV.2. Deduce the so-called circuit relations for H_n^1 and H_n^2 for integral n from the integral representations (19.22).

IV.3. Compute the logarithmic singularity of $H_0(\varrho)$ at the origin from the integral representations (19.22).

IV.4. An elementary process for the asymptotic approximation to $H_n^1(\varrho)$. Verify the asymptotic limiting value of $H_n^1(\varrho)$ by successively neglecting $1/\varrho$ and the higher powers of $1/\varrho$ already in the differential equation. This method is of course dubious from a mathematical point of view.

IV.5. Expansion of a function $f(\vartheta, \varphi)$ on the sphere.

a) Expand f first in a trigonometric series in φ , and then in spherical harmonics in $\cos \vartheta$. That is, find the expansions

$$f(\vartheta, \varphi) = \sum_{m=-\infty}^{+\infty} C_m e^{im\varphi}, \quad C_m = \sum_{n=m}^{\infty} A_{nm} P_n^m(\cos \vartheta)$$

and as combination of both these expansions

$$(1) \quad f(\vartheta, \varphi) = \sum_m \sum_n A_{nm} P_n^m(\cos \vartheta) e^{im\varphi} \begin{cases} -\infty < m < +\infty, \\ |m| \leq n < \infty. \end{cases}$$

b) Construct f from general surface spherical harmonics Y_n and determine the coefficients from the orthogonality relation for

$$Y_{nm} = P_n^m(\cos \vartheta) e^{-im\varphi},$$

that is, find

$$f(\vartheta, \varphi) = \sum_{n=0}^{\infty} Y_n, \quad Y_n = \sum_{m=-n}^{+n} A_{nm} P_n^m(\cos \vartheta) e^{im\varphi};$$

and as combination of both expansions:

$$(2) \quad f(\vartheta, \varphi) = \sum_n \sum_m A_{nm} P_n^m(\cos \vartheta) e^{im\varphi} \begin{cases} 0 < n < \infty, \\ -n \leq m \leq +n. \end{cases}$$

Clarify the apparent dissimilarity in the order of summation in (1) and (2) by a figure (lattice in the m, n -plane) and show that A_{mn} and A_{nm} in (1) and (2) formally have the same meaning (by interchanging the order of summation and integrating).

IV.6. Mapping of the wedge arrangement of Fig. 17 into circular crescents. Transform the 60° -angle wedge of Fig. 17 by reciprocal radii with a suitable position of the center of inversion C (see the text for this figure); the three straight lines 1,-1; 2,-2; 3,-3 then go into three circular arcs and the angles formed by them go into circular crescents. Examine the association of these regions and consider the fact that the Green's function of potential theory can be obtained for each of

these regions (spatially speaking they are spherical crescents) by five repeated reflections.

IV.7. Mapping a) of the plane parallel plate into two tangent spheres, b) of two concentric spheres into a plane and a sphere. Investigate the three-dimensional figure into which the plane parallel plate of p. 74 together with its mirror images are transformed upon inversion. The sphere of inversion is best situated so that it is tangent to the boundary planes of the plate. Show that the plate is thus mapped into the exterior of two tangent spheres; and its mirror images are mapped into the space between two interior tangent spheres. b) Show that two concentric spheres can be inverted into a plane and a sphere. Hence, conversely, we can transform the potential of a sphere towards a plane into the much simpler boundary value problem for two concentric spheres. The same holds for the potential of two arbitrary non-intersecting spheres.

IV.8. Evaluation of two expressions involving Bessel functions. In equation (5) of Appendix I to Chapter IV determine

$$(I) \quad H_n(\varrho) I'_n(\varrho) - H'_n(\varrho) I_n(\varrho)$$

and in equation (20b) of the same appendix determine

$$(II) \quad \zeta_n(\varrho) \psi'_n(\varrho) - \zeta'_n(\varrho) \psi_n(\varrho).$$

EXERCISES FOR CHAPTER V

V.1. Normalization questions. Normalize the functions $I_n(\lambda r)$ and $\psi_n(kr)$ of (26.3) and (26.2) to 1 for the basic interval $0 < r < a$ with the boundary conditions $I'_n(\lambda a) = 0$ and $\psi'_n(ka) = 0$ in analogy to equation (20.19).

V.2. The Gauss theorem of arithmetic mean in potential theory. Prove the theorem: The value of a potential function \bar{u} at any point P of its domain of regularity S is equal to the arithmetic mean u of its values on an arbitrary sphere K_a , which has radius a and center P and which lies entirely in S .

V.3. Summation formulas over the roots of Bessel functions. Verify that the coefficients A_{nm} in the expansion (27.13) equal those of (27.14), and derive interesting identities for the Ψ_n from the equality of the coefficients of $I_n^m(\cos \theta_0) e^{-im\varphi_0}$ in these two expansions. These identities can be rewritten as identities for the ψ_n .

EXERCISES FOR CHAPTER VI

VI.1. Vertical and horizontal antenna at the altitude h over an infinitely conductive ground. Show that the formulas (31.16) and (31.17) for the two *electric* antennas:

$$\text{a) } \Pi = \Pi_z = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \quad \text{b) } \Pi = \Pi_x = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'},$$

satisfy the conditions (31.15) for the vanishing of the tangential component of \mathbf{E} in the entire plane $z = 0$.

In the same manner show that the formulas (31.19) and (31.20) for the two *magnetic* antennas:

$$\text{c) } \Pi = \Pi_z = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}, \quad \text{d) } \Pi = \Pi_x = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}$$

satisfy the conditions (35.1) for the vanishing of the tangential electric component.

VI.2. Behavior of the electric force lines for a Zenneck wave in the neighborhood of the earth's surface. Show that the lines of force in the air are bent forward, i.e., in the direction of propagation and that the lines of force in the earth are dragged behind.

VI.3. Simplified computation of the power needed for the vertical and horizontal antenna. Prove the expressions (36.16) and (36.16a) by determining the work $\mathbf{E}j\mathbf{l}$ done per unit of time by the field strength \mathbf{E} and the current j for the length l of the antenna.

HINTS FOR SOLVING THE EXERCISES

I.1. The equation which determines the position of the extrema is

$$(1) \quad \cos x + \cos 3x + \dots + \cos (2n-1)x = 0.$$

If we write this in the form

$$(2) \quad \operatorname{Re}(e^{ix} + e^{3ix} + \dots + e^{(2n-1)ix}) = 0,$$

then we can sum this geometric series. We obtain finally

$$(3) \quad \frac{\sin 2nx}{2 \sin x} = 0, \quad \text{hence} \quad (4) \quad x = \frac{\pi}{2n}, \frac{2\pi}{2n}, \dots, \frac{(2n-1)\pi}{2n}.$$

Due to the denominator in (3) we do not count the points $x = 0$ and $x = \pi$. Equation (4) proves the statement in the exercise.

I.2. By integrating from 0 to x we obtain a sine series from the cosine series (2.17); and if in the sine series we set $x = \pi/2$ then we obtained the analogue to the Leibniz series which follows (2.14). Integrating this sine series we obtain a series in terms of $1 - \cos x$, $1 - \cos 3x$, ... and setting $x = \pi/2$ here we obtain the next analogue to (2.16), from which we deduce the value of Σ_n . This process can be continued indefinitely, but it does not seem to lead to a transparent law for the successive analogues.

I.3. The two processes mentioned in the exercise lead to the series

$$\sin x = \frac{2}{\pi} \left(1 - \frac{2}{1 \cdot 3} \cos 2x + \frac{2}{3 \cdot 5} \cos 4x - \dots \right)$$

from which, by setting, e.g., $x = 0$, we obtain a representation of $\frac{1}{2}$ as a series in the reciprocals of odd integers.

I.4. Consider case a). If in (4.8) we replace x by t and perform the integration with respect to ξ we obtain

$$(1) \quad f(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \sin \frac{\omega \tau}{2} e^{i\omega t} = \int_0^{\infty} a(\omega) \cos \omega t d\omega.$$

where $|a(\omega)|$ stands for the amplitude of the spectrum of $f(t)$ with the frequency ω and, according to (1),

$$(2) \quad a(\omega) = \frac{1}{\pi} \frac{\sin \omega \tau/2}{\omega/2}.$$

This function has its principal maximum of altitude τ/π at $\omega = 0$,

followed by secondary maxima of decreasing altitudes at intervals with lengths asymptotically equal to $\Delta\omega = 2\pi/\tau$.

In case b), where according to the figure we are dealing with a function f which is odd in t , we obtain in the same manner:

$$(3) \quad f(t) = \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \left(1 - \cos \frac{\omega \tau}{2}\right) e^{i\omega t} = \int_0^{\infty} b(\omega) \sin \omega t d\omega;$$

$$b(\omega) = -\frac{1}{\pi} \frac{1 - \cos \omega \tau/2}{\omega/2} = -\frac{1}{\pi} \frac{\sin^2 \omega \tau/4}{\omega/4}.$$

We now have $b(\omega) = 0$ for $\omega = 0$; the first maximum lies at $\omega \sim 4.7/\tau$; as before, the subsequent secondary maxima successively decrease in altitude.

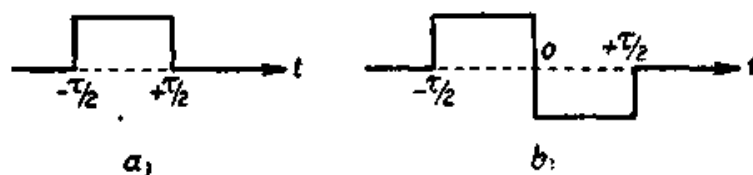


Fig. 33.

$$\begin{aligned} \text{a) } f(t) &= 0 \text{ for } |t| > \tau/2, \\ f(t) &= 1 \text{ for } |t| < \tau/2. \end{aligned} \quad \begin{aligned} \text{b) } f(t) &= 0 \text{ for } |t| > \tau/2, \\ f(t) &= 1 \text{ for } -\tau/2 < t < 0, \\ &= -1 \text{ for } 0 < t < \tau/2. \end{aligned}$$

In both cases a) and b) we are dealing with a "grooved spectrum" that extends to infinity.

In the beginning of the theory of x-rays an attempt was made to interpret them as ether impulses of the type a) or b). From a spectral point of view, which is the only one that is physically justified, this is not a departure from the wave interpretation, but merely an (arbitrary special) assumption about the nature of the x-ray spectrum.

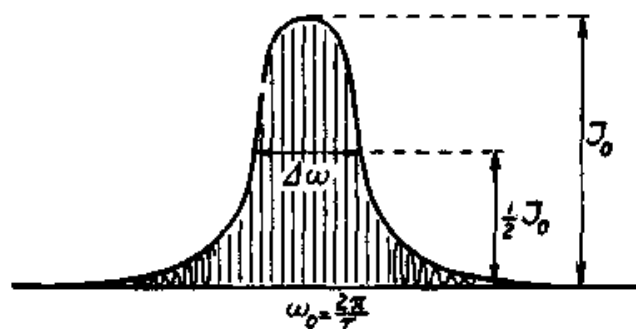


Fig. 33c. Schematic representation of the intensity in the spectrum of a wave process of frequency $\omega_0 = \frac{2\pi}{\tau}$, which breaks off on both sides. Here $\Delta\omega$ is the half-value width of the corresponding spectral line. The ruled middle portion of the spectrum consists of \sin^2 -like oscillations, just like the unruled part.

For the wave process of length $2T = 2\pi\tau$ which breaks off on both sides (Fig. 33c) we start most conveniently from equation (4.11b) and find

$$(4) \quad f(t) = \int_0^{\infty} b(\omega) \sin \omega t d\omega, \quad b(\omega) = \frac{4}{\pi} \frac{\sin \omega T}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2}.$$

The principal maximum of b lies, as expected, at the frequency $\omega_0 = 2\pi/\tau$ and has the altitude $b_0 = \pi\tau/\pi$ corresponding to the intensity $I_0 = \left(\frac{n\tau}{\pi}\right)^2$, and hence increases with increasing length. This principal maximum is flanked on both sides by secondary maxima of successively decreasing altitudes at intervals with length asymptotically approaching $\pi/n\tau$; for all these maxima we have $\sin \omega T \sim 1$. We seek the two maxima for which $I = \frac{1}{2}I_0$, that is, according to (4), those maxima for which

$$(5) \quad \omega^2 - \omega_0^2 = \pm \frac{4\sqrt{2}\pi}{n\tau^2} = \pm \frac{\sqrt{2}}{\pi n} \omega_0^2, \quad \omega^2 = \omega_0^2 \left(1 \pm \frac{\sqrt{2}}{\pi n}\right).$$

The difference of their frequencies is the so-called half-value width of the corresponding spectral line. If we assume $n \gg 1$ then this frequency difference is, according to (5),

$$(6) \quad \Delta\omega = \frac{\sqrt{2}}{\pi n} \omega_0 = \frac{2\sqrt{2}}{T}.$$

Hence the half value width decreases with increasing T as stated in the exercise. Only for $T \rightarrow \infty$ do we obtain an absolutely sharp spectral line.

I.5. The function $f(t)$ of Fig. 33a) for $\tau/2 = 1$ is known in the mathematical literature as the *Dirichlet discontinuous factor*:

$$(1) \quad D = \frac{2}{\pi} \int_0^{\infty} \sin \omega \cos \omega t \frac{d\omega}{\omega} = \begin{cases} 1 & |t| < 1, \\ \frac{1}{2} & |t| = 1, \\ 0 & |t| > 1. \end{cases}$$

If we set $t = 0$ here then we obtain the fundamental integral

$$(2) \quad \int_0^{\infty} \sin \omega \frac{d\omega}{\omega} = \frac{\pi}{2} \quad \text{or} \quad (2a) \quad \int_{-\infty}^{+\infty} \sin \omega \frac{d\omega}{\omega} = \pi.$$

which was used in connection with Fig. 4, p. 11. This can be verified directly through complex integration: since $\sin \omega/\omega$ is analytic on the real axis and in its neighborhood, we can avoid the point $\omega = 0$ (e.g., as in Fig. 34a, p. 301) below the real axis, and we then can decompose (2a)

into the difference of the integrals

$$(3) \quad I = \frac{1}{2i} \int e^{i\omega} \frac{d\omega}{\omega}, \quad II = \frac{1}{2i} \int e^{-i\omega} \frac{d\omega}{\omega},$$

both integrals being taken over the heavy line of the figure. The path of integration in II can be deformed into the infinite part of the *lower* half plane, where the integral vanishes. The path of integration of I must be deformed into the infinite part of the upper half plane since $\exp(i\omega)$ vanishes only there; however, it cannot be deformed across the pole $\omega = 0$. The residue at the pole is $2\pi i$. This proves (2a) and hence (2).

Finally, we easily verify that those parts of the path of integration which in the figure are indicated by short arrows and their dotted continuations also make no contribution to I and II.

The method of complex integration also serves to extend the statements of the preceding exercise for the wave which is bounded on both sides. Such a wave can be considered as the superposition of two waves, which are bounded on one side, of opposite phase, one ranging from $t = -T$ to $t = \infty$, the other from $t = +T$ to $t = \infty$. However these processes cannot be represented individually in the Fourier manner, due to the divergences at $t = \infty$. For this purpose it is necessary to transfer the path of integration in equation (4) of the preceding exercise from the real axis into the complex domain (as shown in Fig. 34b), and then to perform the decomposition. This is explained by the following transformations which start from equation (4) on p. 299:

$$\begin{aligned} f(t) &= \frac{4}{\tau} \int_0^{\infty} \sin \omega t \sin \omega T \frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2} \\ &= \frac{2}{\tau} \int_0^{\infty} \left\{ \cos \omega(t-T) - \cos \omega(t+T) \right\} \frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2} \\ &= \frac{1}{\tau} \int_{-\infty}^{+\infty} (e^{i\omega(t-T)} - e^{i\omega(t+T)}) \frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2} = I - II, \\ \left. \begin{array}{l} I \\ II \end{array} \right\} &= -\frac{1}{\tau} \int e^{i\omega(t \pm T)} \frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2}, \end{aligned}$$

where the integral signs without upper and lower limits are to be taken over the complex path of Fig. 34b.

We claim that I represents the wave process starting at $t = -T$

and II represents that starting at $t = +T$, both continuing to $t = \infty$.

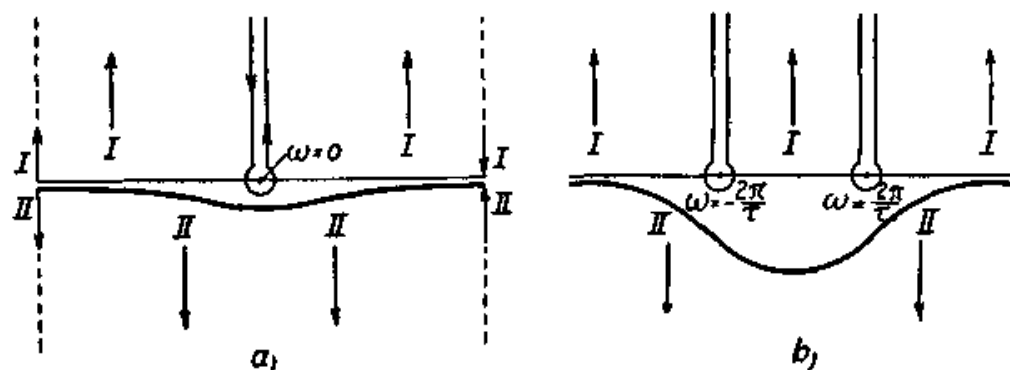


Fig. 34. a) Dirichlet discontinuous factor. We prove equation (3) by deforming the path of integration of II downward, that of I upward. b) Wave process which breaks off on both sides. The complex integrals I and II represent wave processes which are bounded on one side.

In order to prove this we set $T = 0$ for simplicity, and show that

$$(4) \quad f_0(t) = -\frac{1}{\tau} \int e^{t\omega} \frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2}$$

is a sine wave which starts at $t = 0$ and continues to $t = \infty$. The proof is given in a manner similar to that of (3): for $t < 0$ the path of integration can be drawn into the infinite part of the lower half plane and the integral vanishes there. For $t > 0$ the path of integration must be drawn into the upper half plane. Due to the poles $\omega = \pm 2\pi/\tau$ we obtain the residues

$$2\pi i \frac{e^{2\pi i t/\tau}}{4\pi/\tau} \quad \text{and} \quad 2\pi i \frac{e^{-2\pi i t/\tau}}{-4\pi/\tau};$$

so that

$$f_0(t) = -\frac{i}{2} (e^{2\pi i t/\tau} - e^{-2\pi i t/\tau}) = \sin \frac{2\pi t}{\tau}.$$

This completes the proof.

If instead of starting from (4) we start from

$$(5) \quad f_0(t) = -\frac{1}{2\pi} \operatorname{Re} \int e^{t\omega} \frac{d\omega}{\omega - 2\pi/\tau}$$

then we see that for the same choice of the path of integration and the same deformation of this path we obtain the same result as before:

$$(6) \quad f_0(t) = \begin{cases} 0 & t < 0, \\ \sin \frac{2\pi t}{\tau} & t > 0. \end{cases}$$

The interest in this representation lies in the optic theory of dis-

persion. We imagine that perpendicularly to the plane $x = 0$ the wave (6) enters a medium filling the half space $x > 0$, and decompose it according to (5) into partial waves of the form $a(\omega) e^{i\omega t}$. Each of these waves propagates in the direction of increasing x independently of all other waves and we represent it here by $a(\omega) \exp[i(kx - \omega t)]$. The wave number k in a dispersion-free medium would be $k_0 = \omega/c$; due to the induced oscillation of the electrons (numbering N per cm.³) we have

$$(7) \quad k^2 = \frac{\omega^2}{c^2} \left(1 - \frac{Ne^2/m}{\omega^2 - \omega_e^2} \right),$$

where ω_e is the proper frequency of the oscillating electrons (for the sake of simplicity we neglect the damping of these oscillations). Hence in the infinite part of the ω -plane we have:

$$k = k_0, \quad kx - \omega t = k_0 x - \omega t = \frac{\omega}{c} (x - ct).$$

Thus the question of whether the path of integration is to be deformed in the direction of the positive or negative half plane is determined by the sign of $x - ct$. Since this criterion is independent of ω it is the same for all partial waves, so that $x = ct$ stands for the entire light stimulation at the point x of the dispersive medium. *The head of our light signal therefore propagates with the vacuum velocity $dx/dt = c$, not, as one might think, with the phase velocity $V = \omega/k$ which is characteristic for the dispersive medium.* We interpret this in the following way: the electrons are at rest for $t < x/c$ and start plane oscillation for $t = x/c$. The full amplitude corresponding to the incoming oscillation is attained only at a later time that is determined not by the phase velocity V but by the *group velocity* $U = d\omega/dk$. The oscillation processes which precede this time may be called *forerunners of the light signal*.

I.6. a) Hermite polynomials. Due to the even character of $g(x) = e^{-x^2}$ and the fact that the interval is $-\infty < x < +\infty$ we see that the Hermite polynomials, like the spherical harmonics P_n , are even or odd functions of x depending on whether n is even or odd. Considering this fact and the customary normalization of H_n (see exercise) write:

$$H_0 = 1, \quad H_1 = 2x, \quad H_2 = 4x^2 + a, \quad H_3 = 8x^3 + bx, \quad H_4 = 16x^4 + cx^2 + d$$

and compute the coefficients a, b, c, d through a repeated application of the orthogonality condition (result:

$$a = -2, \quad b = -12, \quad c = -48, \quad d = +12).$$

b) *Laguerre polynomials.* Due to the weighting factor $g(x) = e^{-x}$ and the fact that the interval is $0 < x < \infty$ the polynomials are no

longer even or odd. Considering this fact and the customary normalization (see exercise) write:

$L_0 = 1$, $L_1 = ax + 1$, $L_2 = bx^2 + cx + 2$, $L_3 = dx^3 + ex^2 + fx + 6$ and compute the coefficients a, b, \dots, f as in a) result: $a = -1$, $b = 1$, $c = -4$, $d = -1$, $e = 9$, $f = -18$).

II.1. The differential equation (7.8) to which this exercise relates is obtained as follows from the theory of *beam bending*: we start from the differential equation:

$$(1) \quad \frac{\partial^4 u}{\partial x^4} = \frac{1}{EI} \frac{\partial^2 M}{\partial x^2}.$$

The bending moment M of the exterior static load of the beam is to be replaced by the moment of the dynamic inertia resistances

$$- \varrho \frac{\partial^2 u}{\partial t^2}$$

(ϱ = mass per unit of length of the oscillating beam). Let this beam be clamped at $x = 0$ and let the free end be $x = l$. All the cross-sections $x < \xi < l$ contribute to the bending moment at the cross-section x , each cross-section with the lever-arm $\xi - x$. Hence we have

$$(2) \quad M = -\varrho \int_x^l (\xi - x) \frac{\partial^2 u}{\partial t^2} d\xi, \quad \frac{\partial M}{\partial x} = \varrho \int_x^l \frac{\partial^2 u}{\partial t^2} d\xi, \quad \frac{\partial^2 M}{\partial x^2} = -\varrho \frac{\partial^2 u}{\partial t^2}.$$

Substituting this in (1) we obtain (7.8) and for the constant c (of dimension cm^2/sec) we obtain

$$(3) \quad c = \sqrt{\frac{EI}{\varrho}}.$$

According to (2) we have at the free end $M = \frac{\partial M}{\partial x} = 0$. According to equation (1) above this means

$$(4) \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^3 u}{\partial x^3} = 0 \quad x = l.$$

On the other hand the clamping implies

$$(5) \quad u = \frac{\partial u}{\partial x} = 0 \quad \text{for } x = 0.$$

If we write $u = U e^{i\omega t}$, $U = e^{\alpha x}$ then (7.8) yields

$$\alpha^4 = \frac{\omega^2}{c^2}, \quad \alpha = \pm k \text{ and } \pm ik, \quad k = \sqrt{\frac{\omega}{c}}.$$

Hence there are four particular solutions of the differential equation for U :

$$e^{kx}, e^{-kx}, e^{ikx}, e^{-ikx},$$

which for the following can be combined more conveniently into the forms

$$\sinh kx, \cosh kx, \sin kx, \cos kx.$$

Hence the general solution becomes

$$U = A \sinh kx + B \cosh kx + C \sin kx + D \cos kx.$$

According to (4) and (5) there are four relations among the constants of integration A, B, C, D , from which we obtain through elimination the transcendental equation

$$(6) \quad \cos kl = -\frac{1}{\cosh kl}.$$

The graphical treatment of this equation in the manner of Fig. 7 yields an infinity of roots at intervals which asymptotically become

$$(7) \quad k_{n+1} - k_n = \frac{\pi}{l}, \quad \omega_{n+1} - \omega_n = 2cn \frac{\pi^2}{l^2}$$

For the basic oscillation we have

$$(7a) \quad k = k_1 = 1.875/l, \quad \omega = \omega_1 = ck_1^2.$$

The differential equation of a pipe is the same as that of an oscillating string, that is equation (7.6) where u = longitudinal velocity of air, c = velocity of sound. For the pipe which is open on both ends, or one which is covered at $x = 0$ and open at $x = l$ we have the boundary conditions

$$(8a) \quad \frac{\partial u}{\partial x} = 0 \quad \text{for } x = 0 \quad \text{and} \quad x = l \quad (\text{open pipe})$$

or

$$(8b) \quad u = 0 \quad \text{for } x = 0, \quad \frac{\partial u}{\partial x} = 0 \quad \text{for } x = l \quad (\text{covered pipe})$$

(due to the hydrodynamic continuity equation $\partial u / \partial x = 0$ means the same as $\partial p / \partial t = 0$, that is $p = p_0$ = atmospheric pressure which we assume to hold approximately). Writing $u = U e^{i\omega t}$ for the proper oscillations we obtain from (8a, b)

$$(9a) \quad U = A \cos k_n x, \quad k_n = n \frac{\pi}{l}, \quad \omega_n = c k_n, \quad k_1 = 3.14/l$$

$$(9b) \quad U = B \sin k_n x, \quad k_n = \left(n + \frac{1}{2}\right) \frac{\pi}{l}, \quad \omega_n = c k_n, \quad k_1 = 1.57/l.$$

The value (7a) of k_1 lies between the values (9b) and (9a). The sequence of ω is harmonic for both the open and covered pipe; for the elastic rod it becomes harmonic only asymptotically for high overtones (see (7)).

II.2. a) Using the identities

$$v \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial x} \right),$$

$$D v \frac{\partial u}{\partial x} + u \frac{\partial D v}{\partial x} = \frac{\partial}{\partial x} (D u v),$$

and writing $L(u)$ in the normal form (10.1), we obtain

$$(1) \quad v L(u) + A(u, v, \dots) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$$

with

$$(2) \quad A(u, v, \dots) = (\text{grad } u, \text{grad } v) + u \left(\frac{\partial D v}{\partial x} + \frac{\partial E v}{\partial y} - F v \right),$$

$$(3) \quad X = v \frac{\partial u}{\partial x} + D u v, \quad Y = v \frac{\partial u}{\partial y} + E u v.$$

Here A is a *bilinear form* in the u, v and their first derivatives. If L is self-adjoint then, because $D = E = 0$, equation (10.6), we have:

$$(4) \quad A = (\text{grad } u, \text{grad } v) - F u v,$$

which is *symmetric* in u and v .

b) If L has the general form (8.1) then we again have equation (1), but with

$$(5) \quad A(u, v, \dots) = A \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + B \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) + C \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \\ + u \left(\frac{\partial D v}{\partial x} + \frac{\partial E v}{\partial y} - F v \right) + v \left(\frac{\partial A}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial B}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial B}{\partial y} \frac{\partial u}{\partial x} + \frac{\partial C}{\partial y} \frac{\partial u}{\partial y} \right).$$

$$(6) \quad X = v \left(A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} + D u \right), \quad Y = v \left(B \frac{\partial u}{\partial x} + C \frac{\partial u}{\partial y} + E u \right).$$

If L is self-adjoint then, due to (10.6), the expression A simplifies to

$$A = A \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + B \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) + C \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial D u v}{\partial x} + \frac{\partial E u v}{\partial y} - F u v,$$

which is again *symmetric* in the u and v . This can be further simplified by taking the terms

$$\frac{\partial D u v}{\partial x} \quad \text{and} \quad \frac{\partial E u v}{\partial y}$$

over to the X, Y on the right side of (1). We then obtain:

$$(7) \quad \Delta = A \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + B \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) + C \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} - F u v,$$

$$(8) \quad X = v \left(A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} \right), \quad Y = v \left(B \frac{\partial u}{\partial x} + C \frac{\partial u}{\partial y} \right).$$

If we now integrate equation (1) over a region S with the boundary curve C we obtain the most general version of the *second form of Green's theorem*:

$$(9) \quad \int v L(u) d\sigma + \int \Delta(u, v, \dots) d\sigma = \int \{X \cos(n, x) + Y \cos(n, y)\} ds.$$

c) In order to investigate whether the solution of the differential equation for a given boundary condition is *unique* we proceed as in the case of the equation $\Delta u = 0$:

Assuming that for given values of u on the boundary curve C different solutions u_1, u_2 exist, we set as in (9)

$$u = v = u_1 - u_2$$

Then, because of the linearity of L , the first term on the left side of (9) vanishes. Also, because $v = 0$ on the curve C , we see by (6) that $X = Y = 0$ on C , so that the right side of (9) vanishes. Thus (9) becomes

$$(10) \quad \int \Delta(u, u, \dots) d\sigma = 0.$$

If we restrict ourselves to the *self-adjoint* case and introduce the abbreviations $\xi = \partial u / \partial x$, $\eta = \partial u / \partial y$ we obtain

$$(11) \quad \Delta(u, u, \dots) = \begin{cases} \xi^2 + \eta^2 - F u^2 & \text{according to (4)} \\ A \xi^2 + 2 B \xi \eta + C \eta^2 - F u^2 & \text{according to (7)} \end{cases}$$

The upper line of (11) contradicts (10) if $F(x, y)$ is *negative* throughout S ; the second line contradicts (10) if the quadratic form $A \xi^2 + 2 B \xi \eta + C \eta^2$ is *definite* and $F(x, y)$ has the *opposite* sign throughout S . In both these cases we conclude from (10) that

$$(12) \quad u = 0 \quad \text{and hence} \quad u_1 = u_2$$

that is the *uniqueness of the boundary value problem*.

This is identical with the *non-existence of "eigenfunctions"* (see Chapter V). In particular, for the self-adjoint differential equation in the normal form $\Delta u + F u = 0$, where $F = \text{const.} = \pm k^2$, we see that the differential equation

$$(13) \quad \Delta u - k^2 u = 0$$

has no *eigenfunctions* in contrast to the equation of the oscillating membrane

$$(13a) \quad \Delta u + k^2 u = 0,$$

where the eigenfunctions are of basic interest.

The fact that for the self-adjoint differential equation of elliptic type the *uniqueness question for the boundary value problem* or the *question about the existence of eigenfunctions* can be decided quite generally on the basis of the second form of Green's theorem, explains the preferred role of these differential equations in mathematical physics.

In order to stress the physical importance of equation (13) we remark that as the *Yukawa meson equation* it plays the same role in *nuclear physics* as is played by the potential equation in *electron physics*.

II.3. From the conditions a), b), c) of the exercise we obtain

$$(1) \quad \begin{cases} \text{for } 0 < x < \xi \dots G_- = -\left(1 - \frac{\xi}{l}\right)x, \\ \text{for } \xi < x < l \dots G_+ = -\left(1 - \frac{x}{l}\right)\xi. \end{cases}$$

On the other hand the solution of the "boundary value problem" for u is obviously

$$(2) \quad u = u_0 + (u_1 - u_0) \frac{x}{l}.$$

Figure 35 represents the lines for G and u . Verify that Green's equation (10.12) for one dimension

$$(3) \quad u_\xi = u_1 \left(\frac{\partial G_+}{\partial x} \right)_\xi - u_0 \left(\frac{\partial G_-}{\partial x} \right)_0$$

II.4. According to equation (2) of the exercise, $v = v_0(t)$ is given for $x = 0$ and $t > 0$; hence the required Green's function G must satisfy the condition:

$$(1) \quad G = 0 \quad \text{for } x = 0.$$

This function is obtained from the principal solution $V(x, t; \xi, \tau)$, equation (12.16), through reflection on the line $x = 0$ (see also §13):

$$\begin{aligned} G &= V(x, t; \xi, \tau) - V(x, t; -\xi, \tau) \\ &= \{4\pi k(\tau - t)\}^{-\frac{1}{2}} \left(\exp \left\{ \frac{-(x - \xi)^2}{4k(\tau - t)} \right\} - \exp \left\{ \frac{-(x + \xi)^2}{4k(\tau - t)} \right\} \right), \end{aligned}$$

and hence for $x = 0$:

$$(2) \quad \frac{\partial G}{\partial x} = \frac{\xi}{2\sqrt{\pi}} \{k(\tau - t)\}^{-\frac{1}{2}} \exp \left\{ \frac{-\xi^2}{4k(\tau - t)} \right\}.$$

This value is to be substituted in (12.18) for $\partial V/\partial x$, so that we obtain the following simplification. On the right side we have to cancel the first term, since according to equation (4) of the exercise we have

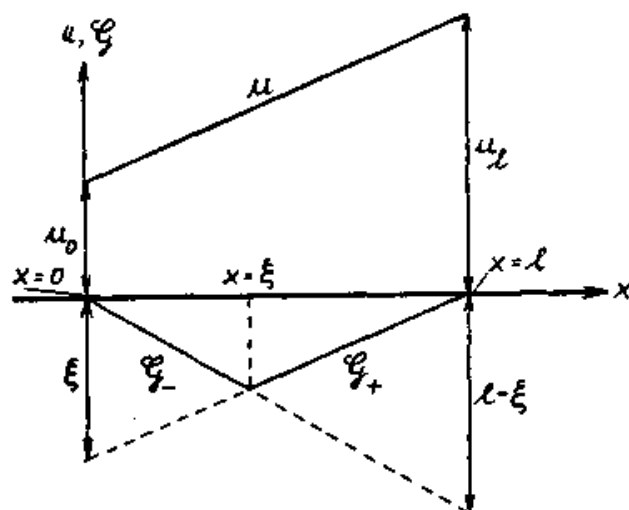


Fig. 35. Behavior of Green's function G with the source point $x = \xi$ and behavior of the potential function u with the boundary values u_0, u_l . The figure at the same time indicates, by the dotted extensions, an elementary construction of the ordinate of G at the point $x = \xi$ coincides with (2) if we use the values of G from (1).

$V = 0$ for $t = 0$. In the second term the part corresponding to $x_1 = \infty$ vanishes, so that only the term corresponding to $x_0 = 0$ remains, which according to (12.18) is to be taken negative; in this term the part multiplied by V vanishes because of (1). Hence, due to equation (2) of the exercise we have:

$$(3) \quad v(\xi, \tau) = \frac{\xi}{2\sqrt{\pi k}} \int_0^\tau v_0(t) \exp\left\{\frac{-\xi^2}{4k(\tau-t)}\right\} \frac{dt}{(\tau-t)^{\frac{3}{2}}}.$$

If instead of t we substitute the variable of integration $p = \xi/\sqrt{4k(\tau-t)}$ then we obtain

$$(4) \quad v(\xi, \tau) = \frac{2}{\sqrt{\pi}} \int_{\xi/\sqrt{4k\tau}}^\infty v_0\left(\tau - \frac{\xi^2}{4kp^2}\right) e^{-p^2} dp$$

as the final solution of the problem.

In order to discuss (4) we expand

$$(5) \quad v_0\left(\tau - \frac{\xi^2}{4kp^2}\right) = v_0(\tau) - \frac{v_0'(\tau)}{1!} \frac{\xi^2}{4kp^2} + \frac{v_0''(\tau)}{2!} \frac{\xi^4}{(4kp^2)^2} - \dots$$

Replacing the variables ξ, τ by x, t then we obtain from (4):

$$(6) \quad v(x, t) = v_0(t) I_0(z) - \frac{x^2}{4k} \frac{v_0'(t)}{1!} I_1(z) + \frac{x^4}{16k^2} \frac{v_0''(t)}{2!} I_2(z) - \dots,$$

with the abbreviations

$$(7) \quad z = \frac{x}{\sqrt{4kt}}, \quad I_0(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-p^2} dp, \quad I_n(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-p^2} \frac{dp}{p^{2n}}.$$

Here $I_0(z)$ is essentially the well known and frequently tabulated "error integral." We have for $z \rightarrow \infty$

$$(8) \quad I_0(0) = 1, \quad I_0(z) \sim \frac{e^{-z^2}}{\sqrt{\pi} z}.$$

The corresponding statements for $I_n(z)$ are

$$(8a) \quad I_n(z) \cdot z^{2n} \rightarrow 0, \quad I_n(z) \cdot z^{2n} \sim \frac{e^{-z^2}}{\sqrt{\pi} z}.$$

The expansion (6) is valid for $z < 1$. For $z \gg 1$ we have:

$$(9a) \quad v(x, t) \sim \frac{e^{-x^2}}{z} \rightarrow 0.$$

The transition between these limiting laws takes place at $z = 1$, that is, for

$$(9b) \quad x \sim \sqrt{4kt}.$$

The plate flow investigated here is a useful preparation for recent investigations of the turbulence problem (see v. II, §38).

III.1. According to (13.1) we write

$$(1) \quad u(x, t) = \int_0^\infty \{f(\xi) U(x, \xi) + f(-\xi) U(x, -\xi)\} d\xi.$$

Here $f(\xi)$ is given and we have to find $f(-\xi)$. For $x = 0$ we have

$$(2) \quad (4\pi kt)^{\frac{1}{2}} u(0, t) = \int_0^\infty \{f(\xi) + f(-\xi)\} e^{-\frac{\xi^2}{4kt}} d\xi,$$

$$(3) \quad (4\pi kt)^{\frac{1}{2}} \left(\frac{\partial u(x, t)}{\partial x} \right)_{x=0} = - \int_0^\infty \{f(\xi) - f(-\xi)\} \frac{d}{d\xi} e^{-\frac{\xi^2}{4kt}} d\xi.$$

If we assume that $f(\xi)$ can be made continuous at $\xi = 0$, that is, that

$$(4) \quad \lim_{\xi \rightarrow 0} f(-\xi) = \lim_{\xi \rightarrow 0} f(+\xi) = f_0, \quad \xi > 0,$$

and if we transform the integral in (3) by integrating by parts, then the

resulting term free of the integral sign vanishes and the right side of (3) becomes

$$(4a) \quad \int_0^{\infty} \frac{d}{d\xi} (f(\xi) - f(-\xi)) e^{-\frac{\xi^2}{4k\xi}} d\xi.$$

The condition imposed in the exercise for $x = 0$ is satisfied by the integrands in (2) and (4a). Considering the fact that here $\partial/\partial n$ is the same as $-\partial/\partial x$, we write:

$$(5) \quad \left(\frac{d}{d\xi} + h\right) f(-\xi) = X, \quad X = \left(\frac{d}{d\xi} - h\right) f(\xi).$$

The differential equation for $f(-\xi)$ obtained in this way becomes integrable if we multiply by $\exp(h\xi)$ and yields

$$(6) \quad f(-\xi) = f(\xi) - 2h e^{-h\xi} \int_0^{\xi} e^{h\eta} f(\eta) d\eta.$$

This expression for $f(-\xi)$ is to be substituted in (1). Verify that we obtain the representation of G in (13.15) if in (1) we now specialize f to a δ -function.

III.2. We are dealing with Green's theorem (16.6) from which we are to deduce the normalizing integral (6.3a) by the limit process $\lambda_n \rightarrow \lambda_n$. Since this integral assumes the form 0/0 we apply de l'Hospital's rule by first differentiating the numerator and denominator with respect to λ_n and then setting $\lambda_n = \lambda_n$:

$$\frac{\frac{du_n}{d\lambda_n} \frac{du_n}{dx} - u_n \frac{d}{d\lambda_n} \frac{du_n}{dx} \Big|_{x=0}}{2k_n \frac{dk_n}{d\lambda_n}} \Big|_{x=0}.$$

The numerator must be computed for $x = l$ only, since for $x = 0$ it vanishes even before we pass to the limit.

According to (16.5) and (16.5a) we obtain

$$\int_0^l u_n^2 dx = \frac{l}{2} \left(\cos^2 \lambda_n \pi + \sin^2 \lambda_n \pi - \frac{1}{\lambda_n \pi} \sin \lambda_n \pi \cos \lambda_n \pi \right),$$

which coincides with (6.4a) if we specialize our present l to π .

We performed this calculation mainly in order to be able to use it as a model in later cases where the normalizing integral cannot be integrated in an elementary manner.

III.3. In the stationary case equation (16.11) becomes

$$(1) \quad \frac{d^2 u}{dx^2} = \lambda^2 u, \quad \lambda = \sqrt{\frac{2h}{b}}.$$

From this we obtain as the general solution

$$u = A e^{\lambda x} + B e^{-\lambda x};$$

The coefficients A and B are computed from the values

$$u = u_1 \text{ for } x = 0, u = u_3 \text{ for } x = l.$$

We obtain

$$(2) \quad u = \frac{u_3 \sinh \lambda x + u_1 \sinh \lambda(l-x)}{\sinh \lambda l}$$

and, setting $x = l/2$,

$$u_2 = \frac{(u_1 + u_3) \sinh \lambda l/2}{\sinh \lambda l} = \frac{u_1 + u_3}{2 \cosh \lambda l/2}.$$

The symbol q , which was introduced in the exercise now becomes:

$$(3) \quad q = \cosh \lambda l/2.$$

From this we obtain a quadratic equation for $\exp(\lambda l/2)$, which yields

$$(4) \quad \frac{\lambda l}{2} = \log(q + \sqrt{q^2 - 1}).$$

According to (1) we also have h expressed in terms of q . According to (13.5) h stands for the ratio of the exterior heat conductivity (which was denoted there by $4\alpha T_0^3$) to the interior heat conductivity κ . For $q = 1$ we obtain from (4) that $h = 0$ as required in the statement of the exercise.

III.4. In the stationary state the energy extracted from the element of the rod (length dx , cross-section q) through heat conduction must equal the Joule heat generated in the same element. If we express the current i in terms of the potential difference V then we obtain as the differential equation of the stationary state:

$$\frac{d^2 u}{dx^2} = -a, \quad a = \frac{V^2}{l^2} \frac{\sigma}{\kappa}.$$

Due to the boundary conditions at the ends of the rod the integration of this differential equation yields a symmetric parabola as the graph of the temperature process. We determine a in terms of the maximal temperature U and obtain:

$$\frac{\kappa}{\sigma} = \frac{1}{8} \frac{V^2}{U}.$$

Thus, through measurement of V and U we can verify the empirical law of Wiedemann and Franz which, according to the theory of metal elec-

trons, asserts that

$$\frac{\kappa}{\sigma} = \frac{\pi^2}{3} \left(\frac{k}{e} \right)^2 T,$$

where T = the absolute temperature, k = the Boltzmann constant, and e = the electron charge.

IV.1. a) For integral n we can expand the function $\exp(i\rho \cos w)$ of (19.18) in the well known power series. As the coefficients of ρ^k we then obtain

$$(1) \quad a_k = \frac{e^{i(k-n)\pi/2}}{2\pi k! 2^k} \int_{-\pi}^{+\pi} (e^{iw} + e^{-iw})^k e^{inw} dw.$$

If we perform the binomial expansion under the integral sign then only one term remains upon integration, and even that term remains only for $k - n \geq 0$. This result agrees with (19.34).

b) For non-integral n (ρ is assumed real) we substitute in (19.14)

$$(2) \quad t = \frac{\rho}{2} e^{-i(w - 3\pi/2)}, \quad dw = i \frac{dt}{t}.$$

The path W_0 , which may be assumed rectangular, is then transformed into the loop of Fig. 37a which starts from $+\infty$, circles the origin in clockwise direction, and returns to $+\infty$ according to the scheme

$$w = i\infty - \frac{\pi}{2}, \quad -\frac{\pi}{2}, \quad 0, \quad +\frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}, \quad i\infty + \frac{3\pi}{2};$$

$$t = +\infty, \quad e^{2\pi i} \frac{\rho}{2}, \quad e^{3\pi i/2} \frac{\rho}{2}, \quad e^{i\pi} \frac{\rho}{2}, \quad e^{i\pi/2} \frac{\rho}{2}, \quad \frac{\rho}{2}, \quad +\infty$$

Equation (19.14) then becomes

$$(3) \quad I_n(\rho) = -\frac{e^{i\pi n}}{2\pi i} \left(\frac{\rho}{2}\right)^n \int_S e^{-t + \frac{\rho^2}{4t}} t^{-n-1} dt.$$

If we now expand in $\exp\left(\frac{\rho^2}{4t}\right)$ we again obtain the series (19.34), provided we use the following general definition of the Γ -function:

$$(4) \quad \frac{1}{\Gamma(x+1)} = \frac{e^{i\pi(x+1)}}{2\pi i} \int_S e^{-t} t^{-x-1} dt;$$

We can easily verify that this definition coincides with the elementary definition $\Gamma(x+1) = x!$ for integral x by forming residues for $t = 0$.

IV.2. In order to complete the investigation of the real part of $i\rho \cos w$ (p. 86) we compute for complex $\rho = |\rho| e^{i\theta}$ the quantity

$$(1) \quad X = \operatorname{Re}(i q \cos w) = \frac{|q|}{2} [\sin(p - \Theta) e^q - \sin(p + \Theta) e^{-q}].$$

In order that X become negative in the infinite part of the upper half of the w -plane ($q \gg 1$) we must have

$$(2) \quad \sin(p - \Theta) < 0.$$

The shaded strip

$$-\pi < p < 0$$

of Figs. 18 and 19 then shifts into the strip,

$$-\pi + \Theta < p < \Theta,$$

that is, the strip is shifted to the right or left in Fig. 18 according as Θ increases or decreases. The opposite holds in the lower half of the w -plane, where according to (1) we must replace (2) by

$$(3) \quad \sin(p + \Theta) > 0.$$

For $0 < \Theta < \pi$ (upper half of the positive q -plane) the shaded regions of the upper and lower half planes have finite segments of the real axis in common, so that the path W_1 can be situated entirely within the shaded region. From this follows, without the use of the asymptotic formula (19.55), that H^1 vanishes for $q \rightarrow \infty$ in the upper half plane. Due to this shift of the shaded regions we also see that for

$0 > \Theta > -\pi$ (lower half of the q -plane) the path W_1 must necessarily lead across the non-shaded region so that H^1 becomes infinite as $q \rightarrow \infty$. The opposites of both these statements holds for W_2 and H^2 .

Figure 36a illustrates the effect of this shift on a full circuit of q around the origin with respect to the path W_1 . The beginning of W_1 has been shifted by 2π , the end by -2π ; thus W_1 has been distorted into W_1' . However W_1' can be decomposed into three partial paths of

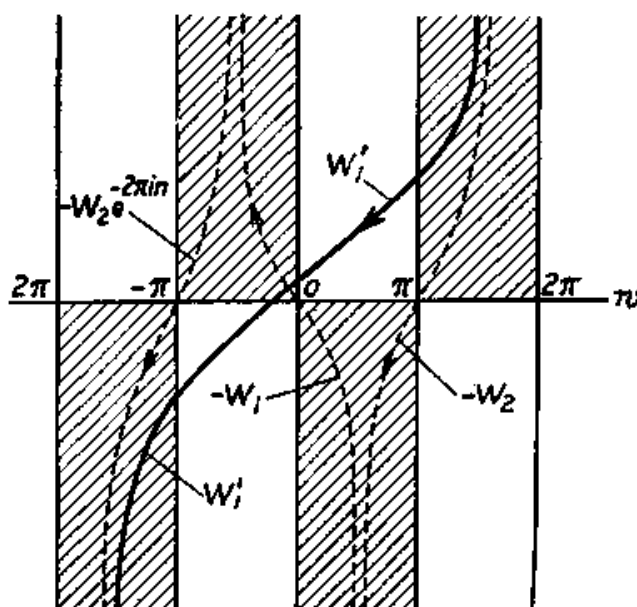


Fig. 36a. Distortion of the path of integration W_1 of H_n^1 into W_1' for a full circuit around the origin.

the same character as the original W_1, W_2 following the symbolic equation

$$(4) \quad W'_1 = -W_2 - W_1 - W_2 e^{-2\pi i n}.$$

Here $-W_1$ is the middle one of the three shaded partial paths of the figure and it differs from W_1 in orientation only; $-W_2$ is the path on the right; the path on the left is obtained from W_2 by replacing w by $w - 2\pi$, thereby changing the factor $\exp(i n w)$ in the integrand by the factor $e^{-2\pi i n}$. Thus, as a result of (4) we obtain:

$$(5) \quad H_n^1(\varrho e^{2\pi i}) = -H_n^1(\varrho) - H_n^2(\varrho)(1 + e^{-2\pi i n}).$$

When n is an integer this becomes

$$(6) \quad H_n^1(\varrho e^{2\pi i}) = -H_n^1(\varrho) - 2H_n^2(\varrho),$$

which we can rewrite in the form:

$$(6a) \quad H_n^1(\varrho e^{2\pi i}) - H_n^1(\varrho) = -2\{H_n^1(\varrho) + H_n^2(\varrho)\} = -4I_n(\varrho).$$

This change in H_n^1 of $4I_n$ together with the relation $H^1 = I + iN$ correspond to the change in $\log \varrho$ of $2\pi i$ in the formula (19.47) for N .

Figure 36b represents the correspondingly distorted path W'_2 of H^2 for a full positive circuit of ϱ around the origin. This path can be decomposed into five partial paths of the same character as W_1, W_2 following the symbolic equation:

$$(7) \quad W'_2 = W_2 + W_1 + W_1 e^{2\pi i n} + W_2 e^{-2\pi i n} + W_2 e^{+2\pi i n}.$$

Here W_2 is the middle one of the five partial paths, W_1 is the path adjacent on the left, $W_1 e^{2\pi i n}$ is the path adjacent on the right, etc. Instead of (5) we now obtain

$$(8) \quad H_n^2(\varrho e^{2\pi i}) = H_n^2(\varrho)(1 + e^{-2\pi i n} + e^{+2\pi i n}) + H_n^1(\varrho)(1 + e^{2\pi i n}).$$

When n is an integer this becomes

$$(9) \quad H_n^2(\varrho e^{2\pi i}) = 3H_n^2(\varrho) + 2H_n^1(\varrho),$$

$$(9a) \quad H_n^2(\varrho e^{2\pi i}) - H_n^2(\varrho) = 2(H_n^1(\varrho) + H_n^2(\varrho)) = 4I_n(\varrho).$$

The change (9a) together with the relation $H^2 = I - iN$ again correspond to the change in $\log \varrho$ of $2\pi i$ in (19.47).

The equations (5) and (8) are the so-called *circuit relations* of the Hankel functions for the angle increment $\Delta\theta = 2\pi$. These relations correspond to the "relationes inter contiguas" which were established by Gauss for the hypergeometric functions (see §24). Just as the equations (6a), (9a) for integral n are obtained from (19.47), the general relations (5) and (8) can be derived from the representations (19.31) and (19.30).

The circuit relations can be generalized from the full circuits $\Delta\Theta = 2\pi\nu$ ($\nu = \text{integer}$) to half-circuits $\Delta\Theta = \pi\nu$. For a later applica-

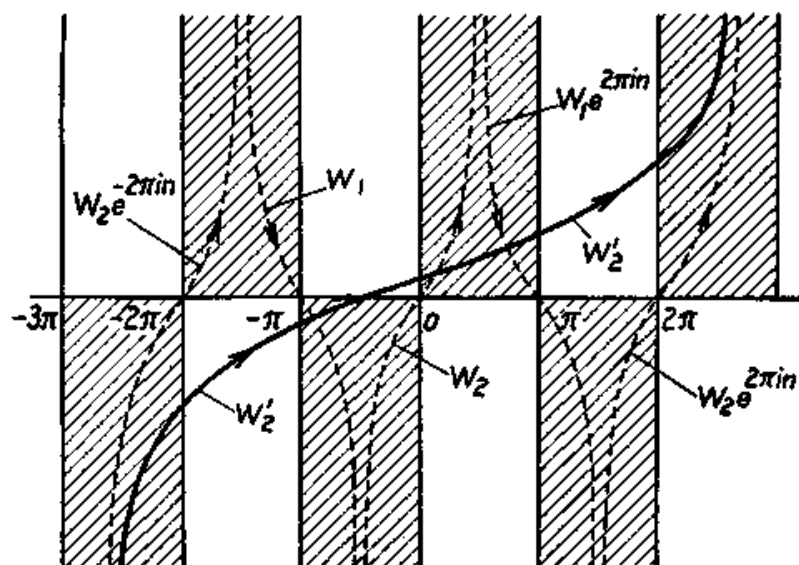


Fig. 36b. Distortion of the path W_2 of H_n^2 into W_2' for a full circuit around the origin.

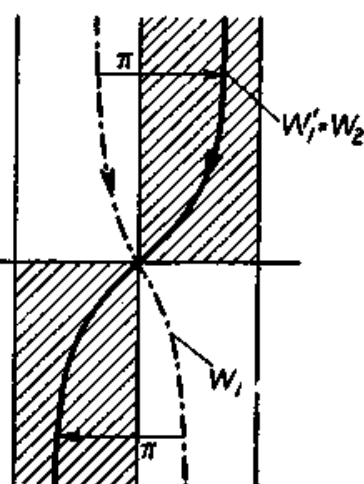


Fig. 36c. The half-circuit relation for H_0^1 .

tion in §32 we discuss the half-circuit relation for $\nu = 1$ and $n = 0$. The relation reads

$$(10) \quad H_0^1(q e^{i\pi}) = -H_0^2(q).$$

For a proof, a look at Fig. 36c suffices. For real q the path W_1 (indicated by dots and dashes in the figure) leads from the region $-\pi < p < 0$ to the region $0 < p < +\pi$; for the present argument $q e^{i\pi}$ this path has been shifted by $+\pi$ in the upper part and by $-\pi$ in the lower part, as indicated by the arrows in the figure. The path W_1' obtained in this manner is identical with the path W_2 for H_0^2 . But this is the statement of equation (10).

A relation which is analogous to (10) is obtained if we replace q by $q e^{-i\pi}$:

$$(10a) \quad H_0^2(q e^{-i\pi}) = -H_0^1(q).$$

Considering the factor $\exp(inw)$ of the integrand we can generalize (10) to

$$(11) \quad H_n^1(q e^{i\pi}) = -e^{-n\pi i} H_n^2(q),$$

for arbitrary n , or (replacing q by $q e^{-i\pi}$)

$$(11a) \quad H_n^2(q e^{-i\pi}) = -e^{+n\pi i} H_n^1(q).$$

These half-circuit relations can also be derived directly from the equations (19.30) and (19.31) with the help of the equations

$$(12) \quad I_n(\varrho \cdot e^{i\pi\nu}) = e^{i\pi n\nu} I_n(\varrho), \quad I_{-n}(\varrho \cdot e^{i\pi\nu}) = e^{-i\pi n\nu} I_{-n}(\varrho),$$

which follow from (19.34).

These relations become very simple if we write them for $\psi_n(\varrho)$ and $\zeta_n(\varrho)$ in which the corresponding Hankel and Bessel functions of index $n + \frac{1}{2}$ with integral n are multiplied by $\sqrt{2\varrho/\pi}$; namely we have

$$(13) \quad \zeta_n^{1,2}(\varrho e^{\pm i\pi}) = (-1)^{n+1} \zeta_n^{2,1}(\varrho), \quad \psi_n(\varrho e^{\pm i\pi}) = (-1)^{n+1} \psi_n(\varrho).$$

In the representation (19.22) for $H_0^1(\varrho)$ we substitute

$$x = i\varrho \cos w, \quad dw = \frac{i dx}{\sqrt{x^2 + \varrho^2}}.$$

The path W_1 , which for convenience is to be taken rectangular, is then transformed into the x -plane according to the scheme

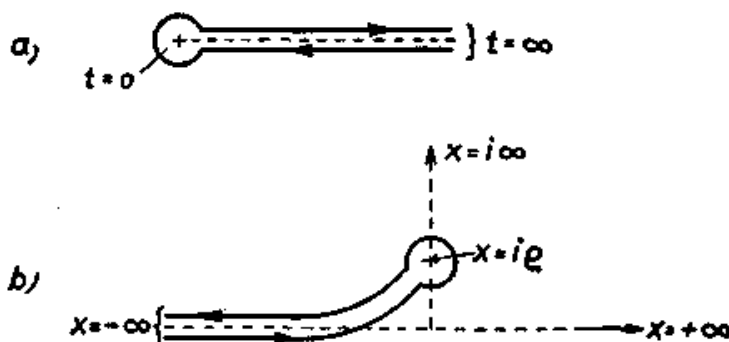


Fig. 37a. The loop integrals for, $1/\Gamma(x+1)$. b. The loop integral for $H_0^1(\varrho)$ for small ϱ .

$$w = -\frac{\pi}{2} + i\infty, \quad -\frac{\pi}{2}, \quad 0, \quad +\frac{\pi}{2}, \quad \frac{\pi}{2} + i\infty,$$

$$x = -\infty, \quad 0, \quad i\varrho, \quad 0, \quad -\infty.$$

We thus have the loop integral of Fig. 37b that begins at the negative infinite end of the real x -axis, circles the point $x = i\varrho$ and returns to the negative infinite end of the real axis; the orientation of this loop is controlled by a small displacement of the real branch of W_1 . From (19.22) we then obtain:

$$(1) \quad H_0^1(\varrho) = \frac{2i}{\pi} \int_{-\infty}^{+i\varrho} \frac{e^x dx}{\sqrt{x^2 + \varrho^2}}.$$

This integral is obtained by the combination of the two branches of Fig. 37b, where the originally negative sign of the returning branch has been reversed through the complete circuit around the branch point

$x = i\varrho$. Through integration by parts with respect to x we obtain

$$(2) \quad H_0^1(\varrho) = \frac{2i}{\pi} \left\{ e^x \log(x + \sqrt{x^2 + \varrho^2}) \right\}_{x=i\varrho} - \frac{2i}{\pi} \int_{-\infty}^{i\varrho} e^x \log(x + \sqrt{x^2 + \varrho^2}) dx.$$

Substituting $x = i\varrho$ in the first term and letting $\varrho \rightarrow 0$ in both terms we obtain

$$(3) \quad \lim_{\varrho \rightarrow 0} H_0^1(\varrho) = \frac{2i}{\pi} \log i\varrho - \frac{2i}{\pi} \int_{-\infty}^0 e^x \log 2x dx.$$

With the substitution $x = -t$ the last integral becomes

$$(4) \quad \log(-2) + \int_0^{\infty} e^{-t} \log t dt = \log(-2) - C = \log\left(\frac{-2}{\gamma}\right).$$

Here C and γ are the quantities defined in (19.41a) (check using the Laplace integral for the Γ -function). Combining (4) and (3) we obtain from (1):

$$(5) \quad \lim_{\varrho \rightarrow 0} H_0^1(\varrho) = \frac{2i}{\pi} \left(\log \frac{\gamma\varrho}{2} - \frac{i\pi}{2} \right) = \frac{2i}{\pi} \log \frac{\gamma\varrho}{2} + 1.$$

Due to the relation

$$H_0 = I_0 + iN_0, \quad \lim_{\varrho \rightarrow 0} H_0 = 1 + i \lim_{\varrho \rightarrow 0} N_0$$

equation (5) coincides with the equation (19.48) for N .

IV.4. 1. If we neglect $1/\varrho$ in the differential equation (19.11) we obtain $H_0^1 = A e^{i\varrho}$ (A = constant of integration; the solution involving $e^{-i\varrho}$ corresponds to H_n^2).

2. We now consider A not as a constant but as a "slowly varying function of ϱ " such that A'' , A'/ϱ and A/ϱ^2 can be neglected. This yields a differential equation for $A(\varrho)$, from which we obtain $A = B/\sqrt{\varrho}$. The normalizing constant B cannot, of course, be determined in this manner.

IV.5. a) According to the equations (1.12), (22.14), (22.31) we obtain as the coefficients C_m and A_{mn} of the exercise:

$$C_m = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(\vartheta_0, \varphi_0) e^{-im\varphi_0} d\varphi_0,$$

$$A_{mn} = \frac{1}{N_n^m} \int_{-\pi}^{+\pi} C_m(\vartheta_0) P_n^m(\cos \vartheta_0) \sin \vartheta_0 d\vartheta_0$$

$$= \frac{n + \frac{1}{2}}{2\pi} \frac{(n-m)!}{(n+m)!} \int_0^\pi \sin \theta_0 d\theta_0 \int_{-\pi}^{+\pi} d\varphi_0 f(\theta_0, \varphi_0) P_n^m(\cos \theta_0) e^{-im\varphi_0}.$$

b) From the scheme for $f(\theta, \varphi)$ in the exercise we see that if we multiply $f(\theta_0, \varphi_0)$ by

$$Y_{\nu, \mu} = P_\nu^\mu(\cos \theta_0) e^{-i\mu\varphi_0}$$

and integrate with respect to φ_0 then we obtain

$$\int_{-\pi}^{+\pi} f(\theta_0, \varphi_0) Y_{\nu, \mu} d\varphi_0 = 2\pi \sum_n A_{n, \mu} P_n^\mu(\cos \theta_0) P_\nu^\mu(\cos \theta_0),$$

Integrating with respect to $\sin \theta_0 d\theta_0$ from (22.14) and (22.31) we obtain the result

$$\int \int f(\theta_0, \varphi_0) Y_{\nu, \mu} d\sigma_0 = 2\pi N_\nu^\mu A_{\nu, \mu}.$$

After a change in notation (ν, μ instead of n, m) this coincides with the expression for $A_{m, n}$ in a) except for the order of summation (see Fig. 38).

In a) the horizontal strips $|m| \leq n < \infty$ are summed in the vertical direction, and in b) the vertical strips $-n \leq m \leq +n$ are summed in the horizontal direction. In both cases the total domain of summation is bounded by the lines $n = \pm m$; thus the sums are the same.

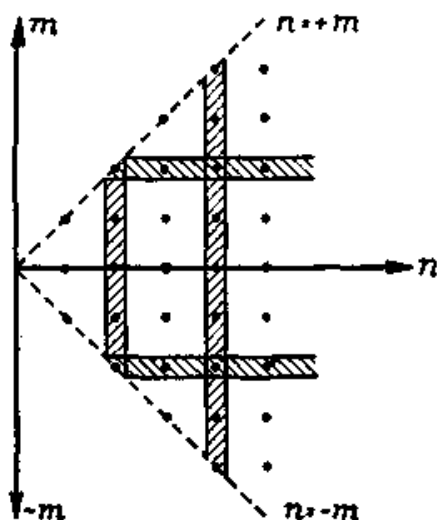


Fig. 38. The double sum in the number lattice of n, m : arranged in horizontal strips for case a) and in vertical strips for case b).

IV.6. Again we draw the 60° wedge of Fig. 17 together with its five reflected images, so that the original wedge lies symmetric with respect to the horizontal plane. For reasons of convenience we situate the center of inversion C on the horizontal line through the vertex O of the wedge and we let the circle of inversion (dotted circle in the figure) pass through O . Since the point at infinity is mapped into C and the points of intersection O and S_1 of the line 1 with the circle of inversion remain fixed, the position of the circle into which the straight line 1, -1 is transformed is determined by the points O, C, S_1 . The arcs of the circle which correspond to the half lines 1 and -1 are again denoted by 1 and -1 . The same holds for the line 2, -2 which is mapped into a circle of the same radius passing through the points O, C, S_2 . The line 3, -3 goes into a circle of diameter OC in which the upper and lower semi-circles correspond to the half lines -3 and 3.

Now the wedge 1,2 is mapped into the exterior of the circular diangle $C, S_1, 1, O, 2, S_2, C$; both regions are indicated by a shading of the boundary. We now seek the images of the reflected wedges. All these images are interiors of certain circular diangles (crescents); e.g., the wedge 2,3 is mapped into the crescent $C, S_2, 2, O, 3, C$, and the wedge $-2, -1$ is mapped into the small lens-like region $C, -2, O, -1, C$ in the center of our figure.

Up to now we have described the drawing as a plane figure and spoken of straight lines, circles, circular diangles, etc. However there is nothing that prevents us from interpreting the figure as *three-dimensional* and to speak of planes and spheres instead of straight lines and circles. These spheres are then situated with their

centers in the plane of the drawing. The original wedge 1,2 then is mapped into the exterior of the two intersecting spheres which belong to the circles 1, -1 and 2, -2 ; in the same manner the reflected wedges correspond to the regions bounded by two of the spheres 1,2,3.

Just as before, we obtained *Green's function* for the wedge from the elementary reflections in Fig. 17, now in the case of the *potential equation* (but only in this case) we obtain Green's function for the corresponding circular or spherical regions by finding the "electric image" of the given pole upon inversion on the boundary circles or spheres 1,2,3 and by giving alternating signs to these poles. For the symmetric structure of our problem it suffices to have five such electric images in order to satisfy the boundary condition $u = 0$ on the boundary of each of the regions under consideration.

IV.7. a) In the inversion in the sphere K (broken line) of Fig. 40 all the infinite points of the reflecting planes $\pm 1, \pm 2, \pm 3, \dots$ are mapped into the center of inversion C ; thus the planes ± 1 go into the spheres $+1$ and -1 which are tangent at C and have diameter equal to the radius a of the sphere of inversion. Here the exterior of the spheres ± 1 corresponds to the interior of the plate and the interior of these spheres

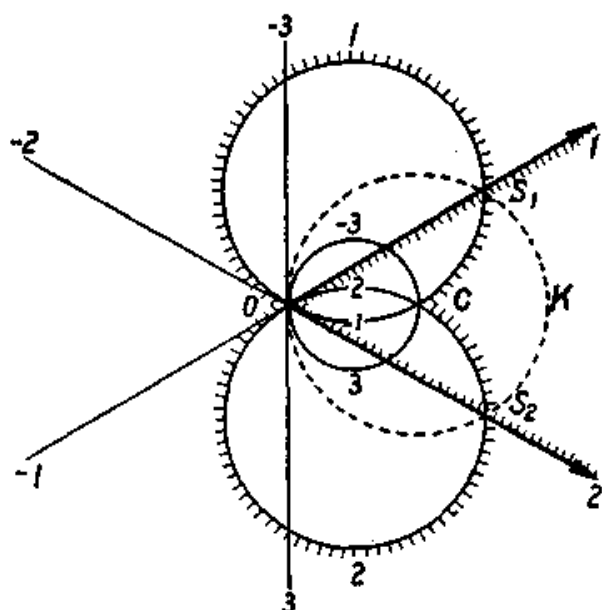


Fig. 39. The wedge 1,2 of the 60° angle is mapped by inversion into the exterior of $C, 1, O, 2, C$ of the intersecting spheres 1, -1 and 2, -2 ; the reflected wedges 2,3; 3, -1 ; ... are mapped into spherical crescents.

corresponds to the exterior of the plate. The planes ± 2 are mapped into spheres which are again tangent at C but have a diameter of only $a/3$. The images of the planes ± 3 are in turn spheres which lie in the interior of the spheres ± 2 , have the diameter $a/5$ and are tangent at C . The region bounded by two consecutive spheres in this sequence corresponds to a reflected image of the original plate.

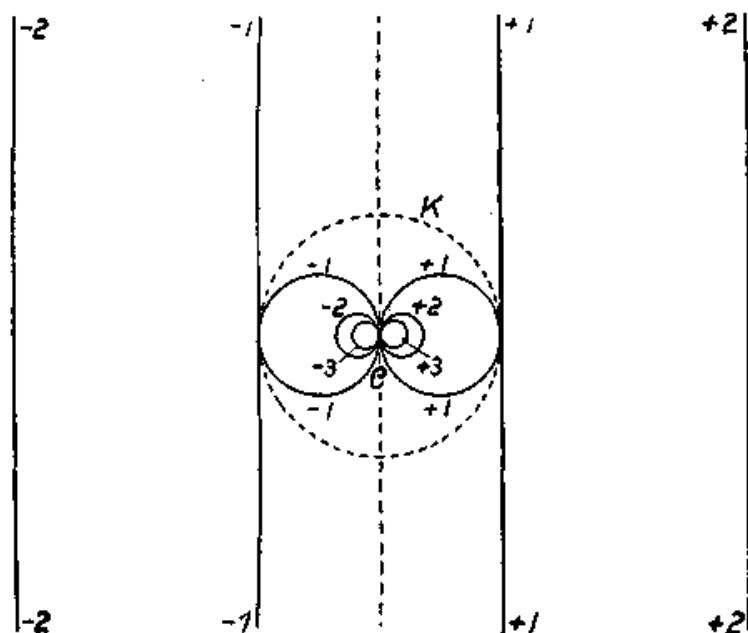


Fig. 40. Inversion of a plane parallel plate and its successive reflected images into a system of spheres tangent at the center of inversion C .

Green's function of potential theory for the exterior of two tangent spheres (e.g., the spheres ± 1 of our figure) can be deduced by inversion from Green's function for the plane parallel plate. The infinitely many image points of the arbitrarily prescribed pole of Green's function that arise in the inversion are situated in the successive spherical regions mentioned above, and they accumulate at the point C .

b) If the radii of the concentric spheres I and II are a and $2a$ then we may choose the radius of the sphere of inversion equal to a and place its center C on sphere I. Then sphere I is mapped into the plane E_I , and II is mapped into a sphere K_{II} of radius $2a/3$; the minimum distance between K_{II} and E_I is $a/6$. Conversely, E_I and K_{II} are mapped into the concentric spheres I and II.

For an arbitrary position of the non-intersecting plane E and sphere K we can proceed in the following way (kindly communicated to me by Caratheodory): from the center of K we drop the perpendicular L to E ; from the foot F of L we draw tangents (of length t) to K and draw the auxiliary sphere H with center F and radius t . As the center of inver-

sion we choose one of the points of intersection S of L and H . Then L is transformed into a straight line, H becomes a plane which is perpendicular to L , and E and K become spheres which are perpendicular to H and L and hence have their center at the point of intersection of H and L , that is, in the center of inversion. The radius of inversion remains arbitrary and determines only the size of the concentric spheres.

Instead of E and K we may also consider two arbitrary non-intersecting spheres K_1 and K_2 (see the last statement in the exercise). In order to transform them into two concentric spheres we start from the pencil of spheres $K_1 + \lambda K_2 = 0$. The pencil contains two spheres of radius 0, namely the two poles of the bipolar coordinate system. If we choose one of these poles as the center of inversion then all the spheres of the pencil, including K_1 and K_2 are mapped into concentric spheres.

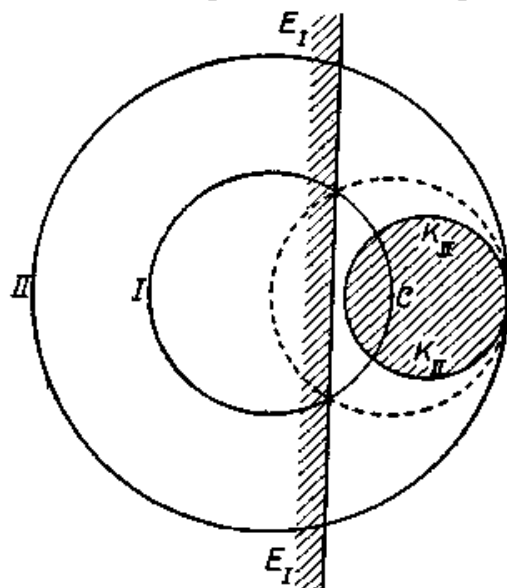


Fig. 41. Two concentric spheres I and II are transformed through inversion into the plane E_I and the sphere K_{II} (which are shaded in the figure).

IV.8. Let u_1 and u_2 be linearly independent solutions of the differential equation

$$L(u) = u'' + pu' + qu = 0$$

of second order, where p and q are arbitrary given functions of the independent variable q . Then for $X = u_1 u_2' - u_2 u_1'$ we have:

$$u_1 L(u_2) - u_2 L(u_1) = \frac{dX}{dq} + pX = 0,$$

hence $X = C e^{-\int p dq}$, $C = \text{constant of integration}$.

a) For the Bessel differential equation (19.11) we have and hence

$$X = C e^{-\log q} = \frac{C}{q}.$$

If we take $u_1 = H_n^1$, $u_2 = H_n^2$ then C is determined most simply from the asymptotic values (19.55), (19.56). We obtain

$$X = \frac{-4i}{q\pi}, \quad C = \frac{-4i}{\pi}.$$

Due to $I_n = \frac{1}{2}(H_n^1 + H_n^2)$, we see that the expression (I) in the exercise, where H is to stand for H^1 , equals half the above X , so that

$$(I) = -\frac{2i}{\varrho\pi} \quad ;$$

the sign is reversed if H is to stand for H^2 .

The determination of C becomes somewhat less simple if we start from $\varrho = 0$ instead of $\varrho = \infty$.

b) In the differential equation (21.11a) we have $p = 2/\varrho$, and hence $X = C/\varrho^2$. If we take $u_1 = \zeta_n^1$, $u_2 = \zeta_n^2$, then according to (21.14) we have, for $\varrho \rightarrow \infty$,

$$\zeta_n^{1,2} = \frac{1}{\varrho} e^{\pm i[\varrho - (n+1)\pi/2]}, \quad X = -\frac{2i}{\varrho^2}, \quad C = -2i.$$

For the expression (II) we then obtain:

$$(II) = \mp \frac{i}{\varrho^2} \quad ,$$

where the sign depends on whether we set ζ equal to ζ^1 or ζ^2 .

V.1. a) Due to $I'_n(\lambda a) = 0$ the limit process of (20.9) yields the normalizing integral

$$N = -a \lim_{\varepsilon \rightarrow 0} \frac{I_n(\lambda a) I'_n(\lambda a + \varepsilon a)}{2\varepsilon} = -\frac{a^2}{2} I_n(\lambda a) I''_n(\lambda a)$$

instead of (20.19). With the help of the Bessel differential equation we obtain

$$N = \frac{1}{2\lambda^2} (\lambda^2 a^2 - n^2) I_n^2(\lambda a).$$

Hence if we "normalize $I_n(\lambda r)$ to 1" we obtain

$$\sqrt{\frac{2\lambda^2}{\lambda^2 a^2 - n^2}} \frac{I_n(\lambda r)}{I_n(\lambda a)}.$$

b) From the relation (21.11) between $\psi_n(\varrho)$ and $I_{n+\frac{1}{2}}(\varrho)$ we obtain for the present normalizing integral

$$N = \int_0^a \psi_n^2(kr) r^2 dr = \frac{\pi}{2k} \int_0^a I_{n+\frac{1}{2}}^2(kr) r dr.$$

For the boundary condition

$$\psi'_n(ka) = \sqrt{\frac{\pi}{2ka}} (I'_{n+\frac{1}{2}}(ka) - \frac{1}{2ka} I_{n+\frac{1}{2}}(ka)) = 0$$

of the exercise we obtain by a limit process analogous to that of a)

$$N = -\frac{\pi}{2} \frac{a^2}{k} J_{n+\frac{1}{2}}(ka) \left\{ J_{n+\frac{1}{2}}''(ka) + \frac{1}{2ka} J_{n+\frac{1}{2}}'(ka) \right\}$$

with the help of the Bessel differential equation we may therefore write

$$N = \frac{a}{2k^2} \psi_n^2(ka) \{k^2 a^2 - n(n+1)\};$$

thus the normalized form of ψ_n is

$$\Psi_n = \sqrt{\frac{2k^2/a}{k^2 a^2 - n(n+1)}} \frac{\psi_n(kr)}{\psi_n(ka)}.$$

V.2. The proof follows from Green's theorem

$$(1) \quad \int (u \Delta v - v \Delta u) d\tau = \int \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\sigma,$$

if we set $v = 1/r$ where r stands for the distance of the point of integration from P . Due to the singularity of v at P we surround P in the customary manner by a sphere K_ϵ of radius $\epsilon \rightarrow 0$. If we extend the integration on the left side over the region bounded by K_a and K_ϵ , then the left side vanishes and the right side becomes the sum of the surface integrals over K_a and K_ϵ (in both cases n stands for the exterior normal to the region). By letting $\epsilon \rightarrow 0$ in the integral over K_ϵ we obtain from (1)

$$(2) \quad 0 = 4\pi u_P - \frac{1}{a^2} \int_{K_a} u d\sigma - \frac{1}{a} \int_{K_a} \frac{\partial u}{\partial n} d\sigma.$$

The third term on the right here vanishes since throughout the interior of S we have $\Delta u = 0$. Thus equation (2) proves the theorem of the arithmetic mean.

V.3. From equation (27.14) and the condition $u = U$ on the sphere $r_0 = a$ we obtain

$$(1) \quad 2\pi U = \sum_n \sum_m A_{nm} P_n^m(\cos \theta_0) e^{-im\varphi_0};$$

Multiplying by $e^{im\varphi_0}$ and integrating with respect to φ_0 from 0 to π we obtain:

$$(2) \quad \int_0^{2\pi} U e^{im\varphi_0} d\varphi_0 = \sum_n A_{nm} P_n^m(\cos \theta_0);$$

multiplying by $P_n^\mu(\cos \theta_0) \sin \theta_0$ and integrating with respect to θ_0 from

0 to π we have:

$$(3) \quad \int_0^\pi \int_0^{2\pi} U \Pi_\nu^\mu(\cos \vartheta_0) e^{i\mu\varphi_0} \sin \vartheta_0 d\vartheta_0 d\varphi_0 = A_{\nu\mu},$$

which coincides with (27.13a) except for notation.

Comparing the r -dependence of (27.13) and (27.14) we obtain

$$(4) \quad -a^2 \sum_l \frac{1}{k_{nl}} \Psi_n(k_{nl} r_0) \Psi'_n(k_{nl} a) = \left(\frac{r_0}{a}\right)^n,$$

The summation extends over all the roots $k = k_n$ of the equation $\Psi_n(ka) = 0$, which are the same as the roots of the equation $\varphi_n(ka) = 0$. In order to determine the Ψ_n in terms of the φ_n we use equation (21.11)

$$(5) \quad \varphi_n(x) = \sqrt{\frac{\pi}{2x}} I_{n+\frac{1}{2}}(x)$$

and the relation (20.19) (which holds for non-integral n , too)

$$(5a) \quad \int_0^a [I_{n+\frac{1}{2}}(kr)]^2 r dr = \frac{a^2}{2} [I'_{n+\frac{1}{2}}(ka)]^2,$$

where k is a root of $I_{n+\frac{1}{2}}(ka) = 0$, and hence a root of $\varphi_n(ka) = 0$. From (5) and (5a) we obtain:

$$(6) \quad \int_0^a [\varphi_n(kr)]^2 r^2 dr = \frac{a^3}{2} [\varphi'_n(ka)]^2.$$

If we now set $\Psi_n = N \varphi_n$ and impose the condition

$$(7) \quad \int_0^a \Psi_n^2(kr) r^2 dr = 1,$$

we obtain

$$N^2 = 2/a^2 [\varphi'_n(ka)]^2.$$

Rewriting the equation (4) in terms of φ_n , and adopting the notation $\alpha = r_0/a$, we obtain:

$$(8) \quad 2 \sum_l \frac{\varphi_n(k_{nl} \alpha a)}{a k_{nl} \varphi'_n(k_{nl} a)} = -\alpha^n.$$

For $n = 0$ we have according to (21.11)

$$\varphi_0(x) = \frac{\sin x}{x};$$

therefore

$$\varphi_0(ka) = 0 \quad \text{for} \quad k = k_{0l} = \frac{l\pi}{a}, \quad l = \pm 1, \pm 2, \dots$$

In this particularly simple case equation (8) becomes

$$(9) \quad 2 \sum_{l=1}^{+\infty} (-1)^l \frac{\sin l \pi \alpha}{l} = -\pi \alpha.$$

For $\alpha = \frac{1}{2}$ this yields

$$1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4},$$

which is the Leibniz series (2.8). In general we obtain from (2.9) for $x = 2\pi\alpha$ a representation of the "saw-tooth profile"

$$\left. \begin{array}{l} +\frac{1}{2}(\pi-x) \\ -\frac{1}{2}(\pi+x) \end{array} \right\} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \text{ for } \begin{cases} 0 < x < +\pi, \\ -\pi < x < 0. \end{cases}$$

The reader is asked to verify this as a further exercise for chapter I. It is apparent, however, that equation (8) for $n > 0$ contains far deeper and more general analytic relations.

VI.1. a) Due to the fact that Π has the z -direction and depends only on z and $r^2 = x^2 + y^2$, we obtain from (31.4)

$$\begin{aligned} \mathbf{E}_x &= \frac{\partial}{\partial x} \frac{\partial \Pi}{\partial z} = \frac{x}{r} \frac{\partial^2 \Pi}{\partial r \partial z}, \\ \mathbf{E}_y &= \frac{\partial}{\partial y} \frac{\partial \Pi}{\partial z} = \frac{y}{r} \frac{\partial^2 \Pi}{\partial r \partial z}. \end{aligned}$$

For the form a) of the exercise we have

$$\frac{\partial \Pi}{\partial z} = \frac{z-h}{R} \frac{d}{dR} \frac{e^{i k R}}{R} + \frac{z+h}{R'} \frac{d}{dR'} \frac{e^{i k R'}}{R'},$$

which vanishes when $z = 0$ since then $R = R'$. Hence the expressions for \mathbf{E}_x and \mathbf{E}_y , which are obtained by differentiation with respect to x, y or r , also vanish for $z = 0$.

b) According to (31.4) we now have

$$\begin{aligned} \mathbf{E}_x &= k^2 \Pi + \frac{\partial}{\partial x} \frac{\partial \Pi}{\partial x}, \\ \mathbf{E}_y &= \frac{\partial}{\partial y} \frac{\partial \Pi}{\partial x}. \end{aligned}$$

But since in the form b) Π vanishes for $z = 0$, we also have \mathbf{E}_x and \mathbf{E}_y vanishing for all points (x, y) on the earth's surface.

c) From (35.1) we obtain

$$\mathbf{E}_x = i \mu_0 \omega \frac{\partial \Pi_z}{\partial y}, \quad \mathbf{E}_y = -i \mu_0 \omega \frac{\partial \Pi_z}{\partial x}.$$

These derivatives vanish for $z = 0$, since in the form c) H_z itself vanishes for $z = 0$.

d) From (35.1) we now have

$$\mathbf{E}_x = 0, \quad \mathbf{E}_y = i \mu_0 \omega \frac{\partial H_z}{\partial z}.$$

But according to the form d) we have

$$\frac{\partial H_z}{\partial z} = \frac{z-h}{R} \frac{d}{dR} \frac{e^{ikR}}{R} + \frac{z+h}{R'} \frac{d}{dR'} \frac{e^{ikR'}}{R'}.$$

which vanishes for $z = 0$ since then $R' = R$.

VI.2. This exercise is instructive not only for the understanding of Zenneck waves, but also for the general knowledge of electromagnetic rotational fields and for their representation using complex operators.

From (32.20) according to the prescription (31.4), we obtain for the air

$$(1) \quad \begin{aligned} \mathbf{E}_x &= -i p \sqrt{p^2 - k^2} A k_z^2 e^{i p z - \sqrt{p^2 - k^2} z}, \\ \mathbf{E}_z &= p^2 A k_z^2 e^{i p z - \sqrt{p^2 - k^2} z}. \end{aligned}$$

These expressions, multiplied by the exponential time factor, represent an elliptic oscillation as known from optics. Due to the complex nature of the right sides of (1) the principal axes of the oscillation ellipse are oblique to the x - and z -direction. If we form the absolute values of \mathbf{E}_x and \mathbf{E}_z together with their negatives, then we obtain the limits between which \mathbf{E}_x and \mathbf{E}_z oscillate, in other words we obtain a rectangle circumscribed about the ellipse. The ratio of the sides of this rectangle is given by the absolute value of

$$(2) \quad \frac{\mathbf{E}_x}{\mathbf{E}_z} = \frac{\sqrt{k^2 - p^2}}{p} = \frac{1}{n}.$$

The value $1/n$ is obtained from the definitions (32.16a) and (32.2) of p and n . Because $|n| > 1$ the rectangle is tall and narrow (See Fig. 42a).

On the other hand, due to (32.20) and (31.7), in the earth we have

$$(3) \quad \begin{aligned} \mathbf{E}_x &= +i p \sqrt{p^2 - k_z^2} A k^2 e^{i p z + \sqrt{p^2 - k_z^2} z}, \\ \mathbf{E}_z &= p^2 A k^2 e^{i p z + \sqrt{p^2 - k_z^2} z}. \end{aligned}$$

Hence again we have an elliptic oscillation that, is this time, situated in a rectangle with the ratio of sides given by the absolute value of

$$(4) \quad \frac{\mathbf{E}_x}{\mathbf{E}_z} = \frac{\sqrt{k_z^2 - p^2}}{p} = n,$$

where n is again obtained from (32.16a) and (32.2). Because $|n| > 1$ the rectangle is now broad and squat (See Fig. 42b). The present ellipse is traversed in the opposite sense of the former, as is seen from the reciprocity of the values n and $1/n$ in (4) and (2).

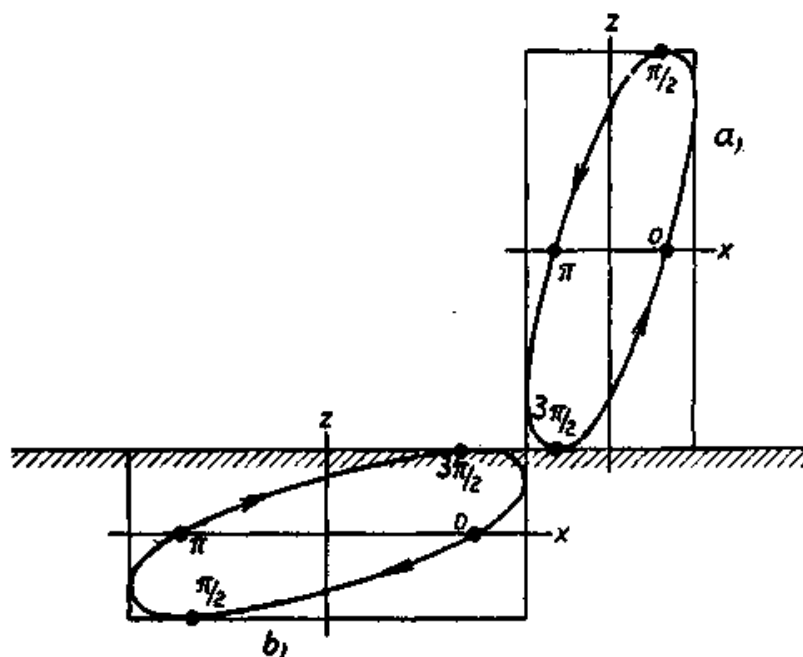


Fig. 42. The rotational field of the Zenneck wave a) in the air, narrow ellipse, b) in the earth, wide ellipse, congruent and of opposite orientation to each other.

If we think of the field in the air as pushing forward with its phase velocity in the positive x -direction, then the field in the earth appears to lag behind against the resistance there.

VI.3. The electric field strength at the antenna and in the direction of the antenna is $\mathbf{E} = \text{Re}\{E e^{-i\omega t}\}$. At an antenna which is short compared to the wavelength this field strength does the work $\mathbf{E} j \, dt$ in the time dt . Hence according to equation (36.20) we have as the time average of work

$$W = j l \int_0^{\tau} \text{Re}\{E e^{-i\omega t}\} \text{Re}\{i e^{-i\omega t}\} \frac{dt}{\tau}.$$

where τ stands for the time of oscillation. With the method given on p. 271 we obtain

$$\begin{aligned} W &= \frac{j l}{4} \int_0^{\tau} (E e^{-i\omega t} + E^* e^{+i\omega t}) (i e^{-i\omega t} - i e^{+i\omega t}) \frac{dt}{\tau} \\ &= \frac{j l}{4} (-i E + i E^*). \end{aligned}$$

and hence also

$$(1) \quad W = \frac{jI}{2} \operatorname{Re}\{-iE\}.$$

a) *Vertical antenna.* From (31.4) and the differential equation of Π we obtain for $\Pi = \Pi_z$ and $\mathbf{E} = \mathbf{E}_z$

$$E = k^2 \Pi + \frac{\partial^2 \Pi}{\partial z^2} = -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Pi}{\partial r}.$$

We express Π with the help of equation (32.9) and both e^{ikR}/R and $e^{ikR'}/R'$ are expressed with the help of (31.14). Since the r -dependence of these three terms is given by $I_0(\lambda r)$, the application of the operator

$$-\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

under the integral sign yields the common factor $+\lambda^2 I_0(\lambda r)$, and hence at the point $r = 0$ of the antenna it yields the factor λ^2 . Thus we obtain from (32.9):

$$(3) \quad \begin{aligned} \operatorname{Re}\{-iE\} = \operatorname{Re}\left\{-i \int_0^\infty (e^{-\mu|z-h|} + e^{-\mu(z+h)}) \frac{\lambda^3 d\lambda}{\mu} \right. \\ \left. + 2i \int_0^\infty e^{-\mu(z+h)} \frac{\mu_E}{n^2 \mu + \mu_E} \frac{\lambda^3 d\lambda}{\mu} \right\}. \end{aligned}$$

Since μ is real for $\lambda > k$ (see p. 273), the integral over $k < \lambda < \infty$ in the first line does not contribute to the real part and we can pass to $z = h$, that is, to the point of the antenna, without encountering difficulties of convergence. Thus we obtain

$$(4) \quad \operatorname{Re}\{-iE\} = \int_0^k -i(1 + e^{-2\mu h}) \frac{\lambda^3 d\lambda}{|\mu|} + 2 \operatorname{Re}\left\{i \int_0^\infty e^{-2\mu h} \frac{\mu_E}{n^2 \mu + \mu_E} \frac{\lambda^3 d\lambda}{\mu} \right\}.$$

If we take the values of these integrals given by (36.13) to (36.17), substitute (4) in (1), and append the factor (36.22) in order to express the result in terms of our units, then we obtain the value of W from (36.23).

b) *Horizontal antenna.* For $\Pi = (\Pi_x, \Pi_z)$ and $\mathbf{E} = \mathbf{E}_x$ we obtain from (31.4):

$$(5) \quad E = k^2 \Pi_x + \frac{\partial^2 \Pi_x}{\partial x^2} + \frac{\partial^2 \Pi_z}{\partial x \partial z}.$$

Now according to (33.12) and (33.15) the x -dependence of Π_x is given by $I_0(\lambda r)$, and the x -dependence of Π_z is given by $\frac{x}{r} I_1(\lambda r)$. Hence

for small x, y we have

$$I_0(\lambda r) = 1 - \frac{\lambda^2}{4} (x^2 + y^2) + \cdots, \quad \frac{x}{r} I_1(\lambda r) = \frac{\lambda}{2} x + \cdots,$$

and for $r = 0$

$$(6) \quad \left(k^2 + \frac{\partial^2}{\partial x^2}\right) I_0 = \frac{1}{2} (2k^2 - \lambda^2), \quad \frac{\partial}{\partial x} \frac{x}{r} I_1 = \frac{\lambda}{2}.$$

These factors $\frac{1}{2}(2k^2 - \lambda^2)$ and $\lambda/2$ appear under the integral signs of the equations (33.12) and (33.15) in the computation of (5), where for the first two terms on the right side of (33.12) we have to use (31.4) and where in (33.15) we have to perform the differentiation with respect to z , in addition to the differentiation with respect to x (this yields a factor $-\mu$ under the integral sign). Thus, instead of (5) we obtain

$$(7) \quad E = \frac{1}{2} \int_0^\infty \frac{2k^2 - \lambda^2}{\mu} (e^{-\mu|z-h|} - e^{-\mu(z+h)}) \lambda^3 d\lambda \\ + \int_0^\infty e^{-\mu(z+h)} \left[\frac{2k^2 - \lambda^2}{\mu + \mu_E} + \frac{\lambda^2}{k^2} \frac{\mu(\mu - \mu_E)}{n^2 \mu + \mu_E} \right] \lambda d\lambda.$$

The first term in [] is due to the third term in (33.12), the second term is due to (33.15). If we form the common denominator of [] and observe that $\mu^2 - \mu_E^2 = k^2(n^2 - 1)$ (see p. 260), then we obtain

$$[] = \frac{\lambda^2 - 2\mu\mu_E}{n^2\mu + \mu_E}.$$

Hence for $z = 0$ the second line of (7) becomes identical with the integral for L in (36.17a). If in the first line of (7) we pass to $\text{Re}\{-iE\}$ according to the procedure of (3), then we can again replace the upper limit ∞ by k and carry out the integration as in (36.16a). If we then pass to our system of units we obtain exactly the expression in (36.23a).

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